1. **Vector** Those quantities which have magnitude as well as direction are called vector quantities or vectors.

   NOTE Those quantities which have only magnitude and no direction, are called **scalar quantities**.

2. **Representation of Vector** A directed line segment has magnitude as well as direction, so it is called vector denoted as $\overrightarrow{AB}$ or simply as $\vec{a}$. Here, the point A from where the vector $\overrightarrow{AB}$ starts is called its initial point and the point B where it ends is called its terminal point.

3. **Magnitude of a Vector** The length of the vector $\overrightarrow{AB}$ or $\vec{a}$ is called magnitude of $\overrightarrow{AB}$ or $\vec{a}$ and it is represented by $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a.

   NOTE Since, the length is never negative, so the notation $|\vec{a}|<0$ has no meaning.

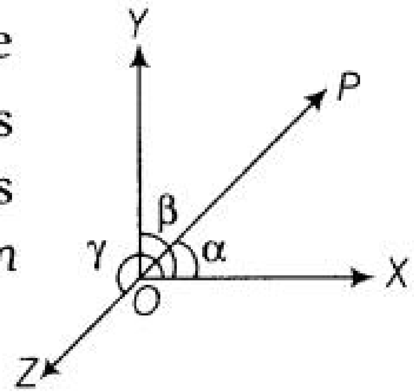
4. **Position Vector** Let $O(0, 0, 0)$ be the origin and P be a point in space having coordinates (x, y, z) with respect to the origin O. Then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point P with respect to O. The magnitude of $\overrightarrow{OP}$ or $\vec{r}$ is given by

$$|\overrightarrow{OP}|=|\vec{r}|=\sqrt{x^2+y^2+z^2}$$

4. **Position Vector** Let $O(0, 0, 0)$ be the origin and P be a point in space having coordinates (x, y, z) with respect to the origin O. Then, the vector $\overrightarrow{OP}$ or $\vec{r}$ is called the position vector of the point P with respect to O. The magnitude of $\overrightarrow{OP}$ or $\vec{r}$ is given by

$$|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

5. **Direction Cosines** If α, β and γ are the angles which a directed line segment OP makes with the positive directions of the coordinate axes OX, OY and OZ respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the direction cosines of OP and are generally denoted by the letters l, m and n, respectively.



i.e. $$l = \cos\alpha, m = \cos\beta, n = \cos\gamma$$

Let l, m and n be the direction cosines of a line and a, b and c be three numbers, such that $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\vec{r}$. Then, direction ratios of the line are proportional to a, b and c.

NOTE $l^2+m^2+n^2=1$

6. Types of Vectors

(i) **Null vector or zero vector** A vector, whose initial and terminal points coincide and magnitude is zero, is called a null vector and denoted as $\vec{0}$.

NOTE Zero vector cannot be assigned a definite direction or it may be regarded as having any direction. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent the zero vector.

(ii) **Unit vector** A vector of unit length is called unit vector. The unit vector in the direction of $\vec{a}$ is $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.

(iii) **Collinear vectors** Two or more vectors are said to be collinear, if they are parallel to the same line, irrespective of their magnitudes and directions. e.g. $\vec{a}$ and $\vec{b}$ are collinear, when $\vec{a}=\pm\lambda\,\vec{b}$ or $|\vec{a}|=\lambda\cdot|\vec{b}|$

(iv) **Coinitial vectors** Two or more vectors having the same initial point are called coinitial vectors.

(v) **Equal vectors** Two vectors are said to be equal, if they have equal magnitudes and same direction regardless of the position of their initial points.

NOTE If $\vec{a} = \vec{b}$, then $|\vec{a}| = |\vec{b}|$ but converse may not be true.

(vi) **Negative vector** Vector having the same magnitude but opposite in direction of the given vector, is called the negative vector. e.g. Vector $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$ and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

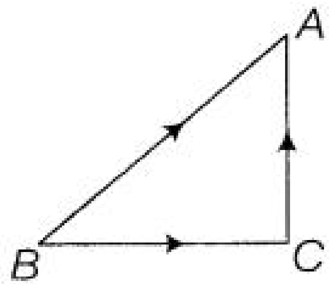
NOTE The vectors defined above are such that any of them may be subject to its parallel displacement without changing its magnitude and direction. Such vectors are called **'free vectors'**.

7. **To Find a Vector when its Position Vectors of End Points are Given** Let $\vec{a}$ and $\vec{b}$ be the position vectors of end points A and B respectively of a line segment AB. Then,

$$\overrightarrow{AB} = \text{Position vector of } \vec{B} - \text{Position vector of } \vec{A}$$
$$= \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$$

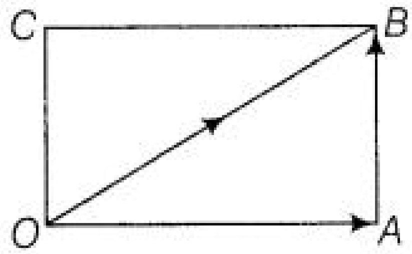
8. **Addition of Vectors**

(i) **Triangle law of vector addition** If two vectors are represented along two sides of a triangle taken in order, then their resultant is represented by the third side taken in opposite direction, i.e. in ΔABC, by triangle law of vector addition, we have



$$\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

(ii) **Parallelogram law of vector addition** If two vectors are represented along the two adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the sides. If the sides OA and OC of parallelogram $OABC$ represent $\overrightarrow{OA}$ and $\overrightarrow{OC}$ respectively, then we get



$$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$$

NOTE Both laws of vector addition are equivalent to each other.

(iii) **Properties of vector addition**

(a) **Commutative** For vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

(b) **Associative** For vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.

NOTE The associative property of vector addition enables us to write the sum of three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ as $\vec{a} + \vec{b} + \vec{c}$ without using brackets.

(c) **Additive identity** For any vector $\vec{a}$, a zero vector $\vec{0}$ is its additive identity as $\vec{a} + \vec{0} = \vec{a}$.

(d) **Additive inverse** For a vector $\vec{a}$, a negative vector of $\vec{a}$ is its additive inverse as $\vec{a} + (-\vec{a}) = \vec{0}$.

9. **Multiplication of a Vector by a Scalar** Let a be a given vector and λ be a scalar, then multiplication of vector $\vec{a}$ by scalar λ, denoted as $\lambda\,\vec{a}$, is also a vector, collinear to the vector $\vec{a}$ whose magnitude is $|\lambda|$ times that of vector $\vec{a}$ and direction is same as $\vec{a}$, if $\lambda > 0$, opposite of $\vec{a}$, if $\lambda < 0$ and zero vector, if $\lambda = 0$.

NOTE For any scalar λ, $\lambda \cdot \vec{0} = \vec{0}$.

Properties of Scalar Multiplication For vectors $\vec{a}$, $\vec{b}$ and scalars p, q, we have

(i) $p(\vec{a}+\vec{b}) = p\,\vec{a}+p\,\vec{b}$

(ii) $(p+q)\,\vec{a} = p\,\vec{a}+q\,\vec{a}$

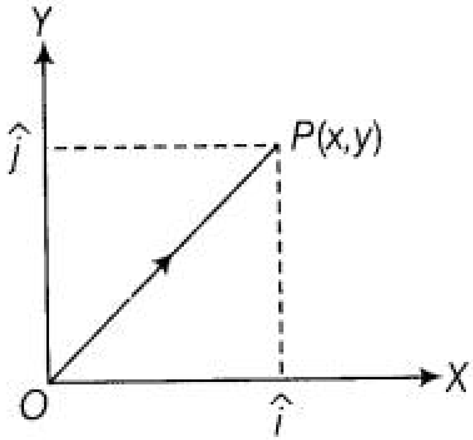
(iii) $p(q\,\vec{a}) = (pq)\,\vec{a}$

NOTE To prove $\vec{a}$ is parallel to $\vec{b}$, we need to show that $\vec{a} = \lambda\,\vec{b}$, where λ is a scalar.

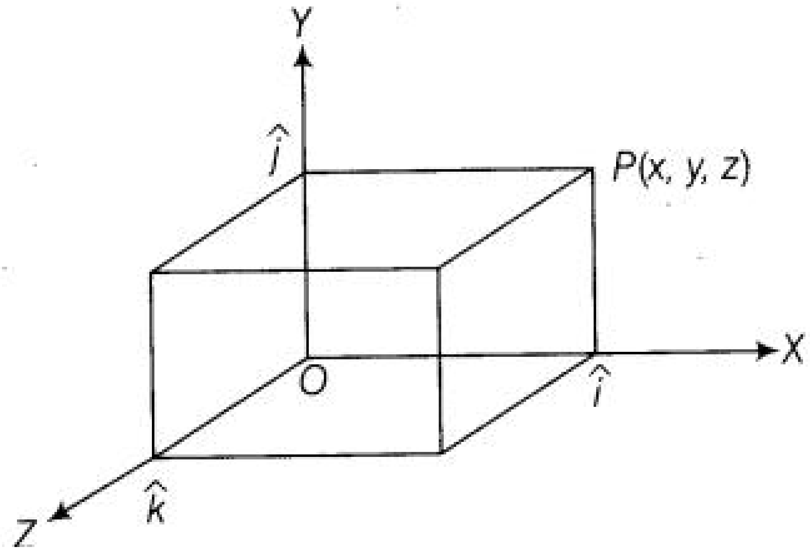
10. **Components of a Vector** Let the position vector of P with reference to O is $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, this form of any vector is called its **component form.** Here, x, y and z are called the scalar components of $\vec{r}$ and $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of $\vec{r}$ along the respective axes.

(i) **Two dimensions** If a point P in a plane has coordinates (x, y), then $\overrightarrow{OP} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY-axes, respectively.

Then, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$.



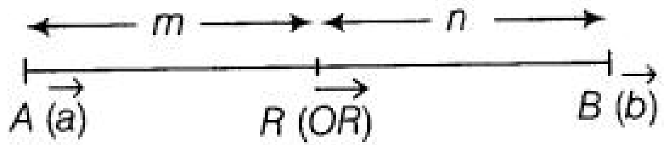
(ii) **Three dimensions** If a point P in a plane has coordinates (x, y, z), then $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, where $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ-axes, respectively. Then, $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$.



11. **Vector Joining of Two Points** If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then the vector joining P_1 and P_2 is the vector $\overrightarrow{P_1P_2}$.

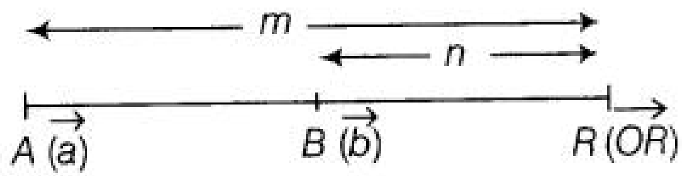
i.e.
$$\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

12. **Section Formula** Position vector $\overrightarrow{OR}$ of point R, which divides the line segment joining the points A and B with position vectors $\vec{a}$ and $\vec{b}$ respectively, internally in the ratio $m:n$ is given by



$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

For external division,



$$\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

NOTE Position vector of mid-point of the line segment joining end points $A(\vec{a})$ and $B(\vec{b})$ is given by
$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$

1. **Dot Product of Two Vectors** If θ is the angle between two vectors a and b, then the scalar or dot product denoted by $\vec{a} \cdot \vec{b}$ is given by $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where $0 \le \theta \le \pi$.

NOTE (i) $\vec{a} \cdot \vec{b}$ is a real number (ii) If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then θ is not defined.

Properties of dot product of two vectors $\vec{a}$ and $\vec{b}$ are as follows:

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ [i.e. dot product is commutative].

(ii) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not defined.

(iii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ [distributive property]

(iv) If $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then $\vec{a} \cdot \vec{b} = 0$, converse is also true.

(v) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ and projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$.

(vi) If $\theta = 0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta = \pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

(vii) If $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

(viii) Angle between two vectors $\vec{a}$ and $\vec{b}$ is

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} \quad \text{or} \quad \theta = \cos^{-1}\left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right]$$

(ix) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

(x) $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = 1$

(xi) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(xii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

(xiii) $(\lambda\cdot\vec{a})\cdot\vec{b} = \lambda(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\lambda\cdot\vec{b})$, where λ is any scalar.

(xiv) If $\theta = \pi$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$; If $\theta = \pi$, then $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

2. **Vector** (or Cross) **Product of Vectors** If θ is the angle between two non-zero, non-parallel vectors $\vec{a}$ and $\vec{b}$, then the cross product of vectors, denoted by $\vec{a}\times\vec{b}$ is given by

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}, \text{ such that } 0 \le \theta \le \pi$$

where, $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

NOTE

(i) $\vec{a}\times\vec{b}$ is a vector quantity, whose magnitude is $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$.

(ii) If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined.

Properties of cross product of two vectors $\vec{a}$ and $\vec{b}$ are as follows:

(i) Angle between two vectors is $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ or $\theta=\sin^{-1}\left[\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right]$

(ii) $\vec{a}\times\vec{a}=0$

(iii) $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$

(iv) In general, $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

(v) $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ [distributive property]

(vi) $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$

(vii) If $\vec{a}$ is parallel to $\vec{b}$, then $\vec{a}\times\vec{b}=\vec{0}$ and converse is also true.

(viii) If $\theta=\dfrac{\pi}{2}$, then $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|$.

(ix) Area of parallelogram whose adjacent sides are along $\vec{a}$ and $\vec{b}=|\vec{a}\times\vec{b}|$.

(x) Area of triangle, whose adjacent sides are along $\vec{a}$ and $\vec{b} = \frac{1}{2}|\vec{a} \times \vec{b}|$.

(xi) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$ and $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(xii) $\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$

(xiii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\Rightarrow \quad (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

(xiv) Unit vector $\hat{n}$, which is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, is given by

$$\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

(xv) For vectors $\vec{a}$ and $\vec{b}$, if $\vec{a} \times \vec{b} = \vec{0}$, then either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a} \parallel \vec{b}$.

3. **Scalar Triple Product of Vectors** Suppose $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors. Then, scalar product of $\vec{a}$ and $\vec{b}\times\vec{c}$, i.e. $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of $\vec{a}\cdot\vec{b}$ and $\vec{c}$ and it is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.

Properties of scalar triple product For vectors $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$, $\vec{b}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$ and $\vec{c}=a_3\hat{i}+b_3\hat{j}+c_3\hat{k}$

(i) $[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ (ii) $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{c}\ \vec{a}]=[\vec{c}\ \vec{a}\ \vec{b}]$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]=-[\vec{b}\ \vec{a}\ \vec{c}]=-[\vec{c}\ \vec{b}\ \vec{a}]=-[\vec{a}\ \vec{c}\ \vec{b}]$ (iv) $[\vec{a}\ \vec{a}\ \vec{b}]=[\vec{b}\ \vec{b}\ \vec{a}]=[\vec{c}\ \vec{c}\ \vec{b}]=0$

(v) $[k\vec{a}\ \vec{b}\ \vec{c}]=k[\vec{a}\ \vec{b}\ \vec{c}]$

4. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, if and only if $\vec{a}\cdot(\vec{b}\times\vec{c})=0$