Name:
Saad Ahmad.

Roll no:
20P-0051.

Class:
BS(CS) - 1D.

Q1a) $f(x) = x^2 - 4x + 5$ $\quad 0 \leq x \leq 3 \quad , \quad n = 6$

We know that

$$\Delta x = \frac{b-a}{n}$$

So

$$\Delta x = \frac{3-0}{6}$$

$$\Delta x = \frac{1}{2} \text{ or } 0.5$$

Now

$x_0 = a + \Delta x(0) = 0$
$x_1 = a + \Delta x(1) = 0.5$
$x_2 = a + \Delta x(2) = 1$
$x_3 = a + \Delta x(3) = 1.5$
$x_4 = a + \Delta x(4) = 2$
$x_5 = a + \Delta x(5) = 2.5$
$x_6 = a + \Delta x(6) = 3.$

So,

$f(x_0) = 5$

$f(x_1) = 3.25$

$f(x_2) = 2$

$f(x_3) = 1.25$

$f(x_4) = 1$

$f(x_5) = 1.25$

$f(x_6) = 2$

And Now

$$x_1^* = \frac{x_0 + x_1}{2}$$

$$= 0.25$$

$$x_2^* = \frac{x_1 + x_2}{2}$$

$$= 0.75.$$

$$x_3^* = \frac{x_2 + x_3}{2}$$

$$= 1.25$$

$$x_4^* = \frac{x_3 + x_4}{2}$$

$$= 1.75.$$

$$x_5^* = \frac{x_4 + x_5}{2}$$

$$= 2.25$$

$$x_6^* = \frac{x_5 + x_6}{2}$$

$$= 2.75.$$

So,

$f(x_1^*) = 4.0625.$
$f(x_2^*) = 2.5625$
$f(x_3^*) = 1.5625$
$f(x_4^*) = 1.0625$
$f(x_5^*) = 1.0625$
$f(x_6^*) = 1.5625$

Now

Left End Points:

$$\sum_{i=0}^{5} f(x_i)\Delta x = \Delta x\left[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\right]$$

$$= 0.5(13.75).$$

$$= 6.875.$$

Right End Points:

$$\sum_{i=1}^{6} f(x_i)\Delta x = \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\right]$$

$= 0.5(10.75)$

$= 5.375.$

Mid Point:

$$\sum_{i=1}^{6} f(x_i^*)\Delta x = \Delta x\left[f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*) + f(x_6^*)\right]$$

$= 0.5(11.875)$

$= 5.9375.$

b) $\int_0^{\pi} x \sin^2 x \, dx \qquad n = 4.$

We know that

$$\Delta x = \frac{\pi - 0}{4}$$

$$\Delta x = \frac{\pi}{4}.$$

Now.

$x_0 = a + \Delta x(0) = 0.$, $x_1 = a + \Delta x(1) = \frac{\pi}{4}$

$x_2 = a + \Delta x(2) = \frac{\pi}{2}$, $x_3 = a + \Delta x(3) = \frac{3\pi}{4}$

$x_4 = a + \Delta x(4) = \pi$

So we have

$$x_1^* = \frac{x_0 + x_1}{2} = \frac{\pi}{8} \quad , \quad x_2^* = \frac{x_1 + x_2}{2} = \frac{3\pi}{8}$$

$$x_3^* = \frac{x_2 + x_3}{2} = \frac{5\pi}{6} \quad , \quad x_4^* = \frac{x_3 + x_4}{2} = \frac{7\pi}{8}$$

So,

$$f(x_1^*) = 0.0575 \quad , \quad f(x_2^*) = 1.0056$$

$$f(x_3^*) = 1.6759 \quad , \quad f(x_4^*) = 0.4026$$

$$\sum_{i=1}^{4} f(x^*) \Delta x = \Delta x [f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)]$$

$$= \frac{\pi}{4}(3.1416)$$

$$= 2.4674.$$

Q2 a) $\int_2^5 (x^2 + \frac{1}{x})\, dx$

Here $a = 2$ & $b = 5$.

So, $\Delta x = \frac{b-a}{n}$

$$= \frac{5-2}{n}$$

$$= \frac{3}{n}$$

So we ~~h~~ have,

$$x_i = a + (\Delta x)(i).$$

$$= a + \Delta x(i).$$

$$= 2 + \frac{3i}{n}.$$

So

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\frac{3}{n}.$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \cancel{f}\left(\left(\frac{6i}{n}\right)^2 + \frac{n}{6i}\right).$$

b) i) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1+x_i}\,\Delta x.$, $[0, \pi]$.

Here $a = 0$, $b = \pi$

So,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sin x_i}{1+x_i} = \int_0^{\pi} \left(\frac{\sin x}{1+x}\right) dx$$

ii) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x$, $[1, 3]$

Here $a = 1$, $b = 3$.

So,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^*}{(x_i^*)^2+4} \Delta x = \int_1^3 \frac{x}{x^2+4}\, dx.$$

Q3 $\int_0^5 f(x)\, dx$ if $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x+e^x & \text{for } x \geq 3 \end{cases}$

$$\int_0^5 f(x)\, dx = \int_0^3 f(x)\, dx + \int_3^5 f(x)\, dx$$

$$= \int_0^3 (3)\, dx + \int_3^5 (x+e^x)\, dx$$

So,

$$\int_0^3 3\, dx = 3\int_0^3 1\, dx$$

$$= 3 \cdot x \Big|_0^3$$

$= 3(3-0)$

$= 9.$

And

$$\int_3^5 (x+e^x)\,dx = \int_3^5 x\,dx + \int_3^5 e^x\,dx$$

$$= \left.\frac{x^2}{2}\right|_3^5 + \left.e^x\right|_3^5$$

$$= \left(\frac{(5)^2-(3)^2}{2}\right) + \left(e^5-e^3\right)$$

$= 8 + 128.3.$

$= 136.3.$

So,

$$\int_0^5 f(x) = 9 + 136.3.$$

$= 145.3.$

Q4 i) $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx.$

$$4\int_{1/2}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\, dx.$$

$$4\int_{1/2}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$$

$$4\left(\sin^{-1}x\right)\Big|_{1/2}^{1/\sqrt{2}}$$

$$4\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$4\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$\cancel{4}\left(\frac{\pi}{\cancel{12}\,3}\right)$$

$$\frac{\pi}{3}.$$

ii) $\int_0^{\pi} f(x)\, dx$, where $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le 2 \end{cases}$

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \cos x\,dx$$

$$= -\cos x \Big|_0^{\pi/2} + \sin x \Big|_{\pi/2}^{\pi}$$

$$= -[\cos(0) - \cos(\pi/2)] + [\sin(\pi/2) - \sin(\pi)]$$

$$= -(1-0) + (1-0).$$

$$= -1 + 1$$

$$= 0.$$

iii) $f(x) = \int_{2\pi}^{0} \sqrt{1+\sec t}\,dt.$

$$- \int_0^{2\pi} \sqrt{1+\sec t}\,dt$$

Multiplying and dividing by $\sqrt{\sec t - 1}$

$$= -\int_0^{2\pi} \sqrt{\sec t + 1} \cdot \frac{\sqrt{\sec t - 1}}{\sqrt{\sec t - 1}}$$

$$= -\int_0^{2\pi} \frac{\sqrt{\sec^2 t - 1}}{\sqrt{\sec t - 1}}.$$

iv) $\int_1^3 \frac{y^3 - 2y^2 - y}{y^2} \cdot dy$

$= \int_1^3 \left(\frac{y^3}{y^2} - \frac{2y^2}{y^2} - \frac{y}{y^2} \right) dy$

$= \int_1^3 y \, dy - 2\int_0^3 1 \, dy - \int_0^3 \frac{1}{y} \, dy$

$= \frac{y^2}{2} \Big|_1^3 - 2y \Big|_1^3 - \ln|y| \Big|_1^3$

$= \frac{1}{2}((3)^2 - (1)^2) - 2(3-1) - (\ln|3| - \ln|1|$

$= \frac{1}{2}(9-1) - 2(2) - 1.1$

$= 4 - 4 - 1.1.$

$= -1.1.$

v) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta \tan^2\theta}{\sec^2\theta} \, d\theta$

$$= \int_0^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/3} \frac{\sin\theta \cdot \cancel{\sec^2\theta}}{\cancel{\sec^2\theta}}\, d\theta$$

$$= \int_0^{\pi/3} \sin\theta \; d\theta$$

$$= -\cos\theta \Big|_0^{\pi/3}.$$

$$= -\left(\cos(\pi/3) - \cos(0).\right).$$

$$= -\left(\frac{1}{2} - 1\right)$$

$$= -(-0.5)$$

$$= 0.5.$$

vi) $$\int_0^2 |2x-1|\, dx$$

$$|2x-1| = \begin{cases} 2x-1 & \text{if} \quad x \geq 1/2. \\ -2x+1 & \text{if} \quad x < 1/2 \end{cases}$$

So,

$$\int_0^2 |2x-1|\,dx = \int_0^{1/2} (-2x+1)\,dx + \int_{1/2}^2 (2x-1)\,dx$$

$$\int_0^{1/2} (-2x+1)\,dx = -2\int_0^{1/2} x\,dx + \int_0^{1/2} dx$$

$$= -2\left(\frac{x^2}{2}\right)\Big|_0^{1/2} + x\Big|_0^{1/2}$$

$$= -\left((1/2)^2 - (0)^2\right) + (1/2 - 0)$$

$$= -\frac{1}{4} + \frac{1}{2}$$

$$= \frac{1}{4}$$

And now:

$$\int_{1/2}^2 (2x+1)\,dx = 2\int_{1/2}^2 x\,dx + \int_{1/2}^2 dx$$

$$= 2\left(\frac{x^2}{2}\right)\Big|_{1/2}^2 + x\Big|_{1/2}^2$$

$= [(2)^2 - (1/2)^2] + [2 - 1/2]$

$= \frac{15}{4} - \frac{3}{2}.$

$= \frac{9}{4}$

So,

$$\int_0^2 |2x-1| = \frac{1}{4} + \frac{9}{4}$$

$$= \frac{5}{2}.$$

Q5 a) i) $g(x) = \int_x^0 \sqrt{t+t^2}\, dt.$

We see this that this function is continuous every where also, $a \le x \le b$. So we can apply fundamental law. so we get

$$g(x) = -\sqrt{x+x^2}$$

ii) $h(x) = \int_0^{e^x} \ln t \, dt.$

As this function is also continuous as logrithmic function is not negative so, also $a \le x \le b$.

So

$$h'(x) = \frac{d}{dx}\int_0^{e^x} \ln t \, dt.$$

let $v = e^x$, so $\frac{d}{dx}v = e^x$ and $dv = e^x dx$. so.

$$h'(x) = \frac{d}{dx} \cdot \frac{dv}{dt}\int_0^{v=e^x} \ln t \, dt$$

$$= \frac{d}{dx}\left[\int_0^{v=e^x} \ln t \, dt\right] \cancel{dt} \frac{dv}{dt}$$

$$= \ln v \cdot e^x$$

$$= \ln e^x \cdot e^x$$

$$= x \ln e \cdot e^x$$

$$= x(1) \cdot e^x$$

$$= xe^x$$

iii) $$h(x) = \int_2^{\sqrt{x}} \frac{z^2}{z^4+1} \, dz.$$

let $v = \sqrt{x}$, so $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

Now

$$h(x) = \int_2^v \frac{z^2}{z^4+1}\, dz.$$

$$h'(x) = \frac{d}{dx} \int_0^v \frac{z^2}{z^4+1}\, dz$$

$$= \frac{d}{dv}\left[\int_2^v \frac{z^2}{z^4+1}\, dz \right] \frac{dv}{dx}$$

$$= \frac{v^2}{v^4+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{x}{x^2+1} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x}}{2(x^2+1)}.$$

iv) $h(x) = \int_{2x}^{3x} \frac{y^2-1}{y^2+1}\, dy.$

$$\int_{2x}^{0} \frac{y^2-1}{y^2+1}\,dy + \int_{0}^{3x} \frac{y^2-1}{y^2+1}\,dy$$

Now.

$$\int_{2x}^{0} \frac{y^2-1}{y^2+1} = -\int_{0}^{2x} \frac{y^2-1}{y^2+1}\,dy$$

let $v = 2x$ then $\frac{dv}{dx} = 2.$

$$= -\frac{d}{dx}\int_{0}^{v} \frac{y^2-1}{y^2+1}\,dy$$

Using chain rule

$$= -\frac{d}{dv}\left[\int_{0}^{v} \frac{y^2-1}{y^2+1}\right]\frac{dv}{dx}$$

$$= -\left[\frac{v^2-1}{v^2+1}\right]2.$$

$$= -2\left[\frac{(2x)^2-1}{(2x)^2+1}\right]$$

$$= -2\left(\frac{4x^2-1}{4x^2+1}\right)$$

$$\frac{-8x^2+2}{4x^2-1}$$

Now.

$$\int_0^{3x} \frac{y^2-1}{y^2+1}\, dy.$$

Let $v=3x$ so $\frac{dv}{dx} = \cancel{d}\, 3.$

Taking $\frac{d}{dx}$.

$$= \frac{d}{dx} \int_0^{v} \frac{y^2-1}{y^2+1}\, dy$$

Using chain rule.

$$= \frac{d}{dv} \left[\int_0^{v} \frac{y^2-1}{y^2+1} \right] \frac{dv}{dx}$$

$$= \frac{v^2-1}{v^2+1} \cdot 3.$$

$$= \frac{9x^2-1}{9x^2+1} \cdot 3.$$

$$\frac{27x^2-3}{9x^2+1}.$$

So,

$$h'(x) = \frac{27x^2-3}{9x^2+1} - \frac{8x^2-2}{4x^2+1}$$

b) i) $\int_{1}^{2} \frac{4}{x^3} dx = -\frac{2}{x^2}\Big]_{-1}^{2} = \frac{2}{3}$.

When $x=0$ this equation tends to go to infinity.
So this function is discountinuous at $x=0$ therefore this solution is not possible.

ii) $\int_{0}^{\pi} \sec^2 x \, dx = \tan x \Big]_{0}^{\pi} = 0$.

When $x = \frac{\pi}{2}$ this function tends towards infinity. So this function is discountinuous at $x = \pi/2$, therefore this solution is not possible.

Q6 i) $\int \sin^5(2t) \cos^2(2t) \, dt$.

Let $v = 2t$ then $\frac{dv}{dt} = 2$, so $dt = \frac{dv}{2}$

$$\int \sin^5(v) \cos^2(v) \frac{dv}{2}$$

$$\frac{1}{2}\int \sin^4(v) \cdot \sin(v) \cdot \cos^2 v \, dv.$$

$$\frac{1}{2}\int (1-(\cos^2 v))^2 \cdot \sin(v) \cdot \cos^2 v \, dv.$$

Let $v = \cos v$, so $\frac{dv}{dv} = -\sin v \Rightarrow \sin v = \frac{-dv}{dv}$

$$\frac{-1}{2}\int (1-v)^2 \cdot \frac{dv}{dv} \cdot v^2 \cdot \cancel{dv}$$

$$-\frac{1}{2}\int v^2(1+v^4-2v^2)\,dv$$

$$-\frac{1}{2}\left[\int v^2 dv + \int v^6 dv - 2\int v^4 dv\right]$$

$$-\frac{1}{2}\left[\frac{v^3}{3} + \frac{v^7}{7} - \frac{2v^5}{5}\right]$$

$$-\frac{1}{2}\left[\frac{\cos^3(2t)}{3} + \frac{\cos^7(2t)}{7} + \frac{2\cos^5(2t)}{5}\right]$$

ii) $\displaystyle\int \frac{\sin^2(1/t)}{t^2}\,dt$

Let $v = \dfrac{1}{t}$ then $\dfrac{dv}{dt} = -\dfrac{1}{t^2} \Rightarrow dt = -t^2\,dv.$

So,

$$\int \frac{\sin v}{(1/v^2)}(-t^2\,dv)$$

$$-\int \cancel{v^2}\sin^2 v\left(\frac{1}{\cancel{v^2}}\,dv\right)$$

$$-\int \sin^2 v\,dv$$

$$-\int \frac{1-\cos^2 v}{2}\,dv$$

$$= -\frac{1}{2}\int dv + \frac{1}{2}\int \cos 2v\,dv$$

$$= -\frac{v}{2} + \frac{\cancel{2}\sin 2v}{\cancel{2}} + C$$

$$= \frac{-v}{2} + \sin 2v + c.$$

$-\frac{1}{2t} + \sin(2/t) + C.$

ii) $\int \tan^3 x \sec^6 x \, dx$

$\int \tan^2 x . \tan x . \sec^6 x \, dx$

$\int (\sec^2 x - 1) . \tan x . \sec^6 x \, dx.$

let $v = \sec x$, so $dv = \tan x \sec x \, dx.$

$\int (v^2 - 1) . v^5 \, dv.$

$\int v^2 \, dv - \int v^5 \, dv.$

$\frac{v^3}{3} - \frac{v^6}{6} + c.$

$\frac{\sec^3 x}{3} - \frac{\sec^6 x}{6} + c.$

iv) $\int_0^{\pi/4} \sqrt{1 - \cos 4\theta} \, d\theta.$

Let $v = 2\theta$ so $dv = 2 d\theta \Rightarrow d\theta = \frac{dv}{2}$

So $\int_0^{\pi/4} \sqrt{1 - \cos 2v} . \frac{dv}{2}$

$$\frac{1}{2}\int_0^{\pi/4} \sqrt{2\sin^2 v}\, dv.$$

$$\frac{1}{2}\int_0^{\pi/4} \sqrt{2} \cdot \sqrt{\sin^2 v}\, dv.$$

$$\frac{\sqrt{2}}{2}\int_0^{\pi/4} \sin v\, dv.$$

$$-\frac{\sqrt{2}}{2} \cos v \Big|_0^{\pi/4}$$

$$-\frac{\sqrt{2}}{2}\left(\cos(\pi/4) - \cos(0)\right)$$

$$-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - 1\right)$$

$$0.207.$$