SECTION 8.5 Equivalence Relations

This section is extremely important. If you do nothing else, do Exercise 9 and understand it, for it deals with the most common instances of equivalence relations. Exercise 16 is interesting—it hints at what fractions really are (if understood properly) and perhaps helps to explain why children (and adults) usually have so much trouble with fractions: they really involve equivalence relations. Spend some time thinking about fractions in this context. (See also Writing Project 4 for this chapter.)

It is usually easier to understand equivalence relations in terms of the associated partition—it's a more concrete visual image. Thus make sure you understand exactly what Theorem 2 says. Look at Exercise 67 for the relationship between equivalence relations and closures.

1. In each case we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair $(1,1)$ is missing. It is also not transitive, since the pairs $(0,2)$ and $(2,3)$ are there, but not $(0,3)$.

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs $(1,3)$ and $(3,2)$ are present, but not $(1,2)$.

e) This relation would be an equivalence relation were the pair $(2,1)$ present. As it is, its absence makes the relation neither symmetric nor transitive.

3. As in Exercise 1, we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation, one of the general form that two things are considered equivalent if they have the same "something" (see Exercise 9 for a formalization of this idea). In this case the "something" is the value at 1.

b) This is not an equivalence relation because it is not transitive. Let $f(x) = 0$, $g(x) = x$, and $h(x) = 1$ for all $x \in \mathbf{Z}$. Then f is related to g since $f(0) = g(0)$, and g is related to h since $g(1) = h(1)$, but f is not related to h since they have no values in common. By inspection we see that this relation is reflexive and symmetric.

c) This relation has none of the three properties. It is not reflexive, since $f(x) - f(x) = 0 \neq 1$. It is not symmetric, since if $f(x) - g(x) = 1$, then $g(x) - f(x) = -1 \neq 1$. It is not transitive, since if $f(x) - g(x) = 1$ and $g(x) - h(x) = 1$, then $f(x) - h(x) = 2 \neq 1$.

d) This is an equivalence relation. Two functions are related here if they differ by a constant. It is clearly reflexive (the constant is 0). It is symmetric, since if $f(x) - g(x) = C$, then $g(x) - f(x) = -C$. It is transitive, since if $f(x) - g(x) = C_1$ and $g(x) - h(x) = C_2$, then $f(x) - h(x) = C_3$, where $C_3 = C_1 + C_2$ (add the first two equations).

e) This relation is not reflexive, since there are lots of functions f (for instance, $f(x) = x$) that do not have the property that $f(0) = f(1)$. It is symmetric by inspection (the roles of f and g are the same). It is not transitive. For instance, let $f(0) = g(1) = h(0) = 7$, and let $f(1) = g(0) = h(1) = 3$; fill in the remaining values arbitrarily. Then f and g are related, as are g and h, but f is not related to h since $7 \neq 3$.

5. Obviously there are many possible answers here. We can say that two buildings are equivalent if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year (as long as there was at least one building opened that year). For another example, we can define two buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on (one class for each n for which there is at least one n-story building). In our third example, partition the set of all buildings into two classes—those in which you do have a class this semester and those in which you don't. (We assume that each of these is nonempty.) Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class (including itself).

7. Two propositions are equivalent if their truth tables are identical. This relation is reflexive, since the truth table of a proposition is identical to itself. It is symmetric, since if p and q have the same truth table, then q and p have the same truth table. There is one technical point about transitivity that should be noted. We need to assume that the truth tables, as we consider them for three propositions p, q, and r, have the same atomic variables in them. If we make this assumption (and it cannot hurt to do so, since adding information about extra variables that do not appear in a pair of propositions does not change the truth value of the propositions), then we argue in the usual way: if p and q have identical truth tables, and if q and r have identical truth tables, then p and r have that same common truth table. The proposition $\mathbf{T}$ is always true; therefore the equivalence class for this proposition consists of all propositions that are always true, no matter what truth values the atomic variables have. Recall that we call such a proposition a tautology. Therefore the equivalence class of $\mathbf{T}$ is the set of all tautologies. Similarly, the equivalence class of $\mathbf{F}$ is the set of all contradictions.

9. This is an important exercise, since very many equivalence relations are of this form. (In fact, all of them are—see Exercise 10.)

a) This relation is reflexive, since obviously $f(x) = f(x)$ for all $x \in A$. It is symmetric, since if $f(x) = f(y)$, then $f(y) = f(x)$ (this is one of the fundamental properties of equality). It is transitive, since if $f(x) = f(y)$ and $f(y) = f(z)$, then $f(x) = f(z)$ (this is another fundamental property of equality).

b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$. This is by definition just the inverse image of $f(x)$. Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

11. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to its first 3 bits.

13. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to the ordered pair (b_1, b_3), where b_1 is the first bit of the string and b_3 is the third bit of the string. Two bit strings agree on their first and third bits if and only if the corresponding ordered pairs for these two strings are equal ordered pairs.

15. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x-y$. Therefore by Exercise 9 this is an equivalence relation. If we want a more explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a+b = b+a$. For symmetry, $((a,b),(c,d)) \in R$ if and only if $a+d = b+c$, which is equivalent to $c+b = d+a$, which is true if and only if $((c,d),(a,b)) \in R$. For transitivity, suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Thus we have $a+d = b+c$ and $c+e = d+f$. Adding, we obtain $a+d+c+e = b+c+d+f$. Simplifying, we have $a+e = b+f$, which tells us that $((a,b),(e,f)) \in R$.

17. a) This follows from Exercise 9, where the function f from the set of differentiable functions (from $\mathbf{R}$ to $\mathbf{R}$) to the set of functions (from $\mathbf{R}$ to $\mathbf{R}$) is the differentiation operator—i.e., f of a function g is the function g'. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) We are asking for all functions that have the same derivative that the function $f(x) = x^2$ has, i.e., all functions of x whose derivative is $2x$. In other words, we are asking for the general antiderivative of $2x$, and we know that $\int 2x = x^2 + C$, where C is any constant. Therefore the functions in the same equivalence class as $f(x) = x^2$ are all the functions of the form $g(x) = x^2 + C$ for some constant C. Indefinite integrals in calculus, then, give equivalence classes of functions as answers, not just functions.

19. This follows from Exercise 9, where the function f from the set of all URLs to the set of all Web pages is the function that assigns to each URL the Web page for that URL.

21. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is not transitive, since the edges (c,d) and (d,c) are missing.

23. As in Exercise 21, this relation is not transitive, since several required edges are missing (such as (a,c)).

25. This follows from Exercise 9, with f being the function from bit strings to nonnegative integers given by $f(s) =$ the number of 1's in s.

27. Only parts **(a)** and **(b)** are relevant here, since the others are not equivalence relations.

a) An equivalence class is the set of all people who are the same age. (To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what one meant by "the same age." For example, we could define two people to be the same age if their official dates of birth were identical. In that case, everybody born on April 25, 1948, for example, would constitute one equivalence class.)

b) For each pair (m,f) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children. (In real life, of course, this is complicated by such things as divorce and remarriage.)

29. The equivalence class of 011 is the set of all bit strings that are related to 011, namely the set of all bit strings that have the same number of 1's as 011. In other words, it is the (infinite) set of all bit strings with exactly 2 1's: $\{11, 110, 101, 011, 1100, 1010, 1001, \ldots\}$.

31. a) We need the first three bits of each string in the equivalence class to agree with the first three bits of 010. Thus this equivalence class is the (infinite) set of all bit strings that begin 010, which we can list as $\{010, 0100, 0101, 01000, 01001, 01010, \ldots\}$.

b) We need the first three bits of each string in the equivalence class to agree with the first three bits of 1011. Thus this equivalence class is the (infinite) set of all bit strings that begin 101, which we can list as $\{101, 1010, 1011, 10100, 10101, 10110, \ldots\}$.

c) We need the first three bits of each string in the equivalence class to agree with the first three bits of 11111. Thus this equivalence class is the (infinite) set of all bit strings that begin 111, which we can list as $\{111, 1110, 1111, 11100, 11101, 11110, \ldots\}$.

d) This string is in the equivalence class given in part **(a)**. Therefore its equivalence class is the same.

33. This is like Example 15. Each bit string of length less than 4 is in an equivalence class by itself ($[\lambda]_{R_4} = \{\lambda\}$, $[0]_{R_4} = \{0\}$, $[1]_{R_4} = \{1\}$, $[00]_{R_4} = \{00\}$, $[01]_{R_4} = \{01\}$, $\ldots$, $[111]_{R_4} = \{111\}$). This accounts for $1+2+4+8 = 15$ equivalence classes. The remaining 16 equivalence classes are determined by the bit strings of length 4:

$$\begin{aligned}
[0000]_{R_4} &= \{0000, 00000, 00001, 000000, 000001, 000010, 000011, 0000000, \ldots\}\\
[0001]_{R_4} &= \{0001, 00010, 00011, 000100, 000101, 000110, 000111, 0001000, \ldots\}\\
[0010]_{R_4} &= \{0010, 00100, 00101, 001000, 001001, 001010, 001011, 0010000, \ldots\}\\
&\ \ \vdots\\
[1111]_{R_4} &= \{1111, 11110, 11111, 111100, 111101, 111110, 111111, 1111000, \ldots\}
\end{aligned}$$

35. We have by definition that $[n]_5 = \{\, i \mid i \equiv n \pmod 5 \,\}$.

a) $[2]_5 = \{\, i \mid i \equiv 2 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

b) $[3]_5 = \{\, i \mid i \equiv 3 \pmod 5 \,\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

c) $[6]_5 = \{\, i \mid i \equiv 6 \pmod 5 \,\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

d) $[-3]_5 = \{\, i \mid i \equiv -3 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ (the same as $[2]_5$)

37. This is very similar to Example 14. There are 6 equivalence classes, namely

$$\begin{aligned}
[0]_6 &= \{\ldots, -12, -6, 0, 6, 12, \ldots\}, \\
[1]_6 &= \{\ldots, -11, -5, 1, 7, 13, \ldots\}, \\
[2]_6 &= \{\ldots, -10, -4, 2, 8, 14, \ldots\}, \\
[3]_6 &= \{\ldots, -9, -3, 3, 9, 15, \ldots\}, \\
[4]_6 &= \{\ldots, -8, -2, 4, 10, 16, \ldots\}, \\
[5]_6 &= \{\ldots, -7, -1, 5, 11, 17, \ldots\}.
\end{aligned}$$

Another way to describe this collection is to say that it is the collection of sets $\{\, 6n + k \mid n \in \mathbf{Z} \,\}$ for $k = 0, 1, 2, 3, 4, 5$.

39. a) We observed in the solution to Exercise 15 that (a, b) is equivalent to (c, d) if $a - b = c - d$. Thus because $1 - 2 = -1$, we have $[(1, 2)] = \{\, (a, b) \mid a - b = -1 \,\} = \{(1, 2), (3, 4), (4, 5), (5, 6), \ldots\}$.

b) Since the equivalence class of (a, b) is entirely *determined* by the integer $a - b$, which can be negative, positive, or zero, we can interpret the equivalences classes as *being* the integers. This is a standard way to *define* the integers once we have defined the whole numbers.

41. The sets in a partition must be nonempty, pairwise disjoint, and have as their union all of the underlying set.

a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

b) This is a partition. c) This is a partition.

d) This is not a partition, since none of the sets includes the element 3.

43. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set of 256 bit strings of length 8.

a) This is a partition, since strings must begin either 1 or 0, and those that begin 0 must continue with either 0 or 1 in their second position. It is clear that the three subsets satisfy the conditions.

b) This is not a partition, since these subsets are not pairwise disjoint. The string 00000001, for example, contains both 00 and 01.

c) This is clearly a partition. Each of these four subsets contains 64 bit strings, and no two of them overlap.

d) This is not a partition, because the union of these subsets is not the entire set. For example, the string 00000010 is in none of the subsets.

e) This is a partition. Each bit string contains some number of 1's. This number can be identified in exactly one way as of the form $3k$, the form $3k+1$, or the form $3k+2$, where k is a nonnegative integer; it really is just looking at the equivalence classes of the number of 1's modulo 3.

45. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set $\mathbf{Z} \times \mathbf{Z}$.

a) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2, 3)$, for example, is in both of the first two subsets listed.

b) This is a partition. Every pair satisfies exactly one of the conditions listed about the parity of x and y, and clearly these subsets are nonempty.

c) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2,3)$, for example, is in both of the first two subsets listed. Also, $(0,0)$ is in none of the subsets.

d) This is a partition. Every pair satisfies exactly one of the conditions listed about the divisibility of x and y by 3, and clearly these subsets are nonempty.

e) This is a partition. Every pair satisfies exactly one of the conditions listed about the positiveness of x and y, and clearly these subsets are nonempty.

f) This is not a partition, because the union of these subsets is not all of $\mathbf{Z} \times \mathbf{Z}$. In particular, $(0,0)$ is in none of the parts.

47. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$

b) $\{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$

c) $\{(0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2),\ (2,0),\ (2,1),\ (2,2),\ (3,3),\ (3,4),\ (3,5),\ (4,3),\ (4,4),\ (4,5),\ (5,3),$ $(5,4),(5,5)\}$

d) $\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$

49. We need to show that every equivalence class modulo 6 is contained in an equivalence class modulo 3. We claim that in fact, for each $n \in \mathbf{Z}$, $[n]_6 \subseteq [n]_3$. To see this suppose that $m \in [n]_6$. This means that $m \equiv n \pmod 6$, i.e., that $m-n$ is a multiple of 6. Then perforce $m-n$ is a multiple of 3, so $m \equiv n \pmod 3$, which means that $m \in [n]_3$.

51. By the definition given in the preamble to Exercise 49, we need to show that every set in the first partition is a subset of some set in the second partition. Let A be a set in the first partition. So A is the set of all bit strings of length 16 that agree on their last eight bits. Pick a particular element x of A, and suppose that the last four bits of x are $abcd$. Then the set of all bit strings of length 16 whose last four bits are $abcd$ is one of the sets in the second partition, and clearly every string in A is in that set, since every string in A agrees with x on the last eight bits, and therefore perforce agrees on the last four bits.

53. We are asked to show that every equivalence class for R_{31} is a subset of some equivalence class for R_8. Let $[x]_{R_{31}}$ be an arbitrary equivalence class for R_{31}. We claim that $[x]_{R_{31}} \subseteq [x]_{R_8}$; proving this claim finishes the proof. To show that one set is a subset of another set, we choose an arbitrary element y in the first set and show that it is also an element of the second set. In this case since $y \in [x]_{R_{31}}$, we know that y is equivalent to x under R_{31}, that is, that either $y = x$ or y and x are each at least 31 characters long and agree on their first 31 characters. Because strings that are at least 31 characters long and agree on their first 31 characters perforce are at least 8 characters long and on their first 8 characters, we know that either $y = x$ or y and x are each at least 8 characters long and agree on their first 8 characters. This means that y is equivalent to x under R_8, that is, that $y \in [x]_{R_8}$.

55. We need first to make the relation symmetric, so we add the pairs (b,a), (c,a), and (e,d). Then we need to make it transitive, so we add the pairs (b,c), (c,b), (a,a), (b,b), (c,c), (d,d), and (e,e). (In other words, we formed the transitive closure of the symmetric closure of the original relation.) It happens that we have already achieved reflexivity, so we are done; if there had been some pairs (x,x) missing at this point, we would have added them as well. Thus the desired equivalence relation is the one consisting of the original 3 pairs and the 10 we have added. There are two equivalence classes, $\{a,b,c\}$ and $\{d,e\}$.

57. **a)** The equivalence class of 1 is the set of all real numbers that differ from 1 by an integer. Obviously this is the set of all integers.

b) The equivalence class of $1/2$ is the set of all real numbers that differ from $1/2$ by an integer, namely $1/2$, $3/2$, $5/2$, etc., and $-1/2$, $-3/2$, etc. These are often called **half-integers**. We could write this set as $\{ (2n+1)/2 \mid n \in \mathbf{Z} \}$, among other ways.

59. This problem actually deals with a branch of mathematics called group theory; the object being studied here is related to a certain dihedral group. If this fascinates you, you might want to take a course with a title like Abstract Algebra or Modern Algebra, in which such things are studied in depth.

In order to have a way to talk about specific colorings, let us agree that a sequence of length four, each element of which is either r or b, represents a coloring of the 2×2 checkerboard, where the first letter denotes the color of the upper left square, the second letter denotes the color of the upper right square, the third letter denotes the color of the lower left square, and the fourth letter denotes the color of the lower right square. For example, the board in which every square is red except the upper right would be represented by $rbrr$. There are really only four different rotations, since after the rotation we need to end up with another checkerboard (and we can assume that the edges of the board are horizontal and vertical). If we rotate our sample coloring $90°$ clockwise, then we obtain the coloring $rrrb$; if we rotate it $180°$, then we obtain the coloring $rrbr$; if we rotate it $270°$ clockwise (or $90°$ counterclockwise), then we obtain the coloring $brrr$; and if we rotate it $360°$ clockwise (or $0°$—i.e., not at all), then we obtain the coloring $rbrr$ itself back. Note also that some colorings are *invariant* (i.e., unchanged) under rotations in addition to the $360°$ one; for example, $bbbb$ is invariant under all rotations, and $brrb$ is invariant under a $180°$ rotation. Similarly there are four reflections: around the center vertical axis of the board, around the center horizontal axis, around the lower-left-to-upper-right diagonal, and around the lower-right-to-upper-left diagonal. For example, applying the vertical axis reflection to $rrbb$ yields itself, while applying the lower-left-to-upper-right diagonal reflection results in $brbr$.

The definition of equivalence for this problem makes the proof rather messy, since both rotations and reflections are involved, and it is required that we reduce everything to just one or two operations. In fact, we claim that there are only eight possible motions of this square: clockwise rotations of $0°$, $90°$, $180°$, or $270°$, and reflections through the vertical, horizontal, lower-left-to-upper-right, and lower-right-to-upper-left diagonals. To verify this, we must show that the composition of every two of these operations is again an operation in our list. Below is the "group table" that shows this, where we use the symbols $r0$, $r90$, $r180$, $r270$, fv, fh, fp, and fn for these operations, respectively. (The mnemonic is that r stands for "rotation," f stands for "flip," and v, h, p, and n stand for "vertical," "horizontal", "positive-sloping," and "negative-sloping," respectively.) It is read just like a multiplication table, with the operation $\circ$ meaning "followed by." For example, if we first perform $r90$ and then perform fh, then we get the same result as if we had just performed fp (try it!).

$\circ$	$r0$	$r90$	$r180$	$r270$	fv	fh	fp	fn
$r0$	$r0$	$r90$	$r180$	$r270$	fv	fh	fp	fn
$r90$	$r90$	$r180$	$r270$	$r0$	fn	fp	fv	fh
$r180$	$r180$	$r270$	$r0$	$r90$	fh	fv	fn	fp
$r270$	$r270$	$r0$	$r90$	$r180$	fp	fn	fh	fv
fv	fv	fp	fh	fn	$r0$	$r180$	$r90$	$r270$
fh	fh	fn	fv	fp	$r180$	$r0$	$r270$	$r90$
fp	fp	fh	fn	fv	$r270$	$r90$	$r0$	$r180$
fn	fn	fv	fp	fh	$r90$	$r270$	$r180$	$r0$

So the result of this computation is that we can consider only these eight moves, and not have to worry about combinations of them—every combination of moves equals just one of these eight.

a) To show reflexivity, we note that every coloring can be obtained from itself via a 0° rotation. In technical terms, the 0° rotation is the *identity element* of our group. To show symmetry, we need to observe that rotations and reflections have inverses: If C_1 comes from C_2 via a rotation of n° clockwise, then C_2 comes from C_1 via a rotation of n° counterclockwise (or equivalently, via a rotation of $(360-n)^\circ$ clockwise); and every reflection applied twice brings us back to the position (and therefore coloring) we began with. And transitivity follows from the fact that the composition of two of these operations is again one of these operations.

b) The equivalence classes are represented by colorings that are truly distinct, in the sense of not being obtainable from each other via these operations. Let us list them. Clearly there is just one coloring using four red squares, and so just one equivalence class, $[rrrr]$. Similarly there is only one using four blues, $[bbbb]$. There is also just one equivalence class of colorings using three reds and one blue, since no matter which corner the single blue occupies in such a coloring, we can rotate to put the blue in any other corner. Thus our third and fourth equivalence classes are $[rrrb]$ and $[bbbr]$. Note that each of them contains four colorings. (For example, $[rrrb] = \{rrrb, rrbr, rbrr, brrr\}$.) This leaves only the colorings with two reds and two blues to consider. In every such coloring, either the red squares are adjacent (i.e., share a common edge), such as in $bbrr$, or they are not (e.g., $brrb$). Clearly the red squares are adjacent if and only if the blue ones are, since the only pairs of nonadjacent squares are (lower-left,upper-right) and (upper-left,lower-right). It is equally clear that there are only two colorings in which the red squares are not adjacent, namely $rbbr$ and $brrb$, and they are equivalent via a 90° rotation (among other transformations). So our fifth equivalence class is $[rbbr] = \{rbbr, brrb\}$. Finally, there is only one more equivalence class, and it contains the remaining four colorings (in which the two red squares are adjacent and the two blue squares are adjacent), namely $\{rrbb, brbr, bbrr, rbrb\}$, since each of these can be obtained from each of the others by a rotation. In summary we have partitioned the set of $2^4 = 16$ colorings (i.e., r-b strings of length four) into six equivalence classes, two of which have cardinality one, three of which have cardinality four, and one of which has cardinality two.

One final comment. We saw in the solution to part **(b)** that only rotations are needed to show the equivalence of every pair of equivalent colorings using just red and blue. This means that we are actually dealing with just part of the dihedral group here. If more colors had been used, then we would have needed to use the reflections as well. A complete discussion would get us into Pólya's theory of enumeration, which is studied in advanced combinatorics classes.

61. It is easier to write down a partition than it is to list the pairs in an equivalence relation, so we will answer the question using this notation. Let the set be $\{1,2,3\}$. We want to write down all possible partitions of this set. One partition is just $\{\{1,2,3\}\}$, i.e., having just one set (this corresponds to the equivalence relation in which every pair of elements are related). At the other extreme, there is the partition $\{\{1\},\{2\},\{3\}\}$, which corresponds to the equality relation (each x is related only to itself). The only other way to split up the elements of this set is into a set with two elements and a set with one element, and there are clearly three ways to do this, depending on which element we decide to put in the set by itself. Thus we get the partitions (pay attention to the punctuation!) $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, and $\{\{2,3\},\{1\}\}$. If we wished to list the ordered pairs, we could; for example, the relation corresponding to $\{\{2,3\},\{1\}\}$ is $\{(2,2),(2,3),(3,2),(3,3),(1,1)\}$. We found five partitions, so the answer to the question is 5.

63. We do get an equivalence relation. The issue is whether the relation formed in this way is reflexive, transitive and symmetric. It is clearly reflexive, since we included all the pairs (a,a) at the outset. It is clearly transitive, since the last thing we did was to form the transitive closure. It is symmetric by Exercise 23 in Section 8.4.

65. We end up with the relation R that we started with. Two elements are related if they are in the same set of the partition, but the partition is made up of the equivalence classes of R, so two elements are related precisely if they are related in R.

67. We make use of Exercise 63. Given the relation R, we first form the reflexive closure R' of R by adding to R each pair (a,a) that is not already there. Next we form the symmetric closure R'' of R', by adding, for each pair $(a,b) \in R'$ the pair (b,a) if it is not already there. Finally we apply Warshall's algorithm (or Algorithm 1) from Section 8.4 to form the transitive closure of R''. This is the smallest equivalence relation containing R.

69. The exercise asks us to compute $p(n)$ for $n = 0, 1, 2, \ldots, 10$. In doing this we will use the recurrence relation, building on what we have already computed (namely $p(n-j-1)$, noting that $n-j-1 < n$), as well as using the binomial coefficients $C(n-1,j) = \dfrac{(n-1)!}{j!(n-1-j)!}$. We organize our computation in the obvious way, using the formula in Exercise 68.

$$\begin{aligned}
p(0) &= 1 \quad \text{(the initial condition)}\\
p(1) &= C(0,0)p(0) = 1\cdot 1 = 1\\
p(2) &= C(1,0)p(1) + C(1,1)p(0) = 1\cdot 1 + 1\cdot 1 = 2\\
p(3) &= C(2,0)p(2) + C(2,1)p(1) + C(2,2)p(0) = 1\cdot 2 + 2\cdot 1 + 1\cdot 1 = 5\\
p(4) &= C(3,0)p(3) + C(3,1)p(2) + C(3,2)p(1) + C(3,3)p(0) = 1\cdot 5 + 3\cdot 2 + 3\cdot 1 + 1\cdot 1 = 15\\
p(5) &= C(4,0)p(4) + C(4,1)p(3) + C(4,2)p(2) + C(4,3)p(1) + C(4,4)p(0)\\
&= 1\cdot 15 + 4\cdot 5 + 6\cdot 2 + 4\cdot 1 + 1\cdot 1 = 52\\
p(6) &= C(5,0)p(5) + C(5,1)p(4) + C(5,2)p(3) + C(5,3)p(2) + C(5,4)p(1) + C(5,5)p(0)\\
&= 1\cdot 52 + 5\cdot 15 + 10\cdot 5 + 10\cdot 2 + 5\cdot 1 + 1\cdot 1 = 203\\
p(7) &= C(6,0)p(6) + C(6,1)p(5) + C(6,2)p(4) + C(6,3)p(3) + C(6,4)p(2) + C(6,5)p(1) + C(6,6)p(0)\\
&= 1\cdot 203 + 6\cdot 52 + 15\cdot 15 + 20\cdot 5 + 15\cdot 2 + 6\cdot 1 + 1\cdot 1 = 877\\
p(8) &= C(7,0)p(7) + C(7,1)p(6) + C(7,2)p(5) + C(7,3)p(4) + C(7,4)p(3) + C(7,5)p(2)\\
&\qquad + C(7,6)p(1) + C(7,7)p(0)\\
&= 1\cdot 877 + 7\cdot 203 + 21\cdot 52 + 35\cdot 15 + 35\cdot 5 + 21\cdot 2 + 7\cdot 1 + 1\cdot 1 = 4140\\
p(9) &= C(8,0)p(8) + C(8,1)p(7) + C(8,2)p(6) + C(8,3)p(5) + C(8,4)p(4) + C(8,5)p(3)\\
&\qquad + C(8,6)p(2) + C(8,7)p(1) + C(8,8)p(0)\\
&= 1\cdot 4140 + 8\cdot 877 + 28\cdot 203 + 56\cdot 52 + 70\cdot 15 + 56\cdot 5 + 28\cdot 2 + 8\cdot 1 + 1\cdot 1 = 21147\\
p(10) &= C(9,0)p(9) + C(9,1)p(8) + C(9,2)p(7) + C(9,3)p(6) + C(9,4)p(5) + C(9,5)p(4)\\
&\qquad + C(9,6)p(3) + C(9,7)p(2) + C(9,8)p(1) + C(9,9)p(0)\\
&= 1\cdot 21147 + 9\cdot 4140 + 36\cdot 877 + 84\cdot 203 + 126\cdot 52\\
&\qquad + 126\cdot 15 + 84\cdot 5 + 36\cdot 2 + 9\cdot 1 + 1\cdot 1 = 115975
\end{aligned}$$