SECTION 3.2 The Growth of Functions

The big-O notation is used extensively in computer science and other areas. Think of it as a crude ruler for measuring functions in terms of how fast they grow. The idea is to treat all functions that are more or less the same as one function—one mark on this ruler. Thus, for example, all linear functions are simply thought of as $O(n)$. Although technically the big-O notation gives an upper bound on the growth of a function, in practice we choose the smallest big-O estimate that applies. (This is made more rigorous with the big-Theta notation, also discussed in this section.) In essence, one finds best big-O estimates by discarding lower order terms and multiplicative constants. Furthermore, one usually chooses the simplest possible representative of the big-O (or big-Theta) class (for example, writing $O(n^2)$ rather than $O(3n^2+5)$). A related concept, used in combinatorics and applied mathematics, is the **little-*o* notation**, *dealt with in Exercises 51–59.*

1. Note that the choices of witnesses C and k are not unique.

a) Yes, since $|10| \leq |x|$ for all $x > 10$. The witnesses are $C = 1$ and $k = 10$.

b) Yes, since $|3x + 7| \leq |4x| = 4|x|$ for all $x > 7$. The witnesses are $C = 4$ and $k = 7$.

c) No. There is no *constant* C such that $|x^2 + x + 1| \leq C|x|$ for all sufficiently large x. To see this, suppose this inequality held for all sufficiently large positive values of x. Then we would have $x^2 \leq Cx$, which would imply that $x \leq C$ for *all* sufficiently large x, an obvious impossibility.

d) Yes. This follows from the fact that $\log x < x$ for all $x > 1$ (which in turn follows from the fact that $x < 2^x$, which can be formally proved by mathematical induction—see Section 4.1). Therefore $|5 \log x| \leq 5|x|$ for all $x > 1$. The witnesses are $C = 5$ and $k = 1$.

e) Yes. This follows from the fact that $\lfloor x \rfloor \leq x$. Thus $|\lfloor x \rfloor| \leq |x|$ for all $x > 0$. The witnesses are $C = 1$ and $k = 0$.

f) Yes. This follows from the fact that $\lceil x/2 \rceil \leq (x/2) + 1$. Thus $|\lceil x/2 \rceil| \leq |(x/2) + 1| \leq |x|$ for all $x > 2$. The witnesses are $C = 1$ and $k = 2$.

3. We need to put some bounds on the lower order terms. If $x > 9$ then we have $x^4 + 9x^3 + 4x + 7 \leq x^4 + x^4 + x^4 + x^4 = 4x^4$. Therefore $x^4 + 9x^3 + 4x + 7$ is $O(x^4)$, taking witnesses $C = 4$ and $k = 9$.

5. We use long division to rewrite this function:

$$\frac{x^2+1}{x+1} = \frac{x^2-1+2}{x+1} = \frac{x^2-1}{x+1} + \frac{2}{x+1} = x - 1 + \frac{2}{x+1}$$

Now this is certainly less than x as long as $x > 1$, so our function is $O(x)$. The witnesses are $C = 1$ and $k = 1$.

7. **a)** Since $\log x$ grows more slowly than x, $x^2 \log x$ grows more slowly than x^3, so the first term dominates. Therefore this function is $O(x^3)$ but not $O(x^n)$ for any $n < 3$. More precisely, $2x^3 + x^2 \log x \leq 2x^3 + x^3 = 3x^3$ for all x, so we have witnesses $C = 3$ and $k = 0$.

b) We know that $\log x$ grows so much more slowly than x that *every* power of $\log x$ grows more slowly than x. Thus the first term dominates, and the best estimate is $O(x^3)$. More precisely, $(\log x)^4 < x^3$ for all $x > 1$, so $3x^3 + (\log x)^4 \leq 3x^3 + x^3 = 4x^3$ for all x, so we have witnesses $C = 4$ and $k = 1$.

c) By long division, we see that $f(x) = x +$ lower order terms. Therefore this function is $O(x)$, so $n = 1$. In fact, $f(x) = x + \frac{1}{x+1} \leq 2x$ for all $x > 1$, so the witnesses can be taken to be $C = 2$ and $k = 1$.

d) Again by long division, this quotient has the form $f(x) = 1 +$ lower order terms. Therefore this function is $O(1)$. In other words, $n = 0$. Since $5 \log x < x^4$ for $x > 1$, we have $f(x) \leq 2x^4/x^4 = 2$, so we can take as witnesses $C = 2$ and $k = 1$.

9. On the one hand we have $x^2 + 4x + 17 \leq x^2 + x^2 + x^2 = 3x^2 \leq 3x^3$ for all $x > 17$, so $x^2 + 4x + 17$ is $O(x^3)$, with witnesses $C = 3$ and $k = 17$. On the other hand, if x^3 were $O(x^2 + 4x + 17)$, then we would have $x^3 \leq C(x^2 + 4x + 17) \leq 3Cx^2$ for all sufficiently large x. But this says that $x \leq 3C$, clearly impossible for the constant C to satisfy for all large x. Therefore x^3 is not $O(x^2 + 4x + 17)$.

11. For the first part we have $3x^4 + 1 \leq 4x^4 = 8|x^4/2|$ for all $x > 1$; we have witnesses $C = 8$, $k = 1$. For the second part we have $x^4/2 \leq 3x^4 \leq 1 \cdot |3x^4 + 1|$ for all x; witnesses are $C = 1$, $k = 0$.

13. To show that 2^n is $O(3^n)$ it is enough to note that $2^n \leq 3^n$ for all $n > 0$. In terms of witnesses we have $C = 1$ and $k = 0$. On the other hand, if 3^n were $O(2^n)$, then we would have $3^n \leq C \cdot 2^n$ for all sufficiently large n. This is equivalent to $C \geq \left(\frac{3}{2}\right)^n$, which is clearly impossible, since $\left(\frac{3}{2}\right)^n$ grows without bound as n increases.

15. A function f is $O(1)$ if $|f(x)| \le C$ for all sufficiently large x. In other words, f is $O(1)$ if its absolute value is **bounded** for all $x > k$ (where k is some constant).

17. Let C_1, C_2, k_1, and k_2 be numbers such that $|f(x)| \le C_1|g(x)|$ for all $x > k_1$ and $|g(x)| \le C_2|h(x)|$ for all $x > k_2$. Let $C = C_1C_2$ and let k be the larger of k_1 and k_2. Then for all $x > k$ we have $|f(x)| \le C_1|g(x)| \le C_1C_2|h(x)| = C|h(x)|$, which is precisely what we needed to show.

19. **a)** The significant terms here are the n^2 being multiplied by the n; thus this function is $O(n^3)$.
b) Since $\log n$ is smaller than n, the significant term in the first factor is n^2. Therefore the entire function is $O(n^5)$.
c) For the first factor we note that $2^n < n!$ for $n \ge 4$, so the significant term is $n!$. For the second factor, the significant term is n^3. Therefore this function is $O(n^3n!)$.

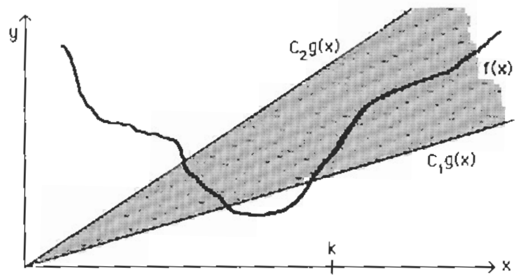
21. **a)** First we note that $\log(n^2+1)$ and $\log n$ are in the same big-O class, since $\log n^2 = 2\log n$. Therefore the second term here dominates the first, and the simplest good answer would be just $O(n^2\log n)$.
b) The first term is in the same big-O class as $O(n^2(\log n)^2)$, while the second is in a slightly smaller class, $O(n^2\log n)$. (In each case, we can throw away the smaller order terms, since they are dominated by the terms we are keeping—this is the essence of doing big-O estimates.) Therefore the answer is $O(n^2(\log n)^2)$.
c) The only issue here is whether 2^n or n^2 is the faster-growing, and clearly it is the former. Therefore the best big-O estimate we can give is $O(n^{2^n})$.

23. We can use the following rule of thumb to determine what simple big-Theta function to use: throw away all the lower order terms (those that don't grow as fast as other terms) and all constant coefficients.
a) This function is $\Theta(x)$, so it is not $\Theta(x^2)$, since x^2 grows faster than x. To be precise, x^2 is not $O(17x+11)$. For the same reason, this function is not $\Omega(x^2)$.
b) This function is $\Theta(x^2)$; we can ignore the "$+1000$" since it is a lower order term. Of course, since $f(x)$ is $\Theta(x^2)$, it is also $\Omega(x^2)$.
c) This function grows more slowly than x^2, since $\log x$ grows more slowly than x. Therefore $f(x)$ is not $\Theta(x^2)$ or $\Omega(x^2)$.
d) This function grows faster than x^2. Therefore $f(x)$ is not $\Theta(x^2)$, but it is $\Omega(x^2)$.
e) Exponential functions (with base larger than 1) grow faster than all polynomials, so this function is not $O(x^2)$ and therefore not $\Theta(x^2)$. But it is $\Omega(x^2)$.
f) For large values of x, this is quite close to x^2, since both factors are quite close to x. Certainly $\lfloor x\rfloor \cdot \lceil x\rceil$ is always between $x^2/2$ and $2x^2$, for $x > 2$. Therefore this function is $\Theta(x^2)$ and hence also $\Omega(x^2)$.

25. If $f(x)$ is $\Theta(g(x))$, then $|f(x)| \le C_2|g(x)|$ and $|g(x)| \le C_1^{-1}|f(x)|$ for all $x > k$. Thus $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. Conversely, suppose that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. Then (with appropriate choice of variable names) we may assume that $|f(x)| \le C_2|g(x)|$ and $|g(x)| \le C|f(x)|$ for all $x > k$. (The k here will be the larger of the two k's involved in the hypotheses.) If $C > 0$ then we can take $C_1 = C^{-1}$ to obtain the desired inequalities in "$f(x)$ is $\Theta(g(x))$." If $C \le 0$, then $g(x) = 0$ for all $x > k$, and hence by the first inequality $f(x) = 0$ for all $x > k$; thus we have $f(x) = g(x)$ for all $x > k$, and we can take $C_1 = C_2 = 1$.

27. The definition of "$f(x)$ is $\Theta(g(x))$" is that $f(x)$ is both $O(g(x))$ and $\Omega(g(x))$. That means that there are positive constants C_1, k_1, C_2, and k_2 such that $|f(x)| \le C_2|g(x)|$ for all $x > k_2$ and $|f(x)| \ge C_1|g(x)|$ for all $x > k_1$. That is practically the same as the statement in this exercise. We need only note that we can take k to be the larger of k_1 and k_2 if we want to prove the "only if" direction, and we can take $k_1 = k_2 = k$ if we want to prove the "if" direction.

29. In the following picture, the wavy line is the graph of the function f. For simplicity we assume that the graph of g is a straight line through the origin. Then the graphs of C_1g and C_2g are also straight lines through the origin, as drawn here. The fact that $f(x)$ is $\Theta(g(x))$ is shown by the fact that for $x > k$ the graph of f is confined to the shaded wedge-shaped space between these latter two lines (see Exercise 27). (We assume that $g(x)$ is positive for positive x, so that $|g(x)|$ is the same as $g(x)$.)



31. Looking at the definition tells us that if $f(x)$ is $\Theta(1)$ then $|f(x)|$ has to be bounded between two positive constants. In other words, $f(x)$ can't get too large (either positive or negative), and it can't get too close to 0.

33. We are given that $|f(x)| \leq C|g(x)|$ for all $x > b$ (we cannot use the variable name k here since we need it later). Hence $|f^k(x)| = |f(x)|^k \leq C^k|g^k(x)|$ for all $x > b$, so $f^k(x)$ is $O(g^k(x))$ (take the constants in the definition to be C^k and b).

35. Since the functions are given to be increasing and unbounded, we may assume that they both take on values greater than 1 for all sufficiently large x. The hypothesis can then be written as $f(x) \leq Cg(x)$ for all $x > k$. If we take the logarithm of both sides, then we obtain $\log f(x) \leq \log C + \log g(x)$. Finally, this latter expression is less than $2 \log g(x)$ for large enough x, since $\log g(x)$ is growing without bound. Note that the converse to this problem is not true.

37. By definition there are positive constants C_1, C_1', C_2, C_2', k_1, k_1', k_2, and k_2' such that $f_1(x) \geq C_1|g(x)|$ for all $x > k_1$, $f_1(x) \leq C_1'|g(x)|$ for all $x > k_1'$, $f_2(x) \geq C_2|g(x)|$ for all $x > k_2$, and $f_2(x) \leq C_2'|g(x)|$ for all $x > k_2'$. We are able to omit the absolute value signs on the $f(x)$'s since we are told that they are positive; we are also told here that the $g(x)$'s are positive, but we do not need that. Adding the first and third inequalities we obtain $f_1(x) + f_2(x) \geq (C_1 + C_2)|g(x)|$ for all $x > \max(k_1, k_2)$; and similarly with the second and fourth inequalities we know $f_1(x) + f_2(x) \leq (C_1' + C_2')|g(x)|$ for all $x > \max(k_1', k_2')$. Thus $f_1(x) + f_2(x)$ meets the definition of being $\Theta(g(x))$.

If the f's can take on negative values, then this is no longer true. For example, let $f_1(x) = x^2 + x$, let $f_2(x) = -x^2 + x$, and let $g(x) = x^2$. Then each $f_i(x)$ is $\Theta(g(x))$, but the sum is $2x$, which is not $\Theta(g(x))$.

39. This is not true. It is similar to Exercise 37, and essentially the same counterexample suffices. Let $f_1(x) = x^2 + 2x$, $f_2(x) = x^2 + x$, and $g(x) = x^2$. Then clearly $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$, but $(f_1 - f_2)(x) = x$ is not.

41. The key here is that if a function is to be big-O of another, then the appropriate inequality has to hold for *all* large inputs. Suppose we let $f(x) = x^2$ for even x and x for odd x. Similarly, we let $g(x) = x^2$ for odd x and x for even x. Then clearly neither inequality $|f(x)| \leq C|g(x)|$ nor $|g(x)| \leq C|f(x)|$ holds for all x, since for even x the first function is much bigger than the second, while for odd x the second is much bigger than the first.

43. We are given that there are positive constants C_1, C_1', C_2, C_2', k_1, k_1', k_2, and k_2' such that $|f_1(x)| \geq C_1|g_1(x)|$ for all $x > k_1$, $|f_1(x)| \leq C_1'|g_1(x)|$ for all $x > k_1'$, $|f_2(x)| \geq C_2|g_2(x)|$ for all $x > k_2$, and $|f_2(x)| \leq C_2'|g_2(x)|$ for all $x > k_2'$. Since f_2 and g_2 never take on the value 0, we can rewrite the last two of these inequalities as $|1/f_2(x)| \leq (1/C_2)|1/g_2(x)|$ and $|1/f_2(x)| \geq (1/C_2')|1/g_2(x)|$. Now we multiply the first inequality and the rewritten fourth inequality to obtain $|f_1(x)/f_2(x)| \geq (C_1/C_2')|g_1(x)/g_2(x)|$ for all $x > \max(k_1, k_2')$. Working with the other two inequalities gives us $|f_1(x)/f_2(x)| \leq (C_1'/C_2)|g_1(x)/g_2(x)|$ for all $x > \max(k_1', k_2)$. Together these tell us that f_1/f_2 is big-Theta of g_1/g_2.

45. We just make the analogous change in the definition of big-Theta that was made in the definition of big-O: there exist positive constants C_1, C_2, k_1, k_2, k_1', k_2' such that $|f(x,y)| \leq C_1|g(x,y)|$ for all $x > k_1$ and $y > k_2$, and $|f(x,y)| \geq C_2|g(x,y)|$ for all $x > k_1'$ and $y > k_2'$.

47. For all values of x and y greater than 1, each term of the expression inside parentheses is less than x^2y, so the entire expression inside parentheses is less than $3x^2y$. Therefore our function is less than $27x^6y^3$ for all $x > 1$ and $y > 1$. By definition this shows that it is big-O of x^6y^3. Specifically, we take $C = 27$ and $k_1 = k_2 = 1$ in the definition.

49. For all positive values of x and y, we know that $\lfloor xy \rfloor \leq xy$ by definition (since the floor function value cannot exceed the argument). Thus $\lfloor xy \rfloor$ is $O(xy)$ from the definition, taking $C = 1$ and $k_1 = k_2 = 0$. In fact, $\lfloor xy \rfloor$ is also $\Omega(xy)$ (and therefore $\Theta(xy)$); this is easy to see since $\lfloor xy \rfloor \geq (x-1)(y-1) \geq (\frac{1}{2}x)(\frac{1}{2}y) = \frac{1}{4}xy$ for all x and y greater than 2.

51. All that we need to do is determine whether the ratio of the two functions approaches 0 as x approaches infinity.

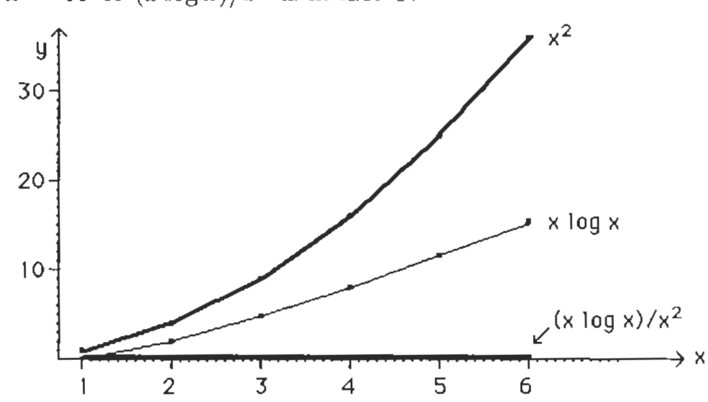
a) $\displaystyle\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0$

b) $\displaystyle\lim_{x\to\infty} \frac{x\log x}{x^2} = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x\ln 2} = 0$ (using L'Hôpital's rule for the second equality)

c) $\displaystyle\lim_{x\to\infty} \frac{x^2}{2^x} = \lim_{x\to\infty} \frac{2x}{2^x\ln 2} = \lim_{x\to\infty} \frac{2}{2^x(\ln 2)^2} = 0$ (with two applications of L'Hôpital's rule)

d) $\displaystyle\lim_{x\to\infty} \frac{x^2+x+1}{x^2} = \lim_{x\to\infty} (1 + \frac{1}{x} + \frac{1}{x^2}) = 1 \neq 0$

53. The picture shows the graph of $y = x^2$ increasing quite rapidly and $y = x\log x$ increasing less rapidly. The ratio is hard to see on the picture; it rises to about $y = 0.53$ at about $x = 2.7$ and then slowly decreases toward 0. The limit as $x \to \infty$ of $(x\log x)/x^2$ is in fact 0.



55. No. As one example, take $f(x) = x^{-2}$ and $g(x) = x^{-1}$. Then $f(x)$ is $o(g(x))$, since $\lim_{x\to\infty} x^{-2}/x^{-1} = \lim_{x\to\infty} 1/x = 0$. On the other hand $\lim_{x\to\infty} (2^{x^{-2}}/2^{x^{-1}}) = \lim_{x\to\infty} 2^{x^{-2}-x^{-1}} = 2^0 = 1 \neq 0$.

57. a) Since the limit of $f(x)/g(x)$ is 0 (as $x \to \infty$), so too is the limit of $|f(x)|/|g(x)|$. In particular, for x large enough, this ratio is certainly less than 1. In other words $|f(x)| \le |g(x)|$ for sufficiently large x, which meets the definition of "$f(x)$ is $O(g(x))$."

b) We can simply let $f(x) = g(x)$ be any function with positive values. Then the limit of their ratio is 1, not 0, so $f(x)$ is not $o(g(x))$, bnt certainly $f(x)$ is $O(g(x))$.

59. This follows immediately from Exercise 57a (whereby we can conclude that $f_2(x)$ is $O(g(x))$) and Corollary 1 to Theorem 2.

61. What we want to show is equivalent to the statement that $\log(n^n)$ is at most a constant times $\log(n!)$, which in turn is equivalent to the statement that n^n is at most a constant power of $n!$ (because of the fact that $C\log A = \log(A^C)$—see Appendix 2). We will show that in fact $n^n \le (n!)^2$ for all $n > 1$. To do this, let us write $(n!)^2$ as $(n\cdot 1)\cdot((n-1)\cdot 2)\cdot((n-2)\cdot 3)\cdots(2\cdot(n-1))\cdot(1\cdot n)$. Now clearly each product pair $(i+1)\cdot(n-i)$ is at least as big as n (indeed, the ones near the middle are significantly bigger than n). Therefore the entire product is at least as big as n^n, as desired.

63. For $n = 5$ we compute that $\log 5! \approx 6.9$ and $(5\log 5)/4 \approx 2.9$, so the inequality holds (it actually holds for all $n > 1$). Therefore we can assume that $n \ge 6$. Since $n!$ is the product of all the integers from n down to 1, we certainly have $n! > n(n-1)(n-2)\cdots\lceil n/2\rceil$ (since at least the term 2 is missing). Note that there are more than $n/2$ terms in this product, and each term is at least as big as $n/2$. Therefore the product is greater than $(n/2)^{(n/2)}$. Taking the log of both sides of the inequality, we have

$$\log n! > \log\left(\frac{n}{2}\right)^{n/2} = \frac{n}{2}\log\frac{n}{2} = \frac{n}{2}(\log n - 1) > (n\log n)/4\,,$$

since $n > 4$ implies $\log n - 1 > (\log n)/2$.

65. In each case we need to evaluate the limit of $f(x)/g(x)$ as $x \to \infty$. If it equals 1, then f and g are asymptotic; otherwise (including the case in which the limit does not exist) they are not. Most of these are straightforward applications of algebra, elementary notions about limits, or L'Hôpital's rule.

a) $f(x) = \log(x^2+1) \ge \log(x^2) = 2\log x$. Therefore $f(x)/g(x) \ge 2$ for all x. Thus the limit is not 1 (in fact, of course, it's 2), so f and g are not asymptotic.

b) By the algebraic rules for exponents, $f(x)/g(x) = 2^{-4} = 1/16$. Therefore the limit of the ratio is 1/16, not 1, so f and g are not asymptotic.

c) $\displaystyle\lim_{x\to\infty}\frac{2^{2^x}}{2^{x^2}} = \lim_{x\to\infty} 2^{2^x - x^2}$. As x gets large, the exponent grows without bound, so this limit is ∞. Thus f and g are not asymptotic.

d) $\displaystyle\lim_{x\to\infty}\frac{2^{x^2+x+1}}{2^{x^2+2x}} = \lim_{x\to\infty} 2^{1-x}$. As x gets large, the exponent grows in the negative direction without bound, so this limit is 0. Thus f and g are not asymptotic.