SECTION 8.6 Partial Orderings

Partial orderings (or "partial orders"—the two phrases are used interchangeably) rival equivalence relations in importance in mathematics and computer science. Again, try to concentrate on the visual image—in this case the Hasse diagram. Play around with different posets to become familiar with the different possibilities; not all posets have to look like the less than or equal relation on the integers. Exercises 32 and 33 are important, and they are not difficult if you pay careful attention to the definitions.

1. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) Clearly this relation is reflexive because each of 0, 1, 2, and 3 is related to itself. The relation is also antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c$ so a is related to c (because the relation is reflexive). This is just the equality relation on $\{0, 1, 2, 3\}$; more generally, the equality relation on any set satisfies all three conditions and is therefore a partial ordering. (It is the smallest partial ordering; reflexivity insures that every partial ordering contains at least all the pairs (a, a).)

b) This is not a partial ordering, because although the relation is reflexive, it is not antisymmetric (we have $2R3$ and $3R2$, but $2 \neq 3$), and not transitive ($3R2$ and $2R0$, but 3 is not related to 0).

c) This is a partial ordering, because it is clearly reflexive; is antisymmetric (we just need to note that $(1, 2)$ is the only pair in the relation with unequal components); and is transitive (for the same reason).

d) This is a partial ordering because it is the "less than or equal to" relation on $\{1, 2, 3\}$ together with the isolated point 0.

e) This is not a partial ordering. The relation is clearly reflexive, but it is not antisymmetric ($0R1$ and $1R0$, but $0 \neq 1$) and not transitive ($2R0$ and $0R1$, but 2 is not related to 1).

3. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since nobody is taller than himself, this relation is not reflexive so (S, R) cannot be a poset.

b) To be not taller means to be exactly the same height or shorter. Two different people x and y could have the same height, in which case $x R y$ and $y R x$ but $x \neq y$, so R is not antisymmetric and this is not a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is an ancestor of b, then b cannot be an ancestor of a (for one thing, an ancestor needs to be born before any descendant), so the relation is vacuously antisymmetric. If a is an ancestor of b, and b is an ancestor of c, then by the way "ancestor" is defined, we know that a is an ancestor of b; thus R is transitive.

d) This relation is not antisymmetric. Let a and b be any two distinct friends of yours. Then $a R b$ and $b R a$, but $a \neq b$.

5. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial partial ordering. (It is the smallest partial partial ordering; reflexivity insures that every partial order contains at least all the pairs (a, a).)

b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

c) This is a poset, as explained in Example 1.

d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that $2 \nmid 2$. (It also is not antisymmetric and not transitive.)

7. a) This relation is $\{(1,1), (1,2), (1,3), (2,1), (2,2), (3,3)\}$. It is not antisymmetric because $(1, 2)$ and $(2, 1)$ are both in the relation, but $1 \neq 2$. We can see this visually by the pair of 1's symmetrically placed around the main diagonal at positions $(1, 2)$ and $(2, 1)$. Therefore this matrix does not represent a partial order.

b) This matrix represents a partial order. Reflexivity is clear. The only other pairs in the relation are $(1, 2)$ and $(1, 3)$, and clearly neither can be part of a counterexample to antisymmetry or transitivity.

c) A little trial and error shows that this relation is not transitive ($(4, 1)$ and $(1, 3)$ are present, but not $(4, 3)$) and therefore not a partial order.

9. This relation is not transitive (there are arrows from a to b and from b to d, but there is no arrow from a to d), so it is not a partial order.

11. This relation is a partial order, since it has all three properties—it is reflexive (there is an arrow at each point), antisymmetric (there are no pairs of arrows going in opposite directions between two different points), and transitive (there is no missing arrow from some x to some z when there were arrows from x to y and y to z).

13. The dual of a poset is the poset with the same underlying set and with the relation defined by declaring a related to b if and only if $b \preceq a$ in the given poset.

a) The dual relation to $\leq$ is $\geq$, so the dual poset is $(\{0,1,2\},\geq)$. Explicitly it is the set $\{(0,0),(1,0),(1,1),(2,0),(2,1),(2,2)\}$.

b) The dual relation to $\geq$ is $\leq$, so the dual poset is $(\mathbf{Z},\leq)$.

c) The dual relation to $\supseteq$ is $\subseteq$, so the dual poset is $(P(\mathbf{Z}),\subseteq)$.

d) There is no symbol generally used for the "is a multiple of" relation, which is the dual to the "divides" relation in this part of the exercise. If we let R be the relation such that aRb if and only if $b \mid a$, then the answer can be written $(\mathbf{Z}^+, R)$.

15. We need to find elements such that the relation holds in neither direction between them. The answers we give are not the only ones possible.

a) One such pair is $\{1\}$ and $\{2\}$. These are both subsets of $\{0,1,2\}$, so they are in the poset, but neither is a subset of the other.

b) Neither 6 nor 8 divides the other, so they are incomparable.

17. We find the first coordinate (from left to right) at which the tuples differ and place first the tuple with the smaller value in that coordinate.

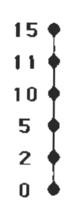
a) Since $1=1$ in the first coordinate, but $1<2$ in the second coordinate, $(1,1,2)<(1,2,1)$.

b) The first two coordinates agree, but $2<3$ in the third, so $(0,1,2,3)<(0,1,3,2)$.

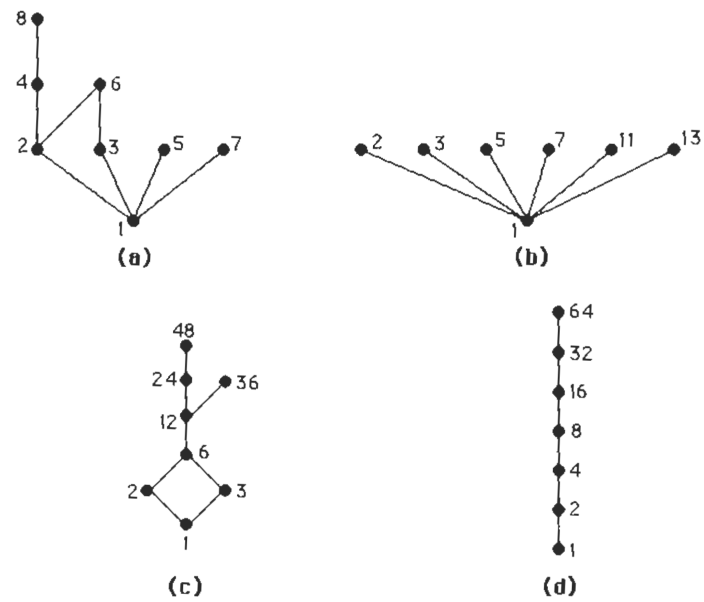
c) Since $0<1$ in the first coordinate, $(0,1,1,1,0)<(1,0,1,0,1)$.

19. All the strings that begin with 0 precede all those that begin with 1. The 0 comes first. Next comes 0001, which begins with three 0's, then 001, which begins with two 0's. Among the strings that begin 01, the order is $01 < 010 < 0101 < 011$. Putting this all together, we have $0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$.

21. This is a totally ordered set, so the Hasse diagram is linear.



23. We put x above y if y divides x. We draw a line between x and y, where y divides x, if there is no number z in our set, other than x or y, such that $y \mid z \wedge z \mid x$. Note that in part (b) the numbers other than 1 are all (relatively) prime, so the Hasse diagram is short and wide, whereas in part (d) the numbers all divide one another, so the Hasse diagram is tall and narrow.



25. We need to include every pair (x, y) for which we can find a path going upward in the diagram from x to y. We also need to include all the reflexive pairs (x, x). Therefore the relation is the following set of pairs: $\{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,d)\}$.

27. The procedure is the same as in Exercise 25: $\{(a,a),(a,d),(a,e),(a,f),(a,g),(b,b),(b,d),(b,e),(b,f),(b,g),(c,c),(c,d),(c,e),(c,f),(c,g),(d,d),(e,e),(f,f),(g,d),(g,e),(g,f),(g,g)\}$.

29. In this problem $X \preceq Y$ when $X \subseteq Y$. For (X, Y) to be in the covering relation, we need X to be a proper subset of Y but we also must have no subset strictly between X and Y. For example, $(\{a\},\{a,b,c\})$ is not in the covering relation, since $\{a\} \subset \{a,b\}$ and $\{a,b\} \subset \{a,b,c\}$. With this understanding it is easy to list the pairs in the covering relation: $(\emptyset,\{a\})$, $(\emptyset,\{b\})$, $(\emptyset,\{c\})$, $(\{a\},\{a,b\})$, $(\{a\},\{a,c\})$, $(\{b\},\{a,b\})$, $(\{b\},\{b,c\})$, $(\{c\},\{a,c\})$, $(\{c\},\{b,c\})$, $(\{a,b\},\{a,b,c\})$, $(\{a,c\},\{a,b,c\})$, and $(\{b,c\},\{a,b,c\})$.

31. Let $(S, \preceq)$ be a finite poset. We claim that this poset is just the reflexive transitive closure of its covering relation. Suppose that (a, b) is in the reflexive transitive closure of the covering relation. Then either $a = b$ or $a \prec b$ (in which cases certainly $a \preceq b$) or else there is a sequence $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$, in which case again $a \preceq b$, by the transitivity of $\preceq$. Conversely, suppose that $a \preceq b$. If $a = b$, then (a, b) is certainly in the reflexive transitive closure of the covering relation. If $a \prec b$ and there is no z such that $a \prec z \prec b$, then $(a, b,)$ is in the covering relation and again therefore in its reflexive transitive closure. Otherwise, let $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$ be a longest possible sequence of this form; since the poset is finite, there must be such a longest sequence. Then no intermediate elements can be inserted into this sequence (to do so would lengthen it), so each pair (a, a_1), (a_1, a_2), $\ldots$, (a_n, b) is in the covering relation, so again (a, b) is in its reflexive transitive closure. This completes the proof. Note how the finiteness of the poset was crucial here. If we let S be the set of all subsets of $\mathbf{N}$ (the set of natural numbers) under the subset relation, then we cannot recover S from its covering relation, since nothing in the covering relation allows us to relate a finite set to an infinite one; thus for example we could not recover the relationship $\{1,2\} \subset \mathbf{N}$.

33. It is helpful in this exercise to draw the Hasse diagram.

a) Maximal elements are those that do not divide any other elements of the set. In this case 24 and 45 are the only numbers that meet that requirement.

b) Minimal elements are those that are not divisible by any other elements of the set. In this case 3 and 5 are the only numbers that meet that requirement.

c) A greatest element would be one that all the other elements divide. The only two candidates (maximal elements) are 24 and 45, and since neither divides the other, we conclude that there is no greatest element.

d) A least element would be one that divides all the other elements. The only two candidates (minimal elements) are 3 and 5, and since neither divides the other, we conclude that there is no least element.

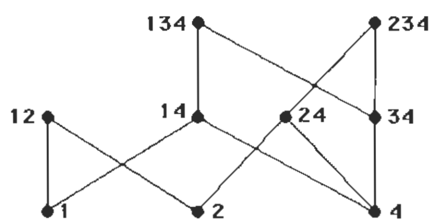
e) We want to find all elements that both 3 and 5 divide. Clearly only 15 and 45 meet this requirement.

f) The least upper bound is 15 since it divides 45 (see part **(e)**).

g) We want to find all elements that divide both 15 and 45. Clearly only 3, 5, and 15 meet this requirement.

h) The number 15 is the greatest lower bound, since both 3 and 5 divide it (see part **(g)**).

35. To help us answer the questions, we will draw the Hasse diagram, with the commas and braces eliminated in the labels, for readability.



a) The maximal elements are the ones without any elements lying above them in the Hasse diagram, namely $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

b) The minimal elements are the ones without any elements lying below them in the Hasse diagram, namely $\{1\}$, $\{2\}$, and $\{4\}$.

c) There is no greatest element, since there is more than one maximal element, none of which is greater than the others.

d) There is no least element, since there is more than one minimal element, none of which is less than the others.

e) The upper bounds are the sets containing both $\{2\}$ and $\{4\}$ as subsets, i.e., the sets containing both 2 and 4 as elements. Pictorially, these are the elements lying above both $\{2\}$ and $\{4\}$ (in the sense of there being a path in the diagram), namely $\{2,4\}$ and $\{2,3,4\}$.

f) The least upper bound is an upper bound that is less than every other upper bound. We found the upper bounds in part **(e)**, and since $\{2,4\}$ is less than (i.e., a subset of) $\{2,3,4\}$, we conclude that $\{2,4\}$ is the least upper bound.

g) To be a lower bound of both $\{1,3,4\}$ and $\{2,3,4\}$, a set must be a subset of each, and so must be a subset of their intersection, $\{3,4\}$. There are only two such subsets in our poset, namely $\{3,4\}$ and $\{4\}$. In the diagram, these are the points which lie below (in the path sense) both $\{1,3,4\}$ and $\{2,3,4\}$.

h) The greatest lower bound is a lower bound that is greater than every other lower bound. We found the lower bounds in part **(g)**, and since $\{3,4\}$ is greater than (i.e., a superset of) $\{4\}$, we conclude that $\{3,4\}$ is the greatest lower bound.

37. First we need to show that lexicographic order is reflexive, i.e., that $(a,b) \preceq (a,b)$; this is true by fiat, since we defined $\preceq$ by adding equality to $\prec$. Next we need to show antisymmetry: if $(a,b) \preceq (c,d)$ and $(a,b) \neq (c,d)$, then $(c,d) \not\preceq (a,b)$. By definition $(a,b) \prec (c,d)$ if and only if either $a \prec c$, or $a = c$ and $b \prec d$. In the first case, by the antisymmetry of the underlying relation, we know that $c \not\prec a$, and similarly in the second case we know that $d \not\prec d$. Thus there is no way that we could have $(c,d) \prec (a,b)$. Finally, for transitivity, let $(a,b) \preceq (c,d) \preceq (e,f)$. We want to show that $(a,b) \preceq (e,f)$. If one of the given inequalities is an equality, then there is nothing to prove, so we may assume that $(a,b) \prec (c,d) \prec (e,f)$. If $a \prec c$, then by the transitivity of the underlying relation, we know that $a \prec e$ and so $(a,b) \prec (e,f)$. Similarly, if $c \prec e$, then again $a \prec e$

and so $(a,b) \prec (e,f)$. The only other way for the given inequalities to hold is if $a = c = e$ and $b \prec d \prec f$. In this case the latter string of inequalities implies that $b \prec f$ and so again by definition $(a,b) \prec (e,f)$.

39. First we must show that $\preceq$ is reflexive. Since $s \preceq_1 s$ and $t \preceq_2 t$ by the reflexivity of these underlying partial orders, $(s,t) \preceq (s,t)$ by definition. For antisymmetry, assume that $(s,t) \preceq (u,v)$ and $(u,v) \preceq (s,t)$. Then by definition $s \preceq_1 u$ and $t \preceq_2 v$, and $u \preceq_1 s$ and $v \preceq_2 t$. By the antisymmetry of the underlying relations, we conclude that $s = u$ and $t = v$, whence $(s,t) = (u,v)$. Finally, for transitivity, suppose that $(s,t) \preceq (u,v) \preceq (w,x)$. This means that $s \preceq_1 u \preceq_1 w$ and $t \preceq_2 v \preceq_2 x$. The transitivity of the underlying partial orders tells us that $s \preceq_1 w$ and $t \preceq_2 x$, whence by definition $(s,t) \preceq (w,x)$.

41. **a)** We argue essentially by contradiction. Suppose that m_1 and m_2 are two maximal elements in a poset that has a greatest element g; we will show that $m_1 = m_2$. Now since g is greatest, we know that $m_1 \preceq g$, and similarly for m_2. But since each m_i is maximal, it cannot be that $m_i \prec g$; hence $m_1 = g = m_2$.

b) The proof is exactly dual to the proof in part **(a)**, so we just copy over that proof, making the appropriate changes in wording. To wit: we argue essentially by contradiction. Suppose that m_1 and m_2 are two minimal elements in a poset that has a least element l; we will show that $m_1 = m_2$. Now since l is least, we know that $l \preceq m_1$, and similarly for m_2. But since each m_i is minimal, it cannot be that $l \prec m_i$; hence $m_1 = l = m_2$.

43. In each case, we need to check whether every pair of elements has both a least upper bound and a greatest lower bound.

a) This is a lattice. If we want to find the l.u.b. or g.l.b. of two elements in the same vertical column of the Hasse diagram, then we simply take the higher or lower (respectively) element. If the elements are in different columns, then to find the g.l.b. we follow the diagonal line upward from the element on the left, and then continue upward on the right, if necessary to reach the element on the right. For example, the l.u.b. of d and c is f; and the l.u.b. of a and e is e. Finding greatest lower bounds in this poset is similar.

b) This is not a lattice. Elements b and c have f, g, and h as upper bounds, but none of them is a l.u.b.

c) This is a lattice. By considering all the pairs of elements, we can verify that every pair of them has a l.u.b. and a g.l.b. For example, b and e have g and a filling these roles, respectively.

45. As usual when trying to extend a theorem from two items to an arbitrary finite number, we will use mathematical induction. The statement we wish to prove is that if S is a subset consisting of n elements from a lattice, where n is a positive integer, then S has a least upper bound and a greatest lower bound. The two proofs are duals of each other, so we will just give the proof for least upper bound here. The basis is $n = 1$, in which case there is really nothing to prove. If $S = \{x\}$, then clearly x is the least upper bound of S. The case $n = 2$ could be singled out for special mention also, since the l.u.b. in that case is guaranteed by the definition of lattice. But there is no need to do so. Instead, we simply assume the inductive hypothesis, that every subset containing n elements has a l.u.b., and prove that every subset S containing $n+1$ elements also has a l.u.b. Pick an arbitrary element $x \in S$, and let $S' = S - \{x\}$. Since S' has only n elements, it has a l.u.b. y, by the inductive hypothesis. Since we are in a lattice, there is an element z that is the l.u.b. of x and y. We will show that in fact z is the least upper bound of S. To do this, we need to show two things: that z is an upper bound, and that every upper bound is greater than or equal to z. For the first statement, let w be an arbitrary element of S; we must show that $w \preceq z$. There are two cases. If $w = x$, then $w \preceq z$ since z is the l.u.b. of x and y. Otherwise, $w \in S'$, and so $w \preceq y$ because y is the l.u.b. of S'. But since z is the l.u.b. of x and y, we also have $y \preceq z$. By transitivity, then, $w \preceq z$. For the second statement, suppose that u is any other upper bound of S; we must show that $z \preceq u$. Since u is an upper bound of S, it is also an upper bound of x and y. But since z is the *least* upper bound of x and y, we know that $z \preceq u$.

47. The needed definitions are in Example 25.

a) No. The authority level of the first pair (1) is less than or equal to (less than, in this case) that of the second (2); but the subset of the first pair is not a subset of that of the second.

b) Yes. The authority level of the first pair (2) is less than or equal to (less than, in this case) that of the second (3); and the subset of the first pair is a subset of that of the second.

c) The classes into which information can flow are those classes whose authority level is at least as high as *Proprietary*, and whose subset is a superset of $\{$ *Cheetah, Puma* $\}$. We can list these classes: (*Proprietary*, $\{$*Cheetah*, *Puma*$\}$), (*Restricted*, $\{$*Cheetah*, *Puma*$\}$), (*Registered*, $\{$*Cheetah*, *Puma*$\}$), (*Proprietary*, $\{$*Cheetah*, *Puma*, *Impala*$\}$), (*Restricted*, $\{$*Cheetah*, *Puma*, *Impala*$\}$), and (*Registered*, $\{$*Cheetah*, *Puma*, *Impala*$\}$).

d) The classes from which information can flow are those classes whose authority level is at least as low as *Restricted*, and whose subset is a subset of $\{$ *Impala, Puma* $\}$, namely (*Nonproprietary*,$\{$*Impala*,*Puma*$\}$), (*Proprietary*,$\{$*Impala*,*Puma*$\}$), (*Restricted*,$\{$*Impala*,*Puma*$\}$), (*Nonproprietary*,$\{$*Impala*$\}$), (*Proprietary*, $\{$*Impala*$\}$), (*Restricted*,$\{$*Impala*$\}$), (*Nonproprietary*,$\{$*Puma*$\}$), (*Proprietary*,$\{$*Puma*$\}$), (*Restricted*,$\{$*Puma*$\}$), (*Nonproprietary*,$\emptyset$), (*Proprietary*,$\emptyset$), and (*Restricted*,$\emptyset$).

49. Let Π be the set of all partitions of a set S, with a relation $\preceq$ defined on Π according to the referenced preamble: a partition P_1 is a refinement of P_2 if every set in P_1 is a subset of one of the sets in P_2. We need to verify all the properties of a lattice. First we need to show that $(\Pi, \preceq)$ is a poset, that is, that $\preceq$ is reflexive, antisymmetric, and transitive. For reflexivity, we need to show that $P \preceq P$ for every partition P. This means that every set in P is a subset of one of the sets in P, and this is trivially true, since every set is a subset of itself. For antisymmetry, suppose that $P_1 \preceq P_2$ and $P_2 \preceq P_1$. We must show that $P_1 = P_2$. By the equivalent roles played here by P_1 and P_2, it is enough to show that every $T \in P_1$ (where $T \subseteq S$) is also an element of P_2. Suppose we have such a T. Then since $P_1 \preceq P_2$, there is a set $T' \in P_2$ such that $T \subseteq T'$. But then since $P_2 \preceq P_1$, there is a set $T'' \in P_1$ such that $T' \subseteq T''$. Putting these together, we have $T \subseteq T''$. But P_1 is a partition, and so the elements of P_1 are nonempty and pairwise disjoint. The only way for this to happen if one is a subset of the other is for the two subsets T and T'' to be the same. But this implies that T' (which is caught in the middle) is also equal to T. Thus $T \in P_2$, which is what we were trying to show. Finally, for transitivity, suppose that $P_1 \preceq P_2$ and $P_2 \preceq P_3$. We must show that $P_1 \preceq P_3$. To this end, we take an arbitrary element $T \in P_1$. Then there is a set $T' \in P_2$ such that $T \subseteq T'$. But then since $P_2 \preceq P_3$, there is a set $T'' \in P_3$ such that $T' \subseteq T''$. Putting these together, we have $T \subseteq T''$. This demonstrates that $P_1 \preceq P_3$.

Next we have to show that every two partitions P_1 and P_2 have a least upper bound and a greatest lower bound in Π. We will show that their greatest lower bound is their "coarsest common refinement", namely the partition P whose subsets are all the nonempty sets of the form $T_1 \cap T_2$, where $T_1 \in P_1$ and $T_2 \in P_2$. As an example, if $P_1 = \{\{1,2,3\},\{4\},\{5\}\}$ and $P_2 = \{\{1,2\},\{3,4\},\{5\}\}$, then the coarsest common refinement is $P = \{\{1,2\},\{3\},\{4\},\{5\}\}$. First, we need to check that this is a partition. It certainly is a set of nonempty subsets of S. It is pairwise disjoint, because the only way an element could be in $T_1 \cap T_2 \cap T_1' \cap T_2'$ if $T_1 \cap T_2 \neq T_1' \cap T_2'$ is for that element to be in both $T_1 \cap T_1'$ and $T_2 \cap T_2'$, which means that $T_1 = T_1'$ and $T_2 = T_2'$, a contradiction. And it covers all of S, because if $x \in S$, then $x \in T_1$ for some $T_1 \in P_1$, and $x \in T_2$ for some $T_2 \in P_2$, and so $x \in T_1 \cap T_2 \in P$. Second, we need to check that P is a refinement of both P_1 and P_2. So suppose $T \in P$. Then $T = T_1 \cap T_2$, for some $T_1 \in P_1$ and $T_2 \in P_2$. It follows that $T \subseteq T_1$ and $T \subseteq T_2$. But then T_1 and T_2 satisfy the requirements in the definition of refinement. Third, we need to check that if P' is any other common refinement of both P_1 and P_2, then P' is also a refinement of P. To this end, suppose that $T \in P'$. Then by definition of refinement, there are subsets $T_1 \in P_1$ and $T_2 \in P_2$ such that $T \subseteq T_1$ and $T \subseteq T_2$. Therefore $T \subseteq T_1 \cap T_2$. But $T_1 \cap T_2 \in P$, and our proof for greatest lower bounds is complete.

It's a little harder to state the definition of the least upper bound (which again we'll call P) of two given partitions P_1 and P_2. Essentially it is just the set of all minimal nonempty subsets of S that do not "split apart" any element of either P_1 or P_2. (In the example above, it is $\{\{1,2,3,4\},\{5\}\}$.) It will be a little easier if we define it in terms of an equivalence relation rather than a partition. Note that from this point of view, one equivalence relation is a refinement of a second equivalence relation if whenever two elements are related by the first relation, then they are related by the second. The equivalence relation determining P is the relation in which $x \in S$ is related to $y \in S$ if there is a "path" (a sequence) $x = x_0, x_1, x_2, \ldots, x_n = y$, for some $n \geq 0$, such that for each i from 1 to n, x_{i-1} and x_i are in the same element of partition P_1 or of partition P_2 (in other words, x_{i-1} and x_i are related either by the equivalence relation corresponding to P_1 or by that corresponding to P_2). It is clear that this is an equivalence relation: it is reflexive by taking $n = 0$; it is symmetric by following the path backwards; and it is transitive by composing paths. It is also clear that P_1 (and P_2 similarly) is a refinement of this partition, since if two elements of S are in the same equivalence class in P_1, then we can take $n = 1$ in our path definition to see that they are in the same equivalence class in P. Thus P is an upper bound of both P_1 and P_2. Finally, we must show that P is the *least* upper bound, that is, a refinement of every other upper bound. This is clear from our construction: we only forced two elements of S to be related (i.e., in the same class of the partition) when they *had* to be related in order to enable P_1 and P_2 to be refinements. Therefore if two elements are related by P, then they have to be related by every equivalence relation (partition) Q of which both P_1 and P_2 are refinements; so P is a refinement of Q.

51. This follows immediately from Exercise 45. To be more specific, according to Exercise 45, there is a least upper bound (respectively, a greatest lower bound) for the entire finite lattice. This element is by definition a greatest element (respectively, a least element).

53. We need to show that every nonempty subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$ has a least element under lexicographic order. Given such a subset S, look at the set S_1 of positive integers that occur as first coordinates in elements of S. Let m_1 be the least element of S_1, which exists since $\mathbf{Z}^+$ is well-ordered under $\leq$. Let S' be the subset of S consisting of those pairs that have m_1 as their first coordinate. Thus S' is clearly nonempty, and by the definition of lexicographic order, every element of S' is less than every element in $S - S'$. Now let S_2 be the set of positive integers that occur as second coordinates in elements of S', and let m_2 be the least element of S_2. Then clearly the element (m_1, m_2) is the least element of S' and hence is the least element of S.

55. If x is an integer in a decreasing sequence of elements of this poset, then at most $|x|$ elements can follow x in the sequence, namely integers whose absolute values are $|x| - 1, |x| - 2, \ldots, 1, 0$. Therefore there can be no infinite decreasing sequence. This is not a totally ordered set, since 5 and -5, for example, are incomparable; from the definition given here, it is neither true that $5 \prec -5$ nor that $-5 \prec 5$, because neither one of $|5|$ or $|-5|$ is less than the other (they are equal).

57. We know from elementary arithmetic that $\mathbf{Q}$ is totally ordered by $<$, and so perforce it is a partially ordered set. To be precise, to find which of two rational numbers is larger, write them with a positive common denominator and compare numerators. To show that this set is dense, suppose $x < y$ are two rational numbers. Let z be their average, i.e., $(x + y)/2$. Since the set of rational numbers is closed under addition and division, z is also a rational number, and it is easy to show that $x < z < y$.

59. Let $(S, \preceq)$ be a partially ordered set. From the definitions of well-ordered, totally ordered, and well-founded, it is clear that what we have to show is that every nonempty subset of S contains a least element if and only if there is no infinite decreasing sequence of elements $a_1, a_2, a_3, \ldots$ in S (i.e., where $a_{i+1} \prec a_i$ for all i). One direction is clear: An infinite decreasing sequence of elements has no least element. Conversely, let A be any nonempty subset of S that has no least element. Since A is nonempty, let a_1 be any element of A. Since a_1 is not the least element of A, there is some $a_2 \in A$ smaller than a_1, i.e., $a_2 \prec a_1$. Since a_2 is not the least