1. **Direction Cosines of a Line** If the directed line OP makes angles α, β and γ with positive X-axis, Y-axis and Z-axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, are called direction cosines of a line. They are denoted by l, m and n. Therefore, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$. Also, sum of squares of direction cosines of a line is always 1, i.e. $l^2 + m^2 + n^2 = 1$ or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

NOTE Direction cosines of a directed line are unique.

2. **Direction Ratios of a Line** Number proportional to the direction cosines of a line, are called direction ratios of a line.

   (i) If a, b and c are direction ratios of a line, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

   (ii) If a, b and c are direction ratios of a line, then its direction cosines are

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

   (iii) Direction ratios of a line PQ passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ and direction cosines are $\frac{x_2 - x_1}{|\overrightarrow{PQ}|}$, $\frac{y_2 - y_1}{|\overrightarrow{PQ}|}, \frac{z_2 - z_1}{|\overrightarrow{PQ}|}$.

NOTE (i) Direction ratios of two parallel lines are proportional.

(ii) Direction ratios of a line are not unique.

3. **Straight line** A straight line is a curve, such that all the points on the line segment joining any two points of it lies on it.

4. Equation of a Line through a Given Point and parallel to a given vector $\vec{b}$

(i) **Vector form** $\vec{r} = \vec{a} + \lambda \vec{b}$

where, $\vec{a}$ = Position vector of a point through which the line is passing

$\vec{b}$ = A vector parallel to a given line

(ii) **Cartesian form** $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

where, (x_1, y_1, z_1) is the point through which the line is passing through and a, b, c are the direction ratios of the line.

If l, m and n are the direction cosines of the line, then the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

Remember point Before we use the DR's of a line, first we have to ensure that coefficients of x, y and z are unity with positive sign.

5. Equation of Line Passing through Two Given Points

(i) **Vector form** $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$, where $\vec{a}$ and $\vec{b}$ are the position vectors of the points through which the line is passing.

(ii) **Cartesian form** $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

where, (x_1, y_1, z_1) and (x_2, y_2, z_2) are the points through which the line is passing.

6. **Angle between Two Lines**

(i) **Vector form** Angle between the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given as

$$\cos\theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|} \right|$$

where, θ is the acute angle between the lines.

(ii) **Cartesian form** If θ is the angle between the lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$, then $\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$

or $$\sin\theta = \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Also, angle (θ) between two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

or $$\sin\theta = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$$

7. **Condition of Perpendicularity** Two lines are said to be perpendicular, when in vector form $\vec{b}_1 \cdot \vec{b}_2 = 0$; in cartesian form $a_1a_2 + b_1b_2 + c_1c_2 = 0$

or $\qquad l_1l_2 + m_1m_2 + n_1n_2 = 0$ [direction cosine form]

8. **Condition that Two Lines are Parallel** Two lines are parallel, when in vector form $\vec{b}_1 \cdot \vec{b}_2 = |\vec{b}_1||\vec{b}_2|$; in cartesian form $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

or $\qquad \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$ [direction cosine form]

9. **Shortest Distance between Two Lines** Two non-parallel and non-intersecting straight lines, are called **skew lines.**

For skew lines, the line of the shortest distance will be perpendicular to both the lines.

(i) **Vector form** If the lines are $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$. Then, shortest distance

$d = \left| \dfrac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$, where $\vec{a}_2$, $\vec{a}_1$ are position vectors of point through which the line is passing and $\vec{b_1}, \vec{b_2}$ are the vectors in the direction of a line.

(ii) **Cartesian form** If the lines are $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$.

Then, shortest distance, $d=\left|\dfrac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$

10. **Distance between two Parallel Lines** If two lines l_1 and l_2 are parallel, then they are coplanar. Let the lines be $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$, then the distance between parallel lines is $\left|\dfrac{\vec{b}\times(\vec{a_2}-\vec{a_1})}{|\vec{b}|}\right|$.

NOTE If two lines are parallel, then they both have same DR's.

11. **Distance between Two Points** The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

12. **Mid-point of a Line** The mid-point of a line joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$.

1. **Plane** A plane is a surface such that a line segment joining any two points of it lies wholly on it. A straight line which is perpendicular to every line lying on a plane, is called a **normal** to the plane.

2. **Equations of a Plane in Normal form**

   (i) **Vector form** The equation of plane in normal form is given by $\vec{r} \cdot \vec{n} = d$, where $\vec{n}$ is a vector which is normal to the plane.

   (ii) **Cartesian form** The equation of plane is given by $ax + by + cz = d$, where a, b and c are the direction ratios of plane and d is the distance of plane from origin.

   Another equation of plane is $lx + my + nz = p$, where l, m and n are direction cosines of perpendicular from origin and p is a distance of a plane from origin.

NOTE If d is the distance from the origin and l, m and n are the direction cosines of the normal to the plane through the origin, then the foot of the perpendicular is (ld, md, nd).

3. Equation of a Plane Perpendicular to a given Vector and Passing Through a given Point

(i) **Vector form** Let a plane passes through a point A with position vector $\vec{a}$ and perpendicular to the vector $\vec{n}$, then $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.

This is the vector equation of the plane.

(ii) **Cartesian form** Equation of plane passing through point (x_1, y_1, z_1) is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where, a, b and c are the direction ratios of normal to the plane.

4. Equation of Plane Passing through Three Non-collinear Points

(i) **Vector form** If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of three given points, then equation of a plane passing through three non-collinear points is $(\vec{r} - \vec{a}) \cdot \{(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\} = 0$.

(ii) **Cartesian form** If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are three non-collinear points, then equation of the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

If above points are collinear, then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$

5. **Equation of Plane in Intercept Form** If a, b and c are x-intercept, y-intercept and z-intercept, respectively made by the plane on the coordinate axes, then equation of plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

6. **Equation of Plane Passing through the Line of Intersection of two given Planes**

   (i) **Vector form** If equation of the planes are $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then equation of any plane passing through the intersection of planes is

   $$\vec{r}\cdot(\vec{n_1}+\lambda\vec{n_2})=d_1+\lambda d_2$$

   where, λ is a constant and calculated from given condition.

   (ii) **Cartesian form** If the equation of planes are $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$, then equation of any plane passing through the intersection of planes is $a_1x+b_1y+c_1z-d_1+\lambda(a_2x+b_2y+c_2z-d_2)=0$
   where, λ is a constant and calculated from given condition.

7. **Coplanarity of Two Lines**

   (i) **Vector form** If two lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, then $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0$.

(ii) **Cartesian form** If two lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ are coplanar, then $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0.$

8. **Angle between Two Planes** Let θ be the angle between two planes.

(i) **Vector form** If $\vec{n_1}$ and $\vec{n_2}$ are normals to the planes and θ be the angle between the planes $\vec{r}\cdot\vec{n_1}=d_1$ and $\vec{r}\cdot\vec{n_2}=d_2$, then θ is the angle between the normals to the planes drawn from some common points.

$$\cos\theta=\left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\;|\vec{n_2}|}\right|.$$

NOTE The planes are perpendicular to each other, if $\vec{n_1}\cdot\vec{n_2}=0$ and parallel, if $\vec{n_1}\cdot\vec{n_2}=|\vec{n_1}||\vec{n_2}|$.

(ii) **Cartesian form** If the two planes are $a_1x+b_1y+c_1z=d_1$ and $a_2x+b_2y+c_2z=d_2$, then $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$.

NOTE Planes are perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0$ and planes are parallel, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

9. Distance of a Point from a Plane

(i) **Vector form** The distance of a point whose position vector is $\vec{a}$ from the plane $\vec{r}\cdot\hat{n}=d$ is $|d-\vec{a}\cdot\hat{n}|$.

NOTE (i) If the equation of the plane is in the form $\vec{r}\cdot\vec{n}=d$, where $\vec{n}$ is normal to the plane, then the perpendicular distance is $\dfrac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$.

(ii) The length of the perpendicular from origin O to the plane $\vec{r}\cdot\vec{n}=d$ is $\dfrac{|d|}{|\vec{n}|}$. $[\because \vec{a}=0]$

(ii) **Cartesian form** The distance of the point (x_1, y_1, z_1) from the plane $Ax+By+Cz=D$ is

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$$

10. Angle between a Line and a Plane

(i) **Vector form** If the equation of line is $\vec{r}=\vec{a}+\lambda\vec{b}$ and the equation of plane is $\vec{r}\cdot\vec{n}=d$, then the angle θ between the line and the normal to the plane is $\cos\theta=\left|\dfrac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$

and so the angle ϕ between the line and the plane is given by $90°-\theta$,
i.e. $\sin(90°-\theta)=\cos\theta$

i.e. $$\sin\phi=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

(ii) **Cartesian form** If a, b and c are the DR's of line and $lx+my+nz+d=0$ be the equation of plane, then

$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}.$$

If a line is parallel to the plane, then $al+bm+cn=0$ and if line is perpendicular to the plane, then $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

11. **Remember Points**

(i) If a line is parallel to the plane, then normal to the plane is perpendicular to the line.

i.e. $$a_1a_2+b_1b_2+c_1c_2=0$$

(ii) If a line is perpendicular to the plane, then DR's of line are proportional to the normal of the plane.

i.e. $$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

where, a_1, b_1 and c_1 are the DR's of a line and a_2, b_2 and c_2 are the DR's of normal to the plane.