SECTION 3.2 The Growth of Functions

2. Note that the choices of C and k witnesses are not unique.

a) Yes, since $17x + 11 \le 17x + x = 18x \le 18x^2$ for all $x > 11$. The witnesses are $C = 18$ and $k = 11$.

b) Yes, since $x^2 + 1000 \le x^2 + x^2 = 2x^2$ for all $x > \sqrt{1000}$. The witnesses are $C = 2$ and $k = \sqrt{1000}$.

c) Yes, since $x \log x \le x \cdot x = x^2$ for all x in the domain of the function. (The fact that $\log x < x$ for all x follows from the fact that $x < 2^x$ for all x, which can be seen by looking at the graphs of these two functions.) The witnesses are $C = 1$ and $k = 0$.

d) No. If there were a constant C such that $x^4/2 \le Cx^2$ for sufficiently large x, then we would have $C \ge x^2/2$. This is clearly impossible for a constant to satisfy.

e) No. If 2^x were $O(x^2)$, then the fraction $2^x/x^2$ would have to be bounded above by some constant C. It can be shown that in fact $2^x > x^3$ for all $x \ge 10$ (using mathematical induction—see Section 4.1—or calculus), so $2^x/x^2 \ge x^3/x^2 = x$ for large x, which is certainly not less than or equal to C.

f) Yes, since $\lfloor x \rfloor \lceil x \rceil \le x(x+1) \le x \cdot 2x = 2x^2$ for all $x > 1$. The witnesses are $C = 2$ and $k = 1$.

4. If $x > 5$, then $2^x + 17 \le 2^x + 2^x = 2 \cdot 2^x \le 2 \cdot 3^x$. This shows that $2^x + 17$ is $O(3^x)$ (the witnesses are $C = 2$ and $k = 5$).

6. We can use the following inequalities, valid for all $x > 1$ (note that making the denominator of a fraction smaller makes the fraction larger).

$$\frac{x^3 + 2x}{2x + 1} \le \frac{x^3 + 2x^3}{2x} = \frac{3}{2}x^2$$

This proves the desired statement, with witnesses $k = 1$ and $C = 3/2$.

8. a) Since $x^3 \log x$ is not $O(x^3)$ (because the $\log x$ factor grows without bound as x increases), $n = 3$ is too small. On the other hand, certainly $\log x$ grows more slowly than x, so $2x^2 + x^3 \log x \le 2x^4 + x^4 = 3x^4$. Therefore $n = 4$ is the answer, with $C = 3$ and $k = 0$.

b) The $(\log x)^4$ is insignificant compared to the x^5 term, so the answer is $n = 5$. Formally we can take $C = 4$ and $k = 1$ as witnesses.

c) For large x, this fraction is fairly close to 1. (This can be seen by dividing numerator and denominator by x^4.) Therefore we can take $n = 0$; in other words, this function is $O(x^0) = O(1)$. Note that $n = -1$ will not do, since a number close to 1 is not less than a constant times n^{-1} for large n. Formally we can write $f(x) \le 3x^4/x^4 = 3$ for all $x > 1$, so witnesses are $C = 3$ and $k = 1$.

d) This is similar to the previous part, but this time $n = -1$ will do, since for large x, $f(x) \approx 1/x$. Formally we can write $f(x) \le 6x^3/x^3 = 6$ for all $x > 1$, so witnesses are $C = 6$ and $k = 1$.

10. Since $x^3 \le x^4$ for all $x > 1$, we know that x^3 is $O(x^4)$ (witnesses $C = 1$ and $k = 1$). On the other hand, if $x^4 \le Cx^3$, then (dividing by x^3) $x \le C$. Since this latter condition cannot hold for all large x, no matter what the value of the constant C, we conclude that x^4 is not $O(x^3)$.

12. We showed that $x \log x$ is $O(x^2)$ in Exercise 2c. To show that x^2 is not $O(x \log x)$ it is enough to show that $x^2/(x \log x)$ is unbounded. This is the same as showing that $x/\log x$ is unbounded. First let us note that $\log x < \sqrt{x}$ for all $x > 16$. This can be seen by looking at the graphs of these functions, or by calculus. Therefore the fraction $x/\log x$ is greater than $x/\sqrt{x} = \sqrt{x}$ for all $x > 16$, and this clearly is not bounded.

14. a) No, by an argument similar to Exercise 10.

b) Yes, since $x^3 \le x^3$ for all x (witnesses $C = 1$, $k = 0$).

c) Yes, since $x^3 \le x^2 + x^3$ for all x (witnesses $C = 1$, $k = 0$).

d) Yes, since $x^3 \leq x^2 + x^4$ for all x (witnesses $C = 1$, $k = 0$).

e) Yes, since $x^3 \leq 2^x \leq 3^x$ for all $x > 10$ (see Exercise 2e). Thus we have witnesses $C = 1$ and $k = 10$.

f) Yes, since $x^3 \leq 2 \cdot (x^3/2)$ for all x (witnesses $C = 2$, $k = 0$).

16. The given information says that $|f(x)| \leq C|x|$ for all $x > k$, where C and k are particular constants. Let k' be the larger of k and 1. Then since $|x| \leq |x^2|$ for all $x > 1$, we have $|f(x)| \leq C|x^2|$ for all $x > k'$, as desired.

18. $1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots + n^k = n \cdot n^k = n^{k+1}$

20. The approach in these problems is to pick out the most rapidly growing term in each sum and discard the rest (including the multiplicative constants).

a) This is $O(n^3 \cdot \log n + \log n \cdot n^3)$, which is the same as $O(n^3 \cdot \log n)$.

b) Since 2^n dominates n^2, and 3^n dominates n^3, this is $O(2^n \cdot 3^n) = O(6^n)$.

c) The dominant terms in the two factors are n^n and $n!$, respectively. Therefore this is $O(n^n n!)$.

22. We can use the following rule of thumb to determine what simple big-Theta function to use: throw away all the lower order terms (those that don't grow as fast as other terms) and all constant coefficients.

a) This function is $\Theta(1)$, so it is not $\Theta(x)$, since 1 (or 10) grows more slowly than x. To be precise, x is not $O(10)$. For the same reason, this function is not $\Omega(x)$.

b) This function is $\Theta(x)$; we can ignore the " $+\,7$" since it is a lower order term, and we can ignore the coefficient. Of course, since $f(x)$ is $\Theta(x)$, it is also $\Omega(x)$.

c) This function grows faster than x. Therefore $f(x)$ is not $\Theta(x)$ but it is $\Omega(x)$.

d) This function grows more slowly than x. Therefore $f(x)$ is not $\Theta(x)$ or $\Omega(x)$.

e) This function has values that are, for all practical purposes, equal to x (certainly $\lfloor x \rfloor$ is always between $x/2$ and x, for $x > 2$), so it is $\Theta(x)$ and therefore also $\Omega(x)$.

f) As in part (e) this function has values that are, for all practical purposes, equal to $x/2$, so it is $\Theta(x)$ and therefore also $\Omega(x)$.

24. a) This follows from the fact that for all $x > 7$, $x \leq 3x + 7 \leq 4x$.

b) For large x, clearly $x^2 \leq 2x^2 + x - 7$. On the other hand, for $x \geq 1$ we have $2x^2 + x - 7 \leq 3x^2$.

c) For $x > 2$ we certainly have $\lfloor x + \frac{1}{2} \rfloor \leq 2x$ and also $x \leq 2\lfloor x + \frac{1}{2} \rfloor$.

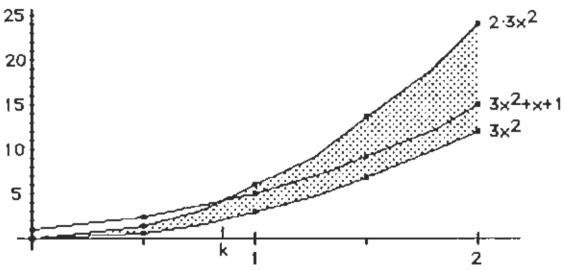
d) For $x > 2$, $\log(x^2 + 1) \leq \log(2x^2) = 1 + 2\log x \leq 3\log x$ (recall that $\log$ means $\log_2$). On the other hand, since $x < x^2 + 1$ for all positive x, we have $\log x \leq \log(x^2 + 1)$.

e) This follows from the fact that $\log_{10} x = C(\log_2 x)$, where $C = 1/\log_2 10$.

26. We just need to look at the definitions. To say that $f(x)$ is $O(g(x))$ means that there are constants C and k such that $|f(x)| \leq C|g(x)|$ for all $x > k$. Note that without loss of generality we may take C and k to be positive. To say that $g(x)$ is $\Omega(f(x))$ is to say that there are positive constants C' and k' such that $|g(x)| \geq C'|f(x)|$ for all $x > k$. These are saying exactly the same thing if we set $C' = 1/C$ and $k' = k$.

28. a) By Exercise 25 we have to show that $3x^2 + x + 1$ is $O(3x^2)$ and that $3x^2$ is $O(3x^2 + x + 1)$. The latter is trivial, since $3x^2 \leq 3x^2 + x + 1$ for $x > 0$. The former is almost as trivial, since $3x^2 + x + 1 \leq 3x^2 + 3x^2 = 2 \cdot 3x^2$ for all $x > 1$. What we have shown is that $1 \cdot 3x^2 \leq 3x^2 + x + 1 \leq 2 \cdot 3x^2$ for all $x > 1$; in other words, $C_1 = 1$ and $C_2 = 2$ in Exercise 27.

b) The following picture shows that graph of $3x^2+x+1$ falls in the shaded region between the graph of $3x^2$ and the graph of $2\cdot 3x^2$ for all $x>1$.



30. Looking at the definition, we see that to say that $f(x)$ is $\Omega(1)$ means that $|f(x)|\geq C$ when $x>k$, for some positive constants k and C. In other words, $f(x)$ keeps at least a certain distance away from 0 for large enough x. For example, $1/x$ is not $\Omega(1)$, since it gets arbitrary close to 0; but $(x-2)(x-10)$ is $\Omega(1)$, since $f(x)\geq 9$ for $x>11$.

32. The n^{th} odd positive integer is $2n-1$. Thus each of the first n odd positive integers is at most $2n$. Therefore their product is at most $(2n)^n$, so one answer is $O\big((2n)^n\big)$. Of course other answers are possible as well.

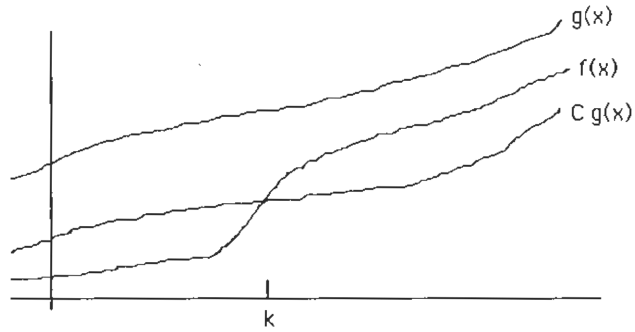
34. This follows from the fact that $\log_b x$ and $\log_a x$ are the same except for a multiplicative constant, namely $d=\log_b a$. Thus if $f(x)\leq C\log_b x$, then $f(x)\leq Cd\log_a x$.

36. This does not follow. Let $f(x)=2x$ and $g(x)=x$. Then $f(x)$ is $O(g(x))$. Now $2^{f(x)}=2^{2x}=4^x$, and $2^{g(x)}=2^x$, and 4^x is not $O(2^x)$. Indeed, $4^x/2^x=2^x$, so the ratio grows without bound as x grows—it is not bounded by a constant.

38. The definition of "$f(x)$ is $\Theta(g(x))$" is that $f(x)$ is both $O(g(x))$ and $\Omega(g(x))$. That means that there are positive constants C_1, k_1, C_2, and k_2 such that $|f(x)|\leq C_2|g(x)|$ for all $x>k_2$ and $|f(x)|\geq C_1|g(x)|$ for all $x>k_1$. Similarly, we have that there are positive constants C_1', k_1', C_2', and k_2' such that $|g(x)|\leq C_2'|h(x)|$ for all $x>k_2'$ and $|g(x)|\geq C_1'|h(x)|$ for all $x>k_1'$. We can combine these inequalities to obtain $|f(x)|\leq C_2C_2'|h(x)|$ for all $x>\max(k_2,k_2')$ and $|f(x)|\geq C_1C_1'|h(x)|$ for all $x>\max(k_1,k_1')$. This means that $f(x)$ is $\Theta(h(x))$.

40. The definitions tell us that there are positive constants C_1, k_1, C_2, and k_2 such that $|f_1(x)|\leq C_2|g_1(x)|$ for all $x>k_2$ and $|f_1(x)|\geq C_1|g_1(x)|$ for all $x>k_1$, and that there are positive constants C_1', k_1', C_2', and k_2' such that $|f_2(x)|\leq C_2'|g_2(x)|$ for all $x>k_2'$ and $|f_2(x)|\geq C_1'|g_2(x)|$ for all $x>k_1'$. We can multiply these inequalities to obtain $|f_1(x)f_2(x)|\leq C_2C_2'|g_1(x)g_2(x)|$ for all $x>\max(k_2,k_2')$ and $|f_1(x)f_2(x)|\geq C_1C_1'|g_1(x)g_2(x)|$ for all $x>\max(k_1,k_1')$. This means that $f_1(x)f_2(x)$ is $\Theta(g_1(x)g_2(x))$.

42. Typically C will be less than 1. From some point onward to the right ($x>k$), the graph of $f(x)$ must be above the graph of $g(x)$ after the latter has been scaled down by the factor C. Note that $f(x)$ does not have to be larger than $g(x)$ itself.



44. We need to show inequalities both ways. First, we show that $|f(x)| \leq Cx^n$ for all $x \geq 1$, as follows, noting that $x^i \leq x^n$ for such values of x whenever $i < n$. We have the following inequalities, where M is the largest of the absolute values of the coefficients and C is $M(n+1)$:

$$\begin{aligned}|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ &\leq |a_n|x^n + |a_{n-1}|x^{n-1} + \cdots + |a_1|x + |a_0| \\ &\leq |a_n|x^n + |a_{n-1}|x^n + \cdots + |a_1|x^n + |a_0|x^n \\ &\leq Mx^n + Mx^n + \cdots + Mx^n + Mx^n = Cx^n\end{aligned}$$

For the other direction, which is a little messier, let k be chosen larger than 1 and larger than $2nm/|a_n|$, where m is the largest of the absolute values of the a_i's for $i < n$. Then each a_{n-i}/x^i will be smaller than $|a_n|/2n$ in absolute value for all $x > k$. Now we have for all $x > k$,

$$\begin{aligned}|f(x)| &= |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ &= x^n \left|a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right| \\ &\geq x^n\,|a_n/2|\,,\end{aligned}$$

as desired.

46. We just make the analogous change in the definition of big-Omega that was made in the definition of big-O: there exist positive constants C, k_1, and k_2 such that $|f(x,y)| \geq C|g(x,y)|$ for all $x > k_1$ and $y > k_2$.

48. For all values of x and y greater than 1, each term of the given expression is greater than x^3y^3, so the entire expression is greater than x^3y^3. In other words, we take $C = k_1 = k_2 = 1$ in the definition given in Exercise 46.

50. For all positive values of x and y, we know that $\lceil xy \rceil \geq xy$ by definition (since the ceiling function value cannot be less than the argument). Thus $\lceil xy \rceil$ is $\Omega(xy)$ from the definition, taking $C = 1$ and $k_1 = k_2 = 0$. In fact, $\lceil xy \rceil$ is also $O(xy)$ (and therefore $\Theta(xy)$); this is easy to see since $\lceil xy \rceil \leq (x+1)(y+1) \leq (2x)(2y) = 4xy$ for all x and y greater than 1.

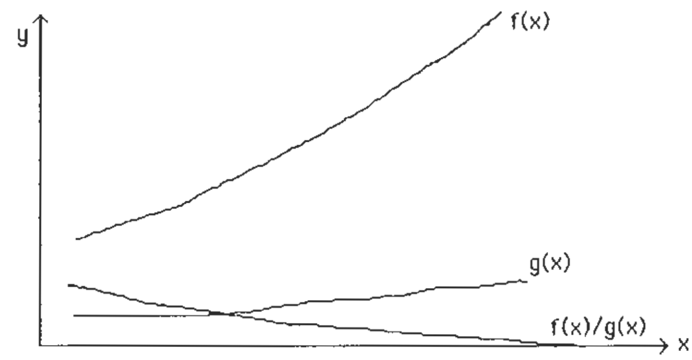
52. **a)** Under the hypotheses,

$$\lim_{x\to\infty} \frac{cf(x)}{g(x)} = c \lim_{x\to\infty} \frac{f(x)}{g(x)} = c \cdot 0 = 0\,.$$

b) Under the hypotheses,

$$\lim_{x\to\infty} \frac{f_1(x) + f_2(x)}{g(x)} = \lim_{x\to\infty} \frac{f_1(x)}{g(x)} + \lim_{x\to\infty} \frac{f_2(x)}{g(x)} = 0 + 0 = 0\,.$$

54. The behaviors of f and g alone are not really at issue; what is important is whether $f(x)/g(x)$ approaches 0 as $x \to \infty$. Thus, as shown in the picture, it might happen that the graphs of f and g rise, but f increases enough more rapidly than g so that the ratio gets small. In the picture, we see that $f(x)/g(x)$ is asymptotic to the x-axis.



56. No. Let $f(x) = x$ and $g(x) = x^2$. Then clearly $f(x)$ is $o(g(x))$, but the ratio of the logs of the absolute values is the constant 2, and 2 does not approach 0. Therefore it is not the case in this example that $\log|f(x)|$ is $o(\log|g(x)|)$.

58. This follows from the fact that the limit of $f(x)/g(x)$ is 0 in this case, as can be most easily seen by dividing numerator and denominator by x^n (the numerator then is bounded and the absolute value of the denominator grows without bound as $x \to \infty$).

60. Since $f(x) = 1/x$ is a decreasing function which has the value $1/x$ at $x = j$, it is clear that $1/j < 1/x$ throughout the interval from $j-1$ to j. Summing over all the intervals for $j = 2, 3, \ldots, n$, and noting that the definite integral is the area under the curve, we obtain the inequality in the hint. Therefore

$$H_n = 1 + \sum_{j=2}^{n} \frac{1}{j} < 1 + \int_1^n \frac{1}{x}\,dx = 1 + \ln n = 1 + C\log n \le 2C\log n$$

for $n > 2$, where $C = \log e$.

62. By Example 6 $\log n!$ is $O(n\log n)$. By Exercise 61 $n\log n$ is $O(\log n!)$. Thus by Exercise 25 $\log n!$ is $\Theta(n\log n)$.

64. In each case we need to evaluate the limit of $f(x)/g(x)$ as $x \to \infty$. If it equals 1, then f and g are asymptotic; otherwise (including the case in which the limit does not exist) they are not. Most of these are straightforward applications of algebra, elementary notions about limits, or L'Hôpital's rule.

a) $\lim\limits_{x\to\infty} \dfrac{x^2+3x+7}{x^2+10} = \lim\limits_{x\to\infty} \dfrac{1+3/x+7/x^2}{1+10/x^2} = 1$, so f and g are asymptotic.

b) $\lim\limits_{x\to\infty} \dfrac{x^2\log x}{x^3} = \lim\limits_{x\to\infty} \dfrac{\log x}{x} = \lim\limits_{x\to\infty} \dfrac{1}{x\cdot\ln 2} = 0$ (we used L'Hôpital's rule for the last equivalence), so f and g are not asymptotic.

c) Here $f(x)$ is dominated by its leading term, x^4, and $g(x)$ is a polynomial of degree 4, so the ratio approaches 1, the ratio of the leading coefficients, as in part (a). Therefore f and g are asymptotic.

d) Here f and g are polynomials of degree 12, so the ratio approaches 1, the ratio of the leading coefficients, as in part (a). Therefore f and g are asymptotic.