Galois Cohomology of Algebraic Groups

Ayushi Tsydendorzhiev

October 17, 2024

Contents

1 Introduction 2
1.1 Galois group actions 2
1.2 The fixed point functor and exact sequences 3
2 Preliminaries from algebraic number theory. 6
2.1 Number fields 6
2.2 Integrality in number fields 7
2.3 The arithmetic of algebraic intgers 8

1 Introduction

1.1 Galois group actions

Lecture 1, 10.10.2024

Let L/K be a Galois extension and $G = \mathrm{Gal}(L/K)$ its Galois group. The Galois group G acts on L via field automorphisms:

- Action on the field extension L: For $\mathbb{Q}(\sqrt{2})$ its Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ acts either by identity or by sending $\sqrt{2}$ to $-\sqrt{2}$.
- Action on the dual of the field extension L^*: For $\mathbb{Q}(\sqrt{2})^*$ its Galois group acts on $f(x_1, x_2) = x_1 \cdot 1 + x_2 \cdot \sqrt{2}$ either by identity or by sending f to $f' = x_1 \cdot 1 - x_2 \cdot \sqrt{2}$.
- Action on the group of nth roots of unity $\mu_n(L)$:
  - In $\mathbb{Q}(\sqrt{2})$, the nth roots of unity consist of $\{-1, 1\}$ if n is even and $\{1\}$ if n is odd. Both automorphisms in $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ leave $\mu_n(\mathbb{Q})$ fixed, so this tells us that they all belong to the base field (are rational, in this case).
  - A more interesting example is the nth cyclotomic field $\mathbb{Q}(\zeta_n)$.In this field $\mu_n(Q(\zeta_n)) = \langle\zeta_n\rangle$, the cyclic group generated by ζ_n. The Galois group $\mathrm{Gal}(Q(\zeta_n)/Q)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$. For $n = 5$ (prime), the Galois group is cyclic and consists of $\{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$. The action of the Galois group then permutes the 5th roots of unity. For $n = 8$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/8\mathbb{Z})^* = \{1, 3, 5, 7\}$ and is cyclic of order 4. The basis of $\mathbb{Q}(\zeta_8)$ over $\mathbb{Q}$ is given by $\{1, \zeta_8, \zeta_8^2, \zeta_8^3\}$. The actions is given as: σ_1 acts trivially, σ_3 maps ζ_8 to ζ_8^3, σ_5 acts by multiplication by -1 and σ_7 maps ζ_8 to ζ_8^7.
- Action on the cyclic group $(\mathbb{Z}/n\mathbb{Z})^*$: same as above.
- Action on a finite abelian group M: trivial action.
- Action on the general linear group $\mathrm{GL}_n(L)$ over a field L of characteristic 0: $\mathrm{GL}_n(L)$ consists of $n \times n$ invertible matrices over L. We have a Galois extension L/K. The Galois group acts by applying the field automorphisms to the entries of the matrices, so $\sigma(A) = \sigma(a_{ij}) \forall 1 \leq ij \leq n$. The fixed points contain $\mathrm{GL}_n(K)$.
  - Backstory: The determinant of a $n \times n$ matrix A is defined as

    sgn(π) is either even or odd. $+1$ if even and -1 if odd.

    $$\det(A) = \sum_{\pi \in S_n} \left(\mathrm{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)} \right)$$

    Consider $\sigma(\det(A))$, where $\sigma \in \mathrm{Gal}(L/K)$ is a field automorphism. It distributes over addition and multiplication:

    $$\sigma(\det(A)) = \sum_{\pi \in S_n} \left(\mathrm{sgn}(\pi) \prod_{i=1}^{n} \sigma(a_{i,\pi(i)}) \right)$$

The signum is either $+1$ or -1, so it is always in the base field K and is fixed by σ. Thus $\sigma(\det(A)) = \det(\sigma(A))$. So the action of the Galois group preserves determinants.

1.2 *The fixed point functor and exact sequences*

All of these examples are special cases of a more general concept: a group G acting on an algebraic group $\mathbb{G} \subseteq \mathrm{GL}_n$.

An algebraic group is a matrix group defined by polynomial conditions, at least this is what "The theory of group schemes of finite type over a field." by Milne says. I guess this is the consequence of Chevalley theorem?

When studying group actions, we're often interested in fixed points

$$A^G = \{a \in A \mid \forall \sigma \in G : \sigma a = a\}$$

Here, A^G represents the set of all elements in A that are fixed by every element of G. To study fixed points more systematically, we introduce the fixed point functor $-^G$. This functor takes a $\mathbb{Z}G$-module and returns its fixed points. We're particularly interested in how this functor behaves with respect to exact sequences.

Note 1.1.
Group action perspective: A $\mathbb{Z}G$-module is an abelian group A endowed with a (left) action $(\sigma, a) \mapsto \sigma a$ of G on A such that for all $\sigma \in G$ the map $\varphi_\sigma : a \mapsto \sigma a$ from A to A is a morphism of abelian groups. This implies that the action of G is distributive, $\varphi_\sigma(ab) = \varphi_\sigma(a) + \varphi_\sigma(b)$.
Ring module perspective: Equivalently, a $\mathbb{Z}G$-module is a module over the group ring $\mathbb{Z}[G]$, where elements consist of formal linear combinations of elements from group G with integer coefficients, so something like $3g_1 + 4g_2 + 10g_3 \in \mathbb{Z}[G]$. It contains both $\mathbb{Z}$ and G as subrings.
The $\mathbb{Z}[G]$-module structure encapsulates both the abelian group structure of A and the G-action on A, which leads to the key insight:

$$\{\text{module over } \mathbb{Z}[G]\} \leftrightarrow \{\text{abelian group } A \text{ with } G\text{-action}\}$$

Lemma 1.2. Consider an exact sequence of $\mathbb{Z}G$-modules:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} 0$$

Applying the fixed point functor $-^G$ to this sequence yields:

$$0 \longrightarrow A^G \xrightarrow{f^G} B^G \xrightarrow{g^G} C^G$$

This new sequence is exact in Ab (the category of abelian groups). Thus the functor $-^G$ is left-exact, meaning it preserves exactness at the left end of the sequence.

- A natural question arises: Is the fixed point functor also right-exact? If such a lifting always exists, then the fixed point functor preserves exactness at C, making it right-exact. If not, we've discovered an obstruction that tells us something about the Galois action and the structure of our groups.

- To investigate this, we need to check if $\ker h^G = \operatorname{im} g^G$, or equivalently, if $\operatorname{im} g^G = C^G$. Breaking this down:
  - Take any $c \in C^G$.
  - Since $C^G \subseteq C$, there exists a $b \in B$ such that $g(b) = c$.
  - If b were fixed by G, we'd be done. But it might not be.

    Why $\sigma b = b$?
    - Consider $\sigma b - b$ for any $\sigma \in G$. We have $g(\sigma b - b) = g(\sigma b) - g(b) = \sigma g(b) - g(b) = \sigma c - c$.
    - Since $c \in C^G$, $\sigma c - c = 0$ and $(\sigma b - b) \in \ker g$.
    - By exactness, $\ker g = \operatorname{im} f$, so $\sigma b - b \in \operatorname{im} f$.
    - We can view this as an element of A (considering f as an inclusion $A \subseteq B$).

    Also, $C \cong B/\operatorname{im} f$. Or consider presentations of groups.

  So the question of right-exactness boils down to whether or not every G-invariant element of C can be lifted to a G-invariant element of B and the obstruction to it lives inside of A.

  And if b were indeed in B^G then $(\sigma b - b) = 0 \in A$.

- This analysis leads us to define a map (for a given $c \in C^G$):

$$\varphi : G \to A, \quad \sigma \mapsto \sigma b - b =: a_\sigma$$

  This map is called a crossed homomorphism (also known as a derivation or 1-cocycle). It measures how far b is from being G-invariant. If b were G-invariant, this map would be identically 0! Note that this is independent of any b taken such that $g(b) = c$. Such cocycles are cohomologous.

Proposition 1.3. The map $\sigma \mapsto a_\sigma$ satisfies:

$$a_{\sigma\tau} = a_\sigma + \sigma a_\tau$$

This property is what defines a crossed homomorphism.

- **In the abelian case**, we define
  - $Z^1(G, A) = \{a' : G \to A \mid a'_{\sigma\tau} = a'_\sigma + \sigma a'_\tau\}$, the set of all crossed homomorphisms from G to A.
  - $B^1(G, A) = \{a : \sigma \in Z^1(G, A) \mid \exists a' \in A : a_\sigma = \sigma a' - a'\}$.
  - The quotient $H^1(G, A) = Z^1(G, A)/B^1(G, A)$ is called the **first cohomology group** of G with coefficients in A. It measures the obstruction to the right-exactness of the fixed point functor.

The functor $A \mapsto H^1(G, A)$ is a derived functor of the $A \mapsto A^G$ functor.

The obstructions for right-exactness: find $\sigma b - b \in A$ such that it is 0 under projection in $Z^1(G, A)/B^1(G, A)$. It is given by $\delta(c) = [a_\sigma] \in H^1(G, A) = Z^1(G, A)/B^1(G, A)$. We can extend our original sequence to a longer exact sequence:

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \longrightarrow 0$$

This sequence is exact in Ab, and the map δ (called the connecting homomorphism) measures the failure of right-exactness of the fixed point functor, since $\ker \delta$ represents all elements of C^G which can be lifted to elements of B^G.

In field theory, $H^1(G, A)$ can represent the obstruction to an element being a norm. In the theory of algebraic groups, $H^1(G, A)$ can represent the obstruction to a torsor having a rational point.

- The key idea of the 1-cocycle is to encode the failure of G-invariance in a way that's compatible with the group structures involved. It allows us to move from concrete elements (b and c) to cohomological objects ($[\varphi]$) that capture essential information about the Galois action and the relationship between our groups A, B, and C. This approach transforms specific lifting problems into more general cohomological questions, allowing us to apply powerful theoretical tools and gain deeper insights into the structures we're studying.

Exercise 1.4. Show that $H^1(G, -)$ is functorial and

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \longrightarrow 0$$

is exact. Find example with $\delta \neq 0$.

- **In the non-abelian case**, we define
  - $H^0(G, A) = A^G$, the fixed points as before.
  - $H^1(G, A) = Z^1(F, A)/\sim$, where $\sim$ is an equivalence relation defined by: $a_\sigma \sim b_\sigma \iff \exists a' \in A : b_\sigma = (a')^{-1} \cdot a_\sigma \cdot {}^\sigma a'$.

We cannot expect $B^1(G, A)$ to be a subgroup. Why?

${}^\sigma a$ denotes the action of σ on a.

In this case, $H^1(G, A)$ doesn't have a group structure, but is a pointed set (a set with a distinguished element). We can still define a notion of exactness for sequences of pointed sets.

Exactness in pointed sets $(A, *)$ is defined as $\operatorname{im} f = \ker g = g^{-1}(*)$

$A \leq_G B$ is G-equivariant inclusion.

Proposition 1.5. For $A \leq_G B$, we obtain $G \curvearrowright B/A$ and

$$1 \longrightarrow H^0(G, A) \longrightarrow H^0(G, B) \longrightarrow H^0(G, C) \longrightarrow H^1(G, A) \longrightarrow H^1(G, B)$$

is exact.

This is the **Galois cohomology**. Why do we care? In the non-commutative case $H^1(G, A)$ classifies "K-objects". In our lecture we will use this to classify simple and simply connected linear algebraic k-groups $\mathbb{G}$.

2 Preliminaries from algebraic number theory.

Lecture 2, 17.10.24
User: GRK, password: 2240.

2.1 Number fields

Definition 2.1. An algebraic number field is a finite field extension $k/\mathbb{Q}$.

- This definition implies the following properties:
  - The field k has characteristic 0.
  - By the Primitive Element Theorem, $k = \mathbb{Q}(a)$ for some $a \in K$.
  - There exists a unique minimal polynomial $f \in \mathbb{Q}[X]$ for a, with $\deg(f) = d = [k : \mathbb{Q}]$.
- Let $(a_1, \ldots, a_d)$ be the roots of f in the algebraic closure of $\mathbb{Q}$ within $\mathbb{C}$. These roots are called the **Galois conjugates** of a. Note that these roots do not lie in $\mathbb{Q}$.
- Properties of embeddings:
  - For each i, the map $a \mapsto a_i$ defines an isomorphism $\mathbb{Q}(a) \cong \mathbb{Q}(a_i)$.
  - Any embedding $k \to \mathbb{C}$ must send a to some a_i.
  - There are exactly d embeddings $k \to \mathbb{C}$, denoted $\sigma_1, \ldots, \sigma_d$.
- Classification of embeddings:
  - Note that $(a_1, \ldots, a_d) = \overline{(a_1, \ldots, a_d)}$, so $\sigma_i(k) \subseteq \mathbb{R}$ if and only if $\overline{a_i} = a_i$.
  - We can thus classify the embeddings as:
    * Real embeddings (real places of K): r_1
    * Complex embeddings (complex places of K): $2r_2$ (counted in pairs due to complex conjugation)
  - This classification implies $d = r_1 + 2r_2$
- Examples:
  - For $k = \mathbb{Q}(\sqrt[3]{2})$: $r_1 = 1, r_2 = 1$
  - For $k = \mathbb{Q}(\exp(2\pi i/n))$, $n \geq 3$: $r_1 = 0, r_2 = \varphi(n)/2$ (odd n)

Definition 2.2. For any $\alpha \in K$, we define two rational numbers:
1. The norm: $N_{K/\mathbb{Q}}(\alpha) = \prod_{i=1}^{d} \sigma_i(\alpha)$
2. The trace: $Tr_{K/\mathbb{Q}}(\alpha) = \sum_{i=1}^{d} \sigma_i(\alpha)$

Note: $N_{K/\mathbb{Q}}(\alpha) = \det(\alpha : K \to K)$, and similarly for the trace.

- Basis criterion: Let $(\alpha_1, \ldots, \alpha_d) \in k$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}$. Then $\sum_{i=1}^{d} \lambda_i \alpha_i = 0 \iff \sum_{i=1}^{d} \lambda_i \sigma_j(\alpha_i) = 0$ for all j. Moreover, $\{\alpha_i\}_{i=1}^{d}$ is a basis of k if and only if $\det(\sigma_i(\alpha_j)) \neq 0$.

Definition 2.3. The **discriminant** of a basis $\{\alpha_1, \ldots, \alpha_d\}$ of a number field k of degree d over $\mathbb{Q}$ is defined as: $\mathrm{discr}(\{\alpha_1, \ldots, \alpha_d\}) = \det^2(\sigma_i(\alpha_j)) \in \mathbb{Q}$, where $\sigma_1, \ldots, \sigma_d$ are the d distinct embeddings of k into $\mathbb{C}$.

Exercise 2.4. Prove that $\mathrm{discr}(\alpha_i) = \det(Tr_{k/\mathbb{Q}}(\alpha_i\alpha_j))_{1\leq i,j\leq d}$. Show that if $k = \mathbb{Q}(a)$ for some $a \in k$, then $\mathrm{discr}(\{1, a, a^2, \ldots, a^{d-1}\}) = \prod_{1\leq i<j\leq d}(\sigma_i(a) - \sigma_j(a))^2$.

To introduce relative versions for an extension l/k, we define the relative discriminant $\mathrm{discr}()_{l/k}$ using only those embeddings $\sigma_i : l \hookrightarrow \mathbb{C}$ which restrict to the identity on k.

2.2 *Integrality in number fields*

Algebraic number theory is not (algebraic) number theory but rather (algebraic number) theory.

Let k be an algebraic number field for the following discussion.

Definition 2.5. The ring of integers in k is defined as:

$$\mathcal{O}_k = \{\alpha \in k : f(\alpha) = 0 \text{ for some monic } f \in \mathbb{Z}[X]\}.$$

- Example: $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$. It is often referred to as the ring of "rational integers".

Proposition 2.6. For $(\alpha_1, \ldots, \alpha_r) \in k$, the following are equivalent:
1. $(\alpha_1, \ldots, \alpha_r) \in \mathcal{O}_k$
2. $\mathbb{Z}[\alpha_1, \ldots, \alpha_r]$ is finitely generated as a $\mathbb{Z}$-module.

The corolllary is that $\mathcal{O}_k$ is a ring. (why?)

Lemma 2.7. For $\alpha \in k$, there exist $\beta \in \mathcal{O}_k, n \in \mathbb{Z}$ such that $\alpha = \frac{\beta}{n}$.

From now on we can assume that our algebraic number field is generated by a primitive element which is an algebraic integer.

Proposition 2.8. We can sandwhich the ring $\mathcal{O}_k$ between $\mathbb{Z}[a]$ and $\frac{1}{discr\{1,\ldots,a^{d-1}\}}\mathbb{Z}[a]$. (This 1/discr is in $\mathbb{Z}$ because it is in the intersection of algebraic integers in k and $\mathbb{Q}$). Because it lies between two free abelian groups of the same rank, it has to be free abelian of the same rank.

Corollary 2.9. $\mathcal{O}_k$ has a $\mathbb{Z}$-basis of rank d.Any such is called an integral basis.

(Z lattice in a Q vector space and you exhaust it by multiplying with the integers? What? Minkowski geometry of numbers (covolumes?))

Corollary 2.10. $\mathcal{O}_k$ is noetherian.

Definition 2.11. The discriminant of k is given by $discr_k = d_k = discr\{\alpha_1, \ldots, \alpha_d\}$ for an integral basis. Well-defined because $\det(T...) = \pm 1$.

More generally, we can also define relative discriminants $fancyd_{l/k} = discr(\beta_i) \subseteq \mathcal{O}_k$. This d is an ideal because in general we might be not in a PID anymore.

Exercise 2.12. $k = \mathbb{Q}(\sqrt{D})$, D square-free integer. If $D \equiv 1(4)$ or $D \equiv 2,3(4)$ then the integral basis is ... and discriminant is D or $4D$.

2.3 *The arithmetic of algebraic intgers*

For $k = \mathbb{Q}(\sqrt{5})$, $\mathcal{O}_k = \mathbb{Z}[\sqrt{-5}]$. In $\mathcal{O}_k$, we have $21 = 3 \cdot 7 = (1 + 2\sqrt{-5}) \cdot (1 - 2\sqrt{-5})$ and these factors are irreducible. (something something norm of a number). So it is not a UFD. Kummer suggested: in an ideal world, there would be ideal numbers $p_1 \cdot p_2 = 3$ and $p_3 \cdot p_4 = 7$, with $p_1 \cdot p_3 = 1 + 2\sqrt{-5}$ and $p_2 p_4 = 1 - 2\sqrt{-5}$, hence $21 = p_1 p_2 p_3 p_4 = p_1 p_3 p_2 p_4$ so they would differ only by a permutation and factorization would be unique. Apparently: $p_1 \mid 3$ and $p_1 \mid 1 + 2\sqrt{-5} \implies p_1 \mid \lambda 3 + \mu(1 + 2\sqrt{5})$ and p_1 should be determined by the set of all $\alpha \in \mathcal{O}_k$ that it divides. So set $p_1 = (3, 1 + 2\sqrt{5})$ and $p_2 = (3, 1 - 2\sqrt{-5})$ and so on... So the idea is that one might get unique factorization in ideals instead.

Theorem 2.13. *The ring $\mathcal{O}_k$ is noetherian, integrally closed and of dimension* 1*.*

The hard thing is to single out that these three properties are key to a ring being a Dedekind domain.

Definition 2.14. An integral domain satisfying these three properties is called a Dedekind domain.

Lecture 3, ...