SECTION 1.5 Rules of Inference

This section lays the groundwork for understanding proofs. You are asked to understand the logical rules of inference behind valid arguments, and you are asked to construct some highly stylized proofs using these rules. The proofs will become more informal in the next section and throughout the remainder of this book (and your mathematical studies).

1. This is modus ponens. The first statement is $p \to q$, where p is "Socrates is human" and q is "Socrates is mortal." The second statement is p. The third is q. Modus ponens is valid. We can therefore conclude that the conclusion of the argument (third statement) is true, because the hypotheses (the first two statements) are true.

3. **a)** This is the addition rule. We are concluding from p that $p \vee q$ must be true, where p is "Alice is a mathematics major" and q is "Alice is a computer science major."
b) This is the simplification rule. We are concluding from $p \wedge q$ that p must be true, where p is "Jerry is a mathematics major" and q is "Jerry is a computer science major.".
c) This is modus ponens. We are concluding from $p \to q$ and p that q must be true, where p is "it is rainy" and q is "the pool will be closed."
d) This is modus tollens. We are concluding from $p \to q$ and $\neg q$ that $\neg p$ must be true, where p is "it will snow today" and q is "the university will close today."
e) This is hypothetical syllogism. We are concluding from $p \to q$ and $q \to r$ that $p \to r$ must be true, where p is "I will go swimming," q is "I will stay in the sun too long," and r is "I will sunburn."

5. Let w be the proposition "Randy works hard," let d be the proposition "Randy is a dull boy," and let j be the proposition "Randy will get the job." We are given premises w, $w \to d$, and $d \to \neg j$. We want to conclude $\neg j$. We set up the proof in two columns, with reasons, as in Example 6.

Step	**Reason**
1. w	Hypothesis
2. $w \to d$	Hypothesis
3. d	Modus ponens using (2) and (3)
4. $d \to \neg j$	Hypothesis
5. $\neg j$	Modus ponens using (3) and (4)

7. First we use universal instantiation to conclude from "For all x, if x is a man, then x is mortal" the special case of interest, "If Socrates is a man, then Socrates is mortal." Then we use modus ponens to conclude that Socrates is mortal.

9. **a)** Because it was sunny on Tuesday, we assume that it did not rain or snow on Tuesday (otherwise we cannot do anything with this problem). If we use modus tollens on the universal instantiation of the given conditional statement applied to Tuesday, then we conclude that I did not take Tuesday off. If we now apply disjunctive syllogism to the disjunction in light of this conclusion, we see that I took Thursday off. Now use modus ponens on the universal instantiation of the given conditional statement applied to Thursday; we conclude that it rained or snowed on Thursday. One more application of disjunctive syllogism tells us that it rained on Thursday.
b) Using modus tollens we conclude two things—that I did not eat spicy food and that it did not thunder. Therefore by the conjunction rule of inference (Table 1), we conclude "I did not eat spicy food and it did not thunder."
c) By disjunctive syllogism from the first two hypotheses we conclude that I am clever. The third hypothesis gives us no useful information.

d) We can apply universal instantiation to the conditional statement and conclude that if Ralph (respectively, Ann) is a CS major, then he (she) has a PC. Now modus tollens tells us that Ralph is not a CS major. There are no conclusions to be drawn about Ann.

e) The first two conditional statements can be phrased as "If x is good for corporations, then x is good for the U.S." and "If x is good for the U.S., then x is good for you." If we now apply universal instantiation with x being "for you to buy lots of stuff," then we can conclude using modus ponens twice that for you to buy lots of stuff is good for the U.S. and is good for you.

f) The given conditional statement is "For all x, if x is a rodent, then x gnaws its food." We can form the universal instantiation of this with x being a mouse, a rabbit, and a bat. Then modus ponens allows us to conclude that mice gnaw their food; and modus tollens allows us to conclude that rabbits are not rodents. We can conclude nothing about bats.

11. We are asked to show that whenever $p_1, p_2, \ldots, p_n$ are true, then $q \to r$ must be true, given that we know that whenever $p_1, p_2, \ldots, p_n$ and q are true, then r must be true. So suppose that $p_1, p_2, \ldots, p_n$ are true. We want to establish that $q \to r$ is true. If q is false, then we are done, vacuously. Otherwise, q is true, so by the validity of the given argument form, we know that r is true.

13. In each case we set up the proof in two columns, with reasons, as in Example 6.

a) Let $c(x)$ be "x is in this class," let $j(x)$ be "x knows how to write programs in JAVA," and let $h(x)$ be "x can get a high paying job." We are given premises $c(\text{Doug})$, $j(\text{Doug})$, and $\forall x(j(x) \to h(x))$, and we want to conclude $\exists x(c(x) \wedge h(x))$.

Step	Reason
1. $\forall x(j(x) \to h(x))$	Hypothesis
2. $j(\text{Doug}) \to h(\text{Doug})$	Universal instantiation using (1)
3. $j(\text{Doug})$	Hypothesis
4. $h(\text{Doug})$	Modus ponens using (2) and (3)
5. $c(\text{Doug})$	Hypothesis
6. $c(\text{Doug}) \wedge h(\text{Doug})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge h(x))$	Existential generalization using (6)

b) Let $c(x)$ be "x is in this class," let $w(x)$ be "x enjoys whale watching," and let $p(x)$ be "x cares about ocean pollution." We are given premises $\exists x(c(x) \wedge w(x))$ and $\forall x(w(x) \to p(x))$, and we want to conclude $\exists x(c(x) \wedge p(x))$. In our proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(c(x) \wedge w(x))$	Hypothesis
2. $c(y) \wedge w(y)$	Existential instantiation using (1)
3. $w(y)$	Simplification using (2)
4. $c(y)$	Simplification using (2)
5. $\forall x(w(x) \to p(x))$	Hypothesis
6. $w(y) \to p(y)$	Universal instantiation using (5)
7. $p(y)$	Modus ponens using (3) and (6)
8. $c(y) \wedge p(y)$	Conjunction using (4) and (7)
9. $\exists x(c(x) \wedge p(x))$	Existential generalization using (8)

c) Let $c(x)$ be "x is in this class," let $p(x)$ be "x owns a PC," and let $w(x)$ be "x can use a word processing program." We are given premises $c(\text{Zeke})$, $\forall x(c(x) \to p(x))$, and $\forall x(p(x) \to w(x))$, and we want to conclnde $w(\text{Zeke})$.

Step	Reason
1. $\forall x(c(x) \to p(x))$	Hypothesis
2. $c(\text{Zeke}) \to p(\text{Zeke})$	Universal instantiation using (1)
3. $c(\text{Zeke})$	Hypothesis
4. $p(\text{Zeke})$	Modus ponens using (2) and (3)
5. $\forall x(p(x) \to w(x))$	Hypothesis
6. $p(\text{Zeke}) \to w(\text{Zeke})$	Universal instantiation using (5)
7. $w(\text{Zeke})$	Modus ponens using (4) and (6)

d) Let $j(x)$ be "x is in New Jersey," let $f(x)$ be "x lives within fifty miles of the ocean," and let $s(x)$ be "x has seen the ocean." We are given premises $\forall x(j(x) \to f(x))$ and $\exists x(j(x) \wedge \neg s(x))$, and we want to conclude $\exists x(f(x) \wedge \neg s(x))$. In our proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(j(x) \wedge \neg s(x))$	Hypothesis
2. $j(y) \wedge \neg s(y)$	Existential instantiation using (1)
3. $j(y)$	Simplification using (2)
4. $\forall x(j(x) \to f(x))$	Hypothesis
5. $j(y) \to f(y)$	Universal instantiation using (4)
6. $f(y)$	Modus ponens using (3) and (5)
7. $\neg s(y)$	Simplification using (2)
8. $f(y) \wedge \neg s(y)$	Conjunction using (6) and (7)
9. $\exists x(f(x) \wedge \neg s(x))$	Existential generalization using (8)

15. **a)** This is correct, using universal instantiation and modus ponens.
b) This is invalid. After applying universal instantiation, it contains the fallacy of affirming the conclusion.
c) This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis.
d) This is valid by universal instantiation and modus tollens.

17. We know that *some* x exists that makes $H(x)$ true, but we cannot conclude that Lola is one such x. Maybe only Suzanne is happy and everyone else is not happy. Then $\exists x\, H(x)$ is true, but $H(\text{Lola})$ is false.

19. **a)** This is the fallacy of affirming the conclusion, since it has the form "$p \to q$ and q implies p."
b) This reasoning is valid; it is modus tollens.
c) This is the fallacy of denying the hypothesis, since it has the form "$p \to q$ and $\neg p$ implies $\neg q$."

21. Let us give an argument justifying the conclusion. By the second premise, there is some lion that does not drink coffee. Let us call him Leo. Thus we know that Leo is a lion and that Leo does not drink coffee. By simplification this allows us to assert each of these statements separately. The first premise says that all lions are fierce; in particular, if Leo is a lion, then Leo is fierce. By modus ponens, we can conclude that Leo is fierce. Thus we conclude that Leo is fierce and Leo does not drink coffee. By the definition of the existential quantifier, this tells us that there exist fierce creatures that do not drink coffee; in other words, that some fierce creatures do not drink coffee.

23. The error occurs in step (5), because we cannot assume, as is being done here, that the c that makes P true is the same as the c that makes Q true.

25. We are given the premises $\forall x(P(x) \to Q(x))$ and $\neg Q(a)$. We want to show $\neg P(a)$. Suppose, to the contrary, that $\neg P(a)$ is not true. Then $P(a)$ is true. Therefore by universal modus ponens, we have $Q(a)$. But this contradicts the given premise $\neg Q(a)$. Therefore our supposition must have been wrong, and so $\neg P(a)$ is true, as desired.

27. We can set this up in two-column format.

Step	Reason
1. $\forall x(P(x) \land R(x))$	Premise
2. $P(a) \land R(a)$	Universal instantiation using (1)
3. $P(a)$	Simplification using (2)
4. $\forall x(P(x) \to (Q(x) \land S(x)))$	Premise
5. $Q(a) \land S(a)$	Universal modus ponens using (3) and (4)
6. $S(a)$	Simplification using (5)
7. $R(a)$	Simplification using (2)
8. $R(a) \land S(a)$	Conjunction using (7) and (6)
9. $\forall x(R(x) \land S(x))$	Universal generalization using (5)

29. We can set this up in two-column format. The proof is rather long but straightforward if we go one step at a time.

Step	Reason
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	Existential instantiation using (1)
3. $\forall x(P(x) \lor Q(x))$	Premise
4. $P(c) \lor Q(c)$	Universal instantiation using (3)
5. $Q(c)$	Disjunctive syllogism using (4) and (2)
6. $\forall x(\neg Q(x) \lor S(x))$	Premise
7. $\neg Q(c) \lor S(c)$	Universal instantiation using (6)
8. $S(c)$	Disjunctive syllogism using (5) and (7), since $\neg\neg Q(c) \equiv Q(c)$
9. $\forall x(R(x) \to \neg S(x))$	Premise
10. $R(c) \to \neg S(c)$	Universal instantiation using (9)
11. $\neg R(c)$	Modus tollens using (8) and (10), since $\neg\neg S(c) \equiv S(c)$
12. $\exists x \neg R(x)$	Existential generalization using (11)

31. Let p be "It is raining"; let q be "Yvette has her umbrella"; let r be "Yvette gets wet." Then our assumptions are $\neg p \lor q$, $\neg q \lor \neg r$, and $p \lor \neg r$. Using resolution on the first two assumptions gives us $\neg p \lor \neg r$. Using resolution on this and the third assumption gives us $\neg r$, so Yvette does not get wet.

33. Assume that this proposition is satisfiable. Using resolution on the first two clauses allows us to conclude $q \lor q$; in other words, we know that q has to be true. Using resolution on the last two clauses allows us to conclude $\neg q \lor \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable.

35. This argument is valid. We argue by contradiction. Assume that Superman does exist. Then he is not impotent, and he is not malevolent (this follows from the fourth sentence). Therefore by (the contrapositives of) the two parts of the second sentence, we conclude that he is able to prevent evil, and he is willing to prevent evil. By the first sentence, we therefore know that Superman does prevent evil. This contradicts the third sentence. Since we have arrived at a contradiction, our original assumption must have been false, so we conclude finally that Superman does not exist.