SECTION 2.4 Sequences and Summations

2. In each case we just plug $n = 8$ into the formula.

a) $2^{8-1} = 128$ b) 7 c) $1 + (-1)^8 = 0$ d) $-(-2)^8 = -256$

4. a) $a_0 = (-2)^0 = 1$, $a_1 = (-2)^1 = -2$, $a_2 = (-2)^2 = 4$, $a_3 = (-2)^3 = -8$

b) $a_0 = a_1 = a_2 = a_3 = 3$

c) $a_0 = 7 + 4^0 = 8$, $a_1 = 7 + 4^1 = 11$, $a_2 = 7 + 4^2 = 23$, $a_3 = 7 + 4^3 = 71$

d) $a_0 = 2^0 + (-2)^0 = 2$, $a_1 = 2^1 + (-2)^1 = 0$, $a_2 = 2^2 + (-2)^2 = 8$, $a_3 = 2^3 + (-2)^3 = 0$

6. These are easy to compute by hand, calculator, or computer.

a) 10, 7, 4, 1, -2, -5, -8, -11, -14, -17

b) We can use the formula in Table 2, or we can just keep adding to the previous term ($1+2=3$, $3+3=6$, $6+4=10$, and so on): 1, 3, 6, 10, 15, 21, 28, 36, 45, 55. These are called the triangular numbers.

c) 1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025

d) 1, 1, 1, 2, 2, 2, 2, 2, 3, 3 (there will be $2k+1$ copies of k) e) 1, 2, 3, 5, 8, 13, 21, 34, 55, 89

f) The largest number whose binary expansion has n bits is $(11\ldots1)_2$, which is $2^n - 1$. So the sequence is 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023.

g) 1, 2, 2, 4, 8, 11, 33, 37, 148, 153 h) 1, 2, 2, 2, 2, 3, 3, 3, 3, 3

8. One rule could be that each term is 2 greater than the previous term; the sequence would be 3, 5, 7, 9, 11, 13, ... Another rule could be that the n^{th} term is the n^{th} odd prime; the sequence would be 3, 5, 7, 11, 13, 17, ... Actually, we could choose any number we want for the fourth term (say 12) and find a third degree polynomial whose value at n would be the n^{th} term; in this case we need to solve for A, B, C, and D in the equations $y = Ax^3 + Bx^2 + Cx + D$ where $(1,3)$, $(2,5)$, $(3,7)$, $(4,12)$ have been plugged in for x and y. Doing so yields $(x^3 - 6x^2 + 15x - 4)/2$. With this formula, the sequence is 3, 5, 7, 12, 23, 43, 75, 122, 187, 273. Obviously many other answers are possible.

10. **a)** The first term is 3, and the n^{th} term is obtained by adding $2n-1$ to the previous term. In other words, we successively add 3, then 5, then 7, and so on. Alternatively, we see that the n^{th} term is n^2+2; we can see this by inspection if we happen to notice how close each term is to a perfect square, or we can fit a quadratic polynomial to the data. The next three terms are $123,\ 146,\ 171$.

b) This is an arithmetic sequence whose first term is 7 and whose difference is 4. Thus the n^{th} term is $7+4(n-1)=4n+3$. Thus the next three terms are $47,\ 51,\ 55$.

c) The n^{th} term is clearly the binary expansion of n. Thus the next three terms are $1100,\ 1101,\ 1110$.

d) The sequence consists of one 1, followed by three 2's, followed by five 3's, followed by seven 5's, and so on, with the number of copies of the next value increasing by 2 each time, and the values themselves following the rule that the first two values are 1 and 2 and each subsequent value is the sum of the previous two values. Obviously other answers are possible as well. By our rule, the next three terms would be $8,\ 8,\ 8$.

e) If we stare at this sequence long enough and compare it with Table 1, then we notice that the n^{th} term is 3^n-1. Thus the next three terms are $59048,\ 177146,\ 531440$.

f) We notice that each term evenly divides the next, and the multipliers are successively 3, 5, 7, 9, 11, and so on. That must be the intended pattern. One notation for this is to use $n!!$ to mean $n(n-2)(n-4)\cdots$; thus the n^{th} term is $(2n-1)!!$. Thus the next three terms are $654729075,\ 13749310575,\ 316234143225$.

g) The sequence consists of one 1, followed by two 0s, then three 1s, four 0s, five 1s, and so on, alternating between 0s and 1s and having one more item in each group than in the previous group. Thus six 0's will follow next, so the next three terms are $0,\ 0,\ 0$.

h) It doesn't take long to notice that each term is the square of its predecessor. The next three terms get very big very fast: $18446744073709551616,\ 340282366920938463463374607431768211456$, and then

$$115792089237316195423570985008687907853269984665640564039457584007913129639936\,.$$

(These were computed using *Maple*.)

12. Let us ask ourselves which is the last term in the sequence whose value is k? Clearly it is $1+2+3+\cdots+k$, which equals $k(k+1)/2$. We can rephrase this by saying that $a_n \le k$ if and only if $k(k+1)/2 \ge n$. Thus, to find k as a function of n, we must find the smallest k such that $k(k+1)/2 \ge n$. This is equivalent to $k^2+k-2n \ge 0$. By the quadratic formula, this tells us that k has to be at least $(-1+\sqrt{1+8n})/2$. Therefore we have $k = \lceil (-1+\sqrt{1+8n})/2 \rceil = \left\lceil -\frac{1}{2} + \sqrt{2n+\frac{1}{4}} \right\rceil$. By Exercise 43 in Section 2.3, this is the same as the integer closest to $\sqrt{2n+\frac{1}{4}}$, where we choose the smaller of the two closest integers if $\sqrt{2n+\frac{1}{4}}$ is a half integer. The desired answer is $\lfloor \sqrt{2n}+\frac{1}{2} \rfloor$, which by Exercise 42 in Section 2.3 is the integer closest to $\sqrt{2n}$ (note that $\sqrt{2n}$ can never be a half integer). To see that these are the same, note that it can never happen that $\sqrt{2n} \le m+\frac{1}{2}$ while $\sqrt{2n+\frac{1}{4}} > m+\frac{1}{2}$ for some positive integer m, since this would imply that $2n \le m^2+m+\frac{1}{4}$ and $2n > m^2+m$, an impossibility. Therefore the integer closest to $\sqrt{2n}$ and the (smaller) integer closest to $\sqrt{2n+\frac{1}{4}}$ are the same, and we are done.

14. **a)** $1+3+5+7=16$ **b)** $1^2+3^2+5^2+7^2=84$
c) $(1/1)+(1/3)+(1/5)+(1/7)=176/105$ **d)** $1+1+1+1=4$

16. **a)** The terms of this sequence alternate between 2 (if j is even) and 0 (if j is odd). Thus the sum is $2+0+2+0+2+0+2+0+2=10$.

b) We can break this into two parts and compute $\left(\sum_{j=0}^{8} 3^j\right) - \left(\sum_{j=0}^{8} 2^j\right)$. Each summation can be computed from the formula for the sum of a geometric progression. Thus the answer is

$$\frac{3^9-1}{3-1} - \frac{2^9-1}{2-1} = 9841 - 511 = 9330\,.$$

c) As in part (b) we can break this into two parts and compute $(\sum_{j=0}^{8} 2\cdot 3^j) + (\sum_{j=0}^{8} 3\cdot 2^j)$. Each summation can be computed from the formula for the sum of a geometric progression. Thus the answer is

$$\frac{2\cdot 3^9 - 2}{3-1} + \frac{3\cdot 2^9 - 3}{2-1} = 19682 + 1533 = 21215\,.$$

d) This could be worked as in part (b), but it is easier to note that the sum telescopes (see Exercise 19). Each power of 2 cancels except for the -2^0 when $j=0$ and the 2^9 when $j=8$. Therefore the answer is $2^9 - 2^0 = 511$. (Alternatively, note that $2^{j+1} - 2^j = 2^j$.)

18. We will just write out the sums explicitly in each case.

a) $(1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2) = 3$

b) $(0+0)+(0+2)+(0+4)+(3+0)+(3+2)+(3+4)+(6+0)+(6+2)+(6+4)+(9+0)+(9+2)+(9+4) = 78$

c) $(0+1+2)+(0+1+2)+(0+1+2) = 9$

d) $(0+0+0+0)+(0+1+8+27)+(0+4+32+108) = 180$

20. We use the suggestion (simple algebra shows that this is indeed an identity) and note that all the terms in the summation cancel out except for the $1/k$ when $k=1$ and the $1/(k+1)$ when $k=n$:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

22. First we note that $k^3 - (k-1)^3 = 3k^2 - 3k + 1$. Then we sum this equation for all values of k from 1 to n. On the left, because of telescoping, we have just n^3; on the right we have

$$3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3\sum_{k=1}^{n} k^2 - \frac{3n(n+1)}{2} + n\,.$$

Equating the two sides and solving for $\sum_{k=1}^{n} k^2$, we obtain the desired formula.

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left(n^3 + \frac{3n(n+1)}{2} - n\right)\\ &= \frac{n}{3}\left(\frac{2n^2+3n+3-2}{2}\right)\\ &= \frac{n}{3}\left(\frac{2n^2+3n+1}{2}\right) = \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

24. This exercise is like Example 15. From Table 2 we know that $\sum_{k=1}^{200} k^3 = 200^2 \cdot 201^2/4 = 404{,}010{,}000$, and $\sum_{k=1}^{98} k^3 = 98^2\cdot 99^2/4 = 23{,}532{,}201$. Therefore the desired sum is $404{,}010{,}000 - 23{,}532{,}201 = 380{,}477{,}799$.

26. If we write down the first few terms of this sum we notice a pattern. It starts $(1+1+1+1+1+1+1)+(2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2+2)+(3+3+3+3+\cdots+3)+\cdots$. There are seven 1s, then 19 2s, then 37 3s, and so on; in general, the number of i's is $(i+1)^3 - i^3 = 3i^2 + 3i + 1$. So we need to sum $i(3i^2+3i+1)$ for an appropriate range of values for i. We must find this range. It gets a little messy at the end if m is such that the sequence stops before a complete range of the last value is present. Let $n = \lfloor \sqrt[3]{m} \rfloor - 1$. Then there are $n+1$ blocks, and $(n+1)^3 - 1$ is where the next-to-last block ends. The sum of those complete blocks is $\sum_{i=1}^{n} i(3i^2+3i+1) = \sum_{i=1}^{n} 3i^3 + 3i^2 + i = n(3n+4)(n+1)^2/4$ (using Table 2 and algebra). The remaining terms in our summation all have the value $n+1$ and the number of them present is $m - ((n+1)^3 - 1)$. Our final answer is therefore $n(3n+4)(n+1)^2/4 + (n+1)(m - (n+1)^3 + 1)$, where, once again, $n = \lfloor \sqrt[3]{m} \rfloor - 1$.

28. $n! = \prod_{i=1}^{n} i$

30. $(0!)(1!)(2!)(3!)(4!) = 1 \cdot 1 \cdot 2 \cdot 6 \cdot 24 = 288$

32. a) This set is countable. The integers in the set are 11, 12, 13, 14, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 11$, $2 \leftrightarrow 12$, $3 \leftrightarrow 13$, and so on; in general $n \leftrightarrow (n+10)$.

b) This set is countable. The integers in the set are -1, -3, -5, -7, and so on. We can list these numbers in that order, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow -1$, $2 \leftrightarrow -3$, $3 \leftrightarrow -5$, and so on; in general $n \leftrightarrow -(2n-1)$.

c) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 21.

d) This set is countable. The integers in the set are 0, ± 10, ± 20, ± 30, and so on. We can list these numbers in the order 0, 10, -10, 20, -20, 30, $\ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 0$, $2 \leftrightarrow 10$, $3 \leftrightarrow -10$, $4 \leftrightarrow 20$, $5 \leftrightarrow -20$, $6 \leftrightarrow 30$, and so on.

34. a) This set is countable. The integers in the set are ± 1, ± 2, ± 4, ± 5, ± 7, and so on. We can list these numbers in the order 1, -1, 2, -2, 4, -4, 5, -5, 7, -7, $\ldots$, thereby establishing the desired correspondence. In other words, the correspondence is given by $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 2$, $4 \leftrightarrow -2$, $5 \leftrightarrow 4$, and so on.

b) This is similar to part (a); we can simply list the elements of the set in order of increasing absolute value, listing each positive term before its corresponding negative: 5, -5, 10, -10, 15, -15, 20, -20, 25, -25, 30, -30, 40, -40, 45, -45, 50, -50, $\ldots$.

c) This set is countable but a little tricky. We can arrange the numbers in a 2-dimensional table as follows:

$$\begin{array}{llllllll}
.\overline{1} & .1 & .11 & .111 & .1111 & .11111 & .111111 & \ldots \\
1.\overline{1} & 1 & 1.1 & 1.11 & 1.111 & 1.1111 & 1.11111 & \ldots \\
11.\overline{1} & 11 & 11.1 & 11.11 & 11.111 & 11.1111 & 11.11111 & \ldots \\
111.\overline{1} & 111 & 111.1 & 111.11 & 111.111 & 111.1111 & 111.11111 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Thus we have shown that our set is the countable union of countable sets (each of the countable sets is one row of this table). Therefore by Exercise 41, the entire set is countable.

d) This set is not countable. We can prove it by the same diagonalization argument as was used to prove that the set of all reals is uncountable in Example 21. All we need to do is choose $d_i = 1$ when $d_{ii} = 9$ and choose $d_i = 9$ when $d_{ii} = 1$ or d_{ii} is blank (if the decimal expansion is finite).

36. If a set A is countable, then we can list its elements, $a_1, a_2, a_3, \ldots, a_n, \ldots$ (possibly ending after a finite number of terms). Every subset of A consists of some (or none or all) of the items in this sequence, and we can list them in the same order in which they appear in the sequence. This gives us a sequence (again, infinite or finite) listing all the elements of the subset. Thus the subset is also countable.

38. The hypothesis gives us a one-to-one and onto function f from A to B. By Exercise 16e in the supplementary exercises for this chapter, the function S_f from $P(A)$ to $P(B)$ defined by $S_f(X) = f(X)$ for all $X \subseteq A$ is one-to-one and onto. Therefore $P(A)$ and $P(B)$ have the same cardinality.

40. Let A and B be the two given countable sets, and let us list their elements as a_1, a_2, $\ldots$, a_n, $\ldots$ and b_1, b_2, $\ldots$, b_n, $\ldots$. Then we can list the elements of their union as a_1, b_1, a_2, b_2, $\ldots$, except that we do not list any element that has already appeared in this list (in case $A \cap B \neq \emptyset$), and if one or both of the original lists stops (in case A or B is finite), then of course we do not list nonexistent terms. Since we have displayed $A \cup B$ as a list, we conclude that it is countable.

42. Exercise 77 in Section 2.3 gave a one-to-one correspondence between $\mathbf{Z}^+ \times \mathbf{Z}^+$ and $\mathbf{Z}^+$. Since $\mathbf{Z}^+$ is countable, so is $\mathbf{Z}^+ \times \mathbf{Z}^+$.

44. There are at most two real solutions of each quadratic equation, so the number of solutions is countable as long as the number of triples (a, b, c), with a, b, and c integers, is countable. But this follows from Exercise 41 in the following way. There are a countable number of pairs (b, c), since for each b (and there are countably many b's) there are only a countable number of pairs with that b as its first coordinate. Now for each a (and there are countably many a's) there are only a countable number of triples with that a as its first coordinate (since we just showed that there are only a countable number of pairs (b, c)). Thus again by Exercise 41 there are only countably many triples.

46. We know from Example 21 that the set of real numbers between 0 and 1 is uncountable. Let us associate to each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as follows: If x is a real number whose decimal representation is $0.d_1d_2d_3\ldots$ (with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to x the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as $0.239999\ldots$.) Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. Bnt the set of all such functions has at least this cardinality, so it, too, must be uncountable (by Exercise 37).

48. We follow the hint. Suppose that f is a function from S to $P(S)$. We must show that f is not onto. Let $T = \{ s \in S \mid s \notin f(s) \}$. We will show that T is not in the range of f. If it were, then we would have $f(t) = T$ for some $t \in S$. Now suppose that $t \in T$. Then because $t \in f(t)$, it follows from the definition of T that $t \notin T$; this is a contradiction. On the other hand, suppose that $t \notin T$. Then because $t \notin f(t)$, it follows from the definition of T that $t \in T$; this is again a contradiction. This completes our proof by contradiction that f is not onto.