Assignment no#4

Q1. Find the absolute maximum and minimum value of the following functions.

1. $f(x) = 12 + 4x - x^2$, $[0, 5]$

Sol:-

$f(x) = 12 + 4x - x^2$

$f'(x) = 0 + 4 - 2x$

$f'(x) = 4 - 2x$

$4 - 2x = 0$

$x = 2$

check value at $x = 2$

$f(0) = 12 + 4(0) - (0)^2$

$= 12$

$f(5) = 12 + 4(5) - (5)^2$

$= 7$

$f(2) = 12 + 4(2) - (2)^2$

$= 16$

so $f(2) = 16$ is absolute maximum value and $f(5) = 7$ is absolute minimum value

2- $f(x) = x + \frac{1}{x}$, $[0.2, 4]$

$$f(x) = x + \frac{1}{x}$$

$$f(x) = x + x^{-1}$$

$$= (1) + (-1)x^{-1-1}$$

$$= 1 + (-1)x^{-2}$$

$$= 1 - x^{-2}$$

$$= 1 - \frac{1}{x^2}$$

$$1 - \frac{1}{x^2} = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$x = -1$ is not in the given interval $[0.2, 4]$

$$f(0.2) = 5.2$$

$$f(1) = 4.5$$

So $f(0.2) = 5.2$ is absolute maximum value

and $f(1) = 2$ is absolute minimum value

3. $\log_e(x^2+x+1) = f(x)$ $[-1, 1]$

$f(x) = \ln(x^2+x+1)$

$f(x) = \dfrac{1}{x^2+x+1} \times 2x+1$

$\Rightarrow \dfrac{2x+1}{x^2+x+1} = 0$

$2x+1 = 0$

$x = -\dfrac{1}{2}$

Checking values at end points and $x = \dfrac{-1}{2}$

$\ln(x^2+x+1) = f(x)$

$f(-1) = \ln[(-1)^2 + \cancel{(-1)} + \cancel{1}]$

$= \ln 1$

$= 0$

$f(1) = \ln[(1)^2 + 1 + 1]$

$= \ln 3$

$= 1.0986$

$$f\left(-\frac{1}{2}\right) = \ln\left[\left[-\frac{1}{2}\right]^2 + \left[-\frac{1}{2}\right] + 1\right]$$

$$= \ln\left[\frac{1}{4} - \frac{1}{2} + 1\right]$$

$$= \ln\left[\frac{1}{4} + \frac{1}{2}\right]$$

$$= \ln\left[\frac{3}{4}\right]$$

$$= -0.288$$

so

$f(1) = 1.0986$ is absolute maximum value

$f\left(-\frac{1}{2}\right) = -0.288$ is absolute minimum value

4- $f(x) = x - 2\tan^{-1}(x)$ $[0,4]$

sol

$$f(x) = x - 2\tan^{-1}(x)$$

$$f'(x) = 1 - 2\left[\frac{1}{1+x^2}\right]$$ As $\tan^{-1}x = \frac{1}{1+x^2}$

$$= 1 - \frac{2}{1+x^2}$$

$$= \frac{1+x^2-2}{1+x^2}$$

$$\Rightarrow \frac{x^2-1}{1+x^2} = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Rejecting 1 as it is not included in interval

checking at end points and at $x=1$

$$f(0) = 0 - 2\tan^{-1}(0)$$
$$= 0 - 2(0)$$
$$f(0) = 0$$

$$f(1) = 1 - 2\tan^{-1}(1)$$
$$= 1 - 2(0.785)$$
$$f(1) = -0.571$$

$$f(4) = 4 - 2\tan^{-1}(4)$$
$$= 4 - 2(1.026)$$
$$= 4 - 2.652 \Rightarrow f(4) = 1.348$$

So,

$f(1) = -0.571$ is the absolute minimum value

$f(4) = 1.348$ is the absolute maximum value

Q2:- Verify that the function satisfies the three hypothoses of Rolle's theorem on the given interval then find the all numbers c that satisfy the conclusion of Rolle's theorem.

i. $f(x) = 2x^2 - 4x + 5$, $[-1, 3]$

- Hypothesis 1:-
  since $f(x)$ is a polynomial and we know that polynomials are continous every where so $f(x)$ is continous in $[-1, 3)$

- Hypothesis 2:
  Every polynomial is differentiable therefore $f(x)$ is differentiable in $[-1, 3]$

- Hypothesis 3:
  $f(a) = f(b)$

$$f(-1) = f(3)$$

$$f(x) = 2x^2 - 4x + 5$$

$$f(-1) = 2(-1)^2 - 4(-1) + 5$$

$$= 2 + 4 + 5$$

$$= 11$$

$$f(3) = 2(3)^2 - 4(3) + 5$$

$$= 2(9) - 12 + 5$$

$$= 18 - 12 + 5$$

$$= 11$$

$$\therefore f(-1) = f(3)$$

Here the conditions of Rolle's Theorem is satisfied, now finding the value of $f(c)$ in the interval $[-1, 3]$

$$f'(c) = 0$$

$$f(x) = 2x^2 - 4x + 5$$

$$f'(x) = 2x - 4$$

$$f'(c) = 2c - 4$$

$$2c - 4 = 0$$

$$2c = 4$$

$$c = 2$$

Thus the value of c is given by $c=2$

ii- $f(x) = \sin\left(\frac{x}{2}\right)$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

- Hypathesis 1:

since its a big function therefore according to theorem its continous every where so $f(x)$ is continuous in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

- Hypathesis 2:-

The function is also differentiable

- Hypathesis 3:-

$$f(a) = f(b)$$
$$f(\pi/2) = f(3\pi/2)$$

$$f(x) = \sin\left(\frac{x}{2}\right)$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} \times \frac{1}{2}\right)$$
$$= \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}}$$

$$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2} \times \frac{1}{2}\right)$$

$$= \sin\left(\frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$

$$f(\pi/2) = f(3\pi/4)$$

Here the condition, of the Rolle's Theorem is satisfied, now finding the value of $f(c)$ in the interval $[\pi/2, 3\pi/2]$.

$$f'(c) = 0$$

$$f(x) = \sin\left(\frac{x}{2}\right)$$

$$f'(x) = \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2}$$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f'(c) = \frac{1}{2}\cos\left(\frac{c}{2}\right)$$

$$\frac{1}{2}\cos\left(\frac{c}{2}\right) = 0$$

$$\cos\left(\frac{c}{2}\right)=0$$

$$\cos\theta = 0$$

when $(2n+1)\frac{\pi}{2} = 0$

So,

$$\frac{c}{\cancel{2}} = (2n+1)\frac{\pi}{\cancel{2}}$$

$$c = (2n+1)\pi$$

$$c = 2n\pi + \pi$$

but only $c = \pi$ is in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Therefore

$$c = \pi$$

Q3:- Verify the hypotheses of mean vallie therefore (MVT) and then find the number c, that satisfies the conclusion of MVT.

i- $f(n) = 2n^2 - 3n + 1 \quad [0,2]$

Sol.

- Hypothesis 1:-

since $f(n)$ is a polynomial

hence, it is continous in $[0, 2]$

- Hypothesis 2:-

$f(x)$ is also differentiable in $(0,2)$ because $f(x)$ is polynomial.

Thus

$$f(b) - f(a) = f'c(b-a)$$
$$f(2) - f(0) = f'(c)(2-0)$$
$$f(2) - f(0) = 2f'(c) \longrightarrow (1)$$

$$f(x) = 2x^2 - 3x + 1$$
$$f(2) = 2(2)^2 - 3(2) + 1$$
$$= 8 - 6 + 1$$
$$= 3$$

$$f(0) = 2(0) - 3(0) + 1$$

$$f(0) = 1$$

$$f'(x) = 4x - 3$$

$$f'(c) = 4c - 3$$

putting the values of $f(2)$, $f(0)$ and $f'(c)$

in eq 1

$$3-1 = (4c-3)(2)$$

$$\cancel{2} = (4c-3)(\cancel{2})$$

$$4c-3 = 0$$

$$c = \frac{3}{4}$$

ii- $f(x) = \ln x \quad [1,4]$

- hypothesis 1:

  $\ln x$ is continuous for all $x \geq 0$

- hypothesis 2:

  $\ln x$ is differentiable for all $x > 0$.

- According to MTV theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(x) = \log_e(x)$$

$$f(1) = \log_e 1$$

$$= 0$$

$$f(4) = \log_e 4$$

$= \ln 4$

$f'(x) = \frac{1}{x}$

$f'(c) = \frac{1}{c}$

putting above values in MVT Theorem,

$$f'(c) = \frac{f(4) - f(1)}{4-1}$$

$$\frac{1}{c} = \frac{\ln 4 - \ln 1}{3}$$

$$\frac{1}{c} = \frac{\ln 4}{3}$$

$$c = \frac{3}{\ln 4}$$

Q4:- Use the mean theorem to prove the inequality $|\sin a - \sin b| \leq |a-b|$ for all a and b.

sol

using Mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

let

$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f'(c) = \cos(c)$$

Thus

$$\cos(c) = \frac{\sin(b) - \sin(a)}{b-a}$$

Since $-1 \leq \cos(c) \leq 1$, we can say that

$$|\cos(c)| \leq 1$$

$$\left| \frac{\sin(b) - \sin(a)}{(b-a)} \right| \leq 1$$

$$|\sin(b) - \sin(a)| \leq |b-a|$$

Multiplying (-1) on both sides.

$|\sin(a) - \sin(b)| \le |a-b|$

QS:- Find the interval of increase or decrease.
b- Find the local max and minimum value
c- Find the interval of concavity and inflection point.

1. $f(x) = x^3 - 12x + 2$

Sol

a) $f(x) = x^3 - 12x + 2$

$f'(x) = 3x^2 - 12$

- when $f(x)$ is increasing $f'(x) > 0$

$3x^2 - 12 > 0$

$3x^2 > 12$

$x^2 > 4$

$|x| > 2$

$x > 2$ or $-x > 2$

$x < -2$

so $f(x)$ is increasing in the interval $(-\infty, -2) \cup (2, \infty)$

• when $f(n)$ is decreasing $f'(n) < 0$

$$3n^2 - 12 < 0$$
$$3n^2 < 12$$
$$n^2 < 4$$
$$|n| < 2$$
$$n < 2 \quad \text{or} \quad -n < 2$$
$$n > -2$$

So $f(n)$ is decreasing in the interval $(-2, 2)$

b) $f(n) = n^3 - 12n + 2$

Increasing then decreasing around $n = -2$

thus put $f(-2) = (-2)^3 - 12(-2) + 2$
$$= 18$$

local maximum $= 18$

Decreasing then increasing around $n = 2$

$$f(2) = (2)^3 - 12(2) + 2$$
$$f(2) = -14$$

local minimum $= -14$

c $f(u) = u^3 - 12u + 2$

$f'(u) = 3u^2 - 12$

$f''(u) = 6u$

$u = 0$

$f''(u) < 0$ on $(-\infty, 0)$, so f is concave downward on $(-\infty, 0)$

$f''(u) > 0$ on $(0, \infty)$, so f is concave upward on $(0, \infty)$

- Point of inflection occur where $f''(u) = 0$

$f''(u) = 6u$

$6u = 0$

$u = 0$

$f(0) = (0)^3 - 12(0) + 2$

$= 2$

so the point of inflection $= (0, 2)$

ii) $h(u) = (u+1)^5 - 5u - 2$

Sol

$$h(x) = (x+1)^5 - 5x - 2$$

$$h'(x) = 5(x+1)^4 - 5$$

when $h(x)$ is increasing $h'(x) > 0$

$$5(x+1)^4 = 5 > 0$$
$$5(x+1)^4 \geqslant 5$$
$$(x+1)4 > 1$$
$$|x+1| > 1$$

$x+1 > 1$ or $-(x+1) > 1$

$x > 0$ or $-x-1 > 1$

$$-x > 2$$
$$x < -2$$

So $h(x)$ is increasing in the interval $(-\infty, -2) \cup (0, \infty)$

when $(h)(x)$ is decreasing $h'(x) < 0$

$$5(x+1)^4 - 5 \angle 0$$
$$5(x+1)4 < 5$$
$$(x+1)4 < 1$$
$$|x+1) < 1$$

$x+1 < 1$ or $-(x+1) < 1$

$-x < 2$

$x < -2$

So $h(x)$ is increasing in the interval $(-2, 0)$

(b) $h(x) = (x+1)^5 - 5x - 2$

find local max and min.

put $x = -2$

$$h(-2) = (-2+1)^5 - 5(-2) - 2$$
$$= (-1)^5 + 10 - 2$$
$$= 7$$

put $x = 0$

$$h(0) = (0+1)^5 - 5(0) - 2$$
$$= (1)^5 - 0 - 2$$
$$= 1 - 2$$
$$= -1$$

So local max is at $h(-2) = 7$
and local min is at $h(0) = -1$

(c) find the interval of concavity and inflection point.

for Inflection point

$h(x) = (x+1)^5 - 5x - 2$

$h'(x) = 5(x+1)^4 - 5$

$h''(x) = 20(x+1)^3$

$20(x+1)^3 = 0$

$(x+1)^3 = 0$

$x+1 = 0$

$x = -1.$

$h(-1) = (-1+1)^5 - 5(-1) - 2$

$= 5 - 2$

$= 3$

flection point is $(-1, 3)$

$h''(x) > 0$ $x = -1$

and $h''(x) < 0$ $\Rightarrow$ $x = -1$

So h is Concavity up on $(-1, \infty$ and h is concave down is $(-\infty, -1)$

(iii) $f(x) = x - \sin x$ $0 < x \leq 4\pi$

(a) find Interval of increase or decrease

$f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$1 - \cos x = 0$

$\cos x = 1$

$x = 0, 2\pi, 4\pi$

$s'(2\pi) = 1 > 0$ s is increase.
$s'(4\pi) = 1 > 0$ s is increasing.

(b) find the local maximum and min.

Solution:

If both the function are increasing thus there will be no local max or minimum

(c) find the inflection point and concavity.

$s(x) = x - \sin x$

$s'(x) = 1 - \cos x$

$s''(x) = \sin x$

$x = 0, 2\pi, 3\pi, 4\pi.$

$s''(x) = \sin x$

$s''(\pi/2) = 1$ concave up

$s''(3\pi/2) = -1$ concave down

$s''(5\pi/2) = 1$ concave up

$s''(2\pi) = 0$ concave down

(iv) $f(\theta) = 2\cos\theta + \cos^2\theta \qquad 0 \le \theta \le 2\pi$

(a) $f'(\theta) = -2\sin\theta + 2\cos\theta(-\sin\theta)$

$= -2\sin\theta(1+\cos\theta)$

$f'(\theta) = 0$

$\theta = 0, \pi, 2\pi$

$f'(\theta) > 0 \Rightarrow \pi < \theta < 2\pi$

$f'(\theta) < 0 \Rightarrow 0 < \theta < \pi$

So f is increasing on $(\pi, 2\pi)$ and decreasing on $(0, \pi)$

(b) $f(\pi) = -1$ is local minimum value

(c) $f'(\theta) = -2\sin\theta(1+\cos\theta)$

$f''(\theta) = -2\sin\theta(-\sin\theta) + (1+\cos\theta)(-2\cos\theta)$

$= 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta$

$= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta$

$= -4\cos^2\theta - 2\cos\theta + 2$

$= 2(-2\cos^2\theta - \cos\theta + 1)$

$= -2\left(2\cos^2\theta + \cos\theta - 1\right)$

$= -2\left(2\cos\theta - 1\right)\left(\cos\theta + 1\right)$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \neq \pi$]. So

$f''(\theta) > 0$.

$2(\cos\theta - 1) < 0 \Rightarrow \cos\theta < 1/2 \Rightarrow \frac{\pi}{3} < \theta < 5\pi/3$

$f'(\theta) < 0$

$\cos\theta > 1/2 = 0 < \theta < \pi/3$ or $5\pi/3 < \theta < 2\pi$.

So f is concave up on $\left(\frac{\pi}{3}, 5\pi/3\right)$ and

f is concave down on $(0, \pi/3) \cup (5\pi/3, 2\pi)$

The point of inflection is

$\left(\pi/3, f(\pi/3)\right) = \left(\pi/3, 5/4\right)$ and $\left(5\pi/3, f(5\pi/3)\right) = \left(5\pi/3, \frac{5}{4}\right)$

Q6: Solve the angle (4, 5, 1)

Example: 4:

If we take x be the distance from c to T

You can do from book.

Q No 7: Find the linerization at $x = -1$

$$f(x) = \sqrt[3]{\left(1 - \frac{x}{2+x}\right)^2}$$

We have Linearization given by

$L(x) = f(a) + f'(a)(x-a)$ and $a = -1$

so,

$$f(a) = \sqrt[3]{\left(1 - \frac{(-1)}{2-(-1)}\right)^2}$$

$$= \sqrt[3]{\frac{(1+1)^2}{1}}$$

$$= \sqrt[3]{4}$$

$$f'(x) = \frac{d}{dx}\left(1 - \frac{x}{2+x}\right)^{2/3}$$

$$= \frac{2}{3}\left(1 - \frac{u}{2+u}\right)^{\frac{2}{3}-1} \frac{d}{du}\left(1 - \frac{u}{2+u}\right)$$

$$= \frac{2}{3}\left(\frac{2+u-u}{2+u}\right)^{-\frac{1}{3}} \left(0 - \frac{d}{du}\left(\frac{u}{2+u}\right)\right)$$

$$= \frac{2}{3}\left(\frac{u+2}{2}\right)^{\frac{1}{3}} \left(\frac{(u+2)\frac{d}{du}u - u\frac{d}{du}(u+2)}{(u+2)^2}\right)$$

$$= \frac{2}{3}\left(\frac{u+2}{2}\right)^{\frac{1}{3}} \left(\frac{u+2-u}{(u+2)^2}\right)$$

$$= \frac{2}{3}\left(\frac{u+2}{2}\right)^{\frac{1}{3}} \left(\frac{u+2-u}{(u+2)^2}\right)$$

$$= \frac{4}{3}\left(\frac{1}{2}(u+2)^{2-\frac{1}{3}}\right)$$

$$f_1(u) = \frac{4}{3}\left(\frac{1}{2}(2)^{\frac{1}{3}}(u+2)^{\frac{5}{3}}\right)$$

$f'(a)$ i.e $f'(-1)$

$$f'(-1) = \frac{4}{3}\left(\frac{1}{(2)^{\frac{1}{3}}(-1+2)^{\frac{5}{3}}}\right)$$

$$= \frac{(2)^2}{3}\left(\frac{1}{(2)^{1/3}}\right)$$

$$= \frac{(2)^{5/3}}{3}$$

$$L(u) = f(a) + f'(a)(u-a)$$

$$= f(-1) + f'(-1)(u-(1))$$

$$= \sqrt[3]{4} + \frac{(2)^{5/3}}{3}(u+1)$$

Q8:- Find the dy of the following

$$y = \frac{2\sqrt{u}}{3(1+\sqrt{u})}$$

Sol

Taking derivative

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{2\sqrt{u}}{3(1+\sqrt{u})}\right)$$

using Division differentiable Rule

$$= \frac{(3+3\sqrt{u}) \cdot \frac{1}{\sqrt{2}} - \left(2\cancel{\sqrt{u}} - \frac{3}{2\cancel{\sqrt{u}}}\right)}{(3+3\sqrt{u})^2}$$

$$= \frac{\frac{3+3\cancel{\sqrt{u}}}{\sqrt{u}\ \ \cancel{\sqrt{u}}} - 3}{(3+3\sqrt{u})^2}$$

$$= \frac{\frac{3}{\sqrt{u}} + \cancel{3} - \cancel{3}}{(3+3\sqrt{2})^2}$$

$$dy = \frac{3}{\sqrt{u}\ (3+3\sqrt{u})^2}$$

Q9- The total - - - - - - -

$$S = 2\pi r^2 + 2\pi r h$$

a) $\frac{ds}{df}$ related to $\frac{du}{df}$ if h is constant.

b. if u is constant.

Sol

for part (a)

$S = 2\pi r^2 + 2\pi r h$

$\frac{ds}{df} = 4\pi r \frac{du}{df} + 2\pi h \frac{du}{df}$

$\frac{ds}{df} = (4\pi r + 2\pi h) \frac{du}{df}$

$\frac{ds}{df} = (4\pi r + 2\pi h) \frac{du}{df}$

Part (b)

we have to find how $\frac{ds}{df}$ is related to $\frac{dh}{dt}$ if r is constant

$S = 2\pi r^2 + 2\pi r h$

$\frac{ds}{df} = 2\pi r \frac{dh}{dt}$