SUPPLEMENTARY EXERCISES FOR CHAPTER 8

2. In each case we will construct a simplest such relation.

a) $\{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b),(d,d)\}$ b) $\emptyset$ c) $\{(a,b),(b,c)\}$

d) $\{(a,a),(b,b),(c,c),(a,b),(b,a),(c,a),(c,b),(d,d)\}$ e) $\{(a,b),(b,a),(c,c),(c,a)\}$

4. Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a,b) \in R_1$ and $(b,a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

6. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \oplus R_2$ for all $a \in A$.

8. Under this hypothesis, $\overline{R}$ must also be symmetric, for if $(a,b) \in \overline{R}$, then $(a,b) \notin R$, whence (b,a) cannot be in R, either (by the symmetry of R); in other words, (b,a) is also in $\overline{R}$.

10. First suppose that R is reflexive and circular. We need to show that R is symmetric and transitive. Let $(a,b) \in R$. Since also $(b,b) \in R$, it follows by circularity that $(b,a) \in R$; this proves symmetry. Now if $(a,b) \in R$ and $(b,c) \in R$, then by circularity $(c,a) \in R$ and so by symmetry $(a,c) \in R$; thus R is transitive. Conversely, transitivity and symmetry immediately imply circularity, so every equivalence relation is reflexive and circular.

12. A primary key in the first relation need not be a primary key in the join. Let the first relation contain the pairs (John, boy) and (Mary, girl); and let the second relation contain the pairs (boy, vain), (girl, athletic), and (girl, smart). Clearly *Name* is a primary key for the first relation. If we take the join on the *Sex* column, then we obtain the relation containing the pairs (John, boy, vain), (Mary, girl, athletic), and (Mary, girl, smart); in this relation *Name* is not a primary key.

14. a) Two mathematicians are related under R^2 if and only if each has written a joint paper with some mathematician c.

b) Two mathematicians are related under R^* if there is a finite sequence of mathematicians $a = c_0$, c_1, c_2, $\ldots$, c_{m-1}, $c_m = b$, with $m \geq 1$, such that for each i from 1 to m, mathematician c_i has written a joint paper with mathematician c_{i-1}.

c) The Erdős number of a is the length of a shortest path in R from a to Erdős, if such a path exists. (Some mathematicians have no Erdős number.)

16. We assume that the notion of calling is a potential one—subroutine **P** is related to subroutine **Q** if it might be possible for **P** to call **Q** during its execution (in other words, there is a call to **Q** as one of the steps in the subroutine **P**). Otherwise this exercise would not be well-defined, since *actual* calls are unpredictable—they depend on what actually happens as the programs execute.

a) Let **P** and **Q** be subroutines. Then **P** is related to **Q** under the transitive closure of R if and only if at some time during an active invocation of **P** it might be possible for **Q** to be called.

b) Routines such as this are usually called recursive—it might be possible for **P** to be called again while it is still active.

c) The reflexive closure of the transitive closure of any relation is just the transitive closure (see part (a)) with all the loops adjoined.

18. We can prove this symbolically, since the symmetric closure of a relation is the union of the relation and its inverse. Thus we have $(R \cup S) \cup (R \cup S)^{-1} = R \cup S \cup R^{-1} \cup S^{-1} = (R \cup R^{-1}) \cup (S \cup S^{-1})$.

20. a) This is an equivalence relation by Exercise 9 in Section 8.5, letting $f(x)$ be the sign of the zodiac under which x was born.

b) This is an equivalence relation by Exercise 9 in Section 8.5, letting $f(x)$ be the year in which x was born.

c) This is not an equivalence relation (it is not transitive).

22. This relation is reflexive, since $x - x = 0 \in \mathbf{Q}$. To see that it is symmetric, suppose that $x - y \in \mathbf{Q}$. Then $y - x = -(x - y)$ is again a rational number. For transitivity, if $x - y \in \mathbf{Q}$ and $y - z \in \mathbf{Q}$, then their sum, namely $x - z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of $1/2$ are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number; in other words it is $\{\, \pi + r \mid r \in \mathbf{Q} \,\}$.

24. Let S be the transitive closure of the symmetric closure of the reflexive closure of R. Then by Exercise 23 in Section 8.4, S is symmetric. Since it is also clearly transitive and reflexive, S is an equivalence relation. Furthermore, every element added to R to produce S was forced to be added in order to insure reflexivity, symmetry, or transitivity; therefore S is the smallest equivalence relation containing R.

26. This follows from the fact (Exercise 54 in Section 8.5) that two partitions are related under the refinement relation if and only if their corresponding equivalence relations are related under the $\subseteq$ relation, together with the fact that $\subseteq$ is a partial order on every collection of sets.

28. A subset of a chain is again a chain, so we list only the maximal chains.

a) $\{a, b, c\}$ and $\{a, b, d\}$ b) $\{a, b, e\}$, $\{a, b, d\}$, and $\{a, c, d\}$

c) In this case there are 9 maximal chains, each consisting of one element from the top row, the element in the middle, and one element in the bottom row.

30. The vertices are arranged in three columns. Each pair of vertices in the same column are clearly comparable. Therefore the largest antichain can have at most three elements. One such antichain is $\{a, b, c\}$.

32. This result is known as Dilworth's Theorem. For a proof, see, for instance, page 58 of *Graph Theory* by Béla Bollobás (Springer-Verlag, 1979).

34. Let x be a minimal element in S. Then the hypothesis $\forall y(y \prec x \to P(y))$ is vacuously true, so the conclusion $P(x)$ is true, which is what we wanted to show.

36. Reflexivity is the statement that f is $O(f)$. This is trivial, by taking $C = 1$ and $k = 1$ in the definition of the big-O relation. Transitivity was proved in Exercise 17 of Section 3.2.

38. It was proved in Exercise 37 that $R \cap R^{-1}$ is an equivalence relation whenever R is a quasi-ordering on a set A. Therefore it makes sense to speak of the equivalence classes of $R \cap R^{-1}$, and the relation S is well-defined from its syntax. To show that S is a partial order, we must show that it is reflexive, anti-symmetric, and transitive. For the first of these, we need to show that (C, C) belongs to S, which means that there are elements $c \in C$ and $d \in C$ such that (c, d) belongs to R. By the definition of equivalence class, C is not empty, so let c be any element of C, and let $d = c$. Then (c, c) belongs to R by the reflexivity of R. Next, for antisymmetry, suppose that (C, D) and (D, C) both belong to S; we must show that $C = D$. We have that (c, d) belongs to R for some $c \in C$ and $d \in D$; and we have that (d', c') belongs to R for some $d' \in D$ and $c' \in C$. If we show that (c, d) also belongs to R^{-1}, then we will know that c and d are in the same equivalence class of $R \cap R'$, and therefore that $C = D$. To do this, we need to show that (d, c) belongs to R. Since d and d' are in the same equivalence class, we know that (d, d') belongs to R; we already mentioned that (d', c') belongs to R; and since c' and c are in the same equivalence class, we know that (c', c) belongs to R. Applying the transitivity of R three times, we conclude that (d, c) belongs to R, as desired.

Finally, to show the transitivity of S, we must show that if (C, D) belongs to S and (D, E) belongs to S, then (C, E) belongs to S. The hypothesis tells us that (c, d) belongs to R for some $c \in C$ and $d \in D$, and that (d', e) belongs to R for some $d' \in D$ and $c \in E$. As in the previous paragraph, we know that (d, d') belongs to R. Therefore by the transitivity of R (thrice), (c, e) belongs to R, and our proof is complete.

40. This follows in essentially one step from part **(c)** of Exercise 39. Suppose that $x \vee y = y$. Then by the first absorption law, $x = x \wedge (x \vee y) = x \wedge y$. Conversely, if $x \wedge y = x$, then by the second absorption law (with the roles of x and y reversed), $y = y \vee (x \wedge y) = y \vee x$. (We are using the commutative law as well, of course.)

42. By Exercise 51 in Section 8.6, every finite lattice has a least element and a greatest element. These elements are the 0 and 1, respectively, discussed in the preamble to this exercise.

44. We learned in Example 24 of Section 8.6 that the meet and join in this lattice are $\cap$ and $\cup$. We know from Section 2.2 that these operations are distributive over each other. There is nothing more to prove.

46. Here is one example. The reader should draw the Hasse diagram to see it more vividly. The elements in the lattice are **0**, **1**, a, b, c, d, and e. The relations are that **0** precedes all other elements; all other elements precede **1**; b, d, and e precede c; and b precedes a. Then both d and e are complements of a, but b has no complement (since $b \vee x \neq 1$ unless $x = 1$).

48. This can be proved by playing around with the symbolism. Suppose that a and b are both complements of x. This means that $x \vee a = 1$, $x \wedge a = \mathbf{0}$, $x \vee b = 1$, and $x \wedge b = 0$. Now using the various identities in Exercises 39 and 41 and the preamble to Exercise 43, we have $a = a \wedge 1 = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b) = \mathbf{0} \vee (a \wedge b) = a \wedge b$. By the same argument, we can also show that $b = a \wedge b$. By transitivity of equality, it follows that $a = b$.

50. Actually all finite games have a winning strategy for one player or the other; one can see this by writing down the game tree and analyzing it from the bottom up, as shown in Section 10.2. What we can show in this case is that the player who goes first has a winning strategy. We give a proof by contradiction.

By the remark above, if the first player does not have a winning strategy, then the second player does. In particular, the second player has a winning response and strategy if the first player chooses b as her first move. Suppose that c is the first move of that winning strategy of the second player. But because $c \preceq b$, if the first player makes the move c at her first turn, then play can proceed exactly as if the first player had chosen b and then the second player had chosen c (because element b would be removed anyway when c is chosen). Thus the first player can win by adopting the strategy that the second player would have adopted. This is a contradiction, because it is impossible for both players to have a winning strategy. Therefore we can conclude that our assumption that the first player does not have a winning strategy is wrong, and therefore the first player does have a winning strategy.