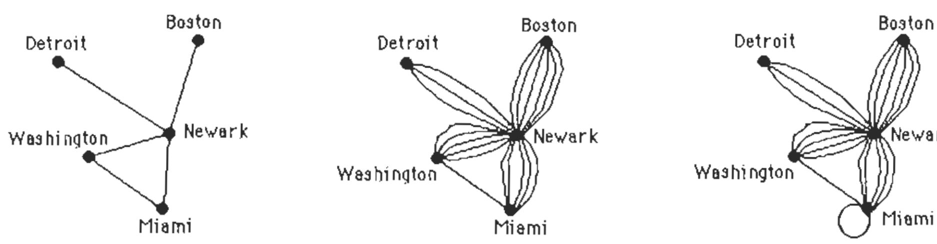
CHAPTER 9
Graphs

SECTION 9.1 Graphs and Graph Models

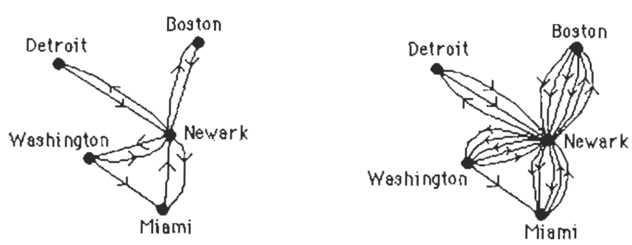
The examples and exercises give a good picture of the ways in which graphs can model various real world applications. In constructing graph models you need to determine what the vertices will represent, what the edges will represent, whether the edges will be directed or undirected, whether loops should be allowed, and whether a simple graph or multigraph is more appropriate.

1. In part **(a)** we have a simple graph, with undirected edges, no loops or multiple edges. In part **(b)** we have a multigraph, since there are multiple edges (making the figure somewhat less than ideal visually). In part **(c)** we have the same picture as in part **(b)** except that there is now a loop at one vertex; thus this is a pseudograph.



(a) (b) (c)

In part **(d)** we have a directed graph, the directions of the edges telling the directions of the flights; note that the **antiparallel edges** (pairs of the form (u, v) and (v, u)) are not parallel. In part **(e)** we have a directed multigraph, since there are parallel edges.



(d) (e)

3. This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

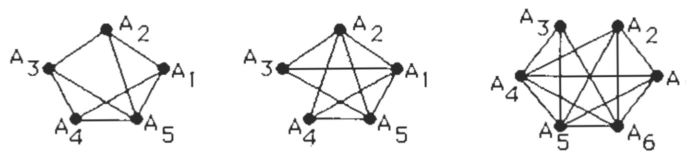
5. This is a pseudograph; the edges are undirected, but there are loops and parallel edges.

7. This is a directed graph; the edges are directed, but there are no parallel edges. (Loops and antiparallel edges—see the solution to Exercise 1d for a definition—are allowed in a directed graph.)

9. This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

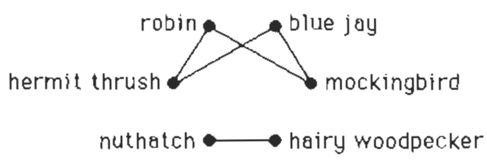
11. In a simple graph, edges are undirected. To show that R is symmetric we must show that if uRv, then vRu. If uRv, then there is an edge associated with $\{u,v\}$. But $\{u,v\} = \{v,u\}$, so this edge is associated with $\{v,u\}$ and therefore vRu. A simple graph does not allow loops; that is if there is an edge associated with $\{u,v\}$, then $u \neq v$. Thus uRu never holds, and so by definition R is irreflexive.

13. In each case we draw a picture of the graph in question. All are simple graphs. An edge is drawn between two vertices if the sets for the two vertices have at least one element in common. For example, in part **(a)** there is an edge between vertices A_1 and A_2 because there is at least one element common to A_1 and A_2 (in fact there are three such elements). There is no edge between A_1 and A_3 since $A_1 \cap A_3 = \emptyset$.

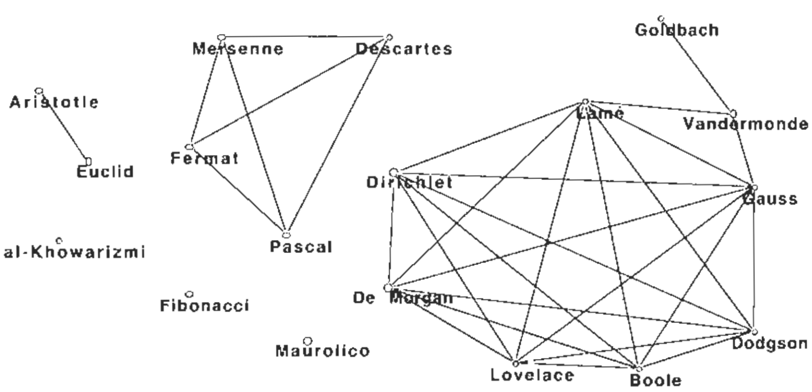


(a) (b) (c)

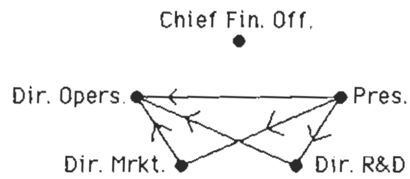
15. We draw a picture of the graph in question, which is a simple graph. Two vertices are joined by an edge if we are told that the species compete (such as robin and mockingbird) but there is no edge between pairs of species that are not given as competitors (such as robin and blue jay).



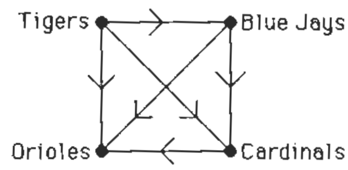
17. Here are the persons to be included, listed in order of birth year: Aristotle (384–322 B.C.E.), Euclid (325–265 B.C.E.), al-Khowarizmi (780–850), Fibonacci (1170–1250), Maurolico (1494–1575), Mersenne (1588–1648), Descartes (1596–1650), Fermat (1601–1665), Pascal (1623–1662), Goldbach (1690–1764), Vandermonde (1735–1796), Gauss (1777–1855), Lamé (1795–1870), Dirichlet (1805–1859), De Morgan (1806–1871), Lovelace (1815–1852), Boole (1815–1864), and Dodgson (1832–1898). We draw the graph by connecting two people if their date ranges overlap. Note that there is a complete subgraph (see Section 9.2) consisting of Mersenne, Descartes, Fermat, and Pascal, and a larger complete subgraph consisting of the last seven people listed. A few of the vertices are isolated (again see Section 9.2). In all our graph has 18 vertices and 31 edges. A graph like this is called an **interval graph**, since each vertex can be associated with an interval of real numbers; it is a special case of an **intersection graph**, where two vertices are adjacent if the sets associated with those vertices have a nonempty intersection (see Exercise 13).



19. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u can influence v. For instance the Chief Financial Officer is an isolated vertex since she is influenced by no one and influences no one.



21. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u beat v.



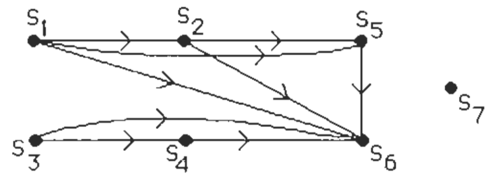
23. We could compile a list of phone numbers (the labels on the vertices) in the February call graph that were not present in January, and a list of the January numbers missing in February. For each number in each list, we could make a list of the numbers they called or were called by, using the edges in the call graphs. Then we could look for February lists that were very similar to January lists. If we found a new February number that had almost the same calling pattern as a defunct January number, then we might suspect that these numbers belonged to the same person, who had recently changed his or her number.

25. For each e-mail address (the labels on the vertices), we could make a list of the other addresses they sent messages to or received messages from. If we see two addresses that had almost the same communication pattern, then we might suspect that these addresses belonged to the same person, who had recently changed his or her e-mail address.

27. The vertices represent the people at the party. Because it is possible that a knows b's name but not vice versa, we need a directed graph. We will include an edge associated with (u, v) if and only if u knows v's name. There is no need for multiple edges (either a knows b's name or he doesn't). One could argue that we should not clutter the model with loops, because obviously everyone knows her own name. On the other hand, it certainly would not be wrong to include loops, especially if we took the instructions literally.

29. For this to be interesting, we want the graph to model all marriages, not just ones that are currently active. (In the latter case, for the Western world, there would be at most one edge incident to each vertex.) So we let the set of vertices be a set of people (for example, all the people in North America who lived at any point in the 20th century), and two vertices are joined by an edge if the two people were ever married. Since laws in the 20th century allowed only marriages between persons of the opposite sex, and ignoring complications caused by sex-change operations, we note that this graph has the property that there are two types of vertices (men and women), and every edge joins vertices of opposite types. In the next section we learn that the word used to describe a graph like this is *bipartite*.

31. We draw a picture of the directed graph in question. There is an edge from u to v if the assignment made in u can possibly influence the assignment made in v. For example, there is an edge from S_3 to S_6, since the assignment in S_3 changes the value of y, which then influences the value of z (in S_4) and hence has a bearing on S_6. We assume that the statements are to be execnted in the given order, so, for example, we do not draw an edge from S_5 to S_2.



33. The vertices in the directed graph represent people in the group. We put a directed edge into our directed graph from every vertex A to every vertex $B \neq A$ (we do not need loops), and furthermore we label that edge with one of the three labels L, D, or N. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with labeled edges. It is defined to be a triple (V, E, f), where (V, E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and f is a function from E to the set $\{L, D, N\}$. Here we are simply thinking of $f(e)$ as the attitude of the person at the tail (initial vertex—see Section 9.2) of e toward the person at the head (terminal vertex) of e.

SECTION 9.2 Graph Terminology and Special Types of Graphs

Graph theory is sometimes jokingly called the "theory of definitions," because so many terms can be—and have been—defined for graphs. A few of the most important concepts are given in this section; others appear in the rest of this chapter and the next, in the exposition and in the exercises. As usual with definitions, it is important to understand exactly what they are saying. You should construct some examples for each definition you encounter—examples both of the thing being defined and of its absence. Some students find it useful to build a dictionary as they read, including their examples along with the formal definitions.

The Handshaking Theorem (that the sum of the degrees of the vertices in a graph equals twice the number of edges), although trivial to prove, is quite handy, as Exercise 49, for example, illustrates. Be sure to look at Exercise 37, which deals with the problem of when a sequence of numbers can possibly be the degrees of the vertices of a simple graph. Some interesting subtleties arise there, as you will discover when you try to draw the graphs. Many arguments in graph theory tend to be rather ad hoc, really getting down to the nitty gritty, and Exercise 37c is a good example. Exercise 45 is really a combinatorial problem; such problems abound in graph theory, and entire books have been written on counting graphs of various types. The notion of **complementary graph**, *introduced in Exercise 53, will appear again later in this chapter, so it would be wise to look at the exercises dealing with it.*

1. There are 6 vertices here, and 6 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 1$ (and hence c is pendant), $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 2$, and $\deg(f) = 3$. Note that the sum of the degrees is $2 + 4 + 1 + 0 + 2 + 3 = 12$, which is twice the number of edges.

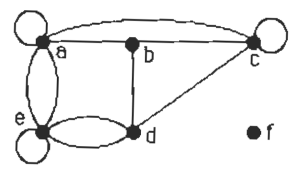
3. There are 9 vertices here, and 12 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 6$, $\deg(f) = 0$ (and hence f is isolated), $\deg(g) = 4$, $\deg(h) = 2$, and $\deg(i) = 3$. Note that the sum of the degrees is $3 + 2 + 4 + 0 + 6 + 0 + 4 + 2 + 3 = 24$, which is twice the number of edges.

5. By Theorem 2 the number of vertices of odd degree must be even. Hence there cannot be a graph with 15 vertices of odd degree 5. (We assume that the problem was meant to imply that the graph contained only these 15 vertices.)

7. This directed graph has 4 vertices and 7 edges. The in-degree of vertex a is $\deg^-(a) = 3$ since there are 3 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$ since only the loop has a as its initial vertex. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 2$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 3$. As a check we see that the sum of the in-degrees and the sum of the out-degrees are equal (both are equal to 7).

9. This directed multigraph has 5 vertices and 13 edges. The in-degree of vertex a is $\deg^-(a) = 6$ since there are 6 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 5$, $\deg^-(c) = 2$, $\deg^+(c) = 5$, $\deg^-(d) = 4$, $\deg^+(d) = 2$, $\deg^-(e) = 0$, and $\deg^+(e) = 0$ (vertex e is isolated). As a check we see that the sum of the in-degrees and the sum of the out-degrees are both equal to the number of edges (13).

11. To form the underlying undirected graph we simply take all the arrows off the edges. Thus, for example, the edges from e to d and from d to e become a pair of parallel edges between e and d.



13. Since a person is joined by an edge to each of his or her collaborators, the degree of v is the number of collaborators v has. An isolated vertex (degree 0) is someone who has never collaborated. A pendant vertex (degree 1) is someone who has just one collaborator.

15. Since there is a directed edge from u to v for each call made by u to v, the in-degree of v is the number of calls v received, and the out-degree of u is the number of calls u made. The degree of a vertex in the undirected version is just the sum of these, which is therefore the number of calls the vertex was involved in.

17. Since there is a directed edge from u to v to represent the event that u beat v when they played, the in-degree of v must be the number of teams that beat v, and the out-degree of u must be the number of teams that u beat. In other words, the pair $(\deg^+(v), \deg^-(v))$ is the win-loss record of v.

19. In order to use Exercise 18, we must find a graph in which the degree of a vertex represents the number of people the given person knows. Therefore we construct the simple graph model in which V is the set of people in the group and there is an edge associated with $\{u, v\}$ if u and v know each other. In this graph the degree of vertex v is the number of people v knows. By the result of Exercise 18, there are two vertices with the same degree. Therefore there are two people who know the same number of other people in the group.

21. To show that this graph is bipartite we can exhibit the parts and note that indeed every edge joins vertices in different parts. Take $\{e\}$ to be one part and $\{a, b, c, d\}$ to be the other (in fact there is no choice in the matter). Each edge joins a vertex in one part to a vertex in the other. This graph is the complete bipartite graph $K_{1,4}$.

23. To show that a graph is not bipartite we must give a proof that there is no possible way to specify the parts. (There is another good way to characterize nonbipartite graphs, but it takes some notions not introduced until Section 9.4.) We can show that this graph is not bipartite by the pigeonhole principle. Consider the vertices b, c, and f. They form a triangle—each is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

An alternative way to look at this is given by Theorem 4. Because of the triangle, it is impossible to color the vertices to satisfy the condition given there.

25. As in Exercise 23, we can show that this graph is not bipartite by looking at a triangle, in this case the triangle formed by vertices b, d, and e. Each of these vertices is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

27. **a)** Following the lead in Example 14, we construct a bipartite graph in which the vertex set consists of two subsets—one for the employees and one for the jobs. Let $V_1 = \{\text{Zamora}, \text{Agraharam}, \text{Smith}, \text{Chou}, \text{Macintyre}\}$, and let $V_2 = \{\text{planning}, \text{publicity}, \text{sales}, \text{marketing}, \text{development}, \text{industry relations}\}$. Then the vertex set for our graph is $V = V_1 \cup V_2$. Given the list of capabilities in the exercise, we must include precisely the following edges in our graph: $\{\text{Zamora}, \text{planning}\}$, $\{\text{Zamora}, \text{sales}\}$, $\{\text{Zamora}, \text{marketing}\}$, $\{\text{Zamora}, \text{industry relations}\}$, $\{\text{Agraharam}, \text{planning}\}$, $\{\text{Agraharam}, \text{development}\}$, $\{\text{Smith}, \text{publicity}\}$, $\{\text{Smith}, \text{sales}\}$, $\{\text{Smith}, \text{industry relations}\}$, $\{\text{Chou}, \text{planning}\}$, $\{\text{Chou}, \text{sales}\}$, $\{\text{Chou}, \text{industry relations}\}$, $\{\text{Macintyre}, \text{planning}\}$, $\{\text{Macintyre}, \text{publicity}\}$, $\{\text{Macintyre}, \text{sales}\}$, $\{\text{Macintyre}, \text{industry relations}\}$.

b) Many assignments are possible. If we take it as an implicit assumption that there will be no more than one employee assigned to the same job, then we want a maximal matching for this graph. So we look for five edges in this graph that share no endpoints. A little trial and error leads us, for example, $\{\text{Zamora}, \text{planning}\}$, $\{\text{Agraharam}, \text{development}\}$, $\{\text{Smith}, \text{publicity}\}$, $\{\text{Chou}, \text{sales}\}$, $\{\text{Macintyre}, \text{industry relations}\}$. We assign the employees to the jobs given in this matching.

29. **a)** Obviously K_n has n vertices. It has $C(n,2) = n(n-1)/2$ edges, since each unordered pair of distinct vertices is an edge.

b) Obviously C_n has n vertices. Just as obviously it has n edges.

c) The wheel W_n is the same as C_n with an extra vertex and n extra edges incident to that vertex. Therefore it has $n+1$ vertices and $n+n=2n$ edges.

d) By definition $K_{m,n}$ has $m+n$ vertices. Since it has one edge for each choice of a vertex in the one part and a vertex in the other part, it has mn edges.

e) Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the Handshaking Theorem), we know that there are $n2^n/2 = n2^{n-1}$ edges.

31. In each case we just record the degrees of the vertices in a list, from largest to smallest.

a) Each of the four vertices is adjacent to each of the other three vertices, so the degree sequence is $3,3,3,3$.

b) Each of the four vertices is adjacent to its two neighbors in the cycle, so the degree sequence is $2,2,2,2$.

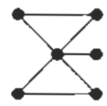
c) Each of the four vertices on the rim of the wheel is adjacent to each of its two neighbors on the rim, as well as to the middle vertex. The middle vertex is adjacent to the four rim vertices. Therefore the degree sequence is $4,3,3,3,3$.

d) Each of the vertices in the part of size two is adjacent to each of the three vertices in the part of size three, and vice versa, so the degree sequence is $3,3,2,2,2$.

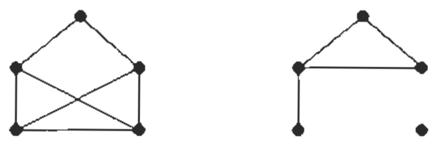
e) Each of the eight vertices in the cube is adjacent to three others (for example, 000 is adjacent to 001, 010, and 100. Therefore the degree sequence is $3,3,3,3,3,3,3,3$.

33. Each of the n vertices is adjacent to each of the other $n-1$ vertices, so the degree sequence is simply $n-1, n-1, \ldots, n-1$, with n terms in the sequence.

35. The number of edges is half the sum of the degrees (Theorem 1). Therefore this graph has $(5+2+2+2+2+1)/2 = 7$ edges. A picture of this graph is shown here (it is essentially unique).



37. There is no such graph in part **(b)**, since the sum of the degrees is odd (and also because a simple graph with 5 vertices cannot have any degrees greater than 4). Similarly, the odd degree sum prohibits the existence of graphs with the degree sequences given in part **(d)** and part **(f)**. There is no such graph in part **(c)**, since the existence of two vertices of degree 4 implies that there are two vertices each joined by an edge to every other vertex. This means that the degree of each vertex has to be at least 2, and there can be no vertex of degree 1. The graphs for part **(a)** and part **(e)** are shown below; one can draw them after just a little trial and error.



(a) (e)

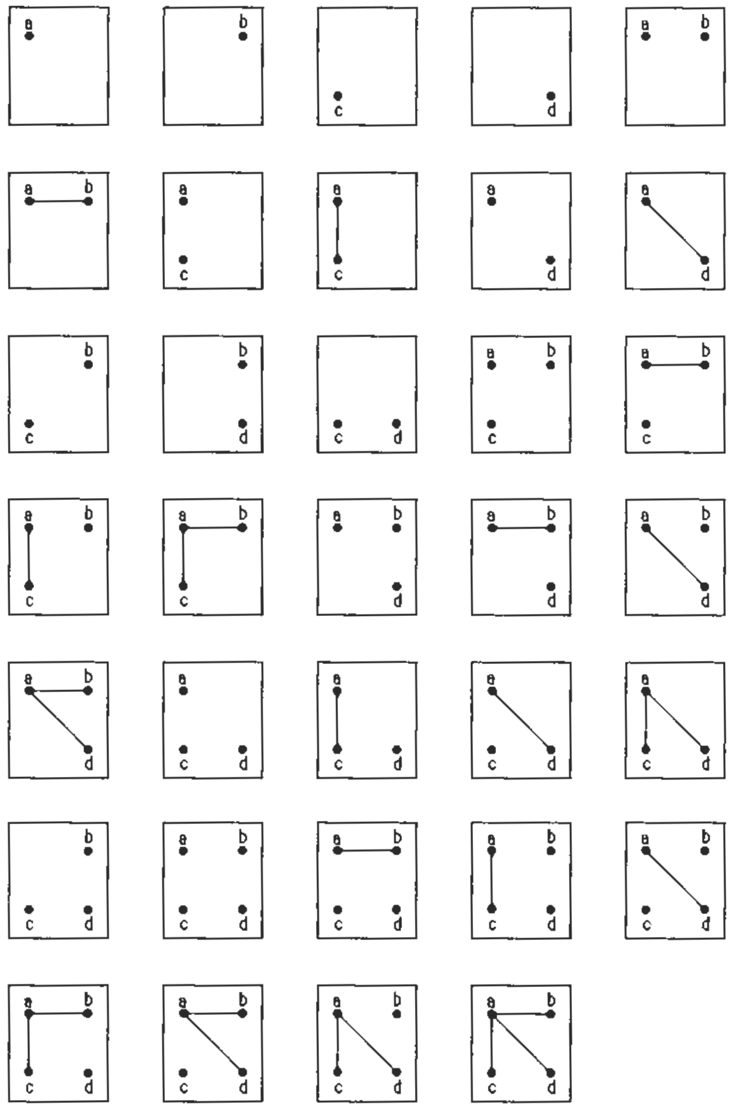
39. We need to prove two conditional statements. First, suppose that $d_1, d_2, \ldots, d_n$ is graphic. We must show that the sequence whose terms are $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. Apparently what we want to do is to remove the vertex of highest degree (d_1) from a graph with the original degree sequence and reduce by 1 the degrees of the vertices to which it is adjacent, but we also want to make sure that those vertices are the ones with the highest degrees among the remaining vertices. In Exercise 38 it is proved that if the original sequence is graphic, then in fact there is a graph having this degree sequence in which the vertex of degree d_1 is adjacent to the vertices of degrees d_2, $d_3, \ldots, d_{d_1+1}$. Thus our plan works, and we have a graph whose degree sequence is as desired.

Conversely, suppose that $d_1, d_2, \ldots, d_n$ is a nonincreasing sequence such that the sequence $d_2 - 1$, $d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, \ldots, d_n$ is graphic once it is put into nonincreasing order. Take a graph with this latter degree seqnence, where vertex v_i has degree $d_i - 1$ for $2 \le i \le d_1 + 1$ and vertex v_i has degree d_i for $d_1 + 2 \le i \le n$. Adjoin one new vertex (call it v_1), and put in an edge from v_1 to each of the vertices $v_2, v_3, \ldots, v_{d_1+1}$. Then clearly the resulting graph has degree sequence $d_1, d_2, \ldots, d_n$.

41. Let $d_1, d_2, \ldots, d_n$ be a nonincreasing sequence of nonnegative integers with an even sum. We want to construct a pseudograph with this as its degree sequence. Even degrees can be achieved using only loops, each of which contributes 2 to the count of its endpoint; vertices of odd degrees will need a non-loop edge, but one will suffice (the rest of the count at that vertex will be made up by loops). Following the hint, we take vertices $v_1, v_2, \ldots, v_n$ and put $\lfloor d_i/2 \rfloor$ loops at vertex v_i, for $i = 1, 2, \ldots, n$. For each i, vertex v_i now has degree either d_i (if d_i is even) or $d_i - 1$ (if d_i is odd). Because the original sum was even, the number of vertices falling into the latter category is even. If there are $2k$ such vertices, pair them up arbitrarily, and put in k more edges, one joining the vertices in each pair. The resulting graph will have degree sequence $d_1, d_2, \ldots, d_n$.

43. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 3 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(3,2) = 3$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $3 \cdot 2 = 6$ subgraphs with two vertices. If a subgraph is to have all three vertices, then there are $2^3 = 8$ ways to decide whether or not to include each of the edges. Thus our answer is $3 + 6 + 8 = 17$.

45. This graph has a lot of subgraphs. First of all, any nonempty subset of the vertex set can be the vertex set for a subgraph, and there are 15 such subsets. If the set of vertices of the subgraph does not contain vertex a, then the subgraph can of course have no edges. If it does contain vertex a, then it can contain or fail to contain each edge from a to whichever other vertices are included. A careful enumeration of all the possibilities gives the 34 graphs shown below.



47. **a)** The complete graph K_n is regular for all values of $n \geq 1$, since the degree of each vertex is $n-1$.

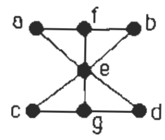
b) The degree of each vertex of C_n is 2 for all n for which C_n is defined, namely $n \geq 3$, so C_n is regular for all these values of n.

c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the "rim" is 3. Therefore W_n is regular if and only if $n = 3$. Of course W_3 is the same as K_4.

d) The cube Q_n is regular for all values of $n \geq 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

49. If a graph is regular of degree 4 and has n vertices, then by the Handshaking Theorem it has $4n/2 = 2n$ edges. Since we are told that there are 10 edges, we just need to solve $2n = 10$. Thus the graph has 5 vertices. The complete graph K_5 is one such graph (and the only simple one).

51. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



53. **a)** The complement of a complete graph is a graph with no edges.
b) Since all the edges between the parts are present in $K_{m,n}$, but none of the edges between vertices in the same part are, the complement must consist precisely of the disjoint union of a K_m and a K_n, i.e., the graph containing all the edges joining two vertices in the same part and no edges joining vertices in different parts.
c) There is really no better way to describe this graph than simply by saying it is the complement of C_n. One representation would be to take as vertex set the integers from 1 to n, inclusive, with an edge between distinct vertices i and j as long as i and j do not differ by ± 1, modulo n.
d) Again, there is really no better way to describe this graph than simply by saying it is the complement of Q_n. One representation would be to take as vertex set the bit strings of length n, with two vertices joined by an edge if the bit strings differ in more than one bit.

55. Since K_v has $C(v,2) = v(v-1)/2$ edges, and since $\overline{G}$ has all the edges of K_v that G is missing, it is clear that $\overline{G}$ has $[v(v-1)/2] - e$ edges.

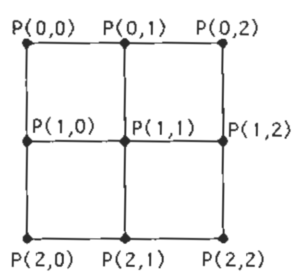
57. If G has n vertices, then the degree of vertex v in $\overline{G}$ is $n-1$ minus the degree of v in G (there will be an edge in $\overline{G}$ from v to each of the $n-1$ other vertices that v is not adjacent to in G). The order of the sequence will reverse, of course, because if $d_i \geq d_j$, then $n-1-d_i \leq n-1-d_j$. Therefore the degree sequence of $\overline{G}$ will be $n-1-d_n$, $n-1-d_{n-1}$, $\ldots$, $n-1-d_2$, $n-1-d_1$.

59. Consider the graph $G \cup \overline{G}$. Its vertex set is clearly the vertex set of G; therefore it has n vertices. If u and v are any two distinct vertices of $G \cup \overline{G}$, then either the edge between u and v is in G, or else by definition it is in $\overline{G}$. Therefore by definition of union, it is in $G \cup \overline{G}$. Thus by definition $G \cup \overline{G}$ is the complete graph K_n.

61. These pictures are identical to the figures in those exercises, with one change, namely that all the arrowheads are turned around. For example, rather than there being a directed edge from a to b in #7, there is an edge from b to a. Note that the loops are unaffected by changing the direction of the arrowhead--a loop from a vertex to itself is the same, whether the drawing of it shows the direction to be clockwise or counterclockwise.

63. It is clear from the definition of converse that a directed graph $G = (V, E)$ is its own converse if and only if it satisfies the condition that $(u, v) \in E$ if and only if $(v, u) \in E$. But this is precisely the definition of symmetry for the associated relation.

65. Our picture is just like Figure 13, but with only three vertices on each side.



67. Suppose $P(i, j)$ and $P(k, l)$ need to communicate. Clearly by using $|i - k|$ hops we can move from $P(i, j)$ to $P(k, j)$. Then using $|j - l|$ hops we can move from $P(k, j)$ to $P(k, l)$. In all we used $|i - k| + |j - l|$ hops. But each of these absolute values is certainly less than m, since all the indices are less than m. Therefore the sum is less than $2m$, so it is $O(m)$.

SECTION 9.3 Representing Graphs and Graph Isomorphism

Human beings can get a good feeling for a small graph by looking at a picture of it drawn with points in the plane and lines or curves joining pairs of these points. If a graph is at all large (say with more than a dozen vertices or so), then the picture soon becomes too crowded to be useful. A computer has little use for nice pictures, no matter how small the vertex set. Thus people and machines need more precise—more discrete—representations of graphs. In this section we learned about some useful representations. They are for the most part exactly what any intelligent person would come up with, given the assignment to do so.

The only tricky idea in this section is the concept of graph isomorphism. It is a special case of a more general notion of isomorphism, or sameness, of mathematical objects in various settings. Isomorphism tries to capture the idea that all that really matters in a graph is the adjacency structure. If we can find a way to superimpose the graphs so that the adjacency structures match, then the graphs are, for all purposes that matter, the same. In trying to show that two graphs are isomorphic, try moving the vertices around in your mind to see whether you can make the graphs look the same. Of course there are often lots of things to help. For example, in every isomorphism, vertices that correspond must have the same degree.

A good general strategy for determining whether two graphs are isomorphic might go something like this. First check the degrees of the vertices to make sure there are the same number of each degree. See whether vertices of corresponding degrees follow the same adjacency pattern (e.g., if there is a vertex of degree 1 adjacent to a vertex of degree 4 in one of the graphs, then there must be the same pattern in the other, if the graphs are isomorphic). Then look for triangles in the graphs, and see whether they correspond. Sometimes, if the graphs have lots of edges, it is easier to see whether the complements are isomorphic (see Exercise 46). If you cannot find a good reason for the graphs not to be isomorphic (an invariant on which they differ), then try to write down a one-to-one and onto function that shows them to be isomorphic (there may be more than one such function); such a function has to have vertices of like degrees correspond, so often the function practically writes itself. Then check each edge of the first graph to make sure that it corresponds to an edge of the second graph under this correspondence.

Unfortunately, no one has yet discovered a really good algorithm for determining graph isomorphism that works on all pairs of graphs. Research in this subject has been quite active in recent years. See Writing Project 7.

1. Adjacency lists are lists of lists. The adjacency list of an undirected graph is simply a list of the vertices of the given graph, together with a list of the vertices adjacent to each. The list for this graph is as follows. Since, for instance, b is adjacent to a and d, we list a and d in the row for b.

Vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

3. To form the adjacency list of a directed graph, we list, for each vertex in the graph, the terminal vertex of each edge that has the given vertex as its initial vertex. The list for this directed graph is as follows. For example, since there are edges from d to each of b, c, and d, we put those vertices in the row for d.

Initial vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

5. For Exercises 5–8 we assume that the vertices are listed in alphabetical order. The matrix contains a 1 as entry (i, j) if there is an edge from vertex i to vertex j; otherwise that entry is 0.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

7. This is similar to Exercise 5. Note that edges have direction here, so that, for example, the $(1,2)$ entry is a 1 since there is an edge from a to b, but the $(2,1)$ entry is a 0 since there is no edge from b to a. Also, the $(1,1)$ entry is a 1 since there is a loop at a, but the $(2,2)$ entry is a 0 since there is no loop at b.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

9. We can solve these problems by first drawing the graph, then labeling the vertices, and finally constructing the matrix by putting a 1 in position (i, j) whenever vertices i and j are joined by an edge. It helps to choose a nice order, since then the matrix will have nice patterns in it.

a) The order of the vertices does not matter, since they all play the same role. The matrix has 0's on the diagonal, since there are no loops in the complete graph.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b) We put the vertex in the part by itself first.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c) We put the vertices in the part of size 2 first. Notice the block structure.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d) We put the vertices in the same order in the matrix as they are around the cycle.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

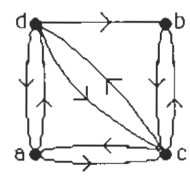
e) We put the center vertex first. Note that the last four columns of the last four rows represent a C_4.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

f) We can label the vertices by the binary numbers from 0 to 7. Thus the first row (also the first column) of this matrix corresponds to the string 000, the second to the string 001, and so on. Since Q_3 has 8 vertices, this is an 8×8 matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

11. This graph has four vertices and is directed, since the matrix is not symmetric. We draw the four vertices as points in the plane, then draw a directed edge from vertex i to vertex j whenever there is a 1 in position (i, j) in the given matrix.



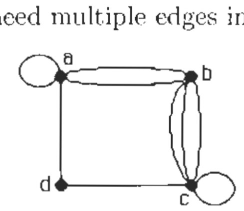
13. We use alphabetical order of the vertices for Exercises 13–15. If there are k parallel edges between vertices i and j, then we put the number k into the $(i, j)^{\text{th}}$ entry of the matrix. In this exercise, there is only one pair of parallel edges.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

15. This is similar to Exercise 13. In this graph there are loops, which are represented by entries on the diagonal. For example, the loop at c is shown by the 1 as the $(3, 3)^{\text{th}}$ entry.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

17. Because of the numbers larger than 1, we need multiple edges in this graph.



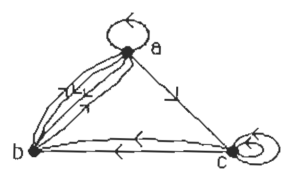
19. We use alphabetical order of the vertices. We put a 1 in position (i,j) if there is a directed edge from vertex i to vertex j; otherwise we make that entry a 0. Note that loops are represented by 1's on the diagonal.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

21. This is similar to Exercise 19, except that there are parallel directed edges. If there are k parallel edges from vertex i to vertex j, then we put the number k into the $(i,j)^{\text{th}}$ entry of the matrix. For example, since there are 2 edges from a to c, the $(1,3)^{\text{th}}$ entry of the adjacency matrix is 2; the loop at c is shown by the 1 as the $(3,3)^{\text{th}}$ entry.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

23. Since the matrix is not symmetric, we need directed edges; furthermore, it must be a directed multigraph because of the entries larger than 1. For example, the 2 in position $(3,2)$ means that there are two parallel edges from vertex c to vertex b.



25. Since the matrix is symmetric, it has to be square, so it represents a graph of some sort. In fact, such a matrix does represent a simple graph. The fact that it is a zero-one matrix means that there are no parallel edges. The fact that there are 0's on the diagonal means that there are no loops. The fact that the matrix is symmetric means that the edges can be assumed to be undirected. Note that such a matrix also represents a directed graph in which all the edges happen to appear in antiparallel pairs (see the solution to Exercise 1d in Section 9.1 for a definition), but that is irrelevant to this question; the answer to the question asked is "yes."

27. In an incidence matrix we have one column for each edge. We use alphabetical order of the vertices. Loops are represented by columns with one 1; other edges are represented by columns with two 1's. The order in which the columns are listed is immaterial.

Exercise 13 $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Exercise 14 $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Exercise 15 $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

29. In an undirected graph, each edge incident to a vertex j contributes 1 in the j^{th} column; thus the sum of the entries in that column is just the number of edges incident to j. Another way to state the answer is that the sum of the entries is the degree of j minus the number of loops at j, since each loop counts 2 toward the degree.

In a directed graph, each edge whose terminal vertex is j contributes 1 in the j^{th} column; thus the sum of the entries in that column is just the number of edges that have j as their terminal vertex. Another way to state the answer is that the sum of the entries is the in-degree of j.

31. Since each column represents an edge, the sum of the entries in the column is either 2, if the edge has 2 incident vertices (i.e., is not a loop), or 1 if it has only 1 incident vertex (i.e., is a loop).

33. **a)** The incidence matrix for K_n has n rows and $C(n,2)$ columns. For each i and j with $1 \le i < j \le n$, there is a column with 1's in rows i and j and 0's elsewhere.

b) The matrix looks like this, with n rows and n columns.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

c) The matrix looks like the matrix for C_n, except with an extra row of 0's (which we have put at the end), since the vertex "in the middle" is not involved in the edges "around the outside," and n more columns for the "spokes." We show some extra space between the rim edge columns and the spoke columns; this is for human convenience only and does not have any bearing on the matrix itself.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & \quad 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & \quad 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & \quad 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \quad 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \quad \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \quad 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & \quad 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \quad 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

d) This matrix has $m+n$ rows and mn columns, one column for each pair (i,j) with $1 \le i \le m$ and $1 \le j \le n$. We have put in some extra spacing for readability of the pattern.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \quad 1 & 1 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 1 & \cdots & 1 \\ \\ 1 & 0 & \cdots & 0 & \quad 1 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \quad 0 & 1 & \cdots & 0 & \quad \cdots & \quad 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \quad 0 & 0 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 1 \end{bmatrix}$$

35. These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$, and $f(u_5) = v_4$.

37. These graphs are isomorphic, since each is the 7-cycle (this is just like Exercise 35).

39. These two graphs are isomorphic. One can see this visually—just imagine "moving" vertices u_1 and u_4 into the inside of the rectangle, thereby obtaining the picture on the right. Formally, one isomorphism is $f(u_1) = v_5$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_6$, $f(u_5) = v_4$, and $f(u_6) = v_1$.

41. These graphs are not isomorphic. In the first graph the vertices of degree 3 are adjacent to a common vertex. This is not true of the second graph.

43. These are isomorphic. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_9$, $f(u_3) = v_4$, $f(u_4) = v_3$, $f(u_5) = v_2$, $f(u_6) = v_8$, $f(u_7) = v_7$, $f(u_8) = v_5$, $f(u_9) = v_{10}$, and $f(u_{10}) = v_6$.

45. We must show that being isomorphic is reflexive, symmetric, and transitive. It is reflexive since the identity function from a graph to itself provides the isomorphism (the one-to-one correspondence)—certainly the identity function preserves adjacency and nonadjacency. It is symmetric, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, then f^{-1} is a one-to-one correspondence that makes G_2 isomorphic to G_1; that is, f^{-1} is a one-to-one and onto function from V_2 to V_1 such that c and d are adjacent in G_2 if and only if $f^{-1}(c)$ and $f^{-1}(d)$ are adjacent in G_1. It is transitive, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, and g is a one-to-one correspondence that makes G_2 isomorphic to G_3, then $g \circ f$ is a one-to-one correspondence that makes G_1 isomorphic to G_3.

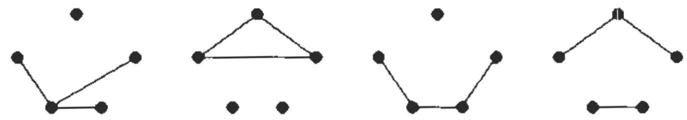
47. If a vertex is isolated, then it has no adjacent vertices. Therefore in the adjacency matrix the row and column for that vertex must contain all 0's.

49. Let V_1 and V_2 be the two parts, say of sizes m and n, respectively. We can number the vertices so that all the vertices in V_1 come before all the vertices in V_2. The adjacency matrix has $m+n$ rows and $m+n$ columns. Since there are no edges between two vertices in V_1, the first m columns of the first m rows must all be 0's. Similarly, since there are no edges between two vertices in V_2, the last n columns of the last n rows must all be 0's. This is what we were asked to prove.

51. There are two such graphs, which can be found by trial and error. (We need only look for graphs with 5 vertices and 5 edges, since a self-complementary graph with 5 vertices must have $C(5,2)/2 = 5$ edges. If nothing else, we can draw them all and find the complement of each. See the pictures for the solution of Exercise 45d in Section 9.4.) One such graph is C_5. The other consists of a triangle, together with an edge from one vertex of the triangle to the fourth vertex, and an edge from another vertex of the triangle to the fifth vertex.

53. If C_n is to be self-complementary, then C_n must have the same number of edges as its complement. We know that C_n has n edges. Its complement has the number of edges in K_n minus the number of edges in C_n, namely $C(n,2)-n = [n(n-1)/2]-n$. If we set these two quantities equal we obtain $[n(n-1)/2]-n = n$, which has $n = 5$ as its only solution. Thus C_5 is the only C_n that *might* be self-complementary—our argument just shows that it has the same number of edges as its complement, not that it is indeed isomorphic to its complement. However, it we draw C_5 and then draw its complement, then we see that the complement is again a copy of C_5. Thus $n = 5$ is the answer to the problem.

55. We need to enumerate these graphs carefully to make sure of getting them all—leaving none out and not duplicating any. Let us organize our catalog by the degrees of the vertices. Since there are only 3 edges, the largest the degree could be is 3, and the only graph with 5 vertices, 3 edges, and a vertex of degree 3 is a $K_{1,3}$ together with an isolated vertex. If all the vertices that are not isolated have degree 2, then the graph must consist of a C_3 and 2 isolated vertices. The only way for there to be two vertices of degree 2 (and therefore also 2 of degree 1) is for the graph to be three edges strung end to end, together with an isolated vertex. The only other possibility is for 2 of the edges to be adjacent and the third to be not adjacent to either of the others. All in all, then, we have the 4 possibilities shown below.



57. a) Both graphs consist of 2 sides of a triangle; they are clearly isomorphic.

b) The graphs are not isomorphic, since the first has 4 edges and the second has 5 edges.

c) The graphs are not isomorphic, since the first has 4 edges and the second has 3 edges.

59. There are at least two approaches we could take here. One approach is to have a correspondence not only of the vertices but also of the edges, with incidence (and nonincidence) preserved. In detail, we say that two pseudographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there are one-to-one and onto functions $f : V_1 \to V_2$ and $g : E_1 \to E_2$ such that for each vertex $v \in V_1$ and edge $e \in E_1$, v is incident to e if and only if $f(v)$ is incident to $g(e)$.

Another approach is simply to count the number of edges between pairs of vertices. Thus we can define $G_1 = (V_1, E_1)$ to be isomorphic to $G_2 = (V_2, E_2)$ if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for every pair of (not necessarily distinct) vertices u and v in V_1, there are exactly the same number of edges in E_1 with $\{u, v\}$ as their set of endpoints as there are edges in E_2 with $\{f(u), f(v)\}$ as their set of endpoints.

61. We can tell by looking at the loop, the parallel edges, and the degrees of the vertices that if these directed graphs are to be isomorphic, then the isomorphism has to be $f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_2$, and $f(u_4) = v_1$. We then need to check that each directed edge (u_i, u_j) corresponds to a directed edge $(f(u_i), f(u_j))$. We check that indeed it does for each of the 7 edges (and there are only 7 edges in the second graph). Therefore the two graphs are isomorphic.

63. If there is to be an isomorphism, the vertices with the same in-degree would have to correspond, and the edge between them would have to point in the same direction, so we would need u_1 to correspond to v_3, and u_2 to correspond to v_1. Similarly we would need u_3 to correspond to v_4, and u_4 to correspond to v_2. If we check all 6 edges under this correspondence, then we see that adjacencies are preserved (in the same direction), so the graphs are isomorphic.

65. If f is an isomorphism from a directed graph G to a directed graph H, then f is also an isomorphism from G^c to H^c. This is clear, because (u, v) is an edge of G^c if and only if (v, u) is an edge of G if and only if $(f(v), f(u))$ is an edge of H if and only if $(f(u), f(v))$ is an edge of H^c.

67. A graph with a triangle will not be bipartite, but cycles of even length are bipartite. So we could let one graph be C_6 and the other be the union of two disjoint copies of C_3.

69. Suppose that the graph has v vertices and e edges. Then the incidence matrix is a $v \times e$ matrix, so its transpose is an $e \times v$ matrix. Therefore the product is a $v \times v$ matrix. Suppose that we denote the typical entry of this product by a_{ij}. Let t_{ik} be the typical entry of the incidence matrix; it is either a 0 or a 1. By definition

$$a_{ij} = \sum_{k=1}^{e} t_{ik} t_{jk} \,.$$

We can now read off the answer from this equation. If $i \neq j$, then a_{ij} is just a count of the number of edges incident to both i and j—in other words, the number of edges between i and j. On the other hand a_{ii} is equal to the number of edges incident to i.

71. Perhaps the simplest example would be to have the graphs have all degrees equaling 2. One way for this to happen is for the graph to be a cycle. But it will also happen if the graph is a disjoint union of cycles. The smallest example occurs when there are six vertices. If G_1 is the 6-cycle and G_2 is the union of two triangles, then the degree seqnences are $(2,2,2,2,2,2)$ for both, but obviously the graphs are not isomorphic. If we want a connected example, then look at Exercise 41, where the degree sequence is $(3,3,2,2,1,1,1,1)$ for each graph.

SECTION 9.4 Connectivity

Some of the most important uses of graphs deal with the notion of path, as the examples and exercises in this and subsequent sections show. It is important to understand the definitions, of course. Many of the exercises here are straightforward. The reader who wants to get a better feeling for what the arguments in more advanced graph theory are like should tackle problems like Exercises 33–36.

1. **a)** This is a path of length 4, but it is not simple, since edge $\{b,c\}$ is used twice. It is not a circuit, since it ends at a different vertex from the one at which it began.
b) This is not a path, since there is no edge from c to a.
c) This is not a path, since there is no edge from b to a.
d) This is a path of length 5 (it has 5 edges in it). It is simple, since no edge is repeated. It is a circuit since it ends at the same vertex at which it began.

3. This graph is not connected— it has three components.

5. This graph is not connected. There is no path from the vertices in one of the triangles to the vertices in the other.

7. A connected component of an acquaintanceship graph represent a maximal set of people with the property that for any two of them, we can find a string of acquaintances that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

9. If a person has Erdős number n, then there is a path of length n from that person to Erdős in the collaboration graph. By definition, that means that that person is in the same component as Erdős. Conversely, if a person is in the same component as Erdős, then there is a path from that person to Erdős, and the length of a shortest such path is that person's Erdős number.

11. **a)** Notice that there is no path from a to any other vertex, because both edges involving a are directed toward a. Therefore the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.
b) Notice that there is no path from c to any other vertex, because both edges involving c are directed toward c. Therefore the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.
c) The underlying nndirected graph is clearly not connected (one component has vertices b, f, and e), so this graph is neither strongly nor weakly connected.

13. The strongly connected components are the maximal sets of phone numbers for which it is possible to find directed paths between every two different numbers in the set, where the existence of a directed path from phone number x to another phone number y means that x called some number, which called another number, $\ldots$, which called y. (The number of intermediary phone numbers in this path can be any natural number.)

15. In each case we want to look for large sets of vertices all which of which have paths to all the others. For these graphs, this can be done by inspection. These will be the strongly connected components.

a) Clearly $\{a, b, f\}$ is a set of vertices with paths between all the vertices in the set. The same can be said of $\{c, d, e\}$. Every edge between a vertex in the first set and a vertex in the second set is directed from the first, to the second. Hence there are no paths from c, d, or e to a, b, or f, and therefore these vertices are not in the same strongly connected component. Therefore these two sets are the strongly connected component.

b) The circuits a, e, d, c, b, a and a, e, d, h, a show that these six vertices are all in the same component. There is no path from f to any of these vertices, and no path from g to any other vertex. Therefore f and g are not in the same strong component as any other vertex. Therefore the strongly connected components are $\{a, b, c, d, e, h\}$, $\{f\}$, and $\{g\}$.

c) It is clear that a and i are in the same strongly connected component. If we look hard, we can also find the circuit b, h, f, g, d, e, d, b, so these vertices are in the same strongly connected component. Because of edges ig and hi, we can get from either of these collections to the other. Thus $\{a, b, d, e, f, g, h, i\}$ is a strong component. We cannot travel from c to any other vertex, so c is in a component by itself.

17. One approach here is simply to invoke Theorem 2 and take successive powers of the adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The answers are the off-diagonal elements of these powers. An alternative approach is to argue combinatorially as follows. Without loss of generality, we assume that the vertices are called $1, 2, 3, 4$, and the path is to run from 1 to 2. A path of length n is determined by choosing the $n - 1$ intermediate vertices. Each vertex in the path must differ from the one immediately preceding it.

a) A path of length 2 requires the choice of 1 intermediate vertex, which must be different from both of the ends. Vertices 3 and 4 are the only ones available. Therefore the answer is 2.

b) Let the path be denoted $1, x, y, 2$. If $x = 2$, then there are 3 choices for y. If $x = 3$, then there are 2 choices for y; similarly if $x = 4$. Therefore there are $3 + 2 + 2 = 7$ possibilities in all.

c) Let the path be denoted $1, x, y, z, 2$. If $x = 3$, then by part **(b)** there are 7 choices for y and z. Similarly if $x = 4$. If $x = 2$, then y and z can be any two distinct members of $\{1, 3, 4\}$, and there are $P(3, 2) = 6$ ways to choose them. Therefore there are $7 + 7 + 6 = 20$ possibilities in all.

d) Let the path be denoted $1, w, x, y, z, 2$. If $w = 3$, then by part **(c)** there are 20 choices for x, y, and z. Similarly if $w = 4$. If $w = 2$, then x must be different from 2, and there are 3 choices for x. For each of these there are by part **(b)** 7 choices for y and z. This gives a total of 21 possibilities in this case. Therefore the answer is $20 + 20 + 21 = 61$.

19. Graph G has a triangle (u_1, u_2, u_3). Graph H does not (in fact, it is bipartite). Therefore G and H are not isomorphic.

21. The drawing of G clearly shows it to be the cube Q_3. Can we see H as a cube as well? Yes—we can view the outer ring as the top face, and the inner ring as the bottom face. We can imagine walking around the top face of G clockwise (as viewed from above), then dropping down to the bottom face and walking around it counterclockwise, finally returning to the starting point on the top face. This is the path $u_1, u_2, u_7, u_6, u_5, u_4, u_3, u_8, u_1$. The corresponding path in H is $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$. We can verify that the edges not in the path do connect corresponding vertices. Therefore $G \cong H$.

23. As explained in the solution to Exercise 17, we could take powers of the adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The answers are found in location $(1,2)$, for instance. Using the alternative approach is much easier than in Exercise 17. First of all, two nonadjacent vertices must lie in the same part, so only paths of even length can join them. Also, there are clearly 3 choices for each intermediate vertex in a path. Therefore we have the following answers:

a) $3^1 = 3$ **b)** 0 **c)** $3^3 = 27$ **d)** 0

25. There are two approaches here. We could use matrix multiplication on the adjacency matrix of this directed graph (by Theorem 2), which is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Thus we can compute $\mathbf{A}^2$ for part **(a)**, $\mathbf{A}^3$ for part **(b)**, and so on, and look at the $(1,5)^{\text{th}}$ entry to determine the number of paths from a to e. Alternately, we can argue in an ad hoc manner, as we do below.

a) There is just 1 path of length 2, namely a, b, e.

b) There are no paths of length 3, since after 3 steps, a path starting at a must be at b, c, or d.

c) For a path of length 4 to end at e, it must be at b after 3 steps. There are only 2 such paths, a, b, a, b, e and a, d, a, b, e.

d) The only way for a path of length 5 to end at e is for the path to go around the triangle bec. Therefore only the path a, b, e, c, b, e is possible.

e) There are several possibilities for a path of length 6. Since the only way to get to e is from b, we are asking for the number of paths of length 5 from a to b. We can go around the square (a, b, e, d, a, b), or else we can jog over to either b or d and back twice—there being 4 ways to choose where to do the jogging. Therefore there are 5 paths in all.

f) As in part **(d)**, it is clear that we have to use the triangle. We can either have a, b, a, b, e, c, b, e or a, d, a, b, e, c, b, e or a, b, e, c, b, a, b, e. Thus there are 3 paths.

27. The definition given here makes it clear that u and v are related if and only if they are in the same component—in other words $f(u) = f(v)$ where $f(x)$ is the component in which x lies. Therefore by Exercise 9 in Section 8.5 this is an equivalence relation.

29. A cut vertex is one whose removal splits the graph into more components than it originally had (which is 1 in this case). Only vertex c is a cut vertex here. If it is removed, then the resulting graph will have two components. If any other vertex is removed, then the graph remains connected.

31. There are several cut vertices here: b, c, e, and i. Removing any of these vertices creates a graph with more than one component. The removal of any of the other vertices leaves a graph with just one component.

33. Without loss of generality, we can restrict our attention to the component in which the cut edge lies; other components of the graph are irrelevant to this proposition. To fix notation, let the cut edge be uv. When the cut edge is removed, the graph has two components, one of which contains v and the other of which contains u.

If v is pendant, then it is clear that the removal of v results in exactly the component containing u—a connected graph. Therefore v is not a cut vertex in this case. On the other hand, if v is not pendant, then there are other vertices in the component containing v—at least one other vertex w adjacent to v. (We are assuming that this proposition refers to a simple graph, so that there is no loop at v.) Therefore when v is removed, there are at least two components, one containing u and another containing w.

35. If every component of G is a single vertex, then clearly no vertex is a cut vertex (the removal of any of them actually decreases the number of components rather than increasing it). Therefore we may as well assume that some component of G has at least two vertices, and we can restrict our attention to that component; in other words, we can assume that G is connected. One clever way to do this problem is as follows. Define the **distance** between two vertices u and v, denoted $d(u, v)$, to be the length of a shortest path joining u and v. Now choose u and v so that $d(u, v)$ is as large as possible. We claim that neither u nor v is a cut vertex. Suppose otherwise, say that u is a cut vertex. Then v is in one component that results after u is removed, and some vertex w is in another. Since there is no path from w to v in the graph with u removed, every path from w to v must have passed through u. Therefore the distance between w and v must have been strictly greater than the distance between u and v. This is a contradiction to the choice of u and v, and our proof by contradiction is complete.

37. This problem is simply asking for the cut edges of these graphs.

a) The link joining Denver and Chicago and the link joining Boston and New York are the cut edges.

b) The following links are the cut edges: Seattle–Portland, Portland–San Francisco, Salt Lake City–Denver, New York–Boston, Boston–Bangor, Boston–Burlington.

39. A vertex basis will be a set of people who collectively can influence everyone, at least indirectly. The set consisting of Deborah is a vertex basis, since she can influence everyone except Yvonne directly, and she can influence Yvonne indirectly through Brian.

41. Since there can be no edges between vertices in different components, G will have the most edges when each of the components is a complete graph. Since K_{n_i} has $C(n_i, 2)$ edges, the maximum number of edges is the sum given in the exercise.

43. Before we give a correct proof here, let us look at an incorrect proof that students often give for this exercise. It goes something like this. "Suppose that the graph is not connected. Then no vertex can be adjacent to every other vertex, only to $n-2$ other vertices. One vertex joined to $n-2$ other vertices creates a component with $n-1$ vertices in it. To get the most edges possible, we must use all the edges in this component. The number of edges in this component is thus $C(n-1,2) = (n-1)(n-2)/2$, and the other component (with only one vertex) has no edges. Thus we have shown that a disconnected graph has at most $(n-1)(n-2)/2$ edges, so every graph with more edges than that has to be connected." The fallacy here is in assuming—without justification—that the maximum number of edges is achieved when one component has $n-1$ vertices. What if, say, there were two components of roughly equal size? Might they not together contain more edges? We will see that the answer is "no," but it is important to realize that this requires proof—it is not obvious without some calculations.

Here is a correct proof, then. Suppose that the graph is not connected. Then it has a component with k vertices in it, for some k between 1 and $n-1$, inclusive. The remaining $n-k$ vertices are in one or more other components. The maximum number of edges this graph could have is then $C(k,2) + C(n-k,2)$, which, after a bit of algebra, simplifies to $k^2 - nk + (n^2 - n)/2$. This is a quadratic function of k. It is minimized when $k = n/2$ (the k coordinate of the vertex of the parabola that is the graph of this function) and maximized at the endpoints of the domain, namely $k = 1$ and $k = n-1$. In the latter cases its value is $(n-1)(n-2)/2$.

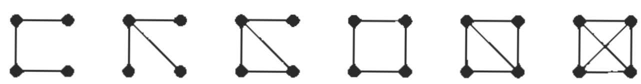
Therefore the largest number of edges that a disconnected graph can have is $(n-1)(n-2)/2$, so every graph with more edges than this must be connected.

45. We have to enumerate carefully all the possibilities.

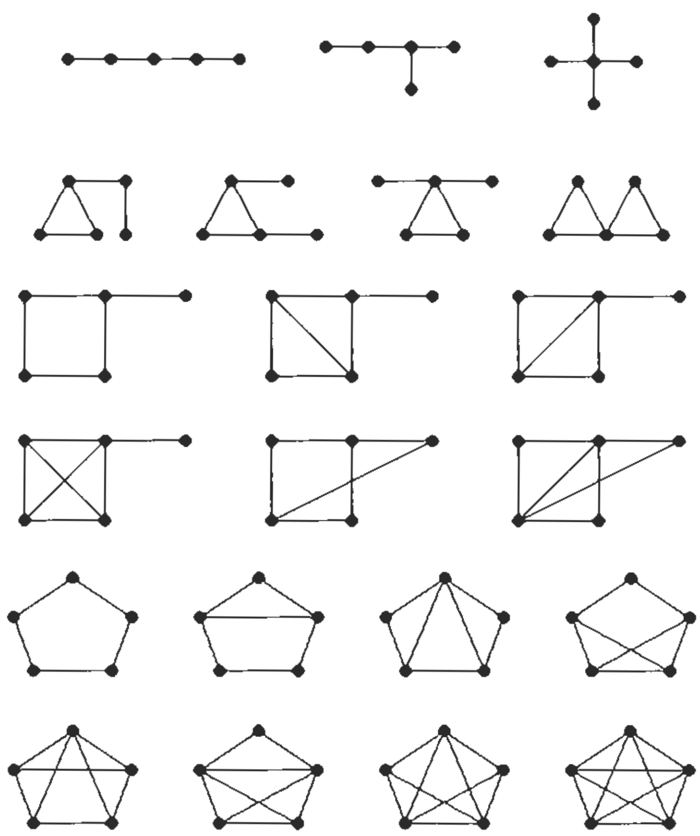
a) There is obviously only 1, namely K_2, the graph consisting of two vertices and the edge between them.

b) There are clearly 2 connected graphs with 3 vertices, namely K_3 and K_3 with one edge deleted, as shown.

c) There are several connected graphs with $n = 4$. If the graph has no circuits, then it must either be a path of length 3 or the "star" $K_{1,3}$. If it contains a triangle but no copy of C_4, then the other vertex must be pendant—only 1 possibility. If it contains a copy of C_4, then neither, one, or both of the other two edges may be present—3 possibilities. Therefore the answer is $2 + 1 + 3 = 6$. The graphs are shown below.



d) We need to enumerate the possibilities in some systematic way, such as by the largest cycle contained in the graph. There are 21 such graphs, as can be seen by such an enumeration, shown below. First we show those graphs with no circuits, then those with a triangle but no C_4 or C_5, then those with a C_4 but no C_5, and finally those with a C_5. In doing this problem we have to be careful not only not to leave out any graphs, but also not to list any twice.



47. We need to look at successive powers of the adjacency matrix until we find one in which the $(1,6)^{\text{th}}$ entry is not 0. Since the matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

we see that the $(1,6)^{\text{th}}$ entry of $\mathbf{A}^2$ is 2. Thus there is a path of length 2 from a to f (in fact 2 of them). On the other hand there is no path of length 1 from a to f (i.e., no edge), so the length of a shortest path is 2.

49. Let the simple paths P_1 and P_2 be $u = x_0, x_1, \ldots, x_n = v$ and $u = y_0, y_1, \ldots, y_m = v$, respectively. The paths thus start out at the same vertex. Since the paths do not contain the same set of edges, they must diverge eventually. If they diverge only after one of them has ended, then the rest of the other path is a simple circuit from v to v. Otherwise we can suppose that $x_0 = y_0$, $x_1 = y_1$, ..., $x_i = y_i$, but $x_{i+1} \neq y_{i+1}$. To form our simple circuit, we follow the path y_i, y_{i+1}, y_{i+2}, and so on, until it once again first encounters a vertex on P_1 (possibly as early as y_{i+1}, no later than y_m). Once we are back on P_1, we follow it along—forwards or backwards, as necessary—to return to x_i. Since $x_i = y_i$, this certainly forms a circuit. It must be a simple circuit, since no edge among the x_k's or the y_l's can be repeated (P_1 and P_2 are simple by hypothesis) and no edge among the x_k's can equal one of the edges y_l that we used, since we abandoned P_2 for P_1 as soon as we hit P_1.

51. Let $\mathbf{A}$ be the adjacency matrix of a given graph G. Theorem 2 tells us that $\mathbf{A}^r$ counts the number of paths of length r between vertices. If an entry in $\mathbf{A}^r$ is greater than 0, then there is a path between the corresponding vertices in G. Suppose that we look at $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^{n-1}$, where n is the number of vertices in G. If there is a path between a pair of distinct vertices in G, then there is a path of length at most $n-1$, so this sum will have a positive integer in the corresponding entry. Conversely, if there is no path, then the corresponding entry in every summand will be 0, and hence the entry in the sum will be 0. Therefore the graph is connected (i.e., there is a path between every pair of distinct vertices in G) if and only if every off-diagonal entry in this sum is strictly positive. To determine whether G is connected, therefore, we just compute this sum and check to see whether this condition holds.

53. We have to prove a statement and its converse here. One direction is fairly easy. If the graph is bipartite, say with parts A and B, then the vertices in every path must alternately lie in A and B. Therefore a path that starts in A, say, will end in B after an odd number of steps and in A after an even number of steps. Since a circuit ends at the same vertex where it starts, the length must be even. The converse is a little harder. We suppose that all circuits have even length and want to show that the graph is bipartite. We can assume that the graph is connected, because if it is not, then we can just work on one component at a time. Let v be a vertex of the graph, and let A be the set of all vertices to which there is a path of odd length starting at v, and let B be the set of all vertices to which there is a path of even length starting at v. Since the component is connected, every vertex lies in A or B. No vertex can lie in both A and B, since then following the odd-length path from v to that vertex and then back along the even-length path from that vertex to v would produce an odd circuit, contrary to the hypothesis. Thus the set of vertices has been partitioned into two sets. Now we just need to show that every edge has endpoints in different parts. If xy is an edge where $x \in A$, then the odd-length path from v to x followed by xy produces an even-length path from v to y, so $y \in B$ (and similarly if $x \in B$).

55. Suppose the couples are Bob and Carol Sanders, and Ted and Alice Henderson (these were characters in a movie from 1969). We represent the initial position by $(BCTA\bullet, \emptyset)$, indicating that all four people are on the left shore along with the boat (the dot). We want to reach the position $(\emptyset, BCTA\bullet)$. Positions will be the vertices of our graph, and legal transitions will be the edges. If Bob and Carol take the boat over, then we reach the position $(TA, BC\bullet)$. The only useful transition at that point is for someone to row back. Let's try Bob; so we have $(BTA\bullet, C)$. If Bob and Ted now row to the right shore, we reach $(A, BCT\bullet)$. Ted can take the boat back to fetch his wife, giving us $(TA\bullet, BC)$ and then $(\emptyset, BCTA\bullet)$. Notice that this path never violates the jealousy conditions imposed in this problem. The entire graph model would have many more positions, but we just need one path.

SECTION 9.5 Euler and Hamilton Paths

An Euler circuit or Euler path uses every edge exactly once. A Hamilton circuit or Hamilton path uses every vertex exactly once (not counting the circuit's return to its starting vertex). Euler and Hamilton circuits and paths have an important place in the history of graph theory, and as we see in this section they have some interesting applications. They provide a nice contrast—there are good algorithms for finding Euler paths (see also Exercises 50–53), but computer scientists believe that there is no good (efficient) algorithm for finding Hamilton paths.

Most of these exercises are straightforward. The reader should at least look at Exercises 16 and 17 to see how the concept of Euler path applies to directed graphs—these exercises are not hard if you understood the proof of Theorem 1 (given in the text before the statement of the theorem).

1. Since there are four vertices of odd degree (a, b, c, and e) and $4 > 2$, this graph has neither an Euler circuit nor an Euler path.

3. Since there are two vertices of odd degree (a and d), this graph has no Euler circuit, but it does have an Euler path starting at a and ending at d. We can find such a path by inspection, or by using the splicing idea explained in this section. One such path is $a, e, c, e, b, e, d, b, a, c, d$.

5. All the vertex degrees are even, so there is an Euler circuit. We can find such a circuit by inspection, or by using the splicing idea explained in this section. One such circuit is $a, b, c, d, c, e, d, b, e, a, e, a$.

7. All the vertex degrees are even, so there is an Euler circuit. We can find such a circuit by inspection, or by using the splicing idea explained in this section. One such circuit is $a, b, c, d, e, f, g, h, i, a, h, b, i, c, e, h, d, g, c, a$.

9. No, an Euler circuit does not exist in the graph modeling the new city either. Vertices A and B have odd degree.

11. Assuming we have just one truck to do the painting, the truck must follow an Euler path through the streets in order to do the job without traveling a street twice. Therefore this can be done precisely when there is an Euler path or circuit in the graph, which means that either zero or two vertices (intersections) have odd degree (number of streets meeting there). We are assuming, of course, that the city is connected.

13. In order for the picture to be drawn under the conditions of Exercises 13–15, the graph formed by the picture must have an Euler path or Euler circuit. Note that all of these graphs are connected. The graph in the current exercise has all vertices of even degree; therefore it has an Euler circuit and can be so traced.

15. See the comments in the solution to Exercise 13. This graph has 4 vertices of odd degree; therefore it has no Euler path or circuit and cannot be so traced.

17. If there is an Euler path, then as we follow it through the graph, each vertex except the starting and ending vertex must have equal in-degree aud out-degree, since whenever we come to the vertex along some edge, we leave it along some edge. The starting vertex must have out-degree 1 greater than its in-degree, since after we have started, using one edge leading out of this vertex, the same argument applies. Similarly, the ending vertex must have in-degree 1 greater than its out-degree, since until we end, using one edge leading into this vertex, the same argument applies. Note that the Euler path itself guarantees weak connectivity; given any two vertices, there is a path from the one that occurs first aloug the Euler path to the other, via the Euler path.

Conversely, suppose that the graph meets the degree conditions stated here. By Exercise 16 it cannot have an Euler circuit. If we add one more edge from the vertex of deficient out-degree to the vertex of deficient in-degree, then the graph now has every vertex with its in-degree equal to its out-degree. Certainly the graph is still weakly connected. By Exercise 16 there is an Euler circuit in this new graph. If we delete the added edge, then what is left of the circuit is an Euler path from the vertex of deficient in-degree to the vertex of deficient out-degree.

19. For Exercises 18–23 we use the results of Exercises 16 and 17. By Exercise 16, we cannot hope to find an Euler circuit since vertex b has different out-degree and in-degree. By Exercise 17, we cannot hope to find an Euler path since vertex b has out-degree and in-degree differing by 2.

21. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from a to e. One such path is $a, d, e, d, b, a, e, c, e, b, c, b, e$.

23. There are more than two vertices whose in-degree and out-degree differ by 1, so by Exercises 16 and 17, there is no Euler path or Euler circuit.

25. The algorithm is very similar to Algorithm 1. The input is a weakly connected directed multigraph in which either each vertex has in-degree equal to its out-degree, or else all vertices except two satisfy this condition and the remaining vertices have in-degree differiug from out-degree by 1 (necessarily once in each direction). We begin by forming a path starting at the vertex whose out-degree exceeds its in-degree by 1 (in the second case) or at any vertex (in the first case). We traverse the edges (never more than once each), forming a path, until we cannot go on. Necessarily we eud up either at the vertex whose in-degree exceeds its out-degree (in the first case) or at the starting vertex (in the second case). From then on we do exactly as in Algorithm 1, finding a simple circuit among the edges not yet used, starting at any vertex on the path we already have; such a vertex exists by the weak connectivity assumption. We splice this circuit into the path, and repeat the process until all edges have been used.

27. **a)** Clearly K_2 has an Euler path but no Euler circuit. For odd $n > 2$ there is an Euler circuit (since the degrees of all the vertices are $n-1$, which is even), whereas for even $n > 2$ there are at least 4 vertices of odd degree and hence no Euler path. Thus for no n other than 2 is there an Euler path but not an Euler circuit.
b) Since C_n has an Euler circuit for all n, there are no valnes of n meeting these conditions.
c) A wheel has at least 3 vertices of degree 3 (around the rim), so there can be no Euler path.
d) The same argument applies here as applied in part **(a)**. In more detail, Q_1 (which is the same as K_2) is the only cube with an Enler path but no Euler circuit, since for odd $n > 1$ there are too many vertices of odd degree, and for even $n > 1$ there is an Euler circuit.

29. Just as a graph with 2 vertices of odd degree can be drawn with one continuous motion, a graph with $2m$ vertices of odd degree can be drawn with m continuous motions. The graph in Exercise 1 has 4 vertices of odd degree, so it takes 2 continuous motions; in other words, the pencil must be lifted once. We could do this, for example, by first tracing a, c, d, e, a, b and then tracing c, b, e. The graphs in Exercises 2–7 all have Euler paths, so no lifting is necessary.

31. It is clear that a, b, c, d, e, a is a Hamilton circuit.

33. There is no Hamilton circuit because of the cut edges ($\{c, e\}$, for instance). Once a purported circuit had reached vertex e, there would be nowhere for it to go.

35. There is no Hamiltonian circuit in this graph. If there were one, then it would have to include all the edges of the graph, because it would have to enter and exit vertex a, enter and exit vertex d, and enter and exit vertex e. But then vertex c would have been visited more than once, a contradiction.

37. This graph has the Hamilton path a, b, c, f, d, e. This simple path hits each vertex once.

39. This graph has the Hamilton path f, e, d, a, b, c.

41. There are eight vertices of degree 2 in this graph. Only two of them can be the end vertices of a Hamilton path, so for each of the other six their two incident edges must be present in the path. Now if either all four of the "outside" vertices of degree 2 (a, c, g, and e) or all four of the "inside" vertices of degree 2 (i, k, l, and n) are not end vertices, then a circuit will be completed that does not include all the vertices—either the outside square or the middle square. Therefore if there is to be a Hamilton path then exactly one of the inside corner vertices must be an end vertex, and each of the other inside corner vertices must have its two incident edges in the path. Without loss of generality we can assume that vertex i is an end, and that the path begins i, o, n, m, l, q, k, j. At this point, either the path must visit vertex p, in which case it gets stuck, or else it must visit b, in which case it will never be able to reach p. Either case gives a contradiction, so there is no Hamilton path.

43. It is easy to write down a Hamiltonian path here; for example, a, d, g, h, i, f, c, e, b.

45. A Hamilton circuit in a bipartite graph must visit the vertices in the parts alternately, returning to the part in which it began. Therefore a necessary condition is certainly $m = n$. Furthermore $K_{1,1}$ does not have a Hamilton circuit, so we need $n \geq 2$ as well. On the other hand, since the complete bipartite graph has all the edges we need, these conditions are sufficient. Explicitly, if the vertices are $a_1, a_2, \ldots, a_n$ in one part and $b_1, b_2, \ldots, b_n$ in the other, with $n \geq 2$, then one Hamilton circuit is $a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_1$.

47. For Dirac's Theorem to be applicable, we need every vertex to have degree at least $n/2$, where n is the number of vertices in the graph. For Ore's Theorem, we need $\deg(x) + \deg(y) \geq n$ whenever x and y are not adjacent.

a) In this graph $n = 5$. Dirac's Theorem does not apply, since there is a vertex of degree 2, and 2 is smaller than $n/2$. Ore's Theorem also does not apply, since there are two nonadjacent vertices of degree 2, so the sum of their degrees is less than n. However, the graph does have a Hamilton circuit—just go around the pentagon. This illustrates that neither of the sufficient conditions for the existence of a Hamilton circuit given in these theorems is necessary.

b) Everything said in the solution to part **(a)** is valid here as well.

c) In this graph $n = 5$, and all the vertex degrees are either 3 or 4, both of which are at least $n/2$. Therefore Dirac's Theorem guarantees the existence of a Hamilton circuit. Ore's Theorem must apply as well, since $(n/2) + (n/2) = n$; in this case, the sum of the degrees of any pair of nonadjacent vertices (there are only two such pairs) is 6, which is greater than or equal to 5.

d) In this graph $n = 6$, and all the vertex degrees are 3, which is (at least) $n/2$. Therefore Dirac's Theorem guarantees the existence of a Hamilton circuit. Ore's Theorem must apply as well, since $(n/2) + (n/2) = n$; in this case, the sum of the degrees of any pair of nonadjacent vertices is 6.

Although not illustrated in any of the examples in this exercise, there are graphs for which Ore's Theorem applies, even though Dirac's does not. Here is one: Take K_4, and then tack on a path of length 2 between two of the vertices, say a, b, c. In all, this graph has five vertices, two with degree 3, two with degree 4, and one with degree 2. Since there is a vertex with degree less than $5/2$, Dirac's Theorem does not apply. However, the sum of the degrees of any two (nonadjacent) vertices is at least $2 + 3 = 5$, so Ore's Theorem does apply and guarantees that there is a Hamilton circuit.

49. The trick is to use a Gray code for n to build one for $n+1$. We take the Gray code for n and put a 0 in front of each term to get half of the Gray code for $n+1$; we put a 1 in front to get the second half. Then we reverse the second half so that the junction at which the two halves meet differ in only the first bit. For a formal proof we use induction on n. For $n = 1$ the code is $0, 1$ (which is not really a Hamilton circuit in Q_1). Assume the inductive hypothesis that $c_1, c_2, \ldots, c_{2^n}$ is a Gray code for n. Then $0c_1, 0c_2, \ldots, 0c_{2^n}, 1c_{2^n}, \ldots, 1c_2, 1c_1$ is a Gray code for $n+1$.

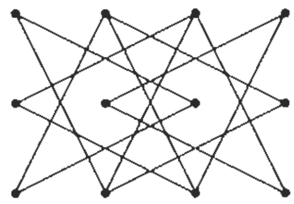
51. Turning this verbal description into pseudocode is straightforward, especially if we allow ourselves lots of words in the pseudocode. We build our *circuit* (which we think of simply as an ordered list of edges) one edge at a time, keeping track of the vertex v we are at; the subgraph containing the edges we have not yet used we will call H. We assume that the vertices of G are listed in some order, so that when we are asked to choose an edge from v meeting certain conditions, we can choose the edge to the vertex that comes first in this order among all those edges meeting the conditions. (This avoids ambiguity, which an algorithm is not supposed to have.)

```
procedure fleury(G : connected multigraph with all degrees even)
v := first vertex of G
circuit := the empty circuit
H := G
while H has edges
begin
        Let e be an edge in H with v as one of its endpoints,
                such that e is not a cut edge of H, if such an edge
                exists; otherwise let e be any edge in H with v as
                one of its endpoints.
        v := other endpoint of e
        Add e to the end of circuit
        Remove e from H
end {circuit is an Euler circuit}
```

53. If every vertex has even degree, then we can simply use Fleury's algorithm to find an Euler circuit, which is by definition also an Euler path. If there are two vertices with odd degree (and the rest with even degree), then we can add an edge between these two vertices and apply Fleury's algorithm (using this edge as the first edge to make it easier to find later), then delete the added edge.

55. A Hamilton circuit in a bipartite graph would have to look like $a_1, b_1, a_2, b_2, \ldots, a_k, b_k, a_1$, where each a_i is in one part and each b_i is in the other part, since the only edges in the graph join vertices in opposite parts. In the Hamilton circuit, no vertex is listed twice (except for the final a_1), and every vertex is listed, so the total number of vertices in the graph must be $2k$, which is not an odd number. Therefore a bipartite graph with an odd number of vertices cannot have a Hamilton circuit.

57. We draw one vertex for each of the 12 squares on the board. We then draw an edge from a vertex to each vertex that can be reached by moving 2 units horizontally and 1 unit vertically or vice versa. The result is as shown.



59. First let us try to find a reentrant knight's tour. Looking at the graph in the solution to Exercise 57 we see that every vertex on the left and right edge has degree 2. Therefore the 12 edges incident to these vertices would have to be in a Hamilton circuit if there were one. If we draw these 12 edges, however, we see that they form two circuits, each with six edges. Therefore there is no re-entrant knight's tour. However, we can splice these two circuits together by using an edge from a middle vertex in the top row to a middle vertex in the bottom row (and omitting two edges adjacent to this edge). The result is the knight's tour shown here.

3	6	9	12
8	11	4	1
5	2	7	10

61. We give an ad hoc argument by contradiction, using the notation shown in the following diagram. We think of the board as a graph and need to decide which edges need to be in a purported Hamilton path.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

There are only two moves from each of the four corner squares. If we put in all of the edges 1-10, 1-7, 16-10, and 16-7, then a circuit is complete too soon, so at least one of these edges must be missing. Without loss of generality, then, we may assume that the endpoints of the path are 1 and either 4 or 13, and that the path contains all of the edges 1-10, 10-16, and 16-7. Now vertex (square) 3 has edges only to squares 5, 10, and 12; and square 10 already has its two incident edges. Therefore 3-5 and 3-12 must be in the Hamilton path. Similarly, edges 8-2 and 8-15 must be in the path. Now square 9 has edges only to squares 2, 7, and 15. If there were to be edges to both 2 and 15, then a circuit would be completed too soon (2-9-15-8-2). Therefore the edge 9-7 must be in the path, thereby giving square 7 its full complemeut of edges. But now square 14 is forced to be joined in the path to squares 5 and 12, and this completes a circuit too soon (5-14-12-3-5). Since we have reached a contradiction, we conclude that there is no Hamilton path.

63. An $m \times n$ board contains mn squares. If both m and n are odd, then it contains an odd number of squares. By Exercise 62, the corresponding graph is bipartite. Exercise 55 told us that the graph does not contain a Hamilton circuit. Therefore there is no re-entrant knight's tour (see Exercise 58b).

65. This is a proof by contradiction. We assume that G satisfies Ore's inequality that $\deg(x) + \deg(y) \geq n$ whenever x and y are nonadjacent vertices in G, but G does not have a Hamilton circuit. We will end up with a contradiction, and therefore conclude that under these conditions, G must have a Hamilton circuit.

a) Since G does not have a Hamilton circuit, we can add missing edges one at a time in such a way that we do not obtain a graph with a Hamilton circuit. We continue this process as long as possible. Clearly it cannot go on forever, because once we've formed the complete graph by adding all missing edges, there is a Hamilton circuit (recall that $n \geq 3$). Whenever the process stops, we have obtained a graph H with the desired property. (Note that H might equal G itself—in other words, we add no edges. However, H cannot be complete, as just noted.)

b) Add one more edge to H. By the construction in part **(a)**, we now have a Hamilton circuit, and clearly this circuit must use the edge we just added. The path consisting of this circuit with the added edge omitted is clearly a Hamilton path in H.

c) Clearly v_1 and v_n are not adjacent in H, since H has no Hamilton circuit. Therefore they are not adjacent in G. But the hypothesis was that the sum of the degrees of vertices not adjacent in G was at least n. This inequality can be rewritten as $n - \deg(v_n) \leq \deg(v_1)$. But $n - \deg(v_n)$ is just the number of vertices not adjacent to v_n.

d) Let's make sure we understand what this means. If, say, v_7 is adjacent to v_1, then v_6 is in S. Note that $v_1 \in S$, since v_2 is adjacent to v_1. Also, v_n is not in S, since there is no vertex following v_n in the Hamilton path. Now each one of the $\deg(v_1)$ vertices adjacent to v_1 gives rise to an element of S, so S contains $\deg(v_1)$ vertices.

e) By part **(c)** there are at most $\deg(v_1) - 1$ vertices other than v_n not adjacent to v_n, and by part **(d)** there are $\deg(v_1)$ vertices in S, none of which is v_n. So S has more vertices other than v_n than there are vertices not adjacent to v_n; in other words, at least one vertex of S is adjacent to v_n. By definition, if v_k is this vertex, then H contains edges $v_k v_n$ and $v_1 v_{k+1}$. Note that $1 < k < n-1$, since we know from part **(c)** that $v_1 v_n$ is not an edge of H.

f) We have shown that all of the edges in the circuit $v_1, v_2, \ldots, v_{k-1}, v_k, v_n, v_{n-1}, \ldots, v_{k+1}, v_1$ are in H, so H has a Hamilton circuit. That is a contradiction to the construction of H. Therefore our assumption that G did not originally have a Hamilton circuit is wrong, and our proof by contradiction is complete.

SECTION 9.6 Shortest-Path Problems

In applying Dijkstra's algorithm for finding shortest paths, it is convenient to keep track, as each vertex is labeled, of where the path comes from. You can put this information on the drawing of the graph itself, by placing a little arrow at the vertex, pointing to the vertex causing the new labeling. Remember that the labels (both the values and these arrows) may change as the algorithm proceeds (and shorter paths are found), so you need an eraser to implement the algorithm in this way. The algorithm is quite simple once you see how it goes, and these exercises are not difficult.

1. In each case we will use a directed weighted graph, since there is no reason to suppose that travel from stop A to stop B should be the same (in whatever respect) as travel from stop B to stop A.

   a) We will put an edge from A to B whenever there is a train that travels from A to B without intermediate stops. The weight of that edge will be the time (in seconds, say) required for the trip, including half the stopping time at each end station. This model is not perfect. For example, the time may depend on the time of day. Also, it is not clear that allocating the waiting time at each station in this way is the best way to model the system (but we should not ignore the waiting time).

   b) We assume that distance refers to the distance along the subway tracks. If so, this model is straightforward and similar to part **(a)**. We put an edge from A to B whenever there is a train that travels from A to B without intermediate stops. The weight of that edge will be the distance the train travels on that trip.

   c) Under the assumption stated, we can model this problem in a manner similar to the previous parts. We put an edge from A to B whenever there is a train that travels from A to B without intermediate stops. The weight of that edge will be the fare required for that trip. Very few subway systems (if any) actually operate under this assumption.

3. We see in the solution to Exercise 5 below that a shortest path has length 16. There really is no better way to find the *length* of a shortest path than by using Dijkstra's algorithm, which for practically the same amount of work actually gives you the path.

5. We can answer these questions by applying Dijkstra's algorithm in each case, with the added feature of indicating, when a vertex is given a new label, where the new path to that vertex comes from. We will denote this by making the vertex a superscript to the distance. Then we can reconstruct the path that produces the minimum distance by tracing these superscripts backward from z to a.

We begin with the graph in Exercise 2. First a is put into S, with label 0, and vertex b is labeled 2^a, and c is labeled 3^a. Since b has the smaller label, b is put into S and d is labeled 7^b, and e is labeled 4^b. Next c is put into S, and no labels are changed. Then e is put into S and the labels of d and z become 5^e and 8^e, respectively. Next d is put into S, and the label of z is changed to 7^d. Finally, z is put into S. Now we know that a shortest path, in reverse, is z, d, e, b, a; we get this by following the superscripts, starting at z. Therefore a shortest path is a, b, e, d, z, with length 7.

We follow the same procedure for the graph in Exercise 3. A shortest path is a, c, d, e, g, z, with length 16.

For the graph in Exercise 4, we follow the same procedure. The graph is bigger, and the process takes longer, but the algorithm is the same. We find a shortest path to be $a, b, e, h, l, m, p, s, z$, having length 16.

7. We apply the variation on Dijkstra's algorithm explained in our solution to Exercise 5. In each case we start at the vertex listed first and can stop once the vertex listed last has been put into S.

a) A shortest path is a, c, d, of length 6.
b) A shortest path is a, c, d, f, of length 11.
c) A shortest path is c, d, f, of length 8.
d) One shortest path is b, d, e, g, z, of length 15.

9. In theory we use the variation on Dijkstra's algorithm explained in our solution to Exercise 5. In each case we start at the vertex listed first and can stop once the vertex listed last has been put into S. In practice for a network of this size with the distances having the geometric significance that they do, we solve the problem by inspection (there are usually at most two conceivable solutions, and we compute the smaller of the two).

a) The shortest trip is, not surprisingly, the direct flight from New York to Los Angeles.
b) The shortest trip is Boston to New York to San Francisco.
c) The shortest trip is Miami to Atlanta to Chicago to Denver.
d) The shortest trip is Miami to New York to Los Angeles.

11. For solution technique, see the comments for Exercise 9.

a) The shortest route is Boston to Chicago to Los Angeles.
b) The shortest route is New York to Chicago to San Francisco.
c) The shortest route is Dallas to Los Angeles to San Francisco.
d) The shortest route is Denver to Chicago to New York.

13. For solution technique, see the comments for Exercise 9.

a) The cheapest route is Boston to Chicago to Los Angeles.
b) One of the cheapest routes is New York to Chicago to San Francisco.
c) The cheapest route is Dallas to Los Angeles to San Francisco.
d) The cheapest route is Denver to Chicago to New York.

15. All we have to do is not stop once z is put into S. Thus we change the condition on the **while** statement to something like "$S \neq V$."

17. For solution technique, see the comments for Exercise 9.

a) The shortest routes are Newark to Woodbridge to Camden, and Newark to Woodbridge to Camden to Cape May. (The map is obviously not drawn to scale.)

b) The cheapest routes (in terms of tolls) are Newark to Woodbridge to Camden, and Newark to Woodbridge to Camden to Cape May.

19. One application, involving directed graphs, is in project scheduling. The vertices can represent parts of the project, and there is a directed edge from A to B if B cannot be started until A is finished. The weight on an edge is the time required to complete the initial vertex of the edge. A longest path from the start of the project to completion represents the total time required to complete the project. Another application would be in trying to find long routes through a city—something a sightseer, political canvasser, or street cleaner might want to do.

21. We can represent the distances with a 6×6 matrix, with alphabetical order. Initially it is

$$\begin{bmatrix} \infty & 4 & 2 & \infty & \infty & \infty \\ 4 & \infty & 1 & 5 & \infty & \infty \\ 2 & 1 & \infty & 8 & 10 & \infty \\ \infty & 5 & 8 & \infty & 2 & 6 \\ \infty & \infty & 10 & 2 & \infty & 3 \\ \infty & \infty & \infty & 6 & 3 & \infty \end{bmatrix}.$$

After completion of the main inner loops for $i = 1$, the matrix looks like this:

$$\begin{bmatrix} \infty & 4 & 2 & \infty & \infty & \infty \\ 4 & 8 & 1 & 5 & \infty & \infty \\ 2 & 1 & 4 & 8 & 10 & \infty \\ \infty & 5 & 8 & \infty & 2 & 6 \\ \infty & \infty & 10 & 2 & \infty & 3 \\ \infty & \infty & \infty & 6 & 3 & \infty \end{bmatrix}.$$

After completion of the main inner loops for $i = 2$, the matrix looks like this:

$$\begin{bmatrix} 8 & 4 & 2 & 9 & \infty & \infty \\ 4 & 8 & 1 & 5 & \infty & \infty \\ 2 & 1 & 2 & 6 & 10 & \infty \\ 9 & 5 & 6 & 10 & 2 & 6 \\ \infty & \infty & 10 & 2 & \infty & 3 \\ \infty & \infty & \infty & 6 & 3 & \infty \end{bmatrix}.$$

After completion of the main inner loops for $i = 3$, the matrix looks like this:

$$\begin{bmatrix} 4 & 3 & 2 & 8 & 12 & \infty \\ 3 & 2 & 1 & 5 & 11 & \infty \\ 2 & 1 & 2 & 6 & 10 & \infty \\ 8 & 5 & 6 & 10 & 2 & 6 \\ 12 & 11 & 10 & 2 & 20 & 3 \\ \infty & \infty & \infty & 6 & 3 & \infty \end{bmatrix}.$$

After completion of the main inner loops for $i = 4$, the matrix looks like this:

$$\begin{bmatrix} 4 & 3 & 2 & 8 & 10 & 14 \\ 3 & 2 & 1 & 5 & 7 & 11 \\ 2 & 1 & 2 & 6 & 8 & 12 \\ 8 & 5 & 6 & 10 & 2 & 6 \\ 10 & 7 & 8 & 2 & 4 & 3 \\ 14 & 11 & 12 & 6 & 3 & 12 \end{bmatrix}.$$

After completion of the main inner loops for $i = 5$, the matrix looks like this:

$$\begin{bmatrix} 4 & 3 & 2 & 8 & 10 & 13 \\ 3 & 2 & 1 & 5 & 7 & 10 \\ 2 & 1 & 2 & 6 & 8 & 11 \\ 8 & 5 & 6 & 4 & 2 & 5 \\ 10 & 7 & 8 & 2 & 4 & 3 \\ 13 & 10 & 11 & 5 & 3 & 6 \end{bmatrix}.$$

There is no change after the final iteration with $i = 6$. Therefore this matrix represents the distances between all pairs.

23. There are two parts to this algorithm. The first part obviously requires $O(n^2)$ operations for bookkeeping and nothing else. The second part obviously requires $O(n^3)$ operations for bookkeeping and the **if**... **then** statement. Therefore the entire procedure takes $O(n^2 + n^3) = O(n^3)$ steps.

25. The following table shows the three different Hamilton circuits and their weights:

Circuit	Weight
a-b-c-d-a	$3+6+7+2=18$
a-b-d-c-a	$3+4+7+5=19$
a-c-b-d-a	$5+6+4+2=17$

Thus we see that the circuit a-c-b-d-a (or the same circuit starting at some other point but traversing the vertices in the same or exactly opposite order) is the one with minimum total weight.

27. The following table shows the twelve different Hamilton circuits and their weights, where we abbreviate the cities with the beginning letter of their name, except that Detroit is M (for Motor City, of course!):

Circuit	Weight
S-M-N-D-L-S	$329+189+279+209+69=1075$
S-M-N-L-D-S	$329+189+379+209+179=1285$
S-M-D-N-L-S	$329+229+279+379+69=1285$
S-M-D-L-N-S	$329+229+209+379+359=1505$
S-M-L-N-D-S	$329+349+379+279+179=1515$
S-M-L-D-N-S	$329+349+209+279+359=1525$
S-N-M-D-L-S	$359+189+229+209+69=1055$
S-N-M-L-D-S	$359+189+349+209+179=1285$
S-N-D-M-L-S	$359+279+229+349+69=1285$
S-N-L-M-D-S	$359+379+349+229+179=1495$
S-D-M-N-L-S	$179+229+189+379+69=1045$
S-D-N-M-L-S	$179+279+189+349+69=1065$

As a check of our arithmetic, we can compute the total weight (price) of all the trips (it comes to 15420) and check that it is equal to 6 times the sum of the weights (which here is 2570), since each edge appears in six paths (and sure enough, $15420 = 6 \cdot 2570$). We see that the circuit S-D-M-N-L-S (or the same circuit starting at some other point but traversing the vertices in the same or exactly opposite order) is the one with minimum total weight, 1045. Note that we might have guessed this route by looking at the drawing (which is more or less to scale in terms of the actual locations of these cities).

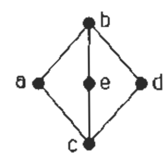
29. If we take a triangle ABC and make one edge, say BC very weighty, then the minimum circuit will avoid that edge. So let's take the weights of AB, AC, and BC to be 1, 2, and 100, respectively. Obviously all Hamilton circuits have total weight 103, but the circuit A-B-A-C-A visits every vertex at least once and has total weight only $1+1+2+2=6$. This circuit visits vertex A an extra time in order to avoid traversing the weighty edge BC.

SECTION 9.7 Planar Graphs

As with Euler and Hamilton circuits and paths, the topic of planar graphs is a classical one in graph theory. The theory (Euler's formula, Kuratowski's Theorem, and their corollaries) is quite beautiful. It is easy to ask extremely difficult questions in this area, however—see Exercise 27, for example. In practice, there are very efficient algorithms for determining planarity that have nothing to do with Kuratowski's Theorem, but they are quite complicated and beyond the scope of this book. For the exercises here, the best way to show that a graph is planar is to draw a planar embedding; the best way to show that a graph is nonplanar is to find a subgraph homeomorphic to K_5 or $K_{3,3}$. (Usually it will be $K_{3,3}$.)

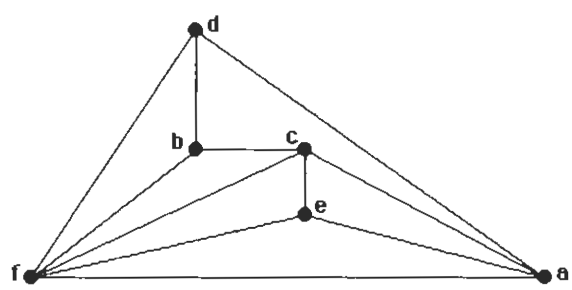
1. The question is whether $K_{5,2}$ is planar. It clearly is so, since we can draw it in the xy-plane by placing the five vertices in one part along the x-axis and the other two vertices on the positive and negative y-axis.

3. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. This graph is just $K_{2,3}$; the picture below shows it redrawn by moving vertex c down.



5. This is $K_{3,3}$, with parts $\{a, d, f\}$ and $\{b, c, e\}$. Therefore it is not planar.

7. This graph can be untangled if we play with it long enough. The following picture gives a planar representation of it.



9. If one has access to software such as *The Geometer's Sketchpad*, then this problem can be solved by drawing the graph and moving the points around, trying to find a planar drawing. If we are unable to find one, then we look for a reason why—either a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$ (always try the latter first). In this case we find that there is a homeomorphic copy of $K_{3,3}$, with vertices b, g, and i in one set and a, f, and h in the other; all the edges are there except for the edge bh, and it is represented by the path beh.

11. We give a proof by contradiction. Suppose that there is a planar representation of K_5, and let us call the vertices v_1, v_2, ..., v_5. There must be an edge from every vertex to every other. In particular, v_1, v_2, v_3, v_4, v_5, v_1 must form a pentagon. The pentagon separates the plane into two regions, an inside and an outside. The edge from v_1 to v_3 must be present, and without loss of generality let us assume it is drawn on the inside. Then there is no way for edges $\{v_2, v_4\}$ and $\{v_2, v_5\}$ to be in the inside, so they must be in the outside region. Now this prevents edges $\{v_1, v_4\}$ and $\{v_3, v_5\}$ from being on the outside. But they cannot both be on the inside without crossing. Therefore there is no planar representation of K_5.

13. We apply Euler's formula $r = e - v + 2$. Here we are told that $v = 6$. We are also told that each vertex has degree 4, so that the sum of the degrees is 24. Therefore by the Handshaking Theorem there are 12 edges, so $e = 12$. Solving, we find $r = 8$.

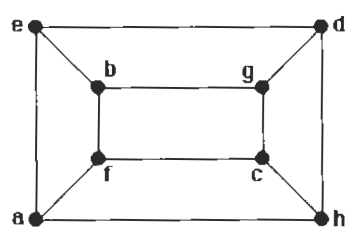
15. The proof is very similar to the proof of Corollary 1. First note that the degree of each region is at least 4. The reason for this is that there are no loops or multiple edges (which would give regions of degree 1 or 2) and no simple circuits of length 3 (which would give regions of degree 3); and the degree of the unbounded region is at least 4 since we are assuming that $v \geq 3$. Therefore we have, arguing as in the proof of Corollary 1, that $2e \geq 4r$, or simply $r \leq e/2$. Plugging this into Euler's formula, we obtain $e - v + 2 \leq e/2$, which gives $e \leq 2v - 4$ after some trivial algebra.

17. The proof is exactly the same as in Exercise 15, except that this time the degree of each region must be at least 5. Thus we get $2e \geq 5r$, which after the same algebra as before, gives the desired inequality.

19. **a)** If we remove a vertex from K_5, then we get K_4, which is clearly planar.
b) If we remove a vertex from K_6, then we get K_5, which is not planar.
c) If we remove a vertex from $K_{3,3}$, then we get $K_{3,2}$, which is clearly planar.
d) We assume the question means "Is it the case that for every v, the removal of v makes the graph planar?" Then the answer is no, since we can remove a vertex in the part of size 4 to leave $K_{3,3}$, which is not planar.

21. This graph is planar and hence cannot be homeomorphic to $K_{3,3}$.

23. The instructions are really not fair. It is hopeless to try to use Kuratowski's Theorem to prove that a graph *is* planar, since we would have to check hundreds of cases to argue that there is no subgraph homeomorphic to K_5 or $K_{3,3}$. Thus we will show that this graph is planar simply by giving a planar representation. Note that it is Q_3.



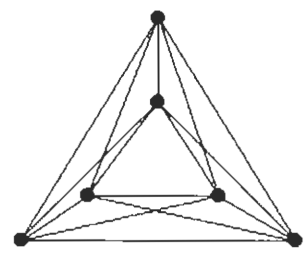
25. This graph is nonplanar, since it contains $K_{3,3}$ as a subgraph: the parts are $\{a, g, d\}$ and $\{b, c, e\}$. (Actually it contains $K_{3,4}$, and it even contains a subgraph homeomorphic to K_5.)

27. This is an extremely hard problem. We will present parts of the solution; the reader should consult a good graph theory book, such as Gary Chartrand and Linda Lesniak's *Graphs & Digraphs*, fourth edition (Chapman & Hall/CRC Press, 2005), for references and further details.

First we will state, without proof, what is known about crossing numbers for complete graphs (much is still not known about crossing numbers). If $n \leq 10$, then the crossing number of K_n is given by the following product

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

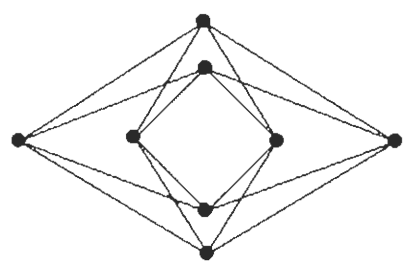
Thus the answers for parts **(a)**, **(b)**, and **(c)** are 1, 3, and 9, respectively. The figure below shows K_6 drawn in the plane with three crossings, which at least proves that the crossing number of K_6 is at most 3. The proof that it is not less than 3 is not easy. The embedding of K_5 with one crossing can be seen in this same picture, by ignoring the vertex at the top.



Second, for the complete bipartite graphs, what is known is that if the smaller of m and n is at most 6, then the crossing number of $K_{m,n}$ is given by the following product

$$\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor.$$

Thus the answers for parts **(d)**, **(e)**, and **(f)** are 2, 4, and 16, respectively. The figure below shows $K_{4,4}$ drawn in the plane with four crossings, which at least proves that the crossing number of $K_{4,4}$ is at most 4. The proof that it is not less than 4 is, again, difficult. It is also easy to see from this picture that the crossing number of $K_{3,4}$ is at most 2 (by ignoring the top vertex).



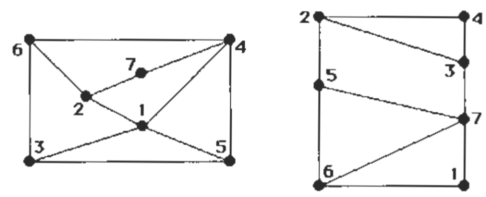
29. Let us follow the hint, and draw all the edges with straight line segments. This clearly produces a drawing of $K_{m,n}$. We will show that the number of crossings is $mn(m-2)(n-2)/16$, and that will complete the proof. (Incidentally, it is not known whether this upper bound is actually the crossing number. No one has found an embedding with fewer crossings, but only in the case in which the smaller of m and n is at most 6 has it been proved that it cannot be done. See the comments in the solution to Exercise 27.) In order to count the crossings, it is enough to count the crossings occurring in the first quadrant and multiply by 4. Let us label the points on the positive x-axis with the numbers 1 through $m/2$, and those on the y-axis with the numbers 1 through $n/2$. If we choose any two distinct numbers, say a and b with $a < b$, from 1 to $m/2$, and any two distinct numbers, say r and s with $r < s$, from 1 through $n/2$, then we get exactly one crossing in our graph, namely between the edges as and br. (There is no crossing between ar and bs.) So the number of crossings in the first quadrant is the same as the number of ways to make these choices, which is clearly $C(m/2,2)\cdot C(n/2,2)$. So the total number of crossings is 4 times this quantity, namely

$$4\cdot C(m/2,2)\cdot C(n/2,2) = 4\cdot\frac{\frac{m}{2}\left(\frac{m}{2}-1\right)}{2}\cdot\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2},$$

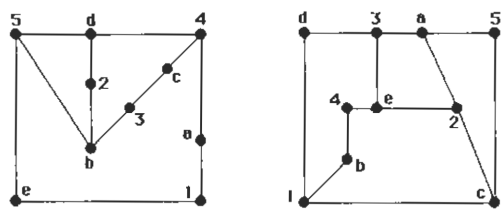
which easily simplifies to

$$\frac{mn(m-2)(n-2)}{16}.$$

31. Each of these graphs is nonplanar; the first three contain K_5, and the last three contain $K_{3,3}$. Thus if we can show how to draw each of the graphs in two planes, then we will have shown that the thickness is 2 in each case. The following picture shows that K_7 can be drawn in 2 planes, so this takes care of part **(a)**, part **(b)**, and part **(c)**.



The following picture shows that $K_{5,5}$ can be drawn in 2 planes, so this takes care of part **(d)**, part **(e)**, and part **(f)**.



33. The formula is certainly valid for $n \leq 4$, so let us assume that $n > 4$. By Exercise 32, the thickness of K_n is at least

$$\frac{C(n,2)}{3n-6} = \frac{n(n-1)/2}{3n-6} = \frac{n(n-1)}{6(n-2)} = \frac{1}{6}\left(n+1+\frac{2}{n-2}\right)$$

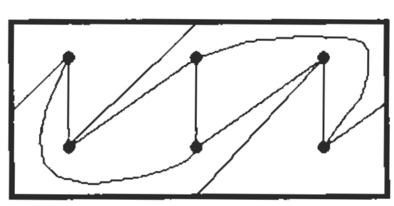
rounded up. Since this quantity is never an integer, it equals one more than itself rounded down, namely

$$\frac{1}{6}\left(n+1+\frac{2}{n-2}\right)+1 = \frac{n+7}{6}+\frac{2}{6(n-2)}$$

rounded down. The last term can be ignored: it is always less than $1/6$ and therefore will not influence the rounding process (since the first term has denominator 6). Thus we have proved that the thickness of K_n is at least $\lfloor (n+7)/6 \rfloor$.

35. This follows immediately from Exercise 34, since $K_{m,n}$ has mn edges and $m+n$ vertices and, being bipartite, has no triangles.

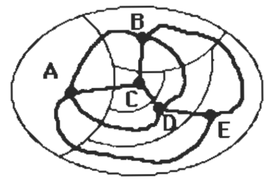
37. We can represent the surface of a torus with a rectangle, thinking of the right-hand edge as being equal to the left-hand edge, and the top edge as being equal to the bottom edge. For example, if we travel out of the rectangle across the right-hand edge about a third of the way from the top, then we immediately reenter the rectangle across the left-hand edge about a third of the way from the top. The picture below shows $K_{3,3}$ drawn on this surface. Note that the edges that seem to leave the rectangle really reenter it from the opposite side.



SECTION 9.8 Graph Coloring

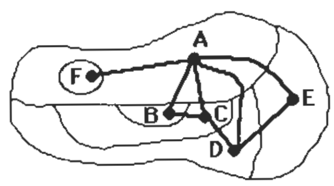
Like the problem of finding Hamilton paths, the problem of finding colorings with the fewest possible colors probably has no good algorithm for its solution. In working these exercises, for the most part you should proceed by trial and error, using whatever insight you can gain by staring at the graph (for instance, finding large complete subgraphs). There are also some interesting exercises here on coloring the edges of graphs—see Exercises 21–22. Exercises 25–27 are worth looking at, as well: they deal with a fast algorithm for coloring a graph that is not guaranteed to produce an optimal coloring.

1. We construct the dual graph by putting a vertex inside each region (but not in the unbounded region), and drawing an edge between two vertices if the regions share a common border. The easiest way to do this is illustrated in our answer. First we draw the map, then we put a vertex inside each region and make the connections. The dual graph, then, is the graph with heavy lines.



The number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Since A, B, C, and D are mutually adjacent, at least four colors are needed. We can color each of the vertices (i.e., regions) A, B, C, and D a different color, and we can give E the same color as we give C.

3. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Three colors are clearly necessary, because of the triangle ABC, for instance. Furthermore three colors suffice, since we can color vertex (region) A red, vertices B, D, and F blue, and vertices C and E green.

5. For Exercises 5 through 11, in order to prove that the chromatic number is k, we need to find a k-coloring and to show that (at least) k colors are needed. Here, since there is a triangle, at least 3 colors are needed. Clearly 3 colors suffice, since we can color a and d the same color.

7. Since there is a triangle, at least 3 colors are needed. Clearly 3 colors suffice, since we can color a and c the same color.

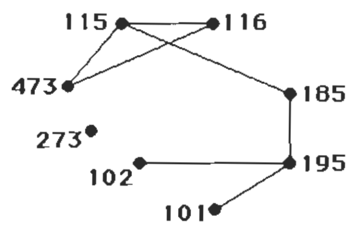
9. Since there is an edge, at least 2 colors are needed. The coloring in which b, d, and e are red and a and c blue shows that 2 colors suffice.

11. Since there is a triangle, at least 3 colors are needed. It is not hard to construct a 3-coloring. We can let a, f, h, j, and n be blue; let b, d, g, k, and m be green; and let c, e, i, l, and o be yellow.

13. If a graph has an edge (not a loop, since we are assuming that the graphs in this section are simple), then its chromatic number is at least 2. Conversely, if there are no edges, then the coloring in which every vertex receives the same color is proper. Therefore a graph has chromatic number 1 if and only if it has no edges.

15. In Example 4 we saw that the chromatic number of C_n is 2 if n is even and 3 if n is odd. Since the wheel W_n is just C_n with one more vertex, adjacent to all the vertices of the C_n along the rim of the wheel, W_n clearly needs exactly one more color than C_n (for that middle vertex). Therefore the chromatic number of W_n is 3 if n is even and 4 if n is odd.

17. Consider the graph representing this problem. The vertices are the 8 courses, and two courses are joined by an edge if there are students taking both of them. Thus there are edges between every pair of vertices except the 7 pairs listed. It is much easier to draw the complement than to draw this graph itself; it is shown below.



We want to find the chromatic number of the graph whose complement we have drawn; the colors will be the time periods for the exams. First note that since Math 185 and the four CS courses form a K_5 (in other words, there are no edges between any two of these in our picture), the chromatic number is at least 5. To show that it equals 5, we just need to color the other three vertices. A little trial and error shows that we can make Math 195 the same color as (i.e., have its final exam at the same time as) CS 101; and we can make Math 115 and 116 the same color as CS 473. Therefore five time slots (colors) are sufficient.

19. We model the problem with the intersection graph of these sets. Note that every pair of these intersect except for C_4 and C_5. Thus the graph is K_6 with that one edge deleted. Clearly its chromatic number is 5, since we need to color all the vertices different colors, except that C_4 and C_5 may have the same color. In other words, 5 meeting times are needed, since only committees C_4 and C_5 can meet simultaneously.

21. Note that the number of colors needed to color the edges is at least as large as the largest degree of a vertex, since the edges at each vertex must all be colored differently. Hence if we can find an edge coloring with that many colors, then we know we have found the answer. In Exercise 5 there is a vertex of degree 3, so the edge chromatic number is at least 3. On the other hand, we can color $\{a,c\}$ and $\{b,d\}$ the same color, so 3 colors suffice. In Exercise 6 the 6 edges incident to g must all get different colors. On the other hand, it is not hard to complete a proper edge coloring with only these colors (for example, color edge $\{a,f\}$ with the same color as used on $\{d,g\}$), so the answer is 6. In Exercise 7 the answer must be at least 3; it is 3 since edges that appear as parallel line segments in the picture can have the same color. In Exercise 8 clearly 4 colors are required, since the vertices have degree 4. In fact 4 colors are sufficient. Here is one proper 4-coloring (we denote edges in the obvious shorthand notation): color 1 for ac, be, and df; color 2 for ae, bd, and cf; color 3 for ab, cd, and ef; and color 4 for ad, bf, and ce. In Exercise 9 the answer must be at least 3; it is easy to construct a 3-coloring of the edges by inspection: $\{a,b\}$ and $\{c,e\}$ have the same color, $\{a,d\}$ and $\{b,c\}$ have the same color, and $\{a,e\}$ and $\{c,d\}$ have the same color. In Exercise 10 the largest degree is 6 (vertex i has degree 6); therefore at least 6 colors are required. By trial and error we come up with this coloring using 6 colors (we use the obvious shorthand notation for edges); there are many others, of course. Assign color 1 to ag, cd, and hi; color 2 to ab, cf, dg, and ei; color 3 to bh, cg, di, and ef; color 4 to ah, ci, and de; color 5 to bi, ch, and fg; and color 6 to ai, bc, and gh. Finally, in Exercise 11 it is easy to construct an edge-coloring with 4 colors; again the edge chromatic number is the maximum degree of a vertex.

Despite the appearances of these examples, it is not the case that the edge chromatic number of a graph is always equal to the maximum degree of the vertices in the graph. The simplest example in which this is not true is K_3. Clearly its edge chromatic number is 3 (since all three edges are adjacent to each other), but its maximum degree is 2. There is a theorem, however, stating that the edge chromatic number is always equal to either the maximum degree or one more than the maximum degree.

23. This problem can be modeled with the intersection graph of the sets of steps during which the variables must be stored. This graph has 7 vertices, t through z; there is an edge between two vertices if the two variables they represent must be stored during some common step. The answer to the problem is the chromatic number of this graph. Rather than considering this graph, we look at its complement (it has a lot fewer edges). Here two vertices are adjacent if the sets (of steps) do not intersect. The only edges are $\{u,w\}$, $\{u,x\}$, $\{u,y\}$, $\{u,z\}$, $\{v,x\}$, $\{x,z\}$. Note that there are no edges in the complement joining any two of $\{t,v,w,y,z\}$, so that these vertices form a K_5 in the original graph. Thus the chromatic number of the original graph is at least 5. To see that it is 5, note that vertex u can have the same color as w, and x can have the same color as z (these pairs appear as edges in the complement). Since the chromatic number is 5, we need 5 registers, with variables u and w sharing a register, and vertices x and z sharing one.

25. First we need to list the vertices in decreasing order of degree. This ordering is not unique, of course; we will pick e,a,b,c,f,h,i,d,g,j. Next we assign color 1 to e, and then to f and d, in that order. Now we assign color 2 to a, c, i, and g, in that order. Finally, we assign color 3 to b, h and j, in that order. Thus the algorithm gives a 3-coloring. Since the graph contains triangles, we know that this is the best possible, so the algorithm "worked" here (but it need not always work—see Exercise 27).

27. A simple example in which the algorithm may fail to provide a coloring with the minimum number of colors is C_6, which of course has chromatic number 2. Since all the vertices are of degree 2, we may order them v_1, v_4, v_2, v_3, v_5, v_6, where the edges are $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_4,v_5\}$, $\{v_5,v_6\}$, and $\{v_1,v_6\}$. Then v_1 gets color 1, as does v_4. Next v_2 and v_5 get color 2; and then v_3 and v_6 must get color 3.

29. We need to show that the wheel W_n when n is an odd integer greater than 1 can be colored with four colors, but that any graph obtained from it by removing one edge can be colored with three colors. Four colors are needed to color this graph, because three colors are needed for the rim (see Example 4), and the center vertex, being adjacent to all the rim vertices, will require a fourth color. To complete the proof that W_n is chromatically 4-critical, we must show that the graph obtained from W_n by deleting one edge can be colored with three colors. There are two cases. If we remove a rim edge, then we can color the rim with two colors, by starting at an endpoint of the removed edge and using the colors alternately around the portion of the rim that remains. The third color is then assigned to the center vertex. On the other hand, if we remove a spoke edge, then we can color the rim by assigning color #1 to the rim endpoint of the removed edge and colors #2 and #3 alternately to the remaining vertices on the rim, and then assign color #1 to the center.

31. We give a proof by contradiction. Suppose that G is chromatically k-critical but has a vertex v of degree $k-2$ or less. Remove from G one of the edges incident to v. By definition of "k-critical," the resulting graph can be colored with $k-1$ colors. Now restore the missing edge and use this coloring for all vertices except v. Because we had a proper coloring of the smaller graph, no two adjacent vertices have the same color. Furthermore, v has at most $k-2$ neighbors, so we can color v with an unused color to obtain a proper $(k-1)$-coloring of G. This contradicts the fact that G has chromatic number k. Therefore our assumption was wrong, and every vertex of G must have degree at least $k-1$.

33. a) Note that vertices d, e, and f are mutually adjacent. Therefore six different colors are needed in a 2-tuple coloring, since each of these three vertices needs a disjoint set of two colors. In fact it is easy to give a coloring with just six colors: Color a, d, and g with $\{1,2\}$; color c and e with $\{3,4\}$; and color b and f with $\{5,6\}$. Thus $\chi_2(G)=6$.

b) This one is trickier than part (a). There is no coloring with just six colors, since if there were, we would be forced (without loss of generality) to color d with $\{1,2\}$; e with $\{3,4\}$; f with $\{5,6\}$; then g with $\{1,2\}$, b with $\{5,6\}$, and c with $\{3,4\}$. This gives no free colors for vertex a. Now this may make it appear that eight colors are required, but a little trial and error shows us that seven suffice: Color a with $\{2,4\}$; color b and f with $\{5,6\}$; color d with $\{1,2\}$; color c with $\{3,7\}$; color e with $\{3,4\}$; and color g with $\{1,7\}$. Thus $\chi_2(H)=7$.

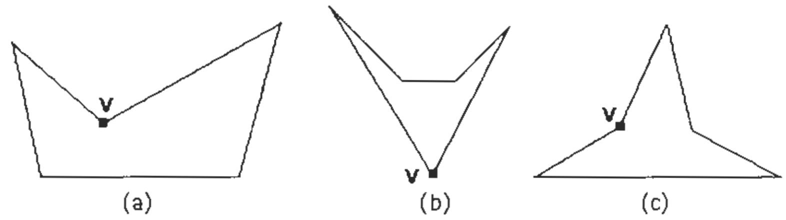
c) This is similar to part (a). Here nine colors are necessary and sufficient, since a, d, and g can get one set of three colors; b and f can get a second set; and c and e can get a third set. Clearly nine colors are necessary to color the triangles.

d) First we construct a coloring with 11 colors: Color a with $\{3,6,11\}$; color b and f with $\{7,8,9\}$; color d with $\{1,2,3\}$; color c with $\{4,5,10\}$; color e with $\{4,6,11\}$; and color g with $\{1,2,5\}$. To prove that $\chi_3(H)=11$, we must show that it is impossible to give a 3-tuple coloring with only ten colors. If such a coloring were possible, without loss of generality we could color d with $\{1,2,3\}$, e with $\{4,5,6\}$, f with $\{7,8,9\}$, and g with $\{1,2,10\}$. Now nine colors are needed for the three vertices a, b, and c, since they form a triangle; but colors 1 and 2 are already used in vertices adjacent to all three of them. Therefore at least $9+2=11$ colors are necessary.

35. The frequencies are the colors, the zones are the vertices, and two zones that are so close that interference would be a problem are joined by an edge in the graph. Then it is clear that a k-tuple coloring is exactly an assignment of frequencies that avoids possible interference.

37. We use induction on the number of vertices of the graph. Every graph with five or fewer vertices can be colored with five or fewer colors, since each vertex can get a different color. That takes care of the basis case(s). So we assume that all graphs with k vertices can be 5-colored and consider a graph G with $k+1$ vertices. By Corollary 2 in Section 9.7, G has a vertex v with degree at most 5. Remove v to form the graph G'. Since G' has only k vertices, we 5-color it by the inductive hypothesis. If the neighbors of v do not use all five colors, then we can 5-color G by assigning to v a color not used by any of its neighbors. The difficulty arises if v has five neighbors, and each has a different color in the 5-coloring of G'. Suppose that the neighbors of v, when considered in clockwise order around v, are a, b, c, m, and p. (This order is determined by the clockwise order of the curves representing the edges incident to v.) Suppose that the colors of the neighbors are azure, blue, chartreuse, magenta, and purple, respectively. Consider the azure-chartreuse subgraph (i.e., the vertices in G colored azure or chartreuse and all the edges between them). If a and c are not in the same component of this graph, then in the component containing a we can interchange these two colors (make the azure vertices chartreuse and vice versa), and G' will still be properly colored. That makes a chartreuse, so we can now color v azure, and G has been properly colored. If a and c are in the same component, then there is a path of vertices alternately colored azure and chartreuse joining a and c. This path together with edges av and vc divides the plane into two regions, with b in one of them and m in the other. If we now interchange blue and magenta on all the vertices in the same region as b, we will still have a proper coloring of G', but now blue is available for v. In this case, too, we have found a proper coloring of G. This completes the inductive step, and the theorem is proved.

39. We follow the hint. Because the measures of the interior angles of a pentagon total $540°$, there cannot be as many as three interior angles of measure more than $180°$ (reflex angles). If there are no reflex angles, then the pentagon is convex, and a guard placed at any vertex can see all points. If there is one reflex angle, then the pentagon must look essentially like figure (a) below, and a guard at vertex v can see all points. If there are two reflex angles, then they can be adjacent or nonadjacent (figures (b) and (c)); in either case, a guard at vertex v can see all points. (In figure (c), choose the reflex vertex closer to the bottom side.) Thus for all pentagons, one guard suffices, so $g(5)=1$.



41. The figure suggested in the hint (generalized to have k prongs for any $k \geq 1$) has $3k$ vertices. Consider the set of points from which a guard can see the tip of the first prong, the set of points from which a guard can see the tip of the second prong, and so on. These are disjoint triangles (together with their interiors). Therefore a separate guard is needed for each of the k prongs, so at least k guards are needed. This shows that $g(3k) \geq k = \lfloor 3k/3 \rfloor$. To handle values of n that are not multiples of 3, let $n = 3k + i$, where $i = 1$ or 2. Then obviously $g(n) \geq g(3k) \geq k = \lfloor n/3 \rfloor$.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 9

1. a) See pp. 589–590 and Table 1 in Section 9.1. b) See Exercise 1 in Section 9.1.

2. See all the examples Section 9.1.

3. See Theorem 1 in Section 9.2.

4. See Theorem 2 in Section 9.2.

5. See Theorem 3 in Section 9.2.

6. a) See Example 5 in Section 9.2. b) See Example 13 in Section 9.2. c) See Example 6 in Section 9.2. d) See Example 7 in Section 9.2. e) See Example 8 in Section 9.2.

7. a) n, $C(n,2)$ b) $m+n$, mn c) n, n d) $n+1$, $2n$ e) 2^n, $n2^{n-1}$

8. a) See p. 602. b) K_2 and C_{2m}
c) (See also Example 12 and Exercise 60 in Section 9.2.) The following algorithm is an efficient way to determine whether a connected graph can be 2-colored (which is the same thing as saying that it is bipartite); apply it to each component of the given graph. First color any vertex red. Then color all vertices adjacent to this vertex blue. Then look at all vertices adjacent to these just-colored blue vertices. If any of them are already colored blue, then stop and declare the graph not to be bipartite; otherwise color all the uncolored ones red. Next look at all vertices adjacent to all the vertices just colored red. If any of them are already colored red, then stop and declare the graph not to be bipartite; otherwise color all the uncolored ones blue. Continue in this way until no more vertices can be colored. If we get this far, then (this component of) the graph is bipartite. (If uncolored vertices remain, they are in a different component, so we can repeat the entire process starting with any uncolored vertex.)

9. a) adjacency lists, adjacency matrices, incidence matrices
b) Look at Figure 1 in Section 9.3, and let G be the graph consisting of the vertices and edges shown there, together with edges $\{b,c\}$ and $\{b,e\}$. Its adjacency lists are shown on p. 612 (Table 1), once we add c and e to the list of adjacent vertices of b, and add b to the list for c and for e. Its adjacency matrix and incidence matrix are as follows (using alphabetical order):

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

10. **a)** See p. 615.

b) See p. 615; number of vertices, number of edges, degrees of vertices, existence of triangle (C_3 as subgraph), and existence of Hamilton circuit are all invariants.

c) See Figure 10 in Section 9.3. **d)** no

11. **a)** See p. 624. **b)** See p. 625.

12. **a)** See p. 612. **b)** See pp. 616–617. **c)** See Theorem 2 in Section 9.4.

13. **a)** See p. 633. **b)** See pp. 633–634.

c) All edges are in the same component and there are at most two vertices of odd degree (see Theorems 1 and 2 in Section 9.5).

d) All edges are in the same component and there are no vertices of odd degree (see Theorem 1 in Section 9.5).

14. **a)** See p. 638. **b)** the existence of a cut edge or a cut vertex

15. finding the shortest highway route between two cities; finding the cheapest way to lease telephone lines to join two communications centers, using intermediate switching centers

16. **a)** See pp. 649–651. **b)** See Exercise 5, third part, in Section 9.6.

17. **a)** See p. 658. **b)** K_6

18. **a)** If the planar graph has v vertices, e edges, and c components, and is embedded in the plane so as to form r regions, then $v - e + r = 1 + c$. (The theorem as stated in the text—Theorem 1 in Section 9.7—assumes that $c = 1$, but this is the more general statement.)

b) By Corollary 1 in Section 9.7, for every planar graph with at least three vertices we know that $e \leq 3v - 6$ (connectivity need not be assumed). Thus we can show that a graph is nonplanar by showing that it has too many edges, namely more than $3v - 6$ edges. For example, K_6 is nonplanar, since it has 15 edges, and $15 \not\leq 3 \cdot 6 - 6$.

19. See Theorem 2 in Section 9.7. A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

20. **a)** See p. 667. **b)** n **c)** 2 if n is even, 3 if n is odd **d)** 2

21. Every planar graph can be 4-colored (i.e., has chromatic number at most 4), but K_6, for instance, requires six colors.

22. See Examples 5–7 in Section 9.8.

SUPPLEMENTARY EXERCISES FOR CHAPTER 9

1. Every vertex has degree 50. Thus the sum of the degrees must be $50 \cdot 100 = 5000$. By the Handshaking Theorem, the graph therefore has $5000/2 = 2500$ edges.

3. Both graphs have a lot of symmetry to them, and the degrees of the vertices are the same, so we might hope that they are isomorphic. Let us try to form the correspondence f. First note that there is a 4-cycle u_1, u_5, u_2, u_6, u_1 in the first graph. Suppose that we try letting it correspond to the 4-cycle v_1, v_2, v_3, v_4, v_1 in the second graph. Thus we let $f(u_1) = v_1$, $f(u_5) = v_2$, $f(u_2) = v_3$, and $f(u_6) = v_4$. The rest of the assignments are forced: since u_7 is the other vertex adjacent to u_1, we must let $f(u_7) = v_6$, since v_6 is the other vertex adjacent to v_1 (which is $f(u_1)$). Similarly, $f(u_3) = v_7$, $f(u_8) = v_8$, and $f(u_4) = v_5$. Now we just have to check that the vertices corresponding to the vertices in the 4-cycle v_5, v_6, v_7, v_8, v_5 in the second graph form a 4-cycle in that order. Since these vertices form the 4-cycle u_4, u_7, u_3, u_8, u_4, our correspondence works.

5. These graphs are isomorphic, although the isomorphism is hard to find. One approach that can lead to the isomorphism is to draw the complement of each graph. The complements are a little simpler than the original graphs, since they have fewer edges. When we do this, it is easy to give a planar representation of each. Then by looking at the sizes of the regions we can find an isomorphism. One such correspondence is $u_1 \leftrightarrow v_5$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_7$, $u_4 \leftrightarrow v_3$, $u_5 \leftrightarrow v_8$, $u_6 \leftrightarrow v_2$, $u_7 \leftrightarrow v_6$, and $u_8 \leftrightarrow v_1$. We just need to check that all the edges are preserved by this correspondence.

7. It follows immediately from the definition that the complete m-partite graph with parts n_1, n_2, ..., n_m has $n_1 + n_2 + \cdots + n_m = \sum_{i=1}^{m} n_i$ vertices. We will organize the count of the edges by looking at which parts the edges join. Fix $1 \leq i < j \leq m$, and consider the edges between the i^{th} part and the j^{th} part. It is easy to see from the product rule that there are $n_i n_j$ edges. Therefore to get all the edges, we have to add all these products, for all possible pairs (i, j). Thus the number of edges is $\sum_{1 \leq i < j \leq m} n_i n_j$.

9. **a)** The subgraph induced by $\{a, b, c\}$ consists of those vertices and all the edges that are in the graph and join pairs of them. Thus the induced subgraph is the entire component H_1.
b) The subgraph induced by $\{a, e, g\}$ consists of those vertices and all the edges joining pairs of them in this graph. Since there are no such edges, the induced subgraph is just the graph with these three vertices and no edges.
c) The induced subgraph consists of these five vertices and the edges $\{b, c\}$, $\{f, g\}$, and $\{g, h\}$ (all the edges joining pairs of these five vertices that were in the original graph).

11. In general it is no easy task to find cliques. We need to be careful not to overlook things. We will denote a clique simply by listing the vertices in it, without punctuation. There is one K_4, namely $bcef$, and it is the largest clique. There are several K_3's not contained in this K_4, and they are all cliques: abg, adg, beg, and deg. Since every edge is contained in one of these five cliques (and there are no isolated vertices), there are no smaller cliques, so this list is complete.

13. See the comments for Exercise 11. We find the cliques by brute force and careful looking, hoping that we do not miss any. Some staring at the graph convinces us that there are no K_6's. There is one K_5, namely the clique $bcdjk$. There are two K_4's not contained in this K_5, which therefore are cliques: $abjk$, and $cfgi$. All the K_3's not contained in any of the cliques listed so far are also cliques. We find abi, aij, bde, bei, bij, ghi, and hij. All the edges are in at least one of the cliques listed so far (and there are no isolated vertices), so we are done.

15. Clearly no single vertex is dominating by itself, but $\{c, d\}$ dominates, so it is a minimum dominating set (there are lots of others).

17. These graphs are quite a mess to draw, since they contain so many edges. Instead of drawing them, we describe them in set-theoretic terms. Note that we do not consider a queen on a square to control that square itself, since to do so would give us loops in the graph.
a) The vertex set consists of all pairs (i, j) with $1 \leq i \leq 3$ and $1 \leq j \leq 3$. Since a queen in any square controls all the squares in the same row and column, there are edges $\{(i, j), (i, j')\}$ and $\{(i, j), (i', j)\}$ for all i, j, i' and j' between 1 and 3 inclusive, with $i \neq i'$ and $j \neq j'$. These are not all, though, since we have to put in the diagonal controls. There are 10 such edges: $(1, 1)$, $(2, 2)$, and $(3, 3)$ are all joined to each other; there is an edge between $(1, 2)$ and $(2, 1)$, and an edge between $(2, 3)$ and $(3, 2)$; $(1, 3)$, $(2, 2)$, and $(3, 1)$ are all joined to each other; and there is an edge between $(1, 2)$ and $(2, 3)$, and an edge between $(2, 1)$ and $(3, 2)$.
b) We do the same sort of thing as in part **(a)**. The vertex set consists of all pairs (i, j) with $1 \leq i \leq 4$ and $1 \leq j \leq 4$. Since a queen in any square controls all the squares in the same row and column, there are edges

$\{(i,j),(i,j')\}$ and $\{(i,j),(i',j)\}$ for all i, j, i' and j' between 1 and 4 inclusive, with $i \neq i'$ and $j \neq j'$. Rather than listing the 28 diagonal control edges explicitly, let us use some analytic geometry. The diagonals over which the queen has control all have slope 1 or -1. Therefore there are edges between (i,j) and (i',j') if these two vertices are distinct and $i - i' = \pm(j - j')$.

19. **a)** A queen in the center controls the entire board, so the answer is 1.

b) One queen cannot control the entire board, as one can verify by considering the possible cases (there are really only three—corner, noncorner edge, or nonedge). On the other hand 2 queens will do: place one in position $(2,2)$ and the other in position $(4,4)$.

c) Three queens can control the entire board. We can place them at positions $(2,2)$, $(3,4)$, and $(5,1)$, for example. To show that two queens are not enough is tedious. One way to do this is with the help of a computer. For each pair of squares (and there are $C(25,2) = 300$ such pairs), check to see that not all the squares are under control. Such a program will show that two queens can control at most 23 of the 25 squares.

21. Suppose that G and H are isomorphic simple graphs, with f the one-to-one and onto function from the vertex set of G to the vertex set of H that gives the isomorphism. By symmetry, it is enough to show in each case that if G has the property, then so does H.

a) Assnme that G is connected; then for every pair of distinct vertices in G there is a path from one to the other. We want to show that H is connected. Let u and v be two distinct vertices of H. Then there is a path in G from $f^{-1}(u)$ to $f^{-1}(v)$, which we can think of as a sequence of vertices. Applying f to each vertex in this path gives us a path from u to v in H.

b) Suppose that G has a Hamilton circuit, which we can think of as a sequence of vertices. Applying f to each vertex in this circuit gives us a Hamilton circuit in H.

c) This is the same as part **(b)**, replacing "Hamilton" by "Euler."

d) The logic of this one is slightly different from the general pattern. Suppose that we can embed G in the plane with C crossings. Then this embedding clearly gives an embedding of H in the plane with C crossings as well: we use the same picture, relabeling u hy $f(u)$. Therefore the crossing number of H is no bigger than the crossing number of G. By symmetry, the crossing number of G can be no bigger than the crossing number of H, either. Therefore the crossing numbers are equal.

e) If i_1, i_2, ..., i_n are n isolated vertices in G, then $f(i_1)$, $f(i_2)$, ..., $f(i_n)$ are n isolated vertices in H.

f) If A and B are the parts for G, then $f(A) = \{\, f(v) \mid v \in A \,\}$ and $f(B) = \{\, f(v) \mid v \in B \,\}$ are the parts for H.

23. We need to consider all the possibilities carefully. First suppose that the parts are of size 1 and 3. The only connected bipartite simple graph with parts of these sizes is $K_{1,3}$. The only other possibility is that the parts are each of size 2. Then the graph could be $K_{2,2}$ or $K_{2,2}$ with one edge missing; if more than one edge is deleted, then the result will not be connected. Therefore the answer is 3.

25. **a)** It is clear from the picture that if this graph is self-converse, then the relevant isomorphism interchanges c and e and interchanges a and b. If we take the given graph, reverse all the arrows, and then apply this correspondence, then we obtain the original graph back. So the graph is self-converse.

b) It is clear from the picture that if this graph were to be self-converse, then the isomorphism must send b and c (the only vertices without loops) back onto themselves, in one order or the other. But b has its incident edges both pointing in the same direction (inwards), whereas c does not, so there is no possible isomorphism between this graph and its converse.

27. This graph is not orientable because of the cut edge $\{b, c\}$. If we orient it from b to c, then there can be no path in the resulting directed graph from c to b; if we orient it from c to b, then there can be no path in the resulting directed graph from b to c.

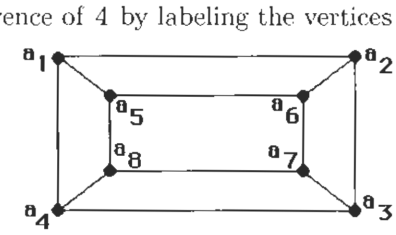
29. This graph is orientable. We can orient the square and each of the triangles in the clockwise direction, for instance. In other words, the edges are (a, b), (b, c), (c, d), (d, a), (c, e), (e, f), (f, c), (c, g), (g, h), and (h, c). There is now a path from every vertex to every other vertex, by traveling clockwise around the appropriate figure to vertex c, and then traveling clockwise around the other appropriate figure.

31. Suppose that $\{a, b\}$ is a cut edge of the undirected graph G. Then a and b are in separate components of G when that edge is removed; in other words, every path from a to b must go through the edge in the a to b direction, and every path from b to a must go through the path in the b to a direction. Suppose that we have an orientation of this graph. If $\{a, b\}$ is oriented as (a, b), then by what we have said, there can be no path in the resulting directed graph from b to a, so the resulting directed graph is not strongly connected. On the other hand, if $\{a, b\}$ is oriented as (b, a), then there can be no path in the resulting directed graph from a to b. Thus by definition G is not orientable. Incidentally, a kind of converse to this result is also true. The ambitious reader should try to construct a proof.

33. Let n be the number of vertices in the tournament. Since for each u different from a given vertex v there is exactly one edge with endpoints u and v (in some order), there are $n - 1$ edges involving vertex v. Thus the sum of the in-degree and out-degree of v is $n - 1$.

35. We make the vertices the chickens in the flock, and for distinct chickens u and v, we have the directed edge (u, v) if and only if u dominates v.

37. **a)** No matter how the vertices are labeled, we must have a_1 adjacent to a_5, so the bandwidth is $5 - 1 = 4$.
b) The best we can do is to have the vertex of degree 3 as a_2. Then the edge between a_2 and a_4 causes the bandwidth to be $4 - 2 = 2$.
c) If we make the vertices in the part with 2 vertices a_2 and a_4, then the maximum of $|i - j|$ with a_i adjacent to a_j occurs when $i = 1$ and $j = 4$, a maximum of 3. If we carefully consider other possibilities, then we see that there is no way to reduce this difference. Therefore the bandwidth is 3.
d) The bandwidth is 4. To see this, assume without loss of generality that a_1 is in part A. Then if a_6 is in part B, the maximum difference is 5, which is larger than 4. On the other hand, if a_6 is also in part A, then the maximum difference is 4, since either a_2 or a_5 must be in part B.
e) We can achieve a maximum difference of 4 by labeling the vertices as shown below.



We must show that there is no way to achieve a difference of 3 or less. Now vertex a_1 is adjacent to three other vertices. If the difference is going to be 3 or less, then these have to be vertices a_2, a_3, and a_4. Now vertex a_2 is adjacent to two vertices besides these four, and therefore the index of one of them must be at least 6, giving a difference of 4.
f) The bandwidth cannot be 1, since at least one of the two vertices adjacent to vertex a_1 must have subscript at least 3, and $3 - 1 = 2$. On the other hand, if we label the vertices around the cycle as a_1, a_2, a_4, a_5, a_3 (and back to a_1), then the maximum value of $|i - j|$ with a_i and a_j adjacent is 2. Thus the bandwidth equals 2.

39. a) Suppose that the diameter of G is at least 4, and let u and v be two vertices whose distance apart in G is at least 4. We want to show that the diameter of $\overline{G}$ is at most 2. Let a and b be two distinct vertices of G. We need to show that the distance between a and b in $\overline{G}$ is at most 2. If $\{a,b\}$ is not an edge of G, then we are done, since then a and b are adjacent (at distance 1) in $\overline{G}$. Thus we assume that a and b are adjacent in G. Now it cannot be that $\{u,v\} = \{a,b\}$, since u and v are not adjacent in G. Without loss of generality assume that u is not in $\{a,b\}$. If u is not adjacent to a and also not adjacent to b in G, then we are done, since the path a,u,b in $\overline{G}$ shows that the distance between a and b is 2 in $\overline{G}$. We now show that the other possibility--that u is adjacent (in G) to at least one of these—leads either to the conclusion that the distance between a and b is at most 2, or to a contradiction. If u is adjacent to at least one of a and b, then v cannot be either a or b, because it would be too close to u in G. Therefore the same reasoning applies to v as applied to u, and either we are done or else we know that v is also adjacent (in G) to at least one of a and b. But this gives us a path from u to v, passing through one or both of a and b, of length less than 4, a contradiction.

b) The proof is rather similar to that in part (a). Let u and v be vertices at a distance of at least 3 in G. Let a and b be arbitrary distinct vertices; we must show that the distance between a and b in $\overline{G}$ is at most 3. Assume not (we will derive a contradiction). Then certainly a and b are adjacent in G. Thus at least one of u and v is not equal to either a or b; say it is u. Now u cannot be adjacent to both a and b in $\overline{G}$, since then the distance between a and b in $\overline{G}$ would be 2. Without loss of generality assume that u is adjacent to a in G. Then v cannot be either a or b (it would be too close to u if it were), and it cannot be adjacent to a (in G), either, for the same reason. If v is not adjacent to b in G, then the path a,v,b in $\overline{G}$ makes the distance between a and b in $\overline{G}$ equal to 2. Thus we can assume that v is adjacent to b in G. It follows that u is not adjacent to b in G, since that adjacency would again make u too close to v. But now we have our contradiction, since there is the path a,v,u,b in $\overline{G}$, of length 3, from a to b.

41. There are two second shortest paths, both of length 8. One is a,b,e,z; the other is a,d,e,z,e,z.

43. Since the shortest path already went through all the vertices, the answer is that same path: a,c,b,d,e,z.

45. First we assume that G has exactly 11 vertices. Suppose that G is planar. By Corollary 1 to Euler's Theorem in Section 9.7, we know that a planar graph with 11 vertices can have at most $3 \cdot 11 - 6 = 27$ edges (if the graph is not connected, then it would have even fewer edges). Therefore G has at most 27 edges. This means that $\overline{G}$ has at least $C(11,2) - 27 = 28$ edges on its 11 vertices. By Corollary 1, again, this means that $\overline{G}$ is nonplanar, as desired. Now if in fact we were dealing with a graph G with more than 11 vertices, then let us restrict ourselves to the first 11 (in some ordering), and let H be the subgraph of G containing those 11 vertices and all the edges of G between pairs of them. Thus H is a subgraph of G, and it is easy to see that $\overline{H}$ is also a subgraph of $\overline{G}$. If G is planar, then so is H; by our argument above this means that $\overline{H}$ is not planar, so $\overline{G}$ cannot be planar.

It is actually the case that this result is still true if we replace the number 11 by the number 9. The proof, however, is very hard. The approach used here does not work, since two planes can contain the required *number* of edges. Indeed if we let G have 18 edges, then $\overline{G}$ will also contain $C(9,2) - 18 = 18$ edges. Corollary 1 to Euler's Theorem says that a planar graph with 9 vertices can have at most $3 \cdot 9 - 6 = 21$ edges, and $18 < 21$. The subtlety comes with trying to embed precisely the right edges for G and the right edges for $\overline{G}$, and it can be proved that this cannot be done, no matter what graph on 9 vertices G is.

47. Let n be the number of vertices in the graph, k its chromatic number, and i its independence number. Then there is a coloring of the graph with k colors. Since no two vertices in the same color class (i.e., colored the same) are adjacent, each color class is an independent set. Thus there are at most i vertices in each color class. This means there are at most ki vertices in all. In other words, $n \le ki$, as desired.

49. Think of the putting in of an edge as a success in a Bernoulli trial (see Section 6.2).

a) By Theorem 2 in Section 6.2, the probability of m successes is $C(n,m)p^m(1-p)^{n-m}$.

b) By Theorem 2 in Section 6.4, this expected value is np.

c) There is a subtle point here. Suppose $n = 3$. Then there are eight different graphs on n vertices, if we view the vertices as labeled 1, 2, and 3. However, many of these graphs are isomorphic. Indeed, there are only four different graphs on three vertices, if we view them as unlabeled, and these four are not equally likely in this random generation process. So for this problem to make sense, we must be speaking of *labeled* graphs. Now we claim that each such graph has probability $1/2^{C(n,2)}$ of arising from this random generation process. Suppose we want to generate labeled graph G. As we apply the process to pairs of vertices, the random number x chosen must be less than or equal to $1/2$ when G has an edge between that pair of vertices and x must be greater than $1/2$ when G has no edge there. So the probability of the process "getting it right" is $1/2$ for each edge. Our claim follows, so all labeled graphs G are equally likely.

51. This is really an exercise in untangling the logic of what is being said. Suppose that P is monotone increasing. We must prove that not having P is monotone decreasing. That is, we must show that the property of not having P is retained whenever edges are removed from a simple graph. If this were not true, then there would be a simple graph G not having P and another simple graph G' with the same vertices but with some of the edges of G removed, which has P. But P is monotone increasing, so since G' has P, so does the graph G obtained by adding edges to G'. This contradicts the assumption that G does not have P. The converse is proved in exactly the same way.

WRITING PROJECTS FOR CHAPTER 9

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. The best source here is [BiLl].

2. A good source for these kinds of applications is [Ro2]. Graph theory is also being used extensively now in the human genome project and other areas of biology.

3. See the comments for Writing Project 2.

4. See [Ba3] for a recent book about research on this subject.

5. See [EaTa] and [Sk].

6. There should be some material in [Sk]. See information on the Web about a program called **nauty**, which has a good many graph algorithms.

7. The definitive work is [KöSc].

8. One of the main researchers in this area is Pavel Pevzner in the Computer Science Department at the University of California at San Diego. Look on his Web page for references.

9. Check out [Ra1].

10. Most advanced graph theory or combinatorics texts will mention this. Look at [Ro1], which also has further references. There is also a relevant chapter in [MiRo].

11. Try general graph theory references, such as [BoMu]. A specialized article on this topic is [Go2].

12. There is a whole book on this topic, namely [La3]. Up-to-date information (such as the size of the largest traveling salesman problem that has been solved) can also be found on the Web; use a search engine.

13. Try [BoMu] or, better yet, books on graph algorithms like [ChOe] or [Ev2].

14. This would be a good time to search for "book number" in *Mathematical Reviews*, which is available on the Web as MathSciNet.

15. Try [Mi] or [SaKa].

16. See [ApHa] and [WoWi]. One of the major critics of computer-based proofs is the philosopher Thomas Tymoczko. You should look at relevant articles by him and others (such as [La2], [Ty1], [Ty2], and [Sw].) Another computer-assisted proof involved projective planes of order 10.

17. One article to look at is B. Manvel's "Extremely greedy coloring algorithms" in [HaMa], which is a conference proceedings. It will be an educational experience to browse through that volume, to see what research mathematicians do. Also, books on algorithmic graph theory, such as [Mc] or those mentioned above in the suggestion for Writing Project 13, will have some material.

18. There is a relevant article in [MiRo].

19. There are entire books on random graphs and, more generally, probabilistic methods in discrete mathematics. Two books to consult are [Bo2] and the more elementary (and fun) [Pa1]. There is also a paper-length introduction, [Bo3]. For something more general, try [AlSp], although it is rather advanced. This is related to the probabilistic method, discussed in Section 6.2.