CHAPTER 9
Graphs

SECTION 9.1 Graphs and Graph Models

2. a) A simple graph would be the model here, since there are no parallel edges or loops, and the edges are undirected.

b) A multigraph would, in theory, be needed here, since there may be more than one interstate highway between the same pair of cities.

c) A pseudograph is needed here, to allow for loops.

4. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

6. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

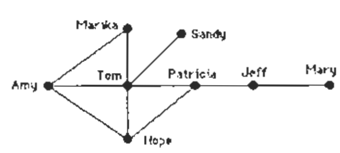
8. This is a directed multigraph; the edges are directed, and there are parallel edges.

10. The graph in Exercise 3 is simple. The multigraph in Exercise 4 can be made simple by removing one of the edges between a and b, and two of the edges between b and d. The pseudograph in Exercise 5 can be made simple by removing the three loops and one edge in each of the three pairs of parallel edges. The multigraph in Exercise 6 can be made simple by removing one of the edges between a and c, and one of the edges between b and d. The other three are not undirected graphs. (Of course removing any supersets of the answers given here are equally valid answers; in particular, we could remove *all* the edges in each case.)

12. If $u\,R\,v$, then there is an edge joining vertices u and v, and since the graph is undirected, this is also an edge joining vertices v and u. This means that $v\,R\,u$. Thus the relation is symmetric. The relation is reflexive because the loops guarantee that $u\,R\,u$ for each vertex u.

14. Since there are edges from Hawk to Crow, Owl, and Raccoon, the graph is telling us that the hawk competes with these three animals.

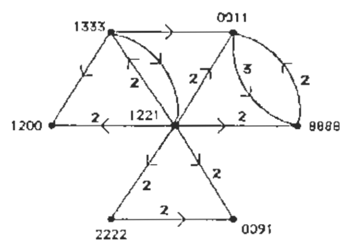
16. Each person is represented by a vertex, with an edge between two vertices if and only if the people are acquainted.



18. Fred influences Brian, since there is an edge from Fred to Brian. Yvonne and Deborah influence Fred, since there are edges from these vertices to Fred.

20. Team four beat the vertices to which there are edges from Team four, namely only Team three. The other teams—Team one, Team two, Team five, and Team six—all beat Team four, since there are edges from them to Team four.

22. This is a directed multigraph with one edge from a to b for each call made by a to b. Rather than draw the parallel edges with parallel lines, we have indicated what is intended by writing a numeral on the edge to indicate how many calls were made, if it was more than one.



24. This is similar to the use of directed graphs to model telephone calls.

a) We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

b) As in part (a) we use a directed edge for each message sent during the week.

26. Vertices with thousands or millions of edges going out from them could be such lists.

28. We make the subway stations the vertices, with an edge from station u to station v if there is a train going from u to v without stopping. It is quite possible that some segments are one-way, so we should use directed edges. (If there are no one-way segments, then we could use undirected edges.) There would be no need for multiple edges, unless we had two kinds of edges, maybe with different colors, to represent local and express trains. In that case, there could be parallel edges of different colors between the same vertices, because both a local and an express train might travel the same segment. There would be no point in having loops, because no passenger would want to travel from a station back to the same station without stopping.

30. The model says that the statements for which there are edges to S_6 must be executed before S_6, namely the statements S_1, S_2, S_3, and S_4.

32. The vertices in the directed graph represent cities. Whenever there is a nonstop flight from city A to city B, we put a directed edge into our directed graph from vertex A to vertex B, and furthermore we label that edge with the flight time. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with weighted edges. It is defined to be a triple (V, E, W), where (V, E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and W is a function from E to the set of nonnegative real numbers. Here we are simply thinking of $W(e)$ as the weight of edge e, which in this case is the flight time.

34. We can let the vertices represent people; an edge from u to v would indicate that u can send a message to v. We would need a directed multigraph in which the edges have labels, where the label on each edge indicates the form of communication (cell phone audio, text messaging, and so on).

SECTION 9.2 Graph Terminology and Special Types of Graphs

2. In this pseudograph there are 5 vertices and 13 edges. The degree of vertex a is 6, since in addition to the 4 nonloops incident to a, there is a loop contributing 2 to the degree. The degrees of the other vertices are $\deg(b) = 6$, $\deg(c) = 6$, $\deg(d) = 5$, and $\deg(e) = 3$. There are no pendant or isolated vertices in this pseudograph.

4. For the graph in Exercise 1, the sum is $2+4+1+0+2+3 = 12 = 2\cdot 6$; there are 6 edges. For the pseudograph in Exercise 2, the sum is $6+6+6+5+3 = 26 = 2\cdot 13$; there are 13 edges. For the pseudograph in Exercise 3, the sum is $3+2+4+0+6+0+4+2+3 = 24 = 2\cdot 12$; there are 12 edges.

6. Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1 the sum of the degrees is even (it is $2e$).

8. In this directed multigraph there are 4 vertices and 8 edges. The degrees are $\deg^-(a) = 2$, $\deg^+(a) = 2$, $\deg^-(b) = 3$, $\deg^+(b) = 4$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 1$.

10. For Exercise 7 the sum of the in-degrees is $3+1+2+1 = 7$, and the sum of the out-degrees is $1+2+1+3 = 7$; there are 7 edges. For Exercise 8 the sum of the in-degrees is $2+3+2+1 = 8$, and the sum of the out-degrees is $2+4+1+1 = 8$; there are 8 edges. For Exercise 9 the sum of the in-degrees is $6+1+2+4+0 = 13$, and the sum of the out-degrees is $1+5+5+2+0 = 13$; there are 13 edges.

12. Since there is an edge from a person to each of his or her acquaintances, the degree of v is the number of people v knows. An isolated vertex would be a person who knew on one, and a pendant vertex would be a person who knew just one other person (it is doubtful that there are many, if any, isolated or pendant vertices). If the average degree is 1000, then the average person knows 1000 other people.

14. Since there is an edge from a person to each of the other actors that person has appeared with in a movie, the degree of v is the number of other actors that person has appeared with. An isolated vertex would be a person who has appeared only in movies in which he or she was the only actor, and a pendant vertex would be a person who has appeared with only one other actor in any movie (it is doubtful that there are many, if any, isolated or pendant vertices).

16. Since there is an edge from a page to each page that it links to, the outdegree of a vertex is the number of links on that page, and the in-degree of a vertex is the number of other pages that have a link to it.

18. This is essentially the same as Exercise 36 in Section 5.2, where the graph models the "know each other" relation on the people at the party. See the solution given for that exercise. The number of people a person knows is the degree of the corresponding vertex in the graph.

20. a) This graph has 7 vertices, with an edge joining each pair of distinct vertices.