CHAPTER 5
Counting

SECTION 5.1 The Basics of Counting

2. By the product rule there are $27 \cdot 37 = 999$ offices.

4. By the product rule there are $12 \cdot 2 \cdot 3 = 72$ different types of shirt.

6. By the product rule there are $4 \cdot 6 = 24$ routes.

8. There are 26 choices for the first initial, then 25 choices for the second, if no letter is to be repeated, then 24 choices for the third. (We interpret "repeated" broadly, so that a string like RWR, for example, is prohibited, as well as a string like RRW.) Therefore by the product rule the answer is $26 \cdot 25 \cdot 24 = 15{,}600$.

10. We have two choices for each bit, so there are $2^8 = 256$ bit strings.

12. We use the sum rule, adding the number of bit strings of each length up to 6. If we include the empty string, then we get $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$ (using the formula for the sum of a geometric progression—see Example 4 in Section 4.1).

14. If $n = 0$, then the empty string—vacuously—satisfies the condition (or does not, depending on how one views it). If $n = 1$, then there is one, namely the string 1. If $n \geq 2$, then such a string is determined by specifying the $n - 2$ bits between the first bit and the last, so there are 2^{n-2} such strings.

16. We can subtract from the number of strings of length 4 of lower case letters the number of strings of length 4 of lower case letters other than x. Thus the answer is $26^4 - 25^4 = 66{,}351$.

18. Because neither 5 nor 31 is divisible by either 3 or 4, whether the ranges are meant to be inclusive or exclusive of their endpoints is moot.

a) There are $\lfloor 31/3 \rfloor = 10$ integers less than 31 that are divisible by 3, and $\lfloor 5/3 \rfloor = 1$ of them is less than 5 as well. This leaves $10 - 1 = 9$ numbers between 5 and 31 that are divisible by 3. They are 6, 9, 12, 15, 18, 21, 24, 27, and 30.

b) There are $\lfloor 31/4 \rfloor = 7$ integers less than 31 that are divisible by 4, and $\lfloor 5/4 \rfloor = 1$ of them is less than 5 as well. This leaves $7 - 1 = 6$ numbers between 5 and 31 that are divisible by 4. They are 8, 12, 16, 20, 24, and 28.

c) A number is divisible by both 3 and 4 if and only if it is divisible by their least common multiple, which is 12. Obviously there are two such numbers between 5 and 31, namely 12 and 24. We could also work this out as we did in the previous parts: $\lfloor 31/12 \rfloor - \lfloor 5/12 \rfloor = 2 - 0 = 2$. Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

20. a) Every seventh number is divisible by 7. Therefore there are $\lfloor 999/7 \rfloor = 142$ such numbers. Note that we use the floor function, because the k^{th} multiple of 7 does not occur until the number $7k$ has been reached.

b) For solving this part and the next four parts, we need to use the principle of inclusion–exclusion. Just as in part (a), there are $\lfloor 999/11 \rfloor = 90$ numbers in our range divisible by 11, and there are $\lfloor 999/77 \rfloor = 12$ numbers in our range divisible by both 7 and 11 (the multiples of 77 are the numbers we seek). If we take these 12 numbers away from the 142 numbers divisible by 7, we see that there are 130 numbers in our range divisible by 7 but not 11.

c) As explained in part (b), the answer is 12.

d) By the principle of inclusion–exclusion, the answer, using the data from part (b), is $142 + 90 - 12 = 220$.

e) If we subtract from the answer to part (d) the number of numbers divisible by both 7 and 11, we will have the number of numbers divisible by neither of them; so the answer is $220 - 12 = 208$.

f) If we subtract the answer to part (d) from the total number of positive integers less than 1000, we will have the number of numbers divisible by exactly one of them; so the answer is $999 - 220 = 779$.

g) If we assume that numbers are written without leading 0's, then we should break the problem down into three cases—one-digit numbers, two-digit numbers and three-digit numbers. Clearly there are 9 one-digit numbers, and each of them has distinct digits. There are 90 two-digit numbers (10 through 99), and all but 9 of them have distinct digits, so there are 81 two-digit numbers with distinct digits. An alternative way to compute this is to note that the first digit must be 1 through 9 (9 choices), and the second digit must be something different from the first digit (9 choices out of the 10 possible digits), so by the product rule, we get $9 \cdot 9 = 81$ choices in all. This approach also tells us that there are $9 \cdot 9 \cdot 8 = 648$ three-digit numbers with distinct digits (again, work from left to right—in the ones place, only 8 digits are left to choose from). So the final answer is $9 + 81 + 648 = 738$.

h) It turns out to be easier to count the odd numbers with distinct digits and subtract from our answer to part (g), so let ns proceed that way. There are 5 odd one-digit numbers. For two-digit numbers, first choose the ones digit (5 choices), then choose the tens digit (8 choices), since neither the ones digit value nor 0 is available); therefore there are 40 such two-digit numbers. (Note that this is not exactly half of 81.) For the three-digit numbers, first choose the ones digit (5 choices), then the hundreds digit (8 choices), then the tens digit (8 choices, giving us 320 in all. So there are $5 + 40 + 320 = 365$ odd numbers with distinct digits. Thus the final answer is $738 - 365 = 373$.

22. It will be useful to note first that there are exactly 9000 numbers in this range.

a) Every ninth number is divisible by 9, so the answer is one ninth of 9000 or 1000.

b) Every other number is even, so the answer is one half of 9000 or 4500.

c) We can reason from left to right. There are 9 choices for the first (left-most) digit (since it cannot be a 0), then 9 choices for the second digit (since it cannot equal the first digit), then, in a similar way, 8 choices for the third digit, and 7 choices for the right-most digit. Therefore there are $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ ways to specify such a number. In other words, there are 4536 such numbers. Note that this coincidentally turns out to be almost exactly half of the numbers in the range.

d) Every third number is divisible by 3, so one third of 9000 or 3000 numbers in this range are divisible by 3. The remaining 6000 are not.

e) For this and the next three parts we need to note first that one fifth of the numbers in this range, or 1800 of them, are divisible by 5, and one seventh of them, or 1286 are divisible by 7. [This last calculation is a little more subtle than we let on, since 9000 is not divisible by 7 (the quotient is $1285.71\ldots$). But 1001 is divisible by 7, and $1001 + 1285 \cdot 7 = 9996$, so there are indeed 1286, and not 1285 such multiples. (By contrast, in the range 1002 to 10001, inclusive, which also includes 9000 numbers, there are only 1285 multiples of 7.)] We also need to know how many of these numbers are divisible by both 5 and 7, which means divisible by 35. The answer, by the similar reasoning, is 257, namely those multiples from $29 \cdot 35 = 1015$ to $285 \cdot 35 = 9975$.

(One more note: We could also have come up with these numbers more formally, using the ideas in Section 7.5, especially Example 2. We could find the number of multiples less than 10,000 and subtract the number of multiples less than 1000.) Now to the problem at hand. The number of numbers divisible by 5 or 7 is the number of numbers divisible by 5, plus the number of numbers divisible by 7, minus (because of having overcounted) the number of numbers divisible by both. So our answer is $1800 + 1286 - 257 = 2829$.

f) Since we just found that 2829 of these numbers are divisible by either 5 or 7, it follows that the rest of them, $9000 - 2829 = 6171$, are not.

g) We noted in the solution to part (e) that 1800 numbers are divisible by 5, and 257 of these are also divisible by 7. Therefore $1800 - 257 = 1543$ numbers in our range are divisible by 5 but not by 7.

h) We found this as part of our solution to part (e), namely 257.

24. **a)** There are 10 ways to choose the first digit, 9 ways to choose the second, and so on; therefore the answer is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

b) There are 10 ways to choose each of the first three digits and 5 ways to choose the last; therefore the answer is $10^3 \cdot 5 = 5000$.

c) There are 4 ways to choose the position that is to be different from 9, and 9 ways to choose the digit to go there. Therefore there are $4 \cdot 9 = 36$ such strings.

26. $10^3 26^3 + 26^3 10^3 = 35{,}152{,}000$

28. $26^3 10^3 + 26^4 10^2 = 63{,}273{,}600$

30. **a)** By the product rule, the answer is $26^8 = 208{,}827{,}064{,}576$.

b) By the product rule, the answer is $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 62{,}990{,}928{,}000$.

c) This is the same as part (a), except that there are only seven slots to fill, so the answer is $26^7 = 8{,}031{,}810{,}176$.

d) This is similar to (b), except that there is only one choice in the first slot, rather than 26, so the answer is $1 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 2{,}422{,}728{,}000$.

e) This is the same as part (c), except that there are only six slots to fill, so the answer is $26^6 = 308{,}915{,}776$.

f) This is the same as part (e); again there are six slots to fill, so the answer is $26^6 = 308{,}915{,}776$.

g) This is the same as part (f), except that there are only four slots to fill, so the answer is $26^4 = 456{,}976$. We are assuming that the question means that the legal strings are BO????BO, where any letters can fill the middle four slots.

h) By part (f), there are 26^6 strings that start with the letters BO in that order. By the same argument, there are 26^6 strings that end that way. By part (g), there are 26^4 strings that both start and end with the letters BO in that order. Therefore by the inclusion–exclusion principle, the answer is $26^6 + 26^6 - 26^4 = 617{,}374{,}576$.

32. In each case the answer is n^{10}, where n is the number of elements in the codomain, since there are n choices for a function value for each of the 10 elements in the domain.

a) $2^{10} = 1024$ **b)** $3^{10} = 59{,}049$ **c)** $4^{10} = 1{,}048{,}576$ **d)** $5^{10} = 9{,}765{,}625$

34. There are 2^n such functions, since there is a choice of 2 function values for each element of the domain.

36. By our solution to Exercise 37, the answer is $(n+1)^5$ in each case, where n is the number of elements in the codomain.

a) $2^5 = 32$ **b)** $3^5 = 243$ **c)** $6^5 = 7776$ **d)** $10^5 = 100{,}000$

38. We know that there are 2^{100} subsets in all. Clearly 101 of them do not have more than one element, namely the empty set and the 100 sets consisting of 1 element. Therefore the answer is $2^{100} - 101 \approx 1.3 \times 10^{30}$.

40. **a)** We first place the bride in any of the 6 positions. Then, from left to right in the remaining positions, we choose the other five people to be in the picture; this can be done in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$ ways. Therefore the answer is $6 \cdot 15120 = 90{,}720$.

b) We first place the bride in any of the 6 positions, and then place the groom in any of the 5 remaining positions. Then, from left to right in the remaining positions, we choose the other four people to be in the picture; this can be done in $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways. Therefore the answer is $6 \cdot 5 \cdot 1680 = 50{,}400$.

c) From part **(a)** there are 90720 ways for the bride to be in the picture. There are (from part **(b)**) 50400 ways for both the bride and groom to be in the picture. Therefore there are $90720 - 50400 = 40320$ ways for just the bride to be in the picture. Symmetrically, there are 40320 ways for just the groom to be in the picture. Therefore the answer is $40320 + 40320 = 80{,}640$.

42. There are 2^5 strings that begin with two 0's (since there are two choices for each of the last five bits). Similarly there are 2^4 strings that end with three 1's. Furthermore, there are 2^2 strings that both begin with two 0's and end with three 1's (since only bits 3 and 4 are free to be chosen). By the inclusion–exclusion principle, there are $2^5 + 2^4 - 2^2 = 44$ such strings in all.

44. First we count the number of bit strings of length 10 that contain five consecutive 0's. We will base the count on where the string of five or more consecutive 0's starts. If it starts in the first bit, then the first five bits are all 0's, but there is free choice for the last five bits; therefore there are $2^5 = 32$ such strings. If it starts in the second bit, then the first bit must be a 1, the next five bits are all 0's, but there is free choice for the last four bits; therefore there are $2^4 = 16$ such strings. If it starts in the third bit, then the second bit must be a 1 but the first bit and the last three bits are arbitrary; therefore there are $2^4 = 16$ such strings. Similarly, there are 16 such strings that have the consecutive 0's starting in each of positions four, five, and six. This gives us a total of $32 + 5 \cdot 16 = 112$ strings that contain five consecutive 0's. Symmetrically, there are 112 strings that contain five consecutive 1's. Clearly there are exactly two strings that contain both (0000011111 and 1111100000). Therefore by the inclusion–exclusion principle, the answer is $112 + 112 - 2 = 222$.

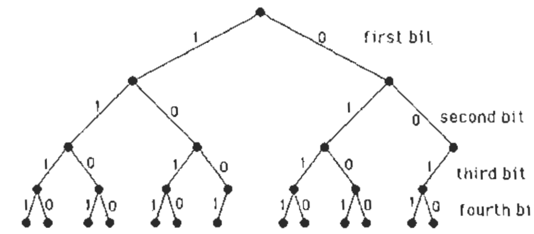
46. This is a straightforward application of the inclusion–exclusion principle: $38+23-7 = 54$ (we need to subtract the 7 double majors counted twice in the sum).

48. Order matters here, since the initials RSZ, for example, are different from the initials SRZ. By the sum rule we can add the number of initials formable with two, three, four, and five letters. By the product rule, these are 26^2, 26^3, 26^4, and 26^5, respectively, so the answer is $676 + 17576 + 456976 + 11881376 = 12{,}356{,}604$.

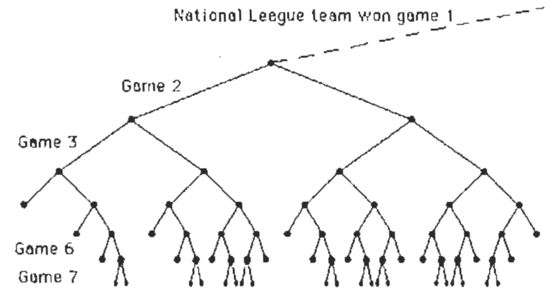
50. We need to compute the number of variable names of length i for $i = 1, 2, \ldots, 8$, and add. A variable name of length i is specified by choosing a first character, which can be done in 53 ways ($2 \cdot 26$ letters and 1 underscore to choose from), and $i - 1$ other characters, each of which can be done in $53 + 10 = 63$ ways. Therefore the answer is

$$\sum_{i=1}^{8} 52 \cdot 63^{i-1} = 52 \cdot \frac{63^8 - 1}{63 - 1} \approx 2.1 \times 10^{14}\,.$$

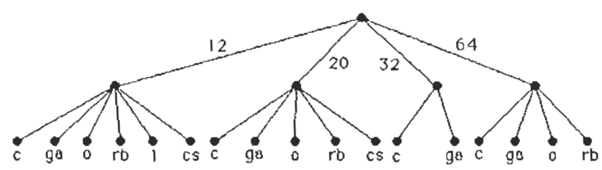
52. We draw the tree, with its root at the top. We show a branch for each of the possibilities 0 and 1, for each bit in order, except that we do not allow three consecutive 0's. Since there are 13 leaves, the answer is 13.



54. The tree is a bit too large to draw in its entirety. We show only half of it, namely the half corresponding to the National League team's having won the first game. By symmetry, the final answer will be twice the number computed with this tree. A branch to the left indicates a win by the National League team; a branch to the right, a win by the American league team. No further branching occurs whenever one team has won four games. Since we see 35 leaves, the answer is 70.



56. a) It is more convenient to branch on bottle size first. Note that there are a different number of branches coming off each of the nodes at the second level. The number of leaves in the tree is 17, which is the answer.



b) We can add the number of different varieties for each of the sizes. The 12-ounce bottle has 6, the 20-ounce bottle has 5, the 32-once bottle has 2, and the 64-ounce bottle has 4. Therefore $6 + 5 + 2 + 4 = 17$ different types of bottles need to be stocked.

58. There are 2^n lines in the truth table, since each of the n propositions can have 2 truth values. Each line can be filled in with T or F, so there are a total of 2^{2^n} possibilities.

60. We want to show that a procedure consisting of m tasks can be done in $n_1 n_2 \cdots n_m$ ways, if the i^{th} task can be done in n_i ways. The product rule stated in the text is the basis step, $m = 2$. Assume the inductive hypothesis. Then to do the procedure we have to do each of the first m tasks, which by the inductive hypothesis can be done in $n_1 n_2 \cdots n_m$ ways, and then the $(m+1)^{\text{st}}$ task, so there are $(n_1 n_2 \cdots n_m) n_{m+1}$ possibilities, as desired.

62. a) The largest value of TOTAL LENGTH is $2^{16} - 1$, since this would be the number represented by a string of 16 1's. So the maximum length of a datagram is 65,535 octets (or bytes).

b) The largest value of HLEN is $2^4 - 1 = 15$, since this would be the number represented by a string of four 1's. So the maximum length of a header is 15 32-bit blocks. Since there are four 8-bit octets (or bytes) in a block, the maximum length of the header is $4 \cdot 15 = 60$ octets.

c) We saw in part (a) that the maximum total length is 65,535 octets. If at least 20 of these must be devoted to the header, the data area can be at most 65,515 octets long.

d) There are $2^8 = 256$ different octets, since each bit of an octet can be 0 or 1. In part (c) we saw that the data area could be at most 65,515 octets long. So the answer is 256^{65515}, which is a huge number (approximately 7×10^{157775}, according to a computer algebra system).