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A planar quaternion approach to the kinematic synthesis of a parallel manipulator

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SUMMARY

In this paper we present a technique for designing planar parallel manipulators with platforms capable of reaching any number of desired poses. The manipulator consists of a platform connected to ground by RPR chains. The set of positions and orientations available to the end-effector of a general RPR chain is mapped into the space of planar quaternions to obtain a quadratic manifold. The coefficients of this *constraint manifold* are functions of the locations of the base and platform R joints and the distance between them. Evaluating the constraint manifold at each desired pose and defining the limits on the extension of the P joint yields a set of equations. Solutions of these equations determine chains that contain the desired poses as part of their workspaces. Parallel manipulators that can reach the prescribed workspace are assembled from these chains. An example shows the determination of three RPR chains that form a manipulator able to reach a prescribed workspace.

KEYWORDS: Kinematic synthesis; Parallel manipulators; RPR chains; Quaternion approach.

1 INTRODUCTION

The fundamentals of parallel robots are now established[1], and the modeling and control of such machines have been widely studied. Even though parallel robots are already in use[2–4], the design of these systems presents many challenges and stays a focus of research. This design process includes considerations of symmetry of the manipulator, isotropy of the Jacobian, velocity and force capabilities, singularity avoidance, redundancy, and size of the workspace to name some examples[5–8]. Designing a parallel manipulator capable of reaching a prescribed workspace is another of these challenges. The complexity of the design problem is increased when the workspace definition includes desired poses for the platform, positions *and* their associated orientations.

We consider the design of planar parallel manipulators constructed from RPR chains (R and P denoting revolute and prismatic joints, respectively) as a first step in solving this problem. Formulated without regard for general joint limitations or platform singularities, this design problem reduces to determining the set of RPR chains that contains the desired poses as a subset of their individual workspaces. This is accomplished by mapping the configuration space of the end-effector of a general RPR chain to a subset of the space of planar quaternions. The desired poses are defined by points in this space. The goal is to determine the dimensions of the RPR chains (and consequently the parallel manipulator) such that the set of end-effector positions and orientations includes the desired poses.

The set of platform configurations of this parallel manipulator is the intersection of the positions and orientations reachable by each of the RPR chains that connect it to ground. The configuration space for an individual RPR chain is a quadratic algebraic manifold in the space of planar quaternions and is called its *constraint manifold.* The coefficients of the quadratic form of this manifold contain the coordinates of the base and platform R joints and the displacement of the P joint as free parameters. Constraint manifolds are also derived in their algebraic forms by Ravani and Roth[9] for the $2C$ spatial chain, by Bodduluri[10] for a wide range of spatial dyads, and Bottema and Roth[11] for a detailed study of the constraint manifold of a planar $4R$ linkage. A convenient parameterization of several of these manifolds is presented by McCarthy.[12] Husty intersects similarly derived manifolds to solve the forward kinematics of the general spatial platform.[13]

To represent a desired workspace for the parallel manipulator to be designed, we discretize the workspace and identify poses both along its boundaries and inside the workspace. These poses are mapped to points in planar quaternion space. The requirement that each of these points lies on the constraint manifold of an RPR chain defines a set of constraint equations for the design parameters. We exploit the structure of these equations and use its coefficients directly rather than the specified design parameters. The result is called the *coefficient representation* of the constraint manifold, and the design problem reduces to intersecting annular regions in two different planes. Any set of N points within these intersections defines an N-chain parallel manipulator that solves the problem. There are several examples of design performed in the space of dual and planar quaternions: Ravani[14] introduced a numerical method for fitting these manifolds to as many as ten points representing rigid body poses,[15] Bodduluri and McCarthy[16] and then

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Larochelle and McCarthy[17] extended Ravani's curve fitting methodology to spherical 4*R* and spatial 4*C* linkages, respectively.

Our effort represents a preliminary result in the direction of the kinematic design of spatial parallel manipulators for a specific workspace. The methodology is generalized by using dual quaternions and the constraint manifold of an *SPS* chain (*S* denoting a spherical joint). At its most basic form, this can be viewed as an extension of Burmester theory.[18]

2 DUAL AND PLANAR QUATERNIONS

Clifford[19] introduced a product operation for vectors in a metric product space that defines a geometric algebra, called Clifford algebra. Complex numbers, quaternions, planar quaternions and dual quaternions are obtained using Clifford's construction.[12] The Clifford algebra associated with spatial displacements encodes the rotation and translation of the displacement in an eight dimensional vector. The components of this vector are defined by the Plücker coordinates of the screw axis and the rotation and translation around and along this axis. Dual quaternions that represent planar displacements have only four non-zero components and form a subalgebra called planar quaternions. Their components are defined by the rotation angle and translation vector defining the planar displacement.

The planar displacement shown in Figure 1 is given by:

$$\boldsymbol{x}_f = \boldsymbol{R}\boldsymbol{x}_m + \boldsymbol{d} \tag{1}$$

where:

- $\boldsymbol{x}_f$ is the coordinate vector of a point in a fixed frame $(\mathscr{F})$
- $\boldsymbol{x}_m$ is the coordinate vector of the same point in a moving frame $(\mathscr{M})$
- $\boldsymbol{R}$ is a 2×2 rotation matrix
- $\boldsymbol{d}$ is a 2×1 displacement vector

The components of the displacement vector, x and y, and the rotation θ are typically used to specify the location of $(\mathscr{M})$ in $(\mathscr{F})$ by defining $\boldsymbol{R}$ and $\boldsymbol{d}$ as follows:

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\boldsymbol{d}(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

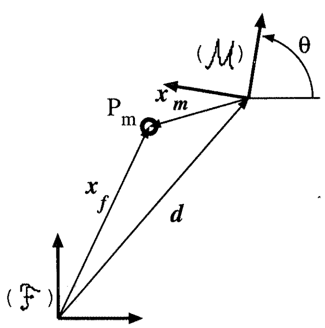

Fig. 1. A planar displacement

An alternate set of parameters for expressing a planar displacement is the *planar quaternion*:[12]

$$Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(x\cos\frac{\theta}{2} + y\sin\frac{\theta}{2}\right) \\ \frac{1}{2}\left(y\cos\frac{\theta}{2} - x\sin\frac{\theta}{2}\right) \\ \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \tag{2}$$

The planar displacement in equation (1) expressed in terms of Q is:

$$\boldsymbol{R}(Q) = \begin{bmatrix} q_4^2 - q_3^2 & -2q_3q_4 \\ 2q_3q_4 & q_4^2 - q_3^2 \end{bmatrix} \tag{3}$$

and

$$\boldsymbol{d}(Q) = \begin{bmatrix} 2(q_1q_4 - q_2q_3) \\ 2(q_1q_3 + q_2q_4) \end{bmatrix} \tag{4}$$

Note the property:

$$q_3^2 + q_4^2 = 1 \tag{5}$$

3 CONSTRAINT MANIFOLD OF AN *RPR* CHAIN

The set of combined positions and orientations available to a moving body constrained by either revolute or prismatic joints is an algebraic submanifold in the space of planar quaternions called a *constraint manifold.* The constraint manifold of the platform of a parallel manipulator is the intersection of the constraint manifolds of the individual serial chains connecting the platform to the base. To perform the kinematic design of a parallel manipulator, each chain is designed to encompass the desired set of poses. The constraint manifold of a general planar *RPR* chain is now determined to accomplish this task.

A planar *RPR* chain is shown in Figure 2. The point P_f is the location of the fixed pivot in the fixed frame, and P_m is the location of the moving pivot also known in the fixed frame. Let the distance between the two pivots be l such that:

$$\|P_mP_f\| = l \tag{6}$$

Let the location of the fixed pivot in the fixed frame, $\mathscr{F}$, be specified by the vector $\boldsymbol{b} = (r, s)^T$. Let the location of the moving pivot in the moving frame, $\mathscr{M}$, be

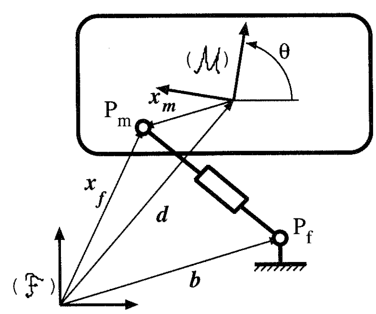

Fig. 2. A planar *RPR* chain

$x_m = (\rho, \sigma)^T$. The location of the moving pivot in the fixed frame is x_f. Then equation (6) yields:

$$(\boldsymbol{b} - \boldsymbol{x}_f) \cdot (\boldsymbol{b} - \boldsymbol{x}_f) = l^2 \qquad (7)$$

Substituting equation (1) into (7):

$$(\boldsymbol{b} - \boldsymbol{R}\boldsymbol{x}_m - \boldsymbol{d}) \cdot (\boldsymbol{b} - \boldsymbol{R}\boldsymbol{x}_m - \boldsymbol{d}) = l^2 \qquad (8)$$

Substituting eqs. (3) and (4) into (8), and using the identity in equation (5) determines the constraint manifold:

$$4(A_0q_1^2 + A_1q_2^2 + A_2q_3^2 + A_3q_4^2 + A_4q_1q_2 + A_5q_1q_3 + A_6q_1q_4 + A_7q_2q_3 + A_8q_2q_4 + A_9q_3q_4) = l^2 \quad (9)$$

where:

$$\begin{vmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ \frac{1}{4}((\rho + r)^2 + (\sigma + s)^2) \\ \frac{1}{4}((\rho - r)^2 + (\sigma - s)^2) \\ 0 \\ -(\sigma + s) \\ \rho - r \\ \rho + r \\ \sigma - s \\ \sigma r - \rho s \end{vmatrix} \qquad (10)$$

A relationship between the design parameters (ρ, σ, r and s) and four of the ten coefficients of the quadratic equation (9) is derived from equation (10):

$$\begin{vmatrix} \rho \\ \sigma \\ r \\ s \end{vmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}\begin{vmatrix} A_5 \\ A_6 \\ A_7 \\ A_8 \end{vmatrix} \qquad (11)$$

A quadric corresponding to an $\dot{R}PR$ chain is distinguished from a general quadric by the coefficients A_i satisfying the following:

$$A_0 = 1 \qquad (12)$$

$$A_1 = 1 \qquad (13)$$

$$A_4 = 0 \qquad (14)$$

$$2A_9 - A_5A_6 - A_7A_8 = 0 \qquad (15)$$

$$4A_2 - A_5^2 - A_7^2 = 0 \qquad (16)$$

$$4A_3 - A_6^2 - A_8^2 = 0 \qquad (17)$$

Equations (12)–(17) are called the *coefficient representation of an RPR chain*. The relationships in the coefficient representation allow the constraint manifold of equation (9) to be written:

$$(2q_1 + A_5q_3 + A_6q_4)^2 + (2q_2 + A_7q_3 + A_8q_4)^2 = l^2 \quad (18)$$

Equation (18) is the desired form for the constraint manifold.

4 THE CONSTRAINT MANIFOLD IN DESIGN

To design RPR chains capable of reaching the desired set of poses, we must solve for the parameters A_5, A_6, A_7, and A_8 such that equation (18) is true for all points Q in the workspace. We make two observations to accomplish this task and produce reasonable designs:

First, the desired workspace is assumed, without loss of generality, to contain poses where the orientation $\theta = 0$. That is, the location of the fixed frame can always be arbitrarily chosen such that it is parallel to some desired platform pose. From equation (2), note that if $\theta = 0$, then $q_3 = 0$, $q_4 = 1$ and equation (18) simplifies to:

$$(2q_1 + A_6)^2 + (2q_2 + A_8)^2 = l^2 \qquad (19)$$

Second, notice that if the distance, l, between the revolute joints is unbounded then any workspace is achievable. To eliminate this possibility and to add a reasonable design consideration, we place bounds on the extension of the prismatic joint:

$$0 \le l_{\min} \le l \le l_{\max} \qquad (20)$$

Equations (18) and (19) can now be written, respectively, as:

$$l_{\min}^2 \le (2q_1 + A_5q_3 + A_6q_4)^2 + (2q_2 + A_7q_3 + A_8q_4)^2 \le l_{\max}^2 \qquad (21)$$

and

$$l_{\min}^2 \le (2q_1 + A_6)^2 + (2q_2 + A_8)^2 \le l_{\max}^2 \qquad (22)$$

Let the prescribed workspace be discretized into p poses. Let n be the number of poses in the $\theta = 0$ subset of the workspace. Calculate the planar quaternion Q that corresponds to each of these poses. For each Q, equation (22) defines an annulus in the (A_6, A_8) plane. The intersection of the annuli define the possible values for A_6 and A_8, noting this as a necessary but not sufficient condition on these values. For equation (22) to be satisfied for all Q of the n poses:

$$-2\min(q_1) - l_{\max} \le A_6 \le -2\max(q_1) + l_{\max} \quad (23)$$

Select A_{6_i}, a candidate value from this range.

We can also determine limits on A_8 from equation (22):

$$\max(-K_1 - 2q_2) \le A_8 \le \min(K_1 - 2q_2) \qquad (24)$$

over all Q of the $p - n$ poses where:

$$K_1 = \sqrt{l_{\max}^2 - (2q_1 + A_{6_i})^2} \qquad (25)$$

Each Q also defines a range of **unusable** values for A_8:

$$-\sqrt{l_{\min}^2 - (2q_1 + A_{6_i})^2} - 2q_2 \le A_8 \le \sqrt{l_{\min}^2 - (2q_1 + A_{6_i})^2} - 2q_2 \quad (26)$$

where, in the event that these limits have imaginary parts, they can simply be ignored. Select a candidate value, A_{8_i}, that satisfies these inequalities noting that equations (24)–(26) may define more than one range of acceptable values for A_8.

For any choice $(A_6, A_8) = (A_{6_i}, A_{8_i})$ determined by equations (23)–(26), and for each of the remaining $p - n$ workspace poses in the $\theta \ne 0$ subset of the discretized workspace, equation (21) defines $p-n$ annuli in the (A_5, A_7) plane. Points in the intersection of these annuli are available as values of A_5 and A_7 that correspond to A_{6_i} and A_{8_i}. An intersection may not exist for a choice of A_{6_i} and A_{8_i}. For equation (21) to be satisfied for these $p-n$ Q's:

$$\max\left(\frac{-l_{\max}-2q_1-A_{6_i}q_4}{q_3}\right)\le A_5\le \min\left(\frac{l_{\max}-2q_1-A_{6_i}q_4}{q_3}\right) \tag{27}$$

Select a candidate value from this range called A_{5_i}.

We can now determine the limits on A_7 from equation (21):

$$\max\left(\frac{-K_2-2q_2-A_{8_i}, q_4}{q_3}\right)\le A_7\le \min\left(\frac{K_2-2q_2-A_{8_i}q_4}{q_3}\right) \tag{28}$$

over all Q of the $p-n$ poses where:

$$K_2=\sqrt{l_{\max}^2-(2q_1+A_{5_i}q_3+A_{6_i}q_4)^2} \tag{29}$$

Each Q also defines a range of **unusable** values for A_7:

$$\frac{-K_3-2q_2-A_{8_i}q_4}{q_3}\le A_7\le\frac{K_3-2q_2-A_{8_i}q_4}{q_3} \tag{30}$$

with:

$$K_3=\sqrt{l_{\min}^2-(2q_1+A_{5_i}q_3+A_{6_i}q_4)^2} \tag{31}$$

Imaginary limits can be dismissed. Equations (27)–(31) assume that $q_3>0$, which can always be achieved by the fact that Q and $-Q$ correspond to the same displacement. Select a candidate value, A_{7_i}, that satisfies these inequalities.

We implemented a procedure in *Matlab* that calculates and grids the usable ranges for the parameters A_5, A_6, A_7 and A_8. Each set of values $(A_{5_i}, A_{6_i}, A_{7_i}, A_{8_i})$ determined via this procedure corresponds to an *RPR* chain whose pivots locations are determined using equation (11). Assembling N such chains produces an N-chain planar manipulator with the p prescribed poses of the workspace reachable within the limits on l. The poses in the continuous workspace are not necessarily reachable within the desired limits on the prismatic joint.

5 CASE STUDY

As an application of this approach, we design a 3-chain fully parallel non-redundant manipulator. The desired workspace is $0<x<0.6$, $0<y<0.5$ and $-15°<\theta<20°$.

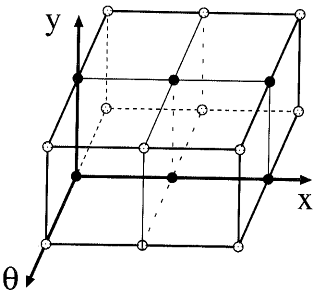

Fig. 3. The discretized workspace

We discretize this workspace into the 18 poses listed in Table I and shown in Figure 3.

This example problem, run as a *Matlab* function under *Windows95*, produces 350 separate *RPR* chains in less than 2 seconds, resulting in approximately 42 million parallel manipulator designs. We implemented a graphical user interface to provide for convenient selection of the chains.

As a first example, three chains are assembled into the manipulator shown in Figure 4. A *Matlab* function allows the animation of the manipulator between the specified workspace poses. A cursory visual inspection for singular positions can be performed. This mechanism is shown in the pose ($x=0$, $y=0.5$, $\theta=20°$), near a singularity.

Trial-and-error is used to produce an acceptable solution. In the second example, three different chains are combined to form the manipulator in Figure 5, shown in the pose of ($x=0$, $y=0$, $\theta=0°$). The values of the coefficients that correspond to these chains are $(A_5, A_6, A_7, A_8)=(-2.05, -2.80, 2.80, -1.55)$, $(-1.38, 1.88, -2.32, -1.62)$ and $(2.82, 0.82, -0.38, 2.58)$.

6 CONCLUSION

In this paper we formulated an initial step in the design of parallel manipulators for the discretization of a prescribed workspace. Defining the problem in the space of planar quaternions results in a definition of the parallel manipulator workspace as the intersection of quadratic equations. The consideration of the quadratic equations for the portion of the workspace parallel to the fixed frame produces annular regions. Discretizing the workspace produces a finite number of annular regions. Candidate values are chosen from the intersection of these annular regions and a second set of annuli is generated. The result is an automated procedure for

Table I: The 18 points in the discretized workspace

Pos.	1	2	3	4	5	6	7	8	9
x	0	0	0·3	0·3	0·6	0·6	0	0	0·3
y	0	0·5	0	0·5	0	0·5	0	0·5	0
θ	0	0	0	0	0	0	20	20	20
Pos.	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
x	0·3	0·6	0·6	0	0	0·3	0·3	0·6	0·6
y	0·5	0	0·5	0	0·5	0	0·5	0	0·5
θ	20	20	20	−15	−15	−15	−15	−15	−15

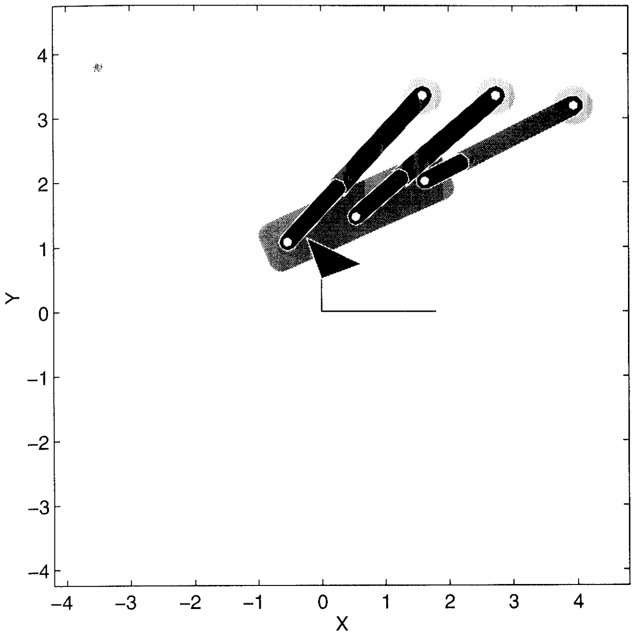

Fig. 4. Three chains define the parallel manipulator shown near a singularity in the desired workspace

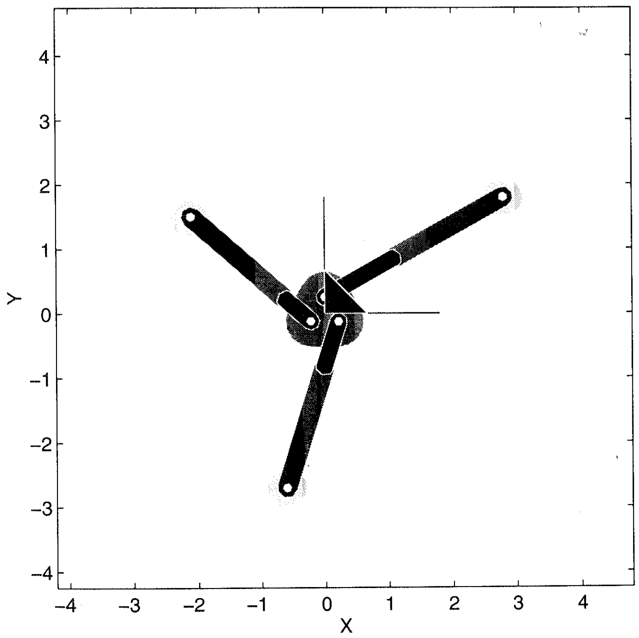

Fig. 5. Three different chains define a parallel manipulator that has no singularities inside the desired workspace

generating the sets of *RPR* chains that combine to form a parallel manipulator encompasing the desired workspace. The method presented here rapidly generates a large number of solutions to the design problem. This procedure only guarantees results at the poses of the discretized workspace. Developing the design procedure to eliminate manipulators with singular configurations in the workspace is a direction for future work, as is the extension of this approach to the design of more general parallel robots.

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