SUPPLEMENTARY EXERCISES FOR CHAPTER 5

2. a) There are no ways to do this, since there are not enough items. b) $6^{10} = 60{,}466{,}176$

c) There are no ways to do this, since there are not enough items.

d) $C(6+10-1,10) = C(15,10) = C(15,5) = 3003$

4. There are 2^7 bit strings of length 10 that start 000, since each of the last 7 bits can be chosen in either of two ways. Similarly, there are 2^6 bit strings of length 10 that end 1111, and there are 2^3 bit strings of length 10 that both start 000 and end 1111 (since only the 3 middle bits can be freely chosen). Therefore by the inclusion–exclusion principle, the answer is $2^7 + 2^6 - 2^3 = 184$.

6. $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90{,}000$

8. a) All the integers from 100 to 999 have three decimal digits, and there are $999 - 100 + 1 = 900$ of these.

b) In addition to the 900 three-digit numbers, there are 9 one-digit positive integers, for a total of 909.

c) There is 1 one-digit number with a 9. Among the two-digit numbers, there are the 10 numbers from 90 to 99, together with the 8 numbers 19, 29, $\ldots$, 89, for a total of 18. Among the three-digit numbers, there are the 100 from 900 to 999; and there are, for each century from the 100's to the 800's, again $1 + 18 = 19$ numbers with at least one 9; this gives a total of $100 + 8 \cdot 19 = 252$. Thus our final answer is $1 + 18 + 252 = 271$. Alternately, we can compute this as $10^3 - 9^3 = 271$, since we want to subtract from the number of three-digit nonnegative numbers (with leading 0's allowed) the number of those that use only the nine digits 0 through 8.

d) Since we can use only even digits, there are $5^3 = 125$ ways to specify a three-digit number, allowing leading 0's. Since, however, the number $0 = 000$ is not in our set, we need to subtract 1, obtaining the answer 124.
e) The numbers in question are either of the form $d55$ or $55d$, with $d \neq 5$, or 555. Since d can be any of nine digits, there are $9 + 9 + 1 = 19$ such numbers.
f) All 9 one-digit numbers are palindromes. The 9 two-digit numbers 11, 22, ..., 99 are palindromes. For three-digit numbers, the first digit (which must equal the third digit) can be any of the 9 nonzero digits, and the second digit can be any of the 10 digits, giving $9 \cdot 10 = 90$ possibilities. Therefore the answer is $9 + 9 + 90 = 108$.

10. Using the generalized pigeonhole principle, we see that we need $5 \times 12 + 1 = 61$ people.

12. There are $7 \times 12 = 84$ day-month combinations. Therefore we need 85 people to ensure that two of them were born on the same day of the week and in the same month.

14. We need at least 551 cards to ensure that at least two are identical. Since the cards come in packages of 20, we need $\lceil 551/20 \rceil = 28$ packages.

16. Partition the set of numbers from 1 to $2n$ into the n pigeonholes $\{1,2\}$, $\{3,4\}$, ..., $\{2n-1, 2n\}$. If we have $n+1$ numbers from this set (the pigeons), then two of them must be in the same hole. This means that among our collection are two consecutive numbers. Clearly consecutive numbers are relatively prime (since every common divisor must divide their difference, 1).

18. Divide the interior of the square, with lines joining the midpoints of opposite sides, into four 1×1 squares. By the pigeonhole principle, at least two of the five points must be in the same small square. The furthest apart two points in a square could be is the length of the diagonal, which is $\sqrt{2}$ for a square 1 unit on a side.

20. If the worm never gets sent to the same computer twice, then it will infect 100 computers on the first round of forwarding, $100^2 = 10{,}000$ other computers on the second round of forwarding, and so on. Therefore the maximum number of different computers this one computer can infect is $100 + 100^2 + 100^3 + 100^4 + 100^5 = 10{,}101{,}010{,}100$. This figure of ten billion is probably comparable to the total number of computers in the world.

22. **a)** We want to solve $n(n-1) = 110$, or $n^2 - n - 110 = 0$. Simple algebra gives $n = 11$ (we ignore $n = -10$, since we need a positive integer for our answer).
b) We recall that $7! = 5040$, so the answer is 7.
c) We need to solve the equation $n(n-1)(n-2)(n-3) = 12n(n-1)$. Since we have $n \geq 4$ in order for $P(n,4)$ to be defined, this equation reduces to $(n-2)(n-3) = 12$, or $n^2 - 5n - 6 = 0$. Simple algebra gives $n = 6$ (we ignore the solution $n = -1$ since n needs to be a positive integer).

24. An algebraic proof is straightforward. We will give a combinatorial proof of the equivalent identity $P(n+1,r)(n+1-r) = (n+1)P(n,r)$ (and in fact both of these equal $P(n+1,r+1)$). Consider the problem of writing down a permutation of $r+1$ objects from a collection of $n+1$ objects. We can first write down a permutation of r of these objects ($P(n+1,r)$ ways to do so), and then write down one more object (and there are $n+1-r$ objects left to choose from), thereby obtaining the left-hand side; or we can first choose an object to write down first ($n+1$ to choose from), and then write down a permutation of length r using the n remaining objects ($P(n,r)$ ways to do so), thereby obtaining the right-hand side.

26. First note that Corollary 2 of Section 5.4 is equivalent to the assertion that the sum of the numbers $C(n,k)$ for even k is equal to the sum of the numbers $C(n,k)$ for odd k. Since $C(n,k)$ counts the number of subsets of size k of a set with n elements, we need to show that a set has as many even-sized subsets as it has odd-sized subsets. Define a function f from the set of all subsets of A to itself (where A is a set with n elements, one of which is a), by setting $f(B) = B \cup \{a\}$ if $a \notin B$, and $f(B) = B - \{a\}$ if $a \in B$. It is clear that f takes even-sized subsets to odd-sized subsets and vice versa, and that f is one-to-one and onto (indeed, $f^{-1} = f$). Therefore f restricted to the set of subsets of odd size gives a one-to-one correspondence between that set and the set of subsets of even size.

28. The base case is $n = 2$, in which case the identity simply states that $1 = 1$. Assume the inductive hypothesis, that $\sum_{j=2}^{n} C(j,2) = C(n+1,3)$. Then

$$\begin{aligned}\sum_{j=2}^{n+1} C(j,2) &= \left(\sum_{j=2}^{n} C(j,2)\right) + C(n+1,2)\\ &= C(n+1,3) + C(n+1,2) = C((n+1)+1,3)\,,\end{aligned}$$

as desired. The last equality made use of Pascal's identity.

30. **a)** For a fixed k, a triple is totally determined by picking i and j; since each can be picked in k ways (each can be any number from 0 to $k-1$, inclusive), there are k^2 ways to choose the triple. Adding over all possible values of k gives the indicated sum.

b) A triple of this sort is totally determined by knowing the *set* of numbers $\{i,j,k\}$, since the order is fixed. Therefore the number of triples of each kind is just the number of sets of 3 elements chosen from the set $\{0,1,2,\ldots,n\}$, and that is clearly $C(n+1,3)$.

c) In order for i to equal j (with both less than k), we need to pick two elements from $\{0,1,2,\ldots,n\}$, using the larger one for k and the smaller one for both i and j. Therefore there are as many such choices as there are 2-element subsets of this set, namely $C(n+1,2)$.

d) This part is its own proof. The last equality follows from elementary algebra.

32. **a)** If we 2-color the $2d-1$ elements of S, then there must be at least d elements of one color (if there were $d-1$ or fewer elements of both colors, then only $2d-2$ elements would be colored); this is just an application of the generalized pigeonhole principle. Thus there is a d-element subset that does not contain both colors, in violation of the condition for being 2-colorable.

b) We must show that every collection of fewer than three sets each containing two elements is 2-colorable, and that there is a collection of three sets each containing two elements that is not 2-colorable. The second statement follows from part **(a)**, with $d = 2$ (the three sets are $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$). On the other hand, if we have two (or fewer) sets each with two elements, then we can color the two elements of the first set with different colors, and we cannot be prevented from properly coloring the second set, since it must contain an element not in the first set.

c) First we show that the given collection is not 2-colorable. Without loss of generality, assume that 1 is red. If 2 is red, then 6 must be blue (second set). Thus either 4 or 5 must be red (seventh set), which means that 3 must be blue (first or fourth set). This would force 7 to be red (sixth set), which would force both 4 and 5 to be blue (third and fifth sets), a contradiction. Thus 2 is blue. If 3 is red, then we can conclude that 5 is blue, 7 is red, 6 is blue, and 4 is blue, making the last set improperly colored. Thus 3 is blue. This implies that 4 is red, hence 7 is blue, hence 5 and 6 are red, another contradiction. So the given collection cannot be 2-colored. Next we must show that all collections of six sets with three elements each are 2-colorable. Since having more elements in S at our disposable only makes it easier to 2-color the collection, we can assume that S has only five elements; let $S = \{a,b,c,d,e\}$. Since there are 18 occurrences of elements in the collection, some element, say a, must occur at least four times (since $3 \cdot 5 < 18$). If a occurs in six of the sets, then we can color a red and the rest of the elements blue. If a occurs in five of the sets, suppose without loss of generality that b and c occur in the sixth set. Then we can color a and b red and the remaining elements blue. Finally, if a occurs in only four of the sets, then that leaves only four elements for the last two sets, and therefore a pair of elements must be shared by them, say b and c. Again coloring a and b red and the remaining elements blue gives the desired coloring.

34. We might as well assume that the first person sits in the northernmost seat. Then there are $P(7,7)$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is $7! = 5040$.

36. We need to know the number of solutions to $d + m + g = 12$, where d, m, and g are integers greater than or equal to 3. This is equivalent to the number of nonnegative integer solutions to $d' + m' + g' = 3$, where $d' = d-3$, $m' = m-3$, and $g' = g-3$. By Theorem 2 of Section 5.5, the answer is $C(3+3-1,3) = C(5,3) = 10$.

38. **a)** By Theorem 3 of Section 5.5, the answer is $10!/(3!2!2!) = 151{,}200$.
b) If we fix the start and the end, then the question concerns only 8 letters, and the answer is $8!/(2!2!) = 10{,}080$.
c) If we think of the three P's as one letter, then the answer is seen to be $8!/(2!2!) = 10{,}080$.

40. There are 26 choices for the third letter. If the digit part of the plate consists of the digits 1, 2, and d, where d is different from 1 or 2, then there are 8 choices for d and $3! = 6$ choices for a permutation of these digits. If $d = 1$ or 2, then there are 2 choices for d and 3 choices for a permutation. Therefore the answer is $26(8 \cdot 6 + 2 \cdot 3) = 1404$.

42. Let us look at the girls first. There are $P(8,8) = 8! = 40320$ ways to order them relative to each other. This much work produces 9 gaps between girls (including the ends), in each of which at most one boy may sit. We need to choose, in order without repetition, 6 of these gaps, and this can be done in $P(9,6) = 60480$ ways. Therefore the answer is, by the product rule, $40320 \cdot 60480 = 2{,}438{,}553{,}600$.

44. We are given no restrictions, so any number of the boxes can be occupied once we have distributed the objects.
a) This is a straightforward application of the product rule; there are $6^5 = 7776$ ways to do this, because there are 6 choices for each of the 5 objects.
b) This is similar to Exercise 50 in Section 5.5. We compute this using the formulae:

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32-2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243 - 96 + 3) = 25$$

$$S(5,4) = \frac{1}{4!}\left(\binom{4}{0}4^5 - \binom{4}{1}3^5 + \binom{4}{2}2^5 - \binom{4}{3}1^5\right) = \frac{1}{4!}(1024 - 972 + 192 - 4) = 10$$

$$S(5,5) = \frac{1}{5!}\left(\binom{5}{0}5^5 - \binom{5}{1}4^5 + \binom{5}{2}3^5 - \binom{5}{3}2^5 + \binom{5}{4}1^5\right) = \frac{1}{5!}(3125 - 5120 + 2430 - 320 + 5) = 1$$

$$\sum_{j=1}^{5} S(5,j) = 1 + 15 + 25 + 10 + 1 = 52$$

c) This is asking for the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5$ in nonnegative integers. By Theorem 2 (see also Example 5) in Section 5.5, the answer is $C(6+5-1,5) = C(10,5) = 252$.

d) This is asking for the number of partitions of 5 (into at most six parts, but that is moot). We list them: $5 = 5$, $5 = 4 + 1$, $5 = 3 + 2$, $5 = 3 + 1 + 1$, $5 = 2 + 2 + 1$, $5 = 2 + 1 + 1 + 1$, $5 = 1 + 1 + 1 + 1 + 1$. Therefore the answer is 7.

46. Assume without loss of generality that we wish to form r-combinations from the set $\{1, 2, \ldots, n\}$. We modify Algorithm 3 in Section 5.6 for generating the next r-combination in lexicographic order, allowing for repetition. Then we generate all such combinations by starting with $11\ldots1$ and calling this modified algorithm $C(n + r - 1, r) - 1$ times (this will give us $nn\ldots n$ as the last one).

> **procedure** *next r-combination*$(a_1, a_2, \ldots, a_r$: integers)
> {We assume that $1 \le a_1 \le a_2 \le \cdots \le a_r \le n$, with $a_1 \ne n$}
> $i := r$
> **while** $a_i = n$
> $i := i - 1$
> $a_i := a_i + 1$
> **for** $j := i + 1$ **to** r
> $a_j := a_i$

48. One needs to play around with this enough to eventually discover a situation satisfying the conditions. Here is a way to do it. Suppose the group consists of three men and three women, and suppose that people of the same sex are always enemies and people of the opposite sex are always friends. Then clearly there can be no set of four mutual enemies, because any set of four people must include at least one man and one woman (since there are only three of each sex in the whole group). Also there can be no set of three mutual friends, because any set of three people must include at least two people of the same sex (since there are only two sexes).