1. **Determinant** Determinant is the numerical value of the square matrix. So, to every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A. It is denoted by det A or $|A|$.

   NOTE

   (i) Read $|A|$ as determinant A not absolute value of A.

   (ii) Determinant gives numerical value but matrix do not give numerical value.

   (iii) A determinant always has equal number of rows and columns, i.e. only square matrix have determinants.

2. Value of a Determinant

(i) Value of determinant of a matrix of order 2, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

(ii) Value of determinant of a matrix of order 3, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by expressing

it in terms of second order determinant. This is known as expansion of a determinant along a row (or column).

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ (expansion along first row } R_1\text{)}$$

NOTE

(i) For easier calculations of determinant, we shall expand the determinant along that row or column which contains maximum number of zeroes.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or −1 according as $(i + j)$ is even or odd.

3. Let A be a matrix of order n and let $|A| = x$. Then, $|kA| = k^n \cdot |A| = k^n \cdot x$, where $n = 1, 2, 3, \ldots$

4. **Minor** Minor of an element a_{ij} of a determinant, is a determinant obtained by deleting the ith row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

NOTE Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $(n - 1)$.

5. **Cofactor** Cofactor of an element a_{ij} of a determinant, denoted by A_{ij} or C_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of an element a_{ij}.

NOTE

(i) For expanding the determinant, we can use minors and cofactors as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

and

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

(ii) If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

6. **Singular and non-singular Matrix** If the value of determinant corresponding to a square matrix is zero, then matrix is said to be a singular matrix, otherwise it is non-singular matrix, i.e. for a square matrix A, if $|A| \neq 0$, then it is said to be a non-singular matrix and of $|A| = 0$, then it is said to be a singular matrix.

Theorems

(i) If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

(ii) The determinant of the product of matrices is equal to product of their respective determinants, i.e. $|AB| = |A||B|$, where A and B are square matrix of same order.

7. **Adjoint of a Matrix** The adjoint of a square matrix 'A' is transpose of the matrix which obtained by cofactors of each element of a determinant corresponding to that given matrix. It is denoted by adj (A).

In general, adjoint of a matrix $A = [a_{ij}]_{n \times n}$ is a matrix $[A_{ji}]_{n \times n}$, where A_{ji} is a cofactor of element a_{ji}.

8. Properties of Adjoint of a Matrix

If A is a square matrix of order $n \times n$, then

(i) $A(\operatorname{adj}A) = (\operatorname{adj}A)A = |A|I_n$

(ii) $|\operatorname{adj}A| = |A|^{n-1}$

(iii) $\operatorname{adj}(A^T) = (\operatorname{adj}A)^T$

9. The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

NOTE Since, area is a positive quantity we always take the absolute value of the determinant.

To find the value of the determinant, we try to make the maximum possible zero in a row (or a column) by using properties given below and then expand the determinant corresponding that row (or column).

Following are the various properties of determinants:

1. If all the elements of any row or column of a determinant are zero, then the value of a determinant is zero.

2. If each element of any one row or one column of a determinant is a multiple of scalar k, then the value of the determinant is a multiple of k.

i.e.
$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

3. If in a determinant any two rows or columns are interchanged, then the value of the determinant obtained is negative of the value of the given determinant. If we make n such changes of rows (columns) in determinant Δ and obtain determinant Δ , then $\Delta_1 = (-1)^n \Delta$.

i.e.
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

4. If all corresponding elements of any two rows or columns of a deter..iinant are identical or proportional, then the value of the determinant is zero.

i.e.
$$\begin{vmatrix} a & b & c \\ b & e & f \\ a & b & c \end{vmatrix} = 0 \qquad\qquad [\because R_1 \text{ and } R_3 \text{ are identical.}]$$

5. The value of a determinant remains unchanged on changing rows into columns and columns into rows. It follows that, if A is a square matrix, then $|A'| = |A|$.

i.e.
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

NOTE $\det(A) = \det(A')$, where $A' = $ transpose of A.

6. If some or all elements of a row or column of a determinant are expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants, i.e.

$$\begin{vmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix}.$$

7. In the elements of any row or column of a determinant, if we add or subtract the multiples of corresponding elements of any other row or column, then the value of determinant remains unchanged, i.e.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix} (C_1 \rightarrow C_1 + kC_2)$$

In other words, the value of determinants remain same, if we apply the operation $R_i \rightarrow R_i + kE_j$ or $C_i \rightarrow C_i + kC_j$.

1. **Inverse of a Square Matrix** If A is a non-singular matrix (i.e. $|A| \neq 0$), then

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the inverse is $A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of A.

NOTE Inverse of a matrix, if exists, is unique.

Properties of a Inverse Matrix

(i) $(A^{-1})^{-1} = A$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$

(iv) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(v) $\text{adj}(A^{-1}) = (\text{adj} A)^{-1}$

2. **Solution of system of linear equations using inverse of a matrix.**

Let the given system of equations be $a_1 x + b_1 y + c_1 z = d_1$; $a_2 x + b_2 y + c_2 z = d_2$ and $a_3 x + b_3 y + c_3 z = d_3$.

We write the following system of linear equations in matrix form as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Case **I** If $|A| \neq 0$, then system is consistent and has a unique solution which is given by $X = A^{-1}B$.

Case **II** If $|A| = 0$ and $(\text{adj } A) B \neq O$, then system is inconsistent and has no solution.

Case **III** If $|A| = 0$ and $(\text{adj } A) B = O$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solutions