Important note about notation for proofs by mathematical induction: *In performing the inductive step, it really does not matter what letter we use. We see in the text the proof of* $(\forall j{\le}k\, P(j)) \to P(k+1)$*; but it would be just as valid to prove* $(\forall j{\le}n\, P(j)) \to P(n+1)$*, since the* k *in the first case and the* n *in the second case are just dummy variables. Furthermore, we could also take the inductive hypothesis to be* $\forall j{<}n\, P(j)$ *and then prove* $P(n)$*. We will use all three notations in this* Guide.

2. Let $P(n)$ be the statement that the n^{th} domino falls. We want to prove that $P(n)$ is true for all positive integers n. For the basis step we note that the given conditions tell us that $P(1)$, $P(2)$, and $P(3)$ are true. For the inductive step, fix $k \ge 3$ and assume that $P(j)$ is true for all $j \le k$. We want to show that $P(k+1)$ is true. Since $k \ge 3$, $k-2$ is a positive integer less than or equal to k, so by the inductive hypothesis we know that $P(k-2)$ is true. That is, we know that the $(k-2)^{\text{nd}}$ domino falls. We were told that "when a domino falls, the domino three farther down in the arrangement also falls," so we know that the domino in position $(k-2)+3 = k+1$ falls. This is $P(k+1)$.

Note that we didn't use strong induction exactly as stated on page 284. Instead, we considered all the cases $n = 1$, $n = 2$, and $n = 3$ as part of the basis step. We could have more formally included $n = 2$ and $n = 3$ in the inductive step as a special case. Writing our proof this way, the basis step is just to note that the first domino falls, so $P(1)$ is true. For the inductive step, if $k = 1$ or $k = 2$, then we are already told that the second and third domino fall, so $P(k+1)$ is true in those cases. If $k > 2$, then the inductive hypothesis tells us that the $(k-2)^{\text{nd}}$ domino falls, so the domino in position $(k-1)+2 = k+1$ falls.

4. a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.

b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \le j \le k$, where we assume that $k \ge 21$.

c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.

d) We want to form $k+1$ cents of postage. Since $k \ge 21$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

6. a) We can form the following amounts of postage as indicated: $3 = 3$, $6 = 3+3$, $9 = 3+3+3$, $10 = 10$, $12 = 3+3+3+3$, $13 = 10+3$, $15 = 3+3+3+3+3$, $16 = 10+3+3$, $18 = 3+3+3+3+3+3$, $19 = 10+3+3+3$, $20 = 10+10$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 18 cents using just 3-cent and 10-cent stamps.

b) Let $P(n)$ be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps. We want to prove that $P(n)$ is true for all $n \ge 18$. The basis step, $n = 18$, is handled above. Assume that we can form k cents of postage (the inductive hypothesis); we will show how to form $k+1$ cents of postage. If the k cents included two 10-cent stamps, then replace them by seven 3-cent stamps ($7 \cdot 3 = 2 \cdot 10 + 1$). Otherwise, k cents was formed either from just 3-cent stamps, or from one 10-cent stamp and $k - 10$ cents in 3-cent stamps. Because $k \geq 18$, there must be at least three 3-cent stamps involved in either case. Replace three 3-cent stamps by one 10-cent stamp, and we have formed $k+1$ cents in postage ($10 = 3 \cdot 3 + 1$).

c) $P(n)$ is the same as in part **(b)**. To prove that $P(n)$ is true for all $n \geq 18$, we note for the basis step that from part **(a)**, $P(n)$ is true for $n = 18, 19, 20$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $18 \leq j \leq k$, where k is a fixed integer greater than or equal to 20. We want to show that $P(k+1)$ is true. Because $k - 2 \geq 18$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 18 and k inclusive, and that enabled us to jump back three steps to a value for which we knew how to form the desired postage.

8. Since both 25 and 40 are multiples of 5, we cannot form any amount that is not a multiple of 5. So let's determine for which values of n we can form $5n$ dollars using these gift certificates, the first of which provides 5 copies of \$5, and the second of which provides 8 copies. We can achieve the following values of n: $5 = 5$, $8 = 8$, $10 = 5+5$, $13 = 8+5$, $15 = 5+5+5$, $16 = 8+8$, $18 = 8+5+5$, $20 = 5+5+5+5$, $21 = 8+8+5$, $23 = 8+5+5+5$, $24 = 8+8+8$, $25 = 5+5+5+5+5$, $26 = 8+8+5+5$, $28 = 8+5+5+5+5$, $29 = 8+8+8+5$, $30 = 5+5+5+5+5+5$, $31 = 8+8+5+5+5$, $32 = 8+8+8+8$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form total amounts of the form $5n$ for all $n \geq 28$ using these gift certificates. (In other words, \$135 is the largest multiple of \$5 that we cannot achieve.)

To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all $n \geq 28$. From our work above, we know that $P(n)$ is true for $n = 28, 29, 30, 31, 32$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \leq j \leq k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ is true. Because $k - 4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more \$25-dollar certificate, and we have formed $5(k+1)$ dollars, as desired.

10. We claim that it takes exactly $n-1$ breaks to separate a bar (or any connected piece of a bar obtained by horizontal or vertical breaks) into n pieces. We use strong induction. If $n = 1$, this is trivially true (one piece, no breaks). Assume the strong inductive hypothesis, that the statement is true for breaking into k or fewer pieces, and consider the task of obtaining $k+1$ pieces. We must show that it takes exactly k breaks. The process must start with a break, leaving two smaller pieces. We can view the rest of the process as breaking one of these pieces into $i+1$ pieces and breaking the other piece into $k-i$ pieces, for some i between 0 and $k-1$, inclusive. By the inductive hypothesis it will take exactly i breaks to handle the first piece and $k-i-1$ breaks to handle the second piece. Therefore the total number of breaks will be $1 + i + (k-i-1) = k$, as desired.

12. The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to k can be written as a sum of distinct powers of 2. We must show that $k+1$ can be written as a sum of distinct powers of 2. If $k+1$ is odd, then k is even, so 2^0 was not part of the sum for k. Therefore the sum for $k+1$ is the same as the sum for k with the extra term 2^0 added. If $k+1$ is even, then $(k+1)/2$ is a positive integer, so by the inductive hypothesis $(k+1)/2$ can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for $k+1$.

14. We prove this using strong induction. It is clearly true for $n = 1$, because no splits are performed, so the sum computed is 0, which equals $n(n-1)/2$ when $n = 1$. Assume the strong inductive hypothesis, and suppose that our first splitting is into piles of i stones and $n-i$ stones, where i is a positive integer less than n. This gives a product $i(n-i)$. The rest of the products will be obtained from splitting the piles thus formed, and so by the inductive hypothesis, the sum of the products will be $i(i-1)/2 + (n-i)(n-i-1)/2$. So we must show that
$$i(n-i) + \frac{i(i-1)}{2} + \frac{(n-i)(n-i-1)}{2} = \frac{n(n-1)}{2}$$
no matter what i is. This follows by elementary algebra, and our proof is complete.

16. We follow the hint to show that there is a winning strategy for the first player in Chomp played on a $2 \times n$ board that starts by removing the rightmost cookie in the bottom row. Note that this leaves a board with n cookies in the top row and $n-1$ cookies in the bottom row. It suffices to prove by strong induction on n that a player presented with such a board will lose if his opponent plays properly. We do this by showing how the opponent can return the board to this form following any nonfatal move this player might make. The basis step is $n = 1$, and in that case only the poisoned cookie remains, so the player loses. Assume the inductive hypothesis (that the statement is true for all smaller values of n). If the player chooses a nonpoisoned cookie in the top row, then that leaves another board with two rows of equal length, so again the opponent chooses the rightmost cookie in the bottom row, and we are back to the hopeless situation, for some board with fewer than n cookies in the top row. If the player chooses the cookie in the m^{th} column from the left in the bottom row (where necessarily $m < n$), then the opponent chooses the cookie in the $(m+1)^{\text{st}}$ column from the left in the top row, and once again we are back to the hopeless situation, with m cookies in the top row.

18. We prove something slightly stronger: If a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, $\ldots$, v_{m+n-1} is triangulated, then the triangles can be numbered from m to $m+n-3$ so that v_i is a vertex of triangle i for $i = m, m+1, \ldots, m+n-3$. (The statement we are asked to prove is the case $m = 1$.) The basis step is $n = 3$, and there is nothing to prove. For the inductive step, assume the inductive hypothesis that the statement is true for polygons with fewer than n vertices, and consider any triangulation of a convex n-gon whose vertices are labeled consecutively as v_m, v_{m+1}, $\ldots$, v_{m+n-1}. One of the diagonals in the triangulation must have either v_{m+n-1} or v_{m+n-2} as an endpoint (otherwise, the region containing v_{m+n-1} would not be a triangle). So there are two cases. If the triangulation uses diagonal $v_k v_{m+n-1}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-1} as v_{k+1} in the polygon that contains v_m. This gives us the desired numbering of the triangles, with numbers v_m through v_{k-1} in the first polygon and numbers v_k through v_{m+n-3} in the second polygon. If the triangulation uses diagonal $v_k v_{m+n-2}$, then we apply the inductive hypothesis to the two polygons formed by this diagonal, renumbering v_{m+n-2} as v_{k+1} and v_{m+n-1} as v_{k+2} in the polygon that contains v_{m+n-1}, and renumbering all the vertices by adding 1 to their indices in the other polygon. This gives us the desired numbering of the triangles, with numbers v_m through v_k in the first polygon and numbers v_{k+1} through v_{m+n-3} in the second polygon. Note that we did not need the convexity of our polygons.

20. The proof takes several pages and can be found in an article entitled "Polygons Have Ears" by Gary H. Meisters in *The American Mathematical Monthly*, Volume 82, Number 6, June–July, 1975, pages 648–651.

22. The basis step for this induction is no problem, because for $n = 3$, there can be no diagonals and therefore there are two vertices that are not endpoints of the diagonals. (Note, though, that $Q(3)$ is not true.) For $n = 4$, there can be at most one diagonal, and the two vertices that are not its endpoints satisfy the requirements for both $P(4)$ and $Q(4)$. We look at the inductive steps.

a) The proof would presumably try to go something like this. Given a polygon with its set of nonintersecting diagonals, think of one of those diagonals as splitting the polygon into two polygons, each of which then has a set of nonintersecting diagonals. By the inductive hypothesis, each of the two polygons has at least two vertices that are not endpoints of any of these diagonals. We would hope that these two vertices would be the vertices we want. However, one or both of them in each case might actually be endpoints of that separating diagonal, which is a side, not a diagonal, of the smaller polygons. Therefore we have no guarantee that *any* of the points we found do what we want them to do in the original polygon.

b) As in part (a), given a polygon with its set of nonintersecting diagonals, think of one of those diagonals—let's call it uv—as splitting the polygon into two polygons, each of which then has a set of nonintersecting diagonals. By the inductive hypothesis, each of the two polygons has at least two nonadjacent vertices that are not endpoints of any of these diagonals. Furthermore, the two vertices in each case cannot both be u and v, because u and v are adjacent. Therefore there is a vertex w in one of the smaller polygons and a vertex x in the other that differ from u and v and are not endpoints of any of the diagonals. Clearly w and x do what we want them to do in the original polygon—they are not adjacent and they are not the endpoints of any of the diagonals.

24. Call a suitee w and a suitor m "possible" for each other if there exists a stable assignment in which m and w are paired. We will prove that if a suitee w rejects a suitor m, then w is impossible for m. Since the suitors propose in their preference order, the desired conclusion follows. The proof is by induction on the round in which the rejection happens. We will let m be Bob and w be Alice in our discussion. If it is the first round, then say that Bob and Ted both propose to Alice (necessarily the first choice of each of them), and Alice rejects Bob because she prefers Ted. There can be no stable assignment in which Bob is paired with Alice, because then Alice and Ted would form an unstable pair (Alice prefers Ted to Bob, and Ted prefers Alice to everyone else so in particular prefers her to his mate). So assume the inductive hypothesis, that every suitor who has been rejected so far is impossible for every suitee who has rejected him. At this point Bob proposes to Alice and Alice rejects him in favor of, say, Ted. The reason that Ted has proposed to Alice is that she is his favorite among everyone who has not already rejected him; but by the inductive hypothesis, all the suitees who have rejected him are impossible for him. But now there can be no stable assignment in which Bob and Alice are paired, because such an assignment would again leave Alice and Ted unhappy—Alice because she prefers Ted to Bob, and Ted because he prefers Alice to the person he ended up with (remember that by the inductive hypothesis, he cannot have ended up with anyone he prefers to Alice). This completes the inductive step.

For more information, see the seminal article on this topic ("College Admissions and the Stability of Marriage" by David Gale and Lloyd S. Shapley in *The American Mathematical Monthly*, Volume 69, Number 1, January, 1962, pages 9–15) or a definitive book (*The Stable Marriage Problem: Structure and Algorithms* by Dan Gusfield and Robert W. Irving (MIT Press, 1989)).

26. a) Clearly these conditions tell us that $P(n)$ is true for the even values of n, namely, $0, 2, 4, 6, 8, \ldots$. Also, it is clear that there is no way to be sure that $P(n)$ is true for other values of n.

b) Clearly these conditions tell us that $P(n)$ is true for the values of n that are multiples of 3, namely, $0, 3, 6, 9, 12, \ldots$. Also, it is clear that there is no way to be sure that $P(n)$ is true for other values of n.

c) These conditions are sufficient to prove by induction that $P(n)$ is true for all nonnegative integers n.

d) We immediately know that $P(0)$, $P(2)$, and $P(3)$ are true, and clearly there is no way to be sure that $P(1)$ is true. Once we have $P(2)$ and $P(3)$, the inductive step $P(n) \to P(n+2)$ gives us the truth of $P(n)$ for all $n \geq 2$.

28. We prove by strong induction on n that $P(n)$ is true for all $n \geq b$. The basis step is $n = b$, which is true by the given conditions. For the inductive step, fix an integer $k \geq b$ and assume the inductive hypothesis that if $P(j)$ is true for all j with $b \leq j \leq k$, then $P(k+1)$ is true. There are two cases. If $k+1 \leq b+j$, then

$P(k+1)$ is true by the given conditions. On the other hand, if $k+1>b+j$, then the given conditional statement has its antecedent true by the inductive hypothesis and so again $P(k+1)$ follows.

30. The flaw comes in the inductive step, where we are implicitly assuming that $k \geq 1$ in order to talk about a^{k-1} in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). Our basis step was $n=0$, so we are not justified in assuming that $k \geq 1$ when we try to prove the statement for $k+1$ in the inductive step. Indeed, it is precisely at $n=1$ that the proposition breaks down.

32. The proof is invalid for $k=4$. We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

34. We use the technique from part (b) of Exercise 33. We are thinking of k as fixed and using induction on n. If $n=1$, then the sum contains just one term, which is just $k!$, and the right-hand side is also $k!$, so the proposition is true in this case. Next we assume the inductive hypothesis,

$$\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1)=\frac{n(n+1)(n+2)\cdots(n+k)}{k+1},$$

and prove the statement for $n+1$, namely,

$$\sum_{j=1}^{n+1} j(j+1)(j+2)\cdots(j+k-1)=\frac{(n+1)(n+2)\cdots(n+k)(n+k+1)}{k+1}.$$

We have

$$\begin{aligned}
\sum_{j=1}^{n+1} j(j+1)(j+2)\cdots(j+k-1) &= \left(\sum_{j=1}^{n} j(j+1)(j+2)\cdots(j+k-1)\right)+(n+1)(n+2)\cdots(n+k)\\
&= \frac{n(n+1)(n+2)\cdots(n+k)}{k+1}+(n+1)(n+2)\cdots(n+k)\\
&= (n+1)(n+2)\cdots(n+k)\left(\frac{n}{k+1}+1\right)\\
&= (n+1)(n+2)\cdots(n+k)\cdot\frac{n+k+1}{k+1},
\end{aligned}$$

as desired.

36. a) That S is nonempty is trivial, since letting $s=1$ and $t=1$ gives $a+b$, which is certainly a positive integer in S.

b) The Well Ordering Property asserts that every nonempty set of positive integers has a least element. Since we just showed that S is a nonempty set of positive integers, it has a least element, which we will call c.

c) If d is a divisor of a and of b, then it is also a divisor of as and bt, and hence of their sum. Since c is such a sum, d is a divisor of c.

d) This is the hard part. By symmetry it is enough to show one of these, say that $c \mid a$. Assume (for a proof by contradiction) that $c \nmid a$. Then by the Division Algorithm (Section 3.4), we can write $a=qc+r$, where $0<r<c$. Now $c=as+bt$ (for appropriate choices of s and t), since $c \in S$, so we can compute that $r=a-qc=a-q(as+bt)=a(1-qs)+b(-qt)$. This expresses the positive integer r as a linear combination with integer coefficients of a and b and hence tells us that $r \in S$. But since $r<c$, this contradicts the choice of c. Therefore our assumption that $c \nmid a$ is wrong, and $c \mid a$, as desired.

e) We claim that the c found in this exercise is the greatest common divisor of a and b. Certainly by part (d) it is a common divisor of a and b. On the other hand, part (c) tells us that every common divisor of a and b is a divisor of (and therefore no greater than) c. Thus c is a greatest common divisor of a and b. Of course the greatest common divisor is unique, since one cannot have two numbers, each of which is greater than the other.

38. In Exercise 44 of Section 1.7, we found a closed path that snakes its way around an 8×8 checkerboard to cover all the squares, and using that we were able to prove that when one black and one white square are removed, the remaining board can be covered with dominoes. The same reasoning works for any size board, so it suffices to show that any board with an even number of squares has such a snaking path. Note that a board with an even number of squares must have either an even number of rows or an even number of columns, so without loss of generality, assume that it has an even number of rows, say $2n$ rows and m columns. Number the squares in the usual manner, so that the first row contains squares 1 to m from left to right, the second row contains squares $m+1$ to $2m$ from left to right, and so on, with the final row containing squares $(2n-1)m+1$ to $2nm$ from left to right.

We will prove the stronger statement that any such board contains a path that includes the top row traversed from left to right. The basis step is $n = 1$, and in that case the path is simply $1, 2, \ldots, m, 2m, 2m-1, \ldots, m+1, 1$. Assume the inductive hypothesis and consider a board with $2n+2$ rows. By the inductive hypothesis, the board obtained by deleting the top two rows has a closed path that includes its top row from left to right (i.e., $2m+1, 2m+2, \ldots, 3m$). Replace this subsequence by $2m+1, m+1, 1, 2, \ldots, m, 2m, 2m-1, \ldots, m+2, 2m+2, \ldots, 3m$, and we have the desired path.

40. If $x < y$ then $y - x$ is a positive real number, and its reciprocal $1/(y-x)$ is a positive real number, so we can choose a positive integer $A > 1/(y-x)$. (Technically this is the Archimedean property of the real numbers; see Appendix 1.) Now look at $\lfloor x \rfloor + (j/A)$ for positive integers j. Each of these is a rational number. Choose j to be the least positive integer such that this number is greater than x. Such a j exists by the well-ordering principle, since clearly if j is large enough, then $\lfloor x \rfloor + (j/A)$ exceeds x. (Note that $j = 0$ results in a value not greater than x.) So we have $r = \lfloor x \rfloor + (j/A) > x$ but $\lfloor x \rfloor + ((j-1)/A) = r - (1/A) \le x$. From this last inequality, substituting $y - x$ for $1/A$ (which only makes the left-hand side smaller) we have $r - (y - x) < x$, whence $r < y$, as desired.

42. The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \to P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \to P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction it follow that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus we have proved $\forall n Q(n)$ by ordinary induction.