Connectivity in Digraphs

Menger's Theorem, Whitney's Theorem, Network flow

We were discussing:

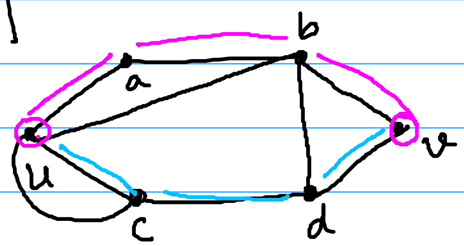
Vertex cut $\longrightarrow$ separating set $S \subseteq V(G)$, $\kappa(G)$
Edge cut $\longrightarrow$ Disconnected set $F \subseteq E(G)$, $\kappa'(G)$

Result (for simple graph)

$$\kappa(G) \leq \kappa'(G) \leq \delta(G)$$

We had k- connectivity (k- connected)

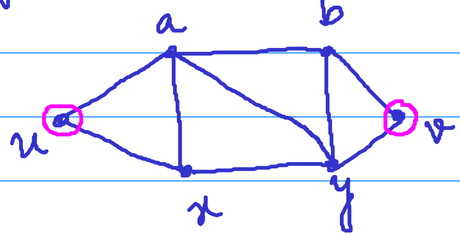
Def. Paths from u to v (uv — non-adjacent) are said to be internally disjoint, if they have no common internal vertices.

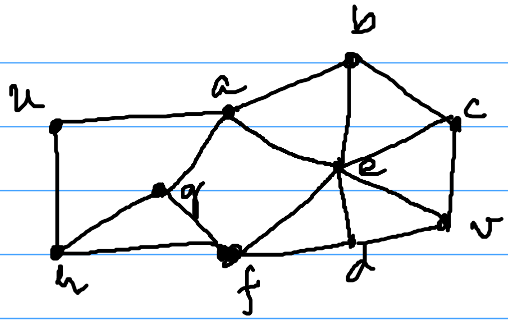


We have following:

1) Separating set $|S|$ its cardinality
2) k- connected
3) Internally disjoint paths.

1. Menger's Theorem:





<u>Statement</u> :- Let u & V be non-adjacent vertices in G. The least no. of vertices in u-v separating set is equal to maximum no. of internally disjoint uv-paths.

Whitney's theorem:

A graph G with at least 3 vertices is 2-connected iff ~~for~~ each pair (uv) of vertices of V(G) there is internally disjoint uv-path in G.

<u>General form</u>:

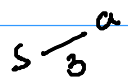
G is k-connected ($k \geq 2$) iff each pair (uv) of distinct vertices $\exists$ at least k-internally disjoint uv paths.

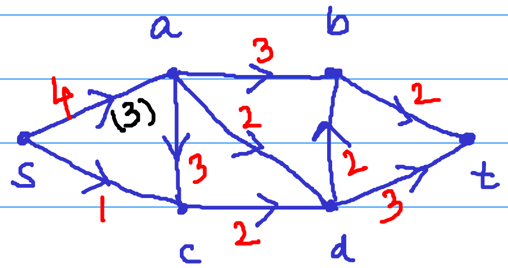
2. Connectivity of Digraphs:

Network flow / flow network

A flow network is directed graph $G(V, E)$ where each edge $e(u v) \in E$ has a capacity $c(u, v) \geq 0$.

A flow network has source vertex $\rightarrow s$
sink vertex $\rightarrow t$





$\rightarrow$ Network.

Node — internal Vertex

Flow:

A flow in a network G is a real valued function:

$$f : V \times V \longrightarrow \mathbb{R}$$

$\left\{ \begin{array}{l} f^{+}{}_{v} \longrightarrow \text{edge leaving } v \quad (\text{outgoing}) \\ f^{-}{}_{v} \longrightarrow \text{edge entering } v \quad (\text{incoming}) \end{array} \right.$

such that

1. $f(u, v) \leq c(u, v)$ {flow can't exceed the capacity

2. $f(u, v) = -f(v, u)$ {Skew symmetry

3. Flow conservation

$$\sum_{u \in V} f(u, v) = \sum_{w \in V} f(v, w)$$

Feasible flow / valid flow

A flow is valid / feasible

if it satisfy capacity constraint

$$0 \leq f(e) \leq c(e)$$

as well as the conservation constraint

$$f^{+}(v) = f^{-}(v)$$ at each node