CHAPTER 2
Basic Structures: Sets, Functions, Sequences, and Sums

SECTION 2.1 Sets

This exercise set (note that this is a "set" in the mathematical sense!) reinforces the concepts introduced in this section—set description, subset and containment, cardinality, power set, and Cartesian product. A few of the exercises (mostly some of the even-numbered ones) are a bit subtle. Keep in mind the distinction between "is an element of" and "is a subset of." Similarly, there is a big difference between $\emptyset$ and $\{\emptyset\}$. In dealing with sets, as in most of mathematics, it is extremely important to say exactly what you mean.

1. **a)** $\{1,-1\}$ **b)** $\{1,2,3,4,5,6,7,8,9,10,11\}$
c) $\{0,1,4,9,16,25,36,49,64,81\}$ **d)** $\emptyset$ ($\sqrt{2}$ is not an integer)

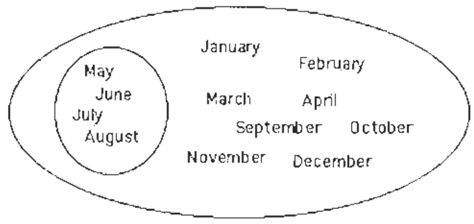
3. **a)** Yes; order and repetition do not matter.
b) No; the first set has one element, and the second has two elements.
c) No; the first set has no elements, and the second has one element (namely the empty set).

5. **a)** Since 2 is an integer greater than 1, 2 is an element of this set.
b) Since 2 is not a perfect square ($1^2 < 2$, but $n^2 > 2$ for $n > 1$), 2 is not an element of this set.
c) This set has two elements, and as we can clearly see, one of those elements is 2.
d) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.
e) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.
f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note that $\{2\}$ is not an element either, since $\{2\} \neq \{\{2\}\}$.

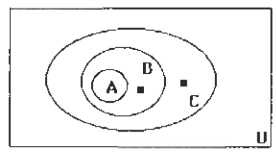
7. **a)** This is false, since the empty set has no elements.
b) This is false. The set on the right has only one element, namely the number 0, not the empty set.
c) This is false. In fact, the empty set has *no* proper subsets.
d) This is true. Every element of the set on the left is, vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not in the set on the left.
e) This is false. The set on the right has only one element, namely the number 0, not the set containing the number 0.
f) This is false. For one set to be a proper subset of another, the two sets cannot be equal.
g) This is true. Every set is a subset of itself.

9. **a)** T (in fact x is the only element) **b)** T (every set is a subset of itself)
c) F (the only element of $\{x\}$ is a letter, not a set) **d)** T (in fact, $\{x\}$ is the only element)
e) T (the empty set is a subset of every set) **f)** F (the only element of $\{x\}$ is a letter, not a set)

11. The four months whose names don't contain the letter R form a subset of the set of twelve months, as shown here.



13. We put the subsets inside the supersets. We also put dots in certain regions to indicate that those regions are not empty (required by the fact that these are proper subset relations). Thus the answer is as shown.



15. We need to show that every element of A is also an element of C. Let $x \in A$. Then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$, as we wished to show.

17. The cardinality of a set is the number of elements it has. The number of elements in its elements is irrelevant.
a) 1 **b)** 1 **c)** 2 **d)** 3

19. **a)** $\{\emptyset, \{a\}\}$ **b)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **c)** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

21. **a)** Since the set we are working with has 3 elements, the power set has $2^3 = 8$ elements.
b) Since the set we are working with has 4 elements, the power set has $2^4 = 16$ elements.
c) The power set of the empty set has $2^0 = 1$ element. The power set of this set therefore has $2^1 = 2$ elements. In particular, it is $\{\emptyset, \{\emptyset\}\}$. (See Example 14.)

23. In each case we need to list all the ordered pairs, and there are $4 \times 2 = 8$ of them.
a) $\{(a, y), (a, z), (b, y), (b, z), (c, y), (c, z), (d, y), (d, z)\}$
b) $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$

25. This is the set of triples (a, b, c), where a is an airline and b and c are cities. For example, (TWA, Rochester Hills Michigan, Middletown New Jersey) is an element of this Cartesian product. A useful subset of this set is the set of triples (a, b, c) for which a flies between b and c. For example, (Northwest, Detroit, New York) is in this subset, but the triple mentioned earlier is not.

27. By definition, $\emptyset \times A$ consists of all pairs (x, a) such that $x \in \emptyset$ and $a \in A$. Since there are no elements $x \in \emptyset$, there are no such pairs, so $\emptyset \times A = \emptyset$. Similar reasoning shows that $A \times \emptyset = \emptyset$.

29. The Cartesian product $A \times B$ has mn elements. (This problem foreshadows the general discussion of counting in Chapter 5.) To see that this answer is correct, note that for each $a \in A$ there are n different elements $b \in B$ with which to form the pair (a, b). Since there are m different elements of A, each leading to n different pairs, there must be mn pairs altogether.

31. The only difference between $A \times B \times C$ and $(A \times B) \times C$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 10, the elements of $A \times B \times C$ consist of 3-tuples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. By Definition 9, the elements of $(A \times B) \times C$ consist of ordered pairs (p, c), where $p \in A \times B$ and $c \in C$, so the typical element of $(A \times B) \times C$ looks like $((a,b),c)$. A 3-tuple is a different creature from a 2-tuple, even if the 3-tuple and the 2-tuple in this case convey exactly the same information. To be more precise, there is a natural one-to-one correspondence between $A \times B \times C$ and $(A \times B) \times C$ given by $(a,b,c) \leftrightarrow ((a,b),c)$.

33. **a)** Every real number has its square not equal to -1. Alternatively, the square of a real number is never -1. This is true, since squares of real numbers are always nonnegative.

b) There exists an integer whose square is 2. This is false, since the only two numbers whose squares are 2, namely $\sqrt{2}$ and $-\sqrt{2}$, are not integers.

c) The square of every integer is positive. This is almost true, but not quite, since $0^2 \not> 0$.

d) There is a real number equal to its own square. This is true, since $x = 1$ (as well as $x = 0$) fits the bill.

35. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.

a) The only integers whose squares are less than 3 are the integers whose absolute values are less than 2. So the truth set is $\{x \in \mathbf{Z} \mid x^2 < 3\} = \{-1, 0, 1\}$.

b) All negative integers satisfy this inequality, as do all nonnegative integers other than 0 and 1. So the truth set is $\{x \in \mathbf{Z} \mid x^2 > x\} = \mathbf{Z} - \{0,1\} = \{\ldots, -2, -1, 2, 3, 4, \ldots\}$.

c) The only real number satisfying this equation is $x = -1/2$. Because this value is not in our domain, the truth set is empty: $\{x \in \mathbf{Z} \mid 2x + 1 = 0\} = \emptyset$.

37. First we prove the statement mentioned in the hint. The "if" part is immediate from the definition of equality. The "only if" part is rather subtle. We want to show that if $\big\{\{a\}, \{a,b\}\big\} = \big\{\{c\}, \{c,d\}\big\}$, then $a = c$ and $b = d$. First consider the case in which $a \neq b$. Then $\big\{\{a\}, \{a,b\}\big\}$ has exactly two elements, both of which are sets; exactly one of them contains one element, and exactly one of them contains two elements. Thus $\big\{\{c\}, \{c,d\}\big\}$ must have the same property; hence c cannot equal d, and so $\{c\}$ is the element containing one element. Hence $\{a\} = \{c\}$, and so $a = c$. Also in this case the two-element elements $\{a,b\}$ and $\{c,d\}$ must be equal, and since $b \neq a = c$, we must have $b = d$. The other possibility is that $a = b$. Then $\big\{\{a\}, \{a,b\}\big\} = \big\{\{a\}\big\}$, a set with one element. Hence $\big\{\{c\}, \{c,d\}\big\}$ must also have only one element, which can only happen when $c = d$ and the set is $\big\{\{c\}\big\}$. It then follows that $a = c$, and hence $b = d$, as well.

Now there is really nothing else to prove. The property that we want ordered pairs to have is precisely the one that we just proved is satisfied by this definition. Furthermore, if we look at the proof, then it is clear how to "recover" both a and b from $\big\{\{a\}, \{a,b\}\big\}$. If this set has two elements, then a is the unique element in the one-element element of this set, and b is the unique member of the two-element element of this set other than a. If this set has only one element, then a and b are both equal to the unique element of the unique element of this set.

39. We can do this recursively, using the idea from Section 4.4 of reducing a problem to a smaller instance of the same problem. Suppose that the elements of the set in question are listed: $A = \{a_1, a_2, a_3, \ldots, a_n\}$. First we will write down all the subsets that do not involve a_n. This is just the same problem we are talking about all over again, but with a smaller set—one with just $n - 1$ elements. We do this by the process we are currently describing. Then we write these same subsets down again, but this time adjoin a_n to each one. Each subset of A will have been written down, then—first all those that do not include a_n, and then all those that do.

For example, using this procedure the subsets of $\{p, d, q\}$ would be listed in the order $\emptyset$, $\{p\}$, $\{d\}$, $\{p, d\}$, $\{q\}$, $\{p, q\}$, $\{d, q\}$, $\{p, d, q\}$.