SECTION 3.8 Matrices

2. We just add entry by entry.

a) $\begin{bmatrix} 0 & 3 & 9 \\ 1 & 4 & -1 \\ 2 & -5 & -3 \end{bmatrix}$ b) $\begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}$

4. To multiply matrices $\mathbf{A}$ and $\mathbf{B}$, we compute the $(i,j)^{\text{th}}$ entry of the product $\mathbf{AB}$ by adding all the products of elements from the i^{th} row of $\mathbf{A}$ with the corresponding element in the j^{th} column of $\mathbf{B}$, that is $\sum_{k=1}^{n} a_{ik}b_{kj}$. This can only be done, of course, when the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$ (called n in the formula shown here).

a) $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 4 & -1 & -7 & 6 \\ -7 & -5 & 8 & 5 \\ 4 & 0 & 7 & 3 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix}$

6. First note that $\mathbf{A}$ must be a 3×3 matrix in order for the sizes to work out as shown. If we name the elements of $\mathbf{A}$ in the usual way as $[a_{ij}]$, then the given equation is really nine equations in the nine unknowns a_{ij}, obtained simply by writing down what the matrix multiplication on the left means:

$$\begin{aligned}
1\cdot a_{11}+3\cdot a_{21}+2\cdot a_{31}&=7\\
1\cdot a_{12}+3\cdot a_{22}+2\cdot a_{32}&=1\\
1\cdot a_{13}+3\cdot a_{23}+2\cdot a_{33}&=3\\
2\cdot a_{11}+1\cdot a_{21}+1\cdot a_{31}&=1\\
2\cdot a_{12}+1\cdot a_{22}+1\cdot a_{32}&=0\\
2\cdot a_{13}+1\cdot a_{23}+1\cdot a_{33}&=3\\
4\cdot a_{11}+0\cdot a_{21}+3\cdot a_{31}&=-1\\
4\cdot a_{12}+0\cdot a_{22}+3\cdot a_{32}&=-3\\
4\cdot a_{13}+0\cdot a_{23}+3\cdot a_{33}&=7
\end{aligned}$$

This is really not as bad as it looks, since each variable only appears in three equations. For example, the first, fourth, and seventh equations are a system of three equations in the three variables a_{11}, a_{21}, and a_{31}. We can solve them using standard algebraic techniques to obtain $a_{11}=-1$, $a_{21}=2$ and $a_{31}=1$. By similar reasoning we also obtain $a_{12}=0$, $a_{22}=1$ and $a_{32}=-1$; and $a_{13}=1$, $a_{23}=0$ and $a_{33}=1$. Thus our answer is

$$\mathbf{A}=\begin{bmatrix}-1&0&1\\2&1&0\\1&-1&1\end{bmatrix}.$$

As a check we can carry out the matrix multiplication and verify that we obtain the given right-hand side.

8. Since the entries of $\mathbf{A}+\mathbf{B}$ are $a_{ij}+b_{ij}$ and the entries of $\mathbf{B}+\mathbf{A}$ are $b_{ij}+a_{ij}$, that $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ follows from the commutativity of addition of real numbers.

10. a) This product is a 3×5 matrix.
b) This is not defined since the number of columns of $\mathbf{B}$ does not equal the number of rows of $\mathbf{A}$.
c) This product is a 3×4 matrix.
d) This is not defined since the number of columns of $\mathbf{C}$ does not equal the number of rows of $\mathbf{A}$.
e) This is not defined since the number of columns of $\mathbf{B}$ does not equal the number of rows of $\mathbf{C}$.
f) This product is a 4×5 matrix.

12. We use the definition of matrix addition and multiplication. All summations here are from 1 to k.
a) $(\mathbf{A}+\mathbf{B})\mathbf{C}=[\sum(a_{iq}+b_{iq})c_{qj}]=[\sum a_{iq}c_{qj}+\sum b_{iq}c_{qj}]=\mathbf{AC}+\mathbf{BC}$
b) $\mathbf{C}(\mathbf{A}+\mathbf{B})=[\sum c_{iq}(a_{qj}+b_{qj})]=[\sum c_{iq}a_{qj}+\sum c_{iq}b_{qj}]=\mathbf{CA}+\mathbf{CB}$

14. Let $\mathbf{A}$ and $\mathbf{B}$ be two diagonal $n\times n$ matrices. Let $\mathbf{C}=[c_{ij}]$ be the product $\mathbf{AB}$. From the definition of matrix multiplication, $c_{ij}=\sum a_{iq}b_{qj}$. Now all the terms a_{iq} in this expression are 0 except for $q=i$, so $c_{ij}=a_{ii}b_{ij}$. But $b_{ij}=0$ unless $i=j$, so the only nonzero entries of $\mathbf{C}$ are the diagonal entries $c_{ii}=a_{ii}b_{ii}$.

16. The $(i,j)^{\text{th}}$ entry of $(\mathbf{A}^t)^t$ is the $(j,i)^{\text{th}}$ entry of $\mathbf{A}^t$, which is the $(i,j)^{\text{th}}$ entry of $\mathbf{A}$.

18. We need to multiply these two matrices together in both directions and check that both products are $\mathbf{I}_3$. Indeed, they are.

20. **a)** Using Exercise 19, noting that $ad - bc = -5$, we write down the inverse immediately:

$$\begin{bmatrix} -3/5 & 2/5 \\ 1/5 & 1/5 \end{bmatrix}.$$

b) We multiply to obtain $\mathbf{A}^2 = \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix}$ and then $\mathbf{A}^3 = \begin{bmatrix} 1 & 18 \\ 9 & 37 \end{bmatrix}$.

c) We multiply to obtain $(\mathbf{A}^{-1})^2 = \begin{bmatrix} 11/25 & -4/25 \\ -2/25 & 3/25 \end{bmatrix}$ and then $(\mathbf{A}^{-1})^3 = \begin{bmatrix} -37/125 & 18/125 \\ 9/125 & -1/125 \end{bmatrix}$.

d) Applying the method of Exercise 19 for obtaining inverses to the answer in part (b), we obtain the answer in part (c). Therefore $(\mathbf{A}^3)^{-1} = (\mathbf{A}^{-1})^3$.

22. A matrix is symmetric if and only if it equals its transpose. So let us compute the transpose of $\mathbf{A}\mathbf{A}^t$ and see if we get this matrix back. Using Exercise 17b and then Exercise 16, we have $(\mathbf{A}\mathbf{A}^t)^t = ((\mathbf{A}^t)^t)\mathbf{A}^t = \mathbf{A}\mathbf{A}^t$, as desired.

24. **a)** If we compute the product as $\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3)$, then by the result of Exercise 23 it will take $50 \cdot 10 \cdot 40$ multiplications for the first product and then $20 \cdot 50 \cdot 40$ for the second. This is a total of 60,000 multiplications. If we compute the product as $(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3$, then it will take $20 \cdot 50 \cdot 10$ multiplications for the first product and then $20 \cdot 10 \cdot 40$ for the second. This is a total of 18,000 multiplications. Therefore the second method is more efficient.

b) If we compute the product as $\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3)$, then by the result of Exercise 23 it will take $5 \cdot 50 \cdot 1$ multiplications for the first product and then $10 \cdot 5 \cdot 1$ for the second. This is a total of 300 multiplications. If we compute the product as $(\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3$, then it will take $10 \cdot 5 \cdot 50$ multiplications for the first product and then $10 \cdot 50 \cdot 1$ for the second. This is a total of 1000 multiplications. Therefore the first method is more efficient.

26. **a)** We simply note that under the given definitions of $\mathbf{A}$, $\mathbf{X}$, and $\mathbf{B}$, the definition of matrix multiplication is exactly the system of equations shown.

b) The given system is the matrix equation $\mathbf{AX} = \mathbf{B}$. If $\mathbf{A}$ is invertible with inverse $\mathbf{A}^{-1}$, then we can multiply both sides of this equation by $\mathbf{A}^{-1}$ to obtain $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$. The left-hand side simplifies to $\mathbf{IX}$, however, by the definition of inverse, and this is simply $\mathbf{X}$. Thus the given system is equivalent to the system $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, which obviously tells us exactly what $\mathbf{X}$ is (and therefore what all the values x_i are).

28. We follow the definitions.

a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

30. We follow the definition and obtain $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

32. **a)** $\mathbf{A} \vee \mathbf{A} = [a_{ij} \vee a_{ij}] = [a_{ij}] = \mathbf{A}$ **b)** $\mathbf{A} \wedge \mathbf{A} = [a_{ij} \wedge a_{ij}] = [a_{ij}] = \mathbf{A}$

34. **a)** $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C} = [(a_{ij} \vee b_{ij}) \vee c_{ij}] = [a_{ij} \vee (b_{ij} \vee c_{ij})] = \mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$

b) This is identical to part (a), with $\wedge$ replacing $\vee$.

36. Since the i^{th} row of $\mathbf{I}$ consists of all 0's except for a 1 in the $(i,i)^{\text{th}}$ position, we have $\mathbf{I} \odot \mathbf{A} = [(0 \wedge a_{1j}) \vee \cdots \vee (1 \wedge a_{ij}) \vee \cdots \vee (0 \wedge a_{nj})] = [a_{ij}] = \mathbf{A}$. Similarly, since the j^{th} column of $\mathbf{I}$ consists of all 0's except for a 1 in the $(j,j)^{\text{th}}$ position, we have $\mathbf{A} \odot \mathbf{I} = [(a_{i1} \wedge 0) \vee \cdots \vee (a_{ij} \wedge 1) \vee \cdots \vee (a_{in} \wedge 0)] = [a_{ij}] = \mathbf{A}$.