SECTION 7.6 Applications of Inclusion–Exclusion

2. $1000 - 450 - 622 - 30 + 111 + 14 + 18 - 9 = 32$

4. $C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) = 20$

6. Square-free numbers are those not divisible by the square of a prime. We count them as follows: $99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61$.

8. $5^7 - C(5,1)4^7 + C(5,2)3^7 - C(5,3)2^7 + C(5,4)1^7 = 16{,}800$

10. This problem is asking for the number of onto functions from a set with 8 elements (the balls) to a set with 3 elements (the urns). Therefore the answer is $3^8 - C(3,1)2^8 + C(3,2)1^8 = 5796$.

12. 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

14. We use Theorem 2 with $n = 10$, which gives us

$$\frac{D_{10}}{10!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{10!} = \frac{1334961}{3628800} = \frac{16481}{44800} \approx 0.3678794643\,,$$

which is almost exactly $e^{-1} \approx 0.3678794412\ldots$.

16. There are $n!$ ways to make the first assignment. We can think of this first seating as assigning student n to a chair we will label n. Then the next seating must be a derangement with respect to this numbering, so there are D_n second seatings possible. Therefore the answer is $n!D_n$.

18. In a derangement of the numbers from 1 to n, the number 1 cannot go first, so let $k \neq 1$ be the number that goes first. There are $n-1$ choices for k. Now there are two ways to get a derangement with k first. One way is to have 1 in the k^{th} position. If we do this, then there are exactly D_{n-2} ways to derange the rest of the numbers. On the other hand, if 1 does not go into the k^{th} position, then think of the number 1 as being temporarily relabeled k. A derangement is completed in this case by finding a derangement of the numbers 2 through n in positions 2 through n, so there are D_{n-1} of them. Combining all this, by the product rule and the sum rule, we obtain the desired recurrence relation. The initial conditions are $D_0 = 1$ and $D_1 = 0$.

20. We apply iteration to the formula $D_n = nD_{n-1} + (-1)^n$, obtaining

$$\begin{aligned}
D_n &= n\big((n-1)D_{n-2} + (-1)^{n-1}\big) + (-1)^n \\
&= n(n-1)D_{n-2} + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\big((n-2)D_{n-3} + (-1)^{n-2}\big) + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)(n-2)D_{n-3} + n(n-1)(-1)^{n-2} + n(-1)^{n-1} + (-1)^n \\
&\ \ \vdots \\
&= n(n-1)\cdots 2D_1 + n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n\,,
\end{aligned}$$

which yields the formula in Theorem 2 after factoring out $n!$.

22. The numbers not relatively prime to pq are the ones that have p and/or q as a factor. Thus we have

$$\phi(pq) = pq - \frac{pq}{p} - \frac{pq}{q} + \frac{pq}{pq} = pq - q - p + 1 = (p-1)(q-1)\,.$$

24. The left-hand side of course counts the number of permutations of the set of integers from 1 to n. The right-hand side counts it, too, by a two-step process: first decide how many and which elements are to be fixed (this can be done in $C(n,k)$ ways, for each of $k = 0, 1, \ldots, n$), and in each case derange the remaining elements (which can be done in D_{n-k} ways).

26. This permutation starts with $4, 5, 6$ in some order ($3! = 6$ ways to choose this), followed by $1, 2, 3$ in some order ($3! = 6$ ways to decide this). Therefore the answer is $6 \cdot 6 = 36$.

34. We will count the number of bit strings that do contain four consecutive 1's. Bits 1 through 4 could be 1's, or bits 2 through 5, or bits 3 through 6, and in each case there are 4 strings meeting those conditions (since the other two bits are free). This gives a total of 12. However we overcounted, since there are ways in which more than one of these can happen. There are 2 strings in which bits 1 through 4 and bits 2 through 5 are 1's, 2 strings in which bits 2 through 5 and bits 3 through 6 are 1's, and 1 string in which bits 1 through 4 and bits 3 through 6 are 1's. Finally, there is 1 string in which all three substrings are 1's. Thus the number of bit strings with 4 consecutive 1's is $12 - 2 - 2 - 1 + 1 = 8$. Therefore the answer to the exercise is $2^6 - 8 = 56$.