SECTION 5.3 Permutations and Combinations

In this section we look at counting problems more systematically than in Section 5.1. We have some formulae that apply in many instances, and the trick is to recognize the instances. If an ordered arrangement without repetitions is asked for, then the formula for permutations usually applies; if an unordered selection without repetition is asked for, then the formula for combinations usually applies. Of course the product rule and the sum rule (and common sense and cleverness) are still needed to solve some of these problems—having formulae for permutations and combinations definitely does not reduce the solving of counting problems to a mechanical algorithm.

Again the general comments of Section 5.1 apply. Try to solve problems more than one way and come up with the same answer—you will learn from the process of looking at the same problem from two or more different angles, and you will be (almost) sure that your answer is correct.

1. Permutations are ordered arrangements. Thus we need to list all the ordered arrangements of all 3 of these letters. There are 6 such: a, b, c; a, c, b; b, a, c; b, c, a; c, a, b; and c, b, a. Note that we have listed them in alphabetical order. Algorithms for generating permutations and combinations are discussed in Section 5.6.

3. If we want the permutation to end with a, then we may as well forget about the a, and just count the number of permutations of $\{b, c, d, e, f, g\}$. Each permutation of these 6 letters, followed by a, will be a permutation of the desired type, and conversely. Therefore the answer is $P(6,6) = 6! = 720$.

5. We simply plug into the formula $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$, given in Theorem 1. Note that there are r terms in this product, starting with n. This is the same as $P(n,r) = n!/(n-r)!$, but the latter formula is not as nice for computation, since it ignores the fact that each of the factors in the denominator cancels one factor in the numerator. Thus to compute $n!$ and $(n-r)!$ and then to divide is to do a lot of extra arithmetic. Of course if the denominator is 1, then there is no extra work, so we note that $P(n,n) = P(n,n-1) = n!$.

a) $P(6,3) = 6 \cdot 5 \cdot 4 = 120$ b) $P(6,5) = 6! = 720$ c) $P(8,1) = 8$

d) $P(8,5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ e) $P(8,8) = 8! = 40{,}320$ f) $P(10,9) = 10! = 3{,}628{,}800$

7. This is $P(9,5) = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15{,}120$ by Theorem 1.

9. We need to pick 3 horses from the 12 horses in the race, and we need to arrange them in order (first, second, and third), in order to specify the win, place, and show. Thus there are $P(12,3) = 12 \cdot 11 \cdot 10 = 1320$ possibilities.

11. **a)** To specify a bit string of length 10 that contains exactly four 1's, we simply need to choose the four positions that contain the 1's. There are $C(10,4) = 210$ ways to do that.

b) To contain at most four 1's means to contain four 1's, three 1's, two 1's, one 1, or no 1's. Reasoning as in part **(a)**, we see that there are $C(10,4)+C(10,3)+C(10,2)+C(10,1)+C(10,0) = 210+120+45+10+1 = 386$ such strings.

c) To contain at least four 1's means to contain four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, or ten 1's. Reasoning as in part **(b)**, we see that there are $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10) = 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848$ such strings. A simpler approach would be to figure out the number of ways not to have at least four 1's (i.e., to have three 1's, two 1's, one 1, or no 1's) and then subtract that from 2^{10}, the total number of bit strings of length 10. This way we get $1024 - (120 + 45 + 10 + 1) = 848$, fortunately the same answer as before. Solving a combinatorial problem in more than one way is a useful check on the correctness of the answer.

d) To have an equal number of 0's and 1's in this case means to have five 1's. Therefore the answer is $C(10,5) = 252$. Incidentally, this gives us another way to do part **(b)**. If we don't have an equal number of 0's and 1's, then we have either at most four 1's or at least six 1's. By symmetry, having at most four 1's occurs in half of these cases. Therefore the answer to part **(b)** is $(2^{10} - C(10,5))/2 = 386$, as above.

13. We assume that the row has a distinguished head. Consider the order in which the men appear relative to each other. There are n men, and all of the $P(n,n) = n!$ arrangements is allowed. Similarly, there are $n!$ arrangements in which the women can appear. Now the men and women must alternate, and there are the same number of men and women; therefore there are exactly two possibilities: either the row starts with a man and ends with a woman ($MWMW\ldots MW$) or else it starts with a woman and ends with a man ($WMWM\ldots WM$). We have three tasks to perform, then: arrange the men among themselves, arrange the women among themselves, and decide which sex starts the row. By the product rule there are $n!\cdot n!\cdot 2 = 2(n!)^2$ ways in which this can be done.

15. We assume that a combination is called for, not a permutation, since we are told to *select a set*, not *form an arrangement*. We need to choose 5 things from 26, so there are $C(26,5) = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22/5! = 65{,}780$ ways to do so.

17. We know that there are 2^{100} subsets of a set with 100 elements. All of them have more than two elements except the empty set, the 100 subsets consisting of one element each, and the $C(100,2) = 4950$ subsets with two elements. Therefore the answer is $2^{100} - 5051 \approx 1.3 \times 10^{30}$.

19. **a)** Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are $C(10,2) = 45$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part **(b)**, we see that there are $C(10,3) + C(10,2) + C(10,1) + C(10,0) = 120 + 45 + 10 + 1 = 176$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is $C(10,5) = 252$.

21. **a)** If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items—the letters A, E, F, and G, and the superletter BCD. Therefore the answer is $P(5,5) = 5! = 120$.

b) Reasoning as in part **(a)**, we see that the answer is $P(4,4) = 4! = 24$.

c) As in part (a), we glue BA into one item and glue GF into one item. Therefore we need to permute five items, and there are $P(5,5) = 5! = 120$ ways to do it.

d) This is similar to part (c). Glue ABC into one item and glue DE into one item, producing four items, so the answer is $P(4,4) = 4! = 24$.

e) If both ABC and CDE are substrings, then $ABCDE$ has to be a substring. So we are really just permuting three items: $ABCDE$, F, and G. Therefore the answer is $P(3,3) = 3! = 6$.

f) There are no permutations with both of these substrings, since B cannot be followed by both A and E at the same time.

23. First position the men relative to each other. Since there are eight men, there are $P(8,8)$ ways to do this. This creates nine slots where a woman (but not more than one woman) may stand: in front of the first man, between the first and second men, $\ldots$, between the seventh and eight men, and behind the eighth man. We need to choose five of these positions, in order, for the first through fifth woman to occupy (order matters, because the women are distinct people). This can be done is $P(9,5)$ ways. Therefore the answer is $P(8,8) \cdot P(9,5) = 8! \cdot 9!/4! = 609{,}638{,}400$.

25. a) Since the prizes are different, we want an ordered arrangement of four numbers from the set of the first 100 positive integers. Thus there are $P(100,4) = 94{,}109{,}400$ ways to award the prizes.

b) If the grand prize winner is specified, then we need to choose an ordered set of three tickets to win the other three prizes. This can be done is $P(99,3) = 941{,}094$ ways.

c) We can first determine which prize the person holding ticket 47 will win (this can be done in 4 ways), and then we can determine the winners of the other three prizes, exactly as in part (b). Therefore the answer is $4P(99,3) = 3{,}764{,}376$.

d) This is the same calculation as in part (a), except that there are only 99 viable tickets. Therefore the answer is $P(99,4) = 90{,}345{,}024$. Note that this answer plus the answer to part (c) equals the answer to part (a), since the person holding ticket 47 either wins the grand prize or does not win the grand prize.

e) This is similar to part (c). There are $4 \cdot 3 = 12$ ways to determine which prizes these two lucky people will win, after which there are $P(98,2) = 9506$ ways to award the other two prizes. Therefore the answer is $12 \cdot 9506 = 114{,}072$.

f) This is like part (e). There are $P(4,3) = 24$ ways to choose the prizes for the three people mentioned, and then 97 ways to choose the other winner. This gives $24 \cdot 97 = 2328$ ways in all.

g) Here it is just a matter of ordering the prizes for these four people, so the answer is $P(4,4) = 24$.

h) This is similar to part (d), except that this time the pool of viable numbers has only 96 numbers in it. Therefore the answer is $P(96,4) = 79{,}727{,}040$.

i) There are four ways to determine the grand prize winner under these conditions. Then there are $P(99,3)$ ways to award the remaining prizes. This gives an answer of $4P(99,3) = 3{,}764{,}376$.

j) First we need to choose the prizes for the holder of 19 and 47. Since there are four prizes, there are $P(4,2) = 12$ ways to do this. Then there are 96 people who might win the remaining prizes, and there are $P(96,2) = 9120$ ways to award these prizes. Therefore the answer is $12 \cdot 9120 = 109{,}440$.

27. a) Since the order of choosing the members is not relevant (the offices are not differentiated), we need to use a combination. The answer is clearly $C(25,4) = 12{,}650$.

b) In contrast, here we need a permutation, since the order matters (we choose first a president, then a vice president, then a secretary, then a treasurer). The answer is clearly $P(25,4) = 303{,}600$.

29. a) In this part the permutation $5, 6, 32, 7$, for example, is to be counted, since it contains the consecutive numbers 5, 6, and 7 in their correct order (even thongh separated by the 32). In order to specify such a

4-permutation, we first need to choose the 3 consecutive integers. They can be anything from $\{1,2,3\}$ to $\{98,99,100\}$; thus there are 98 possibilities. Next we need to decide which slot is to contain a number not in this set; there are 4 possibilities. Finally, we need to decide which of the 97 other positive integers not exceeding 100 is to fill this slot, and there are of course 97 choices. Thus our first attempt at an answer gives us, by the product rule, $98 \cdot 4 \cdot 97$.

Unfortunately, this answer is not correct, because we have counted some 4-permutations more than once. Consider the 4-permutation $4,5,6,7$, for example. We cannot tell whether it arose from choosing 4, 5, and 6 as the consecutive numbers, or from choosing 5, 6, and 7. (These are the only two ways it could have arisen.) In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted. Therefore to correct our count, we need to subtract the number of such 4-permutations. Clearly there are 97 of them (they can begin with any number from 1 to 97). Further thought shows that every other 4-permutation in our collection arises in a unique way (in other words, there is a unique subsequence of three consecutive integers). Thus our final answer is $98 \cdot 4 \cdot 97 - 97 = 37{,}927$.

b) In this part we are insisting that the consecutive numbers be consecutive in the 4-permutation as well. The analysis in part **(a)** works here, except that there are only 2 places to put the fourth number—in slot 1 or in slot 4. Therefore the answer is $98 \cdot 2 \cdot 97 - 97 = 18{,}915$.

31. We need to be careful here, because strings can have repeated letters.

a) We need to choose the position for the vowel, and this can be done in 6 ways. Next we need to choose the vowel to use, and this can be done in 5 ways. Each of the other five positions in the string can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string. Therefore the answer is $6 \cdot 5 \cdot 21^5 = 122{,}523{,}030$.

b) We need to choose the position for the vowels, and this can be done in $C(6,2) = 15$ ways (we need to choose two positions out of six). We need to choose the two vowels (5^2 ways). Each of the other four positions in the string can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string. Therefore the answer is $15 \cdot 5^2 \cdot 21^4 = 72{,}930{,}375$.

c) The best way to do this is to count the number of strings with no vowels and subtract this from the total number of strings. We obtain $26^6 - 21^6 = 223{,}149{,}655$.

d) As in part **(c)**, we will do this by subtracting from the total number of strings, the number of strings with no vowels and the number of strings with one vowel (this latter quantity having been computed in part **(a)**). We obtain $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 = 223149655 - 122523030 = 100{,}626{,}625$.

33. We are told that we must select three of the 10 men and three of the 15 women. This can be done is $C(10,3)C(15,3) = 54{,}600$ ways.

35. To implement the condition that every 0 be immediately followed by a 1, let us think of "gluing" a 1 to the right of each 0. Then the objects we have to work with are eight blocks consisting of the string 01 and two 1's. The question is, then, how many strings are there consisting of these ten objects? This is easy to calculate, for we simply have to choose two of the "positions" in the string to contain the 1's and fill the remaining "positions" with the 01 blocks. Therefore the answer is $C(10,2) = 45$.

37. Perhaps the most straightforward way to do this is to look at the several cases. The string might contain three 1's and seven 0's, four 1's and six 0's, five of each, six 1's and four 0's, or seven 1's and three 0's. In each case we can determine the number of strings by calculating a binomial coefficient, since we simply need to choose the positions for the 1's. Therefore the answer is $C(10,3)+C(10,4)+C(10,5)+C(10,6)+C(10,7) = 120+210+252+210+120 = 912$.

39. To specify such a license plate we need to write down a 3-permutation of the set of 26 letters and follow it by a 3-permutation of the set of 10 digits. By the product rule the answer is therefore $P(26,3) \cdot P(10,3) = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11{,}232{,}000$.

41. If there are no ties, then there are $3! = 6$ possible finishes. If two of the horses tie and the third has a different time, then there are 3 ways to decide which horse is not tied and then 2 ways to decide whether that horse finishes first or last. That gives $3 \cdot 2 = 6$ possibilities. Finally, all three horses can tie. So the answer is $6 + 6 + 1 = 13$.

43. We can solve this problem by breaking it down into cases depending on the ties. There are four basic cases. (1) If there are unique gold and silver winners, then we can choose these winners in $6 \cdot 5 = 30$ ways. Any nonempty subset of the remaining four runners can win the bronze medal. There are $2^4 - 1 = 15$ ways to choose these people, giving us $30 \cdot 15 = 450$ ways in all for this case. (2) If there is a 2-way tie for first place, then there are $C(6,2) = 15$ ways to choose the gold medalists. Any nonempty subset of the remaining four runners can win the bronze medal, so there are $2^4 - 1 = 15$ ways to choose these people, giving us $15 \cdot 15 = 225$ ways in all for this case. (3) If there is a k-way tie for first with $k \geq 3$, then there are $C(6,k)$ ways to choose the gold medalists (there are no other medals in this case). This gives us $C(6,3) + C(6,4) + C(6,5) + C(6,6) = 20 + 15 + 6 + 1 = 42$ more possibilities. (4) The only other case is that there is a single gold medal winner and a k-way tie for second with $k \geq 2$. We can choose the winner in 6 ways and the silver medalists in $2^5 - C(5,1) - C(5,0) = 32 - 5 - 1 = 26$ ways. This gives us $6 \cdot 26 = 156$ possibilities. Putting this all together, the answer is $450 + 225 + 42 + 156 = 873$.