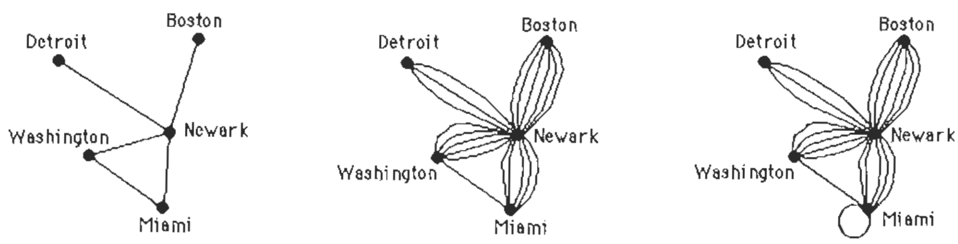
CHAPTER 9
Graphs

SECTION 9.1 Graphs and Graph Models

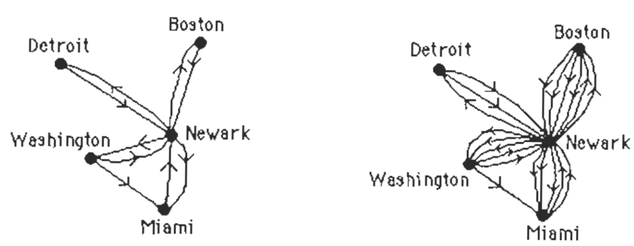
The examples and exercises give a good picture of the ways in which graphs can model various real world applications. In constructing graph models you need to determine what the vertices will represent, what the edges will represent, whether the edges will be directed or undirected, whether loops should be allowed, and whether a simple graph or multigraph is more appropriate.

1. In part **(a)** we have a simple graph, with undirected edges, no loops or multiple edges. In part **(b)** we have a multigraph, since there are multiple edges (making the figure somewhat less than ideal visually). In part **(c)** we have the same picture as in part **(b)** except that there is now a loop at one vertex; thus this is a pseudograph.



(a) (b) (c)

In part **(d)** we have a directed graph, the directions of the edges telling the directions of the flights; note that the **antiparallel edges** (pairs of the form (u, v) and (v, u)) are not parallel. In part **(e)** we have a directed multigraph, since there are parallel edges.



(d) (e)

3. This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

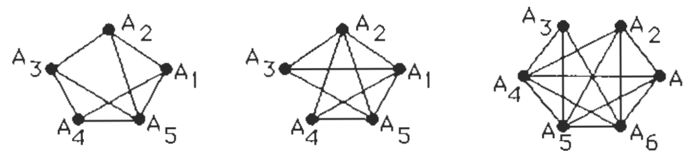
5. This is a pseudograph; the edges are undirected, but there are loops and parallel edges.

7. This is a directed graph; the edges are directed, but there are no parallel edges. (Loops and antiparallel edges—see the solution to Exercise 1d for a definition—are allowed in a directed graph.)

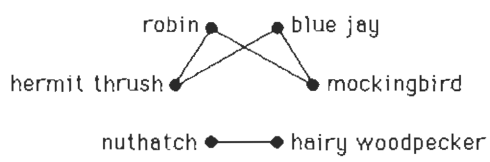
9. This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

11. In a simple graph, edges are undirected. To show that R is symmetric we must show that if uRv, then vRu. If uRv, then there is an edge associated with $\{u,v\}$. But $\{u,v\} = \{v,u\}$, so this edge is associated with $\{v,u\}$ and therefore vRu. A simple graph does not allow loops; that is if there is an edge associated with $\{u,v\}$, then $u \neq v$. Thus uRu never holds, and so by definition R is irreflexive.

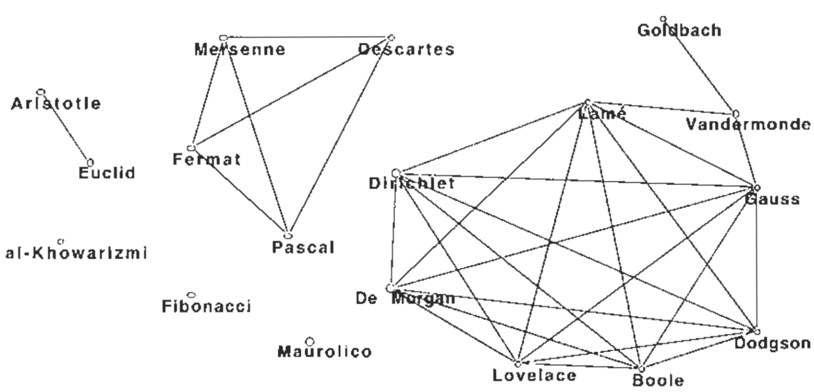
13. In each case we draw a picture of the graph in question. All are simple graphs. An edge is drawn between two vertices if the sets for the two vertices have at least one element in common. For example, in part **(a)** there is an edge between vertices A_1 and A_2 because there is at least one element common to A_1 and A_2 (in fact there are three such elements). There is no edge between A_1 and A_3 since $A_1 \cap A_3 = \emptyset$.



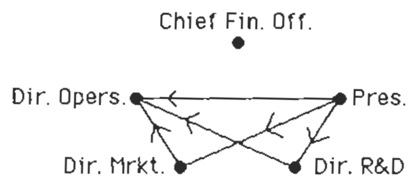
15. We draw a picture of the graph in question, which is a simple graph. Two vertices are joined by an edge if we are told that the species compete (such as robin and mockingbird) but there is no edge between pairs of species that are not given as competitors (such as robin and blue jay).



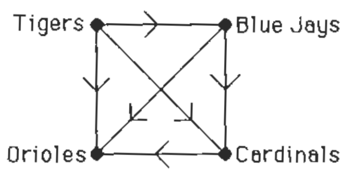
17. Here are the persons to be included, listed in order of birth year: Aristotle (384–322 B.C.E.), Euclid (325–265 B.C.E.), al-Khowarizmi (780–850), Fibonacci (1170–1250), Maurolico (1494–1575), Mersenne (1588–1648), Descartes (1596–1650), Fermat (1601–1665), Pascal (1623–1662), Goldbach (1690–1764), Vandermonde (1735–1796), Gauss (1777–1855), Lamé (1795–1870), Dirichlet (1805–1859), De Morgan (1806–1871), Lovelace (1815–1852), Boole (1815–1864), and Dodgson (1832–1898). We draw the graph by connecting two people if their date ranges overlap. Note that there is a complete subgraph (see Section 9.2) consisting of Mersenne, Descartes, Fermat, and Pascal, and a larger complete subgraph consisting of the last seven people listed. A few of the vertices are isolated (again see Section 9.2). In all our graph has 18 vertices and 31 edges. A graph like this is called an **interval graph**, since each vertex can be associated with an interval of real numbers; it is a special case of an **intersection graph**, where two vertices are adjacent if the sets associated with those vertices have a nonempty intersection (see Exercise 13).



19. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u can influence v. For instance the Chief Financial Officer is an isolated vertex since she is influenced by no one and influences no one.



21. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u beat v.



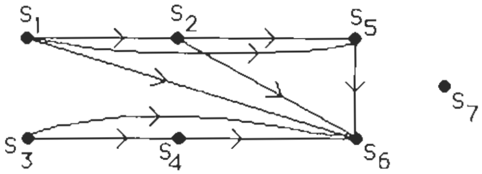
23. We could compile a list of phone numbers (the labels on the vertices) in the February call graph that were not present in January, and a list of the January numbers missing in February. For each number in each list, we could make a list of the numbers they called or were called by, using the edges in the call graphs. Then we could look for February lists that were very similar to January lists. If we found a new February number that had almost the same calling pattern as a defnnct January number, then we might suspect that these numbers belonged to the same person, who had recently changed his or her number.

25. For each e-mail address (the labels on the vertices), we could make a list of the other addresses they sent messages to or received messages from. If we see two addresses that had almost the same communication pattern, then we might suspect that these addresses belonged to the same person, who had recently changed his or her e-mail address.

27. The vertices represent the people at the party. Because it is possible that a knows b's name but not vice versa, we need a directed graph. We will include an edge associated with (u, v) if and only if u knows v's name. There is no need for multiple edges (either a knows b's name or he doesn't). One could argue that we should not clutter the model with loops, because obviously everyone knows her own name. On the other hand, it certainly would not be wrong to include loops, especially if we took the instructions literally.

29. For this to be interesting, we want the graph to model all marriages, not just ones that are currently active. (In the latter case, for the Western world, there would be at most one edge incident to each vertex.) So we let the set of vertices be a set of people (for example, all the people in North America who lived at any point in the 20th century), aud two vertices are joined by an edge if the two people were ever married. Since laws in the 20th century allowed only marriages between persons of the opposite sex, and ignoring complications caused by sex-change operations, we note that this graph has the property that there are two types of vertices (men and women), and every edge joins vertices of opposite types. In the next section we learn that the word used to describe a graph like this is *bipartite*.

31. We draw a picture of the directed graph in question. There is an edge from u to v if the assignment made in u can possibly influence the assignment made in v. For example, there is an edge from S_3 to S_6, since the assignment in S_3 changes the value of y, which then influences the value of z (in S_4) and hence has a bearing on S_6. We assume that the statements are to be executed in the given order, so, for example, we do not draw an edge from S_5 to S_2.



33. The vertices in the directed graph represent people in the group. We put a directed edge into our directed graph from every vertex A to every vertex $B \neq A$ (we do not need loops), and furthermore we label that edge with one of the three labels L, D, or N. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with labeled edges. It is defined to be a triple (V, E, f), where (V, E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and f is a function from E to the set $\{L, D, N\}$. Here we are simply thinking of $f(e)$ as the attitude of the person at the tail (initial vertex—see Section 9.2) of e toward the person at the head (terminal vertex) of e.