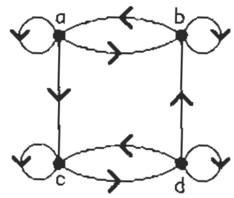
SECTION 8.4 Closures of Relations

This section is harder than the previous ones in this chapter. Warshall's algorithm, in particular, is fairly tricky, and Exercise 27 should be worked carefully, following Example 8. It is easy to forget to include the loops (a,a) when forming transitive closures "by hand."

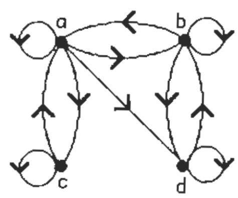
1. a) The reflexive closure of R is R together with all the pairs (a,a). Thus in this case the closure of R is $\{(0,0),(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(3,3)\}$.
   b) The symmetric closure of R is R together with all the pairs (b,a) for which (a,b) is in R. For example, since $(1,2)$ is in R, we need to add $(2,1)$. Thus the closure of R is $\{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,0)\}$.

3. To form the symmetric closure we need to add all the pairs (b,a) such that (a,b) is in R. In this case, that means that we need to include pairs (b,a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a,b) such that b divides a. Thus the closure is $\{\,(a,b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\,\}$.

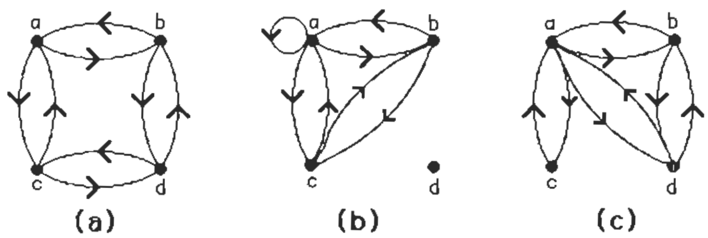
5. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



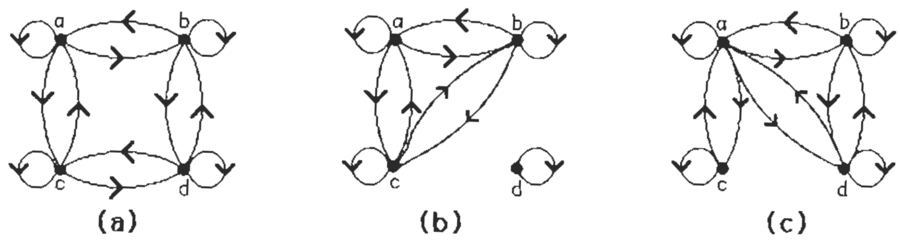
7. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



9. We form the symmetric closure by taking the given directed graph and appending an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there); in other words, we append the edge (b,a) whenever we see the edge (a,b). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



11. We are asked for the symmetric and reflexive closure of the given relation. We form it by taking the given directed graph and appending (1) a loop at each vertex at which there is not already a loop and (2) an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



13. The symmetric closure of R is $R \cup R^{-1}$. The matrix for R^{-1} is $\mathbf{M}_R^t$, as we saw in Exercise 12 in Section 8.3. The matrix for the union of two relations is the join of the matrices for the two relations, as we saw in Section 8.3. Therefore the matrix representing the symmetric closure of R is indeed $\mathbf{M}_R \vee \mathbf{M}_R^t$.

15. If R is already irreflexive, then it is clearly its own irreflexive closure. On the other hand if R is not irreflexive, then there is no relation containing R that is irreflexive, since the loop or loops in R prevent any such relation from being irreflexive. Thus in this case R has no irreflexive closure. This exercise shows essentially that the concept of "irreflexive closure" is rather useless, since no relation has one unless it is already irreflexive (in which case it is its own "irreflexive closure").

17. A circuit of length 3 can be written as a sequence of 4 vertices, each joined to the next by an edge of the given directed graph, ending at the same vertex at which it began. There are several such circuits here, and we just have to be careful and systematically list them all. There are the circuits formed entirely by the loops: $aaaa$, $cccc$, and $eeee$. The triangles $abea$ and $adea$ also qualify. Two circuits start at b: $bccb$ and $beab$. There are two more circuits starting at c, namely $ccbc$ and $cbcc$. Similarly there are the circuits $deed$, $eede$ and $edee$, as well as the other trips around the triangle: $eabe$, $dead$, and $eade$.

19. The way to form these powers is first to form the matrix representing R, namely

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

and then take successive Boolean powers of it to get the matrices representing R^2, R^3, and so on. Finally, for part **(f)** we take the join of the matrices representing R, R^2, ..., R^5. Since the matrix is a perfectly good way to express the relation, we will not list the ordered pairs.

a) The matrix for R^2 is the Boolean product of the matrix displayed above with itself, namely

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

b) The matrix for R^3 is the Boolean product of the first matrix displayed above with the answer to part **(a)**, namely

$$\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

c) The matrix for R^4 is the Boolean product of the first matrix displayed above with the answer to part (b), namely

$$\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

d) The matrix for R^5 is the Boolean product of the first matrix displayed above with the answer to part (c), namely

$$\mathbf{M}_{R^5} = \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

e) The matrix for R^6 is the Boolean product of the first matrix displayed above with the answer to part (d), namely

$$\mathbf{M}_{R^6} = \mathbf{M}_R^{[6]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

f) The matrix for R^* is the join of the first matrix displayed above and the answers to parts (a) through (d), namely

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \mathbf{M}_R^{[4]} \vee \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

21. a) The pair (a,b) is in R^2 if there is a person c other than a or b who is in a class with a and a class with b. Note that it is almost certain that (a,a) is in R^2, since as long as a is taking a class that has at least one other person in it, that person serves as the "c."

b) The pair (a,b) is in R^3 if there are persons c (different from a) and d (different from b and c) such that c is in a class with a, c is in a class with d, and d is in a class with b.

c) The pair (a,b) is in R^* if there is a sequence of persons, $c_0, c_1, c_2, \ldots, c_n$, with $n \geq 1$, such that $c_0 = a$, $c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is in at least one class with c_i.

23. Suppose that $(a,b) \in R^*$; then there is a path from a to b in (the digraph for) R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R (following the given path backwards). This means that (b,a) is in R^* whenever (a,b) is, exactly what we needed to prove.

25. Algorithm 1 finds the transitive closure by computing the successive powers and taking their join. We exhibit our answers in matrix form as $\mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \ldots \vee \mathbf{M}_R^{[n]} = \mathbf{M}_{R^*}$.

a) $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b) $$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

c) $$\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \vee \begin{bmatrix}0&0&1&1\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \vee \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \vee \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} = \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Note that the relation was already transitive, so its transitive closure is itself.

d) $$\begin{bmatrix}1&0&0&1\\1&0&1&0\\1&1&0&1\\0&1&0&0\end{bmatrix} \vee \begin{bmatrix}1&1&0&1\\1&1&0&1\\1&1&1&1\\1&0&1&0\end{bmatrix} \vee \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&0&1\end{bmatrix} \vee \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} = \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

27. In Warshall's algorithm (Algorithm 2 in this section), we compute a sequence of matrices $\mathbf{W}_0$ (the matrix representing R), $\mathbf{W}_1$, $\mathbf{W}_2$, ..., $\mathbf{W}_n$, the last of which represents the transitive closure of R. Each matrix $\mathbf{W}_k$ comes from the matrix $\mathbf{W}_{k-1}$ in the following way. The $(i,j)^{\text{th}}$ entry of $\mathbf{W}_k$ is the "$\vee$" of the $(i,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$ with the "$\wedge$" of the $(i,k)^{\text{th}}$ entry and the $(k,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$. We will exhibit our solution by listing the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$, in that order; $\mathbf{W}_4$ represents the answer. In each case $\mathbf{W}_0$ is the matrix of the given relation. To compute the next matrix in the solution, we need to compute it one entry at a time, using the eqnation jnst discussed (the "$\vee$" of the corresponding entry in the previous matrix with the "$\wedge$" of two entries in the old matrix), i.e., as i and j each go from 1 to 4, we need to write down the $(i,j)^{\text{th}}$ entry using this formula. Note that in computing $\mathbf{W}_k$ the k^{th} row and the k^{th} column are unchanged, but some of the entries in other rows and columns may change.

a) $$\begin{bmatrix}0&1&0&0\\1&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&0&0\\1&1&1&0\\0&0&0&1\\1&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&1&0\\1&1&1&0\\0&0&0&1\\1&1&1&0\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\\1&1&1&1\\0&0&0&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

b) $$\begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&0\\1&0&0&1\\1&0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&1\\1&0&0&1\\1&0&1&1\end{bmatrix} \quad \begin{bmatrix}0&0&0&0\\1&0&1&1\\1&0&1&1\\1&0&1&1\end{bmatrix}$$

c) $$\begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&1&1&1\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Note that the relation was already transitive, so each matrix in the sequence was the same.

d) $$\begin{bmatrix}1&0&0&1\\1&0&1&0\\1&1&0&1\\0&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&0&1&1\\1&1&0&1\\0&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&0&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}1&0&0&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$$

29. **a)** We need to include at least the transitive closure, which we can compute by Algorithm 1 or Algorithm 2 to be (in matrix form) $\begin{bmatrix}1&1&0&1\\0&0&0&0\\0&0&1&0\\1&1&0&1\end{bmatrix}$. All we need in addition is the pair $(2,2)$ in order to make the relation reflexive. Note that the result is still transitive (the addition of a pair (a,a) cannot make a transitive relation no longer transitive), so our answer is $\begin{bmatrix}1&1&0&1\\0&1&0&0\\0&0&1&0\\1&1&0&1\end{bmatrix}$.

b) The symmetric closure of the original relation is represented by $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. We need at least the transitive closure of this relation, namely $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Since it is also symmetric, we are done. Note that it would not have been correct to find first the transitive closure of the original matrix and then make it symmetric, since the pair $(2,2)$ would be missing. What is going on here is that the transitive closure of a symmetric relation is still symmetric, but the symmetric closure of a transitive relation might not be transitive.

c) Since the answer to part **(b)** was already reflexive, it must be the answer to this part as well.

31. Algorithm 1 has a loop executed $O(n)$ times in which the primary operation is the Boolean product computation (the join operation is fast by comparison). If we can do the product in $O(n^{2.8})$ bit operations, then the number of bit operations in the entire algorithm is $O(n \cdot n^{2.8}) = O(n^{3.8})$. Since Algorithm 2 does not use the Boolean product, a fast Boolean product algorithm is irrelevant, so Algorithm 2 still requires $O(n^3)$ bit operations.

33. There are two ways to go. One approach is to take the output of Algorithm 1 as it stands and then make sure that all the pairs (a,a) are included by forming the join with the identity matrix (specifically set $\mathbf{B} := \mathbf{B} \vee \mathbf{I}_n$). See the discussion in Exercise 29a for the justification. The other approach is to insure the reflexivity at the beginning by initializing $\mathbf{A} := \mathbf{M}_r \vee \mathbf{I}_n$; if we do this, then only paths of length strictly less than n need to be looked at, so we can change the n in the loop to $n-1$.

35. **a)** No relation that contains R is not reflexive, since R already contains all the pairs $(0,0)$, $(1,1)$, and $(2,2)$. Therefore there is no "nonreflexive" closure of R.

b) Suppose S were the closure of R with respect to this property. Since R does not have an odd number of elements, $S \neq R$, so S must be a proper superset of R. Clearly S cannot have more than 5 elements, for if it did, then any subset of S consisting of R and one element of $S - R$ would be a proper subset of S with the property; this would violate the requirement that S be a subset of every superset of R with the property. Thus S must have exactly 5 elements. Let T be another superset of R with 5 elements (there are $9 - 4 = 5$ such sets in all). Thus T has the property, but S is not a subset of T. This contradicts the definition. Therefore our original assumption was faulty, and the closure does not exist.