MODULE - II

Number System :

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

$\mathbb{Q} = \{\frac{p}{q} \mid q \neq 0, \gcd(p,q) = 1\}$

$\mathbb{Q}^c$ = Set of irrational nos

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$

$\rightarrow \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}, i^2 = -1\}$

$Re(a + ib) = a$

$Im(a + ib) = b$

$z_1 = a_1 + ib_1, \quad z_2 = a_2 + ib_2$

$z_1 \pm z_2 = (a_1 \pm a_2) + i(b_1 \pm b_2)$

$z_1 z_2 =$

$\frac{z_1}{z_2} = \qquad z_2 \neq 0$

Complex numbers cannot be compared.
Only can be equal.

Conjugate of a complex number

If $z = a + ib$ then its conjugate will be $\bar{z} = a - ib$

Properties of conjugate no.

(i) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

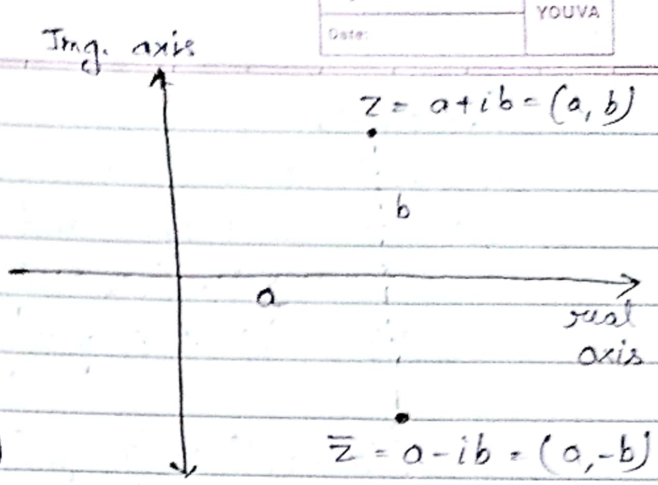
(ii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(iv) $z + \overline{z} = 2a$ (pure real no.)

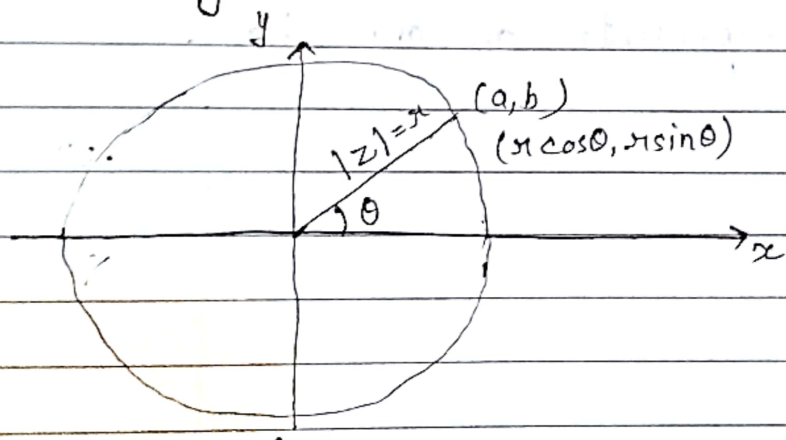
(v) $z - \overline{z} = 2ib$ (pure img.)

(vi) $z\overline{z} = a^2 + b^2 \geq 0$ (Pure real no.)



Polar form of a Complex no.

Euler's identity: $e^{i\theta} = \cos\theta + i\sin\theta$



$z = a + ib = r(\cos\theta) + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\boxed{z = re^{i\theta}}$ $\qquad \theta \in \operatorname{Arg}(z)$

$\arg(z) = \{\theta + 2n\pi \mid n \in \mathbb{Z}\}$

$\operatorname{Arg}(z) \in (-\pi, \pi]$ $\qquad$ Principal argument

Note :- (i) $|z - z_0| = r$ represents a circle of radius r with centre at z_0.

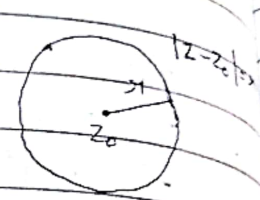
Let $z = x + iy$ and $z_0 = x_0 + iy_0$

$$|(x + iy) - (x_0 + iy_0)| = r$$

$$|(x - x_0) + i(y - y_0)| = r$$

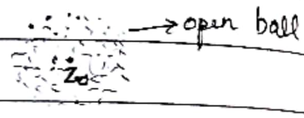
$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

$$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$$

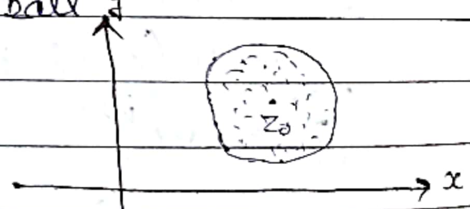


Ex. : $|z - (2 + 3i)| = 2$

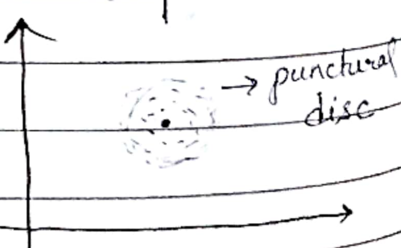
(ii) $|z - z_0| < r$ represents an open ball.



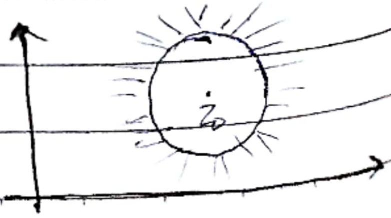
(iii) $|z - z_0| \leq r$ represents a closed ball



(iv) $\{z \mid |z - z_0| < r\} - \{z_0\}$ represents a punctured disc.



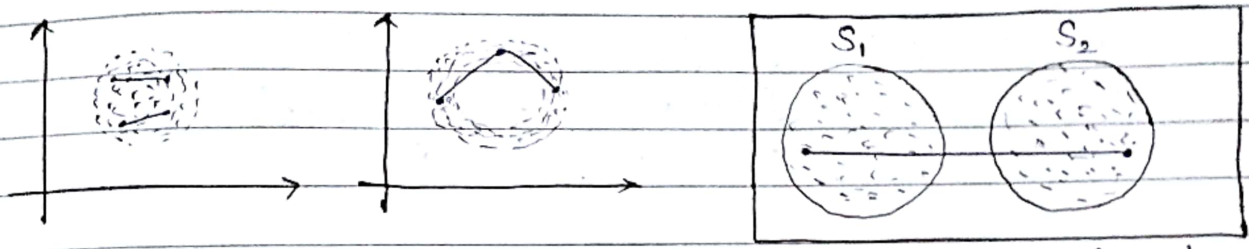
(v) $|z - z_0| > r$



$|z - z_0| \geq r$ (boundary included)

Connected Set

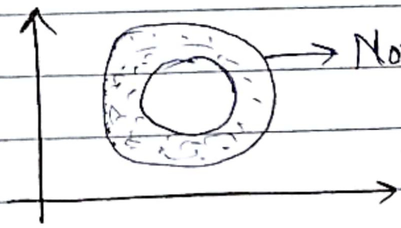
A set S is connected if any two points of the set can be joined by numbers of segments whose all points belong to the set.



$(1,2) \cup (3,4)$

DOMAIN

Open + Connected Set $\Rightarrow$ Domain (open region)



$\rightarrow$ Not a domain (open × connected ✓)

Ques. $|z - z_0| > r$ is a domain

Complex Functions

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

For ex: z^2, $\sin z$, $\cos z$, e^z, $\log z$, etc.

Note: Every complex function can be written as in the following form.

$$f(z) = \underbrace{u(x,y)}_{\text{real part}} + i\underbrace{v(x,y)}_{\text{imaginary part}}$$

(iii) $\log z = \log(x+iy) = \log(re^{i\theta})$

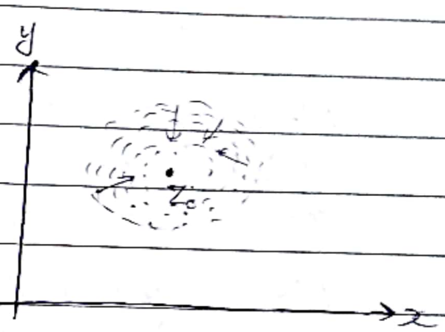
$$= \log r + i\theta$$

$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

LIMIT OF A FUNCTION

$$\lim_{z\to z_0} f(z) = l$$



Note: $\lim\limits_{z\to z_0} f(z)$ exists $\Leftrightarrow$ (iff) $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y)$ exists

Limit in Two Variables

① $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$

Along $y = x$

$$\lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Along $y = 2x$

$$\lim_{x\to 0} \frac{x^2}{x^2+4x^2} = \frac{1}{5}$$

General equation of line for if it goes

$y = mx$, $m \in R$

$\lim\limits_{x\to 0} \dfrac{x^2}{x^2+mx^2} = \dfrac{1}{1+m^2}$ (depends on m) [limit does not exist

Note : $N_\delta(x_0) = \{x \mid |x - x_0| < \delta\}$

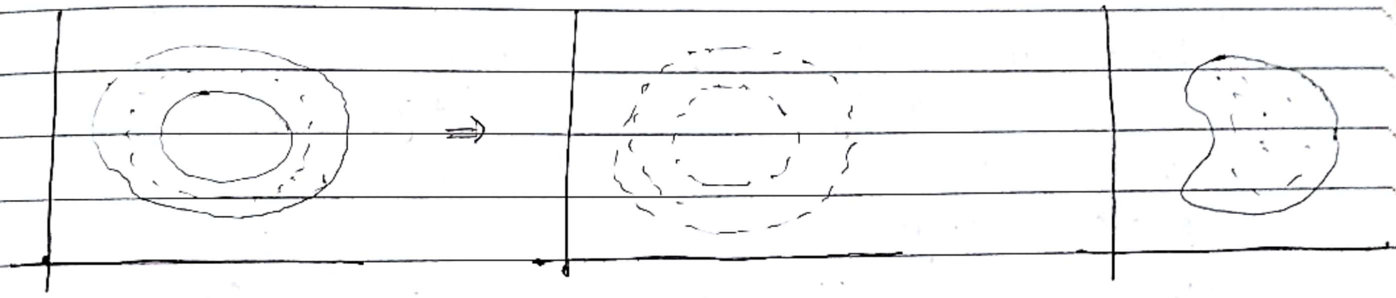
$= \{x \mid -\delta < x - x_0 < \delta\}$

$= \{x \mid x_0 - \delta < x < x_0 + \delta\}$

$= (x_0 - \delta, x_0 + \delta)$

In $\mathbb{C}$

$N_\delta(z_0) = \{z \mid |z - z_0| < \delta\}$



Open set $\Rightarrow$ $A^\circ = A$

If all the points of a set are interior then set is open.

Closed Set : A^c is open so the A is closed.

Limit Point : Every $N_\delta(x) \mid \{x\} \cap S \neq \phi$, then x is limit point of S.

$\Rightarrow$ All boundary points are limit points.

Eg. : $8 = \{2\} \cup [3/7]$

$S = \{\frac{1}{n} \mid n \in N\}$

LIMIT OF A FUNCTION

$$\lim_{z \to z_0} f(z) = l$$

Ex. ① $\lim_{z \to 0} \frac{Im(z)}{z}$

$$\frac{Im(z)}{z} = \frac{y}{(x+iy)} \frac{(x-iy)}{(x-iy)} = \frac{xy - iy^2}{x^2+y^2}$$

$$= \frac{xy}{x^2+y^2} - i\frac{y^2}{x^2+y^2}$$

Here $u = \frac{xy}{x^2+y^2}$, $v = \frac{-y^2}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Along $y = mx$, $\lim_{x \to 0} \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1+m^2}$ (depends on m)

Ex. ② $\lim_{z \to 0} \frac{z}{\bar{z}}$; $u = \frac{x^2-y^2}{x^2+y^2}$

Continuity at a point

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0)}$$

Note : ① $f(z) = u + iv$ is continuous at z_0

$\Rightarrow$ u and v are continuous at (x_0, y_0).

② Continuity $\Rightarrow$ $\lim_{z \to z_0} f(z)$ exists

$p \Rightarrow q$

then $\sim q \Rightarrow \sim p$

or $\lim_{z \to z_0} f(z)$ does not exist $\Rightarrow$ not continuous.

Ex.: $f(z) = \begin{cases} \dfrac{2\,Re(z)}{|z|} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

$$u = \frac{2x}{\sqrt{x^2+y^2}} \quad , \quad v = 0$$

This function is not continuous at $(0,0)$.

DIFFERENTIATION

A function $f: D \longrightarrow \mathbb{C}$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or $\displaystyle\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$ exists

where h is a complex number

Ex.: z^2, e^z, $\sin z$, $\cos z$, $\sin(z^2+1)$, etc. are differentiable function.

The functions which differentiable at every point of complex number is called entire function.

Note: Differentiability $\Rightarrow$ Continuity $\Rightarrow$ limit exists

Ques. Show that $f(z) \Rightarrow \bar{z}$ is a continuous function at $z_0 =$ but not differentiable.

Proof: $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{(x - iy)}{(x + iy)}$$

Along $y = mx$

$$\lim_{x \to 0} \frac{x(1 - im)}{x(1 + im)} = \frac{1 - im}{1 + im} \quad \text{(depend on } m\text{)}$$

$f(z) = \bar{z} = x - iy$

$$\lim_{(x,y) \to (0,0)} x = 0$$

$$\lim_{(x,y) \to (0,0)} (-y) = 0$$

This function is continuous but not differentiable.

Note : PARTIAL DERIVATIVES

If $u = u(x, y)$

$$\frac{\partial u}{\partial x}(x_0, y_0) = u_x(x_0, y_0) = \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = u_y(x_0, y_0) = \lim_{k \to 0} \frac{u(x_0, y_0 + k) - u(x_0, y_0)}{k}$$

CAUCHY - REIMANN EQUATIONS (CR-Equations)

Let $f(z) = u + iv$, the CR-equations are

$u_x = v_y$ &

$u_y = -v_x$

Q.1) $f(z) = z^2 + 4z = (x^2 - y^2 + 4x) + i(2xy + 4y)$

Here $u = x^2 - y^2 + 4x$, $v = 2xy + 4y$

$u_x = 2x + 4$ $\quad v_x = 2y$

$u_y = -2y$ $\quad v_y = 2x + 4$

CR equations are satisfied at every point of $\mathbb{C}$.

Note : (i) Necessary Condition

(ii) Sufficient Condition

u_x, u_y, v_x, v_y continuous + CR equations satisfied $\Rightarrow$ diff.

Ex. $f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

$$u(x,y) = \frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2}$$

$$v(x,y) = \frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^2}$$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^5}{h^4}}{h} = 1$$

$$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = 0$$

$v_x(0,0)$

Note (I) CR equations are necessary conditions for differentiability.
(II) u_x, u_y, v_x, v_y exist, continuous and satisfies CR equations then f is differentiable.

ANALYTIC FUNCTION

A function $f(z)$ is said to be analytic at a particular point z_0 if $\exists$ (there exists) at least one open ball, i.e. $|z - z_0| < \delta$ such that $f(z)$ is differentiable at every point of the open ball. (infinitely many points).

Ques. $f(z) = |z|^2$. Is this function analytic?

$= x^2 + y^2$

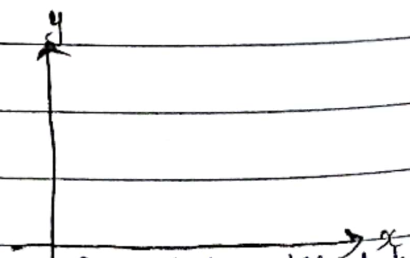
$u = x^2 + y^2$,	$v = 0$	$u_x = v_y$	$u_y = -v_x$
$u_x = 2x$	$v_x = 0$	$2x = 0$	$2y = 0$
$u_y = 2y$	$v_y = 0$	$x = 0$	$y = 0$

It may be differentiable at only one point $(0,0)$. So, it is not an analytic function.

Ques. $f(z) = x^2 + y^2 + i\,2xy$

$u = x^2 + y^2$	$v = 2xy$	$u_x = v_y$	$u_y = -v_x$
$u_x = 2x$	$v_x = 2y$	$2x = 2x$	$2y = -2y$
$u_y = 2y$	$v_y = 2x$		$y = 0$

This function may be differentiable at x-axis.
Not analytic.



Open ball will take space in $+y$ & $-y$ but it is differentiable

Date:

★ Entire function is always analytic.

Th : Sufficient Condition

If u_x, u_y, v_x, v_y exist, continuous and satisfy CR equations at each point of domain D, then $f(z) = u + iv$ is analytic at each point of D.

Also, $f'(z) = u_x + iv_x$

$f'(z) = v_y - iu_y$

Ques. $f(z) = 3x + y + i(3y - x)$

$u = 3x + y$	$v = 3y - x$	$u_x = v_y$	$u_y = -v_x$
$u_x = 3$	$v_x = -1$	$3 = 3$	$1 = -(-1)$
$u_y = 1$	$v_y = 3$	CR equation gets satisfied.	

Partial derivatives exists, CR equation satisfies, continuous. Hence, given function is an analytic function.

Construction of Analytic function

(by Milne-Thomson Method)

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz \quad \left[\text{If } u(x,y) \text{ is given}\right]$$

$$f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz \quad \left[\text{If } v(x,y) \text{ is given}\right]$$

Q.1) Construct the analytic function f for which $u = e^x \cos y$

$u_x = e^x \cos y \qquad u_y = -e^x \sin y$

$u_x(z,0) = e^z \qquad u_y = 0$

$f(z) = \int e^z\,dz - i \times 0 = e^z$

$= e^x \cos y + i e^x \sin y \quad \left[e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)\right.$

Q.2) $u = e^{x^2-y^2}\cos(2xy) + i e^{x^2-y^2}\sin(2xy)$

$u_x =$

Harmonic Function

A function $f: \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic in D if it has continuous partial derivatives of 1^{st} and 2^{nd} order and satisfies Laplace equation i.e.,

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{in } D}$$

Ex. (1) $f(x,y) = x + y$ is harmonic in $\mathbb{C}$

$f_{xx} = 0$, $f_{yy} = 0$

$f_{xx} + f_{yy} = 0$ in $\mathbb{C}$, Domain $\to$ Complete Complex System

(2) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

$$u = e^{-x}x\sin y - e^{-x}y\cos y$$

$$u_x = e^{-x}\sin y + x\sin y\, e^{-x} - e^{-x}y\cos y$$

$$= -2e^{-x}\sin y + e^{-x}x\sin y - e^{-x}y\cos y$$

$$u_y = e^{-x}x\cos y - \left[e^{-x}\left[y(-\sin y) + \cos y\right]\right]$$

$$= e^{-x}x\cos y + e^{-x}y\sin y - e^{-x}\cos y$$

$u_{yy} = e^{-x}x(-\sin y) + e^{-x}[y\cos y + \sin y] - e^{-x}(-\sin y)$
$= -e^{-x}x\sin y + e^{-x}y\cos y + e^{-x}\sin y + e^{-x}\sin y$
$= -e^{-x}x\sin y + e^{-x}y\cos y + 2e^{-x}\sin y$

Note:

$u_{xx} + u_{yy} = 0$, in $\mathbb{C}$

(3) $u = x^3 - 3xy^2$
$u_x = 3x^2 - 3y^2$ $u_y = -6xy$
$u_{xx} = 6x$ $u_{yy} = -6x$
$u_{xx} + u_{yy} = 0$

Properties of Analytic Function

(1) Harmonic Property: Let $f(z) = u + iv$ be an analytic function then, $u(x,y)$ and $v(x,y)$ are harmonic function in D.
i.e. $u_{xx} + u_{yy} = 0$
& $v_{xx} + v_{yy} = 0$ in D.

Ex. (i) $f(z) = e^z$
$= \underbrace{e^x\cos y}_{u} + i\underbrace{e^x\sin y}_{v}$

It satisfies CR equation & hence analytic so, also harmonic.

Proof: $u = e^x\cos y$ $v = e^x\sin y$
$u_x = \cos y\, e^x$ $v_x = e^x\sin y$
$u_{xx} = e^x\cos y$ $v_{xx} = e^x\sin y$
$u_y = -e^x\sin y$ $v_y = e^x\cos y$
$u_{yy} = -e^x\cos y$ $v_{yy} = -e^x\sin y$
$u_{xx} + u_{yy} = e^x\cos y - e^x\cos y = 0$
$v_{xx} + v_{yy} = e^x\sin y - e^x\sin y = 0$

Note: If u and v are harmonic functions then $f(z) = u + iv$ may or may not be analytic.

$f(z) = \bar{z}$ is not analytic

$= x - iy$ but harmonic

Harmonic v function

$u_{xx} + u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

Harmonic Conjugate

If u and v are harmonic functions and satisfies CR equations then u is Harmonic conjugate v.

Note: If $f(z) = u + iv$ is analytic
$\Leftrightarrow$ u and v are harmonic conjugate.

Ques. Show that
$u(x,y) = e^{-x}(x\sin y - y\cos y)$ is harmonic and find the harmonic conjugate $v(x,y)$ such that $f(z) = u + iv$ is analytic.

Sol: $u_{xx} + u_{yy} = 0$

$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + c$

$= ize^{-z} + c$

$= i(x+iy)e^{-x-iy} + c$

$= (-y + ix)e^{-x}.e^{-iy} + c$

$= (-y + ix)e^{-x}[\cos y - i\sin y]$

$= (-ye^{-x} + ixe^{-x})(\cos y - i\sin y)$

$= -ye^{-x}\cos y + iye^{-x}\sin y + ixe^{-x}\cos y + xe^{-x}\sin y$

$= -ye^{-x}\cos y + xe^{-x}\sin y + i(ye^{-x}\sin y + xe^{-x}\cos y)$

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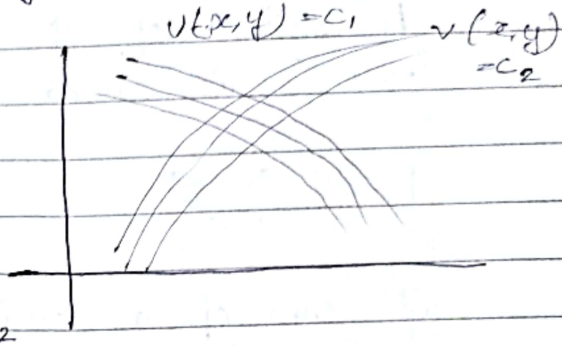
Ques - If $f(z) = u + iv$ is an analytic function then the curves of the families $u(x,y) = c_1$ and $v(x,y) = c_2$ cuts orthogonally.



Sol. Since $f(z) = u + iv$ is analytic
then $u_x = v_y$ & $u_y = -v_x$

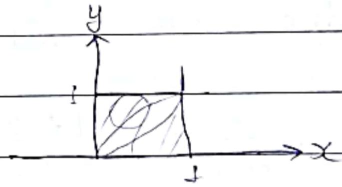
Given, $u(x,y) = c_1$, $v(x,y) = c_2$

$$m_1 = \frac{dy}{dx} = \frac{-u_x}{u_y} , \quad m_2 = \frac{dy}{dx} = \frac{-v_x}{v_y}$$

$$m_1 m_2 = \frac{u_x v_x}{u_y v_y} = \frac{-v_y u_y}{u_y v_y} = -1$$

COMPLEX INTEGRATION

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_C f(z)\,dz$$



Parametric representation of Path C

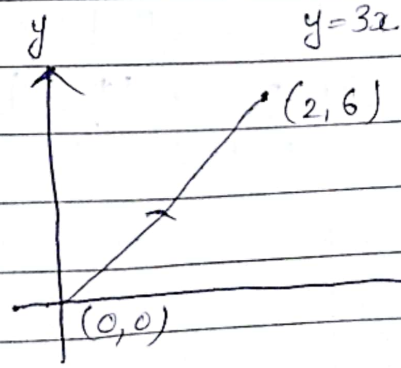
$$z(t) = x(t) + iy(t) , \quad t \in [a,b]$$

Ex. $z(t) = t + i3t$, $t \in [0,2]$

$x(t) = t$, $y(t) = 3t$

$$\frac{x}{y} = \frac{1}{3}$$

$\Rightarrow y = 3x$



Power Series

A series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$ is called a point z_0.

Ex. ① $\sum_{n=0}^{\infty} z^n$ ② $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ ③ $\sum_{n=0}^{\infty} n!\,(z-2)^n$

If sum of a series exist, then it is called convergent series. If $|r| < 1$, $-1 < r < 1$

If sum of a series does not exist, then it is called divergent.

Radius of convergence

Region of convergence

Ques. Find the sum $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$, $\sum_{n=1}^{\infty} 2^n$

$\sum \frac{1}{n!}$, $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots\ldots + \infty$

Note: $\sum_{n=1}^{\infty} z^n$ converges when $|z| < 1$. $R = 1$

where R is radius of convergence.

Ex.: ① $\sum_{n=1}^{\infty} (3+i)^n$ is a divergent series. $|3+i| > 1$

② $\sum_{n=1}^{\infty} \left(\frac{1+i}{2}\right)^n$ is a convergent series. $\left|\frac{1+i}{2}\right| < 1$

Radius of Convergence (R)

Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that

(i) If $|z| < R$, the series converges.

(ii) If $|z| > R$, the series diverges.

Then R is called the radius of convergence (ROC) of a power series.

Ex (1) For $\sum_{n=0}^{\infty} n! z^n$, $R = 0$ $\quad |z| < R$, $|z| = 0$

(2) $\sum_{n=0}^{\infty} z^n$, $R = 1$, $|z| < 1$

(3) $\sum \frac{z^n}{n}$, $R = 1$

(4) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$, $R = \infty$

Note: No conclusion about convergence can be drawn if $|z| = R$

Formula Used:

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Ex.: $\sum_{n=1}^{\infty} z^n$ $\qquad a_n = 1 \quad a_{n+1} = 1$

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{1} = 1$$

$\boxed{R = 1}$ $\qquad |z| < 1 \Rightarrow$ convergence

$|z| \ge 1 \Rightarrow$ divergent

Ques $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)!}$$

$$\frac{1}{R} = \lim_{n \to \infty} \frac{1}{n+1}$$

$$\frac{1}{R} = 0$$

$$\boxed{R = \infty}$$

Ques. Find the radius and circle of convergence of the following power series.

(i) $\sum_{n=0}^{\infty} (z+1)^n$ (ii) $\sum_{n=2}^{\infty} \frac{4^n}{n-1} (z - \pi i)^n$

(iii) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{3n}$ (iv) $\sum_{n=1}^{\infty} n!\, z^n$

Sol: (i) $a_n = 1$

$a_{n+1} = 1$

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\frac{1}{R} = \lim_{n \to \infty} \frac{1}{1} = 1$$

$R = 1$

$|z+1| < R \Rightarrow |z+1| < 1$

Circle of convergence : $|z+1| < 1$

(ii) $a_n = \frac{4^n}{n-1}$

$a_{n+1} = \frac{4^{n+1}}{n}$

$$\frac{1}{R} = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{4n-4}{n} = \lim_{n \to \infty} \left(4 - \frac{4}{n}\right) = 4$$

$$\boxed{R = \frac{1}{4}}$$

Circle of convergence, $|z-\pi i| < \frac{1}{4}$ $(0, \pi)$

(iii) $a_n = \dfrac{(-3)^{n+1}}{3n}$

$a_{n+1} = \dfrac{(-3)^{n+2}}{3(n+1)}$

$\dfrac{1}{R} = \lim\limits_{n\to\infty} \left| \dfrac{(-3)^{n+2}}{3(n+1)} \times \dfrac{3n}{(-3)^{n+1}} \right|$

$= \lim\limits_{n\to\infty} \dfrac{-3\times n}{(n+1)}$

$\dfrac{1}{R} = 3$

$R = \dfrac{1}{3}$

Circle of convergence : $|z| < \frac{1}{3}$

(iv) $a_n = n!$

$a_{n+1} = (n+1)!$

$\dfrac{1}{R} = \lim\limits_{n\to\infty} \left| \dfrac{(n+1)!}{n!} \right|$

$\dfrac{1}{R} = \infty$

$\boxed{R = 0}$

Circle of convergence : $|z| = 0$

Ques. $\lim\limits_{n\to\infty} n^{1/n} = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Let $y = \lim\limits_{n\to\infty} n^{1/n}$

$\log y = \lim\limits_{n\to\infty} \dfrac{\log n}{n} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

$$\log y = 0$$
$$y = e^0 = 1$$

Ques.

Ques. $\sum_{n=1}^{\infty} \frac{3^n}{4^n + 5^n} z^n$

$a_n = \frac{3^n}{4^n + 5^n}$ $\quad a_{n+1} = \frac{3^{(n+1)}}{4^{(n+1)} + 5^{(n+1)}}$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{3^{(n+1)}}{4^{(n+1)} + 5^{(n+1)}} \times \frac{4^n + 5^n}{3^n} = \lim_{n\to\infty} \frac{3(4^n + 5^n)}{4^{(n+1)} + 5^{(n+1)}}$$

$$\star \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{3 \cdot 5^n \left[\left(\frac{4}{5}\right)^n + 1\right]}{5^{n+1}\left[\left(\frac{4}{5}\right)^{n+1} + 1\right]}$$

$\lim_{n\to\infty} x^n = 0$ if $|x| < 1$

If $|x| > 1$ then ∞

$$\frac{1}{R} = \lim_{n\to\infty} \frac{3}{5} \left[\frac{\left(\frac{4}{5}\right)^n + 1}{\left(\frac{4}{5}\right)^{n+1} + 1}\right]$$

$$\frac{1}{R} = \frac{3}{5} \qquad \boxed{R = \frac{5}{3}}$$

Circle of convergence: $|z| < \frac{5}{3}$

① $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |a_n|^{1/n}$

$$\lim_{n\to\infty} n^{1/n} = 1$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

and $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}$

Ques. $\sum_{n=1}^{\infty} (n^{1/n}-1)^n z^n$

$a_n = (n^{1/n}-1)^n$

$(a_n)^{1/n} = n^{1/n}-1$

$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} n^{1/n} - 1$

$\frac{1}{R} = 0$ $\boxed{R=\infty}$

Ques. $\sum_{n=1}^{\infty} \left(\frac{n}{n-1}\right)^{n^2} z^n$

$a_n = \left(\frac{n}{n-1}\right)^{n^2}$

$(a_n)^{1/n} = \left(\frac{n}{n-1}\right)^n$

$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{\left(1-\frac{1}{n}\right)^n} = e$

$\boxed{R = \frac{1}{e}}$

Ques. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n} z^n$

$a_n = \frac{1}{(\log n)^n}$

$(a_n)^{1/n} = \frac{1}{\log n}$

$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$

$\boxed{R = \infty}$

Ques. $\sum_{n=1}^{\infty} \left(\frac{2n^3+34n+5}{n^3+7}\right)^n z^n$

$$a_n = \left(\frac{2n^3+34n+5}{n^3+7}\right)^n$$

$$(a_n)^{1/n} = \left(\frac{2n^3+34n+5}{n^3+7}\right)$$

$$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{2n^3+34n+5}{n^3+7}\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+34}{3n^2}$$

$$= \lim_{n\to\infty} \frac{12\cancel{n}+}{6\cancel{n}} = \lim_{n\to\infty} 2$$

$$\frac{1}{R} = \cancel{\lim}\, 2$$

$$\boxed{R = \frac{1}{2}}$$

Ques. $\sum \frac{(3n)!}{(n!)^3} z^n$

$$a_n = \frac{(3n)!}{(n!)^3}$$

Ques. $\sum_{n=1}^{\infty} \frac{[\ln(n^2)]^n}{n^{2n}} z^n$

$$a_n = \frac{[\ln(n^2)]^n}{n^{2n}} \quad (a_n)^{1/n} = \frac{\ln(n^2)}{n^2}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\ln(n^2)}{n^2}$$

Ques. $\frac{2}{3}z + \left(\frac{3}{5}\right)^2 z^2 + \left(\frac{4}{7}\right)^3 z^3 + \ldots$

$\sum_{n=1} \left(\frac{n+1}{2n+1}\right)^n z^n \Rightarrow a_n = \left(\frac{n+1}{2n+1}\right)^n$

$(a_n)^{1/n} = \left(\frac{n+1}{2n+1}\right)$

$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

$\boxed{R = 2}$

Ques. $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n \quad = \frac{1}{e}$

$a_n = \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!}$

$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!} \times \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

$\frac{1}{R} = e$

$\boxed{R = \frac{1}{e}}$

Ques. $\sum_{n=0}^{\infty} (3+4i)^n z^n$

$a_n = (3+4i)^n \qquad (a_n)^{1/n} = 3+4i$

$\frac{1}{R} = \lim_{n\to\infty} (3+4i) = |3+4i| = 5$

$\boxed{R = \frac{1}{5}}$

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TAYLOR'S SERIES

Suppose that $f(z)$ is analytic throughout a disk, i.e. $|z-z_0|<R$. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n , \quad \{|z-z_0|<R\}$$

where $a_n = \dfrac{f^n(z_0)}{n!}$ ——— (1)

That is, series (1) converges to $f(z)$ when z lies in $|z-z_0|<R$.

Note: If $z_0 = 0$ then (1) will become Maclaurin's series, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \quad |z|<R$$

$$a_n = \frac{f^n(0)}{n!}$$

Ex.: $f(z) = e^z$, $z_0 = 0$

$f^n(z) = e^z$ $\qquad a_n = \dfrac{f^n(0)}{n!} = \dfrac{1}{n!}$

$f^n(0) = 1$ $\qquad f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad \forall\, z \in \mathbb{C}$$

Ques. $1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dots = e$

$$1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots = e^2$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

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Ques. $f(z) = e^z$, $z_0 = 7$

$$e^z = e^{z-7+7} = e^{z-7}e^7$$

$$= e^7\left(1 + (z-7) + \frac{(z-7)^2}{2!} + \ldots\right)$$

Ques. $f(z) = z^2e^z$, $z_0 = 0$

$$= z^2\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right)$$

$$z^2e^z = z^2 + z^3 + \frac{z^4}{2!} + \frac{z^5}{3!} + \ldots$$

$$\boxed{z^2e^z = \sum_{n=2}^{\infty} \frac{z^n}{(n-2)!}}$$

Ques. $f(z) = \sin z$, $z_0 = 0$

$$f^n(z) = \sin\left(z + \frac{n\pi}{2}\right)$$

$$f^n(0) = \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \text{ is even} \\ -1, & n = 3, 7, 11 \\ 1, & n = 1, 5, 9 \end{cases}$$

$$a_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{n!}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n!} z^n$$

$$= 0 + z + 0 + (-)\frac{1}{3!}z^3 + \frac{z^5}{5!} - \frac{z^7}{7!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$$

Similarly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots$$

Ques. $f(z) = \sin z$, $z_0 = \frac{\pi}{2}$

$$\sin\left(z - \frac{\pi}{2} + \frac{\pi}{2}\right) = \cos\left(z - \frac{\pi}{2}\right)$$

$$\sin z = 1 - \frac{\left(z - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(z - \frac{\pi}{2}\right)^4}{4!} \ldots$$

Ques. $f(z) = \frac{1}{1-z}$, $z_0 = 0$

$$y = \frac{1}{ax+b} \quad , \quad y^n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

$$f^n(z) = \frac{(-1)^n n! (-1)^n}{(1-z)^{n+1}}$$

$$f^n(0) = n!$$

$$a_n = \frac{f^n(0)}{n!} = \frac{n!}{n!} = 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots \qquad |z| < 1$$

Similarly,

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - z^3 + \ldots$$

$$|z| < 1$$

Ques. $f(z) = \log(1+z)$, $z_0 = 0$

$$f^n(z) = \frac{(-1)^{n-1}(n-1)!}{(1+z)^n}$$

$$f^n(0) = (-1)^{n-1}(n-1)!$$

$$a_n = \frac{f^n(0)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!} = \frac{(-1)^{n-1}}{n}$$

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \quad , \quad |z| < 1$$

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots\cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots\cdots \qquad |z| < 1$$

$$\frac{1}{1+z} = 1 - z + z^2 \cdots\cdots \qquad |z| < 1$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \cdots\cdots$$

$$\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3}$$

Ques. Expand the Maclaurin's series of the function $f(z) = \sin h z$

(11)

Sol^n.: $f(z) = -i\sin iz$

$$= -i\left[iz + \frac{iz^3}{3!} + \frac{iz^5}{5!} + \dots\right]$$

$\cos iz = \cosh z$

$-i\sin iz = \sin h z$

$$\sin h z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)!} z^{(2n-1)}$$

Ques. Expand the function $\frac{\sin z}{z-\pi}$ about $z=\pi$.

$$\frac{1}{z-\pi}\sin[z-\pi+\pi] = \frac{-1}{z-\pi}\sin(z-\pi)$$

$$= \frac{-1}{z-\pi}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} + \dots\right]$$

$$= -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(z-\pi)^{2n}}{(2n+1)!}$$

Ques. Expand $\frac{1}{z-2}$ in the region.

(i) $|z| < 1$

(ii) $|z-1| < 1$

$$\frac{1}{z-2} = \frac{1}{-2+z} = \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$\left|\frac{z}{2}\right| < 1$

$$= \frac{-1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \dots\right]$$

$$= \frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

(ii) $\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{-1}{1-(z-1)} = -1\left[1+(z-1)+(z-1)^2+\dots\right]$

Ques. Expand $\frac{1}{(z+1)(z+3)}$ in the region.

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(1+z)} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right]$$

$$= \frac{1}{2}\left[1-z+z^2-z^3+\dots - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}\dots\right)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{-z}{3}\right)^n\right]$$

Ques. Expand $\frac{1}{1-z}$ in $|z-i| < \sqrt{2}$ $\Rightarrow \frac{|z-i|}{\sqrt{2}} < 1$

$$f(z) = \frac{1}{1-z}$$ $\Rightarrow \frac{|z-i|}{\sqrt{2}} < 1$

$\left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{\sqrt{2}}$

$$\Rightarrow \frac{1}{1-i-z+i} \Rightarrow \frac{1}{(1-i)\left[1-\frac{z-i}{1-i}\right]}$$

$$= \frac{1}{(1-i)}\left[1+\frac{z-i}{1-i}+\left(\frac{z-i}{1-i}\right)^2+\dots\right]$$

Note: ① $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots$

$$= 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots$$

② $\frac{e^z}{z^2} = \frac{1}{z^2}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right]$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \dots$$

③ $\frac{\sin z^2}{z^4} = \frac{1}{z^4}\left[z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} \dots\right]$

$$= \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} \dots$$

④ $z^3 \cos\left(\frac{1}{z}\right) = z^3\left[1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} \dots\right]$

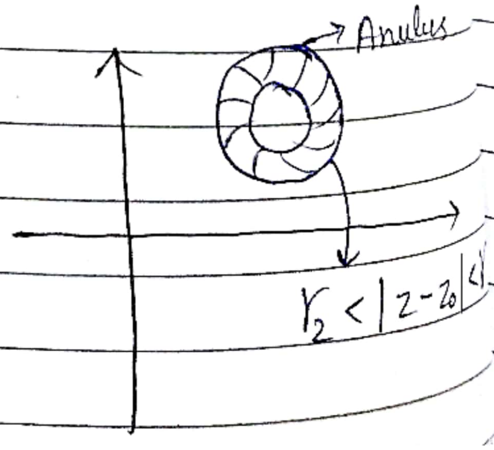
$$= z^3 - \frac{z}{2!} + \frac{1}{z.4!} + \dots$$

LAURENT'S SERIES

If $f(x)$ is analytic in $r_2 < |z - z_0| < r_1$

Then,

$$\boxed{f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{Analytic Part}} + \underbrace{\sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\text{Principal part (p.p)}}}$$



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Ques. $f(z) = \dfrac{1}{(z-1)(z-2)}$

(i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

(i) $f(z) = \dfrac{1}{(z-2)} - \dfrac{1}{(z-1)} = \dfrac{1}{2\left(1-\frac{z}{2}\right)} + \dfrac{1}{(1-z)}$

$= \dfrac{1}{2}\left[1 + \dfrac{z}{2} + \left(\dfrac{z}{2}\right)^2 + \dots\right] + 1 + z + z^2 + \dots$

$= -\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n$

(ii) $1 < |z| < 2$ or $|z| > 1$ and $|z| < 2$

$\Rightarrow \dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{2} < 1$

$f(z) = \dfrac{1}{z-2} - \dfrac{1}{z-1} = \dfrac{1}{-2\left(1-\frac{z}{2}\right)} - \dfrac{1}{z\left(1-\frac{1}{z}\right)}$

$= -\dfrac{1}{2}\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n - \dfrac{1}{z}\sum_{n=0}^{\infty}\left(\dfrac{1}{z}\right.$

(iii) $|z| > 2$ $\dfrac{|z|}{2} > 1 \Rightarrow \dfrac{2}{|z|} < 1 \Rightarrow \dfrac{1}{|z|} < \dfrac{1}{2} < 1$

$f(z) = \dfrac{1}{(z-2)} - \dfrac{1}{(z-1)} = \dfrac{1}{z\left(1-\frac{2}{z}\right)} - \dfrac{1}{z\left(1-\frac{1}{z}\right)}$

$= \dfrac{1}{z}\sum_{n=0}^{\infty}\left(\dfrac{2}{z}\right)^n - \dfrac{1}{z}\sum_{n=0}^{\infty}\left(\dfrac{1}{z}\right)^n$

Ques. Find the Laurent's series (i) $|z+1| < 1$ (ii) $1 < |z+1| <$

$f(z) = \dfrac{7z-2}{z^3 - z^2 - 2z}$ (iii) $|z+1| > 3$

Sol^n: $\dfrac{7z-2}{z(z+1)(z+2)} = \dfrac{1}{z} - \dfrac{3}{z+1} - \dfrac{2}{z+2}$

ues

$\rightarrow$

$-z_0| < r$

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$$= \frac{1}{z+1-1} - \frac{3}{z+1} - \frac{2}{z+1-3}$$

$$= \frac{1}{-1[1-(z+1)]} - \frac{3}{z+1} + \frac{2}{3\left(1-\frac{z+1}{3}\right)}$$

$$= -\sum_{n=0}^{\infty}(z+1)^n - 3(z+1)^{-1} + \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

Vector of $z^2+1 = (z+i)(z-i)$

(ii) $|z+1| > 1$ and $|z+1| < 3$

$$\frac{1}{|z+1|} < 1 \qquad \frac{|z+1|}{3} < 1$$

$$\frac{1}{(z+1)\left(1-\frac{1}{z+1}\right]} - \frac{3}{z+1} + \frac{2}{3}\frac{1}{\left[1-\frac{z+1}{3}\right]}$$

(iii) $|z+1| > 3 \Rightarrow \frac{1}{|z+1|} < \frac{1}{3} < 1$

$$\frac{3}{|z+1|} < 1$$

SINGULARITY OR SINGULAR PAINT

A point z_0 is singular point of $f(z)$ if
(i) $f(z)$ is not analytic at z_0, but
(ii) there is some neighbourhood of z_0 at which $f(z)$ is analytic.

Ex. (i) $\frac{1}{z-1}$, $z = 1$ is singular point

(ii) $f(z) = \bar{z}$ has no singular point.

Singular Point

- Isolated Singular Point
  - (i) Removable Singular Point (No P.P i.e. $b_n = 0$)
  - (ii) Essential (Infinite no. of terms of P.P.)
  - (iii) Pole (finite no. of terms of P.P.)
- Non-isolated Singular Point

Ex. ii) $f(z) = \frac{\sin z}{z}$, $z = 0$ is singular point.

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$\lim_{z \to 0} f(z) = \text{finite}$, $\Rightarrow$ $z = 0$ is removable singular poi

or $\lim_{z \to a} f(z) = \text{finite}$

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Ex. (ii) $f(z) = e^{1/z}$, $z=0$ is singular point

$$= 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\cdots$$

$$\lim_{z\to 0} f(z) = \infty$$

or $\displaystyle\lim_{z\to a} (z-a)^m$

Ex. (iii) $f(z) = \frac{\sin z}{z^3} = \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\right]$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots\cdots$$

$\Rightarrow z=0$ is a pole of order 2.

Simple order is a pole of order 1.

$\displaystyle\lim_{z\to 0} z^2 f(z)$ = finite and non-zero

or $\displaystyle\lim_{z\to a} (z-a)^m f(z)$ = finite and non-zero

Ex. $f(z) = \frac{e^z}{z^6}$, $z=0$ is singular point

$$= \frac{1}{z^6}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{z^6} + \frac{1}{z^5} + \frac{1}{2!\,z^4} + \cdots + \frac{1}{5!\,z} + \frac{1}{6!} + \frac{z}{7!} + $$

Ex. (i) $f(z) = \frac{1}{\sin z}$, $z = n\pi$ is singular point

$$= \frac{1}{z\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right]}$$

(ii) $f(z) = \sin\left(\frac{1}{z-a}\right)$, $z = a$ is singular point

(iii) $f(z) = \frac{1}{(z-1)(z-2)}$, $z = 1, 2$ are singular point.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n}$$

Coefficient of $\frac{1}{z-z_0}$ is b_1

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots$$

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2! z} + \frac{1}{3!} + \dots$$

$$e^z = 1 + z + \frac{1}{2!} z^2$$

$$\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)$$

Ques. $f(z) = \frac{1}{(z-1)(z-2)}$, $z = 1, 2$ are singular points

At $z = 1$ (simple pole)

Ques. $f(z) = \frac{1}{(1-z)(z-2)(z-3)}$

$z = 1, 2, 3$ are singular point

At $z = 1$ (simple pole)

$$\lim_{z \to 1} (z-1) \frac{-1}{(1-z)(z-2)(z-3)} = \frac{-1}{-1 \times -2} = \frac{-1}{2}$$

Removable $\Rightarrow$ $b_n = 0$

Essential $\Rightarrow$ ∞ no. of terms of p.p.

Pole $\Rightarrow$ finite no. of terms of p.p.

Residues at finite poles

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{z-z_0} + a_0 + a_0(z-z_0) + \dots$$

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \dots + b_1 (z-z_0)^{m-1} + (+ve) \text{ power of } (z-z_0)$$

$$\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = b_1 (m-1)! + (+ve) \text{———}$$

$$\lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = b_1$$

Ex. $f(z) = \frac{1}{(z-1)^2 (z-2)^3}$

$z = 1, 2$ are singular point

At $z=1$ (Pole of order 2)

$$\lim_{z\to 1} \frac{1}{1!} \frac{d}{dz} (z-1)^2 \frac{1}{(z-1)^2(z-2)^3}$$

Ques. $f(z) = \frac{\sin h(z)}{z^5}$, find the residue at $z=0$.

$$= \frac{\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots\right)}{z^5}$$

$$= \frac{1}{z^4} + \frac{1}{z^2 3!} + \frac{1}{5!} + \frac{z^2}{7!} + \ldots$$

$\text{Res}(f, 0) = 0$

Ques. $f(z) = \frac{\sin h(z)}{z^5} \cdot e^z$, $z=0$

$$\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots\right)\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right)$$

$$\left(\frac{1}{3!} + \frac{1}{3!}\right) = \frac{1}{3}$$

$$f(z) = \frac{\cot z}{z^2}$$

$$= \frac{\cos z}{z^2 \sin z}$$

$$= \frac{\cos z}{z^3\left(1 - \frac{z^2}{2!}\right)}$$

Ques. $f(z) = \frac{1}{z(z^2+1)(z-2)^2}$

$z = 0, \pm i, 2$

$z = i$ (simple time)

Que. $f(z) = \frac{z^2+4}{\quad}$ (i)

CAUCHY'S RESIDUE THEOREM

Let $f(z)$ is analytic within and on closed contour C except at finite no. of poles $z_1, z_2, \ldots z_n$ inside C then

$$\int_C f(z)\,dz = 2\pi i\,(\text{Sum of residues})$$

Ex.: $f(z) = \frac{\sinh z}{z^5} e^z$

$$\operatorname{Res}(f, 0) = \frac{1}{3!} + \frac{1}{3!} = \frac{1}{3}$$

$$\int_C \frac{\sinh z}{z^5} e^z\,dz = 2\pi i \times \frac{1}{3} \qquad C: |z| = 1$$

C.R.T. $\int_C f(z)\,dz = 2\pi i \times \text{sum of residues}$

(i) Cauchy's Theorem
(ii) Cauchy Integral Formula
(iii) Cauchy Residue Theorem

Que

ii) $\int_C \sin z \, dz = 0$

$\int_C e^z dz = 0$

$\int_C (z^2 + 2z + 3) \, dz = 0$

$\int_C \frac{1}{z - \frac{1}{2}} dz$ $C : |z| = 1$ $\int_C \frac{1}{z-1} dz$

$\frac{1}{2}$ is singular point (simple pole)

$\int_C \frac{1}{z-2} dz = 0$ since it is out of range.

Ques. $\int_C \frac{\sin z}{(z-\pi)^3} dz$ $C : |z| = 4$

[Cauchy Residue Theorem] $\frac{1}{z - z_0}$

$$\frac{\sin z}{(z-\pi)^3} = \frac{\sin(z - \pi + \pi)}{(z-\pi)^3}$$

$$= \frac{-\sin(z-\pi)}{(z-\pi)^3}$$

$$= \frac{-\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} + \dots\right]}{(z-\pi)^3}$$

$$\underset{z=\pi}{\text{Res}} f = 0$$

Ques. $\int_C \frac{3z^3 + 4z^2 - 5z + 1}{(z-2i)(z^3 - z)}$ $C : |z| = 3$

(simple pole not series)

At $z = 2i$

$$R_1 = \lim_{z \to 2i} \cancel{(z-2i)} \times \frac{3z^3 + 4z^2 - 5z + 1}{\cancel{(z-2i)}(z^3 - z)} = -i/2$$

At $z = 0$

$$R_2 = 3\left(\frac{1 + 2i}{10}\right), \quad R_3 = \frac{7(2i - 1)}{10}, \quad R_4 = \frac{34 - 15i}{10}$$

Ans = $6\pi i$

Ques. $\int_{|z|=1} z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i \times -\frac{1}{6} = \frac{-\pi i}{3}$

Residue at $z=0$

$$z^2\left(\frac{1}{z} - \frac{1}{z^3 \cdot 3!} + \cdots\cdots\right)$$

$$\underset{z=0}{\text{Res}}\ f = \frac{-1}{6}$$

Ques. $\int_C \frac{z}{z^2+2z-3} dz$ (a) $|z|=1$ (b) $|z|=2$

$$\int_C \frac{z}{(z+3)(z-1)} dz = \frac{\pi i}{2}$$

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Residue at ∞

$f(z) = e^z$, find $\underset{z=\infty}{\text{res}} f$ $z = \infty$ $\underset{z=\infty}{\text{Res}}\ e^z = 1$

$g(z) = e^{1/z}$, $z = 0$

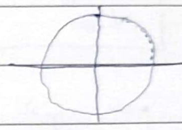
Ques. $f(z) = z$, $z = \infty$

$\underset{z=\infty}{\text{Res}}\ z = 1$

Ques. $f(z) = z^2 + z - 2$, $z = \infty$

$= \frac{1}{z^2} + \frac{1}{z} - 2$

Evaluation of $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$



$|z| = 1$

Let $e^{i\theta} = z$ $\Rightarrow i e^{i\theta} d\theta = dz$

$e^{-i\theta} = \frac{1}{z}$ $\boxed{d\theta = \frac{1}{iz} dz}$

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

$$\boxed{\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)}$$

$$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)}$$

Ques. $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{1}{i}\int_C \frac{\frac{dz}{z}}{5 + \frac{4\times}{2}\left(z + \frac{1}{z}\right)} = \frac{1}{i}\int_C \frac{dz}{2z^2 + 5z + 2}$ $|z| =$

$|z| = 1$

$= \frac{1}{i}\int_C \frac{dz}{(2z+1)(z+2)}$

$= \frac{2\pi}{3}$

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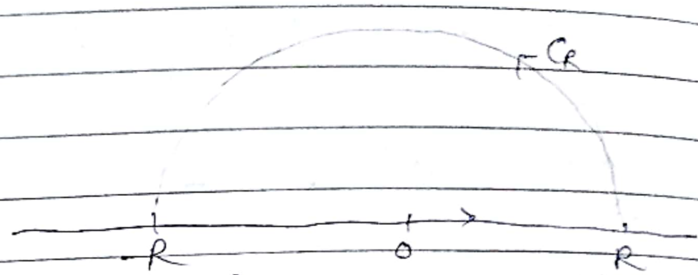
Ques. $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$



$C = [-R, R] + C_R$

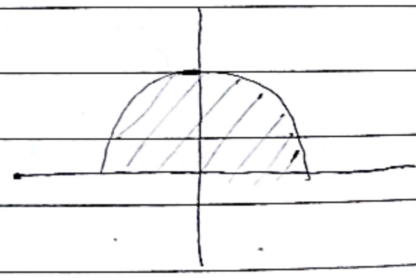
$[-R, R] = C - C_R$

$$\int_{-R}^{R} f(x)\,dx = \int_C f(z)\,dz - \cancel{\int_{C_R} f(z)\,dz} \to 0$$

Taking limit $R \to \infty$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz}$$, C is upper half of circle

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\,dx = \int_C \frac{z^2}{(1+z^2)^3}\,dz$$



$$= \int_C \frac{z^2}{\underset{\times}{(z+i)^3}\,\underline{(z-i)^3}}\,dz$$

$$= \frac{\pi}{4}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\,dx$$

Neighbourhood of a point in $\mathbb{R}$

Let $A \subseteq \mathbb{R}$ is the neighbourhood of x if $\exists$ (there exists) an open interval I such that
$$\boxed{x \in I \subset A}$$

Ex.: (i) Is 10 neighbourhood of $\mathbb{N}$?
(ii) Is 2 neighbourhood of $\{1, 2, 3, 4\}$?
(iii) Is 3 neighbourhood of $\mathbb{Z}$?
(iv) Is 3 neighbourhood of $[1, 2]$?
$3 \in ((3-\delta), (3+\delta)) \not\subset [1, 2]$

1 is nbd of $[1, 2]$
$1 \in (1-\delta, 1+\delta) \not\subset [1, 2]$
$1 \in (0.9, 1.1)$

Interior points: Points except boundary points

$N_\delta(z_0) = \{z \mid |z - z_0| < \delta\}$ = open ball of radius δ centre at z_0

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Sol. Show that $f(z) = xy + iy$ is continuous everywhere but not differentiable anywhere.

Ques. Test the analyticity of the function $f(z) = \sin z$.

Ques. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not?

Since $u = xy$ is continuous and v is continuous hence, $f(z)$ is continuous.

$\sin z = \sin x + i\cos y$

$\sin iy = i\sin hx$

$u = \sin x \cos hy$

$v = \cos x \sin hy$

$\sin hx = \frac{e^{x} - e^{-x}}{2}$

$u_x =$

u_y

v_x

v_y

Que. Find the value of a, b, c if $f(z) = (x + ay) + i(bx + cy)$ is analytic

$u = x + ay$

$v = bx + cy$

$u_x = 1 + 0$

$u_y = a$

$v_x = b$

$v_y = c$

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

$c = 1$

$a = -b$

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Ques. Test the analyticity of the function

$f(z) = \cos hz$

$= \cos iz$

$= \cos i(x+iy)$

$= \cos(ix - y)$

$= \cos ix \cos y + \sin ix \sin y$

$= \cosh x \cos y + i \sinh x \sin y$

$\frac{e^x + e^{-x}}{2}$

$u = \cosh x \cos y$ $\quad v = \sinh x \sin y$

$u_x = \cos y \frac{e^x + e^{-x}}{2}$ $\quad v_x = \sin y \frac{e^x + e^{-x}}{2}$

$u_y = -\cosh x \sin y$ $\quad v_y = \sinh x \cos y$

Ques. (1) Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic

$u_{xx} + u_{yy} = 0$

$u_{xx} =$

$\boxed{m = 2}$

Ques. (2) Find the constants a, b, c if $f(z) = (x + ay) + i(bx + cy)$ is analytic.

Ques.

Ques.

Ques

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(5) Determine whether the function $f(z) = (x+ay) + i(bx+cy)$ is analytic

(6) Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin. Although CR equations are satisfied at that point

Ques (7) Determine the analytic function $f(z) = u + iv$ if

$u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}(\cos 2y) + 2e^{2x}(x\cos 2y - y\sin 2y)$

$u_y = e^{2x}(x(-2\sin 2y) - (2y\cos 2y + \sin 2y))$

$u_x[z, 0] = e^{2z} + 2e^{2z}(z) = e^{2z}(1+2z)$

$u_y[z, 0] = e^{2z}(0 - 0) = 0$

Ques (8) Find c_1 & c_2 such that $f(z) = x^2 + c_1 y^2 - 2xy + i(c_2 x^2 - y^2 + 2xy)$ is analytic. Also find $f'($

Q.6) $f(z) = \sqrt{|xy|}$, $z = 0$

$$u_x(a,b) = \lim_{h\to 0} \frac{u(a+h,b) - u(a,b)}{h}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$u_y(0,0) = 0$

$v_x(0,0) = 0$

$v_y(0,0) = 0$

$\lim_{z\to 0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, then differentiable otherwise not

$\lim_{z\to 0} \frac{\sqrt{|xy|}}{x+iy}$ Along $y = mx$

$$\lim_{x\to 0} \frac{x\sqrt{|m|}}{x(1+mi)}$$

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Ques. $\sum \frac{(n!)^2}{(2n)!} z^n$

$a_n = \frac{(n!)^2}{(2n)!}$ $a_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!}$

$\frac{1}{R} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!} \times \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$

$= \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+2n+4n+2} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}$

$= \lim_{n\to\infty} \frac{2n+2}{8n+6} = \lim_{n\to\infty} \frac{2}{8}$

$\frac{1}{R} = \frac{1}{4}$

$\boxed{R=4}$

Ques. $\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n}$ $2n \to n$ $n \to \frac{n}{2}$

$a_n = \frac{1}{2^n}$

$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$

$R = 2$ for z^{2n}

$R = \sqrt{2}$ for z^n

Ques. $\sum \frac{z^n}{3^n(n^2+1)} =$ $a_n = \frac{1}{3^n(n^2+1)}$ $(a_n)^{1/n} =$

$\frac{1}{R} = \lim_{n\to\infty} \frac{1}{3(n^2+1)^{1/n}}$

$$a_{n+1} = \frac{1}{3^{n+1}\left((n+1)^2+1\right)}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{3^n (n^2+1)}{3^{n+1}\left((n+1)^2+1\right)} = \lim_{n\to\infty} \frac{1}{3} \frac{(n^2+1)}{\left((n+1)^2+1\right)}$$

$$= \lim_{n\to\infty} \frac{1}{3} \left(\frac{2n}{2(n+1)}\right)$$

$$= \frac{1}{3}$$

$$\boxed{R=3}$$

(4) $$\sum \frac{(n+1)}{(n+2)(n+3)} z^n$$

$$a_n = \frac{(n+1)}{(n+2)(n+3)}$$

$$a_{n+1} = \frac{(n+2)}{(n+3)(n+4)}$$

$$\frac{1}{R} = \lim_{n\to\infty} \frac{(n+2)}{\cancel{(n+3)}(n+4)} \times \frac{(n+2)\cancel{(n+3)}}{(n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2+4n+4}{n^2+5n+4} = \lim_{n\to\infty} \frac{2n+4}{2n+5}$$

$$= \lim_{n\to\infty} = \frac{2}{2}$$

$$= 1$$

$$\boxed{R=1}$$

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(1) Show that $\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$, $|z| < 4$.

(2) Expand $\frac{z}{(z+1)(z+2)}$ in powers of $(z-2)$

(3) Expand e^{2z} about $z_0 = 2i$.

(4) Expand $\frac{2z^3+1}{z^2+z}$ about $z = i$.

Solⁿ: 1) $\frac{1}{4z - z^2} = \frac{1}{z(4-z)} = \frac{1}{4z} + \frac{1}{4(4-z)}$ $|z| < 4$

$\frac{4(4-z)+4}{4z \times 4(4-z)}$

$\frac{4}{4z(4-z)}$

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$

$\frac{1}{z(4-z)} = \frac{A}{z} + \frac{B}{(4-z)}$

$1 = A(4-z) + Bz$

$1 = 4A - Az + Bz$

$1 = 4A + (B-A)z$

$4A = 1$ $A = 1/4$

$B - A = 0$ $\boxed{B = 1/4}$

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Ques. $\dfrac{1}{z(z-3)}$, (i) $0 < |z| < 3$
(ii) $0 < |z-3| < 3$

$$\frac{1}{z(z-3)} = \frac{A}{z} + \frac{B}{z-3} = \frac{-1}{3z} + \frac{1}{3(z-3)}$$

$$1 = Az - 3A + Bz$$
$$= (A+B)z - 3A$$

$A + B = 0$ $\boxed{B = \frac{1}{3}}$

$-3A = 1$

$\boxed{A = -\frac{1}{3}}$

Ques. $f(z) = \tan^{-1} z$, $z_0 = 0$

$$f'(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots\ldots$$

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots\ldots$$

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