

# CHAPTER 1

## The Foundations: Logic and Proofs

### SECTION 1.1 Propositional Logic

2. Propositions must have clearly defined truth values, so a proposition must be a declarative sentence with no free variables.
- a) This is not a proposition; it's a command.
  - b) This is not a proposition; it's a question.
  - c) This is a proposition that is false, as anyone who has been to Maine knows.
  - d) This is not a proposition; its truth value depends on the value of  $x$ .
  - e) This is a proposition that is false.
  - f) This is not a proposition; its truth value depends on the value of  $n$ .
4. a) I did not buy a lottery ticket this week.
- b) Either I bought a lottery ticket this week or [in the inclusive sense] I won the million dollar jackpot on Friday.
  - c) If I bought a lottery ticket this week, then I won the million dollar jackpot on Friday.
  - d) I bought a lottery ticket this week and I won the million dollar jackpot on Friday.
  - e) I bought a lottery ticket this week if and only if I won the million dollar jackpot on Friday.
  - f) If I did not buy a lottery ticket this week, then I did not win the million dollar jackpot on Friday.
  - g) I did not buy a lottery ticket this week, and I did not win the million dollar jackpot on Friday.
  - h) Either I did not buy a lottery ticket this week, or else I did buy one and won the million dollar jackpot on Friday.
6. a) The election is not decided.
- b) The election is decided, or the votes have been counted.
  - c) The election is not decided, and the votes have been counted.
  - d) If the votes have been counted, then the election is decided.
  - e) If the votes have not been counted, then the election is not decided.
  - f) If the election is not decided, then the votes have not been counted.
  - g) The election is decided if and only if the votes have been counted.
  - h) Either the votes have not been counted, or else the election is not decided and the votes have been counted.
- Note that we were able to incorporate the parentheses by using the words *either* and *else*.
8. a) If you have the flu, then you miss the final exam.
- b) You do not miss the final exam if and only if you pass the course.
  - c) If you miss the final exam, then you do not pass the course.
  - d) You have the flu, or miss the final exam, or pass the course.
  - e) It is either the case that if you have the flu then you do not pass the course or the case that if you miss the final exam then you do not pass the course (or both, it is understood).
  - f) Either you have the flu and miss the final exam, or you do not miss the final exam and do pass the course.

10. a)  $r \wedge \neg q$     b)  $p \wedge q \wedge r$     c)  $r \rightarrow p$     d)  $p \wedge \neg q \wedge r$     e)  $(p \wedge q) \rightarrow r$     f)  $r \leftrightarrow (q \vee p)$
12. a) This is  $T \leftrightarrow T$ , which is true.  
 b) This is  $T \leftrightarrow F$ , which is false.  
 c) This is  $F \leftrightarrow F$ , which is true.  
 d) This is  $F \leftrightarrow T$ , which is false.
14. a) This is  $F \rightarrow F$ , which is true.  
 b) This is  $F \rightarrow F$ , which is true.  
 c) This is  $T \rightarrow F$ , which is false.  
 d) This is  $T \rightarrow T$ , which is true.
16. a) The employer making this request would be happy if the applicant knew both of these languages, so this is clearly an inclusive *or*.  
 b) The restaurant would probably charge extra if the diner wanted both of these items, so this is an exclusive *or*.  
 c) If a person happened to have both forms of identification, so much the better, so this is clearly an inclusive *or*.  
 d) This could be argued either way, but the inclusive interpretation seems more appropriate. This phrase means that faculty members who do not publish papers in research journals are likely to be fired from their jobs during the probationary period. On the other hand, it may happen that they will be fired even if they do publish (for example, if their teaching is poor).
18. a) The necessary condition is the conclusion: If you get promoted, then you wash the boss's car.  
 b) If the winds are from the south, then there will be a spring thaw.  
 c) The sufficient condition is the hypothesis: If you bought the computer less than a year ago, then the warranty is good.  
 d) If Willy cheats, then he gets caught.  
 e) The "only if" condition is the conclusion: If you access the website, then you must pay a subscription fee.  
 f) If you know the right people, then you will be elected.  
 g) If Carol is on a boat, then she gets seasick.
20. a) If I am to remember to send you the address, then you will have to send me an e-mail message. (This has been slightly reworded so that the tenses make more sense.)  
 b) If you were born in the United States, then you are a citizen of this country.  
 c) If you keep your textbook, then it will be a useful reference in your future courses. (The word "then" is understood in English, even if omitted.)  
 d) If their goaltender plays well, then the Red Wings will win the Stanley Cup.  
 e) If you get the job, then you had the best credentials.  
 f) If there is a storm, then the beach erodes.  
 g) If you log on to the server, then you have a valid password.  
 h) If you do not begin your climb too late, then you will reach the summit.
22. a) You will get an A in this course if and only if you learn how to solve discrete mathematics problems.  
 b) You will be informed if and only if you read the newspaper every day. (It sounds better in this order; it would be logically equivalent to state this as "You read the newspaper every day if and only if you will be informed.")

- c) It rains if and only if it is a weekend day.  
d) You can see the wizard if and only if he is not in.
24. a) Converse: If I stay home, then it will snow tonight. Contrapositive: If I do not stay at home, then it will not snow tonight. Inverse: If it does not snow tonight, then I will not stay home.  
b) Converse: Whenever I go to the beach, it is a sunny summer day. Contrapositive: Whenever I do not go to the beach, it is not a sunny summer day. Inverse: Whenever it is not a sunny day, I do not go to the beach.  
c) Converse: If I sleep until noon, then I stayed up late. Contrapositive: If I do not sleep until noon, then I did not stay up late. Inverse: If I don't stay up late, then I don't sleep until noon.
26. A truth table will need  $2^n$  rows if there are  $n$  variables.  
a)  $2^2 = 4$      b)  $2^3 = 8$      c)  $2^6 = 64$      d)  $2^5 = 32$

28. To construct the truth table for a compound proposition, we work from the inside out. In each case, we will show the intermediate steps. In part (d), for example, we first construct the truth tables for  $p \wedge q$  and for  $p \vee q$  and combine them to get the truth table for  $(p \wedge q) \rightarrow (p \vee q)$ . For parts (a) and (b) we have the following table (column three for part (a), column four for part (b)).

$p$	$\neg p$	$p \rightarrow \neg p$	$p \leftrightarrow \neg p$
T	F	F	F
F	T	T	F

For parts (c) and (d) we have the following table.

$p$	$q$	$p \vee q$	$p \wedge q$	$p \oplus (p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	F	T

For part (e) we have the following table.

$p$	$q$	$\neg p$	$q \rightarrow \neg p$	$p \leftrightarrow q$	$(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

For part (f) we have the following table.

$p$	$q$	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

30. For parts (a) and (b) we have the following table (column two for part (a), column four for part (b)).

$p$	$p \oplus p$	$\neg p$	$p \oplus \neg p$
T	F	F	T
F	F	T	T

For parts (c) and (d) we have the following table (columns five and six).

$p$	$q$	$\neg p$	$\neg q$	$p \oplus \neg q$	$\neg p \oplus \neg q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

For parts (e) and (f) we have the following table (columns five and six). This time we have omitted the column explicitly showing the negation of  $q$ . Note that the first is a tautology and the second is a contradiction (see definitions in Section 1.2).

$p$	$q$	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \vee (p \oplus \neg q)$	$(p \oplus q) \wedge (p \oplus \neg q)$
T	T	F	T	T	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	F	T	T	F

32. For parts (a) and (b), we have

$p$	$q$	$r$	$p \vee q$	$(p \vee q) \vee r$	$(p \vee q) \wedge r$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	F
F	F	F	F	F	F

For parts (c) and (d), we have

$p$	$q$	$r$	$p \wedge q$	$(p \wedge q) \vee r$	$(p \wedge q) \wedge r$
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	F	T	F
F	F	F	F	F	F

Finally, for parts (e) and (f) we have

$p$	$q$	$r$	$\neg r$	$p \vee q$	$(p \vee q) \wedge \neg r$	$p \wedge q$	$(p \wedge q) \vee \neg r$
T	T	T	F	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	F
T	F	F	T	T	T	F	T
F	T	T	F	T	F	F	F
F	T	F	T	T	T	F	T
F	F	T	F	F	F	F	F
F	F	F	T	F	F	F	T

34. This time the truth table needs  $2^4 = 16$  rows.

<u><math>p</math></u>	<u><math>q</math></u>	<u><math>r</math></u>	<u><math>s</math></u>	<u><math>p \rightarrow q</math></u>	<u><math>(p \rightarrow q) \rightarrow r</math></u>	<u><math>((p \rightarrow q) \rightarrow r) \rightarrow s</math></u>
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

36. a) Since the condition is true, the statement is executed, so  $x$  is incremented and now has the value 2.  
b) Since the condition is false, the statement is not executed, so  $x$  is not incremented and now still has the value 1.  
c) Since the condition is true, the statement is executed, so  $x$  is incremented and now has the value 2.  
d) Since the condition is false, the statement is not executed, so  $x$  is not incremented and now still has the value 1.  
e) Since the condition is true when it is encountered (since  $x = 1$ ), the statement is executed, so  $x$  is incremented and now has the value 2. (It is irrelevant that the condition is now false.)
38. a)  $1\ 1000 \wedge (0\ 1011 \vee 1\ 1011) = 1\ 1000 \wedge 1\ 1011 = 1\ 1000$   
b)  $(0\ 1111 \wedge 1\ 0101) \vee 0\ 1000 = 0\ 0101 \vee 0\ 1000 = 0\ 1101$   
c)  $(0\ 1010 \oplus 1\ 1011) \oplus 0\ 1000 = 1\ 0001 \oplus 0\ 1000 = 1\ 1001$   
d)  $(1\ 1011 \vee 0\ 1010) \wedge (1\ 0001 \vee 1\ 1011) = 1\ 1011 \wedge 1\ 1011 = 1\ 1011$
40. The truth value of “Fred and John are happy” is  $\min(0.8, 0.4) = 0.4$ . The truth value of “Neither Fred nor John is happy” is  $\min(0.2, 0.6) = 0.2$ , since this statement means “Fred is not happy, and John is not happy,” and we computed the truth values of the two propositions in this conjunction in Exercise 35.
42. This cannot be a proposition, because it cannot have a truth value. Indeed, if it were true, then it would be truly asserting that it is false, a contradiction; on the other hand if it were false, then its assertion that it is false must be false, so that it would be true—again a contradiction. Thus this string of letters, while appearing to be a proposition, is in fact meaningless.
44. No. This is a classical paradox. (We will use the male pronoun in what follows, assuming that we are talking about males shaving their beards here, and assuming that all men have facial hair. If we restrict ourselves to beards and allow female barbers, then the barber could be female with no contradiction.) If such a barber existed, who would shave the barber? If the barber shaved himself, then he would be violating the rule that he shaves only those people who do not shave themselves. On the other hand, if he does not shave himself, then the rule says that he must shave himself. Neither is possible, so there can be no such barber.
46. a) If the explorer (a woman, so that our pronouns will not get confused here—the cannibals will be male) encounters a truth-teller, then he will honestly answer “no” to her question. If she encounters a liar, then the

honest answer to her question is “yes,” so he will lie and answer “no.” Thus everybody will answer “no” to the question, and the explorer will have no way to determine which type of cannibal she is speaking to.

b) There are several possible correct answers. One is the following question: “If I were to ask you if you always told the truth, would you say that you did?” Then if the cannibal is a truth teller, he will answer yes (truthfully), while if he is a liar, then, since in fact he would have said that he did tell the truth if questioned, he will now lie and answer no.

48. a) “But” means “and”:  $r \wedge \neg p$ .  
 b) “Whenever” means “if”:  $(r \wedge p) \rightarrow q$ .  
 c) Access being denied is the negation of  $q$ , so we have  $\neg r \rightarrow \neg q$ .  
 d) The hypothesis is a conjunction:  $(\neg p \wedge r) \rightarrow q$ .
50. We write these symbolically:  $u \rightarrow \neg a$ ,  $a \rightarrow s$ ,  $\neg s \rightarrow \neg u$ . Note that we can make all the conclusion true by making  $a$  false,  $s$  true, and  $u$  false. Therefore if the users cannot access the file system, they can save new files, and the system is not being upgraded, then all the conditional statements are true. Thus the system is consistent.
52. This system is consistent. We use  $L$ ,  $Q$ ,  $N$ , and  $B$  to stand for the basic propositions here, “The file system is locked,” “New messages will be queued,” “The system is functioning normally,” and “New messages will be sent to the message buffer,” respectively. Then the given specifications are  $\neg L \rightarrow Q$ ,  $\neg L \leftrightarrow N$ ,  $\neg Q \rightarrow B$ ,  $\neg L \rightarrow B$ , and  $\neg B$ . If we want consistency, then we had better have  $B$  false in order that  $\neg B$  be true. This requires that both  $L$  and  $Q$  be true, by the two conditional statements that have  $B$  as their consequence. The first conditional statement therefore is of the form  $F \rightarrow T$ , which is true. Finally, the biconditional  $\neg L \leftrightarrow N$  can be satisfied by taking  $N$  to be false. Thus this set of specifications is consistent. Note that there is just this one satisfying truth assignment.
54. This is similar to Example 17, about universities in New Mexico. To search for hiking in West Virginia, we could enter **WEST AND VIRGINIA AND HIKING**. If we enter **(VIRGINIA AND HIKING) NOT WEST**, then we’ll get websites about hiking in Virginia but not in West Virginia, except for sites that happen to use the word “west” in a different context (e.g., “Follow the stream west until you come to a clearing”).
56. If  $A$  is a knight, then his statement that both of them are knights is true, and both will be telling the truth. But that is impossible, because  $B$  is asserting otherwise (that  $A$  is a knave). If  $A$  is a knave, then  $B$ ’s assertion is true, so he must be a knight, and  $A$ ’s assertion is false, as it should be. Thus we conclude that  $A$  is a knave and  $B$  is a knight.
58. We can draw no conclusions. A knight will declare himself to be a knight, telling the truth. A knave will lie and assert that he is a knight. Since everyone will say “I am a knight,” we can determine nothing.
60. a) We look at the three possibilities of who the innocent men might be. If Smith and Jones are innocent (and therefore telling the truth), then we get an immediate contradiction, since Smith said that Jones was a friend of Cooper, but Jones said that he did not even know Cooper. If Jones and Williams are the innocent truth-tellers, then we again get a contradiction, since Jones says that he did not know Cooper and was out of town, but Williams says he saw Jones with Cooper (presumably in town, and presumably if we was with him, then he knew him). Therefore it must be the case that Smith and Williams are telling the truth. Their statements do not contradict each other. Based on Williams’ statement, we know that Jones is lying, since he said that he did not know Cooper when in fact he was with him. Therefore Jones is the murderer.

b) This is just like part (a), except that we are not told ahead of time that one of the men is guilty. Can none of them be guilty? If so, then they are all telling the truth, but this is impossible, because as we just saw, some of the statements are contradictory. Can more than one of them be guilty? If, for example, they are all guilty, then their statements give us no information. So that is certainly possible.

62. This information is enough to determine the entire system. Let each letter stand for the statement that the person whose name begins with that letter is chatting. Then the given information can be expressed symbolically as follows:  $\neg K \rightarrow H$ ,  $R \rightarrow \neg V$ ,  $\neg R \rightarrow V$ ,  $A \rightarrow R$ ,  $V \rightarrow K$ ,  $K \rightarrow V$ ,  $H \rightarrow A$ ,  $H \rightarrow K$ . Note that we were able to convert all of these statements into conditional statements. In what follows we will sometimes make use of the contrapositives of these conditional statements as well. First suppose that  $H$  is true. Then it follows that  $A$  and  $K$  are true, whence it follows that  $R$  and  $V$  are true. But  $R$  implies that  $V$  is false, so we get a contradiction. Therefore  $H$  must be false. From this it follows that  $K$  is true; whence  $V$  is true, and therefore  $R$  is false, as is  $A$ . We can now check that this assignment leads to a true value for each conditional statement. So we conclude that Kevin and Vijay are chatting but Heather, Randy, and Abby are not.

64. Note that Diana's statement is merely that she didn't do it.

a) John did it. There are four cases to consider. If Alice is the sole truth-teller, then Carlos did it; but this means that John is telling the truth, a contradiction. If John is the sole truth-teller, then Diana must be lying, so she did it, but then Carlos is telling the truth, a contradiction. If Carlos is the sole truth-teller, then Diana did it, but that makes John truthful, again a contradiction. So the only possibility is that Diana is the sole truth-teller. This means that John is lying when he denied it, so he did it. Note that in this case both Alice and Carlos are indeed lying.

b) Again there are four cases to consider. Since Carlos and Diana are making contradictory statements, the liar must be one of them (we could have used this approach in part (a) as well). Therefore Alice is telling the truth, so Carlos did it. Note that John and Diana are telling the truth as well here, and it is Carlos who is lying.