Decision Models 1: Optimization & Linear Programming

Business Analytics

NYU TANDON SCHOOL OF ENGINEERING

Review

1. Regression Applied
   a. Model strength
   b. Model interpretation
   c. Dummy variables
   d. Non-linear transformations
2. Prediction
3. Classification

Lesson Objectives

1. Decision Models
   a. Linear Programming
   b. Optimization Models
2. Sensitivity Analysis
3. Integer Linear Programming
4. Non-linear Programming

Optimization

- Field of management science that finds the optimal or most efficient way of using limited resources to achieve business objectives.
- Applications:
    - Capital Budgeting
    - Portfolio Selection
    - Production Planning
    - Inventory Management
    - Logistics and Transportation
    - Pricing and Revenue Management
    - Advertising
    - Product Mix

Linear Programming

- Linear programming (LP; also called linear optimization) describes a broad class of optimization tasks where the requirements are represented by linear relationships.
  - satisfy a set of linear equations and/or inequalities
  - maximize or minimize a given linear objective function

Why use linear programming?

- Linear programs are easy (efficient) to solve
    - Commercial solver: millions variables & constraints
- The best (optimal) solution is guaranteed to be found (if it exists).
- Useful sensitivity analysis information is generated.
- Many problems are essentially linear: a large class of problems can be captured by linear programming.
- Good approximation for some nonlinear problems.

Example

A chocolatier has two products: its flagship assortment of triangular chocolates,called Pyramide, and the more decadent and deluxe Pyramide Nuit.

- How much of each should it produce to maximize profits?

Decision Modeling Process

1. Managerial definition
2. Mathematical formulation
3. Solution methodology
4. Conclusions and recommendations

Managerial Decision

- Decisions to be made – decision variables
- Performance measure – objective
- Things that restrict the manager’s range of choice - constraints

Decision Variables

- Represent parameters under management's control, to be decided by management.
- Should be defined so as to describe all possible alternative decisions.
- Examples:
    - whether to support an R&D project or not
    - number of salespeople to hire
    - production level in a given period
    - amount to buy from a given supplier
- Together, the values of the decision variables define a **policy** or a **plan of action**.

Objective Function

- Defines the goal of the problem, the target to be optimized
- Provides a criterion to compare alternate solutions
- Expressed as a function of the decision variables
- Examples:
    - Profit → *Maximize*
    - Cost → *Minimize*
    - Risk → *Minimize*
    - Market Share → *Maximize*
- Objective function = an <u>outcome</u> variable
- Can be controlled through decision variables, not directly

Constraints

- Describe what is actually feasible:
  - technically, e.g.: *quantity produced cannot exceed production capacity*
  - legally, e.g.: *employees may not work more than 8 hours per shift*
  - logically, e.g.: *all parts of the budget must add up to 100%*
- Must be expressed in terms of the decision variables

Math Formulation

An LP model takes on the general form:

MAX(or MIN): $c_1X_1 + c_2X_2 + \dots + c_nX_n$

Subject to:

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n \leq b_1$$
$$a_{k1}X_1 + a_{k2}X_2 + \dots + a_{kn}X_n \geq b_k$$
$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mn}X_n = b_m$$

Formulation LP Problems

1. Understand the problem.
2. Identify the decision variables.
3. State the objective function as a <u>linear</u> combination of the decision variables.
4. State the constraints as <u>linear</u> combinations of the decision variables.
5. Identify the upper and lower bounds on the decision variables.

Example

How much of each chocolate to produce to maximize profits?

x_1 : # boxes of Pyramide produced per day

x_2 : # boxes of Nuit produced per day

profit from Pyramide = $1 each

profit from Nuit = $6 each

Example

How much of each should it produce to maximize profits?

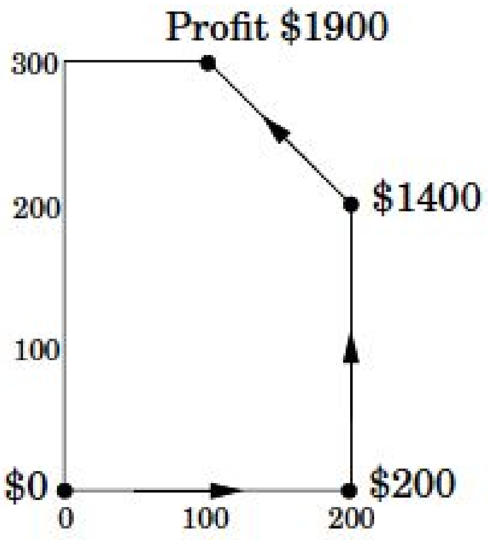
x_1 : # boxes of Pyramide produced per day
x_2 : # boxes of Nuit produced per day
profit from Pyramide = $1 each
profit from Nuit = $6 each

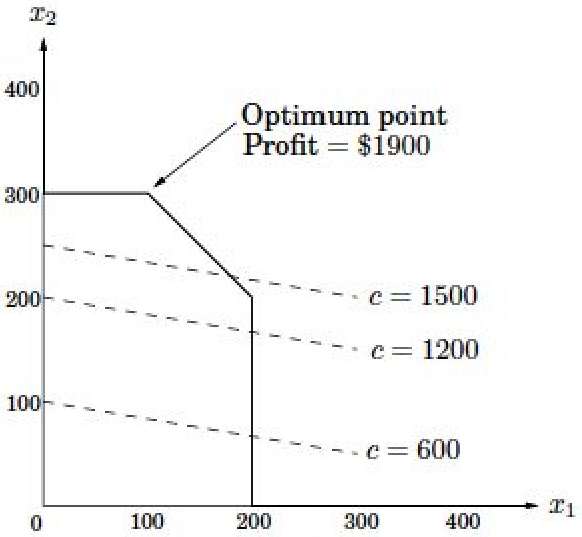
- The daily demand for these chocolates is limited to at most 200 boxes of Pyramide and 300 boxes of Nuit.
- The current workforce can produce a total of at most 400 boxes of chocolate per day.
- What are the optimal levels of production?

Problem Formulation

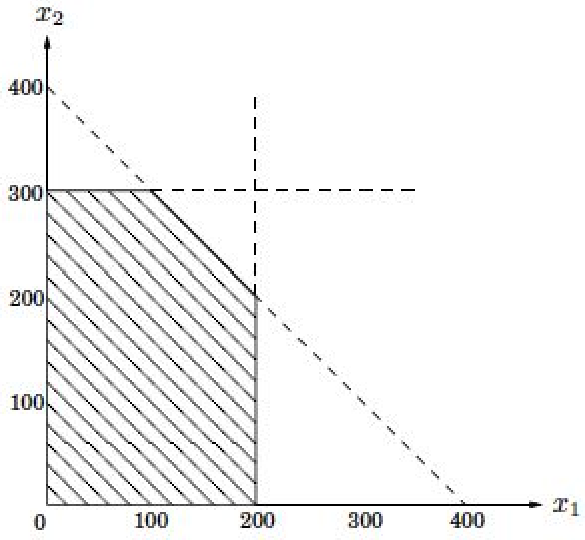
- We represent the situation by a linear program, as follows.

Objective function	$max\ x_1 + 6x_2$
Constraints	$x_1 \leqq 200$
	$x_2 \leqq 300$
	$x_1 + x_2 \leqq 400$
	$x_1; x_2 \geqq 0$


Profit $1900
300
200
100
$1400
$0
$200
0
100
200


x_2
400
300
200
100
Optimum point
Profit = $1900
$c = 1500$
$c = 1200$
$c = 600$
0
100
200
300
400
x_1

Feasible region of the linear problem



Gribbin Brewing Problem

- Regional brewer Andrew Gribbin distributes kegs of his famous beer through 3 warehouses in the greater NYC area, with current supplies as shown in Figure 1.
- On a Thursday morning, he has his usual weekly orders from his 4 loyal customers, as shown in Figure 2.
- Gribbin's shipping manager needs to determine the most cost-efficient plan to deliver beer to these 4 customers, knowing that the costs per keg are different for each possible combination of warehouse and customer (see Figure 3).
- What is the optimal shipping plan?

Gribbin Brewing Problem

Figure 1

Warehouses	Supply
Hoboken	80
Bronx	145
Brooklyn	120

Figure 2

Bars	Demand
Ratkeller	80
McGoldrick's Pub	65
Night Train Bar & Grill	70
Henry Ale's	85

Figure 3

	Ratkeller	McGoldrick's Pub	Night Train Bar & Grill	Henry Ale's
Hoboken	$4.64	$5.13	$6.54	$8.67
Bronx	$3.52	$4.16	$6.90	$7.91

Solution Methodology

Step #1: Data Cells

- Enter all of the data for the problem on the spreadsheet.
- It is a good idea to color code these “data cells” (e.g.: light blue).

Total Cost						
Shipping Plan	Ratkeller	McGoldrick's	Night Train	Henry Ale's	Shipped	Supply
Hoboken						80
Bronx						145
Brooklyn						120
Demand	80	65	70	85		
Costs						
Hoboken	$ 4.64	$ 5.13	$ 6.54	$ 8.67		
Bronx	$ 3.52	$ 4.16	$ 6.90	$ 7.91		
Brooklyn	$ 9.95	$ 6.82	$ 3.88	$ 6.85		

Developing a Spreadsheet Model

Step #2: Changing Cells

- Add a cell in the spreadsheet for every decision that needs to be made.
- If you don't have any particular initial values, just enter 0 in each.
- It is a good idea to color code these "changing cells" (e.g.: yellow).

Developing a Spreadsheet Model

Step #3: Target Cell

- Develop an equation that defines the objective of the model.
- Typically this equation involves the data cells and the changing cells in order to determine a quantity of interest (e.g., total profit or total cost).

Nil Simsek: =SUMPRODUCT(B4:E6,B12:E14)

Total Cost	$ -						
Shipping Plan	Ratkeller	McGoldrick's	Night Train	Henry Ale's	Shipped		Supply
Hoboken							80
Bronx							145
Brooklyn							120
Demand	80	65	70	85			
Costs							
Hoboken	$ 4.64	$ 5.13	$ 6.54	$ 8.67			
Bronx	$ 3.52	$ 4.16	$ 6.90	$ 7.91			
Brooklyn	$ 9.95	$ 6.82	$ 3.88	$ 6.85			

Developing a Spreadsheet Model

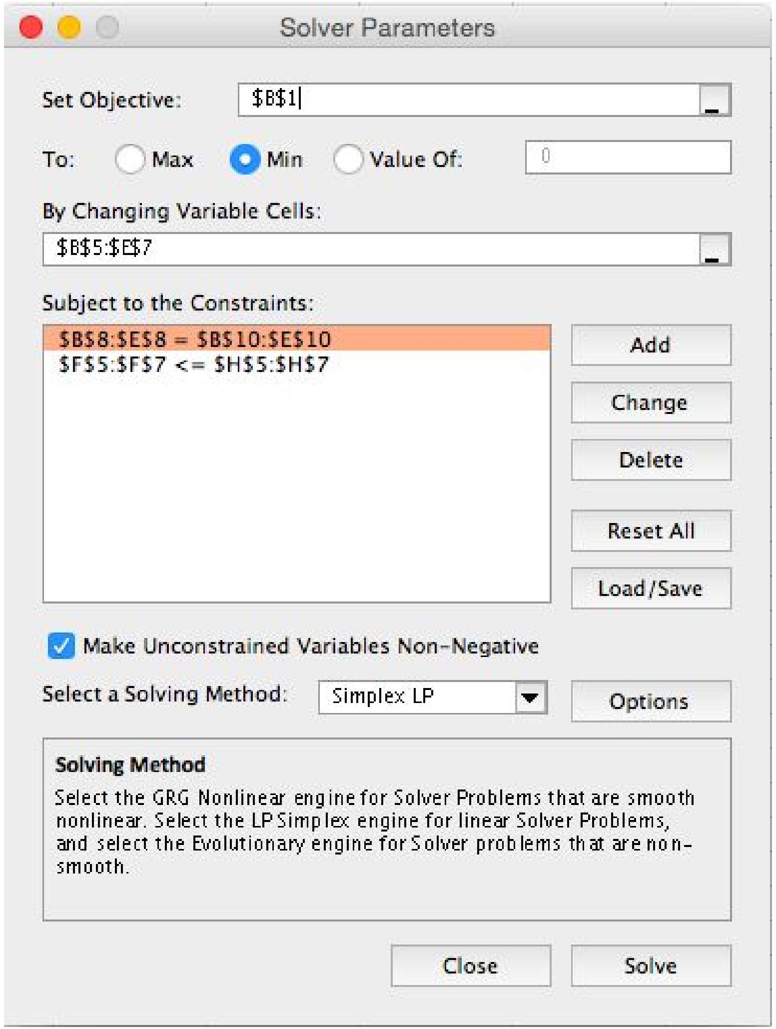
Step #4: Constraints

- For each warehouse, calculate the total amount shipped from it in a cell on the spreadsheet (an output cell).
- Define the supply constraint in three consecutive cells. For example, if Quantity A <= Quantity B, put these three items (Quantity A, <=, Quantity B) in consecutive cells.
- For each customer, calculate the total amount shipped to it in a cell on the spreadsheet (an output cell), and define the demand constraint.

Solver Setup

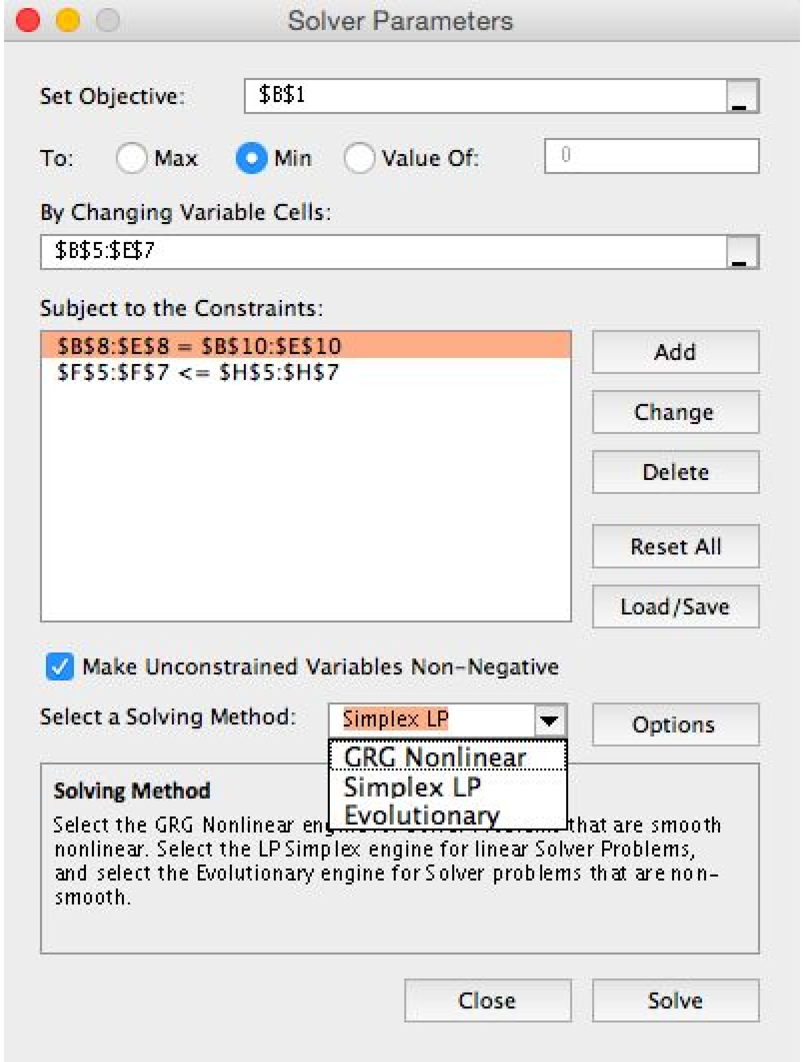
- Choose “Solver” from the Data menu.
- Select the cell you wish to optimize in the “Set Objective” box.
- Choose “Max” or “Min” depending on whether you want to maximize/minimize the target cell.
- Enter all the changing cells in the “By Changing Variable Cells” box.
- To begin entering constraints, click the “Add” button to the right of the constraints box.
- Fill in the entries in the resulting Add Constraint dialogue box

Solver Setup

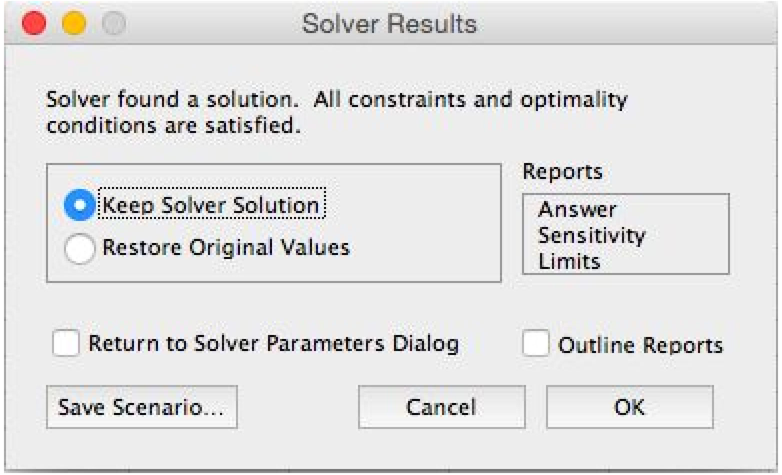


Solver Setup

- Click in the “Make Unconstrained Variables Non-Negative” box.
  - It adds non-negativity constraints to *all* the changing cells.
- Choose “Simplex LP” from “Select a Solving method”
  - It tells the Solver that this is a *linear* programming model.



Solver Results Dialog Box



An Optimal Solution

Total Cost	$ 1,469.55						
Shipping Plan	Ratkeller	McGoldrick's	Night Train	Henry Ale's	Shipped		Supply
Hoboken	0	0	0	35	35	<=	80
Bronx	80	65	0	0	145	<=	145
Brooklyn	0	0	70	50	120	<=	120
	80	65	70	85			
	=	=	=	=			
Demand	80	65	70	85			
Costs							
Hoboken	$ 4.64	$ 5.13	$ 6.54	$ 8.67			
Bronx	$ 3.52	$ 4.16	$ 6.90	$ 7.91			
Brooklyn	$ 9.95	$ 6.82	$ 3.88	$ 6.85			

Nil Simsek:
=SUMPRODUCT(B5:E7,B13:E15)

Nil Simsek:
=SUM(B5:E5)

Nil Simsek:
=SUM(E5:E7)

LP Basic Concepts

- **Feasible solution**: satisfies all constraints
- **Optimal solution**: a feasible solution with the largest objective function value (for a maximization problem) or smallest value for a minimization problem
- **Infeasible linear program**: no feasible solution that satisfies all of the constraints
- **Unbounded LP**: the constraints do not sufficiently restrain the cost function so that for any given feasible solution, another feasible solution can be found that makes a further improvement to the cost function.
- **Binding constraint**: a constraint that restricts the optimal solution and changes to which also change the optimal solution.

LP Basic Concepts

- Redundant constraints are not really a problem. They do not prevent us from finding the optimal solution to an LP problem. However, they do represent “excess baggage” for the computer; so if you know that a constraint is redundant, eliminating it saves the computer this excess work.

- Although it is not unusual to encounter an unbounded solution when solving an LP model, such a solution indicates that there is something wrong with the formulation -- for example, one or more constraints were omitted from the formulation.

- Infeasibility can occur in LP problems due to an error in the formulation of the model -- such as unintentionally making a less than or equal to constraint a greater than or equal to constraint. Or there just might not be a way to satisfy all the constraints in the model. In this case, constraints will have to be eliminated or loosened in order to obtain a feasible solution for the problem.

Decision Models 2: Sensitivity Analysis, Integer Linear & Non-Linear Programming

Business Analytics

Linear Programming

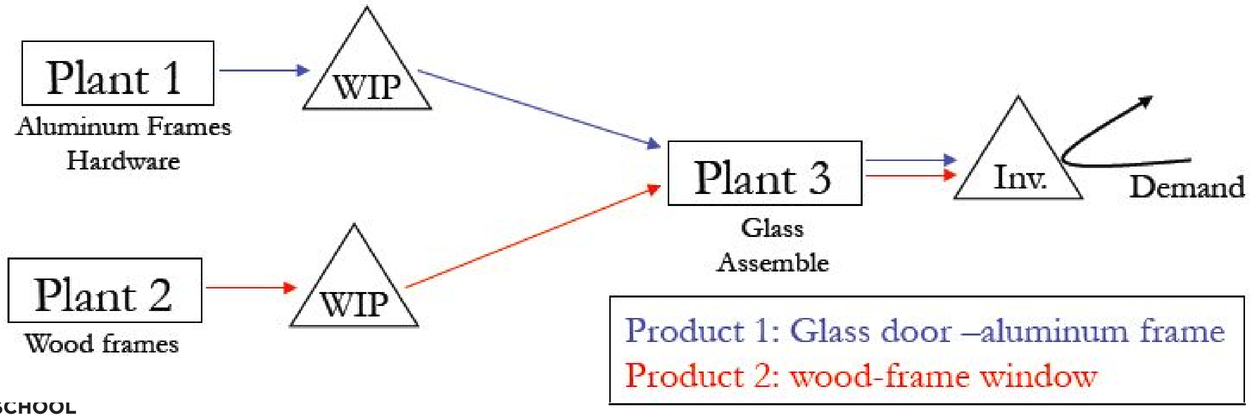
- Linear programming (LP; also called linear optimization) describes a broad class of optimization tasks where the requirements are represented by linear relationships.
  - satisfy a set of linear equations and/or inequalities
  - maximize or minimize a given linear objective function

Sensitivity Analysis

Sensitivity analysis allows us to understand the impact of resource constraints and functional parameters in the total objective value achieved.

Example:

Wyndor Glass Co. manufactures windows and glass doors using three plants, according to the following scheme:



Wyndor Glass Co. Example

Plant	Production Time per Batch (Hours)		
	Product		Production Time Available per Week
	1	2	
1	1	0	4
2	0	2	12
3	3	2	18
Profit per Batch	$3,000	$5,000	

Decision Variables:

$x1$ = # of batches of product 1 to produce per week

$x2$ = # of batches of product 2 to produce per week

Objective Function:

Max $Z = 3x1 + 5x2$

Constraints:

Time availability:

$x1 \leq 4$ (plant 1)

$2x2 \leq 12$ (plant 2)

$3x1 + 2x2 \leq 18$ (plant 3)

Non negativity: $x1 \geq 0$ and $x2 \geq 0$

Wyndor Glass Co. Example

Feasible Solution:

- (0, 0) is feasible
- (5, 1) is not feasible

Objective Function Value (OFV): value of the objective function for a solution

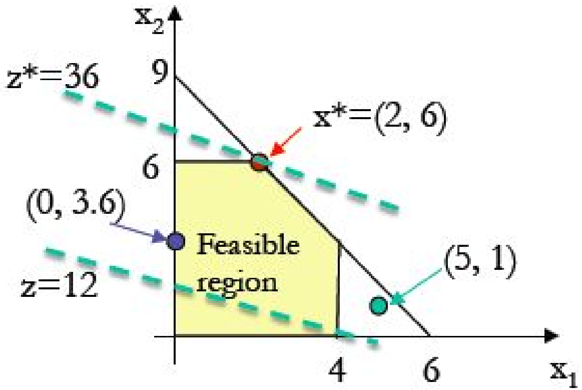
- OFV = 18 for (0, 3.6)

Optimal solution: feasible solution whose OFV cannot be improved upon.

- (2, 6) is optimal for Wyndor.
- There could be more than one optimal solution

max $z = 3x_1 + 5x_2$
subject to:

x_1	≤ 4	(plant 1 time)
$2x_2$	≤ 12	(plant 2 time)
$3x_1 + 2x_2$	≤ 18	(plant 3 time)
x_1, x_2	≥ 0	(non-negativity)



Answer Report

Objective Cell (Max)

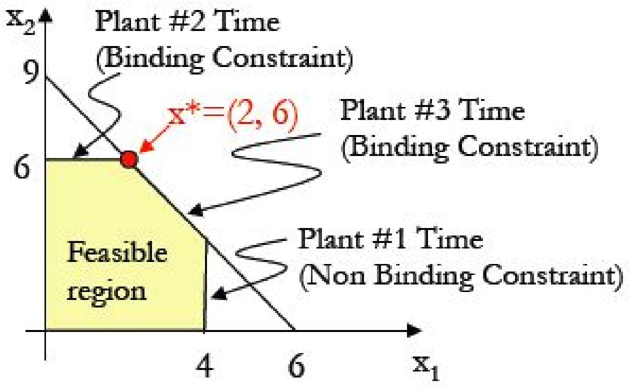
Cell	Name	Original Value	Final Value
$L$21	Total Profit	$0	$36,000

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$L$17	Production Quantity	0	2	Contin
$M$17	Production Quantity	0	6	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$L$29	Total	2	$L$29<=$N$29	Not Binding	2
$L$30	Total	12	$L$30<=$N$30	Binding	0
$L$31	Total	18	$L$31<=$N$31	Binding	0



Sensitivity Report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$L$17	Production Quantity	2	0	3000	4500	3000
$M$17	Production Quantity	6	0	5000	1E+30	3000

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$L$29	Total	2	0	4	1E+30	2
$L$30	Total	12	1500	12	6	6
$L$31	Total	18	1000	18	6	6

Binding and Non-Binding Constraints

- A constraint is binding if it is satisfied as a strict equality in the optimal solution; otherwise, it is nonbinding.
- Binding constraints prevent us from further improving (that is, maximizing or minimizing) the objective function.
- A constraint is binding if its Final Value is equal to its Constraint R.H. Side value.
- Constraint R.H. Side refers to the original constant in the RHS of the constraint.
- Final Value refers to the total value of the LHS when the variables take the optimal values
- By definition, binding constraints have zero slack, and nonbinding constraints have some positive level of slack.
- The values in the Slack column indicate that if this solution is implemented, we will use up the entire available production time at Plants 2 and 3, but will have 2 hours left over in Plant 1.

Changing the level of a resource

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$L$17	Production Quantity	2	0	3000	4500	3000
$M$17	Production Quantity	6	0	5000	1E+30	3000

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$L$29	Total	2	0	4	1E+30	2
$L$30	Total	12	1500	12	6	6
$L$31	Total	18	1000	18	6	6

Shadow Prices

- The shadow price for a constraint indicates the amount by which the objective function value changes given a unit increase in the RHS value of the constraint, assuming all other coefficients remain constant.
- If a shadow price is positive, a unit increase in the RHS value of the associated constraint results in an increase in the optimal objective function value.
- If a shadow price is negative, a unit increase in the RHS value of the associated constraint results in a decrease in the optimal objective function value.
- To analyze the effects of decreases in the RHS values, you reverse the sign on the shadow price.

Optimization Models

- Linear Programming (LP) Models
  - Both objective and constraints are linear functions of the decision variables.
  - Decision variables are not necessarily integers.
- Integer Programming (IP) Models
  - Some variables must take integer values.
  - Two types: integer linear programming (ILP) and integer nonlinear programming (INP)
- Non-linear Programming (NP) Models
  - Either objective or constraints can be non-linear.

Integer Linear Problems

- The entire class of problems referred to as sequencing, scheduling, and routing are inherently integer programs.
- Some ILP models are simply LP models with certain variables constrained to be integers.
  - Building such an ILP model requires just a couple of additional steps beyond an LP model.
  - Some LP models generate integer solutions.

Example - Scheduling Professors

Three professors must be assigned to teach 6 sections of finance. Each professor must teach 2 sections of finance, and each has ranked the 6 time periods during which finance is taught.

	9 A.M.	10 A.M.	11 A.M.	1 P.M.	2 P.M.	3 P.M.
Professor 1	8	7	6	5	7	6
Professor 2	9	9	8	8	4	4
Professor 3	7	6	5	6	9	5

Determine an assignment of professors to sections that maximizes the total satisfaction of the professors.

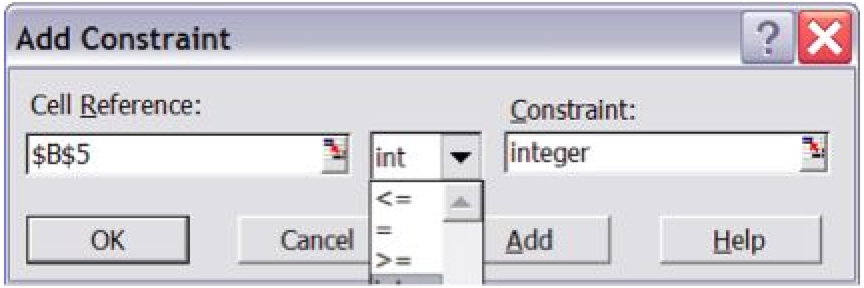
Formulation - Scheduling Professors

- It is not necessary to constrain the decision variables to be binary. This is due to the special structure of the feasible region of this type of problems (called assignment problems).
- Even without the binary constraints, the optimal linear solution will automatically have zeros and ones for the decision variables.

Solver Setup

Solver constraints window allows choices of *int* and *bin* for any variable

- int: variable must be an integer
- bin: variable must be binary, i.e, 0 or 1



Examples of Linear and Non-Linear Formulas

Data cells are located in D1:D6 and **changing cells** are in C1:C6.

Linear Formulas

SUMPRODUCT(D4:D6, C4:C6)

[(D1 + D2) / D3] * C4

IF(D2 >= 2, 2*C3, 3*C4)

SUMIF(D1:D6, 4, C1:C6)

SUM(D4:D6)

2*C1 + 3*C4 + C6

C1 + C2 + C3

Nonlinear Formulas

IF(C2 >= 2, 2*C3, 3*C4)

SUMIF(C1:C6, 4, D1:D6)

ROUND(C1)

MAX(C1, 0)

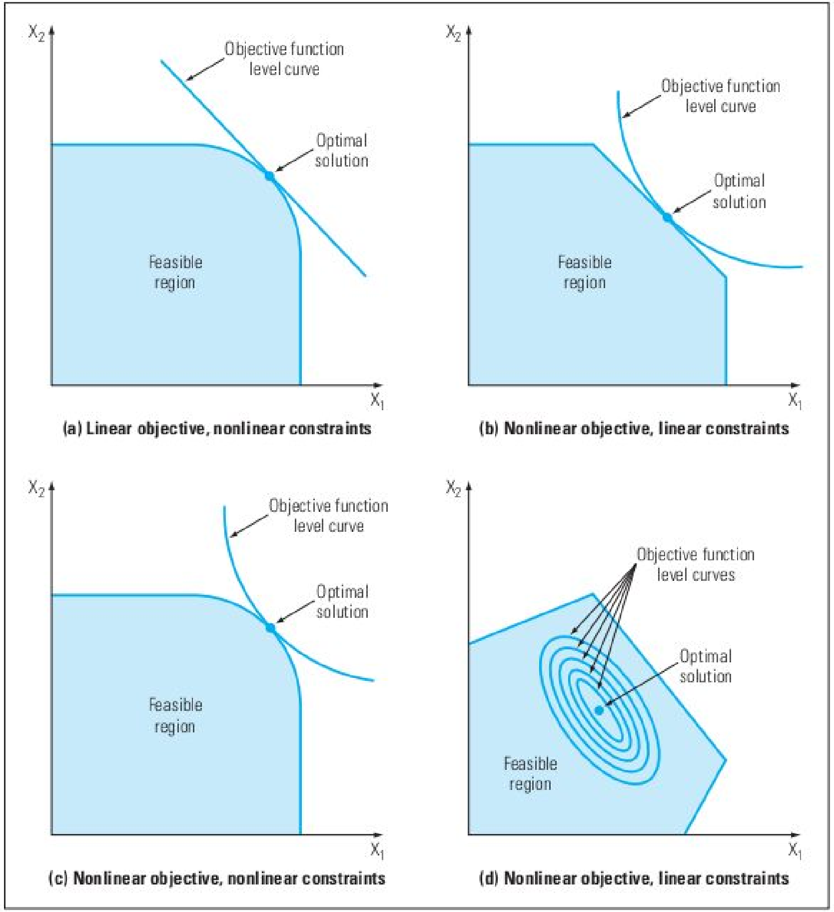
MIN(C1, C2)

ABS(C1)

SQRT(C1)

C1 * C2
C1 / C2
C1 ^2

Non-Linear Problems



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Non-Linear Programming Models

Consider the following model in algebraic form:

Maximize Profit = $0.5x^5 - 6x^4 + 24.5x^3 - 39x^2 + 20x$

Subject to:

$x \leq 5$

$x \geq 0$

Solver Solution Starting at x=0

	A	B	C	D	E
1	**A Simple NLP**				
2					
3					Maximum
4		x =	0.371	<=	5
5					
6		Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
7		=	\$3.19		

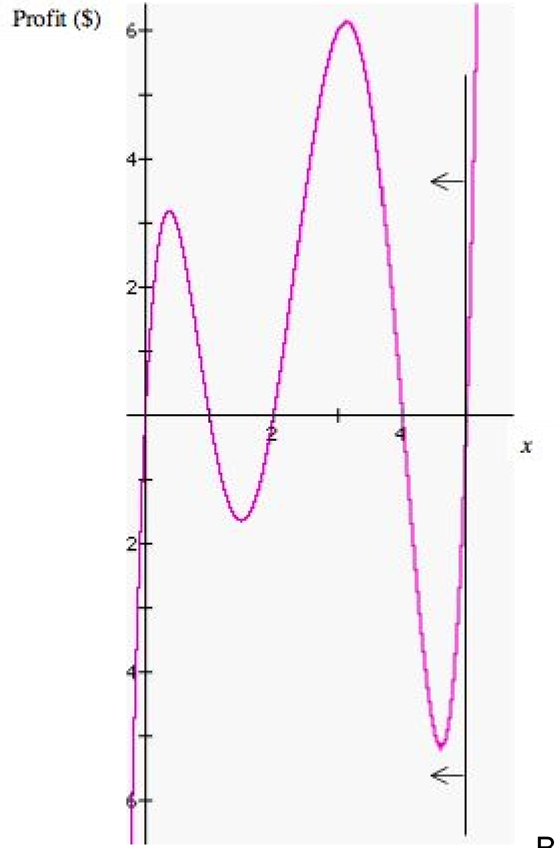
Solver Solution Starting x=3

	A	B	C	D	E
1	**A Simple NLP**				
2					
3					Maximum
4		x =	3.126	<=	5
5					
6		Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
7		=	$6.13		

Solver Solution Starting x=4.7

A Simple NLP				
				Maximum
	x =	5.000	<=	5
	Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$			
	=	$0.00		

The Profit Graph



Example - Beat the Market

A common goal of portfolio managers is to beat the market. If we assume that past performance is somewhat of an indicator of the future, then picking a portfolio that beat the market most often in the past might yield a portfolio that will more than likely beat the market in the future. Consider a portfolio of five large stocks traded on the New York Stock Exchange (NYSE).

The quarterly performance (return) of each of these stocks over a six-year period (2001-2005) is shown in the template file. The performance of the market as a whole, as measured by the NYSE Composite Index, is also shown in the file.

What mix of these five stocks will yield a portfolio that is likely to beat the market in the future?

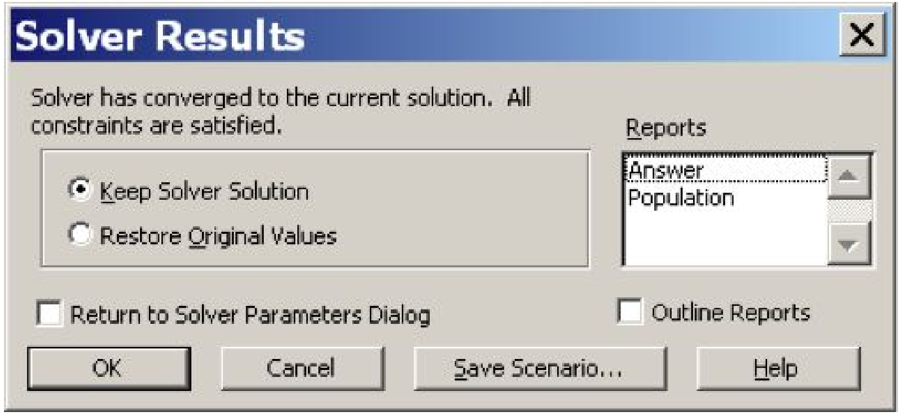
Solver Methods

- Standard GRG Nonlinear is equivalent to the regular Solver *without* choosing "Assume Linear Model".
- Standard Evolutionary uses a genetic evolutionary algorithm that is only available with Premium Solver.

Evolutionary Solver

- The standard non-linear Solver (GRG Nonlinear) has difficulty with problems that are:
    - highly nonlinear
    - are not smooth (have “kinks” in the objective)
    - have discontinuities (the objective jumps in value)
    - have many local optima (many hills and valleys)
- Excel functions like IF, MAX, ABS, ROUND, etc., tend to cause one or more of these problems.

Solution by Evolutionary Solver



Maximum

x = 3.126 <= 5

Profit = $0.5x^5-6x^4+24.5x^3-39x^2+20x$

= $6.13

Add Bounds on Variables

Tips on Using Evolutionary Solver

- Bounding all of the variables greatly aids the Evolutionary Solver by decreasing the search space.
- The limit options should be increased (Max Time, Max Subproblems, and Max Feasible Sols) for challenging problems. Setting Tolerance to 0.0005 and Max Time Without Improvements to 30 will ensure the algorithm will stop if the Target Cell value has improved less than 0.05% in the last 30 seconds.
- Experiment with different populations sizes and mutation rates to see what works well. I have found that higher than default mutation rates can be helpful in problems with lots of local optima.
- The Evolutionary Solver can take a very long time, but it will usually find a good solution.

Tips on Using Evolutionary Solver

- There is no guarantee that Evolutionary Solver will find the *best* solution.
- The Evolutionary Solver performs well even with nasty objective functions, but is not very efficient at handling constraints.
- Much of the solution process is driven by random numbers that direct the search. Thus, two people running Evolutionary Solver on the same model may get different results.
- Once Evolutionary Solver has found a good solution, you may use GRG Nonlinear Solver (the nonlinear algorithm that is included with the Premium Solver software) to try to find a slightly better solution.