$(X, \mathcal{T})$ I + II + III

topology

Members of $\mathcal{T}$ are open sets

$A \subseteq X$

$A \in \mathcal{T}$

A^c

$(A^c)^c = A$

closed sets

$(X, \mathcal{T})$

$\boxed{A \subseteq X}$ is closed if A^c is open OR $A^c \in \mathcal{T}$

$A \in \mathcal{T}$

A^c is closed sets

$(A^c)^c = A \in \mathcal{T}$

Ex $X = \{a, b, c, d, e\}$

$\mathcal{T} = \{\phi, X, \{a, b\}, \{c\}, \{a, b, c\}\}$

closed sets $X \quad \phi \quad \{c, d, e\} \quad \{a, b, d, e\} \quad \{d, e\}$

* $X = \{a, b, c, d, e\}$

$\mathcal{T} = \{\phi, X, \{a\}\}$

closed sets, $X, \phi, \{b, c, d, e\}$

Ex

$X = \{a, b, c\}$

$\mathcal{T} = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$

closed sets. $X, \phi, \{b,c\}, \{a,c\}, \{a,b\}, \{c\}, \{a\}, \{b\}\}$

open sets $\equiv$ closed sets

$(X, \mathcal{T}_{CF}) \longrightarrow$ cofinite topology

$$\mathcal{T}_{CF} = \{ \phi, G \subseteq X ; G^c \text{ is finite} \}$$

closed sets :

$A \subseteq X$ is closed

$\Downarrow$

$A^c \in \mathcal{T}$

$\Downarrow$

$(A^c)^c$ is finite $\Rightarrow$ A is finite

A is finite

$\Downarrow$

$A^c \in \mathcal{T}$

$\Downarrow$

$(A^c)^c = A$

$\downarrow$

finite

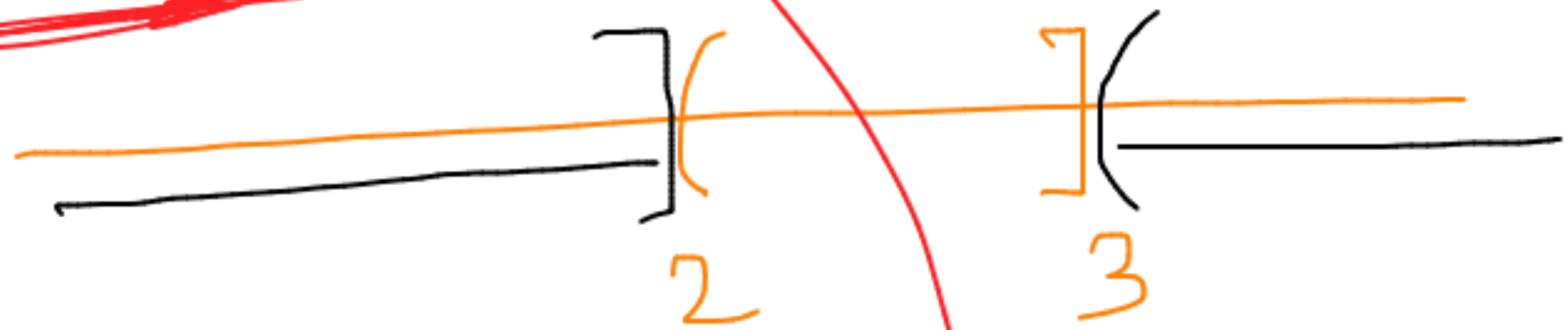
$(X, \mathcal{T}_{cf})$ A is closed $\Leftrightarrow$ A is finite

$(X, \mathcal{T}_{cc})$ A is closed $\Leftrightarrow$ A is countable

$X = \mathbb{R}$

$\mathcal{U}$	$\mathcal{T}_\ell$	$\mathcal{T}_U$	
	$(-\infty, a) \cup (b, \infty)$		$A = [a, b]$ closed

$A = (2,3]$



$A^c = (-\infty, 2] \cup (3, \infty)$

is not open in $\mathcal{U}$

A is not open in $\mathcal{T}_l$

$A^c \in \mathcal{T}_{\mathcal{U}}$

A is closed

& ~~open~~

$A = [a,b]$

open

$(\mathbb{R}, \mathcal{T})$

$\mathcal{D} = P(\mathbb{R})$

$= \{A : A \subseteq \mathbb{R}\}$

$(\mathbb{R}, \mathcal{U})$

$(2,3)$

$\mathcal{T}$

open sets

(i) ϕ & X are open

(ii) Arbitrary union of open sets is open

(iii) Finite intersection of open sets is open

closed

(i) ϕ, X are closed sets

(ii) Arbitrary intersection of closed sets is closed

(iii) Finite union of closed sets is closed

(X, d)

metric space

$d : X \times X \to \underline{R}$

topology

Base

$X = \{a, b, c\}$

$X = \underline{R}$

$X = \mathbb{R}^2$

$\mathbb{R}^3$

Basis

Metric topology:

(X, d) } metric space

$\downarrow$

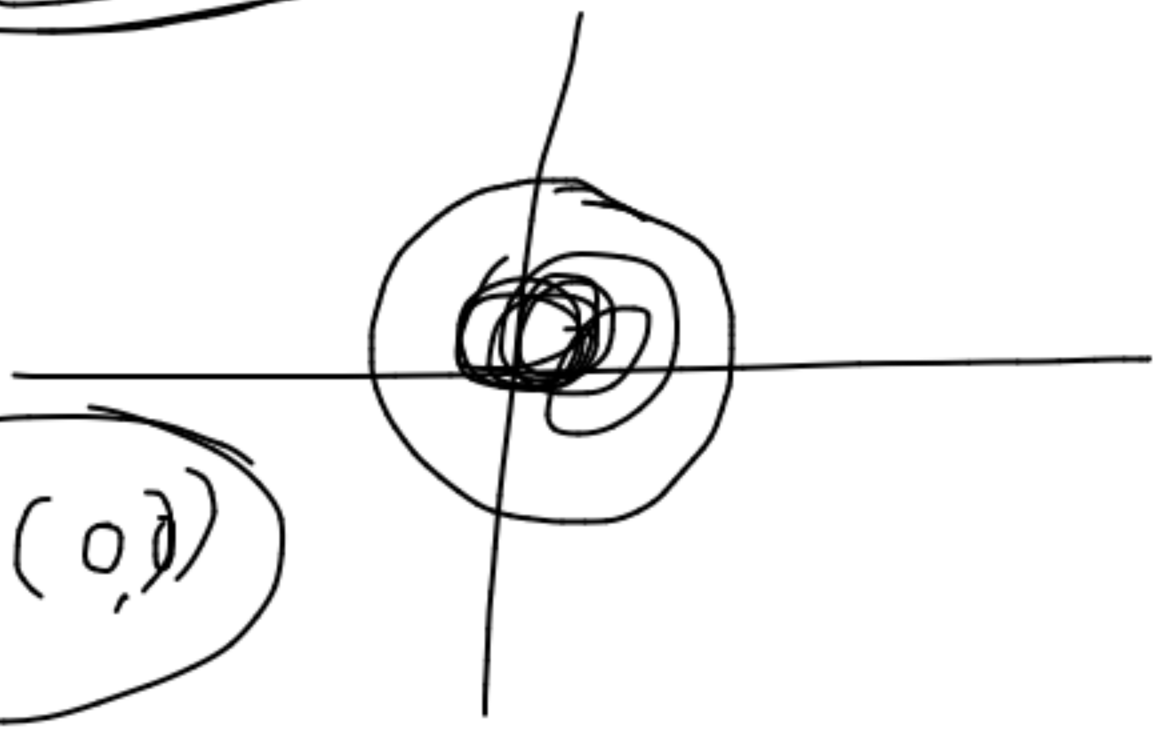
$(X, \mathcal{T})$ $\hookrightarrow$ metric topology

$r > 0$

$S_r(x) = \{y \in X : d(x, y) < r\}$

$r = 1$

$S_1(0,0)$



Discrete Metric

(X, d)

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

$S_{\frac{1}{2}}(x) = ?$

$= \{x\}$

$S_{\frac{1}{2}}(x) = \{y : d(x,y) < \tfrac{1}{2}\}$

$= \{y : d(x,y) = 0\}$

$= \{x\}$

Usual Metric on R

$$d(x, y) = |x - y|$$

$S_r(x) = ?$

$$d(x,y) = |x-y|$$

$$S_r(x) = \{ y \in X ; d(x,y) < r \}$$

$$= \{ y \in X ; \underline{|x-y| < r} \}$$

$$= \{ \underline{y \in X} ; \underline{x-r} < y < \underline{x+r} \}$$

$$\boxed{S_r(x) = (x-r, x+r)}$$

Metric topology $\quad x \in S_r(x) \subseteq S_r(x)$ $\quad S_r(x) \in \mathcal{T}$

For a metric space (X, d),

$$\mathcal{T} = \{ G \subseteq X : \forall x \in G \; \exists r > 0 \text{ s.t } x \in S_r(x) \subseteq G \}$$

(1) is a topology on X and called metric topology $=$

(X, d) $\quad d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y, \end{cases}$ $\quad S_{\frac{1}{2}}(x) = \{x\}$

$\mathcal{T} = \{ \{x\}, \{y\} \ldots \}$ Discrete topology

Discrete Metric $\leadsto$ induces discrete topology

(R, d) $\qquad d(x, y) = |x - y|$ $\qquad S_r(x) = (x - \epsilon, x + \epsilon)$

$X = R$

$\mathcal{T} = \{ G \subseteq R : \forall x \in G \ \exists r > 0$

Usual Metric $\qquad x \in (x - r, x + r) \subseteq G \}$

$\Downarrow$

Usual topology

$$\Rightarrow \begin{array}{c} (X, d) \\ \downarrow \\ (X, \mathcal{T}) \end{array} \qquad \begin{array}{c} (X, \mathcal{T}) \\ \downarrow \\ (X, d) \end{array}$$

(I) Example

(II) Proof. $\mathcal{T}$ is a topol