CHAPTER 6
Discrete Probability

SECTION 6.1 An Introduction to Discrete Probability

Calculating probabilities is one of the most immediate applications of combinatorics. Many people play games in which discrete probability plays a role, such as card games like poker or bridge, board games, casino games, and state lotteries. Probability is also important in making decisions in such areas as business and medicine—for example, in deciding how high a deductible to have on your automobile insurance. This section only scratches the surface, of course, but it is surprising how many useful calculations can be made using just the techniques discussed in this textbook.

The process is basically the same in each problem. First count the number of possible, equally likely, outcomes; this is the denominator of the probability you are seeking. Then count the number of ways that the event you are looking for can happen; this is the numerator. We have given approximate decimal (or percentage) answers to many of the problems, since the human mind can comprehend the magnitude of a number much better this way than by looking at a fraction with large numerator and denominator.

1. There are 52 equally likely cards to be selected, and 4 of them are aces. Therefore the probability is $4/52 = 1/13 \approx 7.7\%$.

3. Among the first 100 positive integers there are exactly 50 odd ones. Therefore the probability is $50/100 = 1/2$.

5. One way to do this is to look at the 36 equally likely outcomes of the roll of two dice, which we can represent by the set of ordered pairs (i, j) with $1 \leq i, j \leq 6$. A better way is to argue as follows. Whatever the number of spots showing on the first die, the sum will be even if and only if the number of spots showing on the second die has the same parity (even or odd) as the first. Since there are 3 even faces (2, 4, and 6) and 3 odd faces (1, 3, and 5), the probability is $3/6 = 1/2$.

7. There are $2^6 = 64$ possible outcomes, represented by all the sequences of length 6 of H's and T's. Only one of those sequences, $HHHHHH$, represents the event under consideration, so the probability is $1/64 \approx 0.016$.

9. We saw in Example 11 of Section 5.3 that there are $C(52,5)$ possible poker hands, and we assume by symmetry that they are all equally likely. In order to solve this problem, we need to compute the number of poker hands that do not contain the queen of hearts. Such a hand is simply an unordered selection from a deck with 51 cards in it (all cards except the queen of hearts), so there are $C(51,5)$ such hands. Therefore the answer to the question is the ratio

$$\frac{C(51,5)}{C(52,5)} = \frac{47}{52} \approx 90.4\%\,.$$

11. This question completely specifies the poker hand, so there is only one hand satisfying the conditions. Since there are $C(52,5)$ equally likely poker hands (see Example 11 of Section 5.3), the probability of drawing this one is $1/C(52,5)$, which is about 1 out of 2.5 million.

13. Let us compute the probability that the hand contains no aces and then subtract from 1 (invoking Theorem 1). A hand with no aces must be drawn from the 48 nonace cards, so there are $C(48,5)$ such hands. Therefore the probability of drawing such a hand is $C(48,5)/C(52,5)$, which works out to about 66%. Thus the probability of holding a hand with at least one ace is $1-(C(48,5)/C(52,5))$, or about 34%.

15. We need to compute the number of ways to hold two pairs. To specify the hand we first choose the kinds (ranks) the pairs will be (such as kings and fives); there are $C(13,2)=78$ ways to do this, since we need to choose 2 kinds from the 13 possible kinds. Then we need to decide which 2 cards of each of the kinds of the pairs we want to include. There are 4 cards of each kind (4 suits), so there are $C(4,2)=6$ ways to make each of these two choices. Finally, we need to decide which card to choose for the fifth card in the hand. We cannot choose any card in either of the 2 kinds that are already represented (we do not want to construct a full house by accident), so there are $52-8=44$ cards to choose from and hence $C(44,1)=44$ ways to make the choice. Putting this all together by the product rule, there are $78\cdot 6\cdot 6\cdot 44=123{,}552$ different hands classified as "two pairs."

Since each hand is equally likely, and since there are $C(52,5)=2{,}598{,}960$ different hands (see Example 11 in Section 5.3), the probability of holding two pairs is $123552/2598960=198/4165\approx 0.0475$.

17. First we need to compute the number of ways to hold a straight. We can specify the hand by first choosing the starting (lowest) kind for the straight. Since the straight can start with any card from the set $\{A,2,3,4,5,6,7,8,9,10\}$, there are $C(10,1)=10$ ways to do this. Then we need to decide which card of each of the kinds in the straight we want to include. There are 4 cards of each kind (4 suits), so there are $C(4,1)=4$ ways to make each of these 5 choices. Putting this all together by the product rule, there are $10\cdot 4^5=10{,}240$ different hands containing a straight. (For poker buffs, it should be pointed out that a hand is classified as a "straight" in poker if it contains a straight but does not contain a straight flush, which is a straight in which all of the cards are in the same suit. Since there are $10\cdot 4=40$ straight flushes, we would need to subtract 40 from our answer above in order to find the number of hands classified as a "straight." Also, some poker books do not count $A,2,3,4,5$ as a straight.)

Since each hand is equally likely, and since there are $C(52,5)=2{,}598{,}960$ different hands (see Example 11 in Section 5.3), the probability of holding a hand containing a straight is $10240/2598960=128/32487\approx 0.00394$.

19. First we can calculate the number of hands that contain cards of five different kinds (this is Exercise 14). All that is required to specify such a hand is to choose the five kinds (which can be done in $C(13,5)$ ways, since there are 13 kinds in all), and then for each of those cards to specify a suit (which can be done in 4^5 ways, since there are four possible suits for each card). Thus there are $C(13,5)\cdot 4^5=1317888$ hands of this type. Now we need to figure out how many of them violate the conditions—in other words, how many of them contain a flush or a straight. To obtain a flush (this is Exercise 16) we need to choose a suit and then choose 5 cards from the 13 in this suit, so there are $4\cdot C(13,5)=5148$ different flushes. We solved the problem of straights in Exercise 17; there are 10240 straights. Furthermore, there are 40 hands that are both straights and flushes (such a hand can have its lowest card be any of the ten kinds A, 2, ..., 10, and be in any of the four suits). Now we are ready to put this all together. The number of hands that are flushes or straights is, by the principle of inclusion–exclusion, $5148+10240-40=15348$. Subtracting this from the number of hands containing five different kinds, we see that there are $1317888-15348=1302540$ hands of the desired type. Therefore the probability of drawing such a hand is $1302540/C(52,5)=1302540/2598960=1277/2548$, which works out to just a hair over 50%.

21. Looked at properly, this is the same as Exercise 7. There are 2 equally likely outcomes for the parity on the roll of a die—even and odd. Of the $2^6=64$ parity outcomes in the roll of a die 6 times, only one consists of

6 odd numbers. Therefore the probability is $1/64$.

23. We need to count the number of positive integers not exceeding 100 that are divisible by 5 or 7. Using an analysis similar to Exercise 21e in Section 5.1, we see that there are $\lfloor 100/5 \rfloor = 20$ numbers in that range divisible by 5 and $\lfloor 100/7 \rfloor = 14$ divisible by 7. However, we have counted the numbers 35 and 70 twice, since they are divisible by both 5 and 7 (i.e., divisible by 35). Therefore there are $20 + 14 - 2 = 32$ such numbers. (We needed to subtract 2 to compensate for the double counting.) Now since there are 100 equally likely numbers in the set, the probability of choosing one of these 32 numbers is $32/100 = 8/25 = 0.32$.

25. In each case, if the numbers are chosen from the integers from 1 to n, then there are $C(n,6)$ possible entries, only one of which is the winning one, so the answer is $1/C(n,6)$.
a) $1/C(50,6) = 1/15890700 \approx 6.3 \times 10^{-8}$ **b)** $1/C(52,6) = 1/20358520 \approx 4.9 \times 10^{-8}$
c) $1/C(56,6) = 1/32468436 \approx 3.1 \times 10^{-8}$ **d)** $1/C(60,6) = 1/50063860 \approx 2.0 \times 10^{-8}$

27. In each case, there are $C(n,6)$ possible choices of winning numbers. If we want to choose exactly one of them correctly, then we have 6 ways to specify which number it is to be, and then $C(n-6,5)$ ways to pick five losing numbers from the $n-6$ losing numbers. Thus the probability is $6C(n-6,5)/C(n,6)$ in each case. We will calculate these numbers for the various values of n. Note that the probability decreases as n increases (it gets harder to choose one of the winning numbers as the pool of numbers grows).
a) $6C(40-6,5)/C(40,6) = 6 \cdot C(34,5)/C(40,6) = 6 \cdot 278256/3838380 = 139128/319865 \approx 0.435$
b) $6C(48-6,5)/C(48,6) = 6 \cdot C(42,5)/C(48,6) = 6 \cdot 850668/12271512 = 212667/511313 \approx 0.416$
c) $6C(56-6,5)/C(56,6) = 6 \cdot C(50,5)/C(56,6) = 6 \cdot 2118760/32468436 = 151340/386529 \approx 0.392$
d) $6C(64-6,5)/C(64,6) = 6 \cdot C(58,5)/C(64,6) = 6 \cdot 4582116/74974368 = 163647/446276 \approx 0.367$

29. There is only one winning choice of numbers, namely the same 8 numbers the computer chooses. Therefore the probability of winning is $1/C(100,8) \approx 1/(1.86 \times 10^{11})$.

31. Since the drawing is done at random, and there are three winners and 97 losers, Michelle's chance of winning is $3/100$. To see this more formally, we reason as follows, thinking of the winners as drawn in order (first, second, third). There are $100 \cdot 99 \cdot 98$ equally likely outcomes of the drawing. The number of possible outcomes in which Michelle wins first prize is $1 \cdot 99 \cdot 98$. The number of possible outcomes in which Michelle wins second prize is $99 \cdot 1 \cdot 98$ (99 people other than Michelle could have won first prize). The number of possible outcomes in which Michelle wins third prize is $99 \cdot 98 \cdot 1$. Adding, we see that there are $3 \cdot 99 \cdot 98$ ways for Michelle to win a prize. Therefore the probability we seek is $(3 \cdot 99 \cdot 98)/(100 \cdot 99 \cdot 98) = 3/100$.

33. **a)** There are $200 \cdot 199 \cdot 198$ equally likely outcomes of the drawings. In only one of these do Abby, Barry, and Sylvia win the first, second, and third prizes, respectively. Therefore the probability is $1/(200 \cdot 199 \cdot 198) = 1/7880400$.
b) There are $200 \cdot 200 \cdot 200$ equally likely outcomes of the drawings. In only one of these do Abby, Barry, and Sylvia win the first, second, and third prizes, respectively. Therefore the probability is $1/(200 \cdot 200 \cdot 200) = 1/8000000$.

35. **a)** There are 18 red numbers and 38 numbers in all, so the probability is $18/38 = 9/19 \approx 0.474$.
b) There are 38^2 equally likely outcomes for two spins, since each spin can result in 38 different outcomes. Of these, 18^2 are a pair of black numbers. Therefore the probability is $18^2/38^2 = 81/361 \approx 0.224$.
c) There are 2 outcomes being considered here, so the probability is $2/38 = 1/19$.
d) There are 38^5 equally likely outcomes in five spins of the wheel. Since 36 outcomes on each spin are not 0 or 00, there are 36^5 outcomes being considered. Therefore the probability is $36^5/38^5 = 1889568/2476099 \approx 0.763$.

e) There are 38^2 equally likely outcomes for two spins. The number of outcomes that meet the conditions specified here is $6 \cdot (38 - 6) = 192$ (by the product rule). Therefore the probability is $192/38^2 = 48/361 \approx 0.133$.

37. Reasoning as in Example 2, we see that there are 4 ways to get a total of 9 when two dice are rolled: $(6,3)$, $(5,4)$, $(4,5)$, and $(3,6)$. There are $6^2 = 36$ equally likely possible outcomes of the roll of two dice, so the probability of getting a total of 9 when two dice are rolled is $4/36 \approx 0.111$. For three dice, there are $6^3 = 216$ equally likely possible outcomes, which we can represent as ordered triples (a,b,c). We need to enumerate the possibilities that give a total of 9. This is done in a more systematic way in Section 5.5, but we will do it here by brute force. The first die could turn out to be a 6, giving rise to the 2 triples $(6,2,1)$ and $(6,1,2)$. The first die could be a 5, giving rise to the 3 triples $(5,3,1)$, $(5,2,2)$, and $(5,1,3)$. Continuing in this way, we see that there are 4 triples giving a total of 9 when the first die shows a 4, 5 triples when it shows a 3, 6 triples when it shows a 2, and 5 triples when it shows a 1 (namely $(1,6,2)$, $(1,5,3)$, $(1,4,4)$, $(1,3,5)$, and $(1,2,6)$). Therefore there are $2+3+4+5+6+5 = 25$ possible outcomes giving a total of 9. This tells us that the probability of rolling a 9 when three dice are thrown is $25/216 \approx 0.116$, slightly larger than the corresponding value for two dice. Thus rolling a total of 9 is more likely when using three dice than when using only two.

39. It's hard to know how to respond to this argument, other than to say that the claim—that the probabilities that the prize is behind each of the doors are equal—is nonsense. For example, if one flips a thumbtack, then it would be silly to say that the probability that it lands with the point up must equal the probability that it lands with the point down—there is no symmetry in the situation to justify such a conclusion. (It is as absurd as claiming that every time you enter a contest you have a 50% probability of winning, since there are only two outcomes—either you win or you lose.) Here, too, there is a lack of symmetry, since the door you chose was chosen by you at random, without knowing where the prize was, and the door chosen by the host was not chosen at random—he carefully avoided opening the door with the prize. (In fact, if the host's algorithm were to choose one of the other two doors at random, then whenever he opens a nonprize door, the probability that the prize is behind your door *does* become $\frac{1}{2}$. Of course in this case, sometimes the game won't advance that far, since the door he chooses might contain the prize.)

41. a) There are 6^4 possible outcomes when a die is rolled four times. There are 5^4 outcomes in which a 6 does not appear, so the probability of not rolling a 6 is $5^4/6^4$. Therefore the probability that at least one 6 does appear is $1 - 5^4/6^4 = 671/1296$, which is about 0.518.

b) There are 36^{24} possible outcomes when a pair of dice is rolled 24 times. There are 35^{24} outcomes in which a double 6 does not appear, so the probability of not rolling a double 6 is $35^{24}/36^{24}$. Therefore the probability that at least one double 6 does appear is $1 - 35^{24}/36^{24}$, which is about 0.491. No, the probability is not greater than $1/2$.

c) From our answers above we see that the answer is yes, since $0.518 > 0.491$.