SECTION 2.4 Sequences and Summations

The first half of this exercise set contains a lot of routine practice with the concept of and notation for sequences. It also discusses **telescoping sums**; *the* **product notation**, *corresponding to the summation notation discussed in the section; the* **factorial function**, *which occurs repeatedly in subsequent chapters; and a few challenging exercises on more complicated sequences. The last part of the exercise set has some fairly challenging exercises involving infinite sets. Do not be surprised if you find this last material very strange and hard to comprehend at first going; mathematicians did not understand it at all until the late nineteenth century, three hundred years after calculus was well understood. To show that an infinite set is countable, you need to find a one-to-one correspondence between the set and the set of positive integers. One way to do this directly is to provide a listing of the elements of the set. (There is no listing, for instance, of the set of real numbers.) Various indirect means are also available, such as showing that the set is a subset of a countable set, or showing that it is the union of a countable collection of countable sets.*

1. a) $a_0 = 2 \cdot (-3)^0 + 5^0 = 2 \cdot 1 + 1 = 3$ b) $a_1 = 2 \cdot (-3)^1 + 5^1 = 2 \cdot (-3) + 5 = -1$
c) $a_4 = 2 \cdot (-3)^4 + 5^4 = 2 \cdot 81 + 625 = 787$ d) $a_5 = 2 \cdot (-3)^5 + 5^5 = 2 \cdot (-243) + 3125 = 2639$

3. In each case we simply evaluate the given function at $n = 0, 1, 2, 3$.
a) $a_0 = 2^0 + 1 = 2$, $a_1 = 2^1 + 1 = 3$, $a_2 = 2^2 + 1 = 5$, $a_3 = 2^3 + 1 = 9$
b) $a_0 = 1^1 = 1$, $a_1 = 2^2 = 4$, $a_2 = 3^3 = 27$, $a_3 = 4^4 = 256$
c) $a_0 = \lfloor 0/2 \rfloor = 0$, $a_1 = \lfloor 1/2 \rfloor = 0$, $a_2 = \lfloor 2/2 \rfloor = 1$, $a_3 = \lfloor 3/2 \rfloor = 1$
d) $a_0 = \lfloor 0/2 \rfloor + \lceil 0/2 \rceil = 0 + 0 = 0$, $a_1 = \lfloor 1/2 \rfloor + \lceil 1/2 \rceil = 0 + 1 = 1$, $a_2 = \lfloor 2/2 \rfloor + \lceil 2/2 \rceil = 1 + 1 = 2$, $a_3 = \lfloor 3/2 \rfloor + \lceil 3/2 \rceil = 1 + 2 = 3$. Note that $\lfloor n/2 \rfloor + \lceil n/2 \rceil$ always equals n.

5. In each case we just follow the instructions.
a) $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$ b) $1, 1, 1, 2, 2, 2, 3, 3, 3, 4$ c) $1, 1, 3, 3, 5, 5, 7, 7, 9, 9$
d) This requires a bit of routine calculation. For example, the fifth term is $5! - 2^5 = 120 - 32 = 88$. The first ten terms are $-1, -2, -2, 8, 88, 656, 4912, 40064, 362368, 3627776$.
e) $3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$ f) $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$
g) For $n = 1$, the binary expansion is 1, which has one bit, so the first term of the sequence is 1. For $n = 2$, the binary expansion is 10, which has two bits, so the second term of the sequence is 2. Continuing in this way we see that the first ten terms are $1, 2, 2, 3, 3, 3, 3, 4, 4, 4$. Note that the sequence has one 1, two 2's, four 3's, eight 4's, as so on, with 2^{k-1} copies of k.
h) The English word for 1 is "one" which has three letters, so the first term is 3. This makes a good brain-teaser; give someone the sequence and ask her or him to find the pattern. The first ten terms are $3, 3, 5, 4, 4, 3, 5, 5, 4, 3$.

7. One pattern is that each term is twice the preceding term. A formula for this would be that the n^{th} term is 2^{n-1}. Another pattern is that we obtain the next term by adding increasing values to the previous term. Thus to move from the first term to the second we add 1; to move from the second to the third we add 2; then add 3, and so on. So the sequence would start out $1, 2, 4, 7, 11, 16, 22, \ldots$. We could also have trivial answers such as the rule that the first three terms are $1, 2, 4$ and all the rest are 17 (so the sequence is $1, 2, 4, 17, 17, 17, \ldots$), or that the terms simply repeat $1, 2, 4, 1, 2, 4, 1, 2, 4, \ldots$. Here is another pattern: Take n points on the unit circle, and connect each of them to all the others by line segments. The inside of the circle will be divided into a number of regions. What is the largest this number can be? Call that value a_n. If there is one point, then there are no lines and therefore just the one original region inside the circle; thus $a_1 = 1$. If $n = 2$, then the one chord divides the interior into two parts, so $a_2 = 2$. Three points give us a triangle, and that makes four regions (the inside of the triangle and the three pieces outside the triangle), so $a_3 = 4$. Careful drawing shows that the sequence starts out $1, 2, 4, 8, 16, 31$. That's right: 31, not 32.

Creative students may well find other rules or patterns with various degrees of appeal.

9. In some sense there are no right answers here. The solutions stated are the most appealing patterns that the author has found.
a) It looks as if we have one 1 and one 0, then two of each, then three of each, and so on, increasing the number of repetitions by one each time. Thus we need three more 1's (and then four 0's) to continue the sequence.
b) A pattern here is that the positive integers are listed in increasing order, with each even number repeated. Thus the next three terms are $9, 10, 10$.
c) The terms in the odd locations (first, third, fifth, etc.) are just the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The n^{th} term is 0 if n is even and is $2^{(n-1)/2}$ if n is odd. Thus the next three terms are $32, 0, 64$.

d) The first term is 3 and each successive term is twice its predecessor. This is a geometric sequence. The n^{th} term is $3 \cdot 2^{n-1}$. Thus the next three terms are $384, 768, 1536$.

e) The first term is 15 and each successive term is 7 less than its predecessor. This is an arithmetic sequence. The n^{th} term is $22 - 7n$. Thus the next three terms are $-34, -41, -48$.

f) The rule is that the first term is 3 and the n^{th} term is obtained by adding n to the $(n-1)^{\text{th}}$ term. One can actually find a quadratic expression for a sequence in which the successive differences form an arithmetic sequence; here it is $(n^2+n+4)/2$. The easiest way to see this is to note that the n^{th} term is $3+2+3+4+5+6+\cdots+n$. Except for the initial 3 instead of a 1, the n^{th} term is the sum of the first n positive integers, which is $n(n+1)/2$ by a formula in Table 2. Therefore the n^{th} term is $(n(n+1)/2)+2$, as claimed. We see that the next three terms are $57, 68, 80$.

g) One should play around with the sequence if nothing is apparent at first. Here we note that all the terms are even, so if we divide by 2 we obtain the sequence $1, 8, 27, 64, 125, 216, 343, \ldots$. This sequence appears in Table 1; it is the cubes. So the n^{th} term is $2n^3$. Thus the next three terms are $1024, 1458, 2000$.

h) These terms look close to the terms of the sequence whose n^{th} term is $n!$ (see Table 1). In fact, we see that the n^{th} term here is $n!+1$. Thus the next three terms are $362881, 3628801, 39916801$.

11. It is pretty clear that a_n should be approximately equal to $n+\sqrt{n}$, since the sequence is just the sequence of positive integers with perfect squares left out. There are about $\sqrt{n}$ perfect squares up to n, so the count needs to get ahead by about this amount. Proving that this plausibility argument gives the correct formula involves some careful counting.

The sequence begins $2, 3, 5, 6, 7, 8, 10, 11, \ldots$. We can write it as the sequence $a_n = n$ plus a sequence b_n that jumps every time a perfect square is encountered. Thus $\{b_n\}$ begins $1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, \ldots$; there are two 1's, four 2's, six 3's, eight 4's, and so on. So we must show that $b_n = \{\sqrt{n}\}$, where $\{\sqrt{n}\}$ means the integer closest to $\sqrt{n}$ (note that there is never an ambiguity here, since this will never be a half-integer). Because of the way the sequence is formed, $b_n \le k$ if and only if $2+4+6+\cdots+2k \ge n$. This is equivalent to $k(k+1) \ge n$. Applying the quadratic formula and recalling that k is an integer, we obtain $b_n = \lceil(-1+\sqrt{1+4n})/2\rceil$. Simplifying, we have $b_n = \left\lceil -\frac{1}{2} + \sqrt{n+\frac{1}{4}} \right\rceil$. Subtracting $\frac{1}{2}$ and then rounding up is the same as rounding to the nearest integer (the smaller one if $\sqrt{n+\frac{1}{4}}$ is a half-integer—see Exercise 43 in Section 2.3), so (with this understanding) $b_n = \left\{\sqrt{n+\frac{1}{4}}\right\}$. But it can never happen that $\sqrt{n} \le m+\frac{1}{2}$ while $\sqrt{n+\frac{1}{4}} > m+\frac{1}{2}$ for some positive integer m—this would imply that $n \le m^2+m+\frac{1}{4}$ and $n > m^2+m$, an impossibility. Therefore $\{\sqrt{n}\} = \left\{\sqrt{n+\frac{1}{4}}\right\}$, and we are done.

An alternative solution is provided in the answer section of the text.

13. **a)** $2+3+4+5+6=20$ **b)** $1-2+4-8+16=11$ **c)** $3+3+\cdots+3=10\cdot 3=30$

d) This series "telescopes": each term cancels part of the term before it (see also Exercise 19). The sum is $(2-1)+(4-2)+(8-4)+\cdots+(512-256) = -1+512 = 511$.

15. We use the formula for the sum of a geometric progression: $\sum_{j=0}^{n} ar^j = a(r^{n+1}-1)/(r-1)$.

a) Here $a=3$, $r=2$, and $n=8$, so the sum is $3(2^9-1)/(2-1) = 1533$.

b) Here $a=1$, $r=2$, and $n=8$. The sum taken over all the values of j from 0 to n is, by the formula, $(2^9-1)/(2-1) = 511$. However, our sum starts at $j=1$, so we must subtract out the term that isn't there, namely 2^0. Hence the answer is $511-1=510$.

c) Again we have to subtract the missing terms, so the sum is $((-3)^9-1)/((-3)-1) - (-3)^0 - (-3)^1 = 4921 - 1 - (-3) = 4923$.

d) $2((-3)^9-1)/((-3)-1) = 9842$

17. The easiest way to do these sums, since the number of terms is reasonably small, is just to write out the summands explicitly. Note that the inside index (j) runs through all of its values for each value of the outside index (i).

a) $(1+1)+(1+2)+(1+3)+(2+1)+(2+2)+(2+3)=21$

b) $(0+3+6+9)+(2+5+8+11)+(4+7+10+13)=78$

c) $(1+1+1)+(2+2+2)+(3+3+3)=18$

d) $(0+0+0)+(1+2+3)+(2+4+6)=18$

19. If we just write out what the sum means, we see that parts of successive terms cancel, leaving only two terms:

$$\sum_{j=1}^{n}(a_j-a_{j-1})=a_1-a_0+a_2-a_1+a_3-a_2+\cdots+a_{n-1}-a_{n-2}+a_n-a_{n-1}=a_n-a_0$$

21. a) We use the hint, where $a_k=k^2$:

$$\sum_{k=1}^{n}(2k-1)=\sum_{k=1}^{n}(k^2-(k-1)^2)=n^2-0^2=n^2$$

b) We can use the distributive law to rewrite $\sum_{k=1}^{n}(2k-1)$ (which we know from part (a) equals n^2) in terms of the sum we want, $S=\sum_{k=1}^{n}k$:

$$n^2=\sum_{k=1}^{n}(2k-1)=2\sum_{k=1}^{n}k-\sum_{k=1}^{n}1=2S-n\,.$$

Now we solve for S, obtaining $S=(n^2+n)/2$, which is usually expressed as $n(n+1)/2$.

23. This exercise is like Example 15. From Table 2 we know that $\sum_{k=1}^{200}k=200\cdot 201/2=20100$, and $\sum_{k=1}^{99}k=99\cdot 100/2=4950$. Therefore the desired sum is $20100-4950=15150$.

25. If we write down the first few terms of this sum we notice a pattern. It starts $(1+1+1)+(2+2+2+2+2)+(3+3+3+3+3+3+3)+\cdots$. There are three 1's, then five 2'ss, then seven 3'ss, and so on; in general there are $(i+1)^2-i^2=2i+1$ copies of i. So we need to sum $i(2i+1)$ for an appropriate range of values for i. We must find this range. It gets a little messy at the end if m is such that the sequence stops before a complete range of the last value is present. Let $n=\lfloor\sqrt{m}\rfloor-1$. Then there are $n+1$ blocks, and $(n+1)^2-1$ is where the next-to-last block ends. The sum of those complete blocks is $\sum_{i=1}^{n}i(2i+1)=\sum_{i=1}^{n}2i^2+i=n(n+1)(2n+1)/3+n(n+1)/2$. The remaining terms in our summation all have the value $n+1$ and the number of them present is $m-((n+1)^2-1)$. Our final answer is therefore $n(n+1)(2n+1)/3+n(n+1)/2+(n+1)(m-(n+1)^2+1)$.

27. a) 0 (anything times 0 is 0) b) $5\cdot 6\cdot 7\cdot 8=1680$

c) Each factor is either 1 or -1, so the product is either 1 or -1. To see which it is, we need to determine how many of the factors are -1. Clearly there are 50 such factors, namely when $i=1,3,5,\ldots,99$. Since $(-1)^{50}=1$, the product is 1.

d) $2\cdot 2\cdots 2=2^{10}=1024$

29. $0!+1!+2!+3!+4!=1+1+1\cdot 2+1\cdot 2\cdot 3+1\cdot 2\cdot 3\cdot 4=1+1+2+6+24=34$

31. a) The negative integers are conntable. Each negative integer can be paired with its absolute value to give the desired one-to-one correspondence: $1\leftrightarrow -1$, $2\leftrightarrow -2$, $3\leftrightarrow -3$, etc.

b) The even integers are countable. We can list the set of even integers in the order $0, 2, -2, 4, -4, 6, -6, \ldots$, and pair them with the positive integers listed in their natural order. Thus $1 \leftrightarrow 0$, $2 \leftrightarrow 2$, $3 \leftrightarrow -2$, $4 \leftrightarrow 4$, etc. There is no need to give a formula for this correspondence—the discussion given is quite sufficient; but it is not hard to see that we are pairing the positive integer n with the even integer $f(n)$, where $f(n) = n$ if n is even and $f(n) = 1 - n$ if n is odd.

c) The proof that the set of real numbers between 0 and 1 is not countable (Example 21) can easily be modified to show that the set of real numbers between 0 and $1/2$ is not countable. We need to let the digit d_i be something like 2 if $d_{ii} \neq 2$ and 3 otherwise. The number thus constructed will be a real number between 0 and $1/2$ that is not in the list.

d) This set is countable, exactly as in part (b); the only difference is that there we are looking at the multiples of 2 and here we are looking at the multiples of 7. The correspondence is given by pairing the positive integer n with $7n/2$ if n is even and $-7(n-1)/2$ if n is odd.

33. a) The bit strings not containing 0 are just the bit strings consisting of all 1's, so this set is $\{\lambda, 1, 11, 111, 1111, \ldots\}$, where λ denotes the empty string (the string of length 0). Thus this set is countable, where the correspondence matches the natural number n with the string of n 1's.

b) This is a subset of the set of rational numbers, so it is countable (see Exercise 36). To find a correspondence, we can just follow the path in Example 20, but omit fractions in the top three rows (as well as continuing to omit those fractions not in lowest terms).

c) This set is uncountable, as can be shown by applying the diagonal argument of Example 21.

d) This set is uncountable, as can be shown by applying the diagonal argument of Example 21.

35. Yes. We need to look at this from the other direction, by noting that $A = B \cup (A - B)$. We are given that B is countable. If $A - B$ were also countable, then, since the union of two countable sets is countable (which we are asked to prove as Exercise 40), we would conclude that A is countable. But we are given that A is not countable. Therefore our assumption that $A - B$ is countable is wrong, and we conclude that $A - B$ is uncountable. (This is an example of a proof by contraposition.)

37. This is just the contrapositive of Exercise 36 and so follows directly from it. In more detail, suppose that B were countable, say with elements $b_1, b_2, \ldots$. Then since $A \subseteq B$, we can list the elements of A using the order in which they appear in this listing of B. Therefore A is countable, contradicting the hypothesis. Thus B is not countable.

39. By what we are given, we know that there are bijections f from A to B and g from C to D. Then we can define a bijection from $A \times C$ to $B \times D$ by sending (a, c) to $(f(a), g(c))$. This is clearly one-to-one and onto, so we have shown that $A \times C$ and $B \times D$ have the same cardinality.

41. Since empty sets do not contribute any elements to unions, we can assume that none of the sets in our given countable collection of countable sets is the empty set. If there are no sets in the collection, then the union is empty and therefore countable. Otherwise let the countable sets be A_1, A_2, ... (If there are only a finite number k of them, then we can still assume that they form an infinite sequence by taking $A_{k+1} = A_{k+2} = \cdots = A_1$.) Since each set A_i is countable and nonempty, we can list its elements in a sequence as a_{i1}, a_{i2}, ...; again, if the set is finite we can list its elements and then list a_{i1} repeatedly to assure an infinite sequence. Now we just need a systematic way to put all the elements a_{ij} into a sequence. We do this by listing first all the elements a_{ij} in which $i + j = 2$ (there is only one such pair, $(1, 1)$), then all the elements in which $i + j = 3$ (there are only two such pairs, $(1, 2)$ and $(2, 1)$), and so on; except that we do not list any element that we have already listed. So, assuming that these elements are distinct, our list starts a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, ... (If any of these terms duplicates a previous term, then it is simply omitted.) The result of this process will be either an infinite sequence or a finite sequence containing all the elements of the union of the sets A_i. Thus that union is countable.

43. There are only a finite number of bit strings of each finite length, so we can list all the bit strings by listing first those of length 0, then those of length 1, etc. The listing might be $\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$. (Recall that λ denotes the empty string.) Actually this is a special case of Exercise 41: the set of all bit strings is the union of a countable number of countable (actually finite) sets, namely the sets of bit strings of length n for $n = 0, 1, 2, \ldots$.

45. We argued in the solution to Exercise 43 that the set of all strings of symbols from the alphabet $\{0,1\}$ is countable, since there are only a finite number of bit strings of each length. There was nothing special about the alphabet $\{0,1\}$ in that argument. For any finite alphabet (for example, the alphabet consisting of all upper and lower case letters, numerals, and punctuation and other mathematical marks typically used in a programming language), there are only a finite number of strings of length 1 (namely the number of symbols in the alphabet), only a finite number of strings of length 2 (namely, the square of this number), and so on. Therefore, using the result of Exercise 41 again, we conclude that there are only countably many strings from any given finite alphabet. Now the set of all computer programs in a particular language is just a subset of the set of all strings over that alphabet (some strings are meaningless jumbles of symbols that are not valid programs), so by Exercise 36, this set, too, is countable.

47. In Exercise 45 we saw that there are only a countable number of computer programs, so there are only a countable number of computable functions. In Exercise 46 we saw that there are an uncountable number of functions. Hence not all functions are computable. Indeed, in some sense, since uncountable sets are so much bigger than countable sets, *almost all* functions are not computable! This is not really so surprising; in real life we deal with only a small handful of useful functions, and these are computable. Note that this is a nonconstructive proof—we have not exhibited even one noncomputable function, merely argued that they have to exist. Actually finding one is much harder, but it can be done. For example, the following function is not computable. Let T be the function from the set of positive integers to $\{0,1\}$ defined by letting $T(n)$ be 0 if the number 0 is in the range of the function computed by the n^{th} computer program (where we list them in alphabetical order by length) and letting $T(n) = 1$ otherwise.