SECTION 6.2 Probability Theory

2. We are told that $p(3) = 2p(x)$ for each $x \neq 3$, but it is implied that $p(1) = p(2) = p(4) = p(5) = p(6)$. We also know that the sum of these six numbers must be 1. It follows easily by algebra that $p(3) = 2/7$ and $p(x) = 1/7$ for $x = 1, 2, 4, 5, 6$.

4. If outcomes are equally likely, then the probability of each outcome is $1/n$, where n is the number of outcomes. Clearly this quantity is between 0 and 1 (inclusive), so (i) is satisfied. Furthermore, there are n outcomes, and the probability of each is $1/n$, so the sum shown in (ii) must equal $n \cdot (1/n) = 1$.

6. We can exploit symmetry in answering these.

a) Since 1 has either to precede 3 or to follow it, and there is no reason that one of these should be any more likely than the other, we immediately see that the answer is $1/2$. We could also simply list all 6 permutations and count that 3 of them have 1 preceding 3, namely 123, 132, and 213.

b) By the same reasoning as in part **(a)**, the answer is again $1/2$.

c) The stated conditions force 3 to come first, so only 312 and 321 are allowed. Therefore the answer is $2/6 = 1/3$.

8. We exploit symmetry in answering many of these.

a) Since 1 has either to precede 2 or to follow it, and there is no reason that one of these should be any more likely than the other, we immediately see that the answer is $1/2$.

b) By the same reasoning as in part **(a)**, the answer is again $1/2$.

c) For 1 immediately to precede 2, we can think of these two numbers as glued together in forming the permutation. Then we are really permuting $n-1$ numbers—the single numbers from 3 through n and the one glued object, 12. There are $(n-1)!$ ways to do this. Since there are $n!$ permutations in all, the probability of randomly selecting one of these is $(n-1)!/n! = 1/n$.

d) Half of the permutations have n preceding 1. Of these permutations, half of them have $n-1$ preceding 2. Therefore one fourth of the permutations satisfy these conditions, so the probability is $1/4$.

e) Looking at the relative placements of 1, 2, and n, we see that one third of the time, n will come first. Therefore the answer is $1/3$.

10. Note that there are $26!$ permutations of the letters, so the denominator in all of our answers is $26!$. To find the numerator, we have to count the number of ways that the given event can happen. Alternatively, in some cases we may be able to exploit symmetry.

a) There are $13!$ possible arrangements of the first 13 letters of the permutation, and in only one of these are they in alphabetical order. Therefore the answer is $1/13!$.

b) Once these two conditions are met, there are $24!$ ways to choose the remaining letters for positions 2 through 25. Therefore the answer is $24!/26! = 1/650$.

c) In effect we are forming a permutation of 25 items—the letters b through y and the double letter combination az or za. There are $25!$ ways to permnte these items, and for each of these permutations there are two choices as to whether a or z comes first. Thus there are $2 \cdot 25!$ ways for form such a permutation, and therefore the answer is $2 \cdot 25!/26! = 1/13$.

d) By part **(c)**, the probability that a and b *are* next to each other is $1/13$. Therefore the probability that a and b are *not* next to each other is $12/13$.

e) There are six ways this can happen: $ax^{24}z$, $zx^{24}a$, $xax^{23}z$, $xzx^{23}a$, $ax^{23}zx$, and $zx^{23}ax$, where x stands for any letter other than a and z (but of course all the x's are different in each permutation). In each of these there are $24!$ ways to permute the letters other than a and z, so there are $24!$ permutations of each type. This gives a total of $6 \cdot 24!$ permutations meeting the conditions, so the answer is $(6 \cdot 24!)/26! = 3/325$.

f) Looking at the relative placements of z, a, and b, we see that one third of the time, z will come first. Therefore the answer is $1/3$.

12. Clearly $p(E \cup F) \geq p(E) = 0.8$. Also, $p(E \cup F) \leq 1$. If we apply Theorem 2 from Section 6.1, we can rewrite this as $p(E) + p(F) - p(E \cap F) \leq 1$, or $0.8 + 0.6 - p(E \cap F) \leq 1$. Solving for $p(E \cap F)$ gives $p(E \cap F) \geq 0.4$.

14. The basis step $n = 1$ is the trivial statement that $p(E_1) \geq p(E_1)$, and the case $n = 2$ was done in Exercise 13. Assume the inductive hypothesis:

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1)$$

Now let $E = E_1 \cap E_2 \cap \cdots \cap E_n$ and let $F = E_{n+1}$, and apply Exercise 13. We obtain

$$p(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1}) \geq p(E_1 \cap E_2 \cap \cdots \cap E_n) + p(E_{n+1}) - 1\,.$$

Substituting from the inductive hypothesis we have

$$\begin{aligned} p(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1}) &\geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1) + p(E_{n+1}) - 1 \\ &= p(E_1) + p(E_2) + \cdots + p(E_n) + p(E_{n+1}) - ((n+1) - 1)\,, \end{aligned}$$

as desired.

16. By definition, to say that $\overline{E}$ and $\overline{F}$ are independent is to say that $p(\overline{E} \cap \overline{F}) = p(\overline{E}) \cdot p(\overline{F})$. By De Morgan's Law, $\overline{E} \cap \overline{F} = \overline{E \cup F}$. Therefore

$$\begin{aligned} p(\overline{E} \cap \overline{F}) &= p(\overline{E \cup F}) = 1 - p(E \cup F) \\ &= 1 - (p(E) + p(F) - p(E \cap F)) \\ &= 1 - p(E) - p(F) + p(E \cap F) \\ &= 1 - p(E) - p(F) + p(E) \cdot p(F) \\ &= (1 - p(E)) \cdot (1 - p(F)) = p(\overline{E}) \cdot p(\overline{F})\,. \end{aligned}$$

(We used the two facts presented in the subsection on combinations of events.)

18. As instructed, we assume that births are independent and the probability of a birth in each day is $1/7$. (This is not exactly true; for example, doctors tend to schedule C-sections on weekdays.)

a) The probability that the second person has the same birth day-of-the-week as the first person (whatever that was) is $1/7$.

b) We proceed as in Example 13. The probability that all the birth days-of-the-week are different is

$$p_n = \frac{6}{7} \cdot \frac{5}{7} \cdot \cdots \cdot \frac{8-n}{7}$$

since each person after the first must have a different birth day-of-the-week from all the previous people in the group. Note that if $n \geq 8$, then $p_n = 0$ since the seventh fraction is 0 (this also follows from the pigeonhole principle). The probability that at least two are born on the same day of the week is therefore $1 - p_n$.

c) We compute $1 - p_n$ for $n = 2, 3, \ldots$ and find that the first time this exceeds $1/2$ is when $n = 4$, so that is our answer. With four people, the probability that at least two will share a birth day-of-the-week is $223/343$, or about 65%.

20. If n people are chosen at random (and we assume 366 equally likely and independent birthdays, as instructed), then the probability that none of them has a birthday today is $(365/366)^n$. The question asks for the smallest n such that this quantity is less than $1/2$. We can determine this by trial and error, or we can solve the equation $(365/366)^n = 1/2$ using logarithms. In either case, we find that for $n \leq 253$, $(365/366)^n > 1/2$, but $(365/366)^{254} \approx .4991$. Therefore the answer is 254.

22. a) Given that we are no longer close to the year 1900, which was not a leap year, let us assume that February 29 occurs one time every four years, and that every other date occurs four times every four years. A cycle of four years contains $4 \cdot 365 + 1 = 1461$ days. Therefore the probability that a randomly chosen day is February 29 is $1/1461$, and the probability that a randomly chosen day is any of the other 365 dates is each $4/1461$.

b) We need to compute the probability that in a group of n people, all of them have different birthdays. Rather than compute probabilities at each stage, let us count the number of ways to choose birthdays from the four-year cycle so that all n people have distinct birthdays. There are two cases to consider, depending on whether the group contains a person born on February 29. Let us suppose that there is such a leap-day person; there are n ways to specify which person he is to be. Then there are 1460 days on which the second person can be born so as not to have the same birthday; then there are 1456 days on which the third person can be born so as not to have the same birthday as either of the first two, as so on, until there are $1468-4n$ days on which the n^{th} person can be born so as not to have the same birthday as any of the others. This gives a total of

$$n \cdot 1460 \cdot 1456 \cdots (1468-4n)$$

ways in all. The other case is that in which there is no leap-day birthday. Then there are 1460 possible birthdays for the first person, 1456 for the second, and so on, down to $1464-4n$ for the n^{th}. Thus the total number of ways to choose birthdays without including February 29 is

$$1460 \cdot 1456 \cdots (1464-4n)\,.$$

The sum of these two numbers is the numerator of the fraction giving the probability that all the birthdays are distinct. The denominator is 1461^n, since each person can have any birthday within the four-year cycle. Putting this all together, we see that the probability that there are at least two people with the same birthday is

$$1-\frac{n \cdot 1460 \cdot 1456 \cdots (1468-4n) + 1460 \cdot 1456 \cdots (1464-4n)}{1461^n}\,.$$

24. There are 16 equally likely outcomes of flipping a fair coin five times in which the first flip comes up tails (each of the other flips can be either heads or tails). Of these only one will result in four heads appearing, namely $THHHH$. Therefore the answer is $1/16$.

26. Intuitively the answer should be yes, because the parity of the number of 1's is a fifty-fifty proposition totally determined by any one of the flips (for example, the last flip). What happened on the other flips is really rather irrelevant. Let us be more rigorous, though. There are 8 bit strings of length 3, and 4 of them contain an odd number of 1's (namely 001, 010, 100, and 111). Therefore $p(E)=4/8=1/2$. Since 4 bit strings of length 3 start with a 1 (namely 100, 101, 110, and 111), we see that $p(F)=4/8=1/2$ as well. Furthermore, since there are 2 strings that start with a 1 and contain an odd number of 1's (namely 100 and 111), we see that $p(E\cap F)=2/8=1/4$. Then since $p(E)\cdot p(F)=(1/2)\cdot(1/2)=1/4=p(E\cap F)$, we conclude from the definition that E and F are independent.

28. These questions are applications of the binomial distribution. Following the lead of King Henry VIII, we call having a boy success. Then $p=0.51$ and $n=5$ for this problem.

a) We are asked for the probability that $k=3$. By Theorem 2 the answer is $C(5,3)0.51^3 0.49^2 \approx 0.32$.

b) There will be at least one boy if there are not all girls. The probability of all girls is 0.49^5, so the answer is $1-0.49^5 \approx 0.972$.

c) This is just like part **(b)**: The probability of all boys is 0.51^5, so the answer is $1-0.51^5 \approx 0.965$.

d) There are two ways this can happen. The answer is clearly $0.51^5+0.49^5 \approx 0.063$.

30. **a)** The probability that all bits are a 1 is $(1/2)^{10}=1/1024$. This is what is being asked for.

b) This is the same as part **(a)**, except that the probability of a 1 bit is 0.6 rather than $1/2$. Thus the answer is $0.6^{10}\approx 0.0060$.

c) We need to multiply the probabilities of each bit being a 1, so the answer is

$$\frac{1}{2}\cdot\frac{1}{2^2}\cdots\frac{1}{2^{10}}=\frac{1}{2^{1+2+\cdots+10}}=\frac{1}{2^{55}}\approx 2.8\times 10^{-17}\,.$$

Note that this is essentially 0.

32. Let E be the event that the bit string begins with a 1, and let F be the event that it ends with 00. In each case we need to calculate the probability $p(E \cup F)$, which is the same as $p(E) + p(F) - p(E) \cdot p(F)$. (The fact that $p(E \cap F) = p(E) \cdot p(F)$ follows from the obvious independence of E and F.) So for each part we will compute $p(E)$ and $p(F)$ and then plug into this formula.

a) We have $p(E) = 1/2$ and $p(F) = (1/2) \cdot (1/2) = 1/4$. Therefore the answer is
$$\frac{1}{2} + \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{5}{8}.$$

b) We have $p(E) = 0.6$ and $p(F) = (0.4) \cdot (0.4) = 0.16$. Therefore the answer is
$$0.6 + 0.16 - 0.6 \cdot 0.16 = 0.664.$$

c) We have $p(E) = 1/2$ and
$$p(F) = (1 - \frac{1}{2^9}) \cdot (1 - \frac{1}{2^{10}}) = 1 - \frac{1}{2^9} - \frac{1}{2^{10}} + \frac{1}{2^{19}}.$$
Therefore the answer is
$$\frac{1}{2} + 1 - \frac{1}{2^9} - \frac{1}{2^{10}} + \frac{1}{2^{19}} - \frac{1}{2} \cdot (1 - \frac{1}{2^9} - \frac{1}{2^{10}} + \frac{1}{2^{19}}) = 1 - \frac{1}{2^9} + \frac{1}{2^{11}} + \frac{1}{2^{19}} - \frac{1}{2^{20}}.$$

34. We need to use the binomial distribution, which tells us that the probability of k successes is
$$b(k; n, p) = C(n, k)p^k(1 - p)^{n-k}.$$

a) Here $k = 0$, since we want all the trials to result in failure. Plugging in and computing, we have $b(0; n, p) = 1 \cdot p^0 \cdot (1 - p)^n = (1 - p)^n$.

b) There is at least one success if and only if it is not the case that there are no successes. Thus we obtain the answer by subtracting the probability in part (**a**) from 1, namely $1 - (1 - p)^n$.

c) There are two ways in which there can be at most one success: no successes or one success. We already computed that the probability of no successes is $(1-p)^n$. Plugging in $k = 1$, we compute that the probability of exactly one success is $b(1; n, p) = n \cdot p^1 \cdot (1 - p)^{n-1}$. Therefore the answer is $(1 - p)^n + np(1 - p)^{n-1}$. This formula only makes sense if $n > 0$, of course; if $n = 0$, then the answer is clearly 1.

d) Since this event is just that the event in part (**c**) does not happen, the answer is $1-[(1-p)^n+np(1-p)^{n-1}]$. Again, this is for $n > 0$; the probability is clearly 0 if $n = 0$.

36. The basis case here can be taken to be $n = 2$, in which case we have $p(E_1 \cup E_2) = p(E_1) + p(E_2)$. The left-hand side is the sum of $p(x)$ for all $x \in E_1 \cup E_2$. Since E_1 and E_2 are disjoint, this is the sum of $p(x)$ for all $x \in E_1$ added to the sum of $p(x)$ for all $x \in E_2$, which is the right-hand side. Assume the strong inductive hypothesis that the statement is true for $n \le k$, and consider the statement for $n = k + 1$, namely $p\left(\bigcup_{i=1}^{k+1}\right) = \sum_{i=1}^{k+1} p(E_i)$. Let $F = \left(\bigcup_{i=1}^{k}\right)$. Then we can rewrite the left-hand side as $p(F \cup E_{k+1})$. By the inductive hypothesis for $n = 2$ (since $F \cap E_{k+1} = \emptyset$) this equals $p(F) + p(E_{k+1})$. Then by the inductive hypothesis for $n = k$ (since the E_i's are pairwise disjoint), this equals $\sum_{i=1}^{k} p(E_i) + p(E_{k+1}) = \sum_{i=1}^{k+1} p(E_i)$, as desired.

38. a) We assume that the observer was instructed ahead of time to tell us whether or not at least one die came up 6 and to provide no more information than that. If we do not make such an assumption, then the following analysis would not be valid. We use the notation (i, j) to represent that the first die came up i and the second die came up j. Note that there are 36 equally likely outcomes.

a) Let S be the event that at least one die came up 6, and let T be the event that sum of the dice is 7. We want $p(T \mid S)$. By Definition 3, this equals $p(S \cap T)/p(S)$. The outcomes in $S \cap T$ are $(1, 6)$ and $(6, 1)$, so $p(S \cap T) = 2/36$. There are $5^2 = 25$ outcomes in $\overline{S}$ (five ways to choose what happened on each die), so $p(S) = (36 - 25)/36 = 11/36$. Therefore the answer is $(2/36)/(11/36) = 2/11$.

b) The analysis is exactly the same as in part (**a**), so the answer is again $2/11$.

40. We assume that n is much greater than k, since otherwise, we could simply compare each element with its successor in the list and know for sure whether or not the list is sorted. We choose two distinct random integers i and j from 1 to n, and we compare the i^{th} and j^{th} elements of the given list; if they are in correct order relative to each other, then we answer "unknown" at this step and proceed. If not, then we answer "true" (i.e., the list is not sorted) and halt. We repeat this for k steps (or until we have found elements out of order), choosing new random indices each time. If we have not found any elements out of order after k steps, we halt and answer "false" (i.e., the original list is probably sorted). Since in a random list the probability that two randomly chosen elements are in correct order relative to each other is $1/2$, the probability that we wrongly answer "false" will be about $1/2^k$ if the list is a random permutation. If k is large, this will be very small; for example, if $k = 100$, then this will be less than one chance in 10^{30}.