GUIDE TO REVIEW QUESTIONS FOR CHAPTER 1

1. a) See p. 3. b) This is not a boring course.

2. a) See pp. 4, 5, 6, and 9.
b) Disjunction: "I'll go to the movies tonight or I'll finish my discrete mathematics homework." Conjunction: "I'll go to the movies tonight and I'll finish my discrete mathematics homework." Exclusive or: "I'll go to the movies tonight or I'll finish my discrete mathematics homework, but not both." Conditional statement: "If I'll go to the movies tonight, then I'll finish my discrete mathematics homework." Biconditional: "I'll go to the movies tonight if and only if I'll finish my discrete mathematics homework."

3. a) See p. 6. b) See p. 8.
c) Converse: "If I go for a walk in the woods tomorrow, then it will be sunny." Contrapositive: "If I don't go for a walk in the woods tomorrow, then it will not be sunny."

4. a) See p. 22.
b) usiug truth tables; symbolically, using identities in Tables 6–8 in Section 1.2; by giving a valid argument about the possihle truth values of the propositional variables involved
c) Use the fact that $r \to \neg q \equiv \neg r \vee \neg q$, or use truth tables.

5. a) Each line of the truth table corresponds to exactly one combination of truth values for the n atomic propositions involved. We can write down a conjnnction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false. If we do this for *each* line of the trnth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form. See Exercise 42 in Section 1.2.
b) See Exercise 43 in Section 1.2.
c) See Exercises 50 and 52 in Section 1.2.

6. The negation of $\forall x P(x)$ is $\exists x \neg P(x)$, and the negation of $\exists x P(x)$ is $\forall x \neg P(x)$.

7. a) In the second, x cau depend on y. In the first, the same x must "work" for every y.
b) See Example 4 in Section 1.4.

8. See pp. 63–64. This is a valid argument because it uses the valid rule of inference called modns tollens.

9. This is a valid argument because it uses the universal modus ponens rule of inference. Therefore if the premises are true, the conclusion must be true.

10. **a)** See pp. 76, 77, and 80.
b) For a direct proof, the hypothesis implies that $n = 2k$ for some k, whence $n+4 = 2(k+2)$, so $n+4$ is even. For a proof by contraposition, suppose that $n+4$ is odd; hence $n+4 = 2k+1$ for some k. Then $n = 2(k-2)+1$, so n is odd, hence not even. For a proof by contradiction, assume that $n = 2k$ and $n+4 = 2l+1$ for some k and l. Subtracting gives $4 = 2(l-k)+1$, which means that 4 is odd, a contradiction.

11. **a)** See p. 82.
b) Suppose that $3n+2$ is odd, so that $3n+2 = 2k+1$ for some k. Multiply both sides by 3 and subtract 1, obtaining $9n+5 = 6k+2 = 2(3k+1)$. This shows that $9n+5$ is even. We prove the converse by contraposition. Suppose that $3n+2$ is not odd, i.e., that it is even. Then $3n+2 = 2k$ for some k. Multiply both sides by 3 and subtract 1, obtaining $9n+5 = 6k-1 = 2(3k-1)+1$. This shows that $9n+5$ is odd.

12. No—we could add to these $p_2 \to p_3$ and $p_1 \to p_4$, for example.

13. **a)** Find a counterexample, i.e., an object c such that $P(c)$ is false. **b)** $n = 1$ is a counterexample.

14. See p. 91.

15. See p. 92.

16. See Example 4 in Section 1.7.

SUPPLEMENTARY EXERCISES FOR CHAPTER 1

1. **a)** $q \to p$ (note that "only if" does not mean "if")
b) $q \wedge p$ **c)** $\neg q \vee \neg p$ (assuming inclusive use of the English word "or" is intended by the speaker)
d) $q \leftrightarrow p$ (this is another way to say "if and only if" in English words)

3. We could use truth tables, but we can also argue as follows.
a) Since q is false but the conditional statement $p \to q$ is true, we must conclude that p is also false.
b) The disjunction says that either p or q is true. Since p is given to be false, it follows that q must be true.

5. The inverse of $p \to q$ is $\neg p \to \neg q$. Therefore the converse of the inverse is $\neg q \to \neg p$. Note that this is the same as the contrapositive of the original statement. The converse of $p \to q$ is $q \to p$. Therefore the converse of the converse is $p \to q$, which was the original statement. The contrapositive of $p \to q$ is $\neg q \to \neg p$. Therefore the converse of the contrapositive is $\neg p \to \neg q$, which is the same as the inverse of the original statement.

7. The straightforward approach is to use disjunctive normal form. There are four cases in which exactly three of the variables are true. The desired proposition is $(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$.

9. Translating these statements into symbols, using the obvious letters, we have $\neg t \to \neg g$, $\neg g \to \neg q$, $r \to q$, and $\neg t \wedge r$. Assume the statements are consistent. The fourth statement tells us that $\neg t$ must be true. Therefore by modus ponens with the first statement, we know that $\neg g$ is true, hence (from the second statement) that $\neg q$ is true. Also, the fourth statement tells us that r must be true, and so again modus ponens (third statement) makes q true. This is a contraction: $q \wedge \neg q$. Thus the statements are inconsistent.

11. We are told that exactly one of these people committed the crime, and exactly one (the guilty party) is a knight. We look at the three cases to determine who the knight is. If Amy were the knight, then her protestations of innocence would be true, but that cannot be, since we know that the knight is guilty. If Claire were the knight, then her statement that Brenda is not a normal is true; and since Brenda cannot be the knight in this situation, Brenda must be a knave. That means that Brenda is lying when she says that Amy was telling the truth; therefore Amy is lying. This means that Amy is guilty, but that cannot be, since Amy isn't the knight. So Brenda must be the knight. Amy is an innocent normal who is telling the truth when she says she is innocent; Brenda is telling the truth when she says that Amy is telling the truth; and Claire is a normal who is telling the truth when she says that Brenda is not a normal. So Brenda committed the crime.

13. The definition of valid argument is an argument in which the truth of all the premises forces the truth of conclusion. In this example, the two premises can never be true simultaneously, because they are contradictory, irrespective of the true status of the tooth fairy. Therefore it is (vacuously) true that whenever both of the premises are true, the conclusion is also true (irrespective of your luck at finding gold at the end of the rainbow). Because the premises are not both true, we cannot conclude that the conclusion is true.

15. **a)** F, since 4 does not divide 5 **b)** T, since 2 divides 4
c) F, by the counterexample in part (**a**) **d)** T, since 1 divides every positive integer
e) F, since no number is a multiple of all positive integers (No matter what positive integer n one chooses, if we take $m = n + 1$, then $P(m, n)$ is false, since $n + 1$ does not divide n.)
f) T, since 1 divides every positive integer

17. The given statement tells us that there are exactly two elements in the domain. Therefore if we let the domain be anything with size other than 2 the statement will be false.

19. For each person we want to assert the existence of two different people who are that person's parents. The most elegant way to do so is $\forall x \exists y \exists z (y \neq z \land \forall w (P(w, x) \leftrightarrow (w = y \lor w = z)))$. Note that we are saying that w is a parent of x if and only if w is one of the two people whose existence we asserted.

21. To express the statement that exactly n members of the domain satisfy P, we need to use n existential quantifiers, express the fact that these n variables all satisfy P and are all different, and express the fact that every other member of the domain that satisfies P must be one of these.

a) This is a special case, however. To say that there are no values of x that make P true we can simply write $\neg \exists x P(x)$ or $\forall x \neg P(x)$.

b) This is the same as Exercise 53 in Section 1.4, because $\exists_1 x P(x)$ is the same as $\exists! x P(x)$. Thus we can write $\exists x (P(x) \land \forall y (P(y) \rightarrow y = x))$.

c) Following the discussion above, we write $\exists x_1 \exists x_2 (P(x_1) \land P(x_2) \land x_1 \neq x_2 \land \forall y (P(y) \rightarrow (y = x_1 \lor y = x_2)))$.

d) We expand the previous answer to one more variable: $\exists x_1 \exists x_2 \exists x_3 (P(x_1) \land P(x_2) \land P(x_3) \land x_1 \neq x_2 \land x_1 \neq x_3 \land x_2 \neq x_3 \land \forall y (P(y) \rightarrow (y = x_1 \lor y = x_2 \lor y = x_3)))$.

23. Suppose that $\exists x (P(x) \rightarrow Q(x))$ is true. Then for some x, either $Q(x)$ is true or $P(x)$ is false. If $Q(x)$ is true for some x, then the conditional statement $\forall x P(x) \rightarrow \exists x Q(x)$ is true (having true conclusion). If $P(x)$ is false for some x, then again the conditional statement $\forall x P(x) \rightarrow \exists x Q(x)$ is true (having false hypothesis). Conversely, suppose that $\exists x (P(x) \rightarrow Q(x))$ is false. That means that for every x, the conditional statement $P(x) \rightarrow Q(x)$ is false, or, in other words, $P(x)$ is true and $Q(x)$ is false. The latter statement implies that $\exists x Q(x)$ is false. Thus $\forall x P(x) \rightarrow \exists x Q(x)$ has a true hypothesis and a false conclusion and is therefore false.

25. No. For each x there may be just one y making $P(x,y)$ true, so that the second proposition will not be true. For example, let $P(x,y)$ be $x+y=0$, where the domain (universe of discourse) is the integers. Then the first proposition is true, since for each x there exists a y, namely $-x$, such that $P(x,y)$ holds. On the other hand, there is no one x such that $x+y=0$ for *every* y.

27. Let $T(s,c,d)$ be the statement that student s has taken class c in department d. Then, with the domains (universes of discourse) being the students in this class, the courses at this university, and the departments in the school of mathematical sciences, the given statement is $\forall s \forall d \exists c T(s,c,d)$.

29. Let $T(x,y)$ mean that stndent x has taken class y, where the domain is all students in this class. We want to say that there exists exactly one student for whom there exists exactly one class that this student has taken. So we can write simply $\exists! x \exists! y T(x,y)$. To do this without quantifiers, we need to expand the uniqueness quantifier using Exercise 52 in Section 1.4. Doing so, we have $\exists x \forall z((\exists y \forall w(T(z,w) \leftrightarrow w=y)) \leftrightarrow z=x)$.

31. By universal instantiation we have $P(a) \rightarrow Q(a)$ and $Q(a) \rightarrow R(a)$. By modus tollens we then conclude $\neg Q(a)$, and again by modus tollens we conclude $\neg P(a)$.

33. We give a proof by contraposition that if $\sqrt{x}$ is rational, then x is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational (q^2 is again nonzero).

35. We can give a constructive proof by letting $m = 10^{500}+1$. Then $m^2 = (10^{500}+1)^2 > (10^{500})^2 = 10^{1000}$.

37. The first three positive cubes are 1, 8, and 27. If we want to find a number that cannot be written as the sum of eight cubes, we would look for a number that is 7 more than a small multiple of 8. Indeed, 23 will do. We can use two 8's but then would have to use seven 1's to reach 23, a total of nine numbers. Clearly no smaller collection will do. This counterexample disproves the statement.

39. The first three positive fifth powers are 1, 32, and 243. If we want to find a number that cannot be written as the sum of 36 fifth powers, we would look for a number that is 31 more than a small multiple of 32. Indeed, $7 \cdot 32 - 1 = 223$ will do. We can use six 32's but then would have to use 31 1's to reach 223, a total of 37 numbers. Clearly no smaller collection will do. This counterexample disproves the statement.

WRITING PROJECTS FOR CHAPTER 1

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. An excellent website for this is `http://www.wordsmith.demon.co.uk/paradoxes`. It includes a bibliography.

2. Search your library's on-line catalog for a book with the word *fuzzy* in the title. You might find [BaGo], [DuPr], [Ka], [Ko3], or [McFr], for example.

3. Even if you can't find a set, you may find some articles about it in materials for high school students and teachers, such as old issues of *Mathematics Teacher*, published by the National Council of Teachers of Mathematics. This journal, and possibly even copies of the game, may exist in the education library at your school (if there is one). The company that currently produces it has a website: `http://www.wff-n-proof.com`. See also `http://thinkers.law.umich.edu/files/WPGames/WFFNPRUF.htm`, which includes the rules.

4. Martin Gardner and others have written some books that annotate Carroll's writings quite extensively. Lewis Carroll has become a cult figure in certain circles. See also [Ca1], [Ca2], and [Ca3], for original material.

5. A textbook on logic programming and/or the language PROLOG, such as [Ho2] or [Sa1], would be a logical place to start. Many bookstores have huge computer science sections these days, so that source should not be ignored.

6. A course on computational logic at Stanford in 2005–2006 had a Web page with class notes: `http://logic.stanford.edu/classes/cs157/2005fall/cs157.html`. Enderton's book on logic [En] would be a possible choice for background information.

7. There are books on this subject, such as [Du].

8. A place to start might be a recent article on this topic in *Science* [Re]. As always, a Web search will also turn up more information.

9. The Web has an encyclopedia made up of articles by contributors. Remarkably, it is usually quite good, with accurate information and useful links and cross-references. See their article on Chomp: `http://en.wikipedia.org/wiki/Chomp`.

10. The references given in the text are the obvious place to start. The mathematics education field has bought into Pólya's ideas, especially as they relate to problem-solving. See what the National Council of Teachers of Mathematics (`http://www.nctm.org`) has to say about it.

11. The classic work in this field is [GrSh].