Notes on Superconductivity Precursory to Relativistic Discussion

Amal Bumbia

PHY 352L Classical Electrodynamics II
Department of Physics
The University of Texas at Austin
Austin, TX 78712, USA

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1 Historical Overview

Superconductivity refers to a macroscopic quantum phenomenon observed in materials known as superconductors. Key characteristics of superconductors include vanishing electrical resistance when they are cooled below a critical temperature and the expulsion of internal magnetic flux. In 1911, Heike Kamerlingh Onnes was the first to observe superconductivity when he cooled mercury below 4.2 K, resulting in its resistivity decreasing to zero. The uncommon aspect of this discovery was not that the resistance decreased with temperature, but that there was a critical temperature (T_c) after which the resistance sharply dropped to zero. The presence of a critical temperature is one distinguishing feature between perfect conductors and superconductors. Another is the Meissner effect — Fritz Meissner and Robert Ochsenfeld discovered in 1933 that when a superconductor is placed in a magnetic field and cooled past its critical temperature, the internal magnetic flux is expelled. It was initially assumed that materials needed to be cooled to a critical temperature below 10 K (note that liquid helium exists at temperatures around 4.15 K), however, in 1986, it was discovered that some cuprate-perovskite ceramic materials have a higher critical temperature, for instance, around or above 90 K. In 1987, the compound $YBa_2Cu_3O_7$ (YCBO) was found to have a critical temperature of around 93 K. Such "high-temperature superconductors" notably had critical temperatures above the boiling point of liquid nitrogen, which is 77 K. This meant that for practical purposes, liquid nitrogen could be used as a refrigerant over liquid helium. Though Fritz and Heinz London attempted to explain superconductivity through the London constitutive equations in 1935, the first microscopic theory of superconductivity was not established until 1957 by John Bardeen, Leon Cooper, and John Schrieffer.

2 London Equations

Before we discuss any microscopic theories, it is important to go over the London constitutive equations. Typically, Ohm's law ($j = \sigma E$) is used to describe electrical conduction in the normal state of a metal. Can we use modify it to describe conduction and the Meissner effect in the superconducting state? Yes we can! We begin with the assumption that in the superconducting state, the current density j is proportional to the vector potential A of the local magnetic field $B = \nabla \times A$. In CGS units, the London equation is

$$j = \frac{-c}{4\pi\lambda_L^2} A$$

or alternatively,

$$\nabla \times j = \frac{-c}{4\pi\lambda_L^2} B$$

which clearly arises with the application of the curl onto the first. Here, we are generalizing the results that one would obtain by beginning the derivation with the equation of motion for a charge carrier. The vector potential A is ascribed the "London gauge" where $\nabla \cdot A = 0$ and $A_n = 0$, the normal component of the potential to the surface through which no external current is fed. From Maxwell's equations, we take $\nabla \times B = \frac{4\pi}{c} j$ and apply a curl on both sides before combining the resulting equation with the second London equations. Thus,

$$\nabla \times \nabla \times B = -\nabla^2 B = \frac{4\pi}{c} \nabla \times j \rightarrow \nabla^2 B = \frac{B}{\lambda_L^2}$$

Regarding this result, $B(r) = B_0$ is not a solution unless the constant $B_0 = 0$. This explains the Meissner effect! To complete this description, note that the solution is $B(r) = B_0 e^{-r/\lambda_L}$ where

$$\lambda_L = (\frac{mc^2}{4\pi n q^2})^{1/2}$$

is the London penetration depth — the depth at which the magnetic field can penetrate a superconductor. This implies that as B penetrates the superconductor, it decreases exponentially.

3 A Largely Qualitative Overview of BCS Theory

The BCS theory by Bardeen, Cooper, and Schrieffer explains superconductivity via electron pairs (called "Cooper pairs") that travel through the lattice of a superconductor without scattering. Because electron scattering within a lattice is a source of electrical resistance (which is typically described by Ohm's law as $R = V/I$ – yes, this is also an Ohm's law), the implication is that Cooper pairs can travel with zero resistance. The movement of Cooper pairs through the lattice of a superconducting material has the effect of generating a persistent current that circulates in the absence of external electric fields.

The electrons comprising Cooper pairs (FIG 1) have opposite spins (+1/2 and -1/2), so Cooper pairs have a net spin of zero and are thus bosons obeying Bose-Einstein statistics. The reason electrons can form Cooper pairs despite the expectation that they would repel each other has to do with phonons (lattice vibrations) enabling attraction between them. The consequent attractive pairwise interaction between electrons enables the formation of bound Cooper pairs near the Fermi surface. Many Cooper pairs can occupy the same ground state and can be described via a wavefunction extending over macroscopic dimensions, hence why superconductivity is a "macroscopic quantum phenomena" (they form a Bose-Einstein condensate).

As a visual for understanding the form of the interaction which yields Cooper pairs, imagine a ring. This ring will be the thin shell surrounding our Fermi surface and have a width proportional to the Debye frequency w_D. Within this ring, a net attractive interaction between pairs of electrons is dominant. Consider a two-electron state $k\uparrow, -k\downarrow$ where the particles have effectively opposite momentum and spin. Scattering due to phonon exchange results in a transfer of momentum p in such a manner that momentum is conserved and results in a new state $(k+p)\uparrow, -(k+p)\downarrow$. The BCS Hamiltonian essentially describes this physics. In this formalism, the critical temperature below which superconductivity is realized can be written as $T_c = w_D e^{-\frac{1}{gv}}$ where g is the electron-phonon interaction strength and v is the density of states at the fermi level. At this point, the Cooper pair binding becomes stronger and is necessary to account for. This is where the Bogoliubov-de Gennes (BdG) extension comes in. We begin by assuming that below the critical temperature, the ground state of the system, Ω_s contains a macroscopic number of Cooper pairs, meaning that adding another shouldn't change much: $c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} \Omega_s \approx \Omega_s$ where the operator $c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow} \Omega_s$ creates a bosonic excitation. Then, the following expectation values will be nonzero:

$$\Delta = \frac{g}{L^d} \sum_k \Omega_s c_{-k\downarrow} c_{k\uparrow} \Omega_s$$

$$\Delta' = \frac{g}{L^d} \sum_k \Omega_s c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \Omega_s$$

which is consistent with the formation of a Bose-Einstein condensate. Below T_c, $\Delta > 0$ and above it, $\Delta = 0$, meaning that it is an order parameter of the superconducting transition. From here, it is possible to obtain the BdG Hamiltonian which results in a proper description of Ω_s as well as the conclusion that T_c is on the order of Δ, which can now be understood to represent an energy gap between Bogoliubov quasi-particles created by superpositions of $c^\dagger_{k\uparrow}$ and $c_{-k\downarrow}$. In particular, T_c and Δ coincide at $T = 0$.

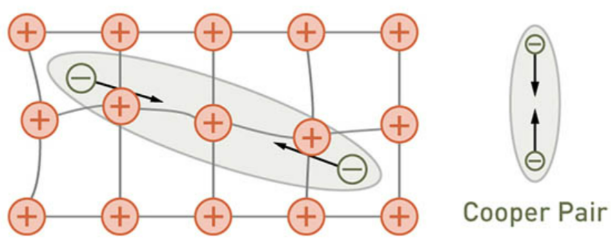

Figure 1: Schematic of Cooper pairs.

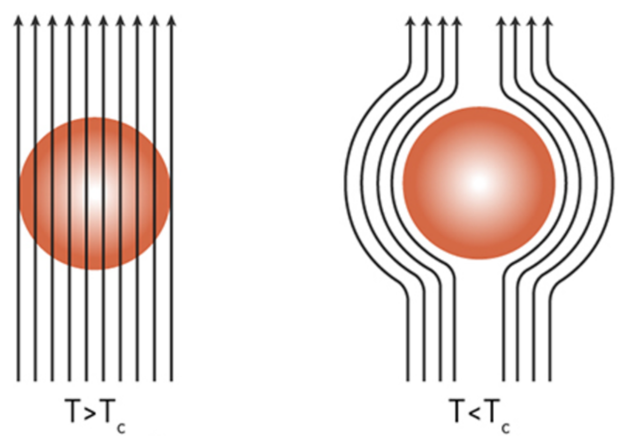

Figure 2: Schematic of magnetic field lines interacting with a sample above and below its T_c — the rightmost graphic shows the Meissner effect.

The energy of Cooper pairs lies below the Fermi energy, making such states energetically favorable and also stable if the gap between the aforementioned energies is greater than the thermal energy of the electrons. A significant number of electrons are bound in Cooper pairs only at low enough temperatures since the energy of the pairing interaction is of the order of 10^3 eV. Thus, the superconductivity of a material is not apparent at temperatures above the material's critical temperature — thermal energy breaks apart Cooper pairs.

Because BCS theory was formulated with conventional type-I superconductors in mind, it may not fully account for high-Tc superconductivity. High-temperature superconductors are type-II superconductors. The difference between type I and type II superconductors lies in the observation of the Meissner effect as shown in FIG 2 and FIG 3.

4 Meissner Effect

As mentioned previously, the Meissner effect is the expulsion of internal magnetic fields as the superconductor cools past its critical temperature. In the case of high-Tc superconductors, a magnet placed nearby can levitate, providing a visualization of the Meissner effect. The induced surface currents produce a magnetic field that cancels with the external field in the interior of the superconductor, and there is another magnetic field outside of the superconductor opposing that of the magnet that results in a lift force. This is the complete Meissner effect as experienced by type I superconductors. Above the critical field H_c, superconductivity breaks. In type II superconductors, the Meissner effect is incomplete in the vortex state since the flux penetrates the material at a field H_{c_1} less than H_c. Above a greater magnetic field denoted H_{c_2}, the sample's electrical properties resemble that of a normal conductor as the flux penetrates the material completely, and in between H_{c_1} and H_{c_2}, the sample is in a stable vortex state, a mixed state.

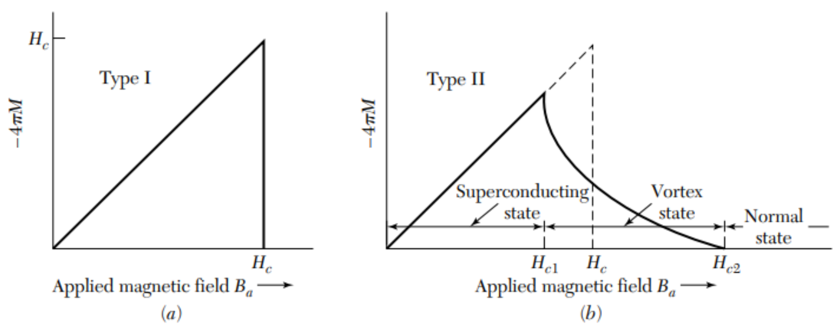

Figure 3: Type I versus Type II superconductors.

The term vortex state describes the circulation of superconducting currents in vortices throughout the material. Flux pinning refers to when these vortices are prevented from moving within the bulk of the material, so magnetic field lines are "pinned" in those regions. High-temperature superconductors tend to be type II.

Acknowledgements

These notes and the rest of the notebooks were compiled with help from Atland and Simon's Many Body Theory [1], Kittel's Solid State, [5], my experimental report [2], and some literature on the relativistic theory of superconductivity [3] [4] [6].

References

[1] Alexander Altland and Ben D. Simons. *Condensed Matter Field Theory*. 2nd ed. Cambridge: Cambridge University Press, 2010. DOI: 10.1017/CBO9780511789984. URL: https://www.cambridge.org/core/books/condensed-matter-field-theory/0A8DE6503ED868D96985D9E7847C63FF (visited on 05/04/2025).

[2] Amal Bumbia. *Demonstration of Zero Resistivity and the Meissner Effect with High Tc Superconductors*. Tech. rep. The University of Texas at Austin, Nov. 2024.

[3] K. Capelle, M. A. L. Marques, and E. K. U. Gross. *Relativistic Theory of Superconductivity*. arXiv:cond-mat/0101405. Jan. 2001. DOI: 10.48550/arXiv.cond-mat/0101405. URL: http://arxiv.org/abs/cond-mat/0101405 (visited on 05/04/2025).

[4] A. Ciechan, A. Donabidowicz, and K.I. Wysokiński. "Relativistic Theory of Superconductivity: BCS vs. Bogolyubov-de Gennes Approaches". en. In: *Acta Physica Polonica A* 106.5 (Nov. 2004), pp. 699–702. ISSN: 0587-4246, 1898-794X. DOI: 10.12693/APhysPolA.106.699. URL: http://przyrbwn.icm.edu.pl/APP/PDF/106/a106z519.pdf (visited on 05/04/2025).

[5] Charles Kittel. *Introduction to solid state physics*. en. 8. ed., [repr.] Hoboken, NJ: Wiley. ISBN: 978-0-471-41526-8.

[6] M. Marques, K. Capelle, and E. K. U. Gross. "Theory of relativistic effects in superconductors". In: *Physica C: Superconductivity* 317-318 (May 1999), pp. 508–510. ISSN: 0921-4534. DOI: 10.1016/S0921-4534(99)00111-2. URL: https://www.sciencedirect.com/science/article/pii/S0921453499001112 (visited on 05/04/2025).