Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Management Science

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Ch 2: Introduction to Optimization and Linear Programming

2.0 Introduction

Our world is filled with limited resources. The amount of oil we can pump out of the earth is limited. The amount of land available for garbage dumps and hazardous waste is limited and, in many areas, diminishing rapidly. On a more personal level, each of us has a limited amount of time in which to accomplish or enjoy the activities we schedule each day. Most of us have a limited amount of money to spend while pursuing these activities. Businesses also have limited resources. A manufacturing organization employs a limited number of workers. A restaurant has a limited amount of space available for seating. Deciding how best to use the limited resources available to an individual or a business is a universal problem. In today's competitive business environment, it is increasingly important to make sure that a company's limited resources are used in the most efficient manner possible. Typically, this involves determining how to allocate the resources in such a way as to maximize profits or minimize costs. Mathematical programming (MP) is a field of management science that finds the optimal, or most effi- cient, way of using limited resources to achieve the objectives of an individual or a business. For this reason, mathematical programming is often referred to as optimization.

2.1 Applications of Mathematical Optimization

To help you understand the purpose of optimization and the types of problems for which it can be used, let's consider several examples of decision-making situations in which MP techniques have been applied.

Determining Product Mix. Most manufacturing companies can make a variety of products. However, each product usually requires different amounts of raw materials and labor. Similarly, the amount of profit generated by the products varies. The manager of such a company must decide how many of each product to produce in order to maximize profits or to satisfy demand at minimum cost.

Manufacturing. Printed circuit boards, like those used in most computers, often have hundreds or thousands of holes drilled in them to accommodate the different electrical components that must be plugged into them. To manufacture these boards, a computer controlled drilling machine must be programmed to drill in a given location, then the drill bit to the next location and drill again. This process is repeated hundreds or thousands of times to complete all the holes on a circuit board. Manufacturers of these boards would benefit from determining the drilling order that minimizes the total distance the drill bit must be moved.

Routing and Logistics. Many retail companies have warehouses around the country that are responsible for keeping stores supplied with merchandise to sell. The amount of merchandise available at the warehouses and the amount needed at each store tends to fluctuate, as does the cost of shipping or delivering merchandise from the warehouses to the retail locations. Determining the least costly method of transferring merchandise from the warehouses to the stores can save large amounts of money.

Financial Planning. The federal government requires individuals to begin withdrawing money from individual retirement accounts (IRAs) and other tax-sheltered retirement programs no later than age 70.5. There are various rules that must be followed to avoid paying penalty taxes on these withdrawals. Most individuals want to withdraw their money in a manner that minimizes the amount of taxes they must pay while still obeying the tax laws.

2.2 Characteristics of Optimization Problems

These examples represent just a few areas in which MP has been used successfully. We will consider many other examples throughout this book. However, these examples give you some idea of the issues involved in optimization. For instance, each example involves one or more decisions that must be made: How many of each product should be produced? Which hole should be drilled next? How much of each product should be shipped from each warehouse to the various retail locations? How much money should an individual withdraw each year from various retirement accounts?

Also, in each example, restrictions, or constraints, are likely to be placed on the alternatives available to the decision maker. In the first example, when determining the number of products to manufacture, a production manager is probably faced with a limited amount of raw materials and a limited amount of labor. In the second example, the drill should never return to a position where a hole has already been drilled. In the third example, there is a physical limitation on the amount of merchandise a truck can carry from one warehouse to the stores on its route. In the fourth example, laws determine the minimum and maximum amounts that can be withdrawn from retirement accounts without incurring a penalty. Many other constraints can also be identified for these examples. It is not unusual for real-world optimization problems to have hundreds or thousands of constraints.

A final common element in each of the examples is the existence of some goal or objective that the decision maker considers when deciding which course of action is best. In the first example, the production manager can decide to produce several different product mixes given the available resources, but the manager will probably choose the mix of products that maximizes profits. In the second example, a large number of possible drilling patterns can be used, but the ideal pattern will probably involve moving the drill bit the shortest total distance. In the third example, there are numerous ways merchandise can be shipped from the warehouses to supply the stores, but the company will probably want to identify the routing that minimizes the total transportation cost. Finally, in the fourth example, individuals can withdraw money from their retirement accounts in many ways without violating tax laws, but they probably want to find the method that minimizes their tax liability.

2.3 Expressing Optimization Problems Mathematically

From the preceding discussion, we know that optimization problems involve three elements: decisions, constraints, and an objective. If we intend to build a mathematical model of an optimization problem, we will need mathematical terms or symbols to represent each of these three elements.

2.3.1 DECISIONS

The decisions in an optimization problem are often represented in a mathematical model by the symbols X1, X2, . . . , Xn. We will refer to X1, X2, . . . , Xn as the decision variables (or simply the variables) in the model. These variables might represent the quantities of different products the production manager can choose to produce. They might represent the amount of different pieces of merchandise to ship from a warehouse to a certain store. They might represent the amount of money to be withdrawn from different retirement accounts.

The exact symbols used to represent the decision variables are not particularly important. You could use Z1, Z2, . . . , Zn or symbols like Dog, Cat, and Monkey to represent the decision variables in the model. The choice of which symbols to use is largely a matter of personal preference and might vary from one problem to the next.

2.3.2 CONSTRAINTS

The constraints in an optimization problem can be represented in a mathematical model in a number of ways. Three general ways of expressing the possible constraint relationships in an optimization problem are:

A less than or equal to constraint:	$f(X1, X2, \ldots, Xn) \leq b$
A greater than or equal to constraint:	$f(X1, X2, \ldots, Xn) \geq b$
An equal to constraint:	$f(X1, X2, \ldots, Xn) = b$

In each case, the constraint is some function of the decision variables that must be less than or equal to, greater than or equal to, or equal to some specific value (represented by the letter b). We will refer to f(X1, X2, . . . , Xn) as the left-hand-side (LHS) of the constraint and to b as the right-hand-side (RHS) value of the constraint.

For example, we might use a less than or equal to constraint to ensure that the total labor used in producing a given number of products does not exceed the amount of available labor. We might use a greater than or equal to constraint to ensure that the total amount of money withdrawn from a person’s retirement accounts is at least the minimum amount required by the IRS. You can use any number of these constraints to represent a given optimization problem depending on the requirements of the situation.

2.3.3 OBJECTIVE

The objective in an optimization problem is represented mathematically by an objective function in the general format:

$$\text{MAX (or MIN):} \quad f(X_1, X_2, \ldots, X_n)$$

The objective function identifies some function of the decision variables that the decision maker wants to either MAXimize or MINimize. In our earlier examples, this function might be used to describe the total profit associated with a product mix, the total distance the drill bit must be moved, the total cost of transporting merchandise, or a retiree's total tax liability.

The mathematical formulation of an optimization problem can be described in the general format:

$$\begin{array}{lll} \text{MAX (or MIN):} & f_0(X_1, X_2, \ldots, X_n) & \textbf{2.1} \\ \text{Subject to:} & f_1(X_1, X_2, \ldots, X_n) \le b_1 & \textbf{2.2} \\ & f_k(X_1, X_2, \ldots, X_n) \ge b_k & \textbf{2.3} \\ & f_m(X_1, X_2, \ldots, X_n) = b_m & \textbf{2.4} \end{array}$$

This representation identifies the objective function (equation 2.1) that will be maximized (or minimized) and the constraints that must be satisfied (equations 2.2 through 2.4). Subscripts added to the f and b in each equation emphasize that the functions describing the objective and constraints can all be different. The goal in optimization is to find the values of the decision variables that maximize (or minimize) the objective function without violating any of the constraints.

2.4 Mathematical Programming Techniques

Our general representation of an MP model is just that—general. There are many kinds of functions you can use to represent the objective function and the constraints in an MP model. Of course, you should always use functions that accurately describe the objective and constraints of the problem you are trying to solve. Sometimes, the functions in a model are linear in nature (that is, form straight lines or flat surfaces); other times, they are nonlinear (that is, form curved lines or curved surfaces). Sometimes, the optimal values of the decision variables in a model must take on integer values (whole numbers); other times, the decision variables can assume fractional values.

Given the diversity of MP problems that can be encountered, many techniques have been developed to solve different types of MP problems. In the next several chapters, we will look at these MP techniques and develop an understanding of how they differ and when each should be used. We will begin by examining a technique called linear programming (LP), which involves creating and solving optimization problems with linear objective functions and linear constraints. LP is a very powerful tool that can be applied in many business situations. It also forms a basis for several other techniques discussed later and is, therefore, a good starting point for our investigation into the field of optimization.

2.5 An Example LP Problem

We will begin our study of LP by considering a simple example. You should not interpret this to mean that LP cannot solve more complex or realistic problems. LP has been used to solve extremely complicated problems, saving companies millions of dollars. However, jumping directly into one of these complicated problems would be like starting a marathon without ever having gone out for a jog—you would get winded and could be left behind very quickly. So we'll start with something simple.

> Blue Ridge Hot Tubs manufactures and sells two models of hot tubs: the Aqua-Spa and the Hydro-Lux. Howie Jones, the owner and manager of the company, needs to decide how many of each type of hot tub to produce during his next production cycle. Howie buys prefabricated fiberglass hot tub shells from a local supplier and adds the pump and tubing to the shells to create his hot tubs. (This supplier has the capacity to deliver as many hot tub shells as Howie needs.) Howie installs the same type of pump into both hot tubs. He will have only 200 pumps available during his next production cycle. From a manufacturing standpoint, the main difference between the two models of hot tubs is the amount of tubing and labor required. Each Aqua-Spa requires 9 hours of labor and 12 feet of tubing. Each Hydro-Lux requires 6 hours of labor and 16 feet of tubing. Howie expects to have 1,566 production labor hours and 2,880 feet of tubing available during the next production cycle. Howie earns a profit of $350 on each Aqua-Spa he sells and $300 on each Hydro-Lux he sells. He is confident that he can sell all the hot tubs he produces. The question is, how many Aqua-Spas and Hydro-Luxes should Howie produce if he wants to maximize his profits during the next production cycle?

2.6 Formulating LP Models

The process of taking a practical problem—such as determining how many Aqua-Spas and Hydro-Luxes Howie should produce—and expressing it algebraically in the form of an LP model is known as formulating the model. Throughout the next several chapters, you will see that formulating an LP model is as much an art as a science.

2.6.1 STEPS IN FORMULATING AN LP MODEL

There are some general steps you can follow to help make sure your formulation of a particular problem is accurate. We will walk through these steps using the hot tub example.

1. **Understand the problem.** This step appears to be so obvious that it hardly seems worth mentioning. However, many people have a tendency to jump into a problem and start writing the objective function and constraints before they really understand the problem. If you do not fully understand the problem you face, it is unlikely that your formulation of the problem will be correct.

The problem in our example is fairly easy to understand: How many Aqua-Spas and Hydro-Luxes should Howie produce to maximize his profit, while using no more than 200 pumps, 1,566 labor hours, and 2,880 feet of tubing?

2. **Identify the decision variables.** After you are sure you understand the problem, you need to identify the decision variables. That is, what are the fundamental decisions that must be made in order to solve the problem? The answers to this question often will help you identify appropriate decision variables for your model. Identifying the decision variables means determining what the symbols $X_1, X_2, \ldots, X_n$ represent in your model.

   In our example, the fundamental decision Howie faces is this: How many AquaSpas and Hydro-Luxes should be produced? In this problem, we will let X_1 represent the number of Aqua-Spas to produce and X_2 represent the number of Hydro-Luxes to produce.

3. **State the objective function as a linear combination of the decision variables.** After determining the decision variables you will use, the next step is to create the objective function for the model. This function expresses the mathematical relationship between the decision variables in the model to be maximized or minimized.

   In our example, Howie earns a profit of $350 on each Aqua-Spa ($X_1$) he sells and $300 on each Hydro-Lux ($X_2$) he sells. Thus, Howie's objective of maximizing the profit he earns is stated mathematically as:

$$\text{MAX:} \qquad 350X_1 + 300X_2$$

   For whatever values might be assigned to X1 and X_2, the previous function calculates the associated total profit that Howie would earn. Obviously, Howie wants to maximize this value.

4. **State the constraints as linear combinations of the decision variables.** As mentioned earlier, there are usually some limitations on the values that can be assumed by the decision variables in an LP model. These restrictions must be identified and stated in the form of constraints. In our example, Howie faces three major constraints. Because only 200 pumps are available, and each hot tub requires one pump, Howie cannot produce more than a total of 200 hot tubs. This restriction is stated mathematically as:

$$1X_1 + 1X_2 \leq 200$$

   This constraint indicates that each unit of X1 produced (that is, each Aqua-Spa built) will use one of the 200 pumps available—as will each unit of X_2 produced (that is, each Hydro-Lux built). The total number of pumps used (represented by $1X_1 + 1X_2$) must be less than or equal to 200.

   Another restriction Howie faces is that he has only 1,566 labor hours available during the next production cycle. Because each Aqua-Spa he builds (each unit of X_1) requires 9 labor hours, and each Hydro-Lux (each unit of X_2) requires 6 labor hours, the constraint on the number of labor hours is stated as:

$$9X_1 + 6X_2 \leq 1{,}566$$

The total number of labor hours used (represented by $9X_1 + 6X_2$) must be less than or equal to the total labor hours available, which is 1,566.

The final constraint specifies that only 2,880 feet of tubing is available for the next production cycle. Each Aqua-Spa produced (each unit of x_1) requires 12 feet of tubing, and each Hydro-Lux produced (each unit of X_2) requires 16 feet of tubing. The following constraint is necessary to ensure that Howie's production plan does not use more tubing than is available:

$$12X_1 + 16X_2 \leq 2{,}880$$

The total number of feet of tubing used (represented by 12X1 + 16X2) must be less than or equal to the total number of feet of tubing available, which is 2,880.

5. **Identify any upper or lower bounds on the decision variables.** Often, simple upper or lower bounds apply to the decision variables. You can view upper and lower bounds as additional constraints in the problem.

   In our example, there are simple lower bounds of zero on the variables X_1 and X_2 because it is impossible to produce a negative number of hot tubs. Therefore, the following two constraints also apply to this problem:

$$X_1 \geq 0$$
$$X_2 \geq 0$$

   Constraints like these are often referred to as nonnegativity conditions and are quite common in LP problems.

2.7 Summary of the LP Model for the Example Problem

The complete LP model for Howie's decision problem can be stated as:

$$
\begin{array}{llrlrlr}
\text{MAX:} & 350X_1 + 300X_2 & & & & & \mathbf{2.5} \\
\text{Subject to:} & 1X_1 + & 1X_2 & \leq & 200 & & \mathbf{2.6} \\
 & 9X_1 + & 6X_2 & \leq & 1{,}566 & & \mathbf{2.7} \\
 & 12X_1 + & 16X_2 & \leq & 2{,}880 & & \mathbf{2.8} \\
 & 1X_1 & & \geq & 0 & & \mathbf{2.9} \\
 & & 1X_2 & \geq & 0 & & \mathbf{2.10}
\end{array}
$$

In this model, the decision variables X_1 and X_2 represent the number of Aqua-Spas and Hydro-Luxes to produce, respectively. Our goal is to determine the values for X_1 and X_2 that maximize the objective in equation 2.5 while simultaneously satisfying all the constraints in equations 2.6 through 2.10.

2.8 The General Form of an LP Model

The technique of linear programming is so-named because the MP problems to which it applies are linear in nature. That is, it must be possible to express all the functions in an LP model as some weighted sum (or linear combination) of the decision variables. So, an LP model takes on the general form:

$$\text{MAX (or MIN):} \quad c_1X_1 + c_2X_2 + \cdots + c_nX_n \qquad \textbf{2.11}$$

$$\text{Subject to:} \quad a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \leq b_1 \qquad \textbf{2.12}$$

$$a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \geq b_k \qquad \textbf{2.13}$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m \qquad \textbf{2.14}$$

Up to this point, we have suggested that the constraints in an LP model represent some type of limited resource. Although this is frequently the case, in later chapters, you will see examples of LP models in which the constraints represent things other than limited resources. The important point here is that any problem that can be formulated in the preceding fashion is an LP problem. The symbols c1, C2, . . . , cn in equation 2.11 are called objective function coefficients and might represent the marginal profits (or costs) associated with the decision variables X1, X2, . . . , Xn, respectively. The symbol aij found throughout equations 2.12 through 2.14 represents the numeric coefficient in the ith constraint for variable Xj. The objective function and constraints of an LP problem represent different weighted sums of the decision variables. The bi symbols in the constraints, once again, represent values that the corresponding linear combination of the decision variables must be less than or equal to, greater than or equal to, or equal to.

You should now see a direct connection between the LP model we formulated for Blue Ridge Hot Tubs in equations 2.5 through 2.10 and the general definition of an LP model given in equations 2.11 through 2.14. In particular, note that the various symbols used in equations 2.11 through 2.14 to represent numeric constants (that is, the cj, aij, and bi) were replaced by actual numeric values in equations 2.5 through 2.10. Also, note that our formulation of the LP model for Blue Ridge Hot Tubs did not require the use of equal to constraints. Different problems require different types of constraints, and you should use whatever types of constraints are necessary for the problem at hand.