SECTION 1.6 Introduction to Proofs

2. We must show that whenever we have two even integers, their sum is even. Suppose that a and b are two even integers. Then there exist integers s and t such that $a = 2s$ and $b = 2t$. Adding, we obtain $a + b = 2s + 2t = 2(s + t)$. Since this represents $a + b$ as 2 times the integer $s + t$, we conclude that $a + b$ is even, as desired.

4. We must show that whenever we have an even integer, its negative is even. Suppose that a is an even integer. Then there exists an integer s such that $a = 2s$. Its additive inverse is $-2s$, which by rules of arithmetic and algebra (see Appendix 1) equals $2(-s)$. Since this is 2 times the integer $-s$, it is even, as desired.

6. An odd number is one of the form $2n + 1$, where n is an integer. We are given two odd numbers, say $2a + 1$ and $2b + 1$. Their product is $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$. This last expression shows that the product is odd, since it is of the form $2n + 1$, with $n = 2ab + a + b$.

8. Let $n = m^2$. If $m = 0$, then $n + 2 = 2$, which is not a perfect square, so we can assume that $m \geq 1$. The smallest perfect square greater than n is $(m + 1)^2$, and we have $(m + 1)^2 = m^2 + 2m + 1 = n + 2m + 1 > n + 2 \cdot 1 + 1 > n + 2$. Therefore $n + 2$ cannot be a perfect square.

10. A rational number is a number that can be written in the form x/y where x and y are integers and $y \neq 0$. Suppose that we have two rational numbers, say a/b and c/d. Then their product is, by the usual rules for multiplication of fractions, $(ac)/(bd)$. Note that both the numerator and the denominator are integers, and that $bd \neq 0$ since b and d were both nonzero. Therefore the product is, by definition, a rational number.

12. This is true. Suppose that a/b is a nonzero rational number and that x is an irrational number. We must prove that the product xa/b is also irrational. We give a proof by contradiction. Suppose that xa/b were rational. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Let us multiply this rational number b/a by the assumed rational number xa/b. By Exercise 26, the product is rational. But the product is $(b/a)(xa/b) = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational, as desired.

14. If x is rational and not zero, then by definition we can write $x = p/q$, where p and q are nonzero integers. Since $1/x$ is then q/p and $p \neq 0$, we can conclude that $1/x$ is rational.

16. We give a proof by contraposition. If it is not true than m is even or n is even, then m and n are both odd. By Exercise 6, this tells us that mn is odd, and our proof is complete.

18. a) We must prove the contrapositive: If n is odd, then $3n + 2$ is odd. Assume that n is odd. Then we can write $n = 2k + 1$ for some integer k. Then $3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$. Thus $3n + 2$ is two times some integer plus 1, so it is odd.

b) Suppose that $3n + 2$ is even and that n is odd. Since $3n + 2$ is even, so is $3n$. If we add subtract an odd number from an even number, we get an odd number, so $3n - n = 2n$ is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

20. We need to prove the proposition "If 1 is a positive integer, then $1^2 \geq 1$." The conclusion is the true statement $1 \geq 1$. Therefore the conditional statement is true. This is an example of a trivial proof, since we merely showed that the conclusion was true.

22. We give a proof by contradiction. Suppose that we don't get a pair of blue socks or a pair of black socks. Then we drew at most one of each color. This accounts for only two socks. But we are drawing three socks. Therefore our supposition that we did not get a pair of blue socks or a pair of black socks is incorrect, and our proof is complete.

24. We give a proof by contradiction. If there were at most two days falling in the same month, then we could have at most $2 \cdot 12 = 24$ days, since there are 12 months. Since we have chosen 25 days, at least three of them must fall in the same month.

26. We need to prove two things, since this is an "if and only if" statement. First let us prove directly that if n is even then $7n + 4$ is even. Since n is even, it can be written as $2k$ for some integer k. Then $7n + 4 = 14k + 4 = 2(7k + 2)$. This is 2 times an integer, so it is even, as desired. Next we give a proof by contraposition that if $7n + 4$ is even then n is even. So suppose that n is not even, i.e., that n is odd. Then n can be written as $2k + 1$ for some integer k. Thus $7n + 4 = 14k + 11 = 2(7k + 5) + 1$. This is 1 more than 2 times an integer, so it is odd. That completes the proof by contraposition.

28. There are two things to prove. For the "if" part, there are two cases. If $m = n$, then of course $m^2 = n^2$; if $m = -n$, then $m^2 = (-n)^2 = (-1)^2n^2 = n^2$. For the "only if" part, we suppose that $m^2 = n^2$. Putting everything on the left and factoring, we have $(m + n)(m - n) = 0$. Now the only way that a product of two numbers can be zero is if one of them is zero. Therefore we conclude that either $m + n = 0$ (in which case $m = -n$), or else $m - n = 0$ (in which case $m = n$), and our proof is complete.

30. We write these in symbols: $a < b$, $(a + b)/2 > a$, and $(a + b)/2 < b$. The latter two are equivalent to $a + b > 2a$ and $a + b < 2b$, respectively, and these are in turn equivalent to $b > a$ and $a < b$, respectively. It is now clear that all three statements are equivalent.

32. We give direct proofs that (i) implies (ii), that (ii) implies (iii), and that (iii) implies (i). That will suffice. For the first, suppose that $x = p/q$ where p and q are integers with $q \neq 0$. Then $x/2 = p/(2q)$, and this is rational, since p and $2q$ are integers with $2q \neq 0$. For the second, suppose that $x/2 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (2p)/q$, so $3x - 1 = (6p)/q - 1 = (6p - q)/q$ and this is rational, since $6p - q$ and q are integers with $q \neq 0$. For the last, suppose that $3x - 1 = p/q$ where p and q are integers with $q \neq 0$. Then $x = (p/q + 1)/3 = (p + q)/(3q)$, and this is rational, since $p + q$ and $3q$ are integers with $3q \neq 0$.

34. No. This line of reasoning shows that *if* $\sqrt{2x^2 - 1} = x$, then we must have $x = 1$ or $x = -1$. These are therefore the only possible solutions, but we have no guarantee that they *are* solutions, since not all of our steps were reversible (in particular, squaring both sides). Therefore we *must* substitute these values back into the original equation to determine whether they do indeed satisfy it.

36. The only conditional statements not shown directly are $p_1 \leftrightarrow p_2$, $p_2 \leftrightarrow p_4$, and $p_3 \leftrightarrow p_4$. But these each follow with one or more intermediate steps: $p_1 \leftrightarrow p_2$, since $p_1 \leftrightarrow p_3$ and $p_3 \leftrightarrow p_2$; $p_2 \leftrightarrow p_4$, since $p_2 \leftrightarrow p_1$ (just established) and $p_1 \leftrightarrow p_4$; and $p_3 \leftrightarrow p_4$, since $p_3 \leftrightarrow p_1$ and $p_1 \leftrightarrow p_4$.

38. We must find a number that cannot be written as the sum of the squares of three integers. We claim that 7 is such a number (in fact, it is the smallest such number). The only squares that can be used to contribute to the sum are 0, 1, and 4. We cannot use two 4's, because their sum exceeds 7. Therefore we can use at most one 4, which means that we must get 3 using just 0's and 1's. Clearly three 1's are required for this, bringing the total number of squares used to four. Thus 7 cannot be written as the sum of three squares.

40. Suppose that we look at the ten groups of integers in three consecutive locations around the circle (first-second-third, second-third-fourth, ..., eighth-ninth-tenth, ninth-tenth-first, and tenth-first-second). Since each number from 1 to 10 gets used three times in these groups, the sum of the sums of the ten groups must equal three times the sum of the numbers from 1 to 10, namely $3 \cdot 55 = 165$. Therefore the average sum is $165/10 = 16.5$. By Exercise 39, at least one of the sums must be greater than or equal to 16.5, and since the sums are whole numbers, this means that at least one of the sums must be greater than or equal to 17.

42. We show that each of these is equivalent to the statement (v) n is odd, say $n = 2k+1$. Example 1 showed that (v) implies (i), and Example 8 showed that (i) implies (v). For $(v) \to (ii)$ we see that $1 - n = 1 - (2k+1) = 2(-k)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $1 - n = 1 - 2m = 2(-m) + 1$, so $1 - n$ would be odd, and this completes the proof by contraposition that $(ii) \to (v)$. For $(v) \to (iii)$, we see that $n^3 = (2k+1)^3 = 8k^3+12k^2+6k+1 = 2(4k^3+6k^2+3k)+1$ is odd. Conversely, if n were even, say $n = 2m$, then we would have $n^3 = 2(4m^3)$, so n^3 would be even, and this completes the proof by contraposition that $(iii) \to (v)$. Finally, for $(v) \to (iv)$, we see that $n^2 + 1 = (2k+1)^2 + 1 = 4k^2 + 4k + 2 = 2(2k^2 + 2k + 1)$ is even. Conversely, if n were even, say $n = 2m$, then we would have $n^2 + 1 = 2(2m^2) + 1$, so $n^2 + 1$ would be odd, and this completes the proof by contraposition that $(iv) \to (v)$.