Semi-classical (2-spin) system with Harmonic oscillator

We have: $H_c = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\hat{H}_q = \frac{\omega_s}{2}\left(\sigma_z \otimes I + I \otimes \sigma_z\right) + \frac{g_1}{2}\left(\alpha\sigma_+ + \alpha^*\sigma_-\right)\otimes I + \frac{g_2}{2} I \otimes \left(\alpha\sigma_+ + \alpha^*\sigma_-\right) + \frac{\lambda}{2}\left(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+\right)$$

where $\alpha \equiv \frac{x + ip}{\sqrt{2\omega}}$

$\dot{x} = \{x, H_{eff}\}$, $\dot{p} = \{p, H_{eff}\}$, with $H_{eff} \equiv H_c + \langle\psi|\hat{H}_q|\psi\rangle$

and $i|\dot{\psi}\rangle = \hat{H}_q|\psi\rangle$

$$\dot{x} = \{x, H_{eff}\} = \frac{\partial x}{\partial x}\frac{\partial H_{eff}}{\partial p} + 0 \;/\; \frac{\partial H_{eff}}{\partial p} = \frac{p}{m} + \langle\psi|\frac{\partial \hat{H}_q}{\partial p}|\psi\rangle .$$

$$\frac{\partial \hat{H}_q}{\partial p} = \frac{g_1}{2}\left(\frac{i}{\sqrt{2\omega}}\sigma_+ - \frac{i}{\sqrt{2\omega}}\sigma_-\right)\otimes I + I\otimes\frac{g_2}{2}\left(\frac{i}{\sqrt{2\omega}}\sigma_+ - \frac{i}{\sqrt{2\omega}}\sigma_-\right)$$

$$= \frac{i}{\sqrt{8\omega}}\left[g_1(\sigma_+ - \sigma_-)\otimes I + g_2 I\otimes(\sigma_+ - \sigma_-)\right]$$

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$$\dot{p} = \{p, H_{eff}\} = 0 - \frac{\partial H_{eff}}{\partial x} = -m\omega^2 x - \langle\psi|\frac{\partial \hat{H}_q}{\partial x}|\psi\rangle$$

$$\frac{\partial \hat{H}_q}{\partial x} = \frac{g_1}{2}\left(\frac{1}{\sqrt{2\omega}}\sigma_+ + \frac{1}{\sqrt{2\omega}}\sigma_-\right)\otimes I + I\otimes\frac{g_2}{2}\left(\frac{1}{\sqrt{2\omega}}\sigma_+ + \frac{1}{\sqrt{2\omega}}\sigma_-\right)$$

$$= \frac{1}{\sqrt{8\omega}}\left[g_1(\sigma_+ + \sigma_-)\otimes I + I\otimes g_2(\sigma_+ + \sigma_-)\right]$$

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$S_{ent} = -\mathrm{Tr}\left[\hat{\rho}_R \ln(\hat{\rho}_R)\right]$, where $\rho_R = \left(\sum_{i=1}^{2} \rho_i\right)\hat{\rho}_{3,4}$