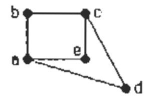
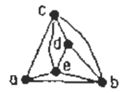
SECTION 9.7 Planar Graphs

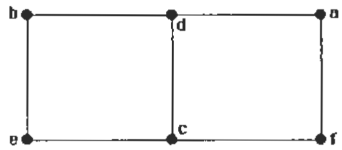
2. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex d down, then the crossings can be avoided.



4. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex b far to the right, and squeeze vertices d and e in a little, then we can avoid crossings.



6. This graph is easily untangled and drawn in the following planar representation.



8. If one has access to software such as Geometer's Sketchpad, then this problem can be solved by drawing the graph and moving the points around, trying to find a planar drawing. If we are unable to find one, then we look for a reason why—either a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$ (always try the latter first). In this case we find that there is in fact an actual copy of $K_{3,3}$, with vertices a, c, and e in one set and b, d, and f in the other.

10. The argument is similar to the argument when v_3 is inside region R_2. In the case at hand the edges between v_3 and v_4 and between v_3 and v_5 separate R_1 into two subregions, R_{11} (bounded by v_1, v_4, v_3, and v_5) and R_{12} (bounded by v_2, v_4, v_3, and v_5). Now again there is no way to place vertex v_6 without forcing a crossing. If v_6 is in R_2, then there is no way to draw the edge $\{v_3, v_6\}$ without crossing another edge. If v_6 is in R_{11}, then the edge between v_2 and v_6 cannot be drawn; whereas if v_6 is in R_{12}, then the edge between v_1 and v_6 cannot be drawn.

12. Euler's formula says that $v - e + r = 2$. We are given $v = 8$, and from the fact that the sum of the degrees equals twice the number of edges, we deduce that $e = (3 \cdot 8)/2 = 12$. Therefore $r = 2 - v + e = 2 - 8 + 12 = 6$.

14. Euler's formula says that $v - e + r = 2$. We are given $e = 30$ and $r = 20$. Therefore $v = 2 - r + e = 2 - 20 + 30 = 12$.

16. A bipartite simple graph has no simple circuits of length three. Therefore the inequality follows from Corollary 3.

18. If we add $k-1$ edges, we can make the graph connected, create no new regions, and still avoid edge crossings. (We just add an edge from one vertex in one component, incident to the unbounded region, to one vertex in each of the other components.) For this new graph, Euler's formula tells us that $v-(e+k-1)+r=2$. This simplifies algebraically to $r=e-v+k+1$.

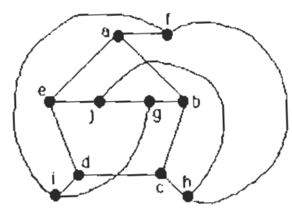
20. This graph is not homeomorphic to $K_{3,3}$, since by rerouting the edge between a and h we see that it is planar.

22. Replace each vertex of degree two and its incident edges by a single edge. Then the result is $K_{3,3}$: the parts are $\{a,e,i\}$ and $\{c,g,k\}$. Therefore this graph is homeomorphic to $K_{3,3}$.

24. This graph is nonplanar. If we delete the five curved edges outside the big pentagon, then the graph is homeomorphic to K_5. We can see this by replacing each vertex of degree 2 and its two edges by one edge.

26. If we follow the proof in Example 3, we see how to construct a planar representation of all of $K_{3,3}$ except for one edge. In particular, if we place vertex v_6 inside region R_{22} of Figure 7(b), then we can draw edges from v_6 to v_2 and v_3 with no crossings, and to v_1 with only one crossing. Furthermore, since $K_{3,3}$ is not planar, its crossing number cannot be 0. Hence its crossing number is 1.

28. First note that the Petersen graph with one edge removed is not planar; indeed, by Example 9, the Petersen graph with three mutually adjacent edges removed is not planar. Therefore the crossing number must be greater than 1. (If it were only 1, then removing the edge that crossed would give a planar drawing of the Petersen graph minus one edge.) The following figure shows a drawing with only two crossings. (This drawing was obtained by a little trial and error.) Therefore the crossing number must be 2. (In this figure, the vertices are labeled as in Figure 14(a).)



30. Since by Exercise 26 we know how to embed all but one edge of $K_{3,3}$ in one plane with no crossings, we can embed all of $K_{3,3}$ in two planes with no crossings simply by drawing the last edge in the second plane.

32. By Corollary 1 to Euler's formula, we know that in one plane we can draw without crossing at most $3v-6$ edges from a graph with v vertices. Therefore if a graph has v vertices and e edges, then it will require at least $e/(3v-6)$ planes in order to draw all the edges without crossing. Since the thickness is a whole number, it must be greater than or equal to the smallest integer at least this large, i.e., $\lceil e/(3v-6)\rceil$.

34. This is essentially the same as Exercise 32, using Corollary 3 in place of Corollary 1.

36. As in the solution to Exercise 37, we represent the torus by a rectangle. The figure below shows how K_5 is embedded without crossings. (The reader might try to embed K_6 or K_7 on a torus.)