SECTION 2.2 Set Operations

Most of the exercises involving operations on sets can be done fairly routinely by following the definitions. It is important to understand what it means for two sets to be equal and how to prove that two given sets are equal—using membership tables, using the definition to reduce the problem to logic, or showing that each is a subset of the other; see, for example, Exercises 5–24. It is often helpful when looking at operations on sets to draw the Venn diagram, even if you are not asked to do so. The **symmetric difference** *is a fairly important set operation not discussed in the section; it is developed in Exercises 32–43. Two other new concepts,* **multisets** *and* **fuzzy sets**, *are also introduced in this set of exercises.*

1. **a)** the set of students who live within one mile of school and walk to class (only students who do both of these things are in the intersection)
b) the set of students who either live within one mile of school or walk to class (or, it goes without saying, both)
c) the set of students who live within one mile of school but do not walk to class
d) the set of students who live more than a mile from school but nevertheless walk to class

3. **a)** We include all numbers that are in one or both of the sets, obtaining $\{0, 1, 2, 3, 4, 5, 6\}$.
b) There is only one number in both of these sets, so the answer is $\{3\}$.
c) The set of numbers in A but not in B is $\{1, 2, 4, 5\}$.
d) The set of numbers in B but not in A is $\{0, 6\}$.

5. By definition $\overline{\overline{A}}$ is the set of elements of the universal set that are not in $\overline{A}$. Not being in $\overline{A}$ means being in A. Thus $\overline{\overline{A}}$ is the same set as A. We can give this proof in symbols as follows:
$\overline{\overline{A}} = \{\, x \mid \neg x \in \overline{A} \,\} = \{\, x \mid \neg\neg x \in A \,\} = \{\, x \mid x \in A \,\} = A$.

7. These identities are straightforward applications of the definitions and are most easily stated using set-builder notation. Recall that $\mathbf{T}$ means the proposition that is always true, and $\mathbf{F}$ means the proposition that is always false.
a) $A \cup U = \{\, x \mid x \in A \lor x \in U \,\} = \{\, x \mid x \in A \lor \mathbf{T} \,\} = \{\, x \mid \mathbf{T} \,\} = U$
b) $A \cap \emptyset = \{\, x \mid x \in A \land x \in \emptyset \,\} = \{\, x \mid x \in A \land \mathbf{F} \,\} = \{\, x \mid \mathbf{F} \,\} = \emptyset$

9. **a)** We must show that every element (of the universal set) is in $A \cup \overline{A}$. This is clear, since every element is either in A (and hence in that union) or else not in A (and hence in that union).
b) We must show that no element is in $A \cap \overline{A}$. This is clear, since $A \cap \overline{A}$ consists of elements that are in A and not in A at the same time, obviously an impossibility.

11. These follow directly from the corresponding properties for the logical operations *or* and *and*.
a) $A \cup B = \{\, x \mid x \in A \lor x \in B \,\} = \{\, x \mid x \in B \lor x \in A \,\} = B \cup A$
b) $A \cap B = \{\, x \mid x \in A \land x \in B \,\} = \{\, x \mid x \in B \land x \in A \,\} = B \cap A$

13. We will show that these two sets are equal by showing that each is a subset of the other. Suppose $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$ by the definition of intersection. Since $x \in A$, we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup B$ as well. Since both of these are true, $x \in A \cap (A \cup B)$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

15. This exercise asks for a proof of one of De Morgan's laws for sets. The primary way to show that two sets are equal is to show that each is a subset of the other. In other words, to show that $X = Y$, we must show that whenever $x \in X$, it follows that $x \in Y$, and that whenever $x \in Y$, it follows that $x \in X$. Exercises 5–7 could also have been done this way, but it was easier in those cases to reduce the problems to the corresponding problems of logic. Here, too, we can reduce the problem to logic and invoke De Morgan's law for logic, but this problem requests specific proof techniques.

a) This proof is similar to the proof of the dual property, given in Example 10. Suppose $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, which means that x is in neither A nor B. In other words, $x \notin A$ and $x \notin B$. This is equivalent to saying that $x \in \overline{A}$ and $x \in \overline{B}$. Therefore $x \in \overline{A} \cap \overline{B}$, as desired. Conversely, if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$. This means $x \notin A$ and $x \notin B$, so x cannot be in the union of A and B. Since $x \notin A \cup B$, we conclude that $x \in \overline{A \cup B}$, as desired.

b) The following membership table gives the desired equality, since columns four and seven are identical.

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

17. This exercise asks for a proof of a generalization of one of De Morgan's laws for sets from two sets to three.

a) This proof is similar to the proof of the two-set property, given in Example 10. Suppose $x \in \overline{A \cap B \cap C}$. Then $x \notin A \cap B \cap C$, which means that x fails to be in at least one of these three sets. In other words, $x \notin A$ or $x \notin B$ or $x \notin C$. This is equivalent to saying that $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$. Therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, as desired. Conversely, if $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, then $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$. This means $x \notin A$ or $x \notin B$ or $x \notin C$, so x cannot be in the intersection of A, B and C. Since $x \notin A \cap B \cap C$, we conclude that $x \in \overline{A \cap B \cap C}$, as desired.

b) The following membership table gives the desired equality, since columns five and nine are identical.

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

19. This is clear, since both of these sets are precisely $\{x \mid x \in A \wedge x \notin B\}$.

21. There are many ways to prove identities such as the one given here. One way is to reduce them to logical identities (some of the associative and distributive laws for $\vee$ and $\wedge$). Alternately, we could argue in each case that the left-hand side is a subset of the right-hand side and vice versa. Another method would be to construct membership tables (they will have eight rows in order to cover all the possibilities). Here we choose the second method. First we show that every element of the left-hand side must be in the right-hand side as well. If $x \in A \cup (B \cup C)$, then x must be either in A or in $B \cup C$ (or both). In the former case, we can conclude that $x \in A \cup B$ and thus $x \in (A \cup B) \cup C$, by the definition of union. In the latter case, x must be either in B or in C (or both). In the former subcase, we can conclude that $x \in A \cup B$ and thus $x \in (A \cup B) \cup C$, by the definition of union; in the latter subcase, we can conclude that $x \in (A \cup B) \cup C$, again using the definition of union.

23. There are many ways to prove identities such as the one given here. One way is to reduce them to logical identities (some of the associative and distributive laws for $\vee$ and $\wedge$). Alternately, we could argue in each case that the left-hand side is a subset of the right-hand side and vice versa. Another method would be to construct membership tables (they will have eight rows in order to cover all the possibilities). Here we choose the third method. We construct the following membership table and note that the fifth and eighth columns are identical.

A	B	C	$B\cap C$	$A\cup(B\cap C)$	$A\cup B$	$A\cup C$	$(A\cup B)\cap(A\cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

25. These are straightforward applications of the definitions.

a) The set of elements common to all three sets is $\{4,6\}$.

b) The set of elements in at least one of the three sets is $\{0,1,2,3,4,5,6,7,8,9,10\}$.

c) The set of elements in C and at the same time in at least one of A and B is $\{4,5,6,8,10\}$.

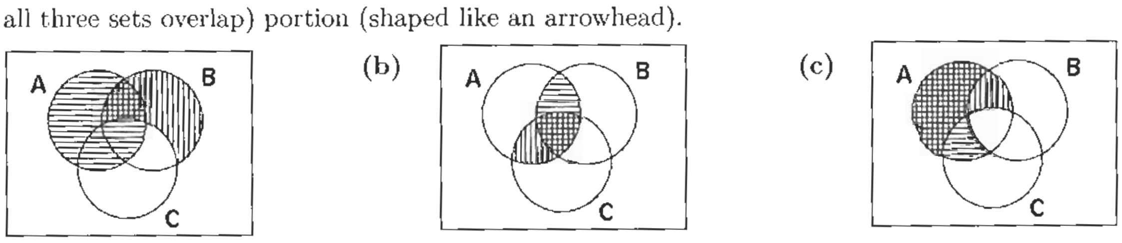
d) The set of elements either in C or in both A and B (or in both of these) is $\{0,2,4,5,6,7,8,9,10\}$.

27. **a)** In the figure we have shaded the A set with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $B-C$ with vertical bars (that portion inside B but outside C. The intersection is where these overlap—the double-shaded portion (shaped like an arrowhead).

b) In the figure we have shaded the set $A\cap B$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A\cap C$ with vertical bars. The union is the entire region that has any shading at all (shaped like a tilted mustache).

c) In the figure we have shaded the set $A\cap\overline{B}$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A\cap\overline{C}$ with vertical bars. The union is the entire region that has any shading at all (everything inside A except the triangular middle portion where all three sets overlap) portion (shaped like an arrowhead).

(a) (b) (c)



29. **a)** If B adds nothing new to A, then we can conclude that all the elements of B were already in A. In other words, $B\subseteq A$.

b) In this case, all the elements of A are forced to be in B as well, so we conclude that $A\subseteq B$.

c) This equality holds precisely when none of the elements of A are in B (if there were any such elements, then $A-B$ would not contain all the elements of A). Thus we conclude that A and B are disjoint ($A\cap B=\emptyset$).

d) We can conclude nothing about A and B in this case, since this equality always holds.

e) Every element in $A-B$ must be in A, and every element in $B-A$ must not be in A. Since no item can be in A and not be in A at the same time, there are no elements in both $A-B$ and $B-A$. Thus the only way for these two sets to be equal is if both of them are the empty set. This means that every element of A must be in B, and every element of B must be in A. Thus we conclude that $A=B$.

31. This is the set-theoretic version of the contrapositive law for logic, which says that $p \to q$ is logically equivalent to $\neg q \to \neg p$. We argue as follows.

$A \subseteq B \equiv \forall x(x \in A \to x \in B) \equiv \forall x(x \notin B \to x \notin A) \equiv \forall x(x \in \overline{B} \to x \in \overline{A}) \equiv \overline{B} \subseteq \overline{A}$

33. Clearly this will be the set of students majoring in computer science or mathematics but not both.

35. This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it is in the union (i.e., in either A or B), but not in the intersection (i.e., not in both A and B).

37. We will use the result of Exercise 36 as well as some obvious identities (some of which are in Exercises 6–10).

a) $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$ **b)** $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$

c) $A \oplus U = (A - U) \cup (U - A) = \emptyset \cup \overline{A} = \overline{A}$ **d)** $A \oplus \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = U$

39. We can conclude that $B = \emptyset$. To see this, suppose that B contains some element b. If $b \in A$, then b is excluded from $A \oplus B$, so $A \oplus B$ cannot equal A. On the other hand, if $b \notin A$, then b must be in $A \oplus B$, so again $A \oplus B$ cannot equal A. Thus in either case, $A \oplus B \neq A$. We conclude that B cannot have any elements.

41. Yes. To show that $A = B$, we need to show that $x \in A$ implies $x \in B$ and conversely. By symmetry, it will be enough to show one direction of this. So assume that $A \oplus C = B \oplus C$, and let $x \in A$ be given. There are two cases to consider, depending on whether $x \in C$. If $x \in C$, then by definition we can conclude that $x \notin A \oplus C$. Therefore $x \notin B \oplus C$. Now if x were *not* in B, then x *would* be in $B \oplus C$ (since $x \in C$ by assumption). Since this is not true, we conclude that $x \in B$, as desired. For the other case, assume that $x \notin C$. Then $x \in A \oplus C$. Therefore $x \in B \oplus C$ as well. Again, if x were *not* in B, then it could not be in $B \oplus C$ (since $x \notin C$ by assumption). Once again we conclude that $x \in B$, and the proof is complete.

43. Yes. Both sides equal the set of elements that occur in an odd number of the sets A, B, C, and D.

45. **a)** The union of these sets is the set of elements that appear in at least one of them. In this case the sets are "increasing": $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. Therefore every element in any of the sets is in A_n, so the union is $A_n = \{1, 2, \ldots, n\}$.

b) The intersection of these sets is the set of elements that appear in all of them. Since $A_1 = \{1\}$, only the number 1 has a chance to be in the intersection. In fact 1 is in the intersection, since it is in all of the sets. Therefore the intersection is $A_1 = \{1\}$.

47. **a)** Here the sets are increasing. A bit string of length not exceeding 1 is also a bit string of length not exceeding 2, so $A_1 \subseteq A_2$. Similarly, $A_2 \subseteq A_3 \subseteq A_4 \subseteq \cdots \subseteq A_n$. Therefore the union of the sets A_1, A_2, ..., A_n is just A_n itself.

b) Since A_1 is a subset of all the A_i's, the intersection is A_1, the set of all nonempty bit strings of length not exceeding 1, namely $\{0, 1\}$.

49. **a)** As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of integers, and every integer is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1\}$.

b) All the sets are subsets of the set of integers, and every nonzero integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z} - \{0\}$. Each pair of these sets are disjoint, so no element is common to all of the sets. Therefore $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

c) This is similar to part **(a)**, the only difference being that here we are working with real numbers. Therefore $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}$ (the set of all real numbers), and $\bigcap_{i=1}^{\infty} A_i = A_1 = [-1, 1]$ (the interval of all real numbers between -1 and 1, inclusive).

d) This time the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_i = A_1 = [1, \infty)$ (all real numbers greater than or equal to 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

51. The i^{th} digit in the string indicates whether the i^{th} number in the universal set (in this case the number i) is in the set in question.
a) $\{1, 2, 3, 4, 7, 8, 9, 10\}$ **b)** $\{2, 4, 5, 6, 7\}$ **c)** $\{1, 10\}$

53. We are given two bit strings, representing two sets. We want to represent the set of elements that are in the first set but not the second. Thus the bit in the i^{th} positiou of the bit string for the difference is 1 if the i^{th} bit of the first string is 1 and the i^{th} bit of the second string is 0, and is 0 otherwise.

55. We represent the sets by bit strings of length 26, using alphabetical order. Thus
A is represented by 11 1110 0000 0000 0000 0000 0000,
B is represented by 01 1100 1000 0000 0100 0101 0000,
C is represented by 00 1010 0010 0000 1000 0010 0111, and
D is represented by 00 0110 0110 0001 1000 0110 0110.
To find the desired sets, we apply the indicated bitwise operations to these strings.
a) 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 =
11 1110 1000 0000 0100 0101 0000, which represents the set $\{a, b, c, d, e, g, p, t, v\}$
b) 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 =
01 1100 0000 0000 0000 0000 0000, which represents the set $\{b, c, d\}$
c) (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$
(01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) =
11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 =
01 1110 0010 0000 1000 0110 0110, which represents the set $\{b, c, d, e, i, o, t, u, x, y\}$
d) 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$
00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 =
11 1110 1110 0001 1100 0111 0111, which represents the set
$\{a, b, c, d, e, g, h, i, n, o, p, t, u, v, x, y, z\}$

57. We simply adjoin the set itself to the list of its elements.
a) $\{1, 2, 3, \{1, 2, 3\}\}$ **b)** $\{\emptyset\}$ **c)** $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

59. a) The multiplicity of a in the union is the maximum of 3 and 2, the multiplicities of a in A and B. Since the maximum is 3, we find that a occurs with multiplicity 3 in the union. Working similarly with b, c (which appears with multiplicity 0 in B), and d (which appears with multiplicity 0 in A), we find that $A \cup B = \{3 \cdot a, 3 \cdot b, 1 \cdot c, 4 \cdot d\}$.

b) This is similar to part **(a)**, with "maximum" replaced by "minimum." Thus $A \cap B = \{2 \cdot a, 2 \cdot b\}$. (In particular, c and d appear with multiplicity 0—i.e., do not appear—in the intersection.)

c) In this case we subtract multiplicities, but never go below 0. Thus the answer is $\{1 \cdot a, 1 \cdot c\}$.

d) Similar to part **(c)** (subtraction in the opposite order); the answer is $\{1 \cdot b, 4 \cdot d\}$.

e) We add multiplicities here, to get $\{5 \cdot a, 5 \cdot b, 1 \cdot c, 4 \cdot d\}$.

61. Assume that the universal set contains just Alice, Brian, Fred, Oscar, and Rita. We subtract the degrees of membership from 1 to obtain the complement. Thus $\overline{F}$ is $\{0.4 \text{ Alice}, 0.1 \text{ Brian}, 0.6 \text{ Fred}, 0.9 \text{ Oscar}, 0.5 \text{ Rita}\}$, and $\overline{R}$ is $\{0.6 \text{ Alice}, 0.2 \text{ Brian}, 0.8 \text{ Fred}, 0.1 \text{ Oscar}, 0.3 \text{ Rita}\}$.

63. Taking the minimums, we obtain $\{0.4 \text{ Alice}, 0.8 \text{ Brian}, 0.2 \text{ Fred}, 0.1 \text{ Oscar}, 0.5 \text{ Rita}\}$ for $F \cap R$.

b) We are told that the domain is $\mathbf{Z}^+$. Since the decimal representation of an integer has to have at least one digit, at most nine digits do not appear, and of course the number of missing digits could be any number less than 9. Thus the range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c) We are told that the domain is the set of bit strings. The block 11 could appear no times, or it could appear any positive number of times, so the range is $\mathbf{N}$.

d) We are told that the domain is the set of bit strings. Since the first 1 can be anywhere in the string, its position can be $1, 2, 3, \ldots$. If the bit string contains no 1's, the value is 0 by definition. Therefore the range is $\mathbf{N}$.

9. The floor function rounds down and the ceiling function rounds up.
a) 1 **b)** 0 **c)** 0 **d)** -1 **e)** 3 **f)** -1 **g)** $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rfloor = \lfloor 2\frac{1}{2} \rfloor = 2$
h) $\lfloor \frac{1}{2} \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot 2 \rfloor = \lfloor 1 \rfloor = 1$

11. We need to determine whether the range is all of $\{a, b, c, d\}$. It is for the function in part **(a)**, but not for the other two functions.

13. a) This function is onto, since every integer is 1 less than some integer. In particular, $f(x + 1) = x$.

b) This function is not onto. Since $n^2 + 1$ is always positive, the range cannot include any negative integers.

c) This function is not onto, since the integer 2, for example, is not in the range. In other words, 2 is not the cube of any integer.

d) This function is onto. If we want to obtain the value x, then we simply need to start with $2x$, since $f(2x) = \lceil 2x/2 \rceil = \lceil x \rceil = x$ for all $x \in \mathbf{Z}$.

15. An onto function is one whose range is the entire codomain. Thus we must determine whether we can write every integer in the form given by the rule for f in each case.

a) Given any integer n, we have $f(0, n) = n$, so the function is onto.

b) Clearly the range contains no negative integers, so the function is not onto.

c) Given any integer m, we have $f(m, 25) = m$, so the function is onto. (We could have used any constant in place of 25 in this argument.)

d) Clearly the range contains no negative integers, so the function is not onto.

e) Given any integer m, we have $f(m, 0) = m$, so the function is onto.

17. Obviously there are an infinite number of correct answers to each part. The problem asked for a "formula." Parts **(a)** and **(c)** seem harder here, since we somehow have to fold the negative integers into the positive ones without overlap. Therefore we probably want to treat the negative integers differently from the positive integers. One way to do this with a formula is to make it a two-part formula. If one objects that this is not "a formula," we can counter as follows. Consider the function $g(x) = \lfloor 2^x \rfloor / 2^x$. Clearly if $x \geq 0$, then 2^x is a positive integer, so $g(x) = 2^x/2^x = 1$. If $x < 0$, then 2^x is a number between 0 and 1, so $g(x) = 0/2^x = 0$. If we want to define a function that has the value $f_1(x)$ when $x \geq 0$ and $f_2(x)$ when $x < 0$, then we can use the formula $g(x) \cdot f_1(x) + (1 - g(x)) \cdot f_2(x)$.

a) We could map the positive integers (and 0) into the positive multiples of 3, say, and the negative integers into numbers that are 1 greater than a multiple of 3, in a one-to-one manner. This will give us a function that leaves some elements out of the range. So let us define our function as follows:

$$f(x) = \begin{cases} 3x + 3 & \text{if } x \geq 0 \\ 3|x| + 1 & \text{if } x < 0 \end{cases}$$

The values of f on the inputs 0 through 4 are then $3, 6, 9, 12, 15$; and the values on the inputs -1 to -4 are $4, 7, 10, 13$. Clearly this function is one-to-one, but it is not onto since, for example, 2 is not in the range.