GUIDE TO REVIEW QUESTIONS FOR CHAPTER 2

1. See p. 114. To prove that A is a subset of B we need to show that an arbitrarily chosen element x of A must also be an element of B.

2. The empty set is the set with no elements. It satisfies the definition of being a subset of every set vacuously.

3. **a)** See p. 116. **b)** See p. 122.

4. **a)** See p. 116. **b)** always **c)** 2^n

5. **a)** See pp. 121 and 123 and the preamble to Exercise 32 in Section 2.2.
b) union: integers that are odd or positive; intersection: odd positive integers; difference: even positive integers; symmetric difference: even positive integers together with odd negative integers

6. **a)** $A = B \equiv (A \subseteq B \wedge B \subseteq A) \equiv \forall x(x \in A \leftrightarrow x \in B)$ **b)** See pp. 124–126.
c) $A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C)$; use Venn diagrams

7. Underlying each set identity is a logical equivalence. See, for instance, Example 11 in Section 2.2.

8. a) See p. 134. b) $\mathbf{Z}$, $\mathbf{Z}$, $\mathbf{Z}^+ = \mathbf{N} - \{0\}$

9. a) See p. 136. b) See p. 137. c) $f(n) = n$ d) $f(n) = 2n$ e) $f(n) = \lceil n/2 \rceil$
f) $f(n) = 42548$

10. a) See p. 139: $f^{-1}(b) = a \equiv f(a) = b$
b) when it is one-to-one and onto
c) yes—itself

11. a) See p. 143. b) integers

12. Hint: subtract 5 from each term and look at the resulting sequence.

13. See p. 155.

14. Set up a one-to-one correspondence between the set of positive integers and the set of all odd integers, such as $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 3$, $4 \leftrightarrow -3$, $5 \leftrightarrow 5$, $6 \leftrightarrow -5$, and so on.

15. See Example 21 in Section 2.4.

SUPPLEMENTARY EXERCISES FOR CHAPTER 2

1. a) $\overline{A} =$ the set of words that are not in A
b) $A \cap B =$ the set of words that are in both A and B
c) $A - B =$ the set of words that are in A but not B
d) $\overline{A} \cap \overline{B} = \overline{(A \cup B)} =$ the set of words that are in neither A nor B
e) $A \oplus B =$ the set of words that are in A or B but not both (can also be written as $(A - B) \cup (B - A)$ or as $(A \cup B) - (A \cap B)$)

3. Yes. We must show that every element of A is also an element of B. So suppose a is an arbitrary element of A. Then $\{a\}$ is a subset of A, so it is an element of the power set of A. Since the power set of A is a subset of the power set of B, it follows that $\{a\}$ is an element of the power set of B, which means that $\{a\}$ is a subset of B. But this means that the element of $\{a\}$, namely a, is an element of B, as desired.

5. We will show that each side is a subset of the other. First suppose $x \in A - (A - B)$. Then $x \in A$ and $x \notin A - B$. Now the only way for x not to be in $A - B$, given that it is in A, is for it to be in B. Thus we have that x is in both A and B, so $x \in A \cap B$. For the other direction, let $x \in A \cap B$. Then $x \in A$ and $x \in B$. It follows that $x \notin A - B$, and so x is in $A - (A - B)$.

7. We need only provide a counterexample to show that $(A - B) - C$ is not necessarily equal to $A - (B - C)$. Let $A = C = \{1\}$, and let $B = \emptyset$. Then $(A - B) - C = (\{1\} - \emptyset) - \{1\} = \{1\} - \{1\} = \emptyset$, whereas $A - (B - C) = \{1\} - (\emptyset - \{1\}) = \{1\} - \emptyset = \{1\}$.

9. This is not necessarily true. For a counterexample, let $A = B = \{1, 2\}$, let $C = \emptyset$, and let $D = \{1\}$. Then $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, but $(A - C) - (B - D) = \{1, 2\} - \{2\} = \{1\}$.

11. a) Since $\emptyset \subseteq A \cap B \subseteq A \subseteq A \cup B \subseteq U$, we have the order $|\emptyset| \le |A \cap B| \le |A| \le |A \cup B| \le |U|$.
b) Note that $A - B \subseteq A \oplus B \subseteq A \cup B$. Also recall that $|A \cup B| = |A| + |B| - |A \cap B|$, so that $|A \cup B|$ is always less than or equal to $|A| + |B|$. Putting this all together, we have $|\emptyset| \le |A - B| \le |A \oplus B| \le |A \cup B| \le |A| + |B|$.

13. a) Yes, f is one-to-one, since each element of the domain $\{1,2,3,4\}$ is sent by f to a different element of the codomain. No, g is not one-to-one, since g sends the two different elements a and d of the domain to the same element, 2.

b) Yes, f is onto, since every element in the codomain $\{a,b,c,d\}$ is the image under f of some element in the domain $\{1,2,3,4\}$. In other words, the range of f is the entire codomain. No, g is not onto, since the element 4 in the codomain is not in the range of g (is not the image under g of any element of the domain $\{a,b,c,d\}$).

c) Certainly f has an inverse, since it is one-to-one and onto. Its inverse is the function from $\{a,b,c,d\}$ to $\{1,2,3,4\}$ that sends a to 3, sends b to 4, sends c to 2, and sends d to 1. (Each element in $\{a,b,c,d\}$ gets sent by f^{-1} to the element in $\{1,2,3,4\}$ that gets sent to it by f.) Since g is not one-to-one and onto, it has no inverse.

15. We need to look at an example in which f is not one-to-one. Suppose we let A be a set with two elements, say 1 and 2, and let B be a set with just one element, say 3. Of course f will be the unique function from A to B. If we let $S=\{1\}$ and $T=\{2\}$, then $f(S\cap T)=f(\emptyset)=\emptyset$, but $f(S)\cap f(T)=\{3\}\cap\{3\}=\{3\}$.

17. The key is to look at sets with just one element. On these sets, the induced functions act just like the original functions. So let x be an arbitrary element of A. Then $\{x\}\in P(A)$, and $S_f(\{x\})=\{\, f(y)\mid y\in\{x\}\,\}=\{f(x)\}$. By the same reasoning, $S_g(\{x\})=\{g(x)\}$. Since $S_f=S_g$, we can conclude that $\{f(x)\}=\{g(x)\}$, and so necessarily $f(x)=g(x)$.

19. This is certainly true if either x or y is an integer, since then this equation is equivalent to the identity (4a) in Table 1 of Section 2.3. Otherwise, write x and y in terms of their integer and fractional parts: $x=n+\epsilon$ and $y=m+\delta$, where $n=\lfloor x\rfloor$, $0<\epsilon<1$, $m=\lfloor y\rfloor$, and $0<\delta<1$. If $\delta+\epsilon<1$, then the equation is true, since both sides equal $m+n$; if $\delta+\epsilon\geq 1$, then the equation is false, since the left-hand side equals $m+n$, but the right-hand side equals $m+n+1$. In summary, the equation is true if and only if either at least one of x and y is an integer or the sum of the fractional parts of x and y is less than 1. (Note that the second condition in the disjunction subsumes the first.)

21. Write x and y in terms of their integer and fractional parts: $x=n+\epsilon$ and $y=m+\delta$, where $n=\lfloor x\rfloor$, $0\leq\epsilon<1$, $m=\lfloor y\rfloor$, and $0\leq\delta<1$. If $\delta=\epsilon=0$, then both sides equal $n+m$. If $\epsilon=0$ but $\delta>0$, then the left-hand side equals $n+m+1$, but the right-hand side equals $n+m$. If $\epsilon>0$, then the right-hand side equals $n+m+1$, so the two sides will be equal if and only if $\epsilon+\delta\leq 1$ (otherwise the left-hand side would be $n+m+2$). In summary, the equation is true if and only if either both x and y are integers, or x is not an integer but the sum of the fractional parts of x and y is less than or equal to 1.

23. If x is an integer, then clearly $\lfloor x\rfloor+\lfloor m-x\rfloor=x+m-x=m$. Otherwise, write x in terms of its integer and fractional parts: $x=n+\epsilon$, where $n=\lfloor x\rfloor$ and $0<\epsilon<1$. In this case $\lfloor x\rfloor+\lfloor m-x\rfloor=\lfloor n+\epsilon\rfloor+\lfloor m-n-\epsilon\rfloor=n+m-n-1=m-1$, because we had to round $m-n-\epsilon$ down to the next smaller integer.

25. Write $n=2k+1$ for some integer k. Then $n^2=4k^2+4k+1$, so $n^2/4=k^2+k+\frac{1}{4}$. Therefore $\lceil n^2/4\rceil=k^2+k+1$. But we also have $(n^2+3)/4=(4k^2+4k+1+3)/4=(4k^2+4k+4)/4=k^2+k+1$.

27. Let us write $x=n+(r/m)+\epsilon$, where n is an integer, r is an nonnegative integer less than m, and ϵ is a real number with $0\leq\epsilon<1/m$. In other words, we are peeling off the integer part of x (i.e., $n=\lfloor x\rfloor$) and the whole multiples of $1/m$ beyond that. Then the left-hand side is $\lfloor nm+r+m\epsilon\rfloor=nm+r$. On the right-hand side, the terms $\lfloor x\rfloor$ through $\lfloor x+(m-r-1)/m\rfloor$ are all just n, and the remaining terms, if any, from $\lfloor x+(m-r)/m\rfloor$ through $\lfloor x+(m-1)/m\rfloor$, are all $n+1$. Therefore the right-hand side is $(m-r)n+r(n+1)=nm+r$ as well.

29. This product telescopes. The numerator in the fraction for k cancels the denominator in the fraction for $k+1$. So all that remains of the product is the numerator for $k = 100$ and the denominator for $k = 1$, namely $101/1 = 101$.

31. There is no good way to determine a nice rule for this kind of problem. One just has to look at the sequence and see what seems to be happening. In this sequence, we notice that $10 = 2 \cdot 5$, $39 = 3 \cdot 13$, $172 = 4 \cdot 43$, and $885 = 5 \cdot 177$. We then also notice that $3 = 1 \cdot 3$ for the second and third terms. So each odd-indexed term (assuming that we call the first term a_1) comes from the term before it, by multiplying by successively larger integers. In symbols, this says that $a_{2n+1} = n \cdot a_{2n}$ for all $n > 0$. Then we notice that the even-indexed terms are obtained in a similar way by adding: $a_{2n} = n + a_{2n-1}$ for all $n > 0$. So the next four terms are $a_{13} = 6 \cdot 891 = 5346$, $a_{14} = 7 + 5346 = 5353$, $a_{15} = 7 \cdot 5353 = 37471$, and $a_{14} = 8 + 37471 = 37479$.

WRITING PROJECTS FOR CHAPTER 2

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. A classic source here is [Wil]. It gives a very readable account of many philosophical issues in the foundations of mathematics, including the topic for this essay.

2. Our list of references mentions several history of mathematics books, such as [Bo4] and [Ev3]. You should also browse the shelves in your library, around QA 21.

3. Go to the Encyclopedia's website, `http://www.research.att.com/~njas/sequences/`.

4. A Web search should turn up some useful references here, including an article in *Science News Online*. It gets its name from the fact that a graph describing it looks like the output of an electrocardiogram.

5. A Web search for this phrase will turn up much information.

6. A classic source here is [Wil]. It gives a very readable account of many philosophical issues in the foundations of mathematics, including the topic for this essay. Of course a Web search will turn up lots of useful material, as well.

CHAPTER 3
The Fundamentals: Algorithms, the Integers, and Matrices

SECTION 3.1 Algorithms

Many of the exercises here are actually miniature programming assignments. Since this is not a book on programming, we have glossed over some of the finer points. For example, there are (at least) two ways to pass variables to procedures—by value and by reference. In the former case the original values of the arguments are not changed. In the latter case they are. In most cases we will assume that arguments are passed by reference. None of these exercises are tricky; they just give the reader a chance to become familiar with algorithms written in pseudocode. The reader should refer to Appendix 3 for more details of the pseudocode being used here.

1. Initially max is set equal to the first element of the list, namely 1. The **for** loop then begins, with i set equal to 2. Immediately i (namely 2) is compared to n, which equals 10 for this sequence (the entire input is known to the computer, including the value of n). Since $2 < 10$, the statement in the loop is executed. This is an **if**...**then** statement, so first the comparison in the **if** part is made: max (which equals 1) is compared to $a_i = a_2 = 8$. Since the condition is true, namely $1 < 8$, the **then** part of the statement is executed, so max is assigned the value 8.

The only statement in the **for** loop has now been executed, so the loop variable i is incremented (from 2 to 3), and we repeat the process. First we check again to verify that i is still less than n (namely $3 < 10$), and then we execute the **if**...**then** statement in the body of the loop. This time, too, the condition is satisfied, since $max = 8$ is less than $a_3 = 12$. Therefore the assignment statement $max := a_i$ is executed, and max receives the value 12.

Next the loop variable is incremented again, so that now $i = 4$. After a comparison to determine that $4 < 10$, the **if**...**then** statement is executed. This time the condition fails, since $max = 12$ is not less than $a_4 = 9$. Therefore the **then** part of the statement is not executed. Having finished with this pass through the loop, we increment i again, to 5. This pass through the loop, as well as the next pass through, behave exactly as the previous pass, since the condition $max < a_i$ continues to fail. On the sixth pass through the loop, however, with $i = 7$, we find again that $max < a_i$, namely $12 < 14$. Therefore max is assigned the value 14.

After three more uneventful passes through the loop (with $i = 8$, 9, and 10), we finally increment i to 11. At this point, when the comparison of i with n is made, we find that i is no longer less than or equal to n, so no further passes through the loop are made. Instead, control passes beyond the loop. In this case there are no statements beyond the loop, so execution halts. Note that when execution halts, max has the value 14 (which is the correct maximum of the list), and i has the value 11. (Actually in many programming languages, the value of i after the loop has terminated in this way is undefined.)

3. We will call the procedure sum. Its input is a list of integers, just as was the case for Algorithm 1. Indeed, we can just mimic the structure of Algorithm 1. We assume that the list is not empty (an assumption made in Algorithm 1 as well).