GUIDE TO REVIEW QUESTIONS FOR CHAPTER 6

1. **a)** See p. 394. **b)** $1/C(50,6)$

2. **a)** $\forall i(0 \leq p(x_i) \leq 1)$ and $\sum_{i=1}^{n} p(x_i) = 1$ **b)** $p(H) = 3/4$, $p(T) = 1/4$

3. **a)** See p. 404. **b)** $1/3$

4. **a)** See p. 405. **b)** yes

5. **a)** See p. 408. **b)** 1, 2, 3, 4, 5, 6

6. **a)** See p. 426. **b)** $1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{161}{36} \approx 4.47$

7. **a)** See p. 431. **b)** $(5n+6)/3$ (see Example 8 in Section 6.4)

8. **a)** See p. 406. **b)** See Theorem 2 in Section 6.2. **c)** See Theorem 2 in Section 6.4.

9. **a)** See p. 429. **b)** See Example 6 in Section 6.4.

10. **a)** See the discussion of Monte Carlo algorithms on p. 411. **b)** See Example 16 in Section 6.2.

11. See p. 418; $p(F \mid E) = \dfrac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})} = \dfrac{(1/3)(2/3)}{(1/3)(2/3) + (1/4)(1/3)} = \dfrac{8}{11}$

12. **a)** See pp. 433–434. **b)** See Theorem 4 in Section 6.4.

13. **a)** See p. 436. **b)** See Example 14 in Section 6.4.

14. **a)** See Theorem 7 in Section 6.4. **b)** See Example 18 in Section 6.4.

15. See pp. 438–439.

SUPPLEMENTARY EXERCISES FOR CHAPTER 6

1. There are 35 outcomes in which the numbers chosen are consecutive, since the first of these numbers can be anything from 1 to 35. There are $C(40,6) = 3{,}838{,}380$ possible choices in all. Therefore the answer is $35/3838380 = 1/109668$.

3. Each probability is of the form s/t where s is the number of hands of the described type and t is the total number of hands, which is clearly $C(52,13) = 635{,}013{,}559{,}600$. Hence in each case we will count the number of hands and divide by this value.

a) There is only one hand with all 13 hearts, so the probability is $1/t$, which is about 1.6×10^{-12}.

b) There are four such hands, since there are four ways to choose the suit, so the answer is $4/t$, which is about 6.3×10^{-12}.

c) To specify such a hand we need to choose 7 spades from the 13 spades available and then choose 6 clubs from the 13 clubs available. Thus there are $C(13,7)C(13,6) = 2944656$ such hands. The probability is therefore $2944656/t \approx 4.6 \times 10^{-6}$.

d) This event is 12 times more likely than the event in part **(c)**, since there are $P(4,2) = 12$ ways to choose the two suits. Thus the answer is $35335872/t \approx 5.6 \times 10^{-5}$.

e) This is similar to part **(c)**, but with four choices to make. The answer is $C(13,4)C(13,6)C(13,2)$ $C(13,1)/t = 1244117160/635013559600 \approx 2.0 \times 10^{-3}$.

f) There are $P(4,4) = 24$ ways to specify the suits, and then there are $C(13,4)C(13,6)C(13,2)C(13,1)$ ways to choose the cards from these suits to construct the desired hand. Therefore the answer is 24 times as big as the answer to part **(e)**, namely $29858811840/635013559600 \approx 0.047$.

5. a) Each of the outcomes 1 through 8 occurs with probability $1/8$, so the expectation is $(1/8)(1+2+3+\cdots+8) = 9/2$.

b) We compute $V(X) = E(X^2) - E(X)^2 = (1/8)(1^2+2^2+3^2+\cdots+8^2) - (9/2)^2 = (51/2) - (81/4) = 21/4$.

7. a) Since expected value is linear, the expected value of the sum is the sum of the expected values, each of which is $9/2$ by Exercise 5a. Therefore the answer is 9.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, each of which is $21/4$ by Exercise 5b. Therefore the answer is $21/2$.

9. a) Since expected value is linear, the expected value of the sum is the sum of the expected values, which are $7/2$ by Example 1 in Section 6.4 and $9/2$ by Exercise 5a. Therefore the answer is $(7/2) + (9/2) = 8$.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, which are $35/12$ by Example 15 in Section 6.4 and $21/4$ by Exercise 5b. Therefore the answer is $(35/12) + (21/4) = 49/6$.

11. a) There are 2^n possible outcomes of the flips. In order for the odd person out to be decided, we must have one head and $n-1$ tails, or one tail and $n-1$ heads. The number of ways for this to happen is clearly $2n$ (choose the odd person and choose whether it is heads or tails). Therefore the probability that there is an odd person out is $2n/2^n = n/2^{n-1}$. Call this value p.

b) Clearly the number of flips has a geometric distribution with parameter $p = n/2^{n-1}$, from part **(a)**. Therefore the probability that the odd person out is decided with the k^{th} flip is $p(1-p)^{k-1}$.

c) By Theorem 4 in Section 6.4, the expectation is $1/p = 2^{n-1}/n$.

13. We start by counting the number of positive integers less than mn that *are* divisible by either m or n. Certainly all the integers m, $2m$, $3m$, $\ldots$, nm are divisible by m. There are n numbers in this list. All but one of them are less than mn. Therefore $n-1$ positive integers less than mn are divisible by m. Similarly, $m-1$ positive integers less than mn are divisible by n. Next we need to see how many numbers are divisible by both m and n. A number is divisible by both m and n if and only if it is divisible by the least common multiple of m and n. Let $L = \text{lcm}(m,n)$. Thus the numbers divisible by both m and n are L, $2L$, ..., mn. This list has $\gcd(m,n)$ numbers in it, since we know that $\text{lcm}(m,n)\cdot\gcd(m,n) = mn$. Therefore $\gcd(m,n)-1$ positive integers strictly less than mn are divisible by both m and n. Using the inclusion-exclusion principle, we deduce that $(n-1)+(m-1)-(\gcd(m,n)-1) = n+m-\gcd(m,n)-1$ positive integers less than mn are divisible by either m or n. Therefore $(mn-1)-(n+m-\gcd(m,n)-1) = mn-n-m+\gcd(m,n) = (m-1)(n-1)+\gcd(m,n)-1$ numbers in this range are not divisible by either m or n. This gives us our answer:

$$\frac{(m-1)(n-1)+\gcd(m,n)-1}{mn-1}$$

15. **a)** Label the faces of the cards $F1$, $B1$, $F2$, $B2$, $F3$, and $B3$ (here F stands for front, B stands for back, and the numeral stands for the card number). Without loss of generality, assume that $F1$, $B1$, and $F2$ are the black faces. There are six equally likely outcomes of this experiment, namely that we are looking at each of these faces. Then the event that we are looking at a black face is the event $E_1 = \{F1, B1, F2\}$. The event that the other side is also black is the event $E_2 = \{F1, B1\}$. We are asked for $p(E_2 \mid E_1)$, which is by definition $p(E_2 \cap E_1)/p(E_1) = p(\{F1, B1\})/p(\{F1, B1, F2\}) = (2/6)/(3/6) = 2/3$.
b) The argument in part **(a)** works for red as well, so the answer is again $2/3$. This seeming paradox comes up in other contexts, such as the Law of Restricted Choice in the game of bridge.

17. There are 2^{10} bit strings. There are 2^5 palindromic bit strings, since once the first five bits are specified arbitrarily, the remaining five bits are forced. If a bit string is picked at random, then, the probability that it is a palindrome is $2^5/2^{10} = 1/32$.

19. **a)** We assume that the coin is fair, so the probability of a head is $1/2$ on each flip, and the flips are independent. The probability that one wins 2^n dollars (i.e., $p(X = 2^n)$) is $1/2^n$, since that happens precisely when the player gets $n-1$ tails followed by a head. The expected value of the winnings is therefore the sum of 2^n times $1/2^n$ as n goes from 1 to infinity. Since each of these terms is 1, the sum is infinite. In other words, one should be willing to wager any amount of money and expect to come out ahead in the long run. The catch, of course, and that is partly why it is a paradox, is that the long run is too long, and the bank could not actually pay 2^n dollars for large n anyway (it would exceed the world's money supply). It would not make sense for someone to pay a million dollars to play this game just once.
b) Now the expectation is $(1/2)(2^1)+(1/2^2)(2^2)+(1/2^3)(2^3)+(1/2^4)(2^4)+(1/2^5)(2^5)+(1/2^6)(2^6)+(1/2^7)(2^7)+(1/2^7)(2^8) = 9$. Therefore a fair wager would be \$9.

21. **a)** The intersection of two sets is a subset of each of them, so the largest $p(A \cap B)$ could be would occur when the smaller is a subset of the larger. In this case, that would mean that we want $B \subseteq A$, in which case $A \cap B = B$, so $p(A \cap B) = p(B) = 1/3$. To construct an example, we find a common denominator of the fractions involved, namely 12, and let the sample space consist of 12 equally likely outcomes, say numbered 1 through 12. We let $B = \{1, 2, 3, 4\}$ and $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The smallest intersection would occur when $A \cup B$ is as large as possible, since $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. The largest $A \cup B$ could ever be is the entire sample space, whose probability is 1, and that certainly can occur here. So we have $1 = (3/4) + (1/3) - p(A \cap B)$, which gives $p(A \cap B) = 1/12$. To construct an example, again we find a common denominator of these fractions, namely 12, and let the sample space consist of 12 equally likely outcomes, say numbered 1 through 12. We let $B = \{1, 2, 3, 4\}$ and $A = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Then $A \cap B = \{4\}$, and $p(A \cap B) = 1/12$.

b) The largest $p(A \cup B)$ could ever be is 1, which occurs when $A \cup B$ is the entire sample space. As we saw in part **(a)**, that is possible here, using the second example above. The union of two sets is a superset of each of them, so the smallest $p(A \cup B)$ could be would occur when the smaller is a subset of the larger. In this case, that would mean that we want $B \subseteq A$, in which case $A \cup B = A$, so $p(A \cup B) = p(A) = 3/4$. This occurs in the first example given above.

23. **a)** We need three conditions for two of the events at once and one condition for all three:

$$\begin{aligned} p(E_1 \cap E_2) &= p(E_1)p(E_2) \\ p(E_1 \cap E_3) &= p(E_1)p(E_3) \\ p(E_2 \cap E_3) &= p(E_2)p(E_3) \\ p(E_1 \cap E_2 \cap E_3) &= p(E_1)p(E_2)p(E_3) \end{aligned}$$

b) Intuitively, it is clear that these three events are independent, since successive flips do not depend on the results of previous flips. Mathematically, we need to look at the various events. There are 8 possible outcomes of this experiment. In four of them the first flip comes up heads, so $p(E_1) = 4/8 = 1/2$. Similarly, $p(E_2) = 1/2$ and $p(E_3) = 1/2$. In two of these outcomes the first flip is a head and the second flip is a tail, so $p(E_1 \cap E_2) = 2/8 = 1/4$. Similarly, $p(E_1 \cap E_3) = 1/4$ and $p(E_2 \cap E_3) = 1/4$. Only one outcome has all three events happening, so $p(E_1 \cap E_2 \cap E_3) = 1/8$. We now need to plug these numbers into the four equations displayed in part **(a)** and check the they are satisfied:

$$\begin{aligned} p(E_1 \cap E_2) &= \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1)p(E_2) \\ p(E_1 \cap E_3) &= \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1)p(E_3) \\ p(E_2 \cap E_3) &= \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_2)p(E_3) \\ p(E_1 \cap E_2 \cap E_3) &= \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = p(E_1)p(E_2)p(E_3) \end{aligned}$$

The first three lines show that E_1, E_2, and E_3 are pairwise independent, and these together with the last line show that they are mutually independent.

c) We need to compute the following quantities, which we do by counting outcomes. $p(E_1) = 4/8 = 1/2$, $p(E_2) = 4/8 = 1/2$, $p(E_3) = 4/8 = 1/2$, $p(E_1 \cap E_2) = 2/8 = 1/4$, $p(E_1 \cap E_3) = 2/8 = 1/4$, $p(E_2 \cap E_3) = 2/8 = 1/4$, and $p(E_1 \cap E_2 \cap E_3) = 1/8$. Note that these are the same values obtained in part **(b)**. Therefore when we plug them into the defining equations for independence, we must again get true statements, so these events are both pairwise and mutually independent.

d) We need to compute the following quantities, which we do by counting outcomes. $p(E_1) = 4/8 = 1/2$, $p(E_2) = 4/8 = 1/2$, $p(E_3) = 4/8 = 1/2$, $p(E_1 \cap E_2) = 2/8 = 1/4$, $p(E_1 \cap E_3) = 2/8 = 1/4$, $p(E_2 \cap E_3) = 2/8 = 1/4$, and $p(E_1 \cap E_2 \cap E_3) = 0/8$. Note that these are the same values obtained in part **(b)**, except that now $p(E_1 \cap E_2 \cap E_3) = 0/8$. Therefore when we plug them into the defining equations for independence, we again get true statements for the first three, but not for the last one. Therefore, these events are pairwise independent, but they are not mutually independent.

e) There will be one condition for each subset of the set of events, other than subsets consisting of no events or just one event. There are 2^n subsets of a set with n elements, and $n+1$ of them have fewer than two elements. Therefore there are $2^n - n - 1$ subsets of interest and that many conditions to check.

25. Using Theorems 3 and 6 of Section 6.4 and the fact that the expectation of a constant is itself (this is easy to prove from the definition), we have

$$\begin{aligned}V(aX+b) &= E((aX+b)^2) - E(aX+b)^2\\ &= E(a^2X^2+2abX+b^2) - (aE(X)+b)^2\\ &= E(a^2X^2)+E(2abX)+E(b^2) - (a^2E(X)^2+2abE(X)+b^2)\\ &= a^2E(X^2)+2abE(X)+b^2 - a^2E(X)^2 - 2abE(X) - b^2)\\ &= a^2(E(X^2)-E(X)^2) = a^2V(X).\end{aligned}$$

27. This is essentially an application of inclusion–exclusion (Section 7.5). To count every element in the sample space exactly once, we want to include every element in each of the sets and then take away the double counting of the elements in the intersections. Thus $p(E_1 \cup E_2 \cup \cdots \cup E_m) = p(E_1)+p(E_2)+\cdots+p(E_m)-p(E_1\cap E_2)-p(E_1\cap E_3)-\cdots-p(E_1\cap E_m)-p(E_2\cap E_3)-p(E_2\cap E_4)-\cdots-p(E_2\cap E_m)-\cdots-p(E_{m-1}\cap E_m) = qm-(m(m-1)/2)r$, since $C(m,2)$ terms are being subtracted. But $p(E_1\cup E_2\cup\cdots\cup E_m)=1$, so we have $qm-(m(m-1)/2)r=1$. Since $r\geq 0$, this equation tells us that $qm\geq 1$, so $q\geq 1/m$. Since $q\leq 1$, this equation also implies that $(m(m-1)/2)r = qm-1\leq m-1$, from which it follows that $r\leq 2/m$.

29. **a)** We purchase the cards until we have gotten one of each type. That means we have purchased X cards in all. On the other hand, that also means that we purchased X_0 cards until we got the first type we got (of course $X_0=1$ in all cases), and then purchased X_1 more cards until we got the second type we got, and so on. Thus X is the sum of the X_j's.
b) Once j distinct types have been obtained, there are $n-j$ new types available out of a total of n types available. Since it is equally likely that we get each type, the probability of success on the next purchase (getting a new type) is $(n-j)/n$.
c) This follows immediately from the definition of geometric distribution, the definition of X_j, and part **(b)**.
d) From part **(c)** it follows that $E(X_j)=n/(n-j)$. Thus by linearity of expectation from part **(a)** we have

$$E(X)=E(X_0)+E(X_1)+\cdots+E(X_{n-1}) = \frac{n}{n}+\frac{n}{n-1}+\cdots+\frac{n}{1} = n\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{1}\right).$$

e) If $n=50$, then

$$E(X)=n\sum_{j=1}^{n}\frac{1}{j}\approx n(\ln n+\gamma)\approx 50(\ln 50+0.57721)\approx 224.46.$$

We can compute the exact answer using a computer algebra system:

$$\frac{13943237577224054960759}{61980890084919934128}\approx 224.96$$

31. We see from Exercise 42 in Section 5.5 (applying the idea in Example 8 of that section) that there are $52!/13!^4$ possible ways to deal the cards. In order to answer this question, we need to find the number of ways to deal them so that each player gets an ace. There are $4!=24$ ways to distribute the aces so that each player receives one. Once this is done, there are 48 cards left, 12 to be dealt to each player, so using the idea in Example 8 in Section 5.5 again, there are $48!/12!^4$ possible ways to deal these cards. Taking the quotient of these two quantities will give us the desired probability:

$$\frac{24\cdot(48!/12!^4)}{52!/13!^4} = \frac{24\cdot 13^4}{52\cdot 51\cdot 50\cdot 49} = \frac{2197}{20825}\approx\frac{1}{10}$$

WRITING PROJECTS FOR CHAPTER 6

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. A general mathematics history text should cover this topic well. Introductory probability books might also have a few words on the subject.

2. It will be instructive to see whether the advice given in popular gambling books (which is where to go for this project) is correct! Your university library might not be a good place to look for this project; try your local bookstore instead.

3. As in the previous project, you should consult popular books on this subject. James Thorpe was one of the first persons to realize that the player can win against the house in blackjack by using the right strategy (which involves keeping track of the cards that have already been used, as well as doing the right thing in terms of drawing additional cards on each hand).

4. See the comments for Writing Project 2.

5. Google gives about five million hits on the phrase "spam filter" (in quotation marks). You might also want to include the word "successful" to narrow the search.

6. There is an article on this in an old issue of *The American Statistician* [Sc].

7. See the classical book on the probabilistic method, in which Erdős has written an appendix, [AlSp].

8. Modern books on algorithms, such as [CoLe], are good sources here.

CHAPTER 7
Advanced Counting Techniques

SECTION 7.1 Recurrence Relations

This section is related to Section 4.3, in that recurrence relations are in some sense really recursive or inductive definitions. Many of the exercises in this set provide practice in setting up such relations from a given applied situation. In each problem of this type, ask yourself how the n^{th} term of the sequence can be related to one or more previous terms; the answer is the desired recurrence relation.

Some of these exercises deal with solving recurrence relations by the iterative approach. The trick here is to be precise and patient. First write down a_n in terms of a_{n-1}. Then use the recurrence relation with $n-1$ plugged in for n to rewrite what you have in terms of a_{n-2}; simplify algebraically. Continue in this manner until a pattern emerges. Then write down what the expression is in terms of a_0 (or a_1, depending on the initial condition), following the pattern that developed in the first few terms. Usually at this point either the answer is what you have just written down, or else the answer can be obtained from what you have by summing a series. The iterative approach is not usually effective for recurrence relations of degree greater than 1 (i.e., those in which a_n depends on previous terms other than just a_{n-1}).

Exercise 37 is interesting and challenging, and shows that the inductive step may be quite nontrivial. Exercise 45 deals with onto functions; another—totally different—approach to counting onto functions is given in Section 7.6. Exercises 46–61 deal with additional interesting applications.

1. We need to compute the terms of the sequence one at a time, since each term is dependent upon one or more of the previous terms.

a) We are given $a_0 = 2$. Then by the recurrence relation $a_n = 6a_{n-1}$ we see (by letting $n = 1$) that $a_1 = 6a_0 = 6\cdot 2 = 12$. Similarly $a_2 = 6a_1 = 6\cdot 12 = 72$, then $a_3 = 6a_2 = 6\cdot 72 = 432$, and $a_4 = 6a_3 = 6\cdot 432 = 2592$.
b) $a_1 = 2$ (given), $a_2 = a_1^2 = 2^2 = 4$, $a_3 = a_2^2 = 4^2 = 16$, $a_4 = a_3^2 = 16^2 = 256$, $a_5 = a_4^2 = 256^2 = 65536$
c) This time each term depends on the two previous terms. We are given $a_0 = 1$ and $a_1 = 2$. To compute a_2 we let $n = 2$ in the recurrence relation, obtaining $a_2 = a_1 + 3a_0 = 2 + 3\cdot 1 = 5$. Then we have $a_3 = a_2 + 3a_1 = 5 + 3\cdot 2 = 11$ and $a_4 = a_3 + 3a_2 = 11 + 3\cdot 5 = 26$.
d) $a_0 = 1$ (given), $a_1 = 1$ (given), $a_2 = 2a_1 + 2^2a_0 = 2\cdot 1 + 4\cdot 1 = 6$, $a_3 = 3a_2 + 3^2a_1 = 3\cdot 6 + 9\cdot 1 = 27$, $a_4 = 4a_3 + 4^2a_2 = 4\cdot 27 + 16\cdot 6 = 204$
e) We are given $a_0 = 1$, $a_1 = 2$, and $a_2 = 0$. Then $a_3 = a_2 + a_0 = 0 + 1 = 1$ and $a_4 = a_3 + a_1 = 1 + 2 = 3$.

3. a) We simply plug in $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$. Thus we have $a_0 = 2^0 + 5\cdot 3^0 = 1 + 5\cdot 1 = 6$, $a_1 = 2^1 + 5\cdot 3^1 = 2 + 5\cdot 3 = 17$, $a_2 = 2^2 + 5\cdot 3^2 = 4 + 5\cdot 9 = 49$, $a_3 = 2^3 + 5\cdot 3^3 = 8 + 5\cdot 27 = 143$, and $a_4 = 2^4 + 5\cdot 3^4 = 16 + 5\cdot 81 = 421$.
b) Using our data from part (a), we see that $49 = 5\cdot 17 - 6\cdot 6$, $143 = 5\cdot 49 - 6\cdot 17$, and $421 = 5\cdot 143 - 6\cdot 49$.
c) This is algebra. The messiest part is factoring out a large power of 2 and a large power of 3. If we substitute $n-1$ for n in the definition we have $a_{n-1} = 2^{n-1} + 5\cdot 3^{n-1}$; similarly $a_{n-2} = 2^{n-2} + 5\cdot 3^{n-2}$. We start with the right-hand side of our desired identity: