CHAPTER 4
Induction and Recursion

SECTION 4.1 Mathematical Induction

Important note about notation for proofs by mathematical induction: *In performing the inductive step, it really does not matter what letter we use. We see in the text the proof of* $P(k) \to P(k+1)$; *but it would be just as valid to prove* $P(n) \to P(n+1)$, *since the* k *in the first case and the* n *in the second case are just dummy variables. We will use both notations in this* Guide; *in particular, we will use* k *for the first few exercises but often use* n *afterwards.*

2. We can prove this by mathematical induction. Let $P(n)$ be the statement that the golfer plays hole n. We want to prove that $P(n)$ is true for all positive integers n. For the basis step, we are told that $P(1)$ is true. For the inductive step, we are told that $P(k)$ implies $P(k+1)$ for each $k \geq 1$. Therefore by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

4. **a)** Plugging in $n = 1$ we have that $P(1)$ is the statement $1^3 = [1 \cdot (1+1)/2]^2$.

b) Both sides of $P(1)$ shown in part **(a)** equal 1.

c) The inductive hypothesis is the statement that

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 .$$

d) For the inductive step, we want to show for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove

$$[1^3 + 2^3 + \cdots + k^3] + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 .$$

e) Replacing the quantity in brackets on the left-hand side of part **(d)** by what it equals by virtue of the inductive hypothesis, we have

$$\left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2 ,$$

as desired.

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n.

6. The basis step is clear, since $1 \cdot 1! = 2! - 1$. Assuming the inductive hypothesis, we then have

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1 , \end{aligned}$$

as desired.

8. The proposition to be proved is $P(n)$:

$$2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2 \cdot (-7)^n = \frac{1 - (-7)^{n+1}}{4} .$$

In order to prove this for all integers $n \geq 0$, we first prove the basis step $P(0)$ and then prove the inductive step, that $P(k)$ implies $P(k+1)$. Now in $P(0)$, the left-hand side has just one term, namely 2, and the right-hand side is $(1-(-7)^1)/4 = 8/4 = 2$. Since $2 = 2$, we have verified that $P(0)$ is true. For the inductive step, we *assume* that $P(k)$ is true (i.e., the displayed equation above), and derive from it the truth of $P(k+1)$, which is the equation

$$2 - 2\cdot 7 + 2\cdot 7^2 - \cdots + 2\cdot(-7)^k + 2\cdot(-7)^{k+1} = \frac{1-(-7)^{(k+1)+1}}{4}.$$

To prove an equation like this, it is usually best to start with the more complicated side and manipulate it until we arrive at the other side. In this case we start on the left. Note that all but the last term constitute precisely the left-hand side of $P(k)$, and therefore by the inductive hypothesis, we can replace it by the right-hand side of $P(k)$. The rest is algebra:

$$\begin{aligned}[2 - 2\cdot 7 + 2\cdot 7^2 - \cdots + 2\cdot(-7)^k] + 2\cdot(-7)^{k+1} &= \frac{1-(-7)^{k+1}}{4} + 2\cdot(-7)^{k+1}\\ &= \frac{1-(-7)^{k+1} + 8\cdot(-7)^{k+1}}{4}\\ &= \frac{1+7\cdot(-7)^{k+1}}{4}\\ &= \frac{1-(-7)\cdot(-7)^{k+1}}{4}\\ &= \frac{1-(-7)^{(k+1)+1}}{4}.\end{aligned}$$

10. **a)** By computing the first few sums and getting the answers $1/2$, $2/3$, and $3/4$, we guess that the sum is $n/(n+1)$.

b) We prove this by induction. It is clear for $n = 1$, since there is just one term, $1/2$. Suppose that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

We want to show that

$$\left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Starting from the left, we replace the quantity in brackets by $k/(k+1)$ (by the inductive hypothesis), and then do the algebra

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2},$$

yielding the desired expression.

12. We proceed by mathematical induction. The basis step ($n = 0$) is the statement that $(-1/2)^0 = (2+1)/(3\cdot 1)$, which is the true statement that $1 = 1$. Assume the inductive hypothesis, that

$$\sum_{j=0}^{k}\left(-\frac{1}{2}\right)^j = \frac{2^{k+1}+(-1)^k}{3\cdot 2^k}.$$

We want to prove that

$$\sum_{j=0}^{k+1}\left(-\frac{1}{2}\right)^j = \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}.$$

Split the summation into two parts, apply the inductive hypothesis, and do the algebra:

$$\begin{aligned}\sum_{j=0}^{k+1}\left(-\frac{1}{2}\right)^j &= \sum_{j=0}^{k}\left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{k+1}\\ &= \frac{2^{k+1}+(-1)^k}{3\cdot 2^k} + \frac{(-1)^{k+1}}{2^{k+1}}\\ &= \frac{2^{k+2}+2(-1)^k}{3\cdot 2^{k+1}} + \frac{3(-1)^{k+1}}{3\cdot 2^{k+1}}\\ &= \frac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}}\,.\end{aligned}$$

For the last step, we used the fact that $2(-1)^k = -2(-1)^{k+1}$.

14. We proceed by induction. Notice that the letter k has been used in this problem as the dummy index of summation, so we cannot use it as the variable for the inductive step. We will use n instead. For the basis step we have $1\cdot 2^1 = (1-1)2^{1+1}+2$, which is the true statement $2=2$. We assume the inductive hypothesis, that

$$\sum_{k=1}^{n} k\cdot 2^k = (n-1)2^{n+1}+2\,,$$

and try to prove that

$$\sum_{k=1}^{n+1} k\cdot 2^k = n\cdot 2^{n+2}+2\,.$$

Splitting the left-hand side into its first n terms followed by its last term and invoking the inductive hypothesis, we have

$$\sum_{k=1}^{n+1} k\cdot 2^k = \left(\sum_{k=1}^{n} k\cdot 2^k\right) + (n+1)2^{n+1} = (n-1)2^{n+1}+2+(n+1)2^{n+1} = 2n\cdot 2^{n+1}+2 = n\cdot 2^{n+2}+2\,,$$

as desired.

16. The basis step reduces to $6=6$. Assuming the inductive hypothesis we have

$$\begin{aligned}&1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3)\\ &\quad= \frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3)\\ &\quad= (k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)\\ &\quad= \frac{(k+1)(k+2)(k+3)(k+4)}{4}\,.\end{aligned}$$

18. **a)** Plugging in $n=2$, we see that $P(2)$ is the statement $2! < 2^2$.

b) Since $2! = 2$, this is the true statement $2<4$.

c) The inductive hypothesis is the statement that $k! < k^k$.

d) For the inductive step, we want to show for each $k\ge 2$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part **(c)**) we can prove that $(k+1)! < (k+1)^{k+1}$.

e) $(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer n greater than 1.

20. The basis step is $n=7$, and indeed $3^7 < 7!$, since $2187 < 5040$. Assume the statement for k. Then $3^{k+1} = 3\cdot 3^k < (k+1)\cdot 3^k < (k+1)\cdot k! = (k+1)!$, the statement for $k+1$.

22. A little computation convinces us that the answer is that $n^2 \le n!$ for $n = 0,\ 1$, and all $n \ge 4$. (Clearly the inequality does not hold for $n = 2$ or $n = 3$.) We will prove by mathematical induction that the inequality holds for all $n \ge 4$. The basis step is clear, since $16 \le 24$. Now suppose that $n^2 \le n!$ for a given $n \ge 4$. We must show that $(n+1)^2 \le (n+1)!$. Expanding the left-hand side, applying the inductive hypothesis, and then invoking some valid bounds shows this:

$$\begin{aligned} n^2 + 2n + 1 &\le n! + 2n + 1 \\ &\le n! + 2n + n = n! + 3n \\ &\le n! + n \cdot n \le n! + n \cdot n! \\ &= (n+1)n! = (n+1)! \end{aligned}$$

24. The basis step is clear, since $1/2 \le 1/2$. We assume the inductive hypothesis (the inequality shown in the exercise) and want to prove the similar inequality for $n+1$. We proceed as follows, using the trick of writing $1/(2(n+1))$ in terms of $1/(2n)$ so that we can invoke the inductive hypothesis:

$$\begin{aligned} \frac{1}{2(n+1)} &= \frac{1}{2n} \cdot \frac{2n}{2(n+1)} \\ &\le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdots 2n} \cdot \frac{2n}{2(n+1)} \\ &\le \frac{1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdots 2n} \cdot \frac{2n+1}{2(n+1)} \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}{2 \cdot 4 \cdots 2n \cdot 2(n+1)} \end{aligned}$$

26. One can get to the proof of this by doing some algebraic tinkering. It turns out to be easier to think about the given statement as $na^{n-1}(a-b) \ge a^n - b^n$. The basis step ($n = 1$) is the true statement that $a - b \ge a - b$. Assume the inductive hypothesis, that $ka^{k-1}(a-b) \ge a^k - b^k$; we must show that $(k+1)a^k(a-b) \ge a^{k+1} - b^{k+1}$. We have

$$\begin{aligned} (k+1)a^k(a-b) &= k \cdot a \cdot a^{k-1}(a-b) + a^k(a-b) \\ &\ge a(a^k - b^k) + a^k(a-b) \\ &= a^{k+1} - ab^k + a^{k+1} - ba^k . \end{aligned}$$

To complete the proof we want to show that $a^{k+1} - ab^k + a^{k+1} - ba^k \ge a^{k+1} - b^{k+1}$. This inequality is equivalent to $a^{k+1} - ab^k - ba^k + b^{k+1} \ge 0$, which factors into $(a^k - b^k)(a-b) \ge 0$, and this is true, because we are given that $a > b$.

28. The base case is $n = 3$. We check that $4^2 - 7 \cdot 4 + 12 = 0$ is nonnegative. Next suppose that $n^2 - 7n + 12 \ge 0$; we must show that $(n+1)^2 - 7(n+1) + 12 \ge 0$. Expanding the left-hand side, we obtain $n^2 + 2n + 1 - 7n - 7 + 12 = (n^2 - 7n + 12) + (2n - 6)$. The first of the parenthesized expressions is nonnegative by the inductive hypothesis; the second is clearly also nonnegative by the assumption that n is at least 3. Therefore their sum is nonnegative, and the inductive step is complete.

30. The statement is true for $n = 1$, since $H_1 = 1 = 2 \cdot 1 - 1$. Assume the inductive hypothesis, that the statement is true for n. Then on the one hand we have

$$\begin{aligned} H_1 + H_2 + \cdots + H_n + H_{n+1} &= (n+1)H_n - n + H_{n+1} \\ &= (n+1)H_n - n + H_n + \frac{1}{n+1} \\ &= (n+2)H_n - n + \frac{1}{n+1}, \end{aligned}$$

and on the other hand

$$\begin{aligned}(n+2)H_{n+1}-(n+1) &= (n+2)\left(H_n+\frac{1}{n+1}\right)-(n+1)\\ &= (n+2)H_n+\frac{n+2}{n+1}-(n+1)\\ &= (n+2)H_n+1+\frac{1}{n+1}-n-1\\ &= (n+2)H_n-n+\frac{1}{n+1}.\end{aligned}$$

That these two expressions are equal was precisely what we had to prove.

32. The statement is true for the base case, $n=0$, since $3 \mid 0$. Suppose that $3 \mid (k^3+2k)$. We must show that $3 \mid \big((k+1)^3+2(k+1)\big)$. If we expand the expression in question, we obtain $k^3+3k^2+3k+1+2k+2 = (k^3+2k)+3(k^2+k+1)$. By the inductive hypothesis, 3 divides k^3+2k, and certainly 3 divides $3(k^2+k+1)$, so 3 divides their sum, and we are done.

34. The statement is true for the base case, $n=0$, since $6 \mid 0$. Suppose that $6 \mid (n^3-n)$. We must show that $6 \mid \big((n+1)^3-(n+1)\big)$. If we expand the expression in question, we obtain $n^3+3n^2+3n+1-n-1 = (n^3-n)+3n(n+1)$. By the inductive hypothesis, 6 divides the first term, n^3-n. Furthermore clearly 3 divides the second term, and the second term is also even, since one of n and $n+1$ is even; therefore 6 divides the second term as well. This tells us that 6 divides the given expression, as desired. (Note that here we have, as promised, used n as the dummy variable in the inductive step, rather than k.)

36. It is not easy to stumble upon the trick needed in the inductive step in this exercise, so do not feel bad if you did not find it. The form is straightforward. For the basis step ($n=1$), we simply observe that $4^{1+1}+5^{2\cdot 1-1} = 16+5 = 21$, which is divisible by 21. Then we assume the inductive hypothesis, that $4^{n+1}+5^{2n-1}$ is divisible by 21, and let us look at the expression when $n+1$ is plugged in for n. We want somehow to manipulate it so that the expression for n appears. We have

$$\begin{aligned}4^{(n+1)+1}+5^{2(n+1)-1} &= 4\cdot 4^{n+1}+25\cdot 5^{2n-1}\\ &= 4\cdot 4^{n+1}+(4+21)\cdot 5^{2n-1}\\ &= 4(4^{n+1}+5^{2n-1})+21\cdot 5^{2n-1}.\end{aligned}$$

Looking at the last line, we see that the expression in parentheses is divisible by 21 by the inductive hypothesis, and obviously the second term is divisible by 21, so the entire quantity is divisible by 21, as desired.

38. The basis step is trivial, as usual: $A_1 \subseteq B_1$ implies that $\bigcup_{j=1}^{1} A_j \subseteq \bigcup_{j=1}^{1} B_j$ because the union of one set is itself. Assume the inductive hypothesis that if $A_j \subseteq B_j$ for $j=1,2,\ldots,k$, then $\bigcup_{j=1}^{k} A_j \subseteq \bigcup_{j=1}^{k} B_j$. We want to show that if $A_j \subseteq B_j$ for $j=1,2,\ldots,k+1$, then $\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$. To show that one set is a subset of another we show that an arbitrary element of the first set must be an element of the second set. So let $x \in \bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1}$. Either $x \in \bigcup_{j=1}^{k} A_j$ or $x \in A_{k+1}$. In the first case we know by the inductive hypothesis that $x \in \bigcup_{j=1}^{k} B_j$; in the second case, we know from the given fact that $A_{k+1} \subseteq B_{k+1}$ that $x \in B_{k+1}$. Therefore in either case $x \in \left(\bigcup_{j=1}^{k} B_j\right) \cup B_{k+1} = \bigcup_{j=1}^{k+1} B_j$.

This is really easier to do directly than by using the principle of mathematical induction. For a noninductive proof, suppose that $x \in \bigcup_{j=1}^{n} A_j$. Then $x \in A_j$ for some j between 1 and n, inclusive. Since $A_j \subseteq B_j$, we know that $x \in B_j$. Therefore by definition, $x \in \bigcup_{j=1}^{n} B_j$.

40. If $n=1$ there is nothing to prove, and the $n=2$ case is the distributive law (see Table 1 in Section 2.2). Those take care of the basis step. For the inductive step, assume that

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B);$$

we must show that

$$(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B) \cap (A_{n+1} \cup B).$$

We have

$$\begin{aligned}(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \cup B &= ((A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}) \cup B \\ &= ((A_1 \cap A_2 \cap \cdots \cap A_n) \cup B) \cap (A_{n+1} \cup B) \\ &= (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B) \cap (A_{n+1} \cup B).\end{aligned}$$

The second line follows from the distributive law, and the third line follows from the inductive hypothesis.

42. If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \overline{B}) \cap (A_2 \cap \overline{B}) = (A_1 \cap A_2) \cap \overline{B}$, which is certainly true, since an element is in each side if and only if it is in all three of the sets A_1, A_2, and $\overline{B}$. Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned}&(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\ &\qquad = ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\ &\qquad = ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1}) - B) \\ &\qquad = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

44. If $n = 1$ there is nothing to prove, and the $n = 2$ case says that $(A_1 \cap \overline{B}) \cup (A_2 \cap \overline{B}) = (A_1 \cup A_2) \cap \overline{B}$, which is the distributive law (see Table 1 in Section 2.2). Those take care of the basis step. For the inductive step, assume that

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) = (A_1 \cup A_2 \cup \cdots \cup A_n) - B;$$

we must show that

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B) = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B.$$

We have

$$\begin{aligned}&(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \cup (A_{n+1} - B) \\ &\qquad = ((A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B)) \cup (A_{n+1} - B) \\ &\qquad = ((A_1 \cup A_2 \cup \cdots \cup A_n) - B) \cup (A_{n+1}) - B) \\ &\qquad = (A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) - B.\end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

46. This proof will be similar to the proof in Example 9. The basis step is clear, since for $n = 3$, the set has exactly one subset containing exactly three elements, and $3(3-1)(3-2)/6 = 1$. Assume the inductive hypothesis, that a set with n elements has $n(n-1)(n-2)/6$ subsets with exactly three elements; we want to prove that a set S with $n+1$ elements has $(n+1)n(n-1)/6$ subsets with exactly three elements. Fix an element a in S, and let T be the set of elements of S other than a. There are two varieties of subsets of S containing exactly three elements. First there are those that do not contain a. These are precisely the three-element subsets of T, and by the inductive hypothesis, there are $n(n-1)(n-2)/6$ of them. Second, there are those that contain a together with two elements of T. Therefore there are just as many of these subsets as there are two-element subsets of T. By Exercise 45, there are exactly $n(n-1)/2$ such subsets of T; therefore there are also $n(n-1)/2$ three-element subsets of S containing a. Thus the total number of subsets of S containing exactly three elements is $(n(n-1)(n-2)/6) + n(n-1)/2$, which simplifies algebraically to $(n+1)n(n-1)/6$, as desired.

48. When $n = 1$ the left-hand side is 1, and the right-hand side is $(1 + \frac{1}{2})^2/2 = 9/8$. Thus the basis step was wrong.

50. The base case is $n = 1$. If we are given a set of two elements from $\{1, 2\}$, then indeed one of them divides the other. Assume the inductive hypothesis, and consider a set A of $n + 2$ elements from $\{1, 2, \ldots, 2n, 2n + 1, 2n+2\}$. We must show that at least one of these elements divides another. If as many as $n+1$ of the elements of A are less than $2n + 1$, then the desired conclusion follows immediately from the inductive hypothesis. Therefore we can assume that both $2n + 1$ and $2n + 2$ are in A, together with n smaller elements. If $n + 1$ is one of these smaller elements, then we are done, since $n + 1 \mid 2n + 2$. So we can assume that $n + 1 \notin A$. Now apply the inductive hypothesis to $B = A - \{2n + 1, 2n + 2\} \cup \{n + 1\}$. Since B is a collection of $n + 1$ numbers from $\{1, 2, \ldots, 2n\}$, the inductive hypothesis guarantees that one element of B divides another. If $n + 1$ is not one of these two numbers, then we are done. So we can assume that $n + 1$ is one of these two numbers. Certainly $n + 1$ can't be the divisor, since its smallest multiple is too big to be in B, so there is some $k \in B$ that divides $n+1$. But now k and $2n+2$ are numbers in A, with k dividing $n+2$, and we are done. An alternative proof of this theorem is given in Example 11 of Section 5.2.

52. There is nothing to prove in the base case, $n = 1$, since $\mathbf{A} = \mathbf{A}$. For the inductive step we just invoke the inductive hypothesis and the definition of matrix multiplication:

$$\begin{aligned}\mathbf{A}^{n+1} = \mathbf{A}\mathbf{A}^n &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \\ &= \begin{bmatrix} a \cdot a^n + 0 \cdot 0 & a \cdot 0 + 0 \cdot b^n \\ 0 \cdot a^n + b \cdot 0 & 0 \cdot 0 + b \cdot b^n \end{bmatrix} = \begin{bmatrix} a^{n+1} & 0 \\ 0 & b^{n+1} \end{bmatrix}\end{aligned}$$

54. The basis step is trivial, since we are already given that $\mathbf{AB} = \mathbf{BA}$. Next we assume the inductive hypothesis, that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$, and try to prove that $\mathbf{AB}^{n+1} = \mathbf{B}^{n+1}\mathbf{A}$. We calculate as follows: $\mathbf{AB}^{n+1} = \mathbf{AB}^n\mathbf{B} = \mathbf{B}^n\mathbf{AB} = \mathbf{B}^n\mathbf{BA} = \mathbf{B}^{n+1}\mathbf{A}$. Note that we used the definition of matrix powers (that $\mathbf{B}^{n+1} = \mathbf{B}^n\mathbf{B}$), the inductive hypothesis, and the basis step.

56. This is identical to Exercise 43, with $\vee$ replacing $\cup$, $\wedge$ replacing $\cap$, and $\neg$ replacing complementation. The basis step is trivial, since it merely says that $\neg p_1$ is equivalent to itself. Assuming the inductive hypothesis, we look at $\neg(p_1 \vee p_2 \vee \cdots \vee p_n \vee p_{n+1})$. By De Morgan's law (grouping all but the last term together) this is the same $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \wedge \neg p_{n+1}$. But by the inductive hypothesis, this equals, $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n \wedge \neg p_{n+1}$, as desired.

58. The statement is true for $n = 1$, since 1 line separates the plane into 2 regions, and $(1^2 + 1 + 2)/2 = 2$. Assume the inductive hypothesis, that n lines of the given type separate the plane into $(n^2 + n + 2)/2$ regions. Consider an arrangement of $n + 1$ lines. Remove the last line. Then there are $(n^2 + n + 2)/2$ regions by the inductive hypothesis. Now we put the last line back in, drawing it slowly, and see what happens to the regions. As we come in "from infinity," the line separates one infinite region into two (one on each side of it); this separation is complete as soon as the line hits one of the first n lines. Then, as we continue drawing from this first point of intersection to the second, the line again separates one region into two. We continue in this way. Every time we come to another point of intersection between the line we are drawing and the figure already present, we lop off another additional region. Furthermore, once we leave the last point of intersection and draw our line off to infinity again, we separate another region into two. Therefore the number of additional regions we formed is equal to the number of points of intersection plus one. Now there are n points of intersection, since our line must intersect each of the other lines in a distinct point (this is where the geometric assumptions get used). Therefore this arrangement has $n + 1$ more points of intersection than the arrangement of n lines, namely $\left((n^2 + n + 2)/2\right) + (n + 1)$, which, after a bit of algebra, reduces to $\left((n + 1)^2 + (n + 1) + 2\right)/2$, exactly as desired.

60. For the base case $n = 1$ there is nothing to prove. Assume the inductive hypothesis, and suppose that we are given $p \mid a_1a_2\cdots a_na_{n+1}$. We must show that $p \mid a_i$ for some i. Let us look at $\gcd(p, a_1a_2\cdots a_n)$. Since the only divisors of p are 1 and p, this is either 1 or p. If it is 1, then by Lemma 1 in Section 3.7, we have $p \mid a_{n+1}$ (here $a = p$, $b = a_1a_2\cdots a_n$, and $c = a_{n+1}$), as desired. On the other hand, if the greatest common divisor is p, this means that $p \mid a_1a_2\cdots a_n$. Now by the inductive hypothesis, $p \mid a_i$ for some $i \le n$, again as desired.

62. Suppose that a statement $\forall n P(n)$ has been proved by this method. Let S be the set of counterexamples to P, i.e., let $S = \{\, n \mid \neg P(n) \,\}$. We will show that $S = \emptyset$. If $S \ne \emptyset$, then let n be the minimum element of S (which exists by the well-ordering property). Clearly $n \ne 1$ and $n \ne 2$, by the basis steps of our proof method. But since n is the least element of S and $n \ge 3$, we know that $P(n-1)$ and $P(n-2)$ are true. Therefore by the inductive step of our proof method, we know that $P(n)$ is also true. This contradicts the choice of n. Therefore $S = \emptyset$, as desired.

64. The basis step is $n = 1$ and $n = 2$. If there is one guest present, then he or she is vacuously a celebrity, and no questions are needed; this is consistent with the value of $3(n-1)$. If there are two guests, then it is certainly true that we can determine who the celebrity is (or determine that neither of them is) with three questions. In fact, two questions suffice (ask each one if he or she knows the other). Assume the inductive hypothesis that if there are k guests present ($k \ge 2$), then we can determine whether there is a celebrity with at most $3(k-1)$ questions. We want to prove the statement for $k+1$, namely, if there are $k+1$ at the party, then we can find the celebrity (or determine that there is none) using $3k$ questions. Let Alex and Britney be two of the guests. Ask Alex whether he knows Britney. If he says yes, then we know that he is not a celebrity. If he says no, then we know that Britney is not a celebrity. Without loss of generality, assume that we have eliminated Alex as a possible celebrity. Now invoke the inductive hypothesis on the k guests excluding Alex, asking $3(k-1)$ questions. If there is no celebrity, then we know that there is no celebrity at our party. If there is, suppose that it is person x (who might be Britney or might be someone else). We then ask two more questions to determine whether x is in fact a celebrity; namely ask Alex whether he knows x, and ask x whether s/he knows Alex. Based on the answers, we will now know whether x is a celebrity for the whole party or there is no celebrity present. We have asked a total of at most $1 + 3(k-1) + 2 = 3k$ questions. Note that in fact we did a little better than $3(n-1)$; because only two questions were needed for $n = 2$, only $3(n-1) - 1 = 3n - 4$ questions are needed in the general case for $n \ge 2$.

66. We prove this by mathematical induction. The basis step, $g(4) = 2 \cdot 4 - 4 = 4$ was proved in Exercise 65. For the inductive step, suppose that when there are k callers, $2k - 4$ calls suffice; we must show that when there are $k+1$ callers, $2(k+1) - 4$ calls suffice, that is, two more calls. It is clear from the hint how to proceed. For the first extra call, have the $(k+1)^{\text{st}}$ person exchange information with the k^{th} person. Then use $2k - 4$ calls for the first k people to exchange information. At that point, each of them knows all the gossip. Finally, have the $(k+1)^{\text{st}}$ person again call the k^{th} person, at which point he will learn the rest of the gossip.

68. We follow the hint. If the statement is true for some value of n, then it is also true for all smaller values of n, because we can use the same arrangement among those smaller numbers. Thus is suffices to prove the statement when n is a power of 2. We use mathematical induction to prove the result for 2^k. If $k = 0$ or $k = 1$, there is nothing to prove. Notice that the arrangement 1324 works for $k = 2$. Assume that we can arrange the positive integers from 1 to 2^k so that the average of any two of these numbers never appears between them. Arrange the numbers from 1 to 2^{k+1} by taking the given arrangement of 2^k numbers, replacing each number by its double, and then following this sequence with the sequence of 2^k numbers obtained from these 2^k even numbers by subtracting 1. Thus for $k = 3$ we use the sequence 1324 to form the sequence 26481537. This clearly is a list of the numbers from 1 to 2^{k+1}. The average of an odd number and an even number is not an integer, so it suffices to shows that the average of two even numbers and the average of two odd numbers in our list never appears between the numbers being averaged. If the average of two even numbers, say $2a$ and $2b$, whose average is $a+b$, appears between the numbers being averaged, then by the way we constructed the sequence, there would have been a similar violation in the 2^k list, namely, $(a+b)/2$ would have appeared between a and b. Similarly, if the average of two odd numbers, say $2c-1$ and $2d-1$, whose average is $c+d-1$, appears between the numbers being averaged, then there would have been a similar violation in the 2^k list, namely, $(c+d)/2$ would have appeared between c and d.

70. a) The basis step works, because for $n=1$ the statement $1/2 < 1/\sqrt{3}$ is true. The inductive step would require proving that

$$\frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)}}.$$

Squaring both sides and clearing fractions, we see that this is equivalent to $4n^2+4n+1 < 4n^2+4n$, which of course is not true.

b) The basis step works, because the statement $3/8 < 1/\sqrt{7}$ is true. The inductive step this time requires proving that

$$\frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3(n+1)+1}}.$$

A little algebraic manipulation shows that this is equivalent to

$$12n^3+28n^2+19n+4 < 12n^3+28n^2+20n+4,$$

which is true.

72. The upper left 4×4 quarter of the figure given in the solution to Exercise 73 gives such a tiling.

74. a) Every $3 \times 2k$ board can be covered in an obvious way: put two pieces together to form a 3×2 rectangle, then lay the rectangles edge to edge. In particular, for all $n \geq 1$ the 3×2^n rectangle can be covered.

b) This is similar to part (a). For all $k \geq 1$ it is easy to cover the $6 \times 2k$ board, using two coverings of the $3 \times 2k$ board from part (a), laid side by side.

c) A little trial and error shows that the $3^1 \times 3^1$ board cannot be covered. Therefore not all such boards can be covered.

d) All boards of this shape can be covered for $n \geq 1$, using reasoning similar to parts (a) and (b).

76. This is too complicated to discuss here. For a solution, see the article by I. P. Chu and R. Johnsonbaugh, "Tiling Deficient Boards with Trominoes," *Mathematics Magazine* **59** (1986) 34–40. (Notice the variation in the spelling of this made-up word.)

78. In order to explain this argument, we label the squares in the 5×5 checkerboard 11, 12, ..., 15, 21, ..., 25, ..., 51, ..., 55, where the first digit stands for the row number and the second digit stands for the column number. Also, in order to talk about the right triomino (L-shaped tile), think of it positioned to look like the letter L; then we call the square on top the head, the square in the lower right the tail, and the square in the corner the corner. We claim that the board with square 12 removed cannot be tiled. First note that in order to cover square 11, the position of one piece is fixed. Next we consider how to cover square 13. There are three possibilities. If we put a head there, then we are forced to put the corner of another piece in square 15. If we put a corner there, then we are forced to put the tail of another piece in 15, and if we put a tail there, then square 15 cannot be covered at all. So we conclude that squares 13, 14, 15, 23, 24, and 25 will have to be covered by two more pieces. By symmetry, the same argument shows that two more pieces must cover squares 31, 41, 51, 32, 42, and 52. This much has been forced, and now we are left with the 3×3 square in the lower left part of the checkerboard to cover with three more pieces. If we put a corner in 33, then we immediately run into an impasse in trying to cover 53 and 35. If we put a head in 33, then 53 cannot be covered; and if we put a tail in 33, then 35 cannot be covered. So we have reached a contradiction, and the desired covering does not exist.