Untwisting the Boerdijk–Coxeter

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Abstract. The Boerdijk–Coxeter helix (BC helix, or tetrahelix) is a face-to-face stack of regular tetrahedra forming a helical column. Considering the edges of these tetrahedra as structural members, the resulting structure is attractive and inherently rigid, and therefore interesting to architects, mechanical engineers, and robotocists. A formula is developed that matches the visually apparent helices forming the outer rails of the BC helix. This formula is generalized to a formula convenient to designers. Formulae for computing the parameters that give edge-length minimax-optimal tetrahelices are given, defining a continuum of tetrahelices of varying torsion. The endpoints of the optimality of this continuum are the BC helix and a structure of zero torsion, the *equitetrabeam*. Numerically finding the rail angle from the equation for pitch allows optimal tetrahelices of any pitch to be designed. An interactive tool for such design and experimentation is provided: https://pubinv.github.io/tetrahelix/. A formula for the inradius of optimal tetrahelices is given. Utility for static and variable geometry truss/space frame design and robotics is discussed.

Key words. Boedijk–Coxeter helix, tetrahelix, robotics, tetrobot, unconventional robots, structural engineering, mechanical engineering, tensegrity, variable-geometry truss

AMS subject classifications. 51M15

1. Introduction.

The Boerdijk–Coxeter helix[3] (BC helix), is a face-to-face stack of tetrahedra that winds about a straight axis. Because architects, structural engineers, and robotocists are inspired by and follow such regular mathematical models but can also build structures and machines of differing or even dynamically changing length, it is useful to develop the mathematics of structures formed from tetrahedra where we relax regularity.

The vertices of the tetrahedra lie upon three helices about the central axis. The glussbot[11] (or Tetrobot)[8] uses the regularity of this geometry to make a tentacle-like robot that can crawl like a slug or mollusc. The Tetrobot uses mechanical actuators which can change their length, connected by special joints, such as the 3D printable Song-Kwon-Kim[15] joint, or the CMS joint[7] which allow many members to meet in a single point. Such machines can follow purely regular mathematical models such as the Boerdijk-Coxeter helix or the Octet Truss[4].

Buckminster Fuller called the BC helix a *tetrahelix*[5], a term now commonly used. In this paper we reserve *BC helix* to mean the purely regular structre and use *tetrahelix* to refer to any structure isomorphic to the BC helix.

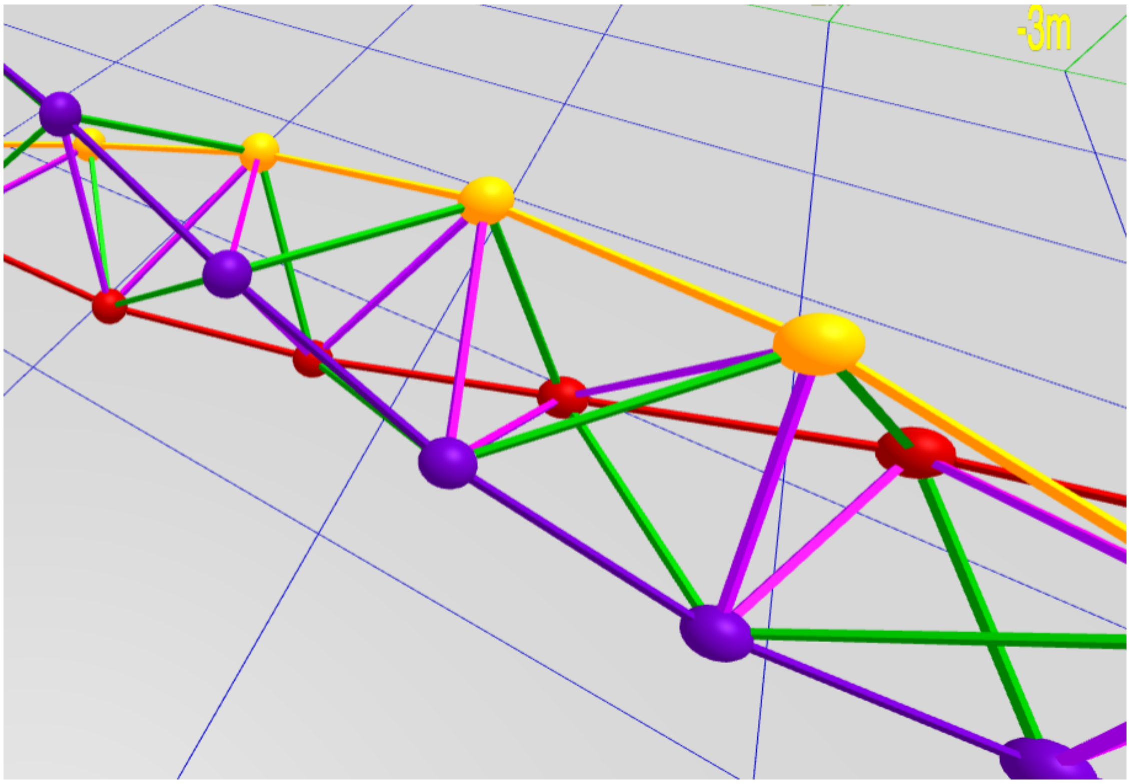

Figure 1. *BC Helix Close-up (partly along axis)*

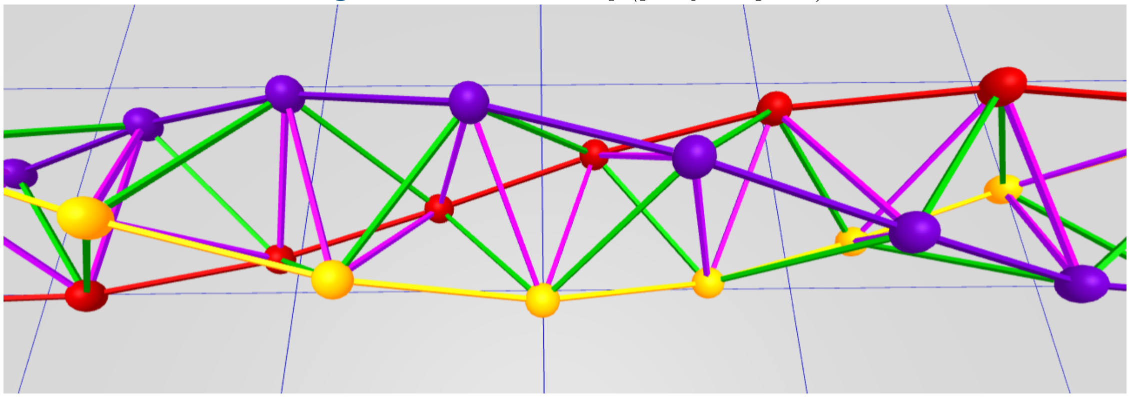

Figure 2. *BC Helix Close-up (orthogonal)*

Imagining Figure 2 as a static mechanical structure, we observe that it is useful to the mechanical engineer or robotocist because the structure remains an inherently rigid, omni-triangulated space frame, which is mechanically strong. Imagine further in Figure 2, that each static edge was replaced with an actuator that could dynamically become shorter or longer in response to electronic control, and the vertices were a joint that supported sufficient angular displacement for this to be possible. An example of such a machine is a glussbot, shown in Figure 11.

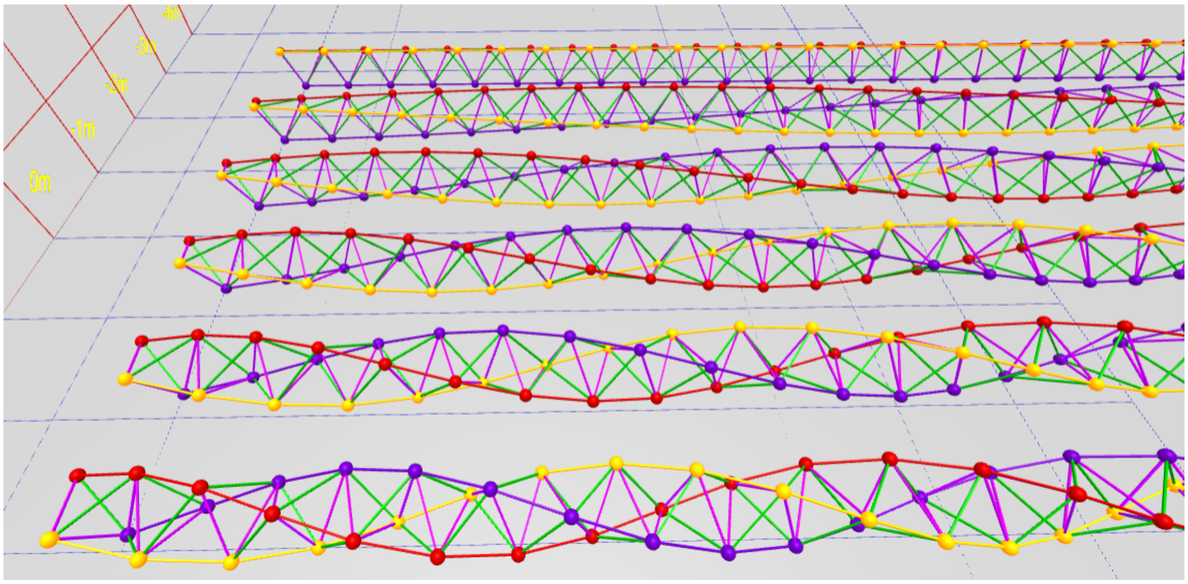

Figure 3. *A Continuum of Tetrahelices*

A BC helix does not rest stably on a plane. It is convenient to be able to "untwist" it and to form a tetrahelix space frame that has a flat planar surface. By making length changes in a certain way, we can untwist a tetrahelix to form a *tetrabeam* which has planar faces and has, for example, an equilateral triangular profile. This paper develops the equations needed to untwist the tetrahelix. All math developed here is available in JavaScript and demonstrated by an interactive design website https://pubinv.github.io/tetrahelix/[12], from which Figure 2 and the figures below are taken.

Figure 3 displays a continuum of tetrahelices optimal in a certain sense, which is the result of this paper. The closest helix is the BC helix, and the furthest is the equitetrabeam, defined in section 6.

2. A Designer's Formulation of the BC Helix.

We would like to design nearly regular tetrahelices with a formula that gives the vertices in space. Eventually we would like to design nearly regular tetrahelices by choosing the lengths of a small set of members. In a space frame, this is a static design choice; in a tetrobot, it is a dynamic choice that can be used to twist[1] the robot and/or exert linear or angular force on the environment.

Ideally we would have a simple formula for defining the nodes based on any curvature or pitch we choose. It is a goal of this paper to relate these two approaches to generating a tetrahelix continuum.

H.S.M Coxeter constructs the BC helix[3] as a repeated rotation and translation of the

[1]The formal definition of twistiness, or *torsion*, is not useful or used in the paper. The formal *curvature* and *torsion* of the helices defined here may be easily computed from the formualae if desired.

tetrahedra, showing the rotation is:

$$\theta_{bc} = \arccos(-2/3)$$

and the translation:

$$h_{bc} = 1/\sqrt{10}$$

θ_{bc} is approximately $0.37 \cdot 2\pi$ radians or 131.81 degrees. The angle θ_{bc} is the rotation of *each* tetrahedron, not the tetraheda along a rail. In Figure 2, each tetrahedron has either a yellow, blue, or red outer edge or rail. That is, a blue-rail tetrahedron is rotated slightly more than a 1/3 of a revolution to match the face of the yellow tetrahedra.

R.W. Gray's site[6], repeating a formula by Coxeter[3] in more accessible form, gives the Cartesian coordinates $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a counter-clockwise BC Helix:

$$(1) \qquad V(n) = \begin{bmatrix} r_{bc} \cos n\theta_{bc} \\ r_{bc} \sin n\theta_{bc} \\ n h_{bc} \end{bmatrix}, \text{where: } \begin{array}{c} r_{bc} = \frac{3\sqrt{3}}{10} \approx 0.5196 \\ h_{bc} = 1/\sqrt{10} \approx 0.3162 \\ \theta_{bc} = \arccos(-2/3) \end{array}$$

where n represents each integer numbered node in succession on every colored rail.

The apparent rotation of a vertex an outer-edge, $V(n)$ relative from $V(n+3)$ for any integer n in (1), is $3\theta_{bc} - 2\pi$).

This formula defines a helix, but it is not any of the apparent helices, or *rail* helices, of the BC helix, but rather one that winds three times as rapidly through all nodes. To a designer of tetrahelices, it is more natural to think of the three helices which are visually apparent, that is, those three which are closely approximated by the outer edges or rails of the BC helix. We think of each of these three rails as being a different color: red, blue, or yellow. This situation is illustrated in Figure 4, wherein the black helix represents that generated by (1), and the colored helices are generated by (2).

It is convenient to have a formula that gives us the nodes of just one rail helix, denoted by color c and integer node number n:

$$(\forall n \in \mathbb{Z}, \forall c \in \{0,1,2\} : H_{BCcolored}(n,c) = V(3n+c))$$

Such a helix can be written:

$$(2) \qquad H_{BCcolored}(n,c) = \begin{bmatrix} r_{bc} \cos\left((3\theta_{bc} - 2\pi)n + c\theta_{bc}\right) \\ r_{bc} \sin\left((3\theta_{bc} - 2\pi)n + c\theta_{bc}\right) \\ 3h_{bc}(n + c/3) \end{bmatrix}, \text{where } \begin{array}{c} r_{bc} = \frac{3\sqrt{3}}{10} \\ h_{bc} = 1/\sqrt{10} \\ \theta_{bc} = \arccos(-2/3) \end{array}$$

In this formula, integral values of n may be taken as a node number for one rail and used to compute its Cartesian coordinates. Allowing n to take non-integer values defines a continuous helix in space which is close to the segmented polyline of the outer tetrahedra edges, and equals them at integer values.

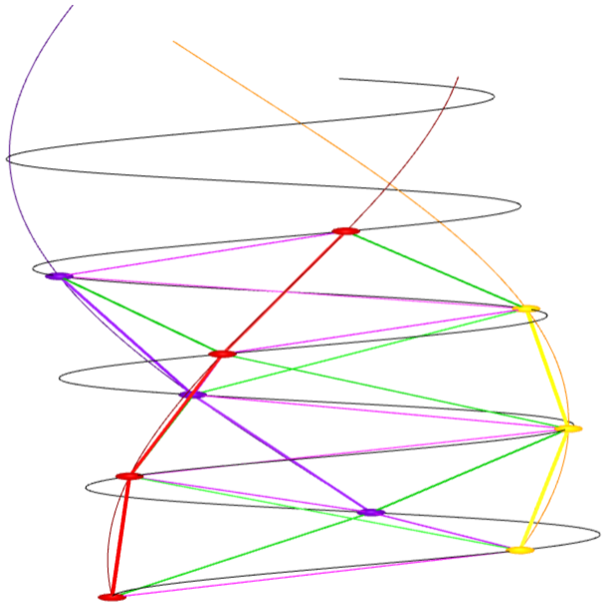

Figure 4. *Rail helices vs. Coxeter/Gray helix*

The quantity $(3\theta_{bc}-2\pi) \approx 35.43°$ is the angular shift between $V(3n+c) = H_{BCcolored}(n, c)$ and $V(3(n+1)+c) = H_{BCcolored}(n+1, c)$. This quantity appears so often that we call it the "rail angle ρ". For the BC helix, $\rho_{bc} = (3\theta_{bc} - 2\pi)$.

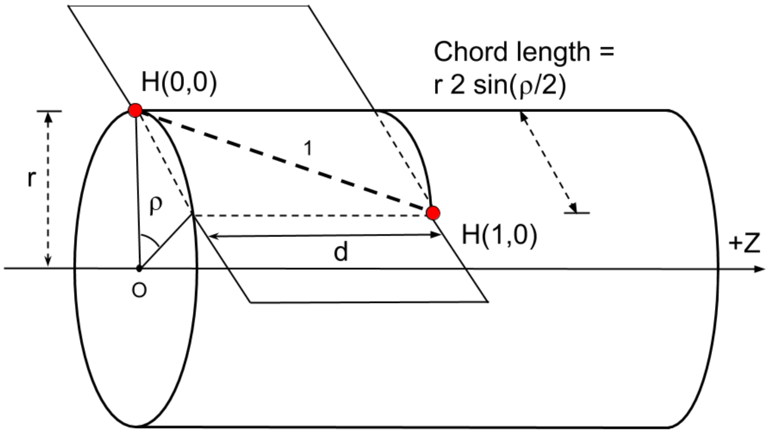

Figure 5. *Rail Angle Geometry*

Note in Figure 5 the z-axis travel for one rail edge is denoted by d. In (1) and (2), the variable h is used for one third of the distance we name d. We will later justify that $d = 3h$. In this paper we assume the length of a rail is always 1 as a simplification. (We make the rail length a parameter in our JavaScript code in https://github.com/PubInv/tetrahelix/blob/master/js/tetrahelix_math.js [12].)

The $H_{BCcolored}(n, c)$ formulation can be further clarified by rewriting directly in terms of the rail angle ρ_{bc} rather than θ_{bc}. Intuitively we seek an expression where $c/3$ is multiplied by a 1/3 rotation plus the rail angle ρ. We expand the expressions θ_{bc} and ρ_{bc} in (2) and seek to isolate the term $c2\pi/3$.

$$\begin{aligned}
c\theta_{bc} &= \{\text{we aim for 3 in denominator, so we split...}\} \\
(c/3)(3\theta_{bc}) &= \{\text{we want } 2\pi \text{ in numerator, so add canceling terms...}\} \\
(c/3)((3\theta_{bc} - 2\pi) + 2\pi) &= \{\text{definition of } \rho_{bc}\}... \\
(c/3)\rho_{bc} + c2\pi/3 &= \text{algebra...} \\
c(\rho_{bc} + 2\pi)/3
\end{aligned}$$

This allows us to redefine:

$$(3) \qquad H_{BCcolored}(n, c) = \begin{bmatrix} r\cos\rho_{bc}n + c(\rho_{bc} + 2\pi)/3 \\ r\sin\rho_{bc}n + c(\rho_{bc} + 2\pi)/3 \\ (n + c/3)h_{bc} \end{bmatrix}, \text{where } \begin{array}{c} \rho_{bc} = (3\theta_{bc} - 2\pi) \\ h_{bc} = 1/\sqrt{10} \end{array}$$

Recall that $c \in \{0, 1, 2\}$, but n is continuous (rational or real-valued.) We can now assert that in Figure 4 the black helix winds at $\frac{3\theta_{bc}}{\rho_{bc}} \approx 11.16$ times the rate of a rail helix.

From this formulation it is easy to see that moving one vertex on a rail ($H_{BCcolored}(n, c)$ to $H_{BCcolored}(n + 1, c)$ for any n and c) moves us ρ_{bc} radians around a circle. Since:

$$\frac{2\pi}{\rho_{bc}} \approx 10.16$$

we can see that there are approximately 10.16 red, blue or yellow tetrahedra on one rail in a single revolution.

The *pitch* of any tetrahelix, defined as the length of a complete revolution where $\rho \neq 0$ is:

$$(4) \qquad p(\rho) = \frac{2\pi \cdot d}{\rho}$$

The pitch of the Boerdijk–Coxeter helix of edge length 1 is the length of three tetrahedra times this number:

$$\frac{3h_{bc} \cdot 2\pi}{\rho_{bc}} = \frac{6\pi}{\sqrt{10}\rho_{bc}} \approx 9.64$$

The pitch is less than the number of tetrahedra because the tetrahedra are not lined up perfectly. It is a famous and interesting result that the pitch is irrational. A BC helix never has two tetrahedra at precisely the same orientation around the z-axis. However, this is inconvenient to designers, who might prefer a rational pitch. The idea of developing a rational period by arranging solid tetrahedra by relaxing the face-to-face matching has been explored[13]. We develop below slightly irregular edge lengths that support, for example, a pitch of precisely 12 tetrahedra in one revolution which would allow an architect to design a column having a basis and a capital in the same relation to the tetrahedra they touch at the bottom and top of the column.

3. Optimal Tetrahelices.

We use the term *tetrahelix* to mean any structure made of vertices and edges which is isomorphic to the BC helix and in which the vertices lie on three helices. We further demand that all edge lengths be finite, as we are only interested in physically constructable tetrahelices. By isomporphic we mean there is a one-to-one mapping between both vertices and edges in the two tetrahelices. One could consider various definitions of optimality for a tetrahelix, but the most useful to us as robotocists working with the Tetrobot concept is to minimize the maximum difference between any two edges, because the Tetrobot uses mechanical linear acutators with limited range of extension.

Suppose that all three rails do not have the same pitch. Starting at any shortest edge, as we move from node to node away from our start node edge lengths between rails must always lengthen without bound, which cannot be optimal. So we are justified in talking about the *pitch* of the optimal tetrahelix as the pitch of its three rail helices, even though there are three such helices of equivalent pitch.

Similarly, if the axes are not parallel, there is an edge of unbounded length in the structure, so we do not consider such cases.

Since the axes are parallel, we may define the *inradius*, represented by the letter i, of a tetrahelix to be the radius of the largest cylinder parallel to this axis contained within the circumscribing cylinder which is penetrated by no edge. Define a *minimax edge-length optimal tetrahelix* or just an *optimal tetrahelix* to be a tetrahelix for which there exists no other tetrahelix of the same inradius and pitch with a lower maximum difference between its edge lengths.

We wish to show that in an optimal tetrahelix, all vertices lie on the cylinder of radius r, regardless of where they lie on the z-axis.

Since all three rails have the same rail length, no matter how we move the rails in the xy plane if we shorten the xy distance between vertices we shorten the total distance. Our tool for thinking about this is to collapse the z dimension to form a two-dimensional figure of nodes and non-rail edges (see Figure 6.) Consider the projection along the z axis of all vertices and non-rail edges into the xy plane, which will be a figure of dots and connecting segments in the xy plane. The convex hull for any one helix projection will be a circle (if its pitch is irrational) or a polygon if rational, or a point if the helix has zero curvature. Each of these figures by definition lies outside the circle of inradius r_{in} in the xy plane.

Theorem 1. *Any optimal tetrahelix with a rail angle less than π has all vertices on a single cylinder.*

Proof. Case 1: Suppose that ρ is zero. Then for any given inradius, an equilateral triangle is the minimax solution for all non-rail edges. Since there is only one rail-edge length, this is the minimax solution for the entire set. Since the vertices of an equilateral triangle lies on a circle, all points lie on a cylinder.

Case 2: Suppose that ρ is positive but less than π. In this case each rail helix has curavature and places points on both sides of any line through the origin of the xy plane (or both coincident on such a line.)

We first show that the inradius is touched by one segment from each pair of rails.

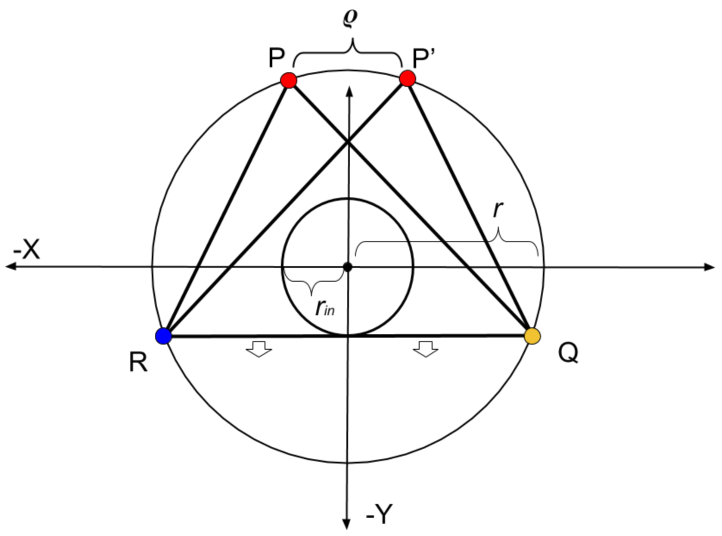

Figure 6. *Untouching Rail*

Figure 6 depicts a projection of only the point P, the point P' which is an adjacent vertex on the same rail, and the intermediary points R and Q on the other two rails.

Supoose there is a rail P which does not have a segment touching the incircle at all in the projection, as depicted in Figure 6. Then a segment connecting the other two Q and R is a longest rail, because any chord touching an incircle is longer than any which does not. These two rails (Q and R) can be moved closer to each other, and further away from P, to form a better minimax solution by shortening the longest rail, until one of the segments from P the previously untouching rail touches one of the incircle. By induction, then all rails have at least one segement touching each vertex on the rail that is tangent to the incircle in an optimal tetrahelix.

Now suppose that you attempt to move the axis of one of the rails. Because $\rho > 0$ and ρ is not a multiple of π, there is some vertex on the two sides of any diameter halving our circle. Moving the axis of a tetrahelix lengthens a longest line while "pinching" the inradius on the other side, so it is not optimal. Since the axes of these helices are the same and they have the same curvature and pitch, the points defined by the helices all lie on a cylinder.

Theorem 1 is not true when ρ is an odd integer multiple of π.

Now that we have shown that any optimal tetrahelix vertices are on helices of the same axes and pitch, we see that the vertices of any optimal tetrahelix will lie on a cylinder, or a circle when axis dimension is projected out. Therefore it is reasonable to now speak of the *radius* r of a tetrahelix as the radius of the cylinder, as distinct from the inradius r_{in}.

Now that we have coincident axes, the same pitch, and the same radius, we can go on to the harder proof about where vertices occur along the z-axis.

We show that in fact the nodes must be distributed in even thirds along the z-axis, as in fact they are in the regular BC helix.

Note that from the point of view of a single edge, we are on a slanted cylinder, when $\rho \neq 0$. This means from its point of view a cross section is an ellipse. So we have to be very careful in comparing lengths of edges relative to the tetrahedron, because a change in position along the edge changes the length of a line, but in a complicated way depending on where it is relative to the ellipse.

In principle in any 3 helices with the same axis of the same radius having any relative displacement along the z axis there are 9 distinct edge classes. If when projecting all vertices on the the z-axis, the interval defined by the z axis value of its endpoints contains no other vertices, we call it a *one-hop* edge, and if it does contain another vertex we call it a *two-hop* edge. Then there are 3 rail edges, 3 one-hop lengths between each pair of 3 rails, and 3 two hop lengths between each pair of 3 rails, where the two-hop length is at least the one-hop length. However, we have already shown the rail lengths are equal in any optimal tetrahelix.

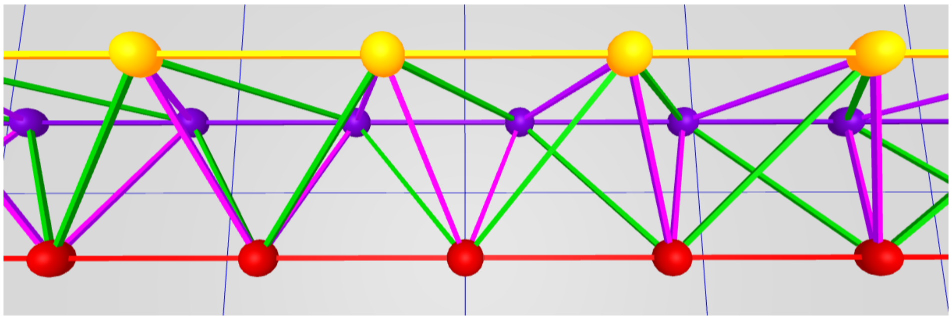

Figure 7. *Equitetrabeam*

Figure 7 shows the equitetrabeam, which is defined in section 6, but also conveniently illustrates the one-hop and two-hop edge defintions. The green edges are the two-hop edges and the purple edges are the one-hop edges. Note that the green edges are slightly longer than the purple edges.

Theorem 2. *An optimal tetrahelix of any rail angle $\rho < \pi$ has all nodes evenly spaced at $d/3$ intervals on the z axis. Any one tetrahedron in a tetrahelix has* 1 *rail edge,* 2 *one-hop edges connected to the rail and* 2 *two-hop edges connected to the rail. The edge opposite of the rail edge is a one-hop edge.*

Proof. Consider the tetrahelix in which the vertices are evenly spaced at $d/3$ intervals on the z axis. Every edge is either a rail edge, or it makes one hop, or it makes two hops. All of the one-hop edges are equal length. All of the two-hop edges are equal length.

Every vertex is connected to 4 non-rail edges. There is a one-hop edge in both the positive and negative z direction. Likewise there is a two-hop edge in both the positive and negative z direction. Let A be the set of edge lengths, which has only 3 members, represented by $A = \{o, t, r\}$ for the one-hop, two-hop, and rail edge lengths.

Any attempt to move any rail in either z direction lengthens one two-hop edge to t', where $t' > t$ and shortens one one-hop edge $o' < o$. Let $B = \{o', t'\} \cup A$ be new edges. The minimax of B is greater than the minimax of A since there is a single rail length which cannot be both greater than t' and o' and less than t' and o'. Therefore, any optimal tetrahelix has all one-hop edges between all rails equal to each other, and all two-hop edges equal to each other, and the z distances between rails equal, and therefore $d/3$ from each other. ■

Note that based on Theorem 2, there are only 3 possible lengths in an optimal tethrahelix, and we are justified in classifying edge lengths as *rail*, *one-hop*, or *two-hop*. The one-hop edges are the edges between rails that are closest on the z-axis, and the two-hop edges are those that skip over a vertex.

By Theorem 2 every optimal tetrahelix has vertices lying on helices expressible in the form:

$$V_{optimal}(n, c) = \begin{bmatrix} r\cos(n\alpha + c2\pi/3) \\ r\sin(n\alpha + c2\pi/3) \\ \frac{d(n+c/3)}{3} \end{bmatrix}, \text{where: } c \in \{0, 1, 2\}$$

where we have not yet investigated in the general case the relationships beteween α, r, and d in this formulation. However, we understand that when $\alpha = 0$, the helices are degenerate, having torsion of 0, and we have the equitetrabeam.

4. Parameterizing Tetrahelices via Rail Angle.

We seek a formula to generate optimal tetrahelices that accepts a parameter that allows us to design the tetrahelix conveniently. Please refer back to Figure 5. The pitch of the helix is an obvious choice, but is not defined when the torsion is 0, an important special case. The radius or the axial distance between two nodes on the same rail are possible choices, but perhaps the clearest choice is to build formulae that takes as their input the "rail angle" ρ. We define ρ to be the angle formed in the X,Y plane $\angle H(0,0)OH(0,1)$ projecting out the z axis and sighting along the positive z axis. In other words, ρ controls how far a rail edge of a tetrahelix deviates from being parallel with the axis, or the "twistiness" of the tetrahelix. We use the parameter $\chi = 1$ to indicate a chirality of counter-clockwise, and $\chi = -1$ for clockwise.

The quantities ρ, r, d are related by the expression:

$$\begin{aligned} 1^2 &= d^2 + (2r\sin\rho/2)^2 \\ d^2 &= 1 - 4r^2(\sin\rho/2)^2 \qquad (5) \end{aligned}$$

Checking the important special case of the BC helix, we find that this equation indeed holds true (treating d in this equation as $3h_{bc}$ as defined by Gray and Coxeter, that is, $d_{bc} = 3h_{bc}$, where they are using h for the axial height from one node to the next of a different color, but we use d to mean distance between nodes of the same color.

The rail angle ρ also has the meaning that $2\pi/\rho$ is the number of tetraheda in a full revolution of the helix.

In choosing ρ, one greatly constrains r and d, but does not completely determine both of them together, so we treat both as parameters.

Rewriting our formulation in terms of ρ:

$$H_{general}(\chi, n, c, \rho, d_\rho, r_\rho) = \begin{bmatrix} r_\rho \cos(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ r_\rho \sin(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ d_\rho(n + c/3) \end{bmatrix} \tag{6}$$

$$\text{where:} \quad \begin{array}{c} 1 = d_\rho^2 + 4r_\rho^2(\sin\rho/2)^2 \\ \chi \in \{-1, 1\} \end{array}$$

$H_{general}$ forces the user to select three values: ρ, r_ρ, and d_ρ satisfying (5).

Note that when $\rho = 0$ then $d_\rho = 1$, but r_ρ is not determined by (5).

Theorem 3. *The tetrahelices generated by $H_{general}$ are optimal in terms of minimum maximum member length when r_ρ is chosen so that the length of the one-hop edge is equal to the rail length.*

Proof. This is proved by a minimax argument.

By Theorem 2, we can compute the (at most) three edge-lengths of an optimal tetrahelix by formula universally quantified by n and c:

$$\begin{aligned} \text{rail} &= dist(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n + 1, c, \rho, d_\rho), r_\rho)) = 1 \\ \text{one-hop} &= dist(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n, c + 1, \rho, d_\rho, r_\rho)) \\ \text{two-hop} &= dist(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n, c + 2, \rho, d_\rho, r_\rho)) \end{aligned}$$

where *dist* is the Cartesian distance function.

$$\begin{aligned} \text{one-hop} &= dist(H_{general}(n, c, \rho, d_\rho), H_{general}(n, c + 1, \rho, d_\rho), r_\rho) \\ \text{one-hop} &= \sqrt{\frac{d_\rho^2}{9} + r_\rho^2(\sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)} \\ &\text{but: } d_\rho^2 = 1 - 4r_\rho^2(\sin(\rho/2)^2 \text{ ...so we substitute:} \\ \text{one-hop} &= \sqrt{\frac{1}{9} + r_\rho^2(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)} \end{aligned}$$

By similar algebra and trigonometry:

$$\text{two-hop} = \sqrt{\frac{4}{9} + r_\rho^2(-\frac{16(\sin^2(\rho/2))}{9} + \sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2)}$$

We would really like to know the partial derivative of the two-hop - one-hop with respect to the radius to be able to understand how to choose the radius to form the minimimax optimum.

Let:

$$f_\rho = -\frac{4(\sin^2(\rho/2))}{9}$$

$$g_\rho = -\frac{16(\sin^2(\rho/2))}{9}$$

$$j_\rho = \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)$$

$$k_\rho = (\sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2)$$

Then:

$$\text{two-hop} - \text{one-hop} = \sqrt{\frac{4}{9} + r_\rho^2(g_\rho + j_\rho)} - \sqrt{\frac{1}{9} + r_\rho^2(f_\rho + k_\rho)}$$

By graph inspection using Mathematica, we see the partial derivative of this with respect to radius r_ρ is always negative. Since the partial derivative of two-hop − one-hop with respect to the radius r_ρ is negative up until ρ_{bc} where it is 0, we opitmize the overall minimax distance by choosing the largest radius up until one-hop = 1, the rail-edge length.

Therefore we decrease the minimax length of the whole system as we increase the radius up to the point that the shorter, one-hop distance is equal to the rail-length (1). Therefore, to optimize the whole system so long as $\rho \leq \rho_{bc}$, we equate one-hop to 1 to find the optimum radius:

$$1 = \sqrt{\frac{1}{9} + r_{opt}^2(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)}$$

$$r_{opt} = \frac{2}{\sqrt{\frac{9}{2} \cdot (\sqrt{3}sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}} \tag{7}$$

We can now give a formula for d_{opt} computed from ρ, r_{opt} via the rail angle equation (5):

$$d_{opt}^2 = 1 - 4\left(\frac{2}{\sqrt{\frac{9}{2}\cdot(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}}\right)^2 (\sin\rho/2)^2$$

$$d_{opt}^2 = 1 - \frac{16(\sin\rho/2)^2}{9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}$$

$$d_{opt} = \sqrt{1 - \frac{16\sin^2(\rho/2)}{\cos(\rho) + 9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + 8}} \tag{8}$$

Thus, by computing r_{opt} and d_{opt} as a function of ρ from this equation, we can construct minimax optimal tetrahelix for an $0 \leq \rho \leq \rho_{bc}$. ■

Writing as robotocists, the “colored” approach of thinking of three rails seems natural to us, in part because our application is focused on the edges, rather than any sort of interpretaion of the tetrahedra as solids. Equation (6) is in terms of helices that are close to what you see when you look at a tetrahelix constructed from edges. Having proved Theorem 3, it becomes plain that one could also use a generalization of the orginal Coxeter/Gray formulation which uses a single helix to define all vertices, and a mathematician, chemist, or solid modeler might prefer that. This formulation is equivalent to Equation (6), but uses $\theta = \frac{\rho+2\pi}{3}$, the rotation between rails. The same constraint on r_{opt} and d_{opt} holds if the user seeks minimax optimality.

$$V_{general}(\chi, k, \theta, d_\theta, r_\theta) = \begin{bmatrix} r_\theta\cos(\chi k\theta) \\ r_\theta\sin(\chi k\theta) \\ d_\theta k/3 \end{bmatrix} \tag{9}$$

$$\text{where:} \quad \begin{array}{c} 1 = d_\theta^2 + 4r_\theta^2(\sin\frac{3\theta-2\pi}{2})^2 \\ \chi \in \{-1, 1\} \end{array}$$

Thus for any choice or d and r : $V_{general}(\chi, k, \theta, d, r) = H_{general}(\chi, n, c, \rho, d, r)$ when $\theta = \frac{\rho+2\pi}{3}$ and $k = 3n + c$, and both produce a minimax optimal tetrahelix for appropriate choices of r and d for $\frac{2\pi}{3} \leq \theta \leq \theta_{bc}$ or $0 \leq \rho \leq \rho_{bc}$.

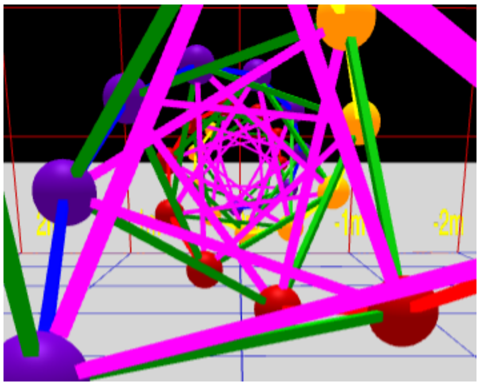

Figure 8. *Axial view of a BC-Helix*

5. The Inradius. If we look down the axis of an optimal tetrahelix as shown in Figure 8, it happens that only one of the one-hop edges (rendered in purple in our software) comes closest to the axis. In other words, they define the radius of the incircle of the projection, or the radius of a cylinder that would just fit inside the tetrahelix. A formula for the inradious of the tetrahelix is useful if you are designing it as a structure that bears something internally, such as a firehose, a pipe, or a ladder for a human. The inradius $r_{in}(\rho)$ of an optimal tetrahelix is a remarkably simple function of the radius r and the rail angle ρ:

$$r_{in}(\rho) = r \sin \frac{\pi - \rho}{6} \tag{10}$$

Which can be seen from the trigonometry of a diagram of the projected one-hop edges connecting four sequentially numbered vertices:

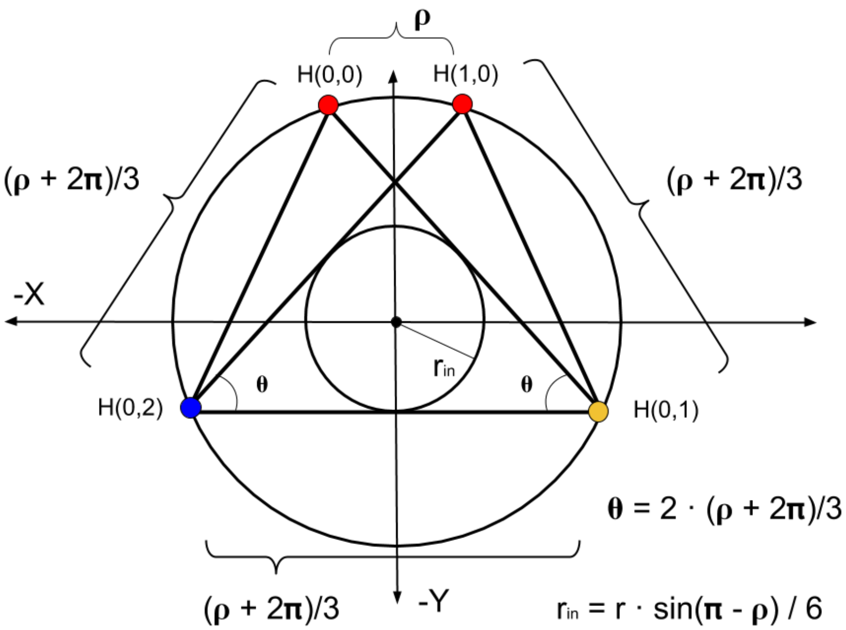

Figure 9. *General One-hop Projection Diagram*

From this equation with the help of symbolic computation we observe that inradius of the BC helix of unit rail length is $r_{in(\rho_{bc})} = \frac{3}{10\sqrt{2}} \approx 0.21$.

6. The Equitetrabeam.

Just as $H_{general}$ constructs the BC helix (with careful and non-obvious choices of parameters) which is an important special case due to its regularity, it constructs an additional special (degenerate) case when the rail angle $\rho = 0$ and $d = 1$ (the edgelength), where the cross sectional area is an equilateral triangle of unchanging orientation, as shown in Figure 7 and at the rear of Figure 3. We call this the *equitetrabeam*.

Corollary 4. *The equitetrabeam with minimal maximal edge difference is produced by $H_{general}$ when $r = \sqrt{\frac{8}{27}}$.*

Proof. Choosing $d = 1$ and $\rho = 0$ we use Equation (7) to find the radius of optimal minimax difference.

Substituting into (6):

$$\text{one-hop} = \sqrt{\frac{1}{9} + 3r^2}$$

Then:

$$1 = \sqrt{\frac{1}{9} + 3r^2} \qquad \text{solved by...}$$

$$r = \sqrt{\frac{8}{27}} \qquad \approx 0.54$$

■

This radius[2] produces a two-hop rail length of $\frac{2}{\sqrt{3}}$. The difference between this and 1 is $\approx 15.47\%$. The inradius of the equitetrabeam of unit rail length from both Equation (10) and the fact that the inradius of an equilateral triangle is half the circumradius is $\sqrt{\frac{8}{27}}/2$, or $\frac{\sqrt{6}}{9}$.

In Figure 3, the furthest tetrahelix is the optimal equitetrabeam.

To the extent that we value tetrabeams (that is, tetrahelices with a rail angle of 0, and therefore zero curvature and torsion) as mathematical or engineering objects, we have motivated the development of $H_{general}$ as a transformation of $V(n)$ defined by Equation (1) from Gray and Coxeter. It is difficult to see how the $V(n)$ formulation could ever give rise to a continuum producing the tetrabeam, since setting the angle in that equation to zero can produce only collinear points.

Note that the equitetrabeam has chirality, which becomes important in our attempt to build a continuum of tetrahelices.

7. An Untwisted Continuum. We observe that Equations (7) and (8) compute r_{opt} and d_{opt} which create an optimal tetrahelix for any rail angle ρ between 0, which gives the equitetrabeam and $\rho_{bc} \approx 35.43°$, which gives the BC helix.

Because the equitetrabeam which has a rail angle of 0 still has chirality, that is, one still must decide to connect the one-hop edge to the clockwise or the counter-clockwise node, it is not possible to build a smooth continuum where ρ transitions from positive to negative which remains optimal. One can use a negative ρ in $H_{general}$ but it does not produce minimax optimal tetrahelices. In other words, untwisting a counter-clockwise tetrahelix to rail angle 0 and then going even further does produce a clockwise tetrahelix, but one in which the one-hop and two-hop lengths in the wrong places (that is, two-hop becomes shorter than one-hop.) Likewise, $\rho > \rho_{bc}$ generates a tetrahelix, but minimax optimality is not guaranteed by $H_{general}$.

The pitch of a helix (see (4), for a fixed z-axis travel d, is trivial. However, if one is computing z-axis travel from (8) the pitch is not simple. It increases monotonically and smoothly with decreasing ρ, so Equation (4) can be easily solved numerically with a Newton-Raphson solver, as we do on our website. For a pitch at least $p \geq \frac{3\sqrt{2}\pi}{\sqrt{5}\rho_{bc}} \approx 9.64$, using (8) produces minimax optimal tetrahelices.

In this way a rail angle can be chosen for any desired (sufficiently large) pitch, yield the optimum radius, one-hop, and two-hop lengths an engineer needs to construct a physical structure.

Perhaps surprisingly, the optimal untwisting is accomplished only by changing the length of the two-hop member, leaving the one-hop member and rail length equivalent within this

[2] Another interesting but non-optimal solution is derived by setting (one-hop + two-hop)/2 = 1, occurs at $r = \sqrt{35}/4$ which produces three length classes of $11/12, 12/12, 13/12$.

continuum.[3] However, it should be noted that an engineer or architect may also use $H_{general}$ directly and interactively, and that minimax length optimality is a mathematical starting point rather than the final word on the beauty and utility of physical structures. For example, a structural engineer might increase radius past optimality in order to resist buckling.

If an equitetrabeam were actually used as a beam, an engineer might start with the optimal tetrabeam and dilate it in one dimension to "deepen" the beam. Similarly, simple length changes curve the equitetrabeam into an "arch". The "colored" approach of (6) exposes these possibilities more than the approach of **??**.

Trusses and space frames remain an important design field in mechanical and structural engineering[10], including deployable and moving trusses[2].

8. Utility for Robotics.

Starting twenty years ago, Sanderson[14], Hamlin,[8], Lee[9], and others created a style of robotics based on changing the lengths of members joined at the center of a joint, thereby creating a connection to pure geometry. More recently NASA has experimented with tensegrities[1], a different point in the same design spectrum. These fields create a need to explore the notion of geometries changing over time, not generally considered directly by pure geometry.

As suggested by Buckminster Fuller, the most convenient geometries to consider are those that have regular member lengths, in order to facilitate the inexpensive manufacture and construction of the robot. In a plane, the octet truss[4] is such a geometry, but in a line, the Boerdijk–Coxeter helix is a regular structure.

However, a robot must move, and so it is interesting to consider the transmutations of these geometries, which was in fact the motivation for creating the equitetrabream.

Theorem 5. *By changing only the length of the longer members that connect two distinct rails (the two-hop members), we can dynamically untwist a tetrobot forming the Boerdijk-Coxeter configuration into the equitetrabeam which rests flat on the plane.*

Proof. Proof by our computer program that does this using Equation (6) applied to the 7-tet Tetrobot/Glussbot.

[3]Before deriving Equation (7), we created a continuum by using a linear interpolation between the optimal radius for the Equitetrabeam and the BC Helix. This minimax optimum of this simpler approach was at most 1% worse than the optimum computed by (7).

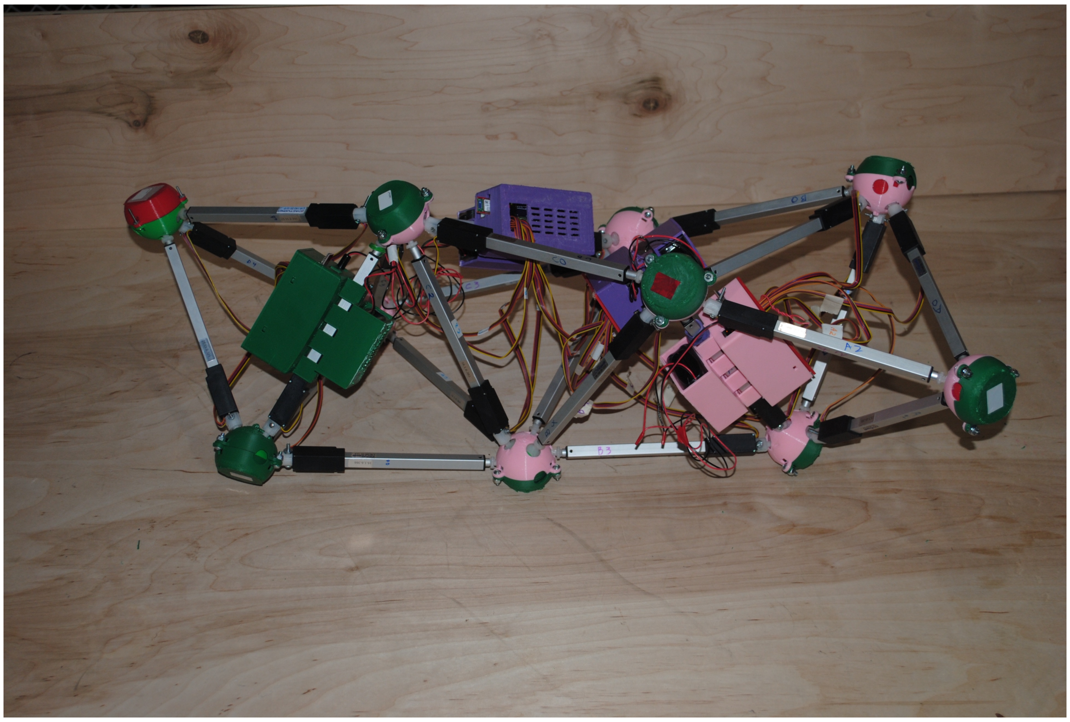

Figure 10. *Glussbot in relaxed, or BC helix configuration*

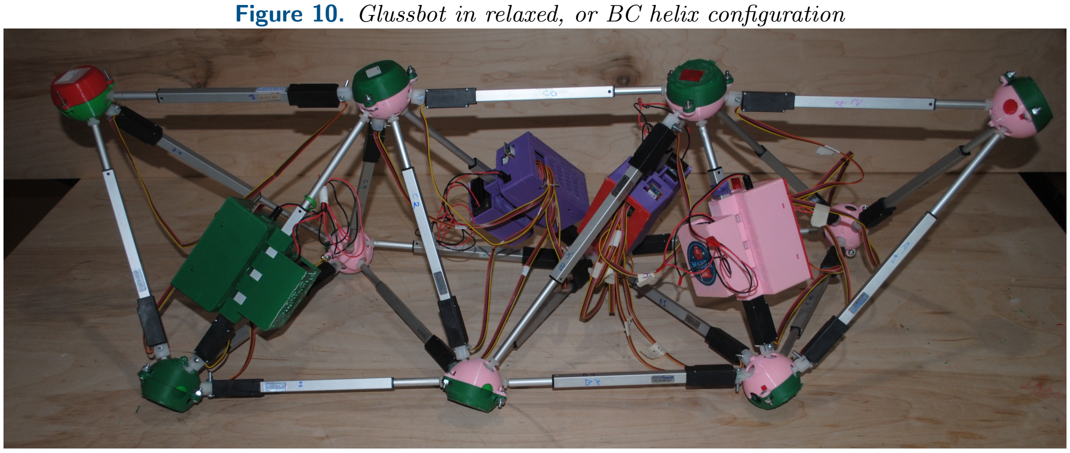

Figure 11. *The Equitetrabream: Fully Untwisted Glussbot in Hexapod Configuration*

By untwisting the tetrahelix so that is has a planar surface resting on the ground, we may consider each vertex touching the ground a foot or pseudopod. A robot can thus become a hexapod or n-pod robot, and the already well-developed approaches to hexapod gaits may be applied to make the robot walk or crawl.

9. Conclusion.

The BC Helix is the end point of a continuum of tetrahelices, the other end point being an untwisted tetrahelix with equilateral cross section, constructed by changing the length of only those members crossing the outside rails after hopping over the nearest vertex.

Under the condition of miminum maximum length difference of all members in the system, all such tetrahelices have vertices evenly spaced along the axis generated by a simple equation. A mechanical machine, such as a robot or a variable-geometry truss, that can change the length of its members, can thus twist and untwist itself by changing the length of the appropriate members to achieve any point in the continuum optimally. With a numeric solution, a design may choose a rotation angle and member lengths to obtain any desired pitch.

10. Contact and Getting Involved. The Gluss Project http://pubinv.github.io/gluss/ is part of Public Invention https://pubinv.github.io/PubInv/, a free-libre, open-source research, hardware, and software project that welcomes volunteers. It is our goal to organize projects for the benefit of all humanity without seeking profit or intellectual property. To assist, contact read.robert@gmail.com.

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