SECTION 1.2 Propositional Equivalences

The solutions to Exercises 1–10 are routine; we use truth tables to show that a proposition is a tautology or that two propositions are equivalent. The reader should do more than this, however; think about what the equivalence is saying. See Exercise 11 for this approach. Some important topics not covered in the text are introduced in this exercise set, including the notion of the **dual** *of a proposition,* **disjunctive normal form** *for propositions,* **functional completeness**, **satisfiability**, *and two other logical connectives,* NAND *and* NOR. *Much of this material foreshadows the study of Boolean algebra in Chapter 11.*

1. First we construct the following truth tables, for the propositions we are asked to deal with.

p	$p \wedge \mathbf{T}$	$p \vee \mathbf{F}$	$p \wedge \mathbf{F}$	$p \vee \mathbf{T}$	$p \vee p$	$p \wedge p$
T	T	T	F	T	T	T
F	F	F	F	T	F	F

The first equivalence, $p \wedge \mathbf{T} \equiv p$, is valid because the second column $p \wedge \mathbf{T}$ is identical to the first column p. Similarly, part **(b)** comes from looking at columns three and one. Since column four is a column of F's, and column five is a column of T's, part **(c)** and part **(d)** hold. Finally, the last two parts follow from the fact that the last two columns are identical to the first column.

3. We construct the followiug truth tables.

p	q	$p \vee q$	$q \vee p$	$p \wedge q$	$q \wedge p$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

Part **(a)** follows from the fact that the third and fourth columns are identical; part **(b)** follows from the fact that the fifth and sixth columns are identical.

5. We construct the following truth table and note that the fifth and eighth columns are identical.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

7. De Morgan's laws tell us that to negate a conjunction we form the disjunction of the negations, and to negate a disjunction we form the conjunction of the negations.

a) This is the conjunction "Jan is rich, and Jan is happy." So the negation is "Jan is not rich, or Jan is not happy."

b) This is the disjunction "Carlos will bicycle tomorrow, or Carlos will run tomorrow." So the negation is "Carlos will not bicycle tomorrow, and Carlos will not run tomorrow." We could also render this as "Carlos will neither bicycle nor run tomorrow."

c) This is the disjunction "Mei walks to class, or Mei takes the bus to class." So the negation is "Mei does not walk to class, and Mei does not take the bus to class." (Maybe she gets a ride with a friend.) We could also render this as "Mei neither walks nor takes the bus to class."

d) This is the conjunction "Ibrahim is smart, and Ibrahim is hard working." So the negation is "Ibrahim is not smart, or Ibrahim is not hard working."

9. We construct a truth table for each conditional statement and note that the relevant column contains only T's. For parts **(a)** and **(b)** we have the following table (column four for part **(a)**, column six for part **(b)**).

p	q	$p \wedge q$	$(p \wedge q) \to p$	$p \vee q$	$p \to (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

For parts **(c)** and **(d)** we have the following table (columns five and seven, respectively).

p	q	$\neg p$	$p \to q$	$\neg p \to (p \to q)$	$p \wedge q$	$(p \wedge q) \to (p \to q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	T

For parts **(e)** and **(f)** we have the following table (this time we have not explicitly shown the columns for negation). Column five shows the answer for part **(e)**, and column seven shows the answer for part **(f)**.

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	F	T	T	T

11. Here is one approach: Recall that the only way a conditional statement can be false is for the hypothesis to be true and the conclusion to be false; hence it is sufficient to show that the conclusion must be true whenever the hypothesis is true. An alternative approach that works for some of these tautologies is to use the equivalences given in this section and prove these "algebraically." We will demonstrate this second method in some of the solutions.

a) If the hypothesis is true, then by the definition of $\wedge$ we know that p is true. Hence the conclusion is also true. For an algebraic proof, we exhibit the following string of equivalences, each one following from one of the laws in this section: $(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p \equiv (\neg q \vee \neg p) \vee p \equiv \neg q \vee (\neg p \vee p) \equiv \neg q \vee \mathbf{T} \equiv \mathbf{T}$. The first logical equivalence is the first equivalence in Table 7 (with $p \wedge q$ playing the role of p, and p playing the role of q); the second is De Morgan's law; the third is the commutative law; the fourth is the associative law; the fifth is the negation law (with the commutative law); and the sixth is the domination law.

b) If the hypothesis p is true, then by the definition of $\vee$, the conclusion $p \vee q$ must also be true.

c) If the hypothesis is true, then p must be false; hence the conclusion $p \to q$ is true, since *its* hypothesis is false. Symbolically we have $\neg p \to (p \to q) \equiv \neg\neg p \vee (\neg p \vee q) \equiv p \vee (\neg p \vee q) \equiv (p \vee \neg p) \vee q \equiv \mathbf{T} \vee q \equiv \mathbf{T}$.

d) If the hypothesis is true, then by the definition of $\wedge$ we know that q must be true. This makes the conclusion $p \to q$ true, since *its* conclusion is true.

e) If the hypothesis is true, then $p \to q$ must be false. But this can happen only if p is true, which is precisely what we wanted to show.

f) If the hypothesis is true, then $p \to q$ must be false. But this can happen only if q is false, which is precisely what we wanted to show.

13. We first construct truth tables and verify that in each case the two propositions give identical columns. The fact that the fourth column is identical to the first column proves part **(a)**, and the fact that the sixth column is identical to the first column proves part **(b)**.

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F

Alternately, we can argue as follows.

a) If p is true, then $p \vee (p \wedge q)$ is true, since the first proposition in the disjunction is true. On the other hand, if p is false, then both parts of the disjunction are false. Hence $p \vee (p \wedge q)$ always has the same truth value as p does, so the two propositions are logically equivalent.

b) If p is false, then $p \wedge (p \vee q)$ is false, since the first proposition in the conjunction is false. On the other hand, if p is true, then both parts of the conjunction are true. Hence $p \wedge (p \vee q)$ always has the same truth value as p does, so the two propositions are logically equivalent.

15. We need to determine whether we can find an assignment of truth values to p and q to make this proposition false. Let us try to find one. The only way that a conditional statement can be false is for the hypothesis to be true and the conclusion to be false. Hence we must make $\neg p$ false, which means we must make p true. Furthermore, in order for the hypothesis to be true, we will need to make q false, so that the first part of the conjunction will be true. But now with p true and q false, the second part of the conjunction is false. Therefore the entire hypothesis is false, so this assignment will not yield a false conditional statement. Since we have argued that no assignment of truth values can make this proposition false, we have proved that this proposition is a tautology. (An alternative approach would be to construct a truth table and see that its final column had only T's in it.) This tautology is telling us that if we know that a conditional statement is true, and that its conclusion is false, then we can conclude that its antecedent is also false.

17. The proposition $\neg(p \leftrightarrow q)$ is true when p and q do not have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). These are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are true in exactly the same instances, and therefore are logically equivalent.

19. The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and q have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). By the same reasoning, these are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are true in exactly the same instances, and therefore are logically equivalent.

21. This is essentially the same as Exercise 17. The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false. Since $p \leftrightarrow q$ is true when p and q have the same truth value, it is false when p and q have different truth values (either p is true and q is false, or vice versa). These are precisely the cases in which $\neg p \leftrightarrow q$ is true.

23. We'll determine exactly which rows of the truth table will have F as their entries. In order for $(p \rightarrow r) \wedge (q \rightarrow r)$ to be false, we must have at least one of the two conditional statements false, which happens exactly when r is false and at least one of p and q is true. But these are precisely the cases in which $p \vee q$ is true and r is false, which is precisely when $(p \vee q) \rightarrow r$ is false. Since the two propositions are false in exactly the same situations, they are logically equivalent.

25. We'll determine exactly which rows of the truth table will have F as their entries. In order for $(p \rightarrow r) \vee (q \rightarrow r)$ to be false, we must have both of the two conditional statements false, which happens exactly when r is false and both p and q are true. But this is precisely the case in which $p \wedge q$ is true and r is false, which is precisely when $(p \wedge q) \rightarrow r$ is false. Since the two propositions are false in exactly the same situations, they are logically equivalent.

27. This fact was observed in Section 1.1 when the biconditional was first defined. Each of these is true precisely when p and q have the same truth values.

29. We will show that if $p \to q$ and $q \to r$ are both true, then $p \to r$ is true. Thus we want to show that if p is true, then so is r. Given that p and $p \to q$ are both true, we conclude that q is true; from that and $q \to r$ we conclude that r is true, as desired. This can also be done with a truth table.

31. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, and r for which the truth values of $(p \to q) \to r$ and $p \to (q \to r)$ differ. One such assignment is F for all three. Then $(p \to q) \to r$ is false and $p \to (q \to r)$ is true.

33. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, r, and s for which the truth values of $(p \to q) \to (r \to s)$ and $(p \to r) \to (q \to s)$ differ. Let us try to make the first one false. That means we have to make $r \to s$ false, so we want r to be true and s to be false. If we let p and q be false, then each of the other three simple conditional statements ($p \to q$, $p \to r$, and $q \to s$) will be true. Then $(p \to q) \to (r \to s)$ will be $\mathrm{T} \to \mathrm{F}$, which is false; but $(p \to r) \to (q \to s)$ will be $\mathrm{T} \to \mathrm{T}$, which is true.

35. We apply the rules stated in the preamble.
a) $p \vee \neg q \vee \neg r$ b) $(p \vee q \vee r) \wedge s$ c) $(p \wedge \mathbf{T}) \vee (q \wedge \mathbf{F})$

37. If we apply the operation for forming the dual twice to a proposition, then every symbol returns to what it originally was. The $\wedge$ changes to the $\vee$, then changes back to the $\wedge$. Similarly the $\vee$ changes to the $\wedge$, then back to the $\vee$. The same thing happens with the $\mathbf{T}$ and the $\mathbf{F}$. Thus the dual of the dual of a proposition s, namely $(s^*)^*$, is equal to the original proposition s.

39. Let p and q be two compound propositions involving only the operators $\wedge$, $\vee$, and $\neg$; we can also allow them to involve the constants $\mathbf{T}$ and $\mathbf{F}$. We want to show that if p and q are logically equivalent, then p^* and q^* are logically equivalent. The trick is to look at $\neg p$ and $\neg q$. They are certainly logically equivalent if p and q are. Now if p is a conjunction, say $r \wedge s$, then $\neg p$ is logically equivalent, by De Morgan's law, to $\neg r \vee \neg s$; a similar statement applies if p is a disjunction. If r and/or s are themselves compound propositions, then we apply De Morgan's laws again to "push" the negation symbol $\neg$ deeper inside the formula, changing $\wedge$ to $\vee$ and $\vee$ to $\wedge$. We repeat this process until all the negation signs have been "pushed in" as far as possible and are now attached to the atomic (i.e., not compound) propositions in the compound propositions p and q. Call these atomic propositions p_1, p_2, etc. Now in this process De Morgan's laws have forced us to change each $\wedge$ to $\vee$ and each $\vee$ to $\wedge$. Furthermore, if there are any constants $\mathbf{T}$ or $\mathbf{F}$ in the propositions, then they will be changed to their opposite when the negation operation is applied: $\neg\mathbf{T}$ is the same as $\mathbf{F}$, and $\neg\mathbf{F}$ is the same as $\mathbf{T}$. In summary, $\neg p$ and $\neg q$ look just like p^* and q^*, except that each atomic proposition p_i within them is replaced by its negation. Now we agreed that $\neg p \equiv \neg q$; this means that for *every* possible assignment of truth values to the atomic propositions p_1, p_2, etc., the truth values of $\neg p$ and $\neg q$ are the same. But assigning T to p_i is the same as assigning F to $\neg p_i$, and assigning F to p_i is the same as assigning T to $\neg p_i$. Thus, for every possible assignment of truth values to the atomic propositions, the truth values of p^* and q^* are the same. This is precisely what we wanted to prove.

41. There are three ways in which exactly two of p, q, and r can be true. We write down these three possibilities as conjunctions and join them by $\vee$ to obtain the answer: $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$. See Exercise 42 for a more general result.

43. Given a compound proposition p, we can construct its truth table and then, by Exercise 42, write down a proposition q in disjunctive normal form that is logically equivalent to p. Since q involves only $\neg$, $\wedge$, and $\vee$, this shows that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators.

45. Given a compound proposition p, we can, by Exercise 43, write down a proposition q that is logically equivalent to p and uses only $\neg$, $\wedge$, and $\vee$. Now by De Morgan's law we can get rid of all the $\wedge$'s by replacing each occurrence of $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ with the equivalent proposition $\neg(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$.

47. The proposition $\neg(p \wedge q)$ is true when either p or q, or both, are false, and is false when both p and q are true; since this was the definition of $p \mid q$, the two are logically equivalent.

49. The proposition $\neg(p \vee q)$ is true when both p and q are false, and is false otherwise; since this was the definition of $p \downarrow q$, the two are logically equivalent.

51. A straightforward approach, using the results of Exercise 50, parts **(a)** and **(b)**, is as follows: $(p \to q) \equiv (\neg p \vee q) \equiv ((p \downarrow p) \vee q) \equiv (((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q))$. If we allow the constant **F** in our expression, then a simpler answer is $\mathbf{F} \downarrow ((\mathbf{F} \downarrow p) \downarrow q)$.

53. This is clear from the definition, in which p and q play a symmetric role.

55. A truth table for a compound proposition involving p and q has four lines, one for each of the following combinations of truth values for p and q: TT, TF, FT, and FF. Now each line of the truth table for the compound proposition can be either T or F. Thus there are two possibilities for the first line; for each of those there are two possibilities for the second line, giving $2 \cdot 2 = 4$ possibilities for the first two lines; for each of those there are two possibilities for the third line, giving $4 \cdot 2 = 8$ possibilities for the first three lines; and finally for each of those, there are two possibilities for the fourth line, giving $8 \cdot 2 = 16$ possibilities altogether. This sort of counting will be studied extensively in Chapter 5.

57. Let do, mc, and in stand for the propositions "The directory database is opened," "The monitor is put in a closed state," and "The system is in its initial state," respectively. Then the given statement reads $\neg in \to (do \to mc)$. By the third line of Table 7 (twice), this is equivalent to $in \vee (\neg do \vee mc)$. In words, this says that it must always be true that either the system is in its initial state, or the data base is not opened, or the monitor is put in a closed state. Another way to render this would be to say that if the database is open, then either the system is in its initial state or the monitor is put in a closed state.

59. Disjunctions are easy to make true, since we just have to make sure that at least one of the things being "or-ed" is true. In this problem, we notice that $\neg p$ occurs in four of the disjunctions, so we can satisfy all of them by making p false. Three of the remaining disjunctions contain r, so if we let r be true, those will be taken care of. That leaves only $p \vee \neg q \vee s$ and $q \vee \neg r \vee \neg s$, and we can satisfy both of those by making q and s both true. This assignment, then, makes all nine of the disjunctions true.

61. A compound proposition c is a tautology if every assignment of truth values to its variables makes c true. That means that every assignment of truth values to its variables makes $\neg c$ false, in other words, that $\neg c$ is not satisfiable. If we had an algorithm to determine whether or not a compound proposition is satisfiable, then we could apply that algorithm to $\neg c$ to determine whether c is a tautology. If the algorithm says that $\neg c$ is satisfiable, then we report that c is not a tautology, and if the algorithm says that $\neg c$ is not satisfiable, then we report that c is a tautology.