SECTION 1.4 Nested Quantifiers

Nested quantifiers are one of the most difficult things for students to understand. The theoretical definition of limit in calculus, for example, is hard to comprehend because it has three levels of nested quantifiers. Study the examples in this section carefully before attempting the exercises, and make sure that you understand the solutions to the exercises you have difficulty with. Practice enough of these until you feel comfortable. The effort will be rewarded in such areas as computer programming and advanced mathematics courses.

1. **a)** For every real number x there exists a real number y such that x is less than y. Basically, this is asserting that there is no largest real number—for any real number you care to name, there is a larger one.

b) For every real number x and real number y, if x and y are both nonnegative, then their product is nonnegative. Or, more simply, the product of nonnegative real numbers is nonnegative.

c) For every real number x and real number y, there exists a real number z such that $xy = z$. Or, more simply, the real numbers are closed under multiplication. (Some authors would include the uniqueness of z as part of the meaning of the word *closed*.)

3. It is useful to keep in mind that x and y can be the same person, so sending messages to oneself counts in this problem.

a) Formally, this says that there exist students x and y such that x has sent a message to y. In other words, there is some student in your class who has sent a message to some student in your class.

b) This is similar to part **(a)** except that x has sent a message to everyone, not just to at least one person. So this says there is some student in your class who has sent a message to every student in your class.

c) Note that this is not the same as part **(b)**. Here we have that for every x there exists a y such that x has sent a message to y. In other words, every student in your class has sent a message to at least one student in your class.

d) Note that this is not the same as part **(c)**, since the order of quantifiers has changed. In part **(c)**, y could depend on x; in other words, the recipient of the messages could vary from sender to sender. Here the existential quantification on y comes first, so it's the same recipient for all the messages. The meaning is that there is a student in your class who has been sent a message by every student in your class.

e) This is similar to part **(c)**, with the role of sender and recipient reversed: every student in your class has been sent a message from at least one student in your class. Again, note that the sender can depend on the recipient.

f) Every student in the class has sent a message to every student in the class.

5. **a)** This simply says that Sarah Smith has visited `www.att.com`.

b) To say that an x exists such that x has visited `www.imdb.org` is just to say that someone (i.e., at least one person) has visited `www.imdb.org`.

c) This is similar to part **(b)**. Jose Orez has visited some website.

d) This is asserting that a y exists that both of these students has visited. In other words, Ashok Puri and Cindy Yoon have both visited the same website.

e) When there are two quantifiers of opposite types, the sentence gets more complicated. This is saying that there is a person (y) other than David Belcher who has visited all the websites that David has visited (i.e., for every website z, if David has visited z, then so has this person). Note that it is not saying that this person has visited only websites that David has visited (that would be the converse conditional statement)—this person may have visited other sites as well.

f) Here the existence of two people is being asserted; they are said to be unequal, and for every website z, one of these people has visited z if and only if the other one has. In plain English, there are two different people who have visited exactly the same websites.

7. **a)** Abdallah Hussein does not like Japanese cuisine.

b) Note that this is the conjunction of two separate quantified statements. Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.

c) There is some cuisine that either Monique Arsenault or Jay Johnson likes.

d) Formally this says that for every x and z, there exists a y such that if x and z are not equal, then it is not the case that both x and z like y. In simple English, this says that for every pair of distinct students at your school, there is some cuisine that at least one them does not like.

e) There are two students at your school who have exactly the same tastes (i.e., they like exactly the same cuisines).

f) For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

9. We need to be careful to put the lover first and the lovee second as arguments in the propositional function L.

a) $\forall x L(x, \text{Jerry})$

b) Note that the "somebody" being loved depends on the person doing the loving, so we have to put the universal quantifier first: $\forall x \exists y L(x, y)$.

c) In this case, one lovee works for all lovers, so we have to put the existential quantifier first: $\exists y \forall x L(x, y)$.

d) We could think of this as saying that there does not exist anyone who loves everybody ($\neg\exists x \forall y L(x, y)$), or we could think of it as saying that for each person, we can find a person whom he or she does not love ($\forall x \exists y \neg L(x, y)$. These two expressions are logically equivalent.

e) $\exists x \neg L(\text{Lydia}, x)$

f) We are asserting the existence of an individual such that everybody fails to love that person: $\exists x \forall y \neg L(y, x)$.

g) In Exercise 52 of Section 1.3, we saw that there is a notation for the existence of a unique object satisfying a certain condition. Employing that device, we could write this as $\exists! x \forall y L(y, x)$. In Exercise 52 of the present section we will discover a way to avoid this notation in general. What we have to say is that the x asserted here exists, and that every z satisfying this condition (of being loved by everybody) must equal x. Thus we obtain $\exists x(\forall y L(y, x) \wedge \forall z((\forall w L(w, z)) \to z = x))$. Note that we could have used y as the bound variable where we used w; since the scope of the first use of y had ended before we came to this point in the formula, reusing y as the bound variable would cause no ambiguity.

h) We want to assert the existence of two distinct people, whom we will call x and y, whom Lynn loves, as well as make the statement that everyone whom Lynn loves must be either x or y: $\exists x \exists y(x \neq y \wedge L(\text{Lynn}, x) \wedge L(\text{Lynn}, y) \wedge \forall z(L(\text{Lynn}, z) \to (z = x \vee z = y)))$.

i) $\forall x L(x, x)$ (Note that nothing in our earlier answers ruled out the possibility that variables or constants with different names might be equal to each other. For example, in part **(a)**, x could equal Jerry, so that statement includes as a special case the assertion that Jerry loves himself. Similarly, in part **(h)**, the two people whom Lynn loves either could be two people other than Lynn (in which case we know that Lynn does not love herself), or could be Lynn herself and one other person.)

j) This is asserting that the one and only one person who is loved by the person being discussed is in fact that person: $\exists x \forall y(L(x, y) \leftrightarrow x = y)$.

11. a) We might want to assert that Lois is a student and Michaels is a faculty member, but the sentence doesn't really say that, so the simple answer is just $A(\text{Lois}, \text{Professor Michaels})$.

b) To say that every student (as opposed to every person) has done this, we need to restrict our universally quantified variable to being a student. The easiest way to do this is to make the assertion being quantified a conditional statement. *As a general rule of thumb, use conditional statements with universal quantifiers and conjunctions with existential quantifiers (see part* **(d)**, *for example).* Thus our answer is $\forall x(S(x) \to A(x, \text{Professor Gross}))$.

c) This is similar to part **(b)**: $\forall x(F(x) \to (A(x, \text{Professor Miller}) \vee A(\text{Professor Miller}, x)))$. Note the need for parentheses in these answers.

d) There is a student such that for every faculty member, that student has not asked that faculty member a question. Note how we need to include the S and F predicates: $\exists x(S(x) \wedge \forall y(F(y) \to \neg A(x, y)))$. We could also write this as $\exists x(S(x) \wedge \neg\exists y(F(y) \wedge A(x, y)))$.

e) This is very similar to part **(d)**, with the role of the players reversed: $\exists x(F(x) \wedge \forall y(S(y) \to \neg A(y, x)))$.

f) This is a little ambiguous in English. If the statement is that there is a very inquisitive student, one who has gone around and asked a question of every professor, then this is similar to part (**d**), without the negation: $\exists x(S(x) \wedge \forall y(F(y) \rightarrow A(x, y)))$. On the other hand, the statement might be intended as asserting simply that for every professor, there exists some student who has asked that professor a question. In other words, the questioner might depend on the questionee. Note how the meaning changes with the change in order of quantifiers. Under the second interpretation the answer is $\forall y(F(y) \rightarrow \exists x(S(x) \wedge A(x, y)))$. The first interpretation is probably the intended one.

g) This is pretty straightforward, except that we have to rule out the possibility that the askee is the same as the asker. Our sentence needs to say that there exists a faculty member such that for every other faculty member, the first has asked the second a question: $\exists x(F(x) \wedge \forall y((F(y) \wedge y \neq x) \rightarrow A(x, y)))$.

h) There is a student such that every faculty member has failed to ask him a question: $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(y, x)))$.

13. Be careful to put in parentheses where needed; otherwise your answer can be either ambiguous or wrong.

a) Clearly this is simply $\neg M(\text{Chou}, \text{Koko})$.

b) We can give two answers, which are equivalent by De Morgan's law: $\neg(M(\text{Arlene}, \text{Sarah}) \vee T(\text{Arlene}, \text{Sarah}))$ or $\neg M(\text{Arlene}, \text{Sarah}) \wedge \neg T(\text{Arlene}, \text{Sarah})$.

c) Clearly this is simply $\neg M(\text{Deborah}, \text{Jose})$.

d) Note that this statement inclndes the assertion that Ken has sent himself a message: $\forall x\, M(x, \text{Ken})$.

e) We can write this in two eqnivalent ways, depending on whether we want to say that everyone has failed to phone Nina or to say that there does not exist someone who has phoned her: $\forall x\, \neg T(x, \text{Nina})$ or $\neg\exists x\, T(x, \text{Nina})$.

f) This is almost identical to part (**d**): $\forall x(T(x, \text{Avi}) \vee M(x, \text{Avi}))$.

g) To get the "else" in there, we have to make sure that y is different from x in our answer: $\exists x \forall y(y \neq x \rightarrow M(x, y))$.

h) This is almost identical to part (**g**): $\exists x \forall y(y \neq x \rightarrow (M(x, y) \vee T(x, y)))$.

i) We need to assert the existence of two distinct people who have sent e-mail both ways: $\exists x \exists y(x \neq y \wedge M(x, y) \wedge M(y, x))$.

j) Only one variable is needed: $\exists x\, M(x, x)$.

k) This poor soul (x in our expression) has a certain thing happen for every person y other than himself: $\exists x \forall y(x \neq y \rightarrow (\neg M(y, x) \wedge \neg T(y, x)))$.

l) Here y is "another student": $\forall x \exists y(x \neq y \wedge (M(y, x) \vee T(y, x)))$.

m) This is almost identical to part (**i**): $\exists x \exists y(x \neq y \wedge M(x, y) \wedge T(y, x))$.

n) Note how the "everyone else" means someone different from both x and y in our expression (and note that there are four possibilities for how each such person z might be contacted): $\exists x \exists y(x \neq y \wedge \forall z((z \neq x \wedge z \neq y) \rightarrow (M(x, z) \vee M(y, z) \vee T(x, z) \vee T(y, z))))$.

15. The answers presented here are not the only ones possible; other answers can be obtained using different predicates and different variables, or by varying the domain (universe of discourse).

a) $\forall x N(x, \text{discrete mathematics})$, where $N(x, y)$ is "computer science x needs a course in subject y"

b) $\exists x O(x, \text{personal compnter})$, where $O(x, y)$ is "x owns y," and the domain for x is students in this class

c) $\forall x \exists y P(x, y)$, where $P(x, y)$ is "x has taken y"; x ranges over students in this class, and y ranges over computer science courses

d) $\exists x \exists y P(x, y)$, with the environment of part (**c**) (i.e., the same definition of P and the same domain)

e) $\forall x \forall y P(x, y)$, where $P(x, y)$ is "x has been in y"; x ranges over students in this class, and y ranges over buildings on campus

f) $\exists x \exists y \forall z (P(z, y) \rightarrow Q(x, z))$, where $P(z, y)$ is "z is in y" and $Q(x, z)$ is "x has been in z"; x ranges over students in this class, y ranges over buildings on campus, and z ranges over rooms

g) $\forall x \forall y \exists z (P(z, y) \wedge Q(x, z))$, with the environment of part (f)

17. a) We need to rule out the possibility that the user has access to another mailbox different from the one that is guaranteed: $\forall u \exists m (A(u, m) \wedge \forall n (n \neq m \rightarrow \neg A(u, n)))$, where $A(u, m)$ means that user u has access to mailbox m.

b) $\exists p \forall e (H(e) \rightarrow S(p, \text{running})) \rightarrow S(\text{kernel}, \text{working correctly})$, where $H(e)$ means that error condition e is in effect and $S(x, y)$ means that the status of x is y. Obviously there are other ways to express this with different choices of predicates. Note that "only if" is the converse of "if," so the kernel's working properly is the conclusion, not the hypothesis.

c) $\forall u \forall s (E(s, .\text{edu}) \rightarrow A(u, s))$, where $E(s, x)$ means that website s has extension x, and $A(u, s)$ means that user u can access website s

d) This is tricky, because we have to interpret the English sentence first, and different interpretations would lead to different answers. We will assume that the specification is that there exist two distinct systems such that they monitor every remote server, and no other system has the property of monitoring every remote system. Thus our answer is $\exists x \exists y (x \neq y \wedge \forall z ((\forall s\, M(z, s)) \leftrightarrow (z = x \vee z = y)))$, where $M(a, b)$ means that system a monitors remote server b. Note that the last part of our expression serves two purposes—it says that x and y do monitor all servers, and it says that no other system does. There are at least two other interpretations of this sentence, which would lead to different legitimate answers.

19. a) $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (x + y < 0))$

b) What does "necessarily" mean in this context? The best explanation is to assert that a certain universal conditional statement is not true. So we have $\neg \forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x - y > 0))$. Note that we do not want to put the negation symbol inside (it is not true that the difference of two positive integers is never positive), nor do we want to negate just the conclusion (it is not true that the sum is always nonpositive). We could rewrite our solution by passing the negation inside, obtaining $\exists x \exists y ((x > 0) \wedge (y > 0) \wedge (x - y \leq 0))$.

c) $\forall x \forall y\, (x^2 + y^2 \geq (x + y)^2)$

d) $\forall x \forall y\, (|xy| = |x||y|)$

21. $\forall x \exists a \exists b \exists c \exists d\, ((x > 0) \rightarrow x = a^2 + b^2 + c^2 + d^2)$, where the domain (universe of discourse) consists of all integers

23. a) $\forall x \forall y ((x < 0) \wedge (y < 0) \rightarrow (xy > 0))$ b) $\forall x\, (x - x = 0)$

c) To say that there are exactly two objects that meet some condition, we must have two existentially quantified variables to represent the two objects, we must say that they are different, and then we must say that an object meets the conditions if and only if it is one of those two. In this case we have $\forall x \exists a \exists b\, (a \neq b \wedge \forall c (c^2 = x \leftrightarrow (c = a \vee c = b)))$.

d) $\forall x\, ((x < 0) \rightarrow \neg \exists y\, (x = y^2))$

25. a) This says that there exists a real number x such that for every real number y, the product xy equals y. That is, there is a multiplicative identity for the real numbers. This is a true statement, since $x = 1$ is the identity.

b) The product of two negative real numbers is always a positive real number.

c) There exist real numbers x and y such that x^2 exceeds y but x is less than y. This is true, since we can take $x = 2$ and $y = 3$, for instance.

d) This says that for every pair of real numbers x and y, there exists a real number z that is their sum. In other words, the real numbers are closed under the operation of addition, another true fact. (Some authors would include the uniqueness of z as part of the meaning of the word *closed*.)

27. Recall that the integers include the positive and negative integers and 0.

a) The import of this statement is that no matter how large n might be, we can always find an integer m bigger than n^2. This is certainly true; for example, we could always take $m = n^2 + 1$.

b) This statement is asserting that there is an n that is smaller than the square of *every* integer; note that n is not allowed to depend on m, since the existential quantifier comes first. This statement is true, since we could take, for instance, $n = -3$, and then n would be less than every square, since squares are always greater than or equal to 0.

c) Note the order of quantifiers: m here is allowed to depend on n. Since we can take $m = -n$, this statement is true (additive inverses exist for the integers).

d) Here one n must work for all m. Clearly $n = 1$ does the trick, so the statement is true.

e) The statement is that the equation $n^2 + m^2 = 5$ has a solution over the integers. This is true; in fact there are eight solutions, namely $n = \pm 1$, $m = \pm 2$, and vice versa.

f) The statement is that the equation $n^2 + m^2 = 6$ has a solution over the integers. There are only a small finite number of cases to try, since if $|m|$ or $|n|$ were bigger than 2 then the left-hand side would be bigger than 6. A few minutes reflection shows that in fact there is no solution, so the existential statement is false.

g) The statement is that the system of equations $\{n + m = 4,\ n - m = 1\}$ has a solution over the integers. By algebra we see that there is a unique solution to this system, namely $n = 2\frac{1}{2}$, $m = 1\frac{1}{2}$. Since there do not exist *integers* that make the equations true, the statement is false.

h) The statement is that the system of equations $\{n + m = 4,\ n - m = 2\}$ has a solution over the integers. By algebra we see that there is indeed an integral solution to this system, namely $n = 3$, $m = 1$. Therefore the statement is true.

i) This statement says that the average of two integers is always an integer. If we take $m = 1$ and $n = 2$, for example, then the only p for which $p = (m + n)/2$ is $p = 1\frac{1}{2}$, which is not an integer. Therefore the statement is false.

29. a) $P(1,1) \land P(1,2) \land P(1,3) \land P(2,1) \land P(2,2) \land P(2,3) \land P(3,1) \land P(3,2) \land P(3,3)$

b) $P(1,1) \lor P(1,2) \lor P(1,3) \lor P(2,1) \lor P(2,2) \lor P(2,3) \lor P(3,1) \lor P(3,2) \lor P(3,3)$

c) $(P(1,1) \land P(1,2) \land P(1,3)) \lor (P(2,1) \land P(2,2) \land P(2,3)) \lor (P(3,1) \land P(3,2) \land P(3,3))$

d) $(P(1,1) \lor P(2,1) \lor P(3,1)) \land (P(1,2) \lor P(2,2) \lor P(3,2)) \land (P(1,3) \lor P(2,3) \lor P(3,3))$

Note the crucial difference between parts (c) and (d).

31. As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \to q) \equiv p \land \neg q$.

a)
$$\begin{aligned}\neg\forall x\exists y\forall z\, T(x,y,z) &\equiv \exists x\neg\exists y\forall z\, T(x,y,z)\\ &\equiv \exists x\forall y\neg\forall z\, T(x,y,z)\\ &\equiv \exists x\forall y\exists z\, \neg T(x,y,z)\end{aligned}$$

b)
$$\begin{aligned}\neg(\forall x\exists y\, P(x,y) \lor \forall x\exists y\, Q(x,y)) &\equiv \neg\forall x\exists y\, P(x,y) \land \neg\forall x\exists y\, Q(x,y)\\ &\equiv \exists x\neg\exists y\, P(x,y) \land \exists x\neg\exists y\, Q(x,y)\\ &\equiv \exists x\forall y\, \neg P(x,y) \land \exists x\forall y\, \neg Q(x,y)\end{aligned}$$

c)
$$\begin{aligned}\neg\forall x\exists y(P(x,y) \land \exists z\, R(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y) \land \exists z\, R(x,y,z))\\ &\equiv \exists x\forall y\neg(P(x,y) \land \exists z\, R(x,y,z))\\ &\equiv \exists x\forall y(\neg P(x,y) \lor \neg\exists z\, R(x,y,z))\\ &\equiv \exists x\forall y(\neg P(x,y) \lor \forall z\, \neg R(x,y,z))\end{aligned}$$

d)

$$\begin{aligned}\neg\forall x\exists y(P(x,y)\to Q(x,y)) &\equiv \exists x\,\neg\exists y(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y\,\neg(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y(P(x,y)\wedge\neg Q(x,y))\end{aligned}$$

33. We need to use the transformations shown in Table 2 of Section 1.3, replacing $\neg\forall$ by $\exists\neg$, and replacing $\neg\exists$ by $\forall\neg$. In other words, we push all the negation symbols inside the quantifiers, changing the sense of the quantifiers as we do so, because of the equivalences in Table 2 of Section 1.3. In addition, we need to use De Morgan's laws (Section 1.2) to change the negation of a conjunction to the disjunction of the negations and to change the negation of a disjunction to the conjunction of the negations. We also use the double negation law.

a) $\exists x\exists y\,\neg P(x,y)$ **b)** $\exists y\forall x\,\neg P(x,y)$

c) We can think of this in two steps. First we transform the expression into the equivalent expression $\exists y\exists x\,\neg(P(x,y)\vee Q(x,y))$, and then we use De Morgan's law to rewrite this as $\exists y\exists x(\neg P(x,y)\wedge\neg Q(x,y))$.

d) First we apply De Morgan's law to write this as a disjunction: $(\neg\exists x\exists y\,\neg P(x,y))\vee(\neg\forall x\forall y\,Q(x,y))$. Then we push the negation inside the quantifiers, and note that the two negations in front of P then cancel out ($\neg\neg P(x,y)\equiv P(x,y)$). So onr final answer is $(\forall x\forall y\,P(x,y))\vee(\exists x\exists y\,\neg Q(x,y))$.

e) First we push the negation inside the outer universal quantifier, then apply De Morgan's law, and finally push it inside the inner quantifiers: $\exists x\,\neg(\exists y\forall z\,P(x,y,z)\wedge\exists z\forall y\,P(x,y,z))\equiv\exists x(\neg\exists y\forall z\,P(x,y,z)\vee\neg\exists z\forall y\,P(x,y,z))\equiv\exists x(\forall y\exists z\,\neg P(x,y,z)\vee\forall z\exists y\,\neg P(x,y,z))$.

35. If the domain (universe of discourse) has at least four members, then no matter what values are assigned to x, y, and z, there will always be another member of the domain, different from those three, that we can assign to w to make the statement true. Thus we can use a domain such as United States Senators. On the other hand, for any domain with three or fewer members, if we assign all the members to x, y, and z (repeating some if necessary), then there will be nothing left to assign to w to make the statement true. For this we can use a domain such as your biological parents.

37. In each case we need to specify some predicates and identify the domain (universe of discourse).

a) To get into the spirit of the problem, we should let $T(x,y)$ be the predicate that x has taken y, where x ranges over students in this class and y ranges over mathematics classes at this school. Then our original statement is $\forall x\exists y\exists z(y\neq z\wedge T(x,y)\wedge T(x,z)\wedge\forall w(T(x,w)\to(w=y\vee w=z)))$. Here y and z are the two math classes that x has taken, and our statement says that these are different and that if x has taken any math class w, then w is one of these two. We form the negation by using Table 2 of Section 1.3 and De Morgan's law to pnsh the negation symbol that we place before the entire expression inwards, to achieve $\exists x\forall y\forall z(y=z\vee\neg T(x,y)\vee\neg T(x,z)\vee\exists w(T(x,w)\wedge w\neq y\wedge w\neq z))$. This can also be expressed as $\exists x\forall y\forall z(y\neq z\to(\neg T(x,y)\vee\neg T(x,z)\vee\exists w(T(x,w)\wedge w\neq y\wedge w\neq z)))$. Note that we formed the negation of a conditional statement by asserting that the hypothesis was true and the conclusion was false. In simple English, this last statement reads "There is someone in this class for whom no matter which two distinct math courses you consider, these are not the two and only two math courses this person has taken."

b) Let $V(x,y)$ be the predicate that x has visited y, where x ranges over people and y ranges over countries. The statement seems to be asserting that the person identified here has visited country y if and only if y is not Libya. So we can write this symbolically as $\exists x\forall y(V(x,y)\leftrightarrow y\neq \text{Libya})$. One way to form the negation of $P\leftrightarrow Q$ is to write $P\leftrightarrow\neg Q$; this can be seen by looking at truth tables. Thus the negation is $\forall x\exists y(V(x,y)\leftrightarrow y=\text{Libya})$. Note that there are two ways for a biconditional to be true; therefore in English this reads "For every person there is a country such that either that country is Libya and the person has visited it, or else that country is not Libya and the person has not visited it." More simply, "For every person, either that person has visited Libya or else that person has failed to visit some country other than Libya." If we are willing to keep the negation in front of the quantifier in English, then of course we could just say "There is nobody who has visited every country except Libya," but that would not be in the spirit of the exercise.

c) Let $C(x,y)$ be the predicate that x has climbed y, where x ranges over people and y ranges over mountains in the Himalayas. Our statement is $\neg\exists x\forall y\, C(x,y)$. Its negation is, of course, simply $\exists x\forall y\, C(x,y)$. In English this reads "Someone has climbed every mountain in the Himalayas."

d) There are different ways to approach this, depending on how many variables we want to introduce. Let $M(x,y,z)$ be the predicate that x has been in movie z with y, where the domains for x and y are movie actors, and for z is movies. The statement then reads: $\forall x((\exists z\, M(x, \text{Kevin Bacon}, z)) \vee (\exists y\exists z_1\exists z_2(M(x,y,z_1)\wedge M(y, \text{Kevin Bacon}, z_2))))$. The negation is formed in the usual manner: $\exists x((\forall z\,\neg M(x, \text{Kevin Bacon}, z)) \wedge (\forall y\forall z_1\forall z_2(\neg M(x,y,z_1)\vee\neg M(y, \text{Kevin Bacon}, z_2))))$. In simple English this means that there is someone who has neither been in a movie with Kevin Bacon nor been in a movie with someone who has been in a movie with Kevin Bacon.

39. **a)** Since the square of a number and its additive inverse are the same, we have many counterexamples, such as $x = 2$ and $y = -2$.

b) This statement is saying that every number has a square root. If x is negative (like $x = -4$), or, since we are working in the domain of the integers, x is not a perfect square (like $x = 6$), then the equation $y^2 = x$ has no solution.

c) Since negative numbers are not larger than positive numbers, we can take something like $x = 17$ and $y = -1$ for our counterexample.

41. We simply want to say that a certain equation holds for all real numbers: $\forall x\forall y\forall z((x\cdot y)\cdot z = x\cdot(y\cdot z))$.

43. We want to say that for each pair of coefficients (the m and the b in the expression $mx+b$), as long as m is not 0, there is a unique x making that expression equal to 0. So we write $\forall m\forall b(m\neq 0\rightarrow\exists x(mx+b = 0\wedge\forall w(mw+b=0\rightarrow w=x)))$. Notice that the uniqueness is expressed by the last part of our proposition.

45. This statement says that every number has a multiplicative inverse.

a) In the universe of nonzero real numbers, this is certainly true. In each case we let $y = 1/x$.

b) Integers usually don't have inverses that are integers. If we let $x = 3$, then no integer y satisfies $xy = 1$. Thus in this setting, the statement is false.

c) As in part **(a)** this is true, since $1/x$ is positive when x is positive.

47. We use the equivalences explained in Table 2 of Section 1.3, twice:
$\neg\exists x\forall y P(x,y)\equiv\forall x\neg\forall y P(x,y)\equiv\forall x\exists y\neg P(x,y)$

49. **a)** We prove this by arguing that whenever the first proposition is true, so is the second; and that whenever the second proposition is true, so is the first. So suppose that $\forall x P(x)\wedge\exists x Q(x)$ is true. In particular, P always holds, and there is some object, call it y, in the domain (universe of discourse) that makes Q true. Now to show that the second proposition is true, suppose that x is any object in the domain. By our assumptions, $P(x)$ is true. Furthermore, $Q(y)$ is true for the particular y we mentioned above. Therefore $P(x)\wedge Q(y)$ is true for this x and y. Since x was arbitrary, we have showed that $\forall x\exists y(P(x)\wedge Q(y))$ is true, as desired. Conversely, suppose that the second proposition is true. Letting x be any member of the domain allows us to assert that there exists a y such that $P(x)\wedge Q(y)$ is true, and therefore $Q(y)$ is true. Thus by the definition of existential quantifiers, $\exists x Q(x)$ is true. Furthermore, our hypothesis tells us in particular that $\forall x P(x)$ is true. Therefore the first proposition, $\forall x P(x)\wedge\exists x Q(x)$ is true.

b) This is similar to part (a). Suppose that $\forall x P(x) \vee \exists x Q(x)$ is true. Thus either P always holds, or there is some object, call it y, in the domain that makes Q true. In the first case it follows that $P(x) \vee Q(y)$ is true for all x, and so we can conclude that $\forall x \exists y (P(x) \vee Q(y))$ is true (it does not matter in this case whether $Q(y)$ is true or not). In the second case, $Q(y)$ is true for this particular y, and so $P(x) \vee Q(y)$ is true regardless of what x is. Again, it follows that $\forall x \exists y (P(x) \vee Q(y))$ is true. Conversely, suppose that the second proposition is true. If $P(x)$ is true for all x, then the first proposition must be true. If not, then $P(x)$ fails for some x, but for this x there must be a y such that $P(x) \vee Q(y)$ is true; hence $Q(y)$ must be true. Therefore $\exists y Q(y)$ holds, and thus the first proposition is true.

51. This will essentially be a proof by (structural) mathematical induction (see Sections 4.1–4.3), where we show how a long expression can be put into prenex normal form if the subexpressions in it can be put into prenex normal form. First we invoke the result of Exercise 45 from Section 1.2 to assume without loss of generality that our given proposition uses only the logical connectives $\vee$ and $\neg$. Then every proposition must either be a single propositional variable (like P), the disjunction of two propositions, the negation of a proposition, or the universal or existential quantification of a predicate. (There is a small technical point that we are sliding over here; disjunction and negation need to be defined for predicates as well as for propositions, since otherwise we would not be able to write down such things as $\forall x (P(x) \wedge Q(x))$. We assume that all that we have done for propositions applies to predicates as well.)

Certainly every proposition that involves no quantifiers is already in prenex normal form; this is the base case of our induction. Next suppose that our proposition is of the form $\mathrm{Q}x P(x)$, where Q is a quantifier. Then $P(x)$ is a shorter expression than the given proposition, so (by the inductive hypothesis) we can put it into prenex form, with all of its quantifiers coming at the beginning. Then $\mathrm{Q}x$ followed by this prenex form is again in prenex form and is equivalent to the original proposition. Next suppose that our proposition is of the form $\neg P$. Again, we can invoke the inductive hypothesis and assume that P is already in prenex form, with all of its quantifiers coming at its front. We now slide the negation symbol past all the quantifiers, using the equivalences in Table 2 of Section 1.3. For example, $\neg \forall x \exists y R(x,y)$ becomes $\exists x \forall y \neg R(x,y)$, which is in prenex form.

Finally, suppose that our given proposition is a disjunction of two propositions, $P \vee Q$, each of which can (again by the inductive hypothesis) be assumed to be in prenex normal form, with their quantifiers at the front. There are several cases. If only one of P and Q has quantifiers, then we invoke the result of Exercise 46 of Section 1.3 to bring the quantifier in front of both. We then apply our process to what remains. For example, $P \vee \forall x Q(x)$ is equivalent to $\forall x (P \vee Q(x))$, and then $P \vee Q(x)$ is put into prenex form. Another case is that the proposition might look like $\exists x R(x) \vee \exists x S(x)$. In this case, by Exercise 45 of Section 1.3, the proposition is equivalent to $\exists x (R(x) \vee S(x))$. Once again, by the inductive hypothesis we can then put $R(x) \vee S(x)$ into prenex form, and so the entire proposition can be put into prenex form. Similarly, using Exercise 48 of the present section we can transform $\forall x R(x) \vee \forall x S(x)$ into the equivalent $\forall x \forall y (R(x) \vee S(y))$; putting $R(x) \vee S(y)$ into prenex form then brings the entire proposition into prenex form. Finally, if the proposition is of the form $\forall x R(x) \vee \exists x Q(x)$, then we invoke Exercise 49b of the present section and apply the same construction.

Note that this proof actually gives us the process for finding the proposition in prenex form equivalent to the given proposition—we just work from the inside out, dealing with one logical operation or quantifier at a time. Here is an example:

$$\begin{aligned}
\forall x P(x) \vee \neg \exists x (Q(x) \vee \forall y R(x,y)) &\equiv \forall x P(x) \vee \neg \exists x \forall y (Q(x) \vee R(x,y)) \\
&\equiv \forall x P(x) \vee \forall x \exists y \neg (Q(x) \vee R(x,y)) \\
&\equiv \forall x \forall z (P(x) \vee \exists y \neg (Q(z) \vee R(z,y))) \\
&\equiv \forall x \forall z \exists y (P(x) \vee \neg (Q(z) \vee R(z,y)))
\end{aligned}$$