SECTION 8.3 Representing Relations

2. In each case we use a 4×4 matrix, putting a 1 in position (i,j) if the pair (i,j) is in the relation and a 0 in position (i,j) if the pair (i,j) is not in the relation.

a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. a) Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,3)^{\text{th}}$ entry is a 0, $(1,3)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,3)$, and $(4,4)$.
b) $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(3,3)$, $(3,4)$, $(4,1)$, and $(1,4)$
c) $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,4)$, $(4,1)$, and $(4,3)$

6. An asymmetric relation (see the preamble to Exercise 16 in Section 8.1) is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i), and there is no pair of 1's symmetrically placed around the main diagonal (i.e., we cannot have $m_{ij} = m_{ji} = 1$ for any values of i and j).

8. For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal); and for transitivity we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.
a) Since some 1's and some 0's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$. Finally, the relation is not transitive; for example, the 1's in positions $(1,2)$ and $(2,3)$ would require a 1 in position $(1,3)$ if the relation were to be transitive.
b) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is not symmetric, the relation is not symmetric (look at positions $(1,2)$ and $(2,1)$, for example). The relation is antisymmetric since there are never two 1's symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is not itself (look at position $(1,4)$ in the square), so the relation is not transitive.
c) Since there are all 0's on the main diagonal, this relation is not reflexive but is irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$, for example. Finally, the Boolean square of this matrix has a 1 in position $(1,1)$, so the relation is not transitive.

10. Note that the total number of entries in the matrix is $1000^2 = 1{,}000{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 100, namely in position (a,b) where $a \le b$, as well as 1's along the diagonal. Thus the answer is the number of subsets of size 2 from a set of 100 elements, plus 1000, i.e., $C(1000,2) + 1000 = 499500 + 1000 = 500{,}500$.

b) There two 1's in each row of the matrix except the first and last rows, in which there is one 1. Therefore the answer is $998 \cdot 2 + 2 = 1998$.

c) There is a 1 in the matrix at each entry just above and to the left of the "anti-diagonal" (i.e., in positions $(1,999)$, $(2,998)$, $\ldots$, $(999,1)$. Therefore the answer is 999.

d) There is a 1 in the matrix at each entry on or above (to the left of) the "anti-diagonal." This is the same number of 1's as in part (a), so the answer is again 500,500.

e) Every row has all 1's except for the first row, so the answer is $999 \cdot 1000 = 999{,}500$.

12. We take the transpose of the matrix, since we want the $(i,j)^{\text{th}}$ entry of the matrix for R^{-1} to be 1 if and only if the $(j,i)^{\text{th}}$ entry of R is 1.

14. a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

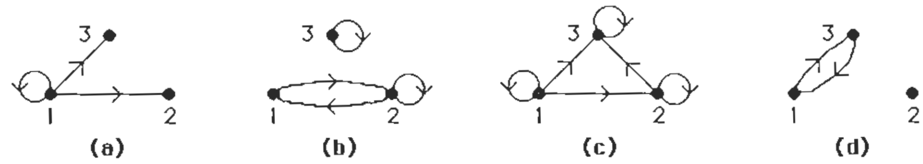
c) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

d) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

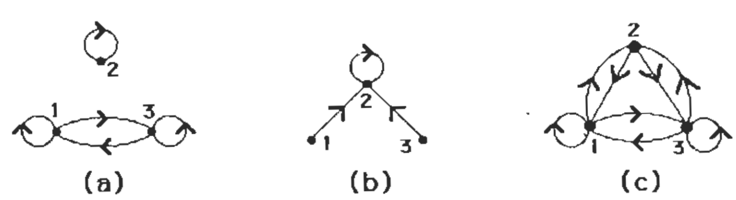
e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

16. Since the matrix for R^{-1} is just the transpose of the matrix for R (see Exercise 12), the entries are the same collection of 0's and 1's, so there are k nonzero entries in $\mathbf{M}_{R^{-1}}$ as well.

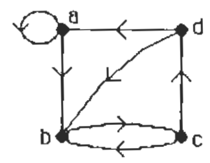
18. We draw the directed graphs, in each case with the vertex set being $\{1,2,3\}$ and an edge from i to j whenever (i,j) is in the relation.



20. In each case we draw a directed graph on three vertices with an edge from a to b for each pair (a,b) in the relation, i.e., whenever there is a 1 in position (a,b) in the matrix. In part (a), for instance, we need an edge from 1 to itself since there is a 1 in position $(1,1)$ in the matrix, and an edge from 1 to 3, but no edge from 1 to 2.



22. We draw the directed graph with the vertex set being $\{a,b,c,d\}$ and an edge from i to j whenever (i,j) is in the relation.



24. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,c)\}$.

26. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(c,d),(d,d)\}$.

28. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$.

30. Clearly R is irreflexive if and only if there are no loops in the directed graph for R.

32. Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in **antiparallel** pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; asymmetric if is both antisymmetric and irreflexive; and transitive if all paths of length 2 (a pair of edges (x,y) and (y,z)) are accompanied by the corresponding path of length 1 (the edge (x,z)). The relation drawn in Exercise 26 is reflexive but not irreflexive since there are loops at each vertex. It is not symmetric, since, for instance, the edge (c,a) is present but not the edge (a,c). It is not antisymmetric, since both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since the path $(c,a),(a,b)$ from c to b is not accompanied by the edge (c,b). The relation drawn in Exercise 27 is neither reflexive nor irreflexive since there are some loops but not a loop at each vertex. It is symmetric, since the edges appear in antiparallel pairs. It is not antisymmetric, since, for instance, both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since edges (c,a) and (a,c) are present, but not (c,c). The relation drawn in Exercise 28 is reflexive and not irreflexive since there are loops at all vertices. It is symmetric but not antisymmetric or asymmetric. It is transitive; the only nontrivial paths of length 2 have the necessary loop shortcuts.

34. For each pair (a,b) of vertices (including the pairs (a,a) in which the two vertices are the same), if there is an edge from a to b, then erase it, and if there is no edge from a to b, put add it in.

36. We assume that the two relations are on the same set. For the union, we simply take the union of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in either of them. For intersection, we simply take the intersection of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there are edges from i to j in both of them. For symmetric difference, we simply take the symmetric difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in one, but not both, of them. Similarly, to form the difference, we take the difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in the first but not the second. To form the directed graph for the composition $S\circ R$ of relations R and S, we draw a directed graph on the same set of vertices and put in an edge from i to j whenever there is a vertex k such that there is an edge from i to k in R, and an edge from k to j in S.