

Untwisting the Boerdijk–Coxeter

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Abstract. The Boerdijk–Coxeter helix (BC helix, or tetrahelix) is a face-to-face stack of regular tetrahedra forming a helical column. Considering the edges of these tetrahedra as structural members, the resulting structure is attractive and inherently rigid, and therefore interesting to architects, mechanical engineers, and robotocists. A formula is developed that matches the visually apparent helices forming the outer rails of the BC helix. This formula is generalized to a formula convenient to designers. Formulae for computing the parameters that give edge-length minimax-optimal tetrahelices are given, defining a continuum of tetrahelices of varying curvature. The endpoints of the optimality of this continuum are the BC helix and a structure of zero curvature, the *equitrapeam*. Numerically finding the rail angle from the equation for pitch allows optimal tetrahelices of any pitch to be designed. An interactive tool for such design and experimentation is provided: <https://pubinv.github.io/tetrahelix/>. A formula for the inradius of optimal tetrahelices is given. Utility for static and variable geometry truss/space frame design and robotics is discussed.

Key words. Boerdijk–Coxeter helix, tetrahelix, robotics, tetrobot, unconventional robots, structural engineering, mechanical engineering, tensegrity, variable-geometry truss

AMS subject classifications. 51M15

1. Introduction. The Boerdijk–Coxeter helix[3] (BC helix), is a face-to-face stack of tetrahedra that winds about a straight axis. Because architects, structural engineers, and robotocists are inspired by and follow such regular mathematical models but can also build structures and machines of differing or even dynamically changing length, it is useful to develop the mathematics of structures formed from tetrahedra where we relax regularity.

The vertices of the tetrahedra lie upon three helices about the central axis. The glussbot[11] (or Tetrobot)[8] uses the regularity of this geometry to make a tentacle-like robot that can crawl like a slug or mollusc. The Tetrobot uses mechanical actuators which can change their length, connected by special joints, such as the 3D printable Song-Kwon-Kim[15] joint, or the CMS joint[7] which allow many members to meet in a single point. Such machines can follow purely regular mathematical models such as the Boerdijk-Coxeter helix or the Octet Truss[4].

Buckminster Fuller called the BC helix a *tetrahelix*[5], a term now commonly used. In this paper we reserve *BC helix* to mean the purely regular structure and use *tetrahelix* to refer to any structure isomorphic to the BC helix.

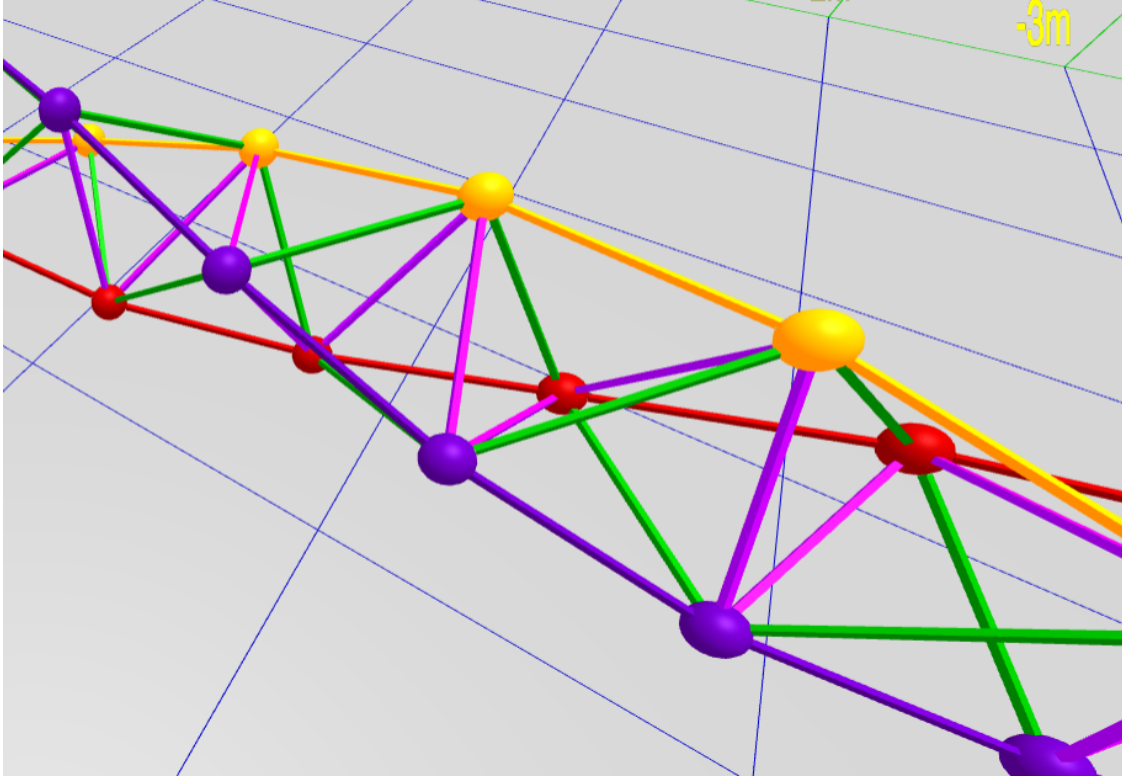


Figure 1. *BC Helix Close-up (partly along axis)*

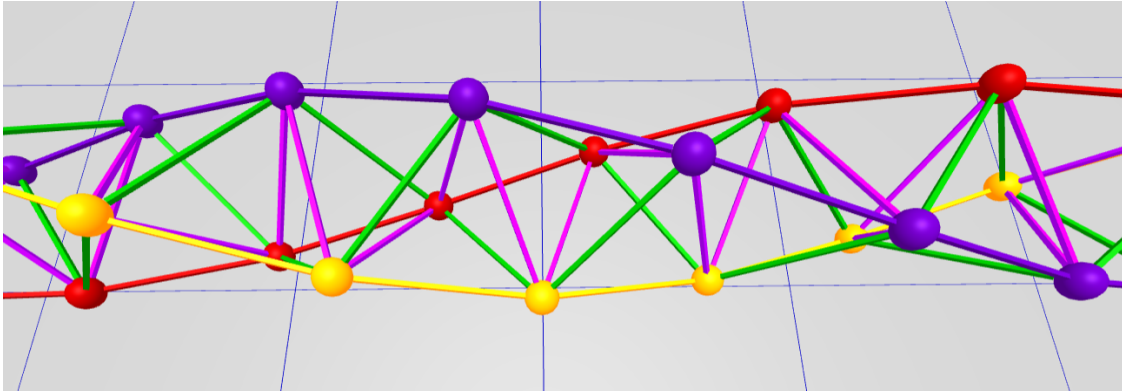


Figure 2. *BC Helix Close-up (orthogonal)*

33 Imagining [Figure 2](#) as a static mechanical structure, we observe that it is useful to the
 34 mechanical engineer or robotocist because the structure remains an inherently rigid, omni-
 35 triangulated space frame, which is mechanically strong. Imagine further in [Figure 2](#), that each
 36 static edge was replaced with an actuator that could dynamically become shorter or longer in
 37 response to electronic control, and the vertices were a joint that supported sufficient angular
 38 displacement for this to be possible. An example of such a machine is a glussbot, shown in
 39 [Figure 11](#).

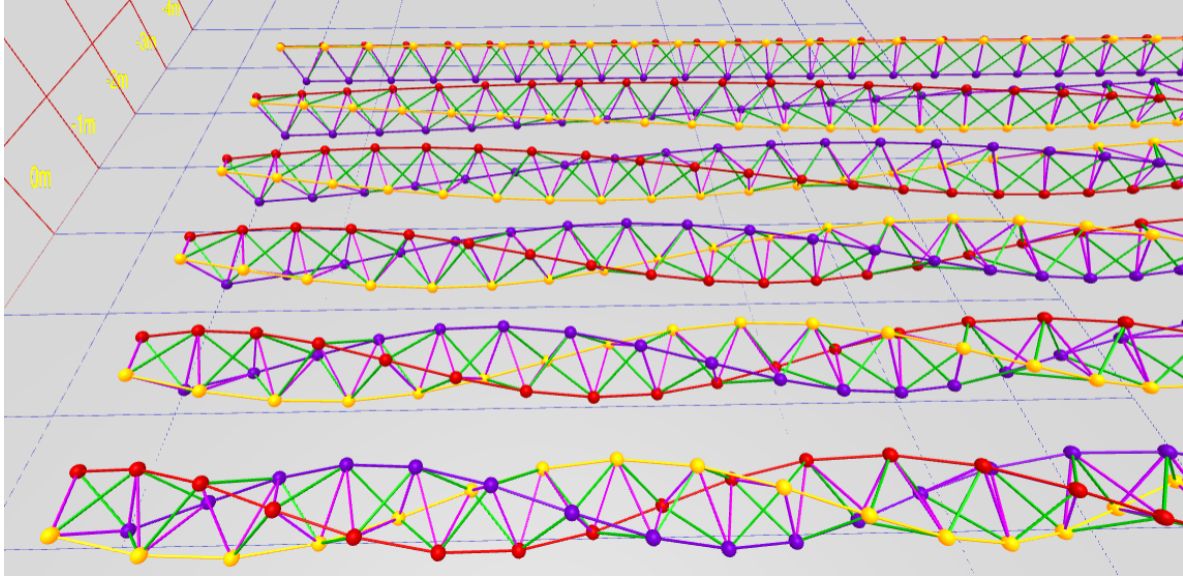


Figure 3. *A Continuum of Tetrahelices*

A BC helix does not rest stably on a plane. It is convenient to be able to “untwist” it and to form a tetrahelix space frame that has a flat planar surface. By making length changes in a certain way, we can untwist a tetrahelix to form a *tetrabeam* which has planar faces and has, for example, an equilateral triangular profile. This paper develops the equations needed to untwist the tetrahelix. All math developed here is available in JavaScript and demonstrated by an interactive design website <https://pubinv.github.io/tetrahelix/>[12], from which Figure 2 and the figures below are taken.

Figure 3 displays a continuum of tetrahelices optimal in a certain sense, which is the result of this paper. The closest helix is the BC helix, and the furthest is the equitetrabeam, defined in section 6.

2. A Designer’s Formulation of the BC Helix. We would like to design nearly regular tetrahelices with a formula that gives the vertices in space. Eventually we would like to design nearly regular tetrahelices by choosing the lengths of a small set of members. In a space frame, this is a static design choice; in a tetrobot, it is a dynamic choice that can be used to twist the robot and/or exert linear or angular force on the environment.

Ideally we would have a simple formula for defining the nodes based on any curvature or pitch we choose. It is a goal of this paper to relate these two approaches to generating a tetrahelix continuum.

H.S.M Coxeter constructs the BC helix[3] as a repeated rotation and translation of the tetrahedra, showing the rotation is:

$$\theta_{bc} = \arccos(-2/3)$$

61 and the translation:

$$62 \quad h_{bc} = 1/\sqrt{10}$$

63 θ_{bc} is approximately $0.37 \cdot 2\pi$ radians or 131.81 degrees. The angle θ_{bc} is the rotation of
 64 *each* tetrahedron, not the tetrahedra along a rail. In [Figure 2](#), each tetrahedron has either a
 65 yellow, blue, or red outer edge or rail. That is, a blue-rail tetrahedron is rotated slightly more
 66 than a 1/3 of a revolution to match the face of the yellow tetrahedra.

67 R.W. Gray's site[6], repeating a formula by Coxeter[3] in more accessible form, gives the

68 Cartesian coordinates $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a counter-clockwise BC Helix:

$$69 \quad (1) \quad V(n) = \begin{bmatrix} r_{bc} \cos n\theta_{bc} \\ r_{bc} \sin n\theta_{bc} \\ nh_{bc} \end{bmatrix}, \text{ where: } \begin{array}{l} r_{bc} = \frac{3\sqrt{3}}{10} \approx 0.5196 \\ h_{bc} = 1/\sqrt{10} \approx 0.3162 \\ \theta_{bc} = \arccos(-2/3) \end{array}$$

70 where n represents each integer numbered node in succession on every colored rail.

71 The apparent rotation of a vertex an outer-edge, $V(n)$ relative from $V(n+3)$ for any
 72 integer n in (1), is $3\theta_{bc} - 2\pi$.

73 This formula defines a helix, but it is not any of the apparent helices, or *rail* helices, of the
 74 BC helix, but rather one that winds three times as rapidly through all nodes. To a designer of
 75 tetrahelices, it is more natural to think of the three helices which are visually apparent, that
 76 is, those three which are closely approximated by the outer edges or rails of the BC helix. We
 77 think of each of these three rails as being a different color: red, blue, or yellow. This situation
 78 is illustrated in [Figure 4](#), wherein the black helix represents that generated by (1), and the
 79 colored helices are generated by (2).

80 In order to develop the continuum of slightly irregular tetrahelices described in [section 7](#),
 81 we need a formula that gives us the nodes of just one rail helix, denoted by color c and integer
 82 node number n :

$$83 \quad (\forall n \in \mathbb{Z}, \forall c \in \{0, 1, 2\} : H_{BC\text{colored}}(n, c) = V(3n + c))$$

84 Such a helix can be written:

$$85 \quad (2) \quad H_{BC\text{colored}}(n, c) = \begin{bmatrix} r_{bc} \cos ((3\theta_{bc} - 2\pi)n + c\theta_{bc}) \\ r_{bc} \sin ((3\theta_{bc} - 2\pi)n + c\theta_{bc}) \\ 3h_{bc}(n + c/3) \end{bmatrix}, \text{ where } \begin{array}{l} r_{bc} = \frac{3\sqrt{3}}{10} \\ h_{bc} = 1/\sqrt{10} \\ \theta_{bc} = \arccos(-2/3) \end{array}$$

86 In this formula, integral values of n may be taken as a node number for one rail and
 87 used to compute its Cartesian coordinates. Allowing n to take non-integer values defines a
 88 continuous helix in space which is close to the segmented polyline of the outer tetrahedra
 89 edges, and equals them at integer values.

90 [Figure 4](#) illustrates this difference. Although the nodes coincide, (1) evaluated at real
 91 values generates the black helix, and (2) defines the red, yellow, and blue helices. (In this
 92 figure these rail helices have been rendered at a slightly higher radius than the nodes; in
 93 actuality the helices are much close to the edges between the nodes that they nearly match.)

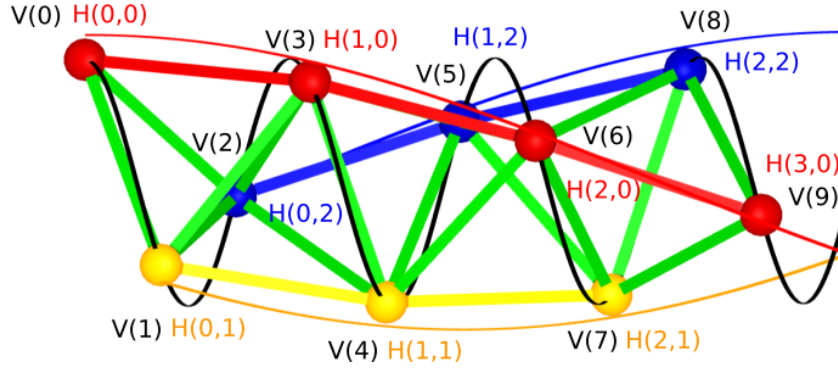


Figure 4. Rail helices (H) vs. Coxeter/Gray helix (V)

94 The quantity $(3\theta_{bc} - 2\pi) \approx 35.43^\circ$ is the angular shift between $V(3n+c) = H_{BCcolored}(n, c)$
 95 and $V(3(n+1)+c) = H_{BCcolored}(n+1, c)$. This quantity appears so often that we call it the
 96 “rail angle ρ ”. For the BC helix, $\rho_{bc} = (3\theta_{bc} - 2\pi)$.

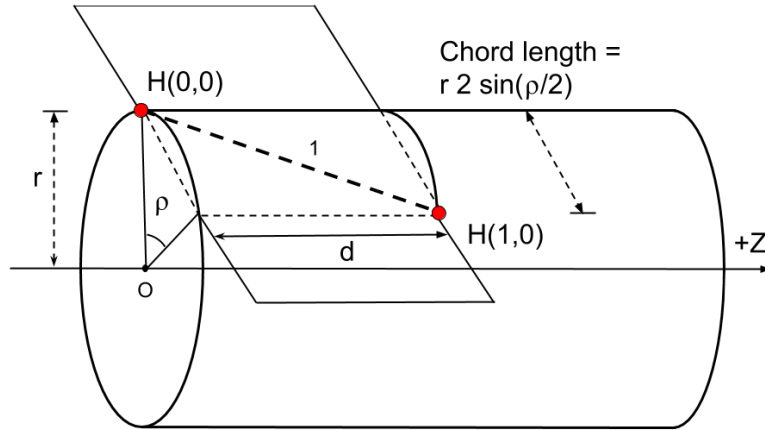


Figure 5. Rail Angle Geometry

97 Note in Figure 5 the z -axis travel for one rail edge is denoted by d . In (1) and (2), the
 98 variable h is used for one third of the distance we name d . We will later justify that $d = 3h$.
 99 In this paper we assume the length of a rail is always 1 as a simplification. (We make the rail
 100 length a parameter in our JavaScript code in [https://github.com/PubInv/tetrahelix/blob/](https://github.com/PubInv/tetrahelix/blob/master/js/tetrahelix_math.js)
 101 [master/js/tetrahelix_math.js](https://github.com/PubInv/tetrahelix/blob/master/js/tetrahelix_math.js) [12].)

102 The $H_{BCcolored}(n, c)$ formulation can be further clarified by rewriting directly in terms of
 103 the rail angle ρ_{bc} rather than θ_{bc} . Intuitively we seek an expression where $c/3$ is multiplied by

104 a $1/3$ rotation plus the rail angle ρ . We expand the expressions θ_{bc} and ρ_{bc} in (2) and seek to
 105 isolate the term $c2\pi/3$.

$$\begin{aligned}
 106 \quad & c\theta_{bc} = \{\text{we aim for 3 in denominator, so we split...}\} \\
 107 \quad & (c/3)(3\theta_{bc}) = \{\text{we want } 2\pi \text{ in numerator, so add canceling terms...}\} \\
 108 \quad & (c/3)((3\theta_{bc} - 2\pi) + 2\pi) = \{\text{definition of } \rho_{bc}\}... \\
 109 \quad & (c/3)\rho_{bc} + c2\pi/3 = \text{algebra...} \\
 110 \quad & c(\rho_{bc} + 2\pi)/3
 \end{aligned}$$

111

113 This allows us to redefine:

$$114 \quad (3) \quad H_{BC\text{colored}}(n, c) = \begin{bmatrix} r \cos \rho_{bc}n + c(\rho_{bc} + 2\pi)/3 \\ r \sin \rho_{bc}n + c(\rho_{bc} + 2\pi)/3 \\ (n + c/3)h_{bc} \end{bmatrix}, \text{ where } \begin{matrix} \rho_{bc} = (3\theta_{bc} - 2\pi) \\ h_{bc} = 1/\sqrt{10} \end{matrix}$$

115 Recall that $c \in \{0, 1, 2\}$, but n is continuous (rational or real-valued.) We can now assert
 116 that in Figure 4 the black helix winds at $\frac{3\theta_{bc}}{\rho_{bc}} \approx 11.16$ times the rate of a rail helix.

117 From this formulation it is easy to see that moving one vertex on a rail ($H_{BC\text{colored}}(n, c)$)
 118 to $H_{BC\text{colored}}(n + 1, c)$ for any n and c) moves us ρ_{bc} radians around a circle. Since:

$$119 \quad \frac{2\pi}{\rho_{bc}} \approx 10.16$$

120 we can see that there are approximately 10.16 red, blue or yellow tetrahedra on one rail in a
 121 single revolution.

122 The *pitch* of any tetrahelix, defined as the length of a complete revolution where $\rho \neq 0$ is:

$$123 \quad (4) \quad p(\rho) = \frac{2\pi \cdot d}{\rho}$$

124 The pitch of the Boerdijk–Coxeter helix of edge length 1 is the length of three tetrahedra
 125 times this number:

$$126 \quad \frac{3h_{bc} \cdot 2\pi}{\rho_{bc}} = \frac{6\pi}{\sqrt{10}\rho_{bc}} \approx 9.64$$

128 The pitch is less than the number of tetrahedra because the tetrahedra are not lined
 129 up perfectly. It is a famous and interesting result that the pitch is irrational. A BC helix
 130 never has two tetrahedra at precisely the same orientation around the z -axis. However, this
 131 is inconvenient to designers, who might prefer a rational pitch. The idea of developing a
 132 rational period by arranging solid tetrahedra by relaxing the face-to-face matching has been
 133 explored[13]. We develop below slightly irregular edge lengths that support, for example, a
 134 pitch of precisely 12 tetrahedra in one revolution which would allow an architect to design a
 135 column having a basis and a capital in the same relation to the tetrahedra they touch at the
 136 bottom and top of the column.

3. Optimal Tetrahelices are Triple Helices.

We use the term *tetrahelix* to mean any structure made of vertices and edges which is isomorphic to the BC helix and in which the vertices lie on three helices. We further demand that all edge lengths be finite, as we are only interested in physically constructable tetrahelices. By isomorphic we mean there is a one-to-one mapping between both vertices and edges in the two tetrahelices. One could consider various definitions of optimality for a tetrahelix, but the most useful to us as robotocists working with the Tetrobot concept is to minimize the maximum difference between any two edges, because the Tetrobot uses mechanical linear acutators with limited range of extension.

A *triple helix* is three congruent helices that share an axis. We show that optimal tetrahelices are triple helices.

Suppose that all three rails do not have the same pitch. Starting at any shortest edge, as we move from node to node away from our start node edge lengths between rails must always lengthen without bound, which cannot be optimal. So we are justified in talking about the *pitch* of the optimal tetrahelix as the pitch of its three rail helices, even though there are three such helices of equivalent pitch.

Similarly, if the axes are not parallel, there is an edge of unbounded length in the structure, so we do not consider such cases.

Since the axes are parallel, we may define the *inradius*, represented by the letter i , of a tetrahelix to be the radius of the largest cylinder parallel to this axis contained within the circumscribing cylinder which is penetrated by no edge. Define a *minimax edge-length optimal tetrahelix* or just an *optimal tetrahelix* to be a tetrahelix for which there exists no other tetrahelix of the same inradius and pitch with a lower maximum difference between its edge lengths.

We wish to show that in an optimal tetrahelix, all vertices lie on the cylinder of radius r , regardless of where they lie on the z -axis.

Since all three rails have the same rail length, no matter how we move the rails in the xy plane if we shorten the xy distance between vertices we shorten the total distance. Our tool for thinking about this is to collapse the z dimension to form a two-dimensional figure of nodes and non-rail edges (see Figure 6.) Consider the projection along the z axis of all vertices and non-rail edges into the xy plane, which will be a figure of dots and connecting segments in the xy plane. The convex hull for any one helix projection will be a circle (if its pitch is irrational) or a polygon if rational, or a point if the helix has zero curvature. Each of these figures by definition lies outside the circle of inradius r_{in} in the xy plane.

Theorem 1. *Any optimal tetrahelix with a rail angle less than π has all vertices on a single cylinder.*

Proof. Case 1: Suppose that ρ is zero. Then for any given inradius, an equilateral triangle is the minimax solution for all non-rail edges. Since there is only one rail-edge length, this is the minimax solution for the entire set. Since the vertices of an equilateral triangle lies on a circle, all points lie on a cylinder.

Case 2: Suppose that ρ is positive but less than π . In this case each rail helix has curavature and places points on both sides of any line through the origin of the xy plane (or both coincident on such a line.)

We first show that the inradius is touched by one segment from each pair of rails.

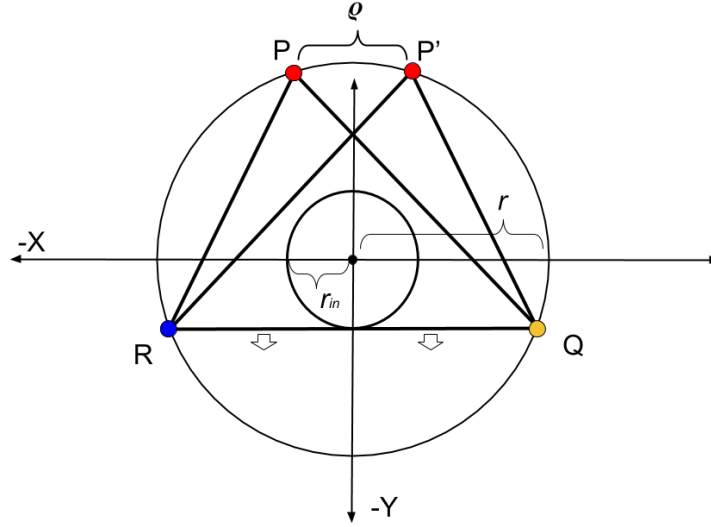


Figure 6. *Untouching Rail*

180 **Figure 6** depicts a projection of only the point P , the point P' which is an adjacent vertex
 181 on the same rail, and the intermediary points R and Q on the other two rails.

182 Suppose there is a rail P which does not have a segment touching the incircle at all in
 183 the projection, as depicted in **Figure 6**. Then a segment connecting the other two Q and R
 184 is a longest rail, because any chord touching an incircle is longer than any which does not.
 185 These two rails (Q and R) can be moved closer to each other, and further away from P , to
 186 form a better minimax solution by shortening the longest rail, until one of the segments from
 187 P the previously untouching rail touches one of the incircle. By induction, then all rails have
 188 at least one segment touching each vertex on the rail that is tangent to the incircle in an
 189 optimal tetrahelix.

190 Now suppose that you attempt to move the axis of one of the rails. Because $\rho > 0$ and
 191 ρ is not a multiple of π , there is some vertex on the two sides of any diameter halving our
 192 circle. Moving the axis of a tetrahelix lengthens a longest line while “pinching” the inradius
 193 on the other side, so it is not optimal. Since the axes of these helices are the same and they
 194 have the same curvature and pitch, the points defined by the helices all lie on a cylinder.

195 Note that **Theorem 1** is not true when ρ is an odd integer multiple of π .

196 Now that we have shown that any optimal tetrahelix vertices are on helices of the same
 197 axes and pitch, we see that the vertices of any optimal tetrahelix will lie on a cylinder, or a
 198 circle when axis dimension is projected out. Therefore it is reasonable to now speak of the
 199 *radius* r of a tetrahelix as the radius of the cylinder, as distinct from the inradius r_{in} .

200 Now that we have coincident axes, the same pitch, and the same radius, we can go on to
 201 the harder proof about where vertices occur along the z -axis.

202 We show that in fact the nodes must be distributed in even thirds along the z -axis, as in

fact they are in the regular BC helix.

Note that from the point of view of a single edge, we are on a slanted cylinder, when $\rho \neq 0$. This means from its point of view a cross section is an ellipse. So we have to be very careful in comparing lengths of edges relative to the tetrahedron, because a change in position along the edge changes the length of a line, but in a complicated way depending on where it is relative to the ellipse.

In principle in any 3 helices with the same axis of the same radius having any relative displacement along the z axis there are 9 distinct edge classes. If when projecting all vertices on the the z -axis, the interval defined by the z axis value of its endpoints contains no other vertices, we call it a *one-hop* edge, and if it does contain another vertex we call it a *two-hop* edge. Then there are 3 rail edges, 3 one-hop lengths between each pair of 3 rails, and 3 two hop lengths between each pair of 3 rails, where the two-hop length is at least the one-hop length. However, we have already shown the rail lengths are equal in any optimal tetrahelix.

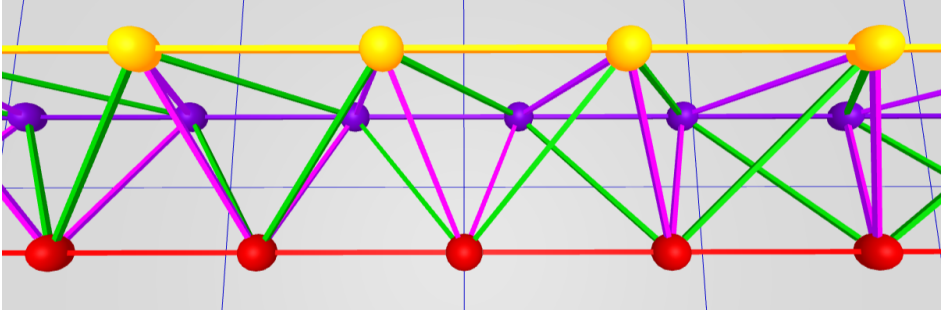


Figure 7. Equitetrahelium

Figure 7 shows the equitetrahelium, which is defined in section 6, but also conveniently illustrates the one-hop and two-hop edge definitions. The green edges are the two-hop edges and the purple edges are the one-hop edges. Note that the green edges are slightly longer than the purple edges.

Theorem 2. *An optimal tetrahelix of any rail angle $\rho < \pi$ is a triple helix all nodes evenly spaced at $d/3$ intervals on the z axis. Any one tetrahedron in a tetrahelix has 1 rail edge, 2 one-hop edges connected to the rail and 2 two-hop edges connected to the rail. The edge opposite of the rail edge is a one-hop edge.*

Proof. Consider the tetrahelix in which the vertices are evenly spaced at $d/3$ intervals on the z axis. Every edge is either a rail edge, or it makes one hop, or it makes two hops. All of the one-hop edges are equal length. All of the two-hop edges are equal length.

Every vertex is connected to 4 non-rail edges. There is a one-hop edge in both the positive and negative z direction. Likewise there is a two-hop edge in both the positive and negative z direction. Let A be the set of edge lengths, which has only 3 members, represented by $A = \{o, t, r\}$ for the one-hop, two-hop, and rail edge lengths.

Any attempt to move any rail in either z direction lengthens one two-hop edge to t' , where $t' > t$ and shortens one one-hop edge $o' < o$. Let $B = \{o', t'\} \cup A$ be new edges. The minimax of B is greater than the minimax of A since there is a single rail length which cannot be both

greater than t' and o' and less than t' and o' . Therefore, any optimal tetrahelix has all one-hop edges between all rails equal to each other, and all two-hop edges equal to each other, and the z distances between rails equal, and therefore $d/3$ from each other.

As each helix is congruent to the others and shares an axis and are evenly spaced, an optimal tetrahelix is a *triple helix*. ■

Note that based on [Theorem 2](#), there are only 3 possible lengths in an optimal tetrahelix, and we are justified in classifying edge lengths as *rail*, *one-hop*, or *two-hop*. The one-hop edges are the edges between rails that are closest on the z -axis, and the two-hop edges are those that skip over a vertex.

By [Theorem 2](#) every optimal tetrahelix has vertices lying on helices expressible in the form:

$$V_{\text{optimal}}(n, c) = \begin{bmatrix} r \cos(n\alpha + c2\pi/3) \\ r \sin(n\alpha + c2\pi/3) \\ \frac{d(n+c/3)}{3} \end{bmatrix}, \text{ where: } c \in \{0, 1, 2\}$$

where we have not yet investigated in the general case the relationships between α , r , and d in this formulation. However, we understand that when $\alpha = 0$, the helices are degenerate, having curvature of 0, and we have the equitetrabeam.

4. Parameterizing Tetrahelices via Rail Angle. We seek a formula to generate optimal tetrahelices that accepts a parameter that allows us to design the tetrahelix conveniently. Please refer back to [Figure 5](#). The pitch of the helix is an obvious choice, but is not defined when the curvature is 0, an important special case. The radius or the axial distance between two nodes on the same rail are possible choices, but perhaps the clearest choice is to build formulae that takes as their input the “rail angle” ρ . We define ρ to be the angle formed in the X,Y plane $\angle H(0,0)OH(0,1)$ projecting out the z axis and sighting along the positive z axis. In other words, ρ controls how far a rail edge of a tetrahelix deviates from being parallel with the axis, or the “twistiness” of the tetrahelix. We use the parameter $\chi = 1$ to indicate a chirality of counter-clockwise, and $\chi = -1$ for clockwise.

The quantities ρ, r, d are related by the expression:

$$\begin{aligned} 1^2 &= d^2 + (2r \sin \rho/2)^2 \\ d^2 &= 1 - 4r^2(\sin \rho/2)^2 \end{aligned} \quad (5)$$

Checking the important special case of the BC helix, we find that this equation indeed holds true (treating d in this equation as $3h_{bc}$ as defined by Gray and Coxeter, that is, $d_{bc} = 3h_{bc}$, where they are using h for the axial height from one node to the next of a different color, but we use d to mean distance between nodes of the same color).

The rail angle ρ also has the meaning that $2\pi/\rho$ is the number of tetrahedra in a full revolution of the helix.

In choosing ρ , one greatly constrains r and d , but does not completely determine both of them together, so we treat both as parameters.

Rewriting our formulation in terms of ρ :

$$(6) \quad H_{general}(\chi, n, c, \rho, d_\rho, r_\rho) = \begin{bmatrix} r_\rho \cos(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ r_\rho \sin(\chi \cdot (n\rho + c(\rho + 2\pi)/3)) \\ d_\rho(n + c/3) \end{bmatrix}$$

where: $1 = d_\rho^2 + 4r_\rho^2(\sin \rho/2)^2$
 $\chi \in \{-1, 1\}$

$H_{general}$ forces the user to select three values: ρ , r_ρ , and d_ρ satisfying (5).

Note that when $\rho = 0$ then $d_\rho = 1$, but r_ρ is not determined by (5).

Theorem 3. *The tetrahelices generated by $H_{general}$ are optimal in terms of minimum maximum member length when r_ρ is chosen so that the length of the one-hop edge is equal to the rail length.*

Proof. This is proved by a minimax argument.

By Theorem 2, we can compute the (at most) three edge-lengths of an optimal tetrahelix by formula universally quantified by n and c :

$$\begin{aligned} \text{rail} &= \text{dist}(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n + 1, c, \rho, d_\rho, r_\rho)) = 1 \\ \text{one-hop} &= \text{dist}(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n, c + 1, \rho, d_\rho, r_\rho)) \\ \text{two-hop} &= \text{dist}(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n, c + 2, \rho, d_\rho, r_\rho)) \end{aligned}$$

where dist is the Cartesian distance function.

$$\begin{aligned} \text{one-hop} &= \text{dist}(H_{general}(n, c, \rho, d_\rho, r_\rho), H_{general}(n, c + 1, \rho, d_\rho, r_\rho)) \\ \text{one-hop} &= \sqrt{\frac{d_\rho^2}{9} + r_\rho^2(\sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)} \\ \text{but: } d_\rho^2 &= 1 - 4r_\rho^2(\sin(\rho/2))^2 \text{ ...so we substitute:} \\ \text{one-hop} &= \sqrt{\frac{1}{9} + r_\rho^2(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)} \end{aligned}$$

By similar algebra and trigonometry:

$$\text{two-hop} = \sqrt{\frac{4}{9} + r_\rho^2(-\frac{16(\sin^2(\rho/2))}{9} + \sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2)}$$

We would really like to know the partial derivative of the two-hop - one-hop with respect to the radius to be able to understand how to choose the radius to form the minimimax optimum.

Let:

$$f_\rho = -\frac{4(\sin^2(\rho/2))}{9}$$

$$g_\rho = -\frac{16(\sin^2(\rho/2))}{9}$$

$$j_\rho = \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2$$

$$k_\rho = (\sin^2(2\rho/3 + \frac{4\pi}{3}) + (1 - \cos(2\rho/3 + \frac{4\pi}{3}))^2)$$

Then:

$$\text{two-hop} - \text{one-hop} = \sqrt{\frac{4}{9} + r_\rho^2(g_\rho + j_\rho)} - \sqrt{\frac{1}{9} + r_\rho^2(f_\rho + k_\rho)}$$

By graph inspection using Mathematica, we see the partial derivative of this with respect to radius r_ρ is always negative. Since the partial derivative of two-hop – one-hop with respect to the radius r_ρ is negative up until ρ_{bc} where it is 0, we optimize the overall minimax distance by choosing the largest radius up until one-hop = 1, the rail-edge length.

Therefore we decrease the minimax length of the whole system as we increase the radius up to the point that the shorter, one-hop distance is equal to the rail-length (1). Therefore, to optimize the whole system so long as $\rho \leq \rho_{bc}$, we equate one-hop to 1 to find the optimum radius:

$$(7) \quad r_{opt} = \frac{\sqrt{\frac{1}{9} + r_{opt}^2(-\frac{4(\sin^2(\rho/2))}{9} + \sin^2(\rho/3 + \frac{2\pi}{3}) + (1 - \cos(\rho/3 + \frac{2\pi}{3}))^2)}}{\sqrt{\frac{9}{2}} \cdot (\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}$$

We can now give a formula for d_{opt} computed from ρ, r_{opt} via the rail angle equation (5):

$$(8) \quad d_{opt}^2 = 1 - 4\left(\frac{2}{\sqrt{\frac{9}{2}} \cdot (\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}\right)^2(\sin \rho/2)^2$$

$$d_{opt}^2 = 1 - \frac{16(\sin \rho/2)^2}{9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + \cos(\rho) + 8}$$

$$d_{opt} = \sqrt{1 - \frac{16 \sin^2(\rho/2)}{\cos(\rho) + 9(\sqrt{3}\sin(\rho/3) + \cos(\rho/3)) + 8}}$$

331 Thus, by computing r_{opt} and d_{opt} as a function of ρ from this equation, we can construct
 332 minimax optimal tetrahelix for an $0 \leq \rho \leq \rho_{bc}$. ■

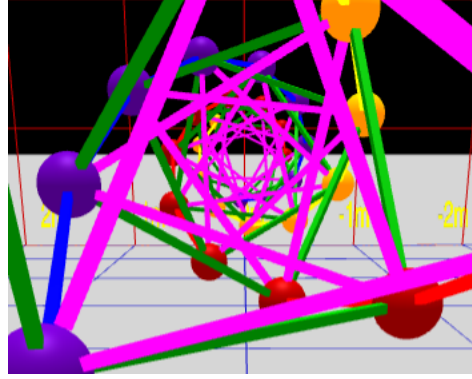


Figure 8. Axial view of a BC-Helix

333 **5. The Inradius.** If we look down the axis of an optimal tetrahelix as shown in Figure 8, it
 334 happens that only one of the one-hop edges (rendered in purple in our software) comes closest
 335 to the axis. In other words, they define the radius of the incircle of the projection, or the
 336 radius of a cylinder that would just fit inside the tetrahelix. A formula for the inradius of the
 337 tetrahelix is useful if you are designing it as a structure that bears something internally, such
 338 as a firehose, a pipe, or a ladder for a human. The inradius $r_{in}(\rho)$ of an optimal tetrahelix is
 339 a remarkably simple function of the radius r and the rail angle ρ :

$$340 \quad (9) \quad r_{in}(\rho) = r \sin \frac{\pi - \rho}{6}$$

341 Which can be seen from the trigonometry of a diagram of the projected one-hop edges con-
 342 necting four sequentially numbered vertices:

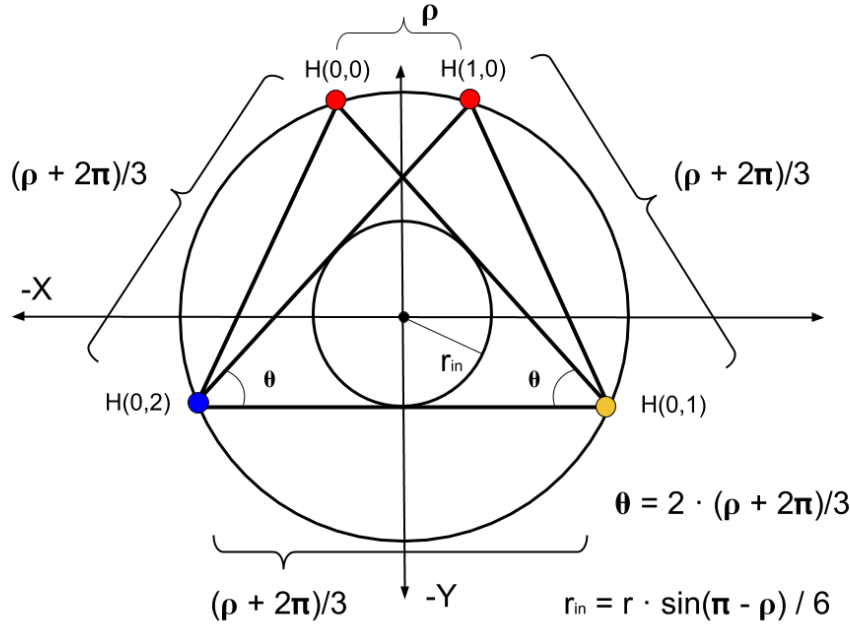


Figure 9. General One-hop Projection Diagram

From this equation with the help of symbolic computation we observe that inradius of the BC helix of unit rail length is $r_{in(\rho_{bc})} = \frac{3}{10\sqrt{2}} \approx 0.21$.

6. The Equitetrabeam. Just as $H_{general}$ constructs the BC helix (with careful and non-obvious choices of parameters) which is an important special case due to its regularity, it constructs an additional special (degenerate) case when the rail angle $\rho = 0$ and $d = 1$ (the edglength), where the cross sectional area is an equilateral triangle of unchanging orientation, as shown in Figure 7 and at the rear of Figure 3. We call this the *equitetrabeam*. It is not possible to generate an equitetrabeam from (1) without the split into three rails introduced by (2) and completed in (6).

Corollary 4. The equitetrabeam with minimal maximal edge difference is produced by $H_{general}$ when $r = \sqrt{\frac{8}{27}}$.

Proof. Choosing $d = 1$ and $\rho = 0$ we use Equation (7) to find the radius of optimal minimax difference.

Substituting into (6):

$$\text{one-hop} = \sqrt{\frac{1}{9} + 3r^2}$$

360 Then:

$$\begin{aligned}
 361 \quad 1 &= \sqrt{\frac{1}{9} + 3r^2} && \text{solved by...} \\
 362 \quad r &= \sqrt{\frac{8}{27}} && \approx 0.54
 \end{aligned}$$

363
364

■

365 This radius¹ produces a two-hop rail length of $\frac{2}{\sqrt{3}}$. The difference between this and 1 is
 366 $\approx 15.47\%$. The inradius of the equitetrabeam of unit rail length from both Equation (9) and
 367 the fact that the inradius of an equilateral triangle is half the circumradius is $\sqrt{\frac{8}{27}}/2$, or $\frac{\sqrt{6}}{9}$.

368 In Figure 3, the furthest tetrahelix is the optimal equitetrabeam. Figure 7 is a closeup of
 369 an equitetrabeam.

370 To the extent that we value tetrabeams (that is, tetrahelices with a rail angle of 0, and
 371 therefore zero curvature and curvature) as mathematical or engineering objects, we have
 372 motivated the development of $H_{general}$ as a transformation of $V(n)$ defined by Equation (1)
 373 from Gray and Coxeter. It is difficult to see how the $V(n)$ formulation could ever give rise
 374 to a continuum producing the tetrabeam, since setting the angle in that equation to zero can
 375 produce only collinear points.

376 The equitetrabeam may possibly be a novel construction. The fact that 6 members meet
 377 in a single point would have been a manufacturing disadvantage that may have dissuaded
 378 structural engineers from using this geometry. However, the advent of additive manufacturing,
 379 such as 3D printing, and the invention of two distinct concentric multimember joints[15, 7] has
 380 improved that situation.

381 Note that the equitetrabeam has chirality, which becomes important in our attempt to
 382 build a continuum of tetrahelices.

383 **7. An Untwisted Continuum.** We observe that Equations (7) and (8) compute r_{opt} and
 384 d_{opt} which create an optimal tetrahelix for any rail angle ρ between 0, which gives the equi-
 385 tetrabeam and $\rho_{bc} \approx 35.43^\circ$, which gives the BC helix.

386 Because the equitetrabeam which has a rail angle of 0 still has chirality, that is, one still
 387 must decide to connect the one-hop edge to the clockwise or the counter-clockwise node, it
 388 is not possible to build a smooth continuum where ρ transitions from positive to negative
 389 which remains optimal. One can use a negative ρ in $H_{general}$ but it does not produce minimax
 390 optimal tetrahelices. In other words, untwisting a counter-clockwise tetrahelix to rail angle 0
 391 and then going even further does produce a clockwise tetrahelix, but one in which the one-hop
 392 and two-hop lengths in the wrong places (that is, two-hop becomes shorter than one-hop.)
 393 Likewise, $\rho > \rho_{bc}$ generates a tetrahelix, but minimax optimality is not guaranteed by $H_{general}$.

394 The pitch of a helix (see (4), for a fixed z -axis travel d , is trivial. However, if one is
 395 computing z -axis travel from (8) the pitch is not simple. It increases monotonically and
 396 smoothly with decreasing ρ , so Equation (4) can be easily solved numerically with a Newton-

¹Another interesting but non-optimal solution is derived by setting $(\text{one-hop} + \text{two-hop})/2 = 1$, occurs at $r = \sqrt{35}/4$ which produces three length classes of 11/12, 12/12, 13/12.

397 Raphson solver, as we do on our website. For a pitch at least $p \geq \frac{3\sqrt{2}\pi}{\sqrt{5}\rho_{bc}} \approx 9.64$, using (8)
 398 produces minimax optimal tetrahelices.

399 In this way a rail angle can be chosen for any desired (sufficiently large) pitch, yield the
 400 optimum radius, one-hop, and two-hop lengths an engineer needs to construct a physical
 401 structure.

402 The curvature of a rail helix is given by:

$$403 \quad (10) \quad \frac{|r_\rho|}{r_\rho^2 + (d_\rho/\rho)^2}$$

404 which goes to 0 as ρ approaches 0 (the equitetrabeam.) As ρ increase up to ρ_{bc} the curvature
 405 increases smoothly until the BC Helix is reached.

406 Perhaps surprisingly, the optimal untwisting is accomplished only by changing the length
 407 of the two-hop member, leaving the one-hop member and rail length equivalent within this
 408 continuum.² However, it should be noted that an engineer or architect may also use $H_{general}$
 409 directly and interactively, and that minimax length optimality is a mathematical starting point
 410 rather than the final word on the beauty and utility of physical structures. For example, a
 411 structural engineer might increase radius past optimality in order to resist buckling.

412 If an equitetrabeam were actually used as a beam, an engineer might start with the
 413 optimal tetrabeam and dilate it in one dimension to “deepen” the beam. Similarly, simple
 414 length changes curve the equitetrabeam into an “arch”. The “colored” approach of (6) exposes
 415 these possibilities more than the approach of ??.

416 Trusses and space frames remain an important design field in mechanical and structural
 417 engineering[10], including deployable and moving trusses[2].

418 **8. Utility for Robotics.** Starting twenty years ago, Sanderson[14], Hamlin,[8], Lee[9], and
 419 others created a style of robotics based on changing the lengths of members joined at the
 420 center of a joint, thereby creating a connection to pure geometry. More recently NASA has
 421 experimented with tensegrities[1], a different point in the same design spectrum. These fields
 422 create a need to explore the notion of geometries changing over time, not generally considered
 423 directly by pure geometry.

424 As suggested by Buckminster Fuller, the most convenient geometries to consider are those
 425 that have regular member lengths, in order to facilitate the inexpensive manufacture and
 426 construction of the robot. In a plane, the octet truss[4] is such a geometry, but in a line, the
 427 Boerdijk–Coxeter helix is a regular structure.

428 However, a robot must move, and so it is interesting to consider the transmutations of
 429 these geometries, which was in fact the motivation for creating the equitetrabeam.

430 **Theorem 5.** *By changing only the length of the longer members that connect two distinct*
 431 *rails (the two-hop members), we can dynamically untwist a tetrobot forming the Boerdijk-*
 432 *Coxeter configuration into the equitetrabeam which rests flat on the plane.*

²Before deriving Equation (7), we created a continuum by using a linear interpolation between the optimal radius for the Equitetrabeam and the BC Helix. This minimax optimum of this simpler approach was at most 1% worse than the optimum computed by (7).

433 *Proof.* Proof by our computer program that does this using Equation (6) applied to the
434 7-tet Tetrobot/Glussbot.

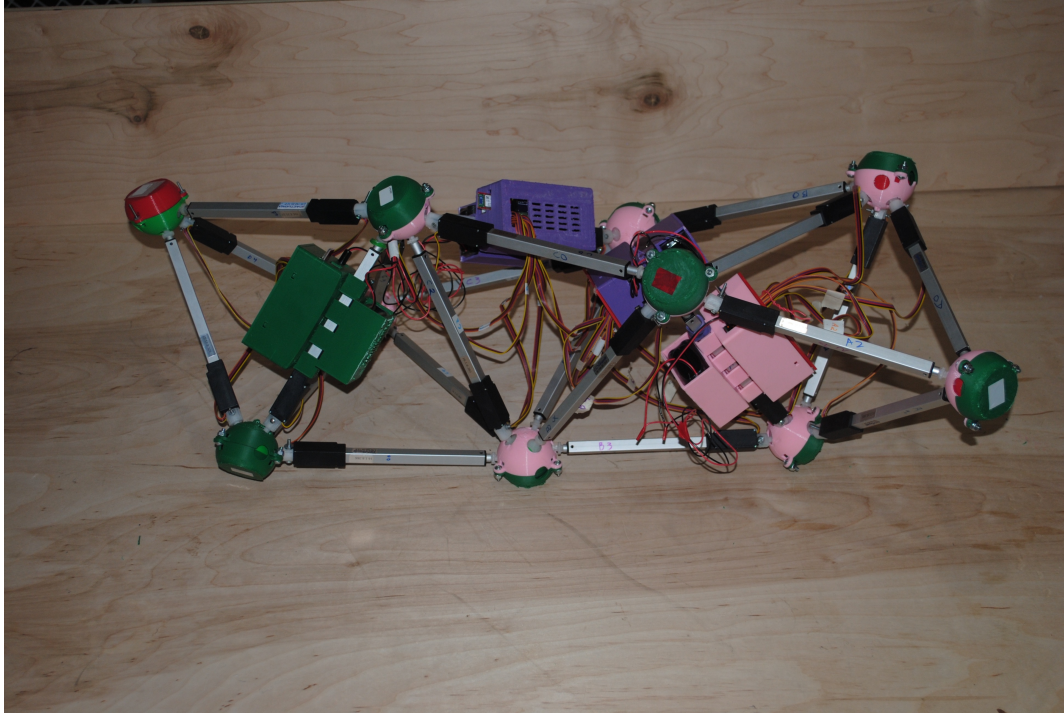


Figure 10. *Glussbot in relaxed, or BC helix configuration*

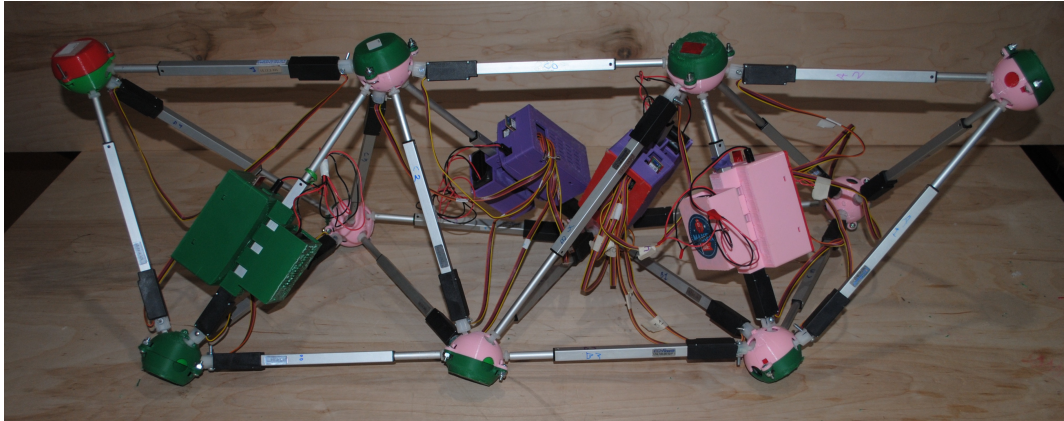


Figure 11. *The Equitetrabream: Fully Untwisted Glussbot in Hexapod Configuration*

435 By untwisting the tetrahelix so that it has a planar surface resting on the ground, we may
436 consider each vertex touching the ground a foot or pseudopod. A robot can thus become a
437 hexapod or n -pod robot, and the already well-developed approaches to hexapod gaits may be
438 applied to make the robot walk or crawl.

9. Conclusion. The BC Helix is the end point of a continuum of tetrahelices, the other end point being an untwisted tetrahelix with equilateral cross section, constructed by changing the length of only those members crossing the outside rails after hopping over the nearest vertex. Under the condition of minimum maximum length difference of all members in the system, all such tetrahelices have vertices evenly spaced along the axis generated by a simple equation. A mechanical machine, such as a robot or a variable-geometry truss, that can change the length of its members, can thus twist and untwist itself by changing the length of the appropriate members to achieve any point in the continuum optimally. With a numeric solution, a design may choose a rotation angle and member lengths to obtain any desired pitch.

10. Contact and Getting Involved. The Gluss Project <http://pubinv.github.io/gluss/> is part of Public Invention <https://pubinv.github.io/PubInv/>, a free-libre, open-source research, hardware, and software project that welcomes volunteers. It is our goal to organize projects for the benefit of all humanity without seeking profit or intellectual property. To assist, contact read.robert@gmail.com.

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