CHAPTER 4
Induction and Recursion

SECTION 4.1 Mathematical Induction

Understanding and constructing proofs by mathematical induction are extremely difficult tasks for most students. Do not be discouraged, and do not give up, because, without doubt, this proof technique is the most important one there is in mathematics and computer science. Pay careful attention to the conventions to be observed in writing down a proof by induction. As with all proofs, remember that a proof by mathematical induction is like an essay - it must have a beginning, a middle, and an end; it must consist of complete sentences, logically and aesthetically arranged; and it must convince the reader. Be sure that your basis step (also called the "base case") is correct (that you have verified the proposition in question for the smallest value or values of n), and be sure that your inductive step is correct and complete (that you have derived the proposition for $k+1$, assuming the inductive hypothesis that the proposition is true for k).

Some, but not all, proofs by mathematical induction are like Exercises 3–17. In each of these, you are asked to prove that a certain summation has a "closed form" representation given by a certain expression. Here the proofs are usually straightforward algebra. For the inductive step you start with the summation for $P(k+1)$, find the summation for $P(k)$ as its first k terms, replace that much by the closed form given by the inductive hypothesis, and do the algebra to get the resulting expression into the desired form. When doing proofs like this, however, remember to include all the words surrounding your algebra—the algebra alone is not the proof. Also keep in mind that $P(n)$ is the proposition that the sum equals the closed-form expression, not just the sum and not just the expression.

Many inequalities can be proved by mathematical induction; see Exercises 18 24, for example. The method also extends to such things as set operations, divisibility, and a host of other applications; a sampling of them is given in other exercises in this set. Some are quite complicated.

One final point about notation. In performing the inductive step, it really does not matter what letter we use. We see in the text the proof of $P(k) \to P(k+1)$; but it would be just as valid to prove $P(n) \to P(n+1)$, since the k in the first case and the n in the second case are just dummy variables. We will use both notations in this Guide; in particular, we will use k for the first few exercises but often use n afterwards.

1. We can prove this by mathematical induction. Let $P(n)$ be the statement that the train stops at station n. We want to prove that $P(n)$ is true for all positive integers n. For the basis step, we are told that $P(1)$ is true. For the inductive step, we are told that $P(k)$ implies $P(k+1)$ for each $k \geq 1$. Therefore by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

3. **a)** Plugging in $n = 1$ we have that $P(1)$ is the statement $1^2 = 1 \cdot 2 \cdot 3/6$.
b) Both sides of $P(1)$ shown in part **(a)** equal 1.
c) The inductive hypothesis is the statement that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we

10. Your essay should mention the RSA-129 project. See *The New York Times*, around the spring of 1994 (use its index). For an expository article, try [Po].

11. There are dozens of books on computer hardware and circuit design that discuss the algorithms and circuits used in performing these operations. If you are a computer science or computer engineering major, you probably have taken (or will take) a course that deals with these topics. See [Ko2] and similar books.

12. A traditional history of mathematics book should be helpful here; try [Bo4] or [Ev3].

13. This topic has taken on a lot of significance recently as randomized algorithms become more and more important. The August/September 1994 issue of *SIAM News* (the newsletter of the Society for Industrial and Applied Mathematics) has a provocative article on the subject. See also [La1], or for older material in a textbook try volume 2 of [Kn].

14. This topic gets into the news on a regular basis. Try *The New York Times* index. See also Writing Project 10, above, and 15, below.

15. If I encrypt my signature with my private key, then I will produce something that will decrypt (using my public key) as my signature. Furthermore, no one else can do this, since no one else knows my private key. Good sources for cryptography include [Be], [MeOo], and [St2].

16. The author's number theory text ([Ro3]) has material on this topic. The amazing mathematician John H. Conway (inventor of the Game of Life, among other things) has devised what he calls the Doomsday Algorithm, and it works quite fast with practice. See [BeCo]. (Conway can determine any day of the week mentally in a second or two.)

17. An advanced algorithms text, such as [Ma2] or [BrBr], is the place to look. One algorithm has the names Schönhage and Strassen associated with it. A related topic is the fast Fourier transform. See also volume 2 of [Kn].

18. This is related to Writing Project 17; see the suggestions there. Strassen also invented (or discovered, depending on your philosophy) a fast matrix multiplication algorithm.

19. Several excellent books have appeared in the past decade on cryptography, such as [MeOo]. Many of them, including that one, will treat this topic.

want to show that assuming the inductive hypothesis (see part (c)) we can show

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

e) The left-hand side of the equation in part (d) equals, by the inductive hypothesis, $k(k+1)(2k+1)/6 + (k+1)^2$. We need only do a bit of algebraic manipulation to get this expression into the desired form: factor out $(k+1)/6$ and then factor the rest. In detail,

$$\begin{aligned}
\left(1^2 + 2^2 + \cdots + k^2\right) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by the inductive hypothesis)} \\
&= \frac{k+1}{6}\big(k(2k+1) + 6(k+1)\big) = \frac{k+1}{6}(2k^2 + 7k + 6) \\
&= \frac{k+1}{6}(k+2)(2k+3) = \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

5. We proceed by induction. The basis step, $n = 0$, is true, since $1^2 = 1 \cdot 1 \cdot 3/3$. For the inductive step assume the inductive hypothesis that

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}.$$

We want to show that

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

(the right-hand side is the same formula with $k+1$ plugged in for n). Now the left-hand side equals, by the inductive hypothesis, $(k+1)(2k+1)(2k+3)/3 + (2k+3)^2$. We need only do a bit of algebraic manipulation to get this expression into the desired form: factor out $(2k+3)/3$ and then factor the rest. In detail,

$$\begin{aligned}
&\left(1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2\right) + (2k+3)^2 \\
&\quad = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2 \quad \text{(by the inductive hypothesis)} \\
&\quad = \frac{2k+3}{3}\big((k+1)(2k+1) + 3(2k+3)\big) = \frac{2k+3}{3}(2k^2 + 9k + 10) \\
&\quad = \frac{2k+3}{3}\big((k+2)(2k+5)\big) = \frac{(k+2)(2k+3)(2k+5)}{3}.
\end{aligned}$$

7. Let $P(n)$ be the proposition $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$. To prove that this is true for all nonnegative integers n, we proceed by mathematical induction. First we verify $P(0)$, namely that $3 = 3(5-1)/4$, which is certainly true. Next we assume that $P(k)$ is true and try to derive $P(k+1)$. Now $P(k+1)$ is the formula

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = \frac{3(5^{k+2} - 1)}{4}.$$

All but the last term of the left-hand side of this equation is exactly the left-hand side of $P(k)$, so by the inductive hypothesis, it equals $3(5^{k+1} - 1)/4$. Thus we have

$$\begin{aligned}
3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^k + 3 \cdot 5^{k+1} &= \frac{3(5^{k+1} - 1)}{4} + 3 \cdot 5^{k+1} \\
&= 5^{k+1}\left(\frac{3}{4} + 3\right) - \frac{3}{4} = 5^{k+1} \cdot \frac{15}{4} - \frac{3}{4} \\
&= 5^{k+2} \cdot \frac{3}{4} - \frac{3}{4} = \frac{3(5^{k+2} - 1)}{4}.
\end{aligned}$$

9. a) We can obtain a formula for the sum of the first n even positive integers from the formula for the sum of the first n positive integers, since $2+4+6+\cdots+2n = 2(1+2+3+\cdots+n)$. Therefore, using the result of Example 1, the sum of the first n even positive integers is $2\big(n(n+1)/2\big) = n(n+1)$.

b) We want to prove the proposition $P(n): 2+4+6+\cdots+2n = n(n+1)$. The basis step, $n=1$, says that $2 = 1\cdot(1+1)$, which is certainly true. For the inductive step, we assume that $P(k)$ is true, namely that

$$2+4+6+\cdots+2k = k(k+1),$$

and try to prove from this assumption that $P(k+1)$ is true, namely that

$$2+4+6+\cdots+2k+2(k+1) = (k+1)(k+2).$$

(Note that the left-hand side consists of the sum of the first $k+1$ even positive integers.) We have

$$\begin{aligned} 2+4+6+\cdots+2k+2(k+1) &= (2+4+6+\cdots+2k)+2(k+1) \\ &= k(k+1)+2(k+1) \quad \text{(by the inductive hypothesis)} \\ &= (k+1)(k+2), \end{aligned}$$

as desired, and our proof by mathematical induction is complete.

11. a) Let us compute the values of this sum for $n \le 4$ to see whether we can discover a pattern. For $n=1$ the sum is $\frac{1}{2}$. For $n=2$ the sum is $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. For $n=3$ the sum is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$. And for $n=4$ the sum is $15/16$. The pattern seems pretty clear, so we conjecture that the sum is always $(2^n-1)/2^n$.

b) We have already verified that this is true in the base case (in fact, in four base cases). So let us assume it for k and try to prove it for $k+1$. More formally, we are letting $P(n)$ be the *statement* that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = \frac{2^n-1}{2^n},$$

and trying to prove that $P(n)$ is true for all n. We have already verified $P(1)$ (as well as $P(2)$, $P(3)$, and $P(4)$ for good measure). We now assume the inductive hypothesis $P(k)$, which is the equation displayed above with k substituted for n, and must derive $P(k+1)$, which is the equation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}}.$$

The "obvious" thing to try is to add $1/2^{k+1}$ to both sides of the inductive hypothesis and see whether the algebra works out as we hope it will. We obtain

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} = \frac{2^k-1}{2^k}+\frac{1}{2^{k+1}} = \frac{2\cdot 2^k-2\cdot 1+1}{2^{k+1}} = \frac{2^{k+1}-1}{2^{k+1}},$$

as desired.

13. The base case of the statement $P(n): 1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$, when $n=1$, is $1^2 = (-1)^0\cdot 1\cdot 2/2$, which is certainly true. Assume the inductive hypothesis $P(k)$, and try to derive $P(k+1)$:

$$1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2+(-1)^k(k+1)^2 = (-1)^k\frac{(k+1)(k+2)}{2}.$$

Starting with the left-hand side of $P(k+1)$, we have

$$\begin{aligned} &\big(1^2-2^2+3^2-\cdots+(-1)^{k-1}k^2\big)+(-1)^k(k+1)^2 \\ &\qquad = (-1)^{k-1}\frac{k(k+1)}{2}+(-1)^k(k+1)^2 \quad \text{(by the inductive hypothesis)} \\ &\qquad = (-1)^k(k+1)\big((-k/2)+k+1\big) \\ &\qquad = (-1)^k(k+1)\left(\frac{k}{2}+1\right) = (-1)^k\frac{(k+1)(k+2)}{2}, \end{aligned}$$

the right-hand side of $P(k+1)$.

15. The base case of the statement $P(n): 1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3$, when $n = 1$, is $1\cdot 2 = 1\cdot 2\cdot 3/3$, which is certainly true. We assume the inductive hypothesis $P(k)$, and try to derive $P(k+1)$:

$$1\cdot 2 + 2\cdot 3 + \cdots + k(k+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

Starting with the left-hand side of $P(k+1)$, we have

$$\begin{aligned}\big(1\cdot 2 + 2\cdot 3 + \cdots + k(k+1)\big) &+ (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by the inductive hypothesis)} \\ &= (k+1)(k+2)\left(\frac{k}{3}+1\right) = \frac{(k+1)(k+2)(k+3)}{3},\end{aligned}$$

the right-hand side of $P(k+1)$.

17. This proof follows the basic pattern of the solution to Exercise 3, but the algebra gets more complex. The statement $P(n)$ that we wish to prove is

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

where n is a positive integer. The basis step, $n = 1$, is true, since $1\cdot 2\cdot 3\cdot 5/30 = 1$. Assume the displayed statement as the inductive hypothesis, and proceed as follows to prove $P(n+1)$:

$$\begin{aligned}(1^4 + 2^4 + \cdots + n^4) + (n+1)^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 \\ &= \frac{n+1}{30}\big(n(2n+1)(3n^2+3n-1) + 30(n+1)^3\big) \\ &= \frac{n+1}{30}(6n^4 + 39n^3 + 91n^2 + 89n + 30) \\ &= \frac{n+1}{30}(n+2)(2n+3)(3(n+1)^2 + 3(n+1) - 1)\end{aligned}$$

The last equality is straightforward to check; it was obtained not by attempting to factor the next to last expression from scratch but rather by knowing exactly what we expected the simplified expression to be.

19. **a)** $P(2)$ is the statement that $1 + \frac{1}{4} < 2 - \frac{1}{2}$.

b) This is true because $5/4$ is less than $6/4$.

c) The indnctive hypothesis is the statement that

$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} < 2 - \frac{1}{k}.$$

d) For the inductive step, we want to show for each $k \geq 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can show

$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}.$$

e) Assume the inductive hypothesis. Then we have

$$\begin{aligned}1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \\ &= 2 - \left(\frac{k^2+2k+1-k}{k(k+1)^2}\right) \\ &= 2 - \frac{k^2+k}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\ &= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2} < 2 - \frac{1}{k+1}.\end{aligned}$$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

21. Let $P(n)$ be the proposition $2^n > n^2$. We want to show that $P(n)$ is true for all $n > 4$. The base case is therefore $n = 5$, and we check that $2^5 = 32 > 25 = 5^2$. Now we assume the inductive hypothesis that $2^k > k^2$ and want to derive the statement that $2^{k+1} > (k+1)^2$. Working from the right-hand side, we have $(k+1)^2 = k^2 + 2k + 1 < k^2 + 2k + k = k^2 + 3k < k^2 + k^2$ (since $k > 3$). Thus we have $(k+1)^2 < 2k^2 < 2 \cdot 2^k$ (by the inductive hypothesis), which in turn equals 2^{k+1}, as desired.

23. We compute the values of $2n+3$ and 2^n for the first few values of n and come to the immediate conjecture that $2n+3 \le 2^n$ for $n \ge 4$ but for no other nonnegative integer values of n. The negative part of this statement is just the fact that $3 > 1$, $5 > 2$, $7 > 4$, and $9 > 8$. We must prove by mathematical induction that $2n+3 \le 2^n$ for all $n \ge 4$. The base case is $n = 4$, in which we see that, indeed, $11 \le 16$. Next assume the inductive hypothesis that $2n+3 \le 2^n$, and consider $2(n+1)+3$. This equals $2n+3+2$, which by the inductive hypothesis is less than or equal to $2^n + 2$. But since $n \ge 1$, this in turn is at most $2^n + 2^n = 2^{n+1}$, precisely the statement we wished to prove.

25. We can assume that $h > -1$ is fixed, and prove the proposition by induction on n. Let $P(n)$ be the proposition $1+nh \le (1+h)^n$. The base case is $n = 0$, in which case $P(0)$ is simply $1 \le 1$, certainly true. Now we assume the inductive hypothesis, that $1 + kh \le (1+h)^k$; we want to show that $1 + (k+1)h \le (1+h)^{k+1}$. Since $h > -1$, it follows that $1+h > 0$, so we can multiply both sides of the inductive hypothesis by $1+h$ to obtain $(1+h)(1+kh) \le (1+h)^{k+1}$. Thus to complete the proof it is enough to show that $1+(k+1)h \le (1+h)(1+kh)$. But the right-hand side of this inequality is the same as $1 + h + kh + kh^2 = 1 + (k+1)h + kh^2$, which is greater than or equal to $1 + (k+1)h$ because $kh^2 \ge 0$.

27. This exercise involves some messy algebra, but the logic is the usual logic for proofs using the principle of mathematical induction. The basis step ($n = 1$) is true, since 1 is greater than $2(\sqrt{2}-1) \approx 0.83$. We assume that

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1}-1)$$

and try to derive the corresponding statement for $n+1$:

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2}-1)$$

Since by the inductive hypothesis we know that

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+1}-1) + \frac{1}{\sqrt{n+1}},$$

we will be finished if we can show that the inequality

$$2(\sqrt{n+1}-1) + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+2}-1)$$

holds. By canceling the -2 from both sides and rearranging, we obtain the equivalent inequality

$$2(\sqrt{n+2}-\sqrt{n+1}) < \frac{1}{\sqrt{n+1}},$$

which in turn is equivalent to

$$2(\sqrt{n+2}-\sqrt{n+1})(\sqrt{n+2}+\sqrt{n+1}) < \frac{\sqrt{n+1}}{\sqrt{n+1}} + \frac{\sqrt{n+2}}{\sqrt{n+1}}.$$

This last inequality simplifies to

$$2 < 1 + \frac{\sqrt{n+2}}{\sqrt{n+1}},$$

which is clearly true. Therefore the original inequality is true, and our proof is complete.

29. Recall that $H_k = 1/1 + 1/2 + \cdots + 1/k$. We want to prove that $H_{2^n} \le 1 + n$ for all natural numbers n. We proceed by mathematical induction, noting that the basis step $n = 0$ is the trivial statement $H_1 = 1 \le 1 + 0$. Therefore we assume that $H_{2^n} \le 1 + n$; we want to show that $H_{2^{n+1}} \le 1 + (n+1)$. We have

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} \quad \text{(by definition; there are } 2^n \text{ fractions here)} \\ &\le (1+n) + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} \quad \text{(by the inductive hypothesis)} \\ &\le (1+n) + \frac{1}{2^n+1} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^n+1} \quad \text{(we made the denominators smaller)} \\ &= 1 + n + \frac{2^n}{2^n+1} < 1 + n + 1 = 1 + (n+1). \end{aligned}$$

31. This is easy to prove without mathematical induction, because we can observe that $n^2 + n = n(n+1)$, and either n or $n+1$ is even. If we want to use the principle of mathematical induction, we can proceed as follows. The basis step is the observation that $1^2 + 1 = 2$ is divisible by 2. Assume the inductive hypothesis, that $k^2 + k$ is divisible by 2; we must show that $(k+1)^2 + (k+1)$ is divisible by 2. But $(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + 2(k+1)$. But now $k^2 + k$ is divisible by 2 by the inductive hypothesis, and $2(k+1)$ is divisible by 2 by definition, so this sum of two multiples of 2 must be divisible by 2.

33. To prove that $P(n) : 5 \mid (n^5 - n)$ holds for all nonnegative integers n, we first check that $P(0)$ is true; indeed $5 \mid 0$. Next assume that $5 \mid (n^5 - n)$, so that we can write $n^5 - n = 5t$ for some integer t. Then we want to prove $P(n+1)$, namely that $5 \mid \big((n+1)^5 - (n+1)\big)$. We expand and then factor the right-hand side to obtain

$$\begin{aligned} (n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + 5(n^4 + 2n^3 + 2n^2 + n) \\ &= 5t + 5(n^4 + 2n^3 + 2n^2 + n) \quad \text{(by the inductive hypothesis)} \\ &= 5(t + n^4 + 2n^3 + 2n^2 + n). \end{aligned}$$

Thus we have shown that $(n+1)^5 - (n+1)$ is also a multiple of 5, and our proof by induction is complete. (Note that here we have used n as the dummy variable in the inductive step, rather than k. It really makes no difference.)

We should point out that using mathematical induction is not the only way to prove this proposition; it can also be proved by considering the five cases determined by the value of n **mod** 5. The reader is encouraged to write down such a proof.

35. First let us rewrite this proposition so that it is a statement about all nonnegative integers, rather than just the odd positive integers. An odd positive integer can be written as $2n - 1$, so let us prove the proposition $P(n)$ that $(2n-1)^2 - 1$ is divisible by 8 for all positive integers n. We first check that $P(1)$ is true; indeed $8 \mid 0$. Next assume that $8 \mid ((2n-1)^2 - 1)$. Then we want to prove $P(n+1)$, namely that $8 \mid ((2n+1)^2 - 1)$. Let us look at the difference of these two expressions: $(2n+1)^2 - 1 - ((2n-1)^2 - 1)$. A little algebra reduces this to $8n$, which is certainly a multiple of 8. But if this difference is a multiple of 8, and if, by the inductive hypothesis, $(2n-1)^2 - 1$ is a multiple of 8, then $(2n+1)^2 - 1$ must be a multiple of 8, and our proof by induction is complete.

37. It is not easy to stumble upon the trick needed in the inductive step in this exercise, so do not feel bad if you did not find it. The form is straightforward. For the basis step ($n = 1$), we simply observe that $11^{1+1} + 12^{2\cdot 1-1} = 121 + 12 = 133$, which is divisible by 133. Then we assume the inductive hypothesis, that

$11^{n+1}+12^{2n-1}$ is divisible by 133, and let us look at the expression when $n+1$ is plugged in for n. We want somehow to manipulate it so that the expression for n appears. We have

$$\begin{aligned} 11^{(n+1)+1}+12^{2(n+1)-1} &= 11\cdot 11^{n+1}+144\cdot 12^{2n-1} \\ &= 11\cdot 11^{n+1}+(11+133)\cdot 12^{2n-1} \\ &= 11(11^{n+1}+12^{2n-1})+133\cdot 12^{2n-1}. \end{aligned}$$

Looking at the last line, we see that the expression in parentheses is divisible by 133 by the inductive hypothesis, and obviously the second term is divisible by 133, so the entire quantity is divisible by 133, as desired.

39. The basis step is trivial, as usual: $A_1\subseteq B_1$ implies that $\bigcap_{j=1}^{1}A_j\subseteq\bigcap_{j=1}^{1}B_j$ because the intersection of one set is itself. Assume the inductive hypothesis that if $A_j\subseteq B_j$ for $j=1,2,\ldots,k$, then $\bigcap_{j=1}^{k}A_j\subseteq\bigcap_{j=1}^{k}B_j$. We want to show that if $A_j\subseteq B_j$ for $j=1,2,\ldots,k+1$, then $\bigcap_{j=1}^{k+1}A_j\subseteq\bigcap_{j=1}^{k+1}B_j$. To show that one set is a subset of another we show that an arbitrary element of the first set must be an element of the second set. So let $x\in\bigcap_{j=1}^{k+1}A_j=\left(\bigcap_{j=1}^{k}A_j\right)\cap A_{k+1}$. Because $x\in\bigcap_{j=1}^{k}A_j$, we know by the inductive hypothesis that $x\in\bigcap_{j=1}^{k}B_j$; because $x\in A_{k+1}$, we know from the given fact that $A_{k+1}\subseteq B_{k+1}$ that $x\in B_{k+1}$. Therefore $x\in\left(\bigcap_{j=1}^{k}B_j\right)\cap B_{k+1}=\bigcap_{j=1}^{k+1}B_j$.

This is really easier to do directly than by using the principle of mathematical induction. For a noninductive proof, suppose that $x\in\bigcap_{j=1}^{n}A_j$. Then $x\in A_j$ for each j from 1 to n. Since $A_j\subseteq B_j$, we know that $x\in B_j$. Therefore by definition, $x\in\bigcap_{j=1}^{n}B_j$.

41. In order to prove this statement, we need to use one of the distributive laws from set theory: $(X\cup Y)\cap Z=(X\cap Z)\cup(Y\cap Z)$ (see Section 2.2). Indeed, the proposition at hand is the generalization of this distributive law, from two sets in the union to n sets in the union. We will also be using implicitly the associative law for set union.

The basis step, $n=1$, is the statement $A_1\cap B=A_1\cap B$, which is obviously true. Therefore we assume the inductive hypothesis that

$$(A_1\cup A_2\cup\cdots\cup A_n)\cap B=(A_1\cap B)\cup(A_2\cap B)\cup\cdots\cup(A_n\cap B).$$

We wish to prove the similar statement for $n+1$, namely

$$(A_1\cup A_2\cup\cdots\cup A_n\cup A_{n+1})\cap B=(A_1\cap B)\cup(A_2\cap B)\cup\cdots\cup(A_n\cap B)\cup(A_{n+1}\cap B).$$

Starting with the left-hand side, we apply the distributive law for two sets:

$$\begin{aligned} \big((A_1\cup A_2\cup\cdots\cup A_n)\cup A_{n+1}\big)\cap B &= \big((A_1\cup A_2\cup\cdots\cup A_n)\cap B\big)\cup(A_{n+1}\cap B) \\ &= \big((A_1\cap B)\cup(A_2\cap B)\cup\cdots\cup(A_n\cap B)\big)\cup(A_{n+1}\cap B) \\ &\qquad\text{(by the inductive hypothesis)} \\ &= (A_1\cap B)\cup(A_2\cap B)\cup\cdots\cup(A_n\cap B)\cup(A_{n+1}\cap B) \end{aligned}$$

43. In order to prove this statement, we need to use one of De Morgan's laws from set theory: $\overline{(A\cup B)}=\overline{A}\cap\overline{B}$ (see Section 2.2). Indeed, the proposition at hand is the generalization of this law, from two sets in the union to n sets in the union. We will also be using implicitly the associative laws for set union and intersection.

The basis step, $n=1$, is the statement $\overline{A_1}=\overline{A_1}$ (since the union or intersection of just one set is the set itself), and this proposition is obviously true. Therefore we assume the inductive hypothesis that

$$\overline{\bigcup_{k=1}^{n}A_k}=\bigcap_{k=1}^{n}\overline{A_k}.$$

We wish to prove the similar statement for $n+1$, namely

$$\overline{\bigcup_{k=1}^{n+1} A_k} = \bigcap_{k=1}^{n+1} \overline{A_k}\,.$$

Starting with the left-hand side, we group, apply De Morgan's law for two sets, and then the inductive hypothesis:

$$\begin{aligned}
\overline{\bigcup_{k=1}^{n+1} A_k} &= \overline{\Big(\bigcup_{k=1}^{n} A_k\Big) \cup A_{n+1}} \\
&= \overline{\bigcup_{k=1}^{n} A_k} \cap \overline{A_{n+1}} \quad \text{(by DeMorgan's law)} \\
&= \Big(\bigcap_{k=1}^{n} \overline{A_k}\Big) \cap \overline{A_{n+1}} \quad \text{(by the inductive hypothesis)} \\
&= \bigcap_{k=1}^{n+1} \overline{A_k}
\end{aligned}$$

45. This proof will be similar to the proof in Example 9. The basis step is clear, since for $n = 2$, the set has exactly one subset containing exactly two elements, and $2(2-1)/2 = 1$. Assume the inductive hypothesis, that a set with n elements has $n(n-1)/2$ subsets with exactly two elements; we want to prove that a set S with $n+1$ elements has $(n+1)n/2$ subsets with exactly two elements. Fix an element a in S, and let T be the set of elements of S other than a. There are two varieties of subsets of S containing exactly two elements. First there are those that do not contain a. These are precisely the two-element subsets of T, and by the inductive hypothesis, there are $n(n-1)/2$ of them. Second, there are those that contain a together with one element of T. Since T has n elements, there are exactly n subsets of this type. Therefore the total number of subsets of S containing exactly two elements is $(n(n-1)/2) + n$, which simplifies algebraically to $(n+1)n/2$, as desired.

47. The one and only flaw in this proof is in this statement, which is part of the inductive step: "the set of the first n horses and the set of the last n horses [in the collection of $n+1$ horses being considered] overlap." The only assumption made about the number n in this argument is that n is a positive integer. When $n = 1$, so that $n+1 = 2$, the statement quoted is obviously nonsense: the set of the first one horse and the set of the last one horse, in this set of two horses, are disjoint.

49. The mistake is in applying tbe inductive hypothesis to look at $\max(x-1, y-1)$, because even though x and y are positive integers, $x-1$ and $y-1$ need not be (one or both could be 0). In fact, that is what happens if we let $x = 1$ and $y = 2$ when $k = 1$.

51. We use the notation (i, j) to mean the square in row i and column j, where we number from the left and from the bottom, starting at $(0, 0)$ in the lower left-hand corner. We use induction on $i + j$ to show that every square can be reached by the knight. There are six base cases, for the cases when $i + j \le 2$. The knight is already at $(0, 0)$ to start, so the empty sequence of moves reaches that square. To reach $(1, 0)$, the knight moves successively from $(0, 0)$ to $(2, 1)$ to $(0, 2)$ to $(1, 0)$. Similarly, to reach $(0, 1)$, the knight moves successively from $(0, 0)$ to $(1, 2)$ to $(2, 0)$ to $(0, 1)$. Note that the knight has reached $(2, 0)$ and $(0, 2)$ in the process. For the last basis step, note this path to $(1, 1)$: $(0, 0)$ to $(1, 2)$ to $(2, 0)$ to $(0, 1)$ to $(2, 2)$ to $(0, 3)$ to $(1, 1)$. We now assume the inductive hypothesis, that the knight can reach any square (i, j) for which $i + j = k$, where k is an integer greater than 1, and we want to show how the knight can reach each square (i, j) when $i + j = k + 1$. Since $k + 1 \ge 3$, at least one of i and j is at least 2. If $i \ge 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i-2, j+1)$, since $i-2+j+1=i+j-1=k$; from there it is just one step to (i,j). Similarly, if $j \geq 2$, then by the inductive hypothesis, there is a sequence of moves ending at $(i+1, j-2)$, since $i+1+j-2=i+j-1=k$; from there it is again just one step to (i,j).

53. The base cases are $n=0$ and $n=1$, and it is a simple matter to evaluate, directly from the "limit of difference quotient" definition, the derivatives of $x^0=1$ and $x^1=x$:

$$\frac{d}{dx}x^0 = \lim_{h\to 0}\frac{(x+h)^0-x^0}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0 = 0\cdot x^{-1}$$

$$\frac{d}{dx}x^1 = \lim_{h\to 0}\frac{(x+h)^1-x^1}{h} = \lim_{h\to 0}\frac{h}{h} = 1 = 1\cdot x^0$$

We are told to assume that the product rule holds:

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

So we work as follows, invoking the inductive hypothesis and the base cases:

$$\begin{aligned}\frac{d}{dx}x^{n+1} &= \frac{d}{dx}(x\cdot x^n) = x\cdot\frac{d}{dx}x^n + x^n\cdot\frac{d}{dx}x \\ &= x\cdot nx^{n-1} + x^n\cdot 1 = nx^n + x^n = (n+1)x^n\end{aligned}$$

55. We prove this by induction on k. The basis step $k=0$ is the trivial statement that $1 \equiv 1 \pmod{m}$. Suppose that the statement is true for k. We must show it for $k+1$. So let $a \equiv b \pmod{m}$. By the inductive hypothesis we know that $a^k \equiv b^k \pmod{m}$. Then we apply Theorem 5 from Section 3.4 to conclude that $a\cdot a^k \equiv b\cdot b^k \pmod{m}$, which by definition says that $a^{k+1} \equiv b^{k+1} \pmod{m}$, as desired.

57. Let $P(n)$ be the proposition

$$[(p_1\to p_2)\wedge(p_2\to p_3)\wedge\cdots\wedge(p_{n-1}\to p_n)] \to [(p_1\wedge p_2\wedge\cdots\wedge p_{n-1})\to p_n].$$

We want to prove this proposition for all $n \geq 2$. The basis step, $(p_1\to p_2)\to(p_1\to p_2)$, is clearly true (a tautology), since every proposition implies itself. Now we assume $P(n)$ and want to show $P(n+1)$, namely

$$\begin{aligned}&[(p_1\to p_2)\wedge(p_2\to p_3)\wedge\cdots\wedge(p_{n-1}\to p_n)\wedge(p_n\to p_{n+1})] \to \\ &\qquad [(p_1\wedge p_2\wedge\cdots\wedge p_{n-1}\wedge p_n)\to p_{n+1}].\end{aligned}$$

To show this, we will assume that the hypothesis (everything in the first square brackets) is true and show that the conclusion (the conditional statement in the second square brackets) is also true.

So we assume $(p_1\to p_2)\wedge(p_2\to p_3)\wedge\cdots\wedge(p_{n-1}\to p_n)\wedge(p_n\to p_{n+1})$. By the associativity of $\wedge$, we can group this as $((p_1\to p_2)\wedge(p_2\to p_3)\wedge\cdots\wedge(p_{n-1}\to p_n))\wedge(p_n\to p_{n+1})$. By the simplification rule, we can conclude that the first group, $(p_1\to p_2)\wedge(p_2\to p_3)\wedge\cdots\wedge(p_{n-1}\to p_n)$, must be true. Now the inductive hypothesis allows us to conclude that $(p_1\wedge p_2\wedge\cdots\wedge p_{n-1})\to p_n$. This together with the rest of the assumption, namely $p_n\to p_{n+1}$, yields, by the hypothetical syllogism rule, $(p_1\wedge p_2\wedge\cdots\wedge p_{n-1})\to p_{n+1}$.

That is almost what we wanted to prove, but not quite. We wanted to prove that $(p_1\wedge p_2\wedge\cdots\wedge p_{n-1}\wedge p_n)\to p_{n+1}$. In order to prove this, let us assume its hypothesis, $p_1\wedge p_2\wedge\cdots\wedge p_{n-1}\wedge p_n$. Again using the simplification rule we obtain $p_1\wedge p_2\wedge\cdots\wedge p_{n-1}$. Now by modus ponens with the proposition $(p_1\wedge p_2\wedge\cdots\wedge p_{n-1})\to p_{n+1}$, which we proved above, we obtain p_{n+1}. Thus we have proved $(p_1\wedge p_2\wedge\cdots\wedge p_{n-1}\wedge p_n)\to p_{n+1}$, as desired.

59. This exercise, as the double star indicates, is quite hard. The trick is to induct not on n itself, but rather on $\log_2 n$. In other words, we write $n=2^k$ and prove the statement by induction on k. This will prove the statement for every n that is a power of 2; a separate argument is needed to extend to the general case.

We take the basis step to be $k=1$ (the case $k=0$ is trivially true, as well), so that $n=2^1=2$. In this case the trick is to start with the true inequality $(\sqrt{a_1}-\sqrt{a_2})^2 \geq 0$. Expanding, we have $a_1 - 2\sqrt{a_1a_2}+a_2 \geq 0$, whence $(a_1+a_2)/2 \geq (a_1a_2)^{1/2}$, as desired. For the inductive step, we assume that the inequality holds for $n=2^k$ and prove that it also holds for $2n = 2^{k+1}$. What we need to show, then, is that

$$\frac{a_1+a_2+\cdots+a_{2n}}{2n} \geq (a_1a_2\cdots a_{2n})^{1/(2n)}.$$

First we observe that

$$\frac{a_1+a_2+\cdots+a_{2n}}{2n} = \left(\frac{a_1+a_2+\cdots+a_n}{n} + \frac{a_{n+1}+a_{n+2}+\cdots+a_{2n}}{n}\right)\Big/2$$

and

$$(a_1a_2\cdots a_{2n})^{1/(2n)} = \left((a_1a_2\cdots a_n)^{1/n}(a_{n+1}a_{n+2}\cdots a_{2n})^{1/n}\right)^{1/2}.$$

Now to simplify notation, let $A(x,y,\ldots)$ denote the arithmetic mean and $G(x,y,\ldots)$ denote the geometric mean of the numbers $x,y,\ldots$. It is clear that if $x \leq x'$, $y \leq y'$, and so on, then $A(x,y,\ldots) \leq A(x',y',\ldots)$ and $G(x,y,\ldots) \leq G(x',y',\ldots)$. Now we have

$$\begin{aligned} A(a_1,\ldots,a_{2n}) &= A\big(A(a_1,\ldots,a_n), A(a_{n+1},\ldots,a_{2n})\big) \quad \text{(by the first observation above)}\\ &\geq A\big(G(a_1,\ldots,a_n), G(a_{n+1},\ldots,a_{2n})\big) \quad \text{(by the inductive hypothesis)}\\ &\geq G\big(G(a_1,\ldots,a_n), G(a_{n+1},\ldots,a_{2n})\big) \quad \text{(this was the case } n=2)\\ &= G(a_1,\ldots,a_{2n}) \quad \text{(by the second observation above)}. \end{aligned}$$

Having proved the inequality in the case in which n is a power of 2, we now turn to the case of n that is not a power of 2. Let m be the smallest power of 2 bigger than m. (For instance, if $n=25$, then $m=32$.) Denote the arithmetic mean $A(a_1,\ldots,a_n)$ by a, and set $a_{n+1}=a_{n+2}=\cdots=a_m$ all equal to a. One effect of this is that then $A(a_1,\ldots,a_m)=a$. Now we have

$$\left((\prod_{i=1}^{n} a_i)a^{m-n}\right)^{1/m} \leq A(a_1,\ldots,a_m)$$

by the case we have already proved, since m is a power of 2. Using algebra on the left-hand side and the observation that $A(a_1,\ldots,a_m)=a$ on the right, we obtain

$$(\prod_{i=1}^{n} a_i)^{1/m} a^{1-n/m} \leq a$$

or

$$(\prod_{i=1}^{n} a_i)^{1/m} \leq a^{n/m}.$$

Finally we raise both sides to the power m/n to give

$$(\prod_{i=1}^{n} a_i)^{1/n} \leq a,$$

as desired.

61. Let us check the cases $n=1$ and $n=2$, both to establish the basis and to try to see what is going on. For $n=1$ there is only one nonempty subset of $\{1\}$, so the left-hand side is just $\frac{1}{1}$, and that equals 1. For $n=2$ there are three nonempty subsets: $\{1\}$, $\{2\}$, and $\{1,2\}$, so the left-hand side is $\frac{1}{1}+\frac{1}{2}+\frac{1}{1\cdot 2}=2$. To prove the inductive step, assume that the statement is true for n, and consider it for $n+1$. Now the set of the first $n+1$ positive integers has many nonempty subsets, but they fall into three categories: a nonempty subset of the first n positive integers, a nonempty subset of the first n positive integers together with $n+1$, or just $\{n+1\}$.

So we need to sum over these three categories. By the inductive hypothesis, the sum over the first category is n. For the second category, we can factor out $\frac{1}{n+1}$ from each term of the sum, note that the remaining factor again gives n by the inductive hypothesis, and so conclude that this part of the sum is $\frac{n}{n+1}$. Finally, the third category simply yields the value $\frac{1}{n+1}$. Therefore the entire summation is $n + \frac{n}{n+1} + \frac{1}{n+1} = n+1$, as desired.

63. The basis step ($n=2$) is clear, because if $A_1 \subseteq A_2$, then A_1 satisfies the condition of being a subset of each set in the collection, and otherwise A_2 does, because in that case, A_2 must be a subset of A_1 (by the stated assumptions). For the inductive step, assume the inductive hypothesis, that the conditional statement is true for k sets, and suppose we are given $k+1$ sets that satisfy the given conditions. By the inductive hypothesis, there must be a set A_i for some $i \le k$ such that $A_i \subseteq A_j$ for $1 \le j \le k$. If $A_i \subseteq A_{k+1}$, then we are done. Otherwise, we know that $A_{k+1} \subseteq A_i$, and this tells us that A_{k+1} satisfies the condition of being a subset of A_j for $1 \le j \le k+1$.

65. Number the people 1, 2, 3, and 4, and let s_i be the scandal originally known only to person i. It is clear that $G(1) = 0$ and $G(2) = 1$. For three people, without loss of generality assume that 1 calls 2 first and 1 calls 3 next. At this point 1 and 3 know all three scandals, but it takes one more call to let 2 know s_3. Thus $G(3) = 3$. For four people, without loss of generality assume that 1 calls 2 first. If now 3 calls 4, then after two calls 1 and 2 both know s_1 and s_2, while 3 and 4 both know s_3 and s_4. It is clear that two more calls (between 1 and 3, and between 2 and 4, say) are necessary and sufficient to complete the exchange. This makes a total of four calls. The only other case to consider (to see whether $G(4)$ might be less than 4) is when the second call, without loss of generality, occurs between 1 and 3. At this point, both 2 and 4 still need to learn s_3, and talking to each other won't give them that information, so at least two more calls would be required. Thus $G(4) = 4$.

67. We need to show that $2n-4$ calls are both necessary and sufficient to exchange all the gossip. Sufficiency is easier. Select four of the people, say 1, 2, 3, and 4, to be the central committee. Every person outside the central committee calls one person on the central committee. This takes $n-4$ calls, and at this point the central committee members *as a group* know all the scandals. They then exchange information among themselves by making the calls 12, 34, 13, and 24 in that order (of course the first two can be done in parallel and the last two can be done in parallel). At this point, *every* central committee member knows all the scandals, and we have used $n-4+4=n$ calls. Finally, again every person outside the central committee calls one person on the central committee, at which point everyone knows all the scandals. This takes $n-4$ more calls, for a total of $2n-4$ calls.

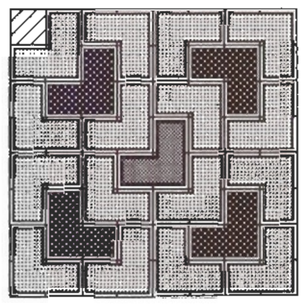
That this cannot be done with fewer than $2n-4$ calls is much harder to prove, and the proof will not be presented here. See the following website for details:

`www.cs.cornell.edu/vogels/Epidemics/gossips-telephones.pdf`

69. We prove this by mathematical induction. The basis step ($n=2$) is true tautologically (if $I_1 \cap I_2 \ne \emptyset$ then $I_1 \cap I_2 \ne \emptyset$). The heart of the argument occurs with three sets, so we will give the proof for $n=3$ explicitly. Recall the notation $(u,v) = \{x \mid u < x < v\}$. Suppose that the intervals are (a,b), (c,d), and (e,f), where without loss of generality we can assume that $a \le c \le e$. Because $(a,b) \cap (e,f) \ne \emptyset$, we must have $e < b$; for a similar reason, $e < d$. It follows that the number halfway between e and the smaller of b and d is common to all three intervals. Now for the inductive step, assume that whenever we have k intervals that have pairwise nonempty intersections then there is a point common to all the intervals, and suppose that we are given intervals $I_1, I_2, \ldots, I_{k+1}$ that have pairwise nonempty intersections. For each i from 1 to k, let $J_i = I_i \cap I_{k+1}$. We claim that the collection $J_1, J_2, \ldots, J_k$ satisfies the inductive hypothesis, that is, that $J_{i_1} \cap J_{i_2} \ne \emptyset$ for each choice of subscripts i_1 and i_2. This follows from the $n=3$ case proved above, using the sets I_{i_1}, I_{i_2}, and I_{k+1}. We can now invoke the inductive hypothesis to conclude that there is a number common to all of the sets J_i for $i = 1, 2, \ldots, k$, which perforce is in the intersection of all the sets I_i for $i = 1, 2, \ldots, k+1$.

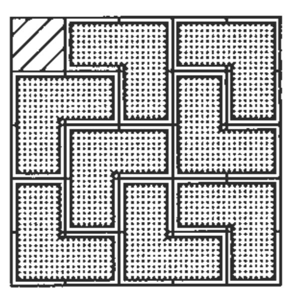
71. Pair up the people. Have the people stand at mutually distinct small distances from their partners but far away from everyone else. Then each person throws a pie at his or her partner, so everyone gets hit.

73. The proof in Example 13 guides us to one solution (it is certainly not unique). We begin by placing a right triomino in the center, with its gap in the same quadrant as the missing square in the upper left corner of the board (this piece is distinctively shaded in our solution below). This reduces the problem to four problems on 4×4 boards. Then we place triominoes in the centers of these four quadrants, using the same principle (shaded somewhat differently below). Finally, we place pieces in the remaining squares to fill up each quadrant.



75. This problem is very similar to Example 13; the only difficulty is in visualizing what's happening in three dimensions. The basis step ($n = 1$) is trivial, since one tile coincides with the solid to be tiled. To make this read a little easier, let us call a $1 \times 1 \times 1$ cube a "cubie"; and let us call the object we are tiling with, namely the $2 \times 2 \times 2$ cube with one cubie removed, a tile. For the inductive step, assume the inductive hypothesis, that the $2^n \times 2^n \times 2^n$ cube with one cubie removed can be covered with tiles, and suppose that a $2^{n+1} \times 2^{n+1} \times 2^{n+1}$ cube with one cubie removed is given. We must show how to cover it with tiles. Think of this large object as split into eight octants through its center, by planes parallel to the faces. The missing cubie occurs in one of these octants. Now position one tile with its center at the center of the large object, so that the missing cubie in the tile lies in the octant in which the large object is missing its cubie. This creates eight $2^n \times 2^n \times 2^n$ cubes, each missing exactly one cubie—one in each octant. By the inductive hypothesis, we can fill each of these smaller objects with tiles. Putting these tilings together gives us the desired tiling of the $2^{n+1} \times 2^{n+1} \times 2^{n+1}$ cube with one cubie removed, as desired.

77. For this specific tiling, the most straightforward proof consists of producing the desired picture. It can be discovered by playing around with the tiles (either make a set, or use paper and pencil).



79. Let $Q(n)$ be $P(n+b-1)$. Thus $Q(1)$ is $P(b)$, $Q(2)$ is $P(b+1)$, and so on. Therefore the statement that $P(n)$ is true for $n = b, b+1, b+2, \ldots$ is the same as the statement that $Q(m)$ is true for all positive integers m.

We are given that $P(b)$ is true (i.e., that $Q(1)$ is true), and that $P(k) \to P(k+1)$ for all $k \geq b$ (i.e., that $Q(m) \to Q(m+1)$ for all positive integers m). Therefore by the principle of mathematical induction, $Q(m)$ is true for all positive integers m, as desired.