

1. Big $O(n)$

$$f(n) \Rightarrow O(g(n))$$

$$\text{if } f(n) \leq g(n) \times c \quad \forall n \geq n_0$$

for some constant, $c > 0$

$g(n) \rightarrow$ tight upper bound of $f(n)$

$$\text{eg } f(n) = n^2 + n$$

$$g(n) = n^3$$

$$n^2 + n \leq c * n^3$$

$$n^2 + n = O(n^3)$$

(ii) Big Omega (Ω)

$$\text{when } f(n) = \Omega(g(n))$$

$g(n) \rightarrow$ tight upperbound of $f(n)$

$$\text{i.e. } f(x) = \Omega g(x)$$

if and only if

$$f(x) \geq c \cdot g(x)$$

$$\forall n_2 > n_0 \text{ and } c = \text{constant}$$

$$\text{eg. } f(n) = n^3 + 4n^2$$

$$g(n) = n^2$$

$$\text{i.e. } f(n) \geq c \cdot g(n)$$

$$n^3 + 4n^2 = \Omega(n^2)$$

(iii) Big Theta (Θ)

When $f(n) = \Theta(g(n))$ gives tight upperbound and lowerbound both.

$$\text{i.e. } f(n) = \Theta(g(n))$$

if and only if

$$c_1 * g(n_1) \leq f(n) \leq c_2 * g(n_2)$$

for all $n > \max(n_1, n_2)$ some constant

$$c_1 > 0 \text{ \& } c_2 > 0$$

i.e. $f(n)$ can never go beyond $c_2 g(n)$ and will never come down of $c_1 g(n)$.

$$\text{eg. } 3n+2 = \Theta(n) \text{ as } 3n+2 > 3n$$

$$3n+2 \leq 4n \text{ for } n, c_1 = 3, c_2 = 4 \text{ \& } n_0 = 2$$

(iv) Small O (o)

When $f(n) = o(g(n))$ gives the upper bound

$$\text{i.e. } f(n) = o(g(n))$$

if and only if

$$f(n) < c * g(n)$$

$$\forall n > n_0 \text{ \& } n > 0$$

$$\text{eg. } f(n) = n^2 ; g(n) = n^3$$

$$f(n) < c * g(n)$$

$$n^2 = o(n^3)$$

(v) Small Omega (ω)

It gives lower bound i.e. $f(n) = \omega(g(n))$

where $g(n) \rightarrow$ lower bound of $f(n)$ if

and only if $f(n) > c * g(n)$

$$\forall n > n_0 \text{ \& } \text{some constant } c > 0$$

Ans 2:- for $r = 1, 2, 4, 6, 8, \dots$ n times

i.e series is a GP

so $a = 1$ $r = 2$

k^{th} value of GP :

$$t_k = ar^{k-1}$$

$$t_k = 1(2)^{k-1}$$

$$2^n = 2^k$$

$$\log_2(2^n) = k(\log 2)$$

$$\log_2 2 + \log_2 n = k$$

$$\log_2 n + 1 = k \quad (\text{neglecting } 1)$$

so TC $T(n) \Rightarrow O(\log_2 n)$

Ans 3:-

$$T(n) = 3T(n-1) \text{ --- (i)}$$

$$T(n) = 1$$

put $n \Rightarrow n-1$ in (i)

$$T(n-1) = 3T(n-2) \text{ --- (ii)}$$

put (ii) in (i)

$$T(n) = 3(3T(n-2))$$

$$T(n) = 9T(n-2) \text{ --- (iii)}$$

put $n = n-2$ in (ii)

$$T(n-2) = 3T(n-3)$$

put in (iii)

$$T(n) = 27T(n-3) \text{ --- (iv)}$$

Generalizing series,

$$T(k) = 3^k + (n-k) \text{ --- (v)}$$

for k^{th} terms, let $n-k=1$ (Base Case)

$$k = n-1$$

put in (v)

$$T(n) = 3^{n-1} + 1$$

$$T(n) = 3^{n-1}$$

$$T(n) = O(3^n)$$

Ans 4:-

$$T(n) = 2T(n-1) - 1 \text{ --- } \textcircled{1}$$

put $n = n-1$

$$T(n-1) = 2T(n-2) - 1 \text{ --- } \textcircled{2}$$

put in $\textcircled{1}$

$$T(n) = 2(2T(n-2) - 1) - 1$$

$$T(n) = 4T(n-2) - 2 - 1 \text{ --- } \textcircled{3}$$

put $n = n-2$ in $\textcircled{1}$

$$T(n-2) = 2T(n-3) - 1$$

put in $\textcircled{3}$

$$T(n) = 4(2T(n-3) - 1) - 2 - 1$$

$$= 8T(n-3) - 4 - 2 - 1 \text{ --- } \textcircled{4}$$

Generalizing series,

$$T(n) = 2^k T(n-k) - 2^{k-1} - 2^{k-2} - \dots - 2^0$$

k^{th} term

$$\text{Let } n-k=1$$

$$k=n-1$$

$$T(n) = 2^{n-1} T(1) - 2^k \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^k} \right)$$

$$= 2^{n-1} - 2^{n-1} \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \right)$$

i.e. Series in GP

$$a = \frac{1}{2}, \quad r = \frac{1}{2}$$

So,

$$T(n) = 2^{n-1} \left(1 - \left(\frac{1}{2} \cdot \frac{(1 - (1/2)^{n-1})}{1 - 1/2} \right) \right)$$

$$= 2^{n-1} \left(1 - 1 + \left(\frac{1}{2} \right)^{n-1} \right)$$

$$= \frac{2^{n-1}}{2^{n-1}}$$

$$T(n) = O(1)$$

Ans.

$$\text{As } i^2 = n$$

$$i = \sqrt{n}$$

$$i = 1, 2, 3, 4, \dots, \sqrt{n}$$

$$\sum_{i=1}^{\sqrt{n}} 1 + 2 + 3 + 4 + \dots + \sqrt{n}$$

$$T(n) = \frac{\sqrt{n} + (\sqrt{n} + 1)}{2}$$

$$T(n) = \frac{n + \sqrt{n}}{2}$$

$$T(n) = O(n)$$

Ans 7:-

Since for $k = k^2$

$$k = 1, 2, 4, 8, \dots, n$$

\therefore Series is in GP

$$\text{So } a = 1, r = 2$$

$$= \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(2^k - 1)}{2 - 1}$$

$$n = 2^k - 1$$

$$n + 1 = 2^k$$

$$\log_2(n) = k$$

i	f	k
1	$\log(n)$	$\log(n) * \log(n)$
2	$\log(n)$	$\log(n) * \log(n)$
\vdots	\vdots	\vdots
n	$\log(n)$	$\log(n) * \log(n)$

$$TC \Rightarrow O(n * \log n * \log n)$$

$$\Rightarrow \underline{O(n \log^2(n))}$$

Ans 8:-

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for $(i=1 \text{ to } n)$

we get $j = n$ times every turn

$$\therefore i * j = n^2$$

k^{th} ,

Now,

$$T(n) = n^2 + T(n-3);$$

$$T(n-3) = (n-3)^2 + T(n-6);$$

$$T(n-6) = (n-6)^2 + T(n-9);$$

$$\text{and } T(1) = 1;$$

Now substitute each value in $T(n)$

$$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \dots + 1$$

let

$$k^{\text{th}} - 3k = 1$$

$$k = (n-1)/3$$

total terms = $k+1$

$$T(n) = n^2 + (n-3)^2 + (n-6)^2 + \dots + 1$$

$$T(n) \approx k n^2$$

$$T(n) \approx (k-1)/3 \cdot n^2$$

$$\text{So } T(n) = O(n^3)$$

Ans 9: -

$$\text{for } i=1 \quad J = 1 + 2 + \dots \quad (n \geq J+i)$$

$$i=2 \quad J = 1 + 3 + 5 + \dots \quad (n \geq J+i)$$

$$i=3 \quad J = 1 + 4 + 7 + \dots \quad (n \geq J+i)$$

n^{th} term of AP is

$$T(n) = a + d \cdot n$$

$$T(n) = 1 + d \cdot n$$

$$(2-n)T + \frac{n(n-1)}{d} = n$$

$$(2-n)T + \frac{n(n-1)}{d} = (2-n)T$$

$$\text{for } i=1 \quad (n-1)/1 \text{ times}$$

$$i=2 \quad (n-1)/2 \text{ times}$$

$$i=n-1$$

We get

$$T(n) = i_1 j_1 + i_2 j_2 + \dots + i_{n-1} j_{n-1}$$

$$= \frac{(n-1)}{1} + \frac{(n-2)}{2} + \frac{(n-3)}{3} + \dots +$$

$$= n + \frac{n}{2} + \frac{n}{3} + \dots + \frac{n}{n-1} - n \times 1$$

$$= n \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right] - n \times 1$$

$$= n \times \log n - n + 1$$

$$\text{Since } \int \frac{1}{x} = \log x$$

$$T(n) = O(n \log n)$$

Anslo :-

As given n^k and c^n

Relationship b/w n^k and c^n is

$$n^k = o(c^n)$$

$$n^k \leq a(c^n)$$

$\forall n \geq n_0$ & constant, $a > 0$

for $n_0 = 1$; $c = 2$

$$\Rightarrow 1^k < a^2$$

$$n_0 = 1 \text{ \& } c = 2$$