Name:
Saad Ahmad.

Roll no:
P20-0051.

Class:
BS(CS) - 1D.

Q1 Find the absolute maximum and minimum value of the following functions.

i) $f(x) = 12 + 4x - x^2$, $[0, 5]$

First we will find the critical number.

So,
Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(12 + 4x - x^2)$$

$$f'(x) = \frac{d}{dx}(12) + \frac{d}{dx}(4x) - \frac{d}{dx}(x^2)$$

$$f'(x) = 0 + 4 - 2x.$$

$$f'(x) = 4 - 2x.$$

$$4 - 2x = 0.$$
$$2x = 4.$$

$2x = 4.$

$x = 2.$

So, critical number is 2.

Now we will find the values at the endpoints of the interval.

The endpoints of the intervals are 0 and 5. So substitute $x = 0$ & $x = 5$ in given equation

$f(x) = 12 + 4x - x^2$

$\underline{\underline{x = 0}}$:

$f(0) = 12 + 4(0) - (0)^2$

$f(0) = 12 + 0 - 0.$

$f(0) = 12.$

$\underline{\underline{x = 5}}$:

$f(5) = 12 + 4(5) - (5)^2.$

$f(5) = 12 + 20 - 25$

$f(5) = 32 - 25$

$f(5) = 7.$

And for $x = 2$:

$f(2) = 12 + 4(2) - (2)^2$

$f(2) = 12 + 8 - 4$

$f(2) = 16$

So,

Absolute maximum value is 16.
Absolute minimum value is 7.

ii) $f(x) = x + \frac{1}{x}$, $[0.2, 4]$

First we will find the critical number.

So,

Taking $\frac{d}{dx}$ on b/s.

$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x + \frac{1}{x}\right).$

$f'(x) = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}).$

$f'(x) = 1 + (-x^{-2}).$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(x) = \frac{x^2 - 1}{x^2}$

$\frac{x^2 - 1}{x^2} = 0.$

$$\frac{x^2-1}{x^2}=0.$$

$$x^2-1=0$$

$$(x-1)(x+1)=0.$$

$$x-1=0 \quad , \quad x+1=0.$$

$$x=1 \quad , \quad x=-1$$

As ~~x~~ $-1 \notin [0.2, 4]$, so.

$$x=1.$$

So,
Critical number is 1.

Now we will find the values at the endpoints of the interval.

The endpoints of the interval are 0.2 & 4.
So, substitute $x=0.2$ and $x=4$ in the given equation $f(x)=x+\frac{1}{x}$

$x=0.2$:

$$f(x)=x+\frac{1}{x}.$$

$$f(0.2)=0.2+\frac{1}{0.2}.$$

$$f(0.2)=0.2+5.$$

$$f(0.2)=5.2.$$

$x = 4$:

$$f(4) = 4 + \frac{1}{4}$$

$$f(4) = 4 + 0.25$$

$$f(4) = 4.25$$

And for $x = 1$:

$$f(1) = 1 + \frac{1}{1}$$

$$f(1) = 2$$

So,

Absolute maximum value is 5.2.
Absolute minimum value is 2.

iii) $f(x) = \ln(x^2+x+1)$, $[-1, 1]$

First we will find the critical number.

So,
Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f(x) = \frac{d}{dx} (\ln(x^2+x+1)).$$

$$f'(x) = \frac{1}{x^2+x+1} \cdot \frac{d}{dx} (x^2+x+1)$$

$$f'(x) = \frac{1}{x^2+x+1} \cdot (2x+1+0)$$

$$f'(x) = \frac{2x+1}{x^2+x+1}$$

Now from the above equation we see that if the numerator is 0, then the derivative is 0 and if the denominator is 0, then the derivative does not exist.
Both cases can generate critical numbers, so we will take them one by one.

So,

$$f'(x) = \frac{2x+1}{x^2+x+1}$$

Now we will find the points where the numerator is 0.

$$2x+1=0.$$

$$2x=-1$$

$$x=-\frac{1}{2}.$$

And now finding the points where the denominator is 0.

$$x^2 + x + 1 = 0$$

~~(Put $x = 1$, so.)~~

$$\cancel{(x^2 + x + 1 = 0)}$$
$$\cancel{((1)^2 + 1 + 1)}$$

The above equation has no solution because its discriminant is.

$$(1)^2 - 4(1)(1)$$

$$1 - 4 < 0.$$

So, the critical number is $-\frac{1}{2}$

Now we will find the values at the endpoints of the intervals.

The endpoints of the intervals are -1 and 1, so substitute $x = -1$ & $x = 1$ in the given equation $f(x) = \ln(x^2 + x + 1)$.

$\underline{x = -1}$:

$$f(-1) = \ln((-1)^2 + (-1) + 1)$$

$$f(-1) = \ln(1)$$

$$f(-1) = 0.$$

$\underline{\underline{x = 1}}$:

$$f(1) = \ln((1)^2 + (1) + 1)$$

$$f(1) = \ln(3).$$

And for $x = \frac{-1}{2}$.

$$f(-1/2) = \ln((-1/2)^2 + (-1/2) + 1)$$

$$= \ln((1/4) + (1/2))$$

$$= \ln(3/4) < 0.$$

So,

Absolute maximum value is $\ln 3$.

Absolute minimum value is $\ln(3/4)$.

iv) $f(x) = x - 2\tan^{-1}x$, $[0, 4]$.

First we will find the critical number.

So,

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x - 2\tan^{-1}x).$$

$$f'(x) = \frac{d}{dx}(x) - \frac{d}{dx}(2\tan^{-1}x).$$

$$f'(x) = \frac{d}{dx}(x) - \frac{d}{dx}(2\tan^{-1}x).$$

$$= 1 - \frac{2}{x^2+1}.$$

Now.

$$1 - \frac{2}{x^2+1} = 0.$$

As $1 = \frac{1(x^2+1)}{x^2+1}$, so.

$$\frac{x^2+1}{x^2+1} - \frac{2}{x^2+1} = 0.$$

As $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$, so.

$$\frac{(x^2+1) - 2}{x^2+1} = 0.$$

$$\frac{x^2+1-2}{x^2+1} = 0.$$

$$\frac{x^2-1}{x^2+1} = 0.$$

$$x^2 - 1 = 0.$$

$$(x-1)(x+1) = 0$$

$(x-1)(x+1)=0.$

$x-1=0 \quad , \quad x+1=0.$

$x=1 \quad , \quad x=-1$

As. $-1 \in [0,4]$, so.

Critical number is 1.

Now we will find the values at the endpoints of the intervals.

The endpoints of the intervals are 0 & 4, so subtitute $x=0$ & $x=4$ in the given equation

$f(x) = x - 2\tan^{-1}x$

$\underline{\underline{x=0:}}$

$$f(0) = 0 - 2\tan^{-1}(0).$$
$$= 0 - 2(0).$$
$$= 0.$$

$\underline{\underline{x=4:}}$

$$f(4) = 4 - 2\tan^{-1}(4).$$

$$= 4 - 2\tan^{-1}(4) \approx 1.35.$$

And for $x=1$:

$$f(1) = 1 - 2\tan^{-1}(1).$$

$$= 1 - 2 \cdot \frac{\pi}{4}$$

$$= 1 - \frac{\pi}{2} \approx -0.57.$$

So,

Absolute maximum value is 1.35.
Absolute minimum value is -0.57.

Q2 Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

i) $f(x) = 2x^2 - 4x + 5$, $[-1, 3]$

The three hypotheses of Rolle's Theorem are:

$f(x)$ is continuous on the closed interval $[a,b]$.

$f(x)$ is differentiable on the open interval (a,b).

$f(a) = f(b)$

Now from the above hypotheses we see that the given function is continuous at all points in the closed interval because it is a polynomial. and the function is differentiable at all points in the open interval because it is a polynomial. The first two conditions are satisfied for the third we will solve the function so.

$$f(x) = 2x^2 - 4x + 5.$$

Now

$\underline{\underline{x = -1}}$:

$$f(-1) = 2(-1)^2 - 4(-1) + 5.$$

$$f(-1) = 2 + 4 + 5$$
$$f(-1) = 11$$

$\underline{\underline{x = 3}}$:

$$f(3) = 2(3)^2 - 4(3) + 5.$$
$$f(3) = 18 - 12 + 5$$
$$f(3) = 11$$

Since all 3 hypotheses are satisfied, the Rolle's Theorem is applicable.

Now we ~~ha~~ have to find all values of $c \in [-1, 3]$, such that $f'(c) = 0$. so.

$$f(x) = 2x^2 - 4x + 5$$

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(2x^2 - 4x + 5).$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(2x^2 - 4x + 5).$$

$$f'(x) = 4x - 4.$$

Now to find the value of c.

Put $x = c$ and $f'(x) = 0$, so we get.

$$0 = 4c - 4.$$

$$4c = 4$$

$$c = 1$$

ii) $f(x) = \sin(x/2)$, $[\pi/2 , 3\pi/2]$

The three hypotheses of Rolle's Theorm are.

$f(x)$ is continuous on the closed interval $[a,b]$.

$f(x)$ is differentiable on the open interval (a,b).

$f(a) = f(b)$.

Now from the above hypotheses we see that the given function is continuous as there are no holes or jumps in its graph and the function is differentiable because the graph is smoth at all points in the interval. The first two conditions are satisfied but for the third condition we will solve the function so,

$$f(x) = \sin(x/2).$$

Now

$\underline{x = \pi/2}$:

$$f(\pi/2) = \sin\left(\frac{\pi/2}{2}\right).$$

$$= \sin(\pi/4)$$

$$= \frac{1}{\sqrt{2}}$$

$x = 3\pi/2$

$$f(3\pi/2) = \sin\left(\frac{3\pi/2}{2}\right)$$

$$= \sin\left(\frac{3\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}$$

Since all 3 hypotheses are satisfied, the Rolle's theorem is applicable.

Now we have to find all values of

$c \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, such that $f'(c) = 0$, so.

$$f(x) = \sin(x/2).$$

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\sin(x/2))$$

$$f'(x) = \cos(x/2)\frac{d}{dx}(x/2).$$

$$= \frac{1}{2}\cos(x/2).$$

Now to find the value of c,

Put $x = c$ and $f'(x) = 0$, we get

$$0 = \frac{1}{2}\cos(c/2).$$

$$\cos(c/2) = 0. \quad —①$$

As we know that

$$\cos\theta = 0, \text{ when } \theta = \frac{(2n+1)\pi}{2} \quad —②$$

So comparing eq ① & ② we get

$$\frac{(2n+1)\pi}{2} = \frac{c}{2}.$$

$$c = (2n+1)\pi.$$

So, $c = \pi$.

Q3 Verify the hypotheses of mean value theorem (MVT) and then find the numbers c that satisfies the conclusion of MVT.

i) $f(x) = 2x^2 - 3x + 1$, $[0,2]$.

Mean Value Theorem can be applied to $f(x)$ on the interval $[a,b]$, If

f is continuous on $[a,b]$

f is differentiable on (a,b)

Mean Value Theorem:

There exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

So, from above discussion we see that $f(x)$ is a polynomial and is always differentiable and continuous. ~~and~~ so mean value theorem is applicable.

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(2x^2 - 3x + 1)$$

$$f'(x) = 4x - 3 \quad -①$$

Now to find value of c:

Put $x = 0$ in $f(x) = 2x^2 - 3x + 1$.

$$f(0) = 2(0)^2 - 3(0) + 1$$

$$f(0) = 0 - 0 + 1$$

$$f(0) = 1$$

Put $x = 2$, in $f(x) = 2x^2 - 3x + 1$.

$$f(2) = 2(2)^2 - 3(2) + 1$$

$$f(2) = 8 - 6 + 1$$

$$= 3.$$

Now putting the values in $f'(c) = \frac{f(b) - f(a)}{b - a}$, so

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{3 - 1}{2 - 0}$$

$$f'(c) = \frac{2}{2}$$

$$f'(c) = 1$$

Now using eq ① and replacing x with c.

$$4c - 3 = 1$$

$$4c = 1 + 3.$$

$$4c = 4$$

$$c = \frac{4}{4}$$

$$c = 1$$

ii) $f(x) = \ln x \quad , \quad [1, 4].$

Mean Value Theorem can be applied to $f(x)$ on the interval $[a, b]$, if

f is continuous on $[a, b]$

f is differentiable on (a, b)

Mean Value Theorem:

There exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

We see that function $f(x)$ is continuous and is differentiable on $(1,4)$. so $f'(x)$ exist

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\ln x).$$

$$f'(x) = \frac{1}{x} \quad \text{--- (1)}$$

Now to find the value of c:

Put $x=1$ in $f(x) = \ln x$.

$$f(1) = \ln 1$$

$$f(1) = 0.$$

Put $x=4$ in $f(x) = \ln x$.

$$f(4) = \ln 4.$$

Now putting the values in $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$f'(c) = \frac{\ln 4 - \ln 1}{4 - 1}$$

$$= \frac{\ln 4 - 0}{3}$$

$$f'(c) = \frac{\ln 4}{3}$$

Now using eq ① and replacing x with c.

$$\frac{1}{c} = \frac{\ln 4}{3}.$$

$$c \ln 4 = 3.$$

$$c = \frac{3}{\ln 4}$$

$$c \approx 2.16.$$

Q4 Use the Mean Value Theorem to prove the inequality

$|\sin a - \sin b| \leq |a - b|$ for all a and b.

According to mean value theorem

If $f(x)$ differentiable, then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Let suppose $f(x) = \sin(x)$.

Since $\sin x$ is differentiable everywhere, we can use mean value theorem on any ~~en~~ interval, so.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\sin(x).$$

$$f'(x) = \cos x.$$

So we can write

$$\cancel{\cos x}$$
$$\cos(c) = \frac{\sin(b) - \sin(a)}{b - a}$$

As we know that

$-1 < \cos(c) < 1$, so we can say that

$$|\cos(c)| \leq 1$$

Therefore

$$\left|\frac{\sin(b) - \sin(a)}{b - a}\right| \leq 1$$

$$\left|\frac{\sin(b)-\sin(a)}{b-a}\right| \leq 1$$

$$|\sin(b)-\sin(a)| \leq |b-a|$$

$$|\sin(a)-\sin(b)| \leq |a-b|$$

Hence Proved.

Q5 a) Find the interval of increase or decrease.
b) Find the local maximum and minimum values.
c) Find the intervals of concavity and the inflection points.

i) $f(x) = x^3 - 12x + 2$.

First we will find critical numbers where

$f' = 0$. so

$F(x) = x^3 - 12x + 2$.

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^3 - 12x + 2).$$

$$f'(x) = 3x^2 - 12.$$
$$f'(x) = 3(x^2 - 4)$$
$$f'(x) = 3(x-2)(x+2).$$

Now.

Put $f'(x) = 0$ as given.

$$0 = 3(x-2)(x+2).$$

$$x-2=0 \quad , \quad x+2=0.$$

$$x=2 \quad , \quad x=-2.$$

As we know that the domain of polynomials is $(-\infty, \infty)$ so using f' we will test numbers on the intervals determined by the domain and the critical numbers.

For $(-\infty, -2)$:

$$f'(x) = 3x^2 - 12.$$

$$f'(-3) = 3(-3)^2 - 12.$$

$$f'(-3) = 27 - 12$$

$$f'(-3) = 15$$

As $15 > 0$ so f is increasing.

For $(-2, 2)$:

$$f'(x) = 3x^2 - 12$$

$$f'(0) = 3(0)^2 - 12.$$

$$f'(0) = -12 < 0$$

As $-12<0$ so f is decreasing.

For $(2,\infty)$:

$$f'(x) = 3x^2-12$$

$$f(3) = 3(3)^2-12.$$

$$f'(3) = 27-12.$$

$$f'(3) = 15$$

As $15>0$ so f is decreasing.

Hence f is increasing at $(-\infty,-2)$ & $(2,\infty)$ and decreasing at $(-2,2)$.

Now we will find local maximum and local minimum.

As we see that it is increasing then decreasing around $x=-2$ so the local maximum will be:

$$f(x) = x^3-12x+2.$$

$$f(-2) = (-2)^3-12(-2)+2.$$

$$f(-2) = -8+24+2$$

$$f(-2) = 18.$$

And it is decreasing then increasing around

$x = 2$ so the local minimum will be:

$$f(x) = x^3 - 12x + 2.$$

$$f(2) = (2)^3 - 12(2) + 2.$$

$$f(2) = 8 - 24 + 2.$$

$$f(2) = -14.$$

And now we will find the intervals of concavity and the inflection points.

First we have to find where $f'' = 0$. so. taking $\frac{d}{dx}$ on b/s.

$$f(x) = x^3 - 12x + 2.$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} (x^3 - 12x + 2).$$

$$f'(x) = 3x^2 - 12.$$

Again taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f'(x) = \frac{d}{dx} (3x^2 - 12).$$

$$f''(x) = 6x.$$

Now putting $f''(x) = 0$ as given so we get.

$$6x = 0.$$

$$x = 0.$$

From the above calculation we see that in the interval $(-\infty, 0)$

$$f'' < 0$$

So f is concave down.

And in the interval $(0, \infty)$

$$f'' > 0$$

So f is concave up.

The point where concavity direction changes is an inflection point.

$$f(0) = 2.$$

ii) $h(x) = (x+1)^5 - 5x - 2$

First we will find critical numbers where $h' = 0$, so

$$h(x) = (x+1)^5 - 5x - 2.$$

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(h(x)) = \frac{d}{dx}\left[(x+1)^5 - 5x - 2\right]$$

$$h'(x) = 5(x+1)^4 - 5.$$

Now

Put $h'(x) = 0$ as given.

$$0 = 5(x+1)^4 - 5.$$

$$5(x+1)^4 = 5$$

$$(x+1)^4 = 1$$

$$x+1 = 1 \quad , \quad x+1 = -1$$

$$x = 0 \quad , \quad x = -2.$$

(Since the)

Now

$h'(x) > 0$ at $x < -2$ and $x > 0$

$h'(x) < 0$ at $-2 < x < 0$

$h'(x) = 0$ at $x = -2$ and $x = 0$

Since the derivative is positive when x is smaller than -2 and greater than 0, the function is increasing at these two points while the derivative is negative when x is between -2 and 0, the function is decreasing at this interval.

Now find the local maximum and minimum.

As we see that it is increasing then decreasing around $x=-2$ so the local maximum will be:

$$h(x) = (x+1)^5 - 5x - 2$$

$$h(-2) = (-2+1)^5 - 5(-2) - 2$$

$$h(-2) = -1 + 10 - 2$$

$$h(-2) = 7$$

And it is decreasing then increasing around $x=0$ so the local minimum will be:

$$h(x) = (x+1)^5 - 5x - 2.$$

$$h(0) = (0+1)^5 - 5(0) - 2.$$

$$h(0) = 1 - 2.$$

$$h(0) = -1.$$

And now we will find the intervals of concavity and the inflection points.

First we have to find where $f''=0$ so taking $\frac{d}{dx}$ on b/s.

$$h(x) = (x+1)^5 - 5x - 2.$$

$$\frac{d}{dx} h(x) = \frac{d}{dx}[(x+1)^5 - 5x - 2]$$

$$h'(x) = 5(x+1)^4 - 5.$$

Again taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx}(h'(x)) = \frac{d}{dx}(5(x+1)^4 - 5).$$

$$h''(x) = 20(x+1)^3$$

Now putting $h''(x) = 0$ as given, so we get:

$$20(x+1)^3 = 0$$

$$x+1 = 0$$

$$x = -1$$

From the above calculation we see that

$$h'' > -1$$

so h is concave up.

And

$$h'' < -1$$

so h is concave down.

And the point where concavity direction changes is an inflection point i.e.

$$h(-1) = 0.$$

iii) $f(\theta) = 2\cos\theta + \cos^2\theta, \quad 0 \le \theta \le 2\pi$

First we will find the critical numbers where

$$f' = 0, \text{ so}$$

$$f(\theta) = 2\cos\theta + \cos^2\theta$$

Taking $\frac{d}{d\theta}$ on b/s.

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}(2\cos\theta + \cos^2\theta).$$

$$f'(\theta) = -2\sin\theta - 2\cos\theta\sin\theta$$

$$f'(\theta) = -2\sin\theta(1 + \cos\theta).$$

Now Put $f'(\theta) = 0$ as given

$$0 = -2\sin\theta(1+\cos\theta).$$

$$(1+\cos\theta = 0$$

$$\cos\theta = -1$$

$$\theta = \cos^{-1}(-1).$$

$$\theta = \pi.)$$

$$-2\sin\theta = 0$$

$\sin\theta = 0.$ where $\theta = 0, \pi, 2\pi.$

$$1+\cos\theta = 0.$$

$\cos\theta = -1$ where $\theta = \pi.$

Now using f', test numbers on the intervals determind by the domain and the critical numbers.

For $(0, \pi)$:

$$f'(\theta) = -2\sin\theta(1+\cos\theta).$$

$$f'(\pi/2) = -2\sin(\pi/2)[1+\cos(\pi/2)]$$

$$f'(\pi/2) = -2.$$

As $-2 < 0$, so f is decreasing.

For $(\pi, 2\pi)$

$$f'(\theta) = -2\sin\theta(1+\cos\theta).$$

$$f'(3\pi/2) = -2\sin(3\pi/2)[1+\cos(3\pi/2)]$$

$$= 2$$

As $2 > 0$, so f is increasing.

Hence f is increasing at $(\pi, 2\pi)$ and decreasing at $(0, \pi)$.

Now we will find local maximum and minimum.

As we find out that f is decreasing on $(0, \pi)$ and increasing on $(\pi, 2\pi)$ so if it changes from decreasing at $x = \pi$ then that must be a local minimum.

$$f(\theta) = 2\cos\theta + \cos^2\theta$$

$$f(\pi) = 2\cos(\pi) + \cos^2(\pi).$$

$$= 2(-1) + (-1)$$

$$f(\pi) = -1$$

It does not change from increasing to

decreasing in the given domain, so there is no local maximum.

And now we will find the intervals of concavity and the inflection points.

First we have to find where $f''=0$ so taking $\frac{d}{d\theta}$ on b/s.

$$f(\theta) = 2\cos\theta + \cos^2\theta \qquad 0 \le \theta \le 2\pi$$

$$\frac{d}{d\theta} f(\theta) = \frac{d}{d\theta}(2\cos\theta + \cos^2\theta).$$

$$f'(\theta) = -2\sin\theta - 2\cos\theta\sin\theta$$

Again taking $\frac{d}{d\theta}$ on b/s.

$$\frac{d}{d\theta} f'(\theta) = \frac{d}{d\theta}(-2\sin\theta - 2\cos\theta\sin\theta).$$

$$f''(\theta) = -2\cos\theta - 2(-\sin\theta\sin\theta + \cos\theta\cos\theta).$$

$$= -2\cos\theta + 2\sin^2\theta - 2\cos^2\theta$$

$$= -2\cos\theta + 2(1-\cos^2\theta) - 2\cos^2\theta$$

$$= -2\cos\theta + 2 - 2\cos^2\theta - 2\cos^2\theta$$

$$= -4\cos^2\theta - 2\cos\theta + 2.$$

$f''(\theta) = -2(2\cos^2\theta + \cos\theta - 1)$

Now putting $f''(x)=0$ as given so we get

$$0 = -2(2\cos^2\theta + \cos\theta - 1)$$

$$2\cos^2\theta + \cos\theta - 1 = 0.$$

$$0 = (2\cos\theta - 1)(\cos\theta + 1)$$

$$2\cos\theta - 1 = 0.$$

$$\cos\theta = \frac{1}{2} \quad \text{where } \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\cos\theta + 1 = 0.$$

$$\cos\theta = -1 \quad \text{where } \theta = \pi.$$

From the above calculation we see that in the interval $(0, \pi/3)$

$$f'' < 0$$

So f is is concave down.

And in the interval $(\pi/3, \pi)$

$$f'' > 0$$

So f is concave ~~do~~ up.

And in interval $(\pi, 5\pi/3)$

$$f'' > 0$$

So f is concave up.

And in interval $(5\pi/3, 2\pi)$

$$f'' < 0$$

So f is concave down.

The point where concavity direction changes is an inflection point.

$(\pi/3, f(\pi/3)) = (\pi/3, 5/4)$

$5\pi/3, f(5\pi/3) = (5\pi/3, 5/4)$.

iv) $S(x) = x - \sin x$, $0 \leq x \leq 4\pi$.

First we will find the critical numbers where

$$S' = 0, \text{ so}$$

$$S(x) = x - \sin x \qquad 0 \leq x \leq 4\pi$$

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} S(x) = \frac{d}{dx}(x - \sin x)$$

$s'(x) = 1 - \cos x$

Now put $s'(x) = 0$ as given

$0 = 1 - \cos x$.

$\cos x = 1$ where $x = 0, 2\pi, 4\pi$

Now using s', test numbers on the intervals determind by the domain and the critical numbers.

For $(0, 2\pi)$:

$s'(x) = 1 - \cos x$.

$s'(\pi/2) = 1 - \cos(\pi/2)$.

$= 1 - 0$

$= 1$

As $1 > 0$, so s is increasing.

For $(2\pi, 4\pi)$:

$s'(x) = 1 - \cos x$

$s'(5\pi/2) = 1 - \cos(5\pi/2)$

$s'(5\pi/2) = 1 - 0$

$= 1$

As $1 > 0$, so S is increasing.

Hence S is increasing at $(0, 2\pi)$ & $(2\pi, 4\pi)$.

As we found that the function is increasing on the given domain, so there are no local extrema.

Now we will find the intervals of concavity and the inflection points.

First we have to find where $S'' = 0$, so,
taking $\frac{d}{dx}$ on b/s

$$\cancel{\frac{d}{}} \; S(x) = x - \sin x$$

$$\frac{d}{dx} \sin(x) =$$

$$\frac{d}{dx} S(x) = \frac{d}{dx} (\sin x - \sin x).$$

$$S'(x) = -\cos x.$$

Again taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} S'(x) = \frac{d}{dx} (-\cos x).$$

$$S''(x) = \sin x.$$

Now put $S''(x) = 0$ as given we get.

$\sin x = 0.$ where $x = 0, \pi, 2\pi, 3\pi, 4\pi.$

From the above calculation we see that in the interval $(0, \pi)$

$S'' > 0$

So S is concave up.

And in interval $(\pi, 2\pi)$

$S'' < 0$

So S is concave down

And in interval $(2\pi, 3\pi)$

$S'' > 0$

So S is concave up

And in interval $(3\pi, 4\pi)$

$S'' < 0$

So S is concave down.

Q8 Find dy of the following.

$$y = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$$

$dy = f(x)\,dx$

First we find $f'(x) = ?$

we know that $y = f(x)$. so.

$$f(x) = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$$

Taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{d(x)} f(x) = \cancel{\frac{2\sqrt{x}}{3\sqrt{1+\sqrt{x}}}} \quad \frac{2\sqrt{x}}{3(1+\sqrt{x})}.$$

Using quotient rule.

$$f'(x) = \frac{[3(1+\sqrt{x})]\frac{d}{dx}[2\sqrt{x}] - [2\sqrt{x}]\frac{d}{dx}[3(1+\sqrt{x})]}{[3(1+\sqrt{x})]^2}$$

$$f'(x) = \frac{(3+3\sqrt{x}) \times \frac{2}{\sqrt{x}} - 2\sqrt{x} \times \frac{3}{\sqrt{x}}}{(3+3\sqrt{x})^2}$$

$$f'(x) = \frac{\frac{6+6\sqrt{x}}{\sqrt{x}} - 6}{(3+3\sqrt{x})^2}$$

$$f'(x) = \frac{\frac{6 + \cancel{6\sqrt{x}} - \cancel{6\sqrt{x}}}{\sqrt{x}}}{(3+3\sqrt{x})^2}$$

$$= \frac{\frac{6}{\sqrt{x}}}{(3+3\sqrt{x})^2}.$$

$$f'(x) = \frac{6}{(\sqrt{x})(3+3\sqrt{x})^2}.$$

As we ~~k~~ have to find dy so.

$$\frac{dy}{dx} = \frac{6}{(\sqrt{x})(3+3\sqrt{x})^2}$$

$$dy = \frac{6}{(\sqrt{x})(3+3\sqrt{x})^2}\, dx.$$ Ans

Q9 The total surface area "S" of a circular cylindar is related to the base radius "r", height (h) by the equation

$$S = 2\pi r^2 + 2\pi r h$$

a) How is $\frac{ds}{dt}$ is related to $\frac{dr}{dt}$ if "h" is constant.

b) And if "r" is constant.

$$S = 2\pi r^2 + 2\pi rh$$

we want to find how $\frac{ds}{dt}$ is related to $\frac{dx}{dt}$ if h is constant.

$$\frac{ds}{dt} = 4\pi r \frac{dx}{dt} + 2\pi h \frac{dx}{dt}$$

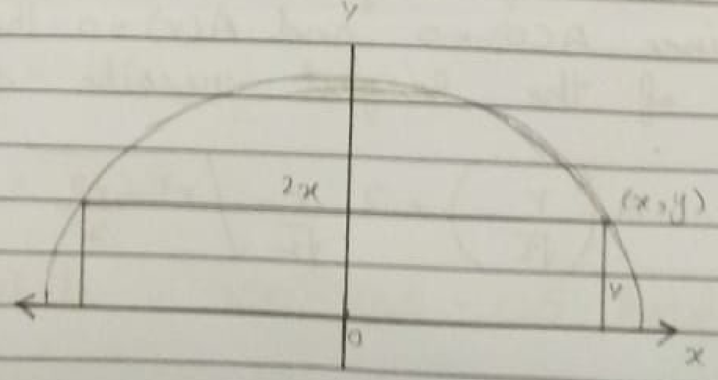
$$= (4\pi r + 2\pi h) \frac{dx}{dt}.$$

Now we want to find how $\frac{ds}{dt}$ is related to $\frac{dh}{dt}$ is r is constant.

$$S = 2\pi r^2 + 2\pi rh.$$

$$\frac{ds}{dt} = 2\pi r \frac{dh}{dt}.$$

Q6 b) Find the area of the largest rectangle ....?



Let take the semi-circle to be the upper half of the circle $x^2+y^2=r^2$ with the centre the origin. Let (x,y) be the vertex that lies in the first quadrant. Then the rectangle has side of $2x$ and y. So the area of the rectangle will be,

$$A = 2xy.$$

To eliminate y we use that $y=\sqrt{r^2-x^2}$ so.

$$A = 2x\sqrt{r^2-x^2}$$

The domain of this function is $0 \le x \le r$. Its derivative is

$$A' = 2\sqrt{x^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}}$$

$$A' = \frac{2(r^2-2x^2)}{\sqrt{r^2-x^2}}$$

which is 0 when $2x^2=r^2$, that is, $x=r/\sqrt{2}$. This value of x gives a maximum value of A since $A(0)=0$ and $A(r)=0$. Therefore the area of the largest inscribe rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2-\frac{r^2}{2}} = r^2.$$

c) A store has been selling .....?

If x is the number of TVs sold per week, then the weekly increase in sales is $x-200$. For each increase of 20 units sold, the price is decreased by \$ 10. So for each additional unit sold, the decrease in price will be $\frac{1}{20} \times 10$ and the demand of function is

$$p(x) = 350 - \frac{10}{20}(x-200)$$

$$p(x) = 450 - \frac{1}{2}x$$

The ~~sen~~ revenue function is

$$R(x) = \cancel{Rx}\; x\,p(x)$$

$$= x(450 - 1/2\, x).$$

$$= 450x - \frac{1}{2}x^2$$

As ~~the~~ $R'(x) = 450 - x$, we see that $R'(x) = 0$ when $x = 450$. This value of x gives an absolute maximum by the First derivative test. The corresponding price is

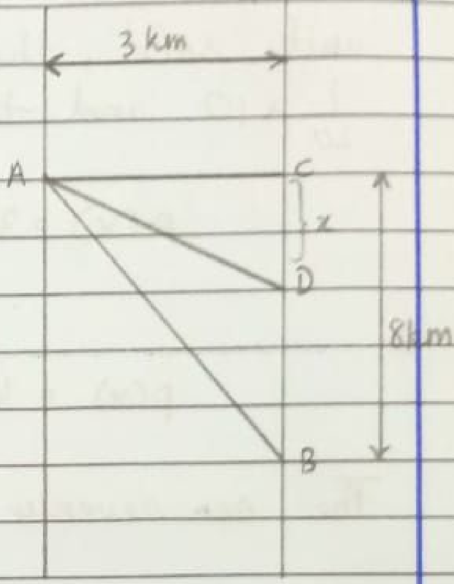
$$p(450) = 450 - \frac{1}{2}(450).$$

$$= 225$$

And the rebate is $350 - 225 = 125$. Therefore, to maximize revenue, the store should offer a rebate of \$125.

a) A man launcher his boat ....?

If we let x be the distance from C to D, then the running distance is $|DB| = 8-x$ and the Pythagorean Theorem gives the rowing distance as: $|AD| = \sqrt{x^2+9}$. we use the equation



$$\text{time} = \frac{\text{distance}}{\text{rate}}.$$

Then the rowing time is $\frac{\sqrt{x^2+9}}{6}$ and the runing time is $(8-x)/8$, so the total time T as a function of x is

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

The domain of this function T is $[0, 8]$. Notice that if $x=0$, he rows to C and if $x=8$ he rows directly to B. The derivative of T is

$$T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

Thus, using the fact that $x \geq 0$, we have

$$T'(x) = 0$$

$$\frac{x}{6\sqrt{x^2+9}} = \frac{1}{8}$$

$$4x = 3\sqrt{x^2+9}$$

$$16x^2 = 9(x^2+9).$$

$$7x^2 = 81.$$

$$x = \frac{9}{\sqrt{7}}.$$

The only critical number is $x = 9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0,8]$, we follow closed interval

Now evaluating T at all three points:

$$T(0) = 1.5.$$

$$T(9/\sqrt{7}) = 1 + \frac{\sqrt{7}}{8} \approx 1.33.$$

$$T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of T

must occur there. Thus the man should land the boat at a point $9/\sqrt{7}$ km (≈ 4.7 (≈ 3.4 km) downstream from his starting point.

Q7 Find the linearization at $x = -1$

$$f(x) = \sqrt[3]{\left(1 - \frac{x}{2+x}\right)^2}$$

First we will find the first derivative of f(x) so taking $\frac{d}{dx}$ on b/s.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\left(1 - \frac{x}{2+x}\right)^2\right)^{1/3}$$

$$f'(x) = \frac{d}{dx}\left(1 - \frac{x}{2+x}\right)^{2/3}.$$

$$f'(x) = \frac{2}{3}\left(1 - \frac{x}{2+x}\right)^{-1/3} \frac{d}{dx}\left(1 + \frac{x}{2+x}\right)$$

$$f'(x) = \frac{2}{3\left(1 - \frac{x}{2+x}\right)^{1/3}} \cdot \frac{1}{(2+x)^2}$$

$$f'(x) = \frac{2}{3(2+x)^2\left(1 - \frac{x}{2+x}\right)^{1/3}}$$

Now as we know that

$$L(x) = f(a) + f'(a)(x-a).$$

Here $a = -1$ so.

$$L(x) = f(-1) + f'(-1)(x-(-1))$$

$$L(x) = \sqrt[3]{\left(1 - \frac{(-1)}{2+(-1)}\right)^2} + \frac{2}{3(2+(-1))^2\left(1 - \frac{(-1)}{2+(-1)}\right)^{1/3}}(x+1)$$

$$L(x) = \sqrt[3]{4} + \frac{2}{3 \times (1.25)}(x+1)$$

$$L(x) = 1.58 + 0.53(x+1).$$

$$L(x) = 1.58 + 0.53x + 0.53.$$

$$= 0.53x + 2.11.$$