Name:
Saad Ahmad.

Roll no:
20P-0051.

Class:
BS(CS) - 1D.

Q1 ~~a)~~ Explain why each of the following integrals is improper.

a) $\int_1^2 \frac{x}{x-1} \, dx.$

As we know that

Improper integrals of Type 1:
At least one of the upper/lower limit of the definite integral is $\pm\infty$.

Improper integrals of Type 2:
The integrand is not continuous at all points between and including the limits of integration.

So, we see that

$$\lim_{x \to 1^+} \frac{x}{x - \cancel{x}1}$$

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$$\frac{1}{0} = +\infty$$

Since the integrand is discontinuous at $x=1$, this is an improper integral of type 2.

b) $\displaystyle\int_0^{\infty} \frac{1}{1+x^3}\,dx.$

As we know that improper integrals of type 1 is that at least one of the the upper/lower limit of the define integral is $\pm\infty$, and improper integrals of type 2 is that the integrand is not continuous at all points between and including the limits of integration. So, we see that since the upper limit is ∞, this is an improper integral of type ~~8~~1.

c) $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx.$

As we know that improper integrals of type 1 is that at least one of the upper/lower limit of the define integral is $\pm\infty$ and improper integrals of type 2 is that the integrand is not continuous at all points between and including the limits of integration. Since both the upper and lower limits are non-finite, this is an improper integral of type 1.

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d) $\int_0^{\pi/4} \cot x \, dx.$

As we know that improper integrals of type 1 is that at least one of the upper/lower limit of the define integral is $\pm\infty$ and improper integrals of type 2 is that the integrand is not continuous at all points between and including the limits of integration. Since the integrand is discontinuous at $x=0$, so this is an improper integral of Type 2.

Q2 Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

i) $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} \, dx.$

As we see that this is an improper integral of type 1, so we can write

$$\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx = \lim_{t\to\infty} \int_3^{t} \frac{1}{(x-2)^{3/2}} dx.$$

Let $u = x-2$

on differentiation, we get

$$dx = du.$$

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The limits of integration will change from

$$\int_3^t \text{ to } \int_{3-2}^{t-2} = \int_1^{t-2}$$

So,

$$= \lim_{t\to\infty} \int_1^{t-2} \frac{1}{u^{3/2}}\, du$$

$$= \lim_{t\to\infty} \int_1^{t-2} u^{-3/2}\, du.$$

$$= \lim_{t\to\infty} \left[\frac{u^{(-3/2)+1}}{((-3/2)+1)} \right]_1^{t-2}$$

$$= \lim_{t\to\infty} \left[\frac{u^{-1/2}}{-1/2} \right]_1^{t-2}$$

$$= \lim_{t\to\infty} \left[-\frac{2}{\sqrt{u}} \right]_1^{t-2}$$

$$= \lim_{t\to\infty} -\frac{2}{\sqrt{t-2}} + \frac{2}{\sqrt{1}}$$

$$= -\frac{2}{\infty} + 2$$

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$= 0 + 2.$

$= 2.$

Since the limit exist, the improper integral is convergent and it is converges to 2.

ii) $\int_{-\infty}^{0} \frac{z}{z^2+4}\, dx.$

Let

$z^2 = 2u$ and

$2z\, dz = 2\, du \Rightarrow z\, dz = du.$

$\int \frac{du}{(2u)^2+4}$

$\frac{1}{4}\int \frac{du}{u^2+1}$

$\frac{1}{4}\tan^{-1} u + C$

Substitute back $u = \frac{z^2}{2}.$

$= \frac{1}{4}\tan^{-1}\frac{z^2}{2} + C$

Therefore,

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$$\int_{-\infty}^{0} \frac{z}{z^4+4}\, dz = \left[\frac{1}{4}\tan^{-1}\frac{z^2}{2}\right]_{-\infty}^{0}$$

$$= \frac{1}{4}\tan^{-1}0 - \frac{1}{4}\tan^{-1}(\infty)$$

$$= 0 - \frac{1}{4}\cdot\frac{\pi}{2}$$

$$= -\frac{\pi}{8}$$

It ~~is~~ converges to $-\frac{\pi}{8}$.

iii) $$\int_{e}^{\infty} \frac{1}{x(\ln x)}\, dx$$

let $\ln x = u$ and

$$\frac{dx}{x} = du$$

$$\int_{e}^{\infty} \frac{du}{u^2}$$

$$\left[-\frac{1}{u}\right]_{e}^{\infty} + C$$

Substitute back $u = \ln x$.

$$= \left[-\frac{1}{\ln x}\right]_e^{\infty} + C$$

$$= -\frac{1}{\ln(\infty)} + \frac{1}{\ln e}$$

$$= -\frac{1}{\infty} + \frac{1}{1}$$

$$= 0 + 1$$

$$= 1.$$

It converges to 1.

iv) $$\int_1^{\infty} \frac{dx}{\sqrt{x} + x\sqrt{x}}.$$

Let $x = u^2$ and

$$dx = 2u\,du$$

$$\int \frac{2u\,du}{\sqrt{u^2} + u^2\sqrt{u^2}}$$

$$\int \frac{2u\,du}{u + u^3}$$

$$\int \frac{2\cancel{u}\,du}{\cancel{u}(1+u^2)}$$

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$$\int \frac{2\,du}{1+u^2}$$

$2\tan^{-1} u + C.$

Substitute back $u = \sqrt{x}$

$2\tan^{-1}\sqrt{x} + C.$

Therefore

$$\int_1^{\infty} \frac{dx}{\sqrt{x}+x\sqrt{x}} = \Big[2\tan^{-1}\sqrt{x}\Big]_1^{\infty}$$

$= 2\tan^{-1}\sqrt{\infty} - 2\tan^{-1}\sqrt{1}$

$= 2\tan^{-1}(\infty) - 2\tan^{-1}(1)$

$= \cancel{2}\cdot\frac{\pi}{\cancel{2}} - 2\cdot\frac{\pi}{\cancel{4}2}$

$= \pi - \frac{\pi}{2}$

$= \frac{\pi}{2}$

It converges to $\frac{\pi}{2}$.

v) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

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As we see that, when.

$$= \lim_{x \to 1^+} \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-1}}$$

$$= \frac{1}{\sqrt{0}}$$

$$= \infty.$$

Since the integrand is discontinuous at $x = 1$, this is an improper integral of type 2. so we can write

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{b \to 1^-} \Big[\sin^{-1} x \Big]_0^b$$

$$= \lim_{b \to 1^-} \sin^{-1} b - \sin^{-1} 0$$

$$= \sin^{-1} 1 - 0.$$

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$= \frac{\pi}{2}$.

It converges to $\frac{\pi}{2}$.

vi) $\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta$.

Let

$x = \sin\theta$ and

$dx = \cos\theta\, d\theta$

Limits of Integration will change from

$\int_0^{\pi/2}$ to $\int_{\sin 0}^{\sin(\pi/2)} = \int_0^1$

$= \int_0^1 \frac{1}{\sqrt{x}}\, dx$

$= \left[2\sqrt{x}\right]_0^1$

$= 2\sqrt{1} - 2\sqrt{0}$

$= 2 - 0$.

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= 2.

It converges to 2.

vii) $\int_0^1 \frac{e^{1/x}}{x^3}\, dx$

Let $t = \frac{1}{x}$ and

$dt = -\frac{dx}{x^2}$

$\int -te^t\, dt$

Now we will perform integration by parts.

As we know that

$\int u\,dv = uv - \int v\,du.$

Let

$u = -t \qquad dv = e^t\, dt.$

Then

$du = -dt \qquad v = e^t$

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So,

$$\int -te^t\,dt = -t\cdot e^t - \int e^t\cdot(-dt)$$

$$= -te^t + \int e^t\,dt$$

$$= -te^t + e^t + C.$$

Substitute back $t = \frac{1}{x}$

$$= -\frac{e^{1/x}}{x} + e^{1/x} + C$$

Therefore

$$= \lim_{t\to 0^+}\left[\frac{-e^{1/x}}{x} + e^{1/x}\right]_t^1$$

$$= \left[\frac{-e^{1/1}}{1} + e^{1/1}\right] - \lim_{t\to 0^+}\left[\frac{e^{1/t}}{t} + e^{1/t}\right]$$

$$= 0 + \lim_{t\to 0^+} e^{1/t}\left[\frac{1}{t} - 1\right]$$

$$= e^{\infty}\left[\infty - 1\right]$$

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$= \infty$

It ~~is~~ diverges.

viii) $\int_0^1 r \ln r \, dr$

As we know that

$$\int u\,dv = uv - \int v\,du.$$

Let

$u = \ln r \quad , \quad dv = r\,dr$

Then

$du = \frac{dr}{r} \quad , \quad v = \frac{r^2}{2}$

So,

$$\int r \ln r \, dr = \ln r \cdot \frac{r^2}{2} - \int \frac{r^2}{2} \cdot \frac{dr}{r}$$

$$= \frac{r^2 \ln r}{2} - \int \frac{r}{2} \, dr.$$

$$= \frac{r^2 \ln r}{2} - \frac{r^2}{4} + C.$$

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Therefore,

$$= \lim_{t\to0^+}\left[\frac{r^2\ln r}{2} - \frac{r^2}{4}\right]_t^1$$

$$= 0 - \frac{1}{4} - \lim_{t\to0^+}\frac{t^2\ln t}{2} + \lim_{t\to0^+}\frac{t^2}{4}$$

$$= -\frac{1}{4} + \lim_{t\to0^+}\frac{-\ln t}{2/t^2}$$

Since the limit is of the form $\frac{\infty}{\infty}$, I will apply L-hospital's Rule.

$$= -\frac{1}{4} + \lim_{t\to0^+}\frac{-1/t}{-4/t^3}$$

$$= \frac{-1}{4} + \lim_{t\to0^+}\frac{t^2}{4}$$

$$= \frac{-1}{4} + 0$$

$$= -\frac{1}{4}$$

It ~~is~~ converges to $-\frac{1}{4}$.

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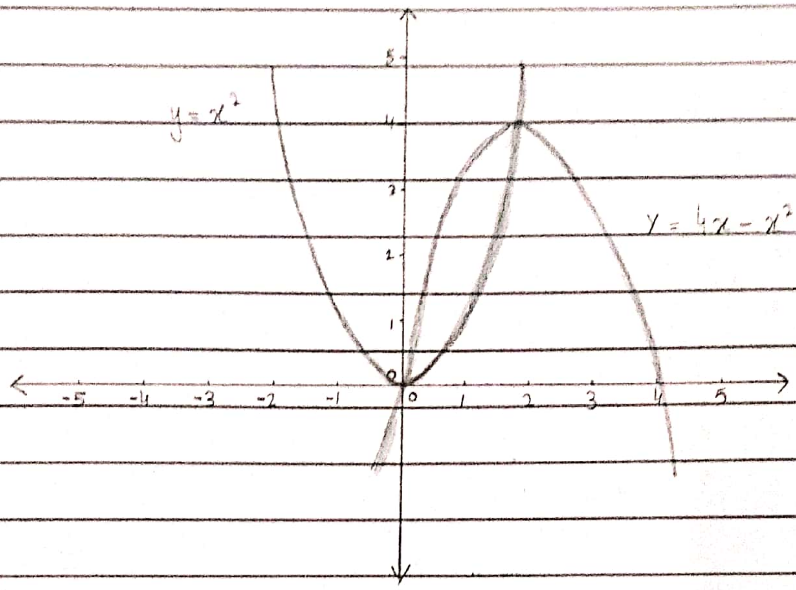
Q3 Sketch the region enclosed by the given curves and find its area.

i) $y = x^2$, $y = 4x - x^2$.



First we will find the intersection points so,

$$4x - x^2 = x^2$$

$$0 = 2x^2 - 4x$$

$$0 = 2x(x-2)$$

$$2x = 0 \quad , \quad x - 2 = 0.$$

$$x = 0 \quad , \quad x = 2.$$

These will be the integral limits. The top function is $y = 4x - x^2$, the bottom is $y = x^2$. So, subtracting the bottom function from the top one.

$$A = \int_0^2 ((4x - x^2) - x^2)\, dx$$

$$= \int_0^2 (4x - 2x^2)\, dx$$

$$= \left[2x^2 - 2 \cdot \frac{1}{3} x^3 \right]_0^2$$

$$= 2(2)^2 - \frac{2}{3}(2)^3 - (0 - 0)$$

$$= 8 - \frac{16}{3}$$

$$= \frac{24 - 16}{3}$$

$$= \frac{8}{3}$$

Area of the enclosed region is $6\sqrt{3}$ square units.

ii) $y = sec^2 x$, $y = 8 \cos x$, $-\frac{\pi}{3} \le x \le \frac{\pi}{3}$

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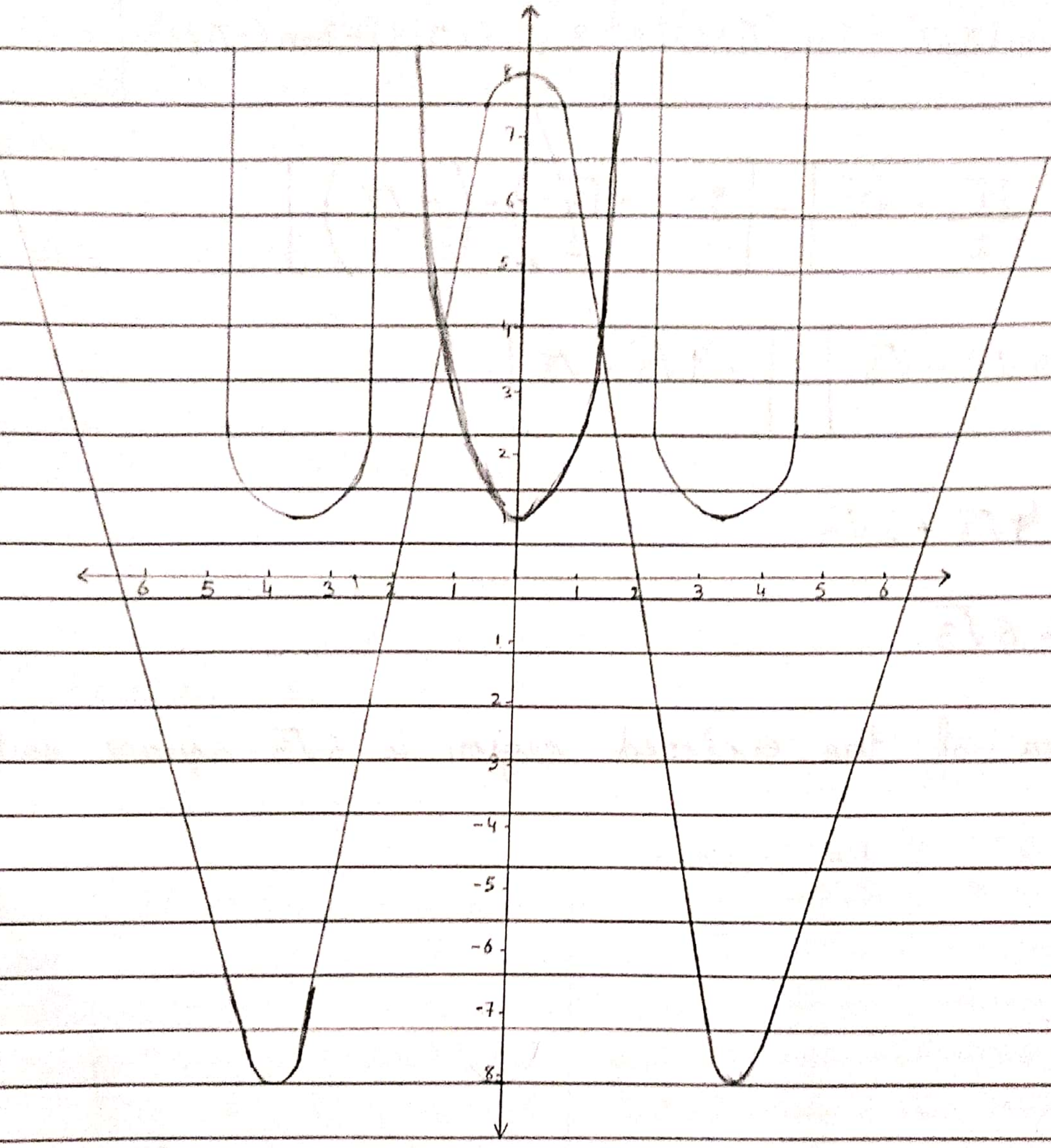


From the above graph, we see that $8\cos x \geq \sec^2 x$ on the interval $[-\pi/3, \pi/3]$. So the area is

$$A = \int_{-\pi/3}^{\pi/3} 8\cos x - \sec^2 x \, dx$$

$$= \Big[8\sin x - \tan x \Big]_{-\pi/3}^{\pi/3}$$

$= \left[8\sin(\pi/3) - \tan(\pi/3)\right] - \left[8\sin(-\pi/3) - \tan(-\pi/3)\right]$

$= \left[8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right] - \left[8 \cdot \left(-\frac{\sqrt{3}}{2}\right) - \left(-\sqrt{3}\right)\right]$

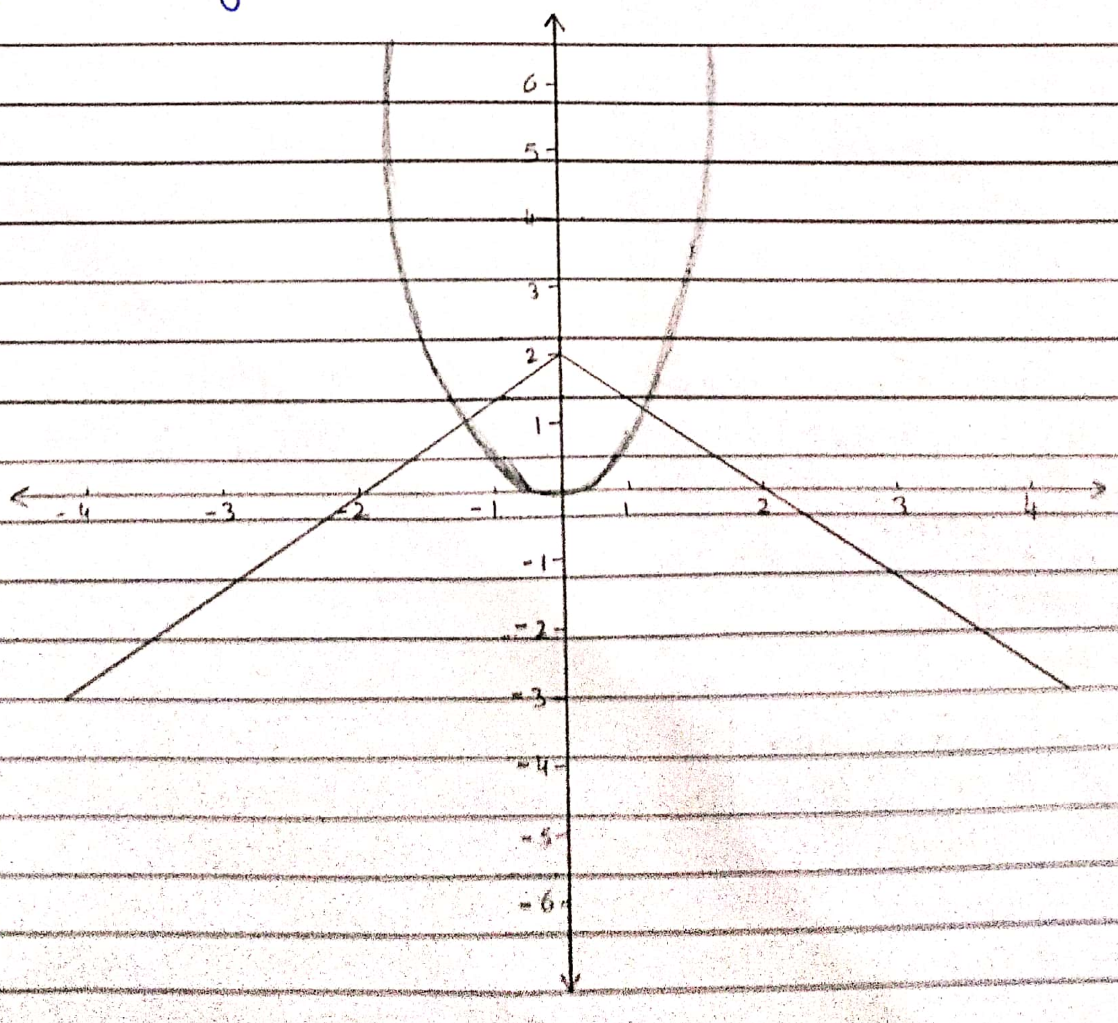
$= \left[4\sqrt{3} - \sqrt{3}\right] - \left[-4\sqrt{3} + \sqrt{3}\right]$

$= 4\sqrt{3} + 2\sqrt{3}$

$= 6\sqrt{3}$

Area of the enclosed region is $6\sqrt{3}$ square units.

iii) $y = x^4$, $y = 2 - |x|$.



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From the above graph we can see that $2 - |x| \geq x^4$ on the interval $[-1, 1]$, so the area is

$$A = \int_{-1}^{1} 2 - |x| - x^4 \, dx$$

$$= \int_{-1}^{0} 2 - (-x) - x^4 dx + \int_{0}^{1} 2 - x - x^4 \, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^5}{5} \right]_{-1}^{0} + \left[2x - \frac{x^2}{2} - \frac{x^5}{5} \right]_{0}^{1}$$

$$= 0 - \left[2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^5}{5} \right] + \left[2(1) - \frac{(1)^2}{2} - \frac{(1)^5}{5} \right] - 0$$

$$= 0 - \left[-2 + \frac{1}{2} + \frac{1}{5} \right] + \left[2 - \frac{1}{2} - \frac{1}{5} \right] - 0$$

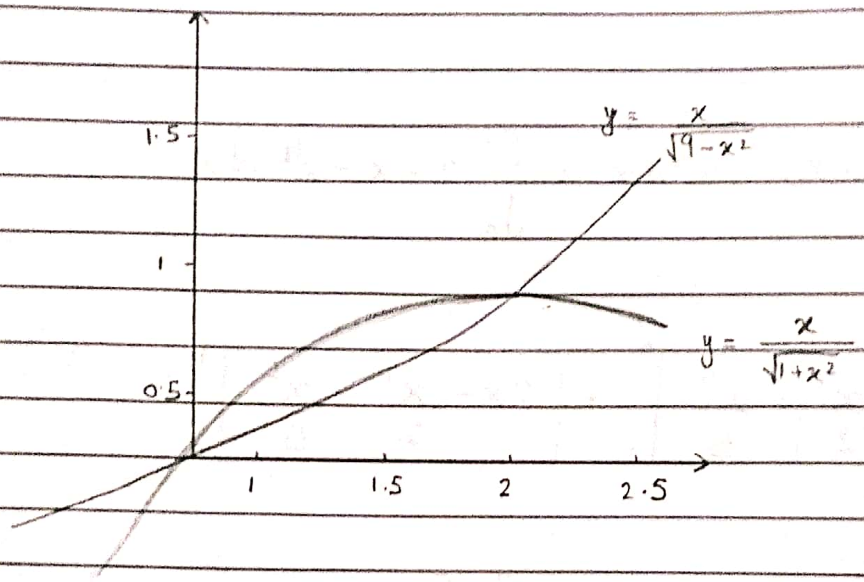
$$= 2 - \frac{1}{2} - \frac{1}{5} + 2 - \frac{1}{2} - \frac{1}{5}$$

$$= \frac{13}{5}$$

Area of the enclosed region is $6\sqrt{3}$ square units.

Q4 Sketch the region enclosed by the given curves and find its area.

i) $y = \frac{x}{\sqrt{1+x^2}}$, $y = \frac{x}{\sqrt{9-x^2}}$, $x \geq 0$



From theory we know that area A bounded by the curves $f(x)$ ang $g(x)$ between line $x = a$ and $x = b$, when f and g are continuous and $f(x) \geq g(x)$ for all x from the interval $[a, b]$ is given by the following formula:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

We can see that our function intersect when $x = 0$ and $x = 2$. We can also see that function $y = \frac{x}{\sqrt{1+x^2}}$ is our upper bound on interval $[0, 2]$.

Therefore we have everything needed to calculate the area.

$$A = \int_0^2 \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{9-x^2}} \, dx$$

Solving the first integral:

$$\int_0^2 \frac{x}{\sqrt{1+x^2}} \, dx.$$

Let $u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow 0 \to 1$ & $2 \to 5$

$$\int_1^5 \frac{1}{2} u^{-1/2} \, du$$

$$u^{1/2} \Big|_1^5$$

$$\sqrt{5} - 1$$

Now solving the second integral:

$$\int_0^2 - \frac{x}{\sqrt{9-x^2}} \, dx$$

$z = 9 - x^2 \Rightarrow dz = -2x\,dx \Rightarrow 0 \to 9$ & $2 \to 5.$

$$\int_9^5 -\frac{1}{2} z^{-1/2} \, dz$$

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$$\int_5^9 \frac{1}{2} z^{-1/2} \, dz$$

$$z^{1/2} \Big|_5^9$$

$$3 - \sqrt{5}$$

So,

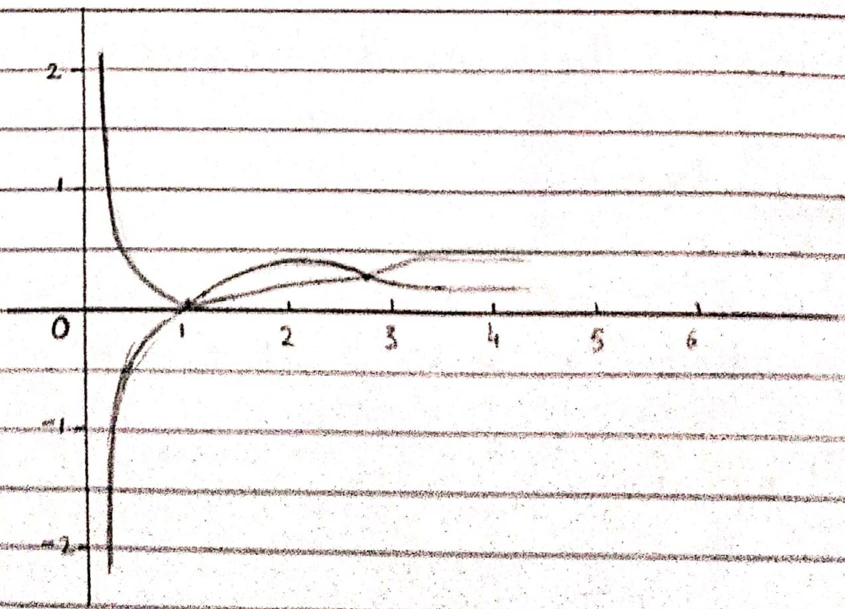
$$\int_0^2 \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{9-x^2}} \, dx.$$

$$A = \sqrt{5} - 1 - (3 - \sqrt{5})$$

$$= 2\sqrt{5} - 4.$$

$$A = 0.4721.$$

ii) $y = \frac{\ln x}{x}$, $y = \frac{(\ln x)^2}{x}$



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As we know that area enclosed between two curve $y = f(x)$ and $y = g(x)$ over $[a, b]$ such that

$$f(x) \geq g(x)$$

So.

$$A = \int_a^b f(x) - g(x)\, dx$$

From the above graph we see that $\frac{\ln x}{x} \geq \frac{(\ln x)^2}{x}$

on the interval $[1, e]$.

Therefore area between the two curves is

$$= \int_1^e \frac{\ln x}{x} - \frac{(\ln x)^2}{x}\, dx$$

$$= \int_1^e \frac{\ln x - (\ln x)^2}{x}\, dx$$

Let $u = \ln x$ and $du = \frac{dx}{x}$

limits of integration will change from

$$\int_1^e \text{ to } \int_{\ln 1}^{\ln e} = \int_0^1$$

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$$= \int_0^1 u - u^2 \, du.$$

$$= \left[\frac{u^2}{2} - \frac{u^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}$$

$\therefore$ Area $= \frac{1}{6}$.

Q5 Find the limit, use L·Hospital's Rule where appropriate.

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$$

$$= \lim_{x \to 4} \frac{x^2 - 2x - 8}{x - 4}$$

$$= \lim_{x \to 4} \frac{(x+2)\cancel{(x-4)}}{\cancel{x-4}}$$

$$= \lim_{x \to 4} x + 2$$

$$= 4 + 2$$

$$= 6$$

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ii) $\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}.$

On direct substitution of $x = \frac{\pi}{2}$ we get $\frac{0}{0}$ form so we will use L-Hospital Rule. So.

$$= \lim_{x \to (\pi/2)^+} \frac{d/dx(\cos x)}{d/dx(1 - \sin x)}$$

$$= \lim_{x \to (\pi/2)^+} \frac{\cancel{-} \sin x}{\cancel{-} \cos x}$$

$$= \lim_{x \to (\pi/2)^+} \tan x.$$

As we know that ~~tan~~ right hand side limit of $\tan x$ at $x = \pi/2$ is $-\infty$, so.

$$= \lim_{x \to (\pi/2)^+} \tan x.$$

$$= \tan(\pi/2).$$

$$= -\infty.$$

iii) $\lim_{x \to \pi} \frac{1 + \cos x}{1 - \cos x}.$

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$$\lim_{\theta \to \pi} \frac{1+\cos\theta}{1-\cos\theta}$$

As we know that

* $\cos\pi = -1$, so.

$$= \frac{1+\cos(\pi)}{1-\cos(\pi)}$$

$$= \frac{1+(-1)}{1-(-1)}$$

$$= \frac{1-1}{1+1}$$

$$= \frac{0}{2}$$

$= 0$

iv) $$\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2}$$

On direct substitution of $x = 0$ we get $\frac{0}{0}$ form so we will use L-Hospital Rule.
So, differentiate the numerator and the denominator:

$$= \lim_{x \to 0} \frac{d/dx(\cos mx - \cos nx)}{d/dx(x^2)}.$$

$$\lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x}$$

$$\lim_{x\to 0} \frac{-m\sin mx + n\sin nx}{2x}.$$

By applying limit we get $\frac{0}{0}$ so using L-Hospital's rule again.

$$= \lim_{x\to 0} \frac{d/dx(-m\sin mx + n\sin nx)}{d/dx(2x)}.$$

$$= \lim_{x\to 0} \frac{-m^2\cos mx + n^2\cos nx}{2}$$

$$= \lim_{x\to 0} \frac{-m^2\cos mx + n^2\cos nx}{2}.$$

$$= \frac{-m^2 + n^2}{2}.$$

$$= \frac{n^2 - m^2}{2}.$$

v) $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$ $\quad b \neq 0$

On direct substitution of $x = 1$ we get $\frac{0}{0}$ form so we will use L-Hospital Rule.

$$\lim_{x\to 1} \frac{d/dx(x^a - 1)}{d/dx(x^b - 1)}$$

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$$= \lim_{x \to 1} \frac{a x^{a-1}}{b x^{b-1}}$$

$$= \frac{a . 1^{a-1}}{b . 1^{b-1}}$$

$$= \frac{a}{b} .$$

vi) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$

On direct substitution of $x = 1$ we get $\frac{0}{0}$ form so we will use L-Hospital Rule.

$$\lim_{x \to 1} \frac{x \ln x - x + 1}{\ln x (x-1)} .$$

$$\lim_{x \to 1} \frac{d/dx\,(x \ln x - x + 1)}{d/dx(\ln x (x-1))}$$

$$\lim_{x \to 1} \frac{\ln x + x \cdot 1/x - 1}{\frac{1}{x}(x-1) - \ln x .}$$

$$\lim_{x \to 1} \frac{\ln x + 1 - 1}{1 - \frac{1}{x} + \ln x}$$

$$\lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} .$$

By applying limit we get $\frac{0}{0}$ so we will

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use L-Hospital rule again. so.

$$= \lim_{x \to 1} \frac{d/dx(\ln x)}{d/dx(1 - \frac{1}{x} + \ln x)}$$

$$= \lim_{x \to 1} \frac{1/x}{1/x^2 + 1/x}$$

$$= \lim_{x \to 1} \frac{1}{\frac{1}{x} + 1}$$

$$= \frac{1}{1/1 + 1}$$

$$= \frac{1}{2}$$

vii) $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

As we know that

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

and

$$\cot x = \frac{\cos x}{\sin x}$$

so,

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

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On direct substitution of $x=0$ we get $\frac{0}{0}$ form so we will use L-Hospital rule. so.

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}.$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \lim_{x \to 0} \frac{d/dx\,(1 - \cos x)}{d/dx\,(\sin x)}.$$

$$= \lim_{x \to 0} \frac{-\sin x}{-\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x}.$$

$$= \frac{\sin(0)}{\cos(0)}.$$

$$= \frac{0}{1}$$

$= 0$. Ans

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