SECTION 9.3 Representing Graphs and Graph Isomorphism

2. This is similar to Exercise 1. The list is as follows.

Vertex	Adjacent vertices
a	b, d
b	a, d, e
c	d, e
d	a, b, c
e	b, c

4. This is similar to Exercise 3. The list is as follows.

Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	c, e

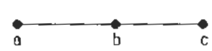
6. This is similar to Exercise 5. The vertices are assumed to be listed in alphabetical order.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

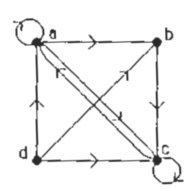
8. This is similar to Exercise 7.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

10. This graph has three vertices and is undirected, since the matrix is symmetric.



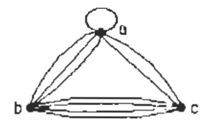
12. This graph is directed, since the matrix is not symmetric.



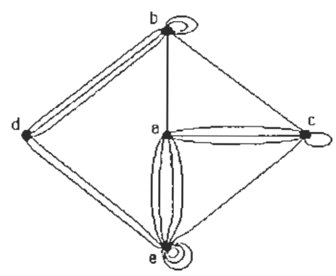
14. This is similar to Exercise 13.

$$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

16. Because of the numbers larger than 1, we need multiple edges in this graph.



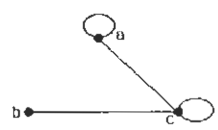
18. This is similar to Exercise 16.



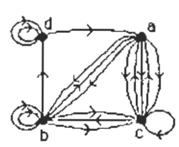
20. This is similar to Exercise 19.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

22. a) This matrix is symmetric, so we can take the graph to be undirected. No parallel edges are present, since no entries exceed 1.



24. This is the adjacency matrix of a directed multigraph, because the matrix is not symmetric and it contains entries greater than 1.



26. Each column represents an edge; the two 1's in the column are in the rows for the endpoints of the edge.

Exercise 1 $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Exercise 2 $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

28. For an undirected graph, the sum of the entries in the i^{th} row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i. (See the solution to Exercise 29.)In a directed graph, the answer is dual to the answer for Exercise 29. The sum of the entries in the i^{th} row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

30. The sum of the entries in the i^{th} row of the incidence matrix is the number of edges incident to vertex i, since there is one column with a 1 in row i for each such edge.

32. a) This is just the matrix that has 0's on the main diagonal and 1's elsewhere, namely

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

b) We label the vertices so that the cycle goes $v_1, v_2, \dots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

c) This matrix is the same as the answer in part (b), except that we add one row and column for the vertex in the middle of the wheel; in our matrix it is the last row and column:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}$$

d) Since the first m vertices are adjacent to none of the first m vertices but all of the last n, and vice versa, this matrix splits up into four pieces:

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

e) It is not convenient to show these matrices explicitly. Instead, we will give a recursive definition. Let $\mathbf{Q}_n$ be the adjacency matrix for the graph Q_n. Then

$$\mathbf{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathbf{Q}_{n+1} = \begin{bmatrix} \mathbf{Q}_n & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{Q}_n \end{bmatrix},$$

where $\mathbf{I}_n$ is the identity matrix (since the corresponding vertices of the two n-cubes are joined by edges in the $(n+1)$-cube).

34. These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$.

36. These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

38. These two graphs are isomorphic. Each consists of a K_4 with a fifth vertex adjacent to two of the vertices in the K_4. Many isomorphisms are possible. One is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

40. These graphs are not isomorphic—the degrees of the vertices are not the same (the graph on the right has a vertex of degree 4, which the graph on the left lacks).

42. These graphs are not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.

44. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.

46. This is immediate from the definition, since an edge is in $\overline{G}$ if and only if it is not in G, if and only if the corresponding edge is not in H, if and only if the corresponding edge is in $\overline{H}$.

48. An isolated vertex has no incident edges, so the row consists of all 0's.

50. The complementary graph consists of edges $\{a,c\}$, $\{c,d\}$, and $\{d,b\}$; it is clearly isomorphic to the original graph (send d to a, a to c, b to d, and c to b).

52. If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

54. a) There are just two graphs with 2 vertices—the one with no edges, and the one with one edge.
b) A graph with three vertices can contain 0, 1, 2, or 3 edges. There is only one graph for each number of edges, up to isomorphism. Therefore the answer is 4.
c) Here we look at graphs with 4 vertices. There is 1 graph with no edges, and 1 (up to isomorphism) with a single edge. If there are two edges, then these edges may or may not be adjacent, giving us 2 possibilities. If there are three edges, then the edges may form a triangle, a star, or a path, giving us 3 possibilities. Since graphs with four, five, or six edges are just complements of graphs with two, one, or no edges (respectively), the number of isomorphism classes must be the same as for these earlier cases. Thus our answer is $1+1+2+3+2+1+1=11$.

56. There are 9 such graphs. Let us first look at the graphs that have a cycle in them. There is only 1 with a 4-cycle. There are 2 with a triangle, since the fourth edge can either be incident to the triangle or not. If there are no cycles, then the edges may all be in one connected component (see Section 9.4), in which case there are 3 possibilities (a path of length four, a path of length three with an edge incident to one of the middle vertices on the path, and a star). Otherwise, there are two components, which are necessarily either two paths of length two, a path of length three plus a single edge, or a star with three edges plus a single edge (3 possibilities in this case as well).

58. a) These graphs are both K_3, so they are isomorphic.
b) These are both simple graphs with 4 vertices and 5 edges. Up to isomorphism there is only one such graph (its complement is a single edge), so the graphs have to be isomorphic.

60. We need only modify the definition of isomorphism of simple graphs slightly. The directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function $f: V_1 \to V_2$ such that for all pairs of vertices a and b in V_1, $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.

62. These two graphs are not isomorphic. In the first there is no edge from the unique vertex of in-degree 0 (u_1) to the unique vertex of out-degree 0 (u_2), whereas in the second graph there is such an edge, namely v_3v_4.

64. We claim that the digraphs are isomorphic. To discover an isomorphism, we first note that vertices u_1, u_2, and u_3 in the first digraph are independent (i.e., have no edges joining them), as are u_4, u_5, and u_6. Therefore these two groups of vertices will have to correspond to similar groups in the second digraph, namely v_1, v_3, and v_5, and v_2, v_4, and v_6, in some order. Furthermore, u_3 is the only vertex among one of these groups of u's to be the only one in the group with out-degree 2, so it must correspond to v_6, the vertex with the similar property in the other digraph; and in the same manner, u_4 must correspond to v_5. Now it is an easy matter, by looking at where the edges lead, to see that the isomorphism (if there is one) must also pair up u_1 with v_2; u_2 with v_4; u_5 with v_1; and u_6 with v_3. Finally, we easily verify that this indeed gives an isomorphism—each directed edge in the first digraph is present precisely when the corresponding directed edge is present in the second digraph.

66. To show that the property that a graph is bipartite is an isomorphic invariant, we need to show that if G is bipartite and G is isomorphic to H, say via the function f, then H is bipartite. Let V_1 and V_2 be the partite sets for G. Then we claim that $f(V_1)$—the images under f of the vertices in V_1—and $f(V_2)$—the images under f of the vertices in V_2—form a bipartition for H. Indeed, since f must preserve the property of not being adjacent, since no two vertices in V_1 are adjacent, no two vertices in $f(V_1)$ are adjacent, and similarly for V_2.

68. **a)** There are 10 nonisomorphic directed graphs with 2 vertices. To see this, first consider graphs that have no edges from one vertex to the other. There are 3 such graphs, depending on whether they have no, one, or two loops. Similarly there are 3 in which there is an edge from each vertex to the other. Finally, there are 4 graphs that have exactly one edge between the vertices, because now the vertices are distinguished, and there can be or fail to be a loop at each vertex.

b) A detailed discussion of the number of directed graphs with 3 vertices would be rather long, so we will just give the answer, namely 104. There are some useful pictures relevant to this problem (and part **(c)** as well) in the appendix to *Graph Theory* by Frank Harary (Addison-Wesley, 1969).

c) The answer is 3069.

70. The answers depend on exactly how the storage is done, of course, but we will give naive answers that are at least correct as approximations.

a) We need one adjacency list for each vertex, and the list needs some sort of name or header; this requires v storage locations. In addition, each edge will appear twice, once in the list of each of its endpoints; this will require $2e$ storage locations. Therefore we need $v + 2e$ locations in all.

b) The adjacency matrix is a $v \times v$ matrix, so it requires v^2 bits of storage.

c) The incidence matrix is a $v \times e$ matrix, so it requires ve bits of storage.