Class 10th

Mathematics

Short Note

QUADRATIC EQUATIONS

- **INTRODUCTION**

When a polynomial f(x) is equated to zero, we get an equation which is known as a polynomial equation.

- **QUADRATIC EQUATIONS**

A polynomial equations of degree two is called a quadratic equation.

$2x^2 - 3x + 1 = 0$, $4x - 3x^2 = 0$ and $1 - x^2 = 0$

General form of quadratic equations : $ax^2 + bx + c = 0$, where a,b,c, are real numbers and $a \neq 0$.

Roots of quadratic equation : $x = \alpha$ is said to be root of the quadratic equation ax2 + bx + c = 0, $a \neq 0$ iff $x = \alpha$ satisfies the quadratic equation i.e. in other words the value of $a\alpha^2 + b\alpha + c$ is zero.

- **METHODS OF SOLVING QUADRATIC EQUATIONS**

Solution by factorization method

Solve the following quadratic equation by factorization method $x^2 - 2ax + a^2 - b^2 = 0$

Factors of the constant term $a^2 - b^2$ are$(a - b)$&$(a + b)$ also coefficient of the middle term $= -2a = -[(a - b)+(a + b)]$

$\Rightarrow \quad x^2 - 2ax + a^2 - b^2 = 0$

$\Rightarrow \quad x^2 - \{(a - b) + (a + b)\} x + (a + b)(a - b) = 0$

$\Rightarrow \quad x^2 - (a - b) x - (a + b) x + (a - b)(a + b) = 0$

$\Rightarrow \quad x [x - (a - b)] - (a + b) [x - (a - b)] = 0$

$\Rightarrow \quad [x - (a - b)] [x - (a + b)] = 0$

$x - (a - b) = 0$ or $x - (a + b) = 0$

$x = a - b$, $x = a + b$

Solution by Completion of Squares Method

In the method of completing the squares, the quadratic equation is expressed in the form $(x\pm k)^2=p^2$.

Consider the quadratic equation $2x^2-8x=10$

(i) Express the quadratic equation in standard form.

$2x^2-8x-10=0$

(ii) Divide the equation by the coefficient of x^2 to make the coefficient of x^2 equal to 1.

$x^2-4x-5=0$

(iii) Add the square of half of the coefficient of x to both sides of the equation to get an expression of the form $x^2\pm2kx+k^2$.

$(x^2-4x+4)-5=0+4$

(iv) Isolate the above expression, $(x\pm k)^2$ on the LHS to obtain an equation of the form $(x\pm k)^2=p^2$

$(x-2)^2=9$

(v) Take the positive and negative square roots.

$x - 2 = \pm 3$

$x = -1$ or $x = 5$

Solution by Quadratic Formula

Quadratic Formula is used to directly obtain the roots of a quadratic equation from the standard form of the equation. For the quadratic equation $ax^2+bx+c=0$,

$$x=\frac{-b \pm \sqrt{b^2-4ac})}{2a}$$

By substituting the values of a,b and c, we can directly get the roots of the equation.

NATURE OF THE ROOTS OF THE QUADRATIC EQUATION:-

Based on the value of the discriminant, $D=b^2-4ac$, the roots of a quadratic equation, $ax^2 + bx + c = 0$, can be of three types.

Case 1: If $D>0$, the equation has two distinct real roots.

Case 2: If $D=0$, the equation has two equal real roots.

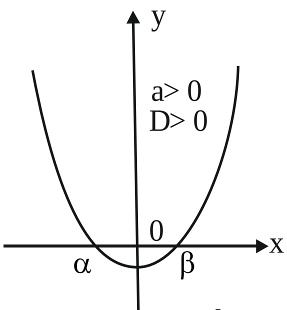
Case 3: If $D<0$, the equation has no real roots.

GEOMETRICAL REPRESENTATION OF QUADRATIC EXPRESSION

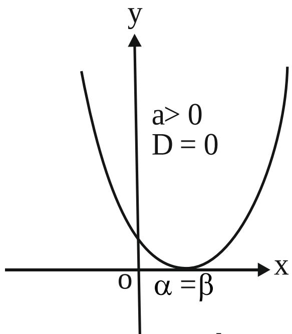
For quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a,b,c \in R$ then :

(i) The graph between x,y is always a parabola. If $a > 0$, then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

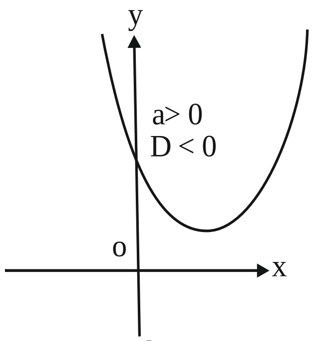
(ii) The graph of $y = ax^2 + bx + c$ can be divided into 6 categories which are as follows :



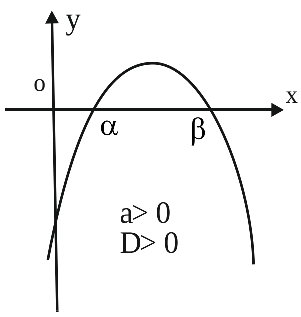
Roots are real and distinct



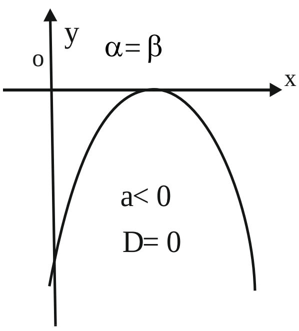
Roots are real and equal



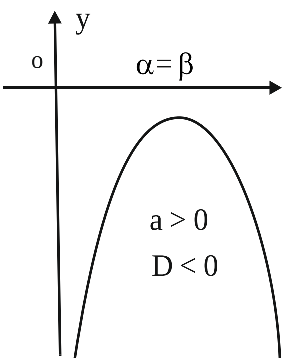
No real roots



Roots are real and distinct



Roots are equal



No real roots

ROOTS UNDER PARTICUALR CASES

(A) Let the quadratic equation $ax^2 + bx + c = 0$ has real roots and

(i) If $b = 0 \Leftrightarrow$ roots are of equal magnitude but of opposite sign.

(ii) If c = 0 $\Leftrightarrow$ one roots is zero and the other is $-\frac{b}{a}$

(iii) If a = c $\Leftrightarrow$ roots are of opposite sign.

(iv) If $\left.\begin{matrix} a > 0 c < 0 \\ a < 0 c > 0 \end{matrix}\right\}$ $\Leftrightarrow$ roots are of opposite sign.|

(v) If $\left.\begin{matrix} a > 0, b > 0, c < 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\}$ $\Leftrightarrow$ both roots are negative $(\alpha + \beta < 0 \& \alpha\beta > 0)$

(vi) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\}$ $\Leftrightarrow$ both roots are positive $(\alpha + \beta < 0 \& \alpha\beta > 0)$

(vii) If a + b + c = 0 $\Leftrightarrow$ One of the roots is 1 and the other roots is $\frac{c}{a}$.

(viii) If a = 1 b, c $\in$ Z and the roots are rational numbers, then these roots must be integers.

(ix) If a, b, c $\in$ Q and D is a perfect square $\Leftrightarrow$ roots are rational.

(x) (A) If a, b, c $\in$ Q and D is positive but not a perfect square $\Leftrightarrow$ roots are irrational.

(B) If $ax^2 + bx + c = 0$ is satisfied by more then two values, it is an identity and a = b = c = 0 and vice versa

(C) The quadratic equation whose roots are reciprocal of the roots of $ax^2 + bx + c = 0$ is $cx^2 + bx + a = 0$.

SUM & PRODUCT OF THE ROOTS:-

Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

Then $\alpha = \frac{-b+\sqrt{+b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{+b^2-4ac}}{2a}$

$\therefore$ The sum of roots $\alpha+\beta = -\frac{b}{a} = -\frac{Ceoff.of\ x}{Ceoff.of\ x^2}$

and product of roots = $\alpha.\beta = \frac{c}{a} = -\frac{\cos\tan t\ term}{coefficient\ of\ x^2}$

FORMATION OF QUADRATIC EQUATION:-

Let α and β be the roots of the quadratic equation

$\therefore$ $\alpha+\beta = -\frac{b}{a}$ and $\alpha.\beta = \frac{c}{a}$

Hence the quadratic equation whose roots are α and β is given by

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$

i.e. x^2 – (sum of the roots) x + product of the roots = 0

CONDITION FOR TWO QUADRATIC EQUATION TO HAVE A COMMON ROOT:-

Suppose that the quadratic equation $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ (where a, a' $\neq$ 0 and ab' – a'b $\neq$ 0) have a common root. Let this common root be α Than $a\alpha^2 b\alpha + c = 0$ and $a'\alpha^2 b'\alpha + c' = 0$

Solving the above equations, we get,

$$\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$$

$$\Rightarrow \alpha^2 = \frac{bc'-b'c}{ab'-a'b} \text{ and } \alpha = \frac{a'c-ac'}{ab'-a'b}$$

Eliminating α, we get : $\frac{(a'c-ac')^2}{(ab-a'b)^2}=\frac{bc'-b'c}{ab'-a'b}$

$\Rightarrow\ (a'c-ac')^2=(bc'-b'c)(ab'-a'b)$

CONDITION FOR TWO QUADRATIC EQUATION TO HAVE THE SAME ROOT:-

Two quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have the same roots if and only if

$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

APPLICATIONS OF QUADRATIC EQUATIONS

Type-I : Problems Based On Numbers.

The difference of two numbers is 3 and their product is 504. Find the numbers.

Let the required numbers be x and (x – 3). Then,

$x(x-3) = 504$

$\Rightarrow\ x^2 - 3x - 504 = 0 \Rightarrow x^2 - 24x + 21x - 504 = 0$

$\Rightarrow\ x(x-24) + 21(x-24) = 0 \Rightarrow (x-24)(x+21) = 0$

$\Rightarrow\ x - 24 = 0$ or $x + 21 = 0 \Rightarrow x = 24$ or $x = -21$

If x = – 21, then the numbers are – 21 and – 24.

Again, if x = 24, then the numbers are 24 and 21.

Hence, the numbers are – 21, – 24 or 24, 21

Type-II :Problems Based On Ages :

Seven years ago Varun's age was five times the square of Swati's age. Three years hence, Swati's age will be two fifth of Varun's age. Find their present ages.

Let the present ages of Varun and Swati be x years and y years respectively.

Seven years ago,

Varun's age = (x – 7) years and Swati's age = (y – 7) years.

$\therefore\ (x-7) = 5(y-7)^2 \Rightarrow x - 7 = 5(y^2 - 14y + 49)$

$\Rightarrow\ x = 5y^2 - 70y + 245 + 7 \Rightarrow x = 5y^2 - 70y + 252$ …(i)

Three years hence,

Varun's age = (x + 3) years and Swati's age = (y + 3) years.

$(y+3)=\frac{2}{5}(x+3)\Rightarrow 5y+15=2x+6\Rightarrow x=\frac{5y+9}{2}$ …(ii)

From (i) and (ii) we get $5y^2-70y+252=\frac{5y+9}{2}$

$\Rightarrow\ 10y^2 - 140y + 504 = 5y + 9 \Rightarrow 10y^2 - 145y + 495 = 0 \Rightarrow 2y^2 - 29y + 99 = 0$

$\Rightarrow\ 2y^2 - 18y - 11y + 99 = 0 \Rightarrow 2y(y-9) - 11(y-9) = 0$

$\Rightarrow\ (y-9)(2y-11) = 0 \Rightarrow y = 9$ or $y=\frac{11}{2}$

$\because\ y=\frac{11}{2}$ is not possible $\left[\because \frac{11}{2}<7\right]$

So, y = 9.

$\therefore \quad \frac{5\times9+9}{2}=27$ [From (ii)]

Hence, the Varu's present age is 27 years and Swati's present age is 9 years..

Type-III : Problems Based On Geometrical Concepts :

The length of the hypotenuse of a right triangle exceeds the length of the base by 2 cm and exceeds twice the length of the altitude by 1 cm. Find the length of each side of the triangle.

Let ΔABC be a right triangle, right angled at B.

Let AB = x. Then

AC = (2x + 1) and BC = (2x + 1) – 2 = 2x – 1

$\Rightarrow$ ΔABC, $AC^2 = AB^2 + BC^2$ [By Pythagoras theorem]

$\Rightarrow$ $(2x + 1)^2 = x^2 + (2x – 1) \Rightarrow 4x^2 + 4x + 1 = x^2 + 4x^2 – 4x + 1$

$\Rightarrow$ $x^2 = 8x \Rightarrow x = 8$ cm,

$\therefore$ BC = 2x – 1 = $2\times8 + 1$ = 15 cm

AC = 2x + 1 = $2\times8 + 1$ = 17 cm

Hence, the sides of the given triangle are 8cm, 15 cm and 17 cm.

Type-IV : Problems Based On Perimeter/Age :

Is it possible to design a rectangular park of perimeter 80 cm and area 400 m^2? If so, find its length and breadth.

Let the length and breadth of the rectangular park be ℓ and b respectively. Then,

$2(\ell + b) = 80$

$\ell + b = 40 \Rightarrow \ell = (40 – b)$

And area of the park = 400 m^2

$\therefore \ell b = 400$

$\Rightarrow$ $(40 – b) b = 400 \Rightarrow 40b – b^2 = 400$

$\Rightarrow$ $b^2 – 40b + 400 = 0 \Rightarrow b^2 – 20b + 400 = 0$

$\Rightarrow$ $b (b – 20) – 20 (b – 20) = 0 \Rightarrow (b – 20) (b – 20) = 0$

$\Rightarrow$ $(b – 20)^2 = 0 \Rightarrow b – 20 \Rightarrow b = 20$ m

$\therefore$ $\ell = 40 – b = 40 – 20 = 20$ m

Hence, length and breadth of the park are 20 m and 20 m respectively.

Thus, it is possible to design a rectangular park of perimeter 80 m and area 400 m^2

Type-V : Problems Based On Time and Distance :

A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less tor the same journey. Find the speed of the train.

Let the speed of the train be x km/h. Then,

Time taken to cover the distance of 360 km = $\frac{360}{x}$ hours.

If the speed of the train increased by 5 km/h. Them,

Time taken to cover the same distance = $\left(\frac{360}{x+5}\right)$h

According to the question, $\frac{360}{x}-\frac{360}{x+5}=1$

$$\Rightarrow \frac{360(x+5)-360x}{x(x+5)}=0 \Rightarrow 360x+1800-360x=x^2+5x$$

$$\Rightarrow x^2+5x-1800=0 \Rightarrow x^2+45x-40x-1800=0$$

$$\Rightarrow x(x+45)-40(x+45)=0 \Rightarrow (x+45)(x-40)=0$$

$$\Rightarrow x=-45 \text{ or } x=40$$

But the speed can not be negative.

Hence, the speed of the train is 40 km/h.

Type-VI : Problems Based On Time and Work :

Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank respectively. Find the time in which each tap can separately fill the tank.

Let the tap of larger diameter takes x hours to fill the tank. Then, the tap of smaller diameter takes (x + 10) hours to fill the tank.

$\therefore$ The portion of tank filled by the larger tap in one hour $=\frac{1}{x}$, the portion of tank filled by the smaller tap in one hour $=\frac{1}{x+10}$

And the portion of tank filled by both the smaller and the larger tap in one hour $=\frac{1}{9\frac{3}{8}}=\frac{8}{75}$

$$\therefore \quad \frac{1}{x}+\frac{1}{x+10}=\frac{8}{75}$$

$$\Rightarrow \quad \frac{x+10+x}{x(x+10)}=\frac{8}{75} \Rightarrow \frac{2x+10}{x^2+10x}=\frac{8}{75}$$

$$\Rightarrow \quad 15x+750=8x^2+80x \Rightarrow 8x^2-70x-750=0$$

$$\Rightarrow \quad 4x^2-35x-375=0 \Rightarrow 4x^2-60x+25x-375=0$$

$$\Rightarrow \quad 4x(x-15)+25(x-15)=0 \Rightarrow (x-15)(4x+25)=0$$

$$\Rightarrow \quad x=15 \text{ or } x=\frac{-25}{4}$$

But the speed can not be negative.

Hence, the larger tap takes 15 hours and the smaller tap takes 25 hours.

Type-VI : Miscellaneous Problems :

300 apples are distributed equally among a certain number of students. Had there been 10 more students, each would have received one apple less. Find the number of students.

Let the number of students be x. Then,

The number of apples received by each student $=\frac{300}{x}$

if there is 10 more students, i.e., (x + 10) students. Then,

The number of apples received by each student = $\frac{300}{x+10}$

According to the question, $\frac{300}{x} - \frac{300}{x+10} = 1$

$\Rightarrow \frac{300x + 3000 - 300x}{x(x+10)} = 1 \Rightarrow 3000 = x^2 + 10x$

$\Rightarrow x^2 + 10x - 3000 = 0 \Rightarrow x^2 + 60x - 50x - 3000 = 0$

$\Rightarrow x(x+60) - 50(x+60) = 0 \Rightarrow (x+60)(x-50) = 0$

$\Rightarrow x = -60$ or $x = 50$

But the number of students can not be negative.

Hence, the number of students is 50.