SECTION 1.7 Proof Methods and Strategy

The preamble to the solutions for Section 1.6 applies here as well, so you might want to reread it at this time. In addition, the section near the back of this Guide, *entitled "A Guide to Proof-Writing," provides an excellent tutorial, with many additional examples. Don't forget to take advantage of the many additional resources on the website for this text, as well.*

If you are majoring in mathematics, then proofs are the bread and butter of your field. Most likely you will take a course devoted entirely to learning how to read and write proofs, using one of the many textbooks available on this subject. For a review of many of them (as well as reviews of hundreds of mathematics books), see this site provided by the Mathematical Association of America: `http://www.maa.org/reviews/`.

1. We give an exhaustive proof—just check the entire domain. For $n = 1$ we have $1^2 + 1 = 2 \geq 2 = 2^1$. For $n = 2$ we have $2^2 + 1 = 5 \geq 4 = 2^2$. For $n = 3$ we have $3^2 + 1 = 10 \geq 8 = 2^3$. For $n = 4$ we have $4^2 + 1 = 17 \geq 16 = 2^4$. Notice that for $n \geq 5$, the inequality is no longer true.

3. Following the hint, we consider the two cases determined by the relative sizes of x and y. First suppose that $x \geq y$. Then by definition $\max(x, y) = x$ and $\min(x, y) = y$. Therefore in this case $\max(x, y) + \min(x, y) = x + y$, exactly as desired. For the second (and final) case, suppose that $x < y$. Then $\max(x, y) = y$ and $\min(x, y) = x$. Therefore in this case $\max(x, y) + \min(x, y) = y + x = x + y$, again the desired conclusion. Hence in all cases, the equality holds.

5. There are several cases to consider. If x and y are both nonnegative, then $|x|+|y| = x+y = |x+y|$. Similarly, if both are negative, then $|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$, since $x + y$ is negative in this case. The complication (and strict inequality) comes if one of the variables is nonnegative and the other is negative. By the symmetry of the roles of x and y here (strictly speaking, by the commutativity of addition), we can assume without loss of generality that it is x that is nonnegative and y that is negative. So we have $x \geq 0$ and $y < 0$.

Now there are two subcases to consider within this case, depending on the relative sizes of the nonnegative numbers x and $-y$. First suppose that $x \geq -y$. Then $x + y \geq 0$. Therefore $|x + y| = x + y$, and this quantity is a nonnegative number smaller than x (since y is negative). On the other hand $|x| + |y| = x + |y|$ is a positive number bigger than x. Therefore we have $|x + y| < x < |x| + |y|$, as desired.

Finally, consider the possibility that $x < -y$. Then $|x + y| = -(x + y) = (-x) + (-y)$ is a positive number less than or equal to $-y$ (since $-x$ is nonpositive). On the other hand $|x| + |y| = |x| + (-y)$ is a positive number greater than or equal to $-y$. Therefore we have $|x + y| \leq -y \leq |x| + |y|$, as desired.

7. We want to find consecutive squares that are far apart. If n is large enough, then $(n + 1)^2$ will be much bigger than n^2, and that will do it. Let's take $n = 100$. Then $100^2 = 10000$ and $101^2 = 10201$, so the 201 consecutive numbers 10001, 10002, ..., 10200 are not perfect squares. The first 100 of these will satisfy the requirements of this exercise. Our proof was constructive, since we actually exhibited the numbers.

9. We try some small numbers and discover that $8 = 2^3$ and $9 = 3^2$. In fact, this is the only solution, but the proof of this fact is not trivial.

11. One way to solve this is the following nonconstructive proof. Let $x = 2$ (rational) and $y = \sqrt{2}$ (irrational). If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, then let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$; x is rational by assumption, and y is irrational (if it were rational, then $\sqrt{2}$ would be rational). But now $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = 2^{1/2} = \sqrt{2}$, which is irrational, as desired.

13. **a)** This statement asserts the existence of x with a certain property. If we let $y = x$, then we see that $P(x)$ is true. If y is anything other than x, then $P(x)$ is not true. Thus x is the unique element that makes P true.

b) The first clause here says that there is an element that makes P true. The second clause says that whenever two elements both make P true, they are in fact the same element. Together this says that P is satisfied by exactly one element.

c) This statement asserts the existence of an x that makes P true and has the further property that whenever we find an element that makes P true, that element is x. In other words, x is the unique element that makes P true. Note that this is essentially the same as the definition given in the text, except that the final conditional statement has been replaced by its contrapositive.

15. The equation $|a-c| = |b-c|$ is equivalent to the disjunction of two equations: $a-c=b-c$ or $a-c=-b+c$. The first of these is equivalent to $a = b$, which contradicts the assumptions made in this problem, so the original equation is equivalent to $a - c = -b + c$. By adding $b + c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = (a + b)/2$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.

17. We are being asked to solve $n = (k - 2) + (k + 3)$ for k. Using the usual, reversible, rules of algebra, we see that this equation is equivalent to $k = (n - 1)/2$. In other words, this is the one and only value of k that makes our equation true. Since n is odd, $n - 1$ is even, so k is an integer.

19. If x is itself an integer, then we can take $n = x$ and $\epsilon = 0$. No other solution is possible in this case, since if the integer n is greater than x, then n is at least $x + 1$, which would make $\epsilon \geq 1$. If x is not an integer, then round it up to the next integer, and call that integer n. We let $\epsilon = n - x$. Clearly $0 \leq \epsilon < 1$, this is the only ϵ that will work with this n, and n cannot be any larger, since ϵ is constrained to be less than 1.

21. If $x = 5$ and $y = 8$, then the harmonic mean is $2\cdot5\cdot8/(5+8) \approx 6.15$, and the geometric mean is $\sqrt{5\cdot8} \approx 6.32$. If $x = 10$ and $y = 100$, then the harmonic mean is $2 \cdot 10 \cdot 100/(10 + 100) \approx 18.18$, and the geometric mean is $\sqrt{10\cdot100} \approx 31.62$. We conjecture that the harmonic mean of x and y is always less than their geometric mean if x and y are distinct positive real numbers (clearly if $x = y$ then both means are this common value). So we want to verify the inequality $2xy/(x + y) < \sqrt{xy}$. Multiplying both sides by $(x + y)/(2\sqrt{xy})$ gives us the equivalent inequality $\sqrt{xy} < (x + y)/2$, which is proved in Example 14.

23. The key point here is that *the parity (oddness or evenness) of the sum of the numbers written on the board never changes.* If j and k are both even or both odd, then their sum and their difference are both even, and we are replacing the even sum $j+k$ by the even difference $|j-k|$, leaving the parity of the total unchanged. If j and k have different parities, then erasing them changes the parity of the total, but their difference $|j-k|$ is odd, so adding this difference restores the parity of the total. Therefore the integer we end up with at the end of the process must have the same parity as $1 + 2 + \cdots + (2n)$. It is easy to compute this sum. If we add the first and last terms we get $2n + 1$; if we add the second and next-to-last terms we get $2 + (2n - 1) = 2n + 1$; and so on. In all we get n sums of $2n + 1$, so the total sum is $n(2n + 1)$. If n is odd, this is the product of two odd numbers and therefore is odd, as desired.

25. Without loss of generality we can assume that n is nonnegative, since the fourth power of an integer and the fourth power of its negative are the same. To get a handle on the last digit of n, we can divide n by 10, obtaining a quotient k and remainder l, whence $n = 10k + l$, and l is an integer between 0 and 9, inclusive. Then we compute n^4 in each of these ten cases. We get the following values, where ?? is some integer that is a multiple of 10, whose exact value we do not care about.

$$\begin{aligned}
(10k+0)^4 &= 10000k^4 = 10000k^4 + 0\\
(10k+1)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 1\\
(10k+2)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 16\\
(10k+3)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 81\\
(10k+4)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 256\\
(10k+5)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 625\\
(10k+6)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 1296\\
(10k+7)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 2401\\
(10k+8)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 4096\\
(10k+9)^4 &= 10000k^4 + ??\cdot k^3 + ??\cdot k^2 + ??\cdot k + 6561
\end{aligned}$$

Since each coefficient indicated by ?? is a multiple of 10, the corresponding term has no effect on the ones digit of the answer. Therefore the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1, respectively, so it is always a 0, 1, 5, or 6.

27. Because $n^3 > 100$ for all $n > 4$, we need only note that $n = 1$, $n = 2$, $n = 3$, and $n = 4$ do not satisfy $n^2 + n^3 = 100$.

29. Since $5^4 = 625$, for there to be positive integer solutions to this equation both x and y must be less than 5. This means that each of x^4 and y^4 is at most $4^4 = 256$, so their sum is at most 512 and cannot be 625.

31. We give a proof by contraposition. Assume that it is not the case that $a \le \sqrt[3]{n}$ or $b \le \sqrt[3]{n}$ or $c \le \sqrt[3]{n}$. Then it must be true that $a > \sqrt[3]{n}$ and $b > \sqrt[3]{n}$ and $c > \sqrt[3]{n}$. Multiplying these inequalities of positive numbers together we obtain $abc < (\sqrt[3]{n})^3 = n$, which implies the negation of our hypothesis that $n = abc$.

33. The idea is to find a small irrational number to add to the smaller of the two given rational numbers. Because we know that $\sqrt{2}$ is irrational, we can use a small multiple of $\sqrt{2}$. Here is our proof: By finding a common denominator, we can assume that the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \le c/b$. Thus $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, in violation of Example 10 in Section 1.6.

35. **a)** Without loss of generality, we may assume that the x sequence is already sorted into nondecreasing order, since we can relabel the indices. There are only a finite number of possible orderings for the y sequence, so if we can show that we can increase the sum (or at least keep it the same) whenever we find y_i and y_j that are out of order (i.e., $i < j$ but $y_i > y_j$) by switching them, then we will have shown that the sum is largest when the y sequence is in nondecreasing order. Indeed, if we perform the swap, then we have added $x_iy_j + x_jy_i$ to the sum and subtracted $x_iy_i + x_jy_j$. The net effect, then, is to have added $x_iy_j + x_jy_i - x_iy_i - x_jy_j = (x_j - x_i)(y_i - y_j)$, which is nonnegative by our ordering assumptions.

b) This is similar to part **(a)**. Again we assume that the x sequence is already sorted into nondecreasing order. If the y sequence is not in nonincreasing order, then $y_i < y_j$ for some $i < j$. By swapping y_i and y_j we increase the sum by $x_iy_j + x_jy_i - x_iy_i - x_jy_j = (x_j - x_i)(y_i - y_j)$, which is nonpositive by our ordering assumptions.

37. In each case we just have to keep applying the function f until we reach 1, where $f(x) = 3x + 1$ if x is odd and $f(x) = x/2$ if x is even.

a) $f(6) = 3$, $f(3) = 10$, $f(10) = 5$, $f(5) = 16$, $f(16) = 8$, $f(8) = 4$, $f(4) = 2$, $f(2) = 1$. We abbreviate this to $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

b) $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

c) $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

d) $21 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$

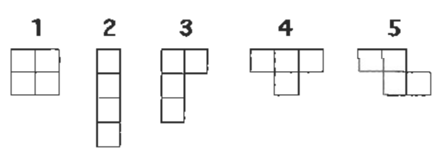
39. We give a constructive proof. Without loss of generality, we can assume that the upper left and upper right corners of the board are removed. We can place three dominoes horizontally to fill the remaining portion of the first row, and we can place four dominoes horizontally in each of the other seven rows to fill them.

41. The number of squares in a rectangular board is the product of the number of squares in each row and the number of squares in each column. We are given that this number is even, so there is either an even number of squares in each row or an even number of squares in each column. In the former case, we can tile the board in the obvious way by placing the dominoes horizontally, and in the latter case, we can tile the board in the obvious way by placing the dominoes vertically.

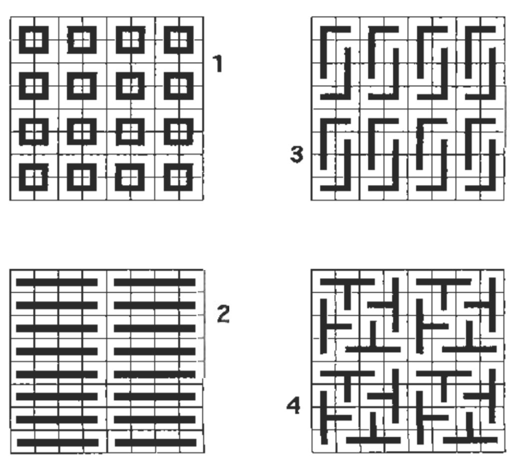
43. We follow the suggested labeling scheme. Clearly we can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15. Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15.

45. Remove the two black squares adjacent to one of the white corners, and remove two white squares other than that corner. Then no domino can cover that white corner, because neither of the squares adjacent to it remains.

47. a) It is not hard to find the five patterns:



b) It is clear that the pattern labeled 1 and the pattern labeled 2 will tile the checkerboard. It is harder to find the tiling for patterns 3 and 4, but a little experimentation shows that it is possible.



It remains to argue that pattern 5 cannot tile the checkerboard. Label the squares from 1 to 64, one row at a time from the top, from left to right in each row. Thus square 1 is the upper left corner, and square 64 is the lower right. Suppose we did have a tiling. By symmetry and without loss of generality, we may suppose that the tile is positioned in the upper left corner, covering squares 1, 2, 10, and 11. This forces a tile to be adjacent to it on the right, coveriug squares 3, 4, 12, and 13. Continue in this manner and we are forced to have a tile covering squares 6, 7, 15, and 16. This makes it impossible to cover square 8. Thus no tiling is possible.