SECTION 9.2 Graph Terminology and Special Types of Graphs

Graph theory is sometimes jokingly called the "theory of definitions," because so many terms can be—and have been—defined for graphs. A few of the most important concepts are given in this section; others appear in the rest of this chapter and the next, in the exposition and in the exercises. As usual with definitions, it is important to understand exactly what they are saying. You should construct some examples for each definition you encounter—examples both of the thing being defined and of its absence. Some students find it useful to build a dictionary as they read, including their examples along with the formal definitions.

The Handshaking Theorem (that the sum of the degrees of the vertices in a graph equals twice the number of edges), although trivial to prove, is quite handy, as Exercise 49, for example, illustrates. Be sure to look at Exercise 37, which deals with the problem of when a sequence of numbers can possibly be the degrees of the vertices of a simple graph. Some interesting subtleties arise there, as you will discover when you try to draw the graphs. Many arguments in graph theory tend to be rather ad hoc, really getting down to the nitty gritty, and Exercise 37c is a good example. Exercise 45 is really a combinatorial problem; such problems abound in graph theory, and entire books have been written on counting graphs of various types. The notion of **complementary graph**, *introduced in Exercise 53, will appear again later in this chapter, so it would be wise to look at the exercises dealing with it.*

1. There are 6 vertices here, and 6 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 1$ (and hence c is pendant), $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 2$, and $\deg(f) = 3$. Note that the sum of the degrees is $2 + 4 + 1 + 0 + 2 + 3 = 12$, which is twice the number of edges.

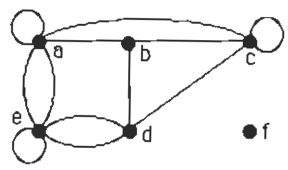
3. There are 9 vertices here, and 12 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 6$, $\deg(f) = 0$ (and hence f is isolated), $\deg(g) = 4$, $\deg(h) = 2$, and $\deg(i) = 3$. Note that the sum of the degrees is $3+2+4+0+6+0+4+2+3 = 24$, which is twice the number of edges.

5. By Theorem 2 the number of vertices of odd degree must be even. Hence there cannot be a graph with 15 vertices of odd degree 5. (We assume that the problem was meant to imply that the graph contained only these 15 vertices.)

7. This directed graph has 4 vertices and 7 edges. The in-degree of vertex a is $\deg^-(a) = 3$ since there are 3 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$ since only the loop has a as its initial vertex. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 2$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 3$. As a check we see that the sum of the in-degrees and the sum of the out-degrees are equal (both are equal to 7).

9. This directed multigraph has 5 vertices and 13 edges. The in-degree of vertex a is $\deg^-(a) = 6$ since there are 6 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 5$, $\deg^-(c) = 2$, $\deg^+(c) = 5$, $\deg^-(d) = 4$, $\deg^+(d) = 2$, $\deg^-(e) = 0$, and $\deg^+(e) = 0$ (vertex e is isolated). As a check we see that the sum of the in-degrees and the sum of the out-degrees are both equal to the number of edges (13).

11. To form the underlying undirected graph we simply take all the arrows off the edges. Thus, for example, the edges from e to d and from d to e become a pair of parallel edges between e and d.



13. Since a person is joined by an edge to each of his or her collaborators, the degree of v is the number of collaborators v has. An isolated vertex (degree 0) is someone who has never collaborated. A pendant vertex (degree 1) is someone who has just one collaborator.

15. Since there is a directed edge from u to v for each call made by u to v, the in-degree of v is the number of calls v received, and the out-degree of u is the number of calls u made. The degree of a vertex in the undirected version is just the sum of these, which is therefore the number of calls the vertex was involved in.

17. Since there is a directed edge from u to v to represent the event that u beat v when they played, the in-degree of v must be the number of teams that beat v, and the out-degree of u must be the number of teams that u beat. In other words, the pair $(\deg^+(v), \deg^-(v))$ is the win-loss record of v.

19. In order to use Exercise 18, we must find a graph in which the degree of a vertex represents the number of people the given person knows. Therefore we construct the simple graph model in which V is the set of people in the group and there is an edge associated with $\{u, v\}$ if u and v know each other. In this graph the degree of vertex v is the number of people v knows. By the result of Exercise 18, there are two vertices with the same degree. Therefore there are two people who know the same number of other people in the group.

21. To show that this graph is bipartite we can exhibit the parts and note that indeed every edge joins vertices in different parts. Take $\{e\}$ to be one part and $\{a, b, c, d\}$ to be the other (in fact there is no choice in the matter). Each edge joins a vertex in one part to a vertex in the other. This graph is the complete bipartite graph $K_{1,4}$.

23. To show that a graph is not bipartite we must give a proof that there is no possible way to specify the parts. (There is another good way to characterize nonbipartite graphs, but it takes some notions not introduced until Section 9.4.) We can show that this graph is not bipartite by the pigeonhole principle. Consider the vertices b, c, and f. They form a triangle—each is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

An alternative way to look at this is given by Theorem 4. Because of the triangle, it is impossible to color the vertices to satisfy the condition given there.

25. As in Exercise 23, we can show that this graph is not bipartite by looking at a triangle, in this case the triangle formed by vertices b, d, and e. Each of these vertices is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

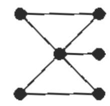
27. **a)** Following the lead in Example 14, we construct a bipartite graph in which the vertex set consists of two subsets—one for the employees and one for the jobs. Let $V_1 = \{\text{Zamora, Agraharam, Smith, Chou, Macintyre}\}$, and let $V_2 = \{\text{planning, publicity, sales, marketing, development, industry relations}\}$. Then the vertex set for our graph is $V = V_1 \cup V_2$. Given the list of capabilities in the exercise, we must include precisely the following edges in our graph: {Zamora, planning}, {Zamora, sales}, {Zamora, marketing}, {Zamora, industry relations}, {Agraharam, planning}, {Agraharam, development}, {Smith, publicity}, {Smith, sales}, {Smith, industry relations}, {Chou, planning}, {Chou, sales}, {Chou, industry relations}, {Macintyre, planning}, {Macintyre, publicity}, {Macintyre, sales}, {Macintyre, industry relations}.
b) Many assignments are possible. If we take it as an implicit assumption that there will be no more than one employee assigned to the same job, then we want a maximal matching for this graph. So we look for five edges in this graph that share no endpoints. A little trial and error leads us, for example, {Zamora, planning}, {Agraharam, development}, {Smith, publicity}, {Chou, sales}, {Macintyre, industry relations}. We assign the employees to the jobs given in this matching.

29. **a)** Obviously K_n has n vertices. It has $C(n,2) = n(n-1)/2$ edges, since each unordered pair of distinct vertices is an edge.
b) Obviously C_n has n vertices. Just as obviously it has n edges.
c) The wheel W_n is the same as C_n with an extra vertex and n extra edges incident to that vertex. Therefore it has $n+1$ vertices and $n+n=2n$ edges.
d) By definition $K_{m,n}$ has $m+n$ vertices. Since it has one edge for each choice of a vertex in the one part and a vertex in the other part, it has mn edges.
e) Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the Handshaking Theorem), we know that there are $n2^n/2 = n2^{n-1}$ edges.

31. In each case we just record the degrees of the vertices in a list, from largest to smallest.
a) Each of the four vertices is adjacent to each of the other three vertices, so the degree sequence is $3,3,3,3$.
b) Each of the four vertices is adjacent to its two neighbors in the cycle, so the degree sequence is $2,2,2,2$.
c) Each of the four vertices on the rim of the wheel is adjacent to each of its two neighbors on the rim, as well as to the middle vertex. The middle vertex is adjacent to the four rim vertices. Therefore the degree sequence is $4,3,3,3,3$.
d) Each of the vertices in the part of size two is adjacent to each of the three vertices in the part of size three, and vice versa, so the degree sequence is $3,3,2,2,2$.
e) Each of the eight vertices in the cube is adjacent to three others (for example, 000 is adjacent to 001, 010, and 100. Therefore the degree sequence is $3,3,3,3,3,3,3,3$.

33. Each of the n vertices is adjacent to each of the other $n-1$ vertices, so the degree sequence is simply $n-1, n-1, \ldots, n-1$, with n terms in the sequence.

35. The number of edges is half the sum of the degrees (Theorem 1). Therefore this graph has $(5+2+2+2+2+1)/2 = 7$ edges. A picture of this graph is shown here (it is essentially unique).



37. There is no such graph in part **(b)**, since the sum of the degrees is odd (and also because a simple graph with 5 vertices cannot have any degrees greater than 4). Similarly, the odd degree sum prohibits the existence of graphs with the degree sequences given in part **(d)** and part **(f)**. There is no such graph in part **(c)**, since the existence of two vertices of degree 4 implies that there are two vertices each joined by an edge to every other vertex. This means that the degree of each vertex has to be at least 2, and there can be no vertex of degree 1. The graphs for part **(a)** and part **(e)** are shown below; one can draw them after just a little trial and error.

(a) **(e)**

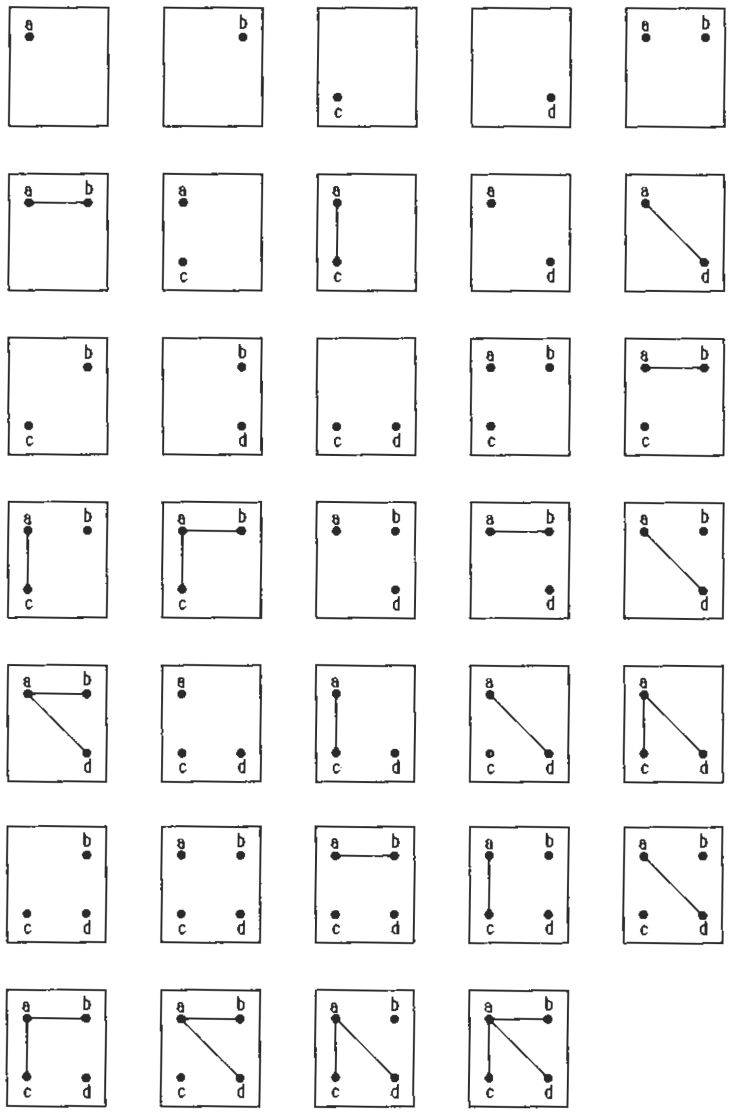
39. We need to prove two conditional statements. First, suppose that $d_1, d_2, \ldots, d_n$ is graphic. We must show that the sequence whose terms are $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. Apparently what we want to do is to remove the vertex of highest degree (d_1) from a graph with the original degree sequence and reduce by 1 the degrees of the vertices to which it is adjacent, but we also want to make sure that those vertices are the ones with the highest degrees among the remaining vertices. In Exercise 38 it is proved that if the original sequence is graphic, then in fact there is a graph having this degree sequence in which the vertex of degree d_1 is adjacent to the vertices of degrees d_2, $d_3, \ldots, d_{d_1+1}$. Thus our plan works, and we have a graph whose degree sequence is as desired.

Conversely, suppose that $d_1, d_2, \ldots, d_n$ is a nonincreasing sequence such that the sequence $d_2 - 1$, $d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. Take a graph with this latter degree sequence, where vertex v_i has degree $d_i - 1$ for $2 \le i \le d_1 + 1$ and vertex v_i has degree d_i for $d_1 + 2 \le i \le n$. Adjoin one new vertex (call it v_1), and put in an edge from v_1 to each of the vertices $v_2, v_3, \ldots, v_{d_1+1}$. Then clearly the resulting graph has degree sequence $d_1, d_2, \ldots, d_n$.

41. Let $d_1, d_2, \ldots, d_n$ be a nonincreasing sequence of nonnegative integers with an even sum. We want to construct a pseudograph with this as its degree sequence. Even degrees can be achieved using only loops, each of which contributes 2 to the count of its endpoint; vertices of odd degrees will need a non-loop edge, but one will suffice (the rest of the count at that vertex will be made up by loops). Following the hint, we take vertices $v_1, v_2, \ldots, v_n$ and put $\lfloor d_i/2 \rfloor$ loops at vertex v_i, for $i = 1, 2, \ldots, n$. For each i, vertex v_i now has degree either d_i (if d_i is even) or $d_i - 1$ (if d_i is odd). Because the original sum was even, the number of vertices falling into the latter category is even. If there are $2k$ such vertices, pair them up arbitrarily, and put in k more edges, one joining the vertices in each pair. The resulting graph will have degree sequence $d_1, d_2, \ldots, d_n$.

43. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 3 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(3,2) = 3$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $3 \cdot 2 = 6$ subgraphs with two vertices. If a subgraph is to have all three vertices, then there are $2^3 = 8$ ways to decide whether or not to include each of the edges. Thus our answer is $3 + 6 + 8 = 17$.

45. This graph has a lot of subgraphs. First of all, any nonempty subset of the vertex set can be the vertex set for a subgraph, and there are 15 such subsets. If the set of vertices of the subgraph does not contain vertex a, then the subgraph can of course have no edges. If it does contain vertex a, then it can contain or fail to contain each edge from a to whichever other vertices are included. A careful enumeration of all the possibilities gives the 34 graphs shown below.



47. **a)** The complete graph K_n is regular for all values of $n \geq 1$, since the degree of each vertex is $n-1$.

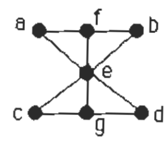
b) The degree of each vertex of C_n is 2 for all n for which C_n is defined, namely $n \geq 3$, so C_n is regular for all these values of n.

c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the "rim" is 3. Therefore W_n is regular if and only if $n = 3$. Of course W_3 is the same as K_4.

d) The cube Q_n is regular for all values of $n \geq 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

49. If a graph is regular of degree 4 and has n vertices, then by the Handshaking Theorem it has $4n/2 = 2n$ edges. Since we are told that there are 10 edges, we just need to solve $2n = 10$. Thus the graph has 5 vertices. The complete graph K_5 is one such graph (and the only simple one).

51. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



53. a) The complement of a complete graph is a graph with no edges.

b) Since all the edges between the parts are present in $K_{m,n}$, but none of the edges between vertices in the same part are, the complement must consist precisely of the disjoint union of a K_m and a K_n, i.e., the graph containing all the edges joining two vertices in the same part and no edges joining vertices in different parts.

c) There is really no better way to describe this graph than simply by saying it is the complement of C_n. One representation would be to take as vertex set the integers from 1 to n, inclusive, with an edge between distinct vertices i and j as long as i and j do not differ by ± 1, modulo n.

d) Again, there is really no better way to describe this graph than simply by saying it is the complement of Q_n. One representation would be to take as vertex set the bit strings of length n, with two vertices joined by an edge if the bit strings differ in more than one bit.

55. Since K_v has $C(v,2) = v(v-1)/2$ edges, and since $\overline{G}$ has all the edges of K_v that G is missing, it is clear that $\overline{G}$ has $[v(v-1)/2] - e$ edges.

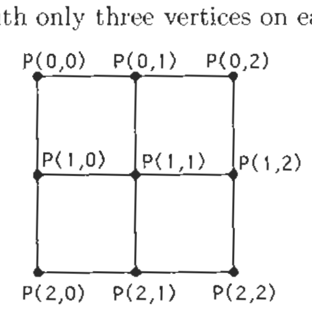
57. If G has n vertices, then the degree of vertex v in $\overline{G}$ is $n-1$ minus the degree of v in G (there will be an edge in $\overline{G}$ from v to each of the $n-1$ other vertices that v is not adjacent to in G). The order of the sequence will reverse, of course, because if $d_i \geq d_j$, then $n-1-d_i \leq n-1-d_j$. Therefore the degree sequence of $\overline{G}$ will be $n-1-d_n$, $n-1-d_{n-1}$, $\ldots$, $n-1-d_2$, $n-1-d_1$.

59. Consider the graph $G \cup \overline{G}$. Its vertex set is clearly the vertex set of G; therefore it has n vertices. If u and v are any two distinct vertices of $G \cup \overline{G}$, then either the edge between u and v is in G, or else by definition it is in $\overline{G}$. Therefore by definition of union, it is in $G \cup \overline{G}$. Thus by definition $G \cup \overline{G}$ is the complete graph K_n.

61. These pictures are identical to the figures in those exercises, with one change, namely that all the arrowheads are turned around. For example, rather than there being a directed edge from a to b in #7, there is an edge from b to a. Note that the loops are unaffected by changing the direction of the arrowhead—a loop from a vertex to itself is the same, whether the drawing of it shows the direction to be clockwise or counterclockwise.

63. It is clear from the definition of converse that a directed graph $G = (V, E)$ is its own converse if and only if it satisfies the condition that $(u,v) \in E$ if and only if $(u,v) \in E$. But this is precisely the definition of symmetry for the associated relation.

65. Our picture is just like Figure 13, but with only three vertices on each side.



67. Suppose $P(i,j)$ and $P(k,l)$ need to communicate. Clearly by using $|i-k|$ hops we can move from $P(i,j)$ to $P(k,j)$. Then using $|j-l|$ hops we can move from $P(k,j)$ to $P(k,l)$. In all we used $|i-k|+|j-l|$ hops. But each of these absolute values is certainly less than m, since all the indices are less than m. Therefore the sum is less than $2m$, so it is $O(m)$.