SUPPLEMENTARY EXERCISES FOR CHAPTER 10

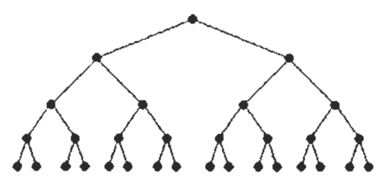
2. There are 20 such trees. We can organize our count by the height of the tree. There is just 1 rooted tree on 6 vertices with height 5. If the height is 4 (so that there is a path from the root containing 5 vertices), then there are 4 choices as to where to attach the sixth vertex. If the height is 3, fix a path of length three from the root. Two more vertices need to be added. If they are both attached directly to the original path, then there are $C(3+2-1,2)=6$ ways to attach them (since there are three possible points of attachment). On the other hand if they form a path of length 2 from their point of attachment, then there are 2 choices. Next suppose the height is 2. If there are not two disjoint paths of length 2 from the root, then there are 4 ways that the other 3 vertices can be attached to a given path of length 2 from the root (0, 1, 2, or 3 of them can be attached to the root). If there are two disjoint paths, then there are 2 choices for the sixth vertex. Finally, there is 1 tree of height 1. Thus we have $1+4+6+2+4+2+1=20$ trees in all.

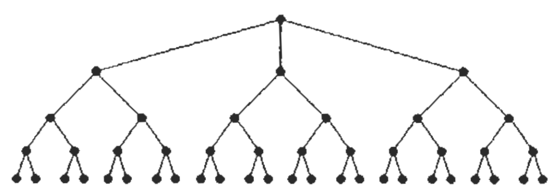
4. We know that the sum of the degrees must be $2(n-1)$. The $n-1$ pendant vertices account for $n-1$ in this sum, so the degree of the other vertex must be $n-1$. This vertex is one part of $K_{1,n-1}$, therefore, and the pendant vertices are the other part.

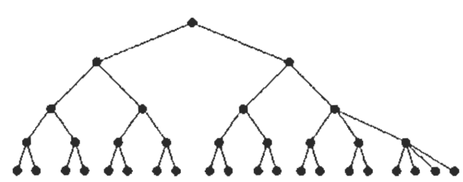
6. We prove this by induction on n. The problem is trivial if $n \leq 2$, so assume that the inductive hypothesis holds and let $n \geq 3$. First note that at least one of the positive integers d_i must equal 1, since the sum of n numbers each greater than or equal to 2 is greater than or equal to $2n$. Without loss of generality assume that $d_n = 1$. Now it is impossible for all the remaining d_i's to equal 1, since $2n - 2 > n$ (we are assuming that $n > 2$); without loss of generality assume that $d_1 > 1$. Now apply the inductive hypothesis to the sequence $d_1 - 1, d_2, d_3, \ldots, d_{n-1}$. There is a tree with these degrees. Add an edge from the vertex with degree $d_1 - 1$ to a new vertex, and we have the desired tree with degrees $d_1, d_2, \ldots, d_n$.

8. We consider the tree as a rooted tree. One part is the set of vertices at even-numbered levels, and the other part is the set of vertices at odd-numbered levels.

10. The following pictures show some B-trees with the desired height and degree. The root must have either 2 or 3 children, and the other internal vertices must have between 2 and 4 children, inclusive. Note that our first example is a complete binary tree.







12. The lower bound for the height of a B-tree of degree k with n leaves comes from the upper bound for the number of leaves in a B-tree of degree k with height h, obtained in Exercise 11. Since there we found that $n \leq k^h$, we have $h \geq \log_k n$. The upper bound for the height of a B-tree of degree k with n leaves comes from the lower bound for the number of leaves in a B-tree of degree k with height h, obtained in Exercise 11. Since there we found that $n \geq 2\lceil k/2 \rceil^{h-1}$, we have $h \leq 1 + \log_{\lceil k/2 \rceil}(n/2)$.

14. Since B_{k+1} is formed from two copies of B_k, the number of vertices doubles as k increases by 1. Since B_0 had $1 = 2^0$ vertices, it follows by induction that B_k has 2^k vertices.

16. Looking at the pictures for B_k leads one to conjecture that the number of vertices at depth j is $C(k,j)$. For example, in B_4 the number of vertices at the various levels form the sequence 1, 4, 6, 4, 1, which are exactly $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, $C(4,4)$. To prove this by mathematical induction (the basis step being trivial), note that by the way B_{k+1} is constructed, the number of vertices at level $j+1$ in B_{k+1} is the sum of the number of vertices at level $j+1$ in B_k and the number of vertices at level j in B_k. By the inductive hypothesis this is $C(k,j+1) + C(k,j)$, which equals $C(k+1,j+1)$ as desired, by Pascal's identity. This holds for $j = k$ as well, and at the 0^{th} level, too, there is clearly just one vertex.

18. Our inductive hypothesis is that the root and the left-most child of the root of B_k have degree k and every other vertex has degree less than k. This is certainly true for B_0 and B_1. Consider B_{k+1}. By Exercise 17, its root has degree $k+1$, as desired. The left-most child of the root is the root of a B_k, which had degree k, and we have added one edge to connect it to the root of B_{k+1}, so its degree is now $k+1$, as desired. Every other vertex of B_{k+1} has the same degree it had in B_k, which was at most k by the inductive hypothesis, and our proof is complete.

20. That an S_k-tree has 2^k vertices is clear by induction, since an S_k-tree has twice as many vertices as an S_{k-1}-tree and an S_0-tree has $2^0 = 1$ vertex. Also by induction we see that there is a unique vertex at level k, since there was a unique vertex at level $k-1$ in the S_{k-1}-tree whose root was made a child of the root of the other S_{k-1}-tree in the construction of the S_k-tree.

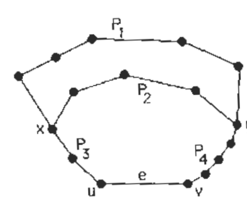
22. The level order in each case is the alphabetical order in which the vertices are labeled.

24. Given the set of universal addresses, we need to check two things. First we need to be sure that no address in our list is the address of an internal vertex. This we can accomplish by checking that no address in our list is a prefix of another address in our list. (Also of course, if the list contains 0, then it must contain no other addresses.) Second we need to make sure that all the internal vertices have a leaf as a descendant. To check this, for each address $a_1.a_2.\cdots.a_r$ in the list, and for each i from 1 to r, inclusive, and for each b with $1 \leq b < a_i$, we check that there is an address in the list with prefix $a_1.a_2.\cdots.a_{i-1}.b$.

26. We assume that the graph in question is connected. (If it is not, then the statement is vacuously true.) If we remove all the edges of a cut set, the resulting graph cannot still be connected. If the resulting graph contained all the edges of a spanning tree, then it would be connected. Therefore there must be at least one edge of the spanning tree in the cut set.

28. A tree is necessarily a cactus, since no edge is in any simple circuit at all.

30. Suppose G is not a cactus; we will show that G contains a very simple circuit with an even number of edges (see the solution to Exercise 27 for the definition of "very simple circuit"). Suppose instead, then, that every very simple circuit of G contains an odd number of edges. Since G is not a cactus, we can find an edge $e = \{u, v\}$ that is in two different very simple circuits. By simplifying the second circuit if necessary, we can assume that the situation is as pictured here, where x might be u and y might be v. Since the circuits $u, P_3, x, P_1, y, P_4, v, e, u$ and $u, P_3, x, P_2, y, P_4, v, e, u$ are both odd, the paths P_1 and P_2 have to have the same parity. Therefore the very simple circuit consisting of P_1 followed by P_2 backwards has even length, as desired.

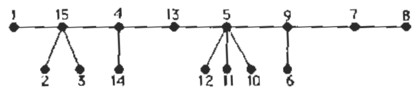


32. The only spanning tree here is the graph itself, and vertex i has degree greater than 3. Thus there is no degree-constrained spanning tree where each vertex has degree less than or equal to 3.

34. Such a tree must be a path (since it is connected and has no vertices of degree greater than 2), and since it includes every vertex in the graph, it is a Hamilton path.

36. The graphs in the first three parts are caterpillars, since every vertex is either in the horizontal path of length 3 or adjacent to a vertex in this path. In part **(d)** it is clear that there is no path that can serve as the "spine" of the caterpillar.

38. **a)** We can gracefully label the vertices in the path in the following manner. Suppose there are n vertices. We label every other vertex, starting with the first, with the numbers $1, 2, \ldots, \lceil n/2 \rceil$; we number the remaining vertices, in the same order, with $n, n-1, \ldots, \lceil n/2 \rceil + 1$. For example, if $n = 7$, then the vertices are labeled $1, 7, 2, 6, 3, 5, 4$. The successive differences are then easily seen to be $n-1, n-2, \ldots, 2, 1$, as desired.
b) We extend the idea in the solution to part **(a)**, allowing for labeling the "feet" as well as the "spine" of the caterpillar. We can assume that the first and last vertices in the spine have no feet. First we label the vertex at the beginning of the spine 1, and, as above, label the vertex adjacent to it n. If there are some feet at this vertex, then we label them $2, 3, \ldots, k$ (where the number of feet there is $k-1$). Then we label the next vertex on the spine with the smallest available number—either 2 or $k+1$ (if there were feet that needed labeling). If this vertex has feet, then we label them $n-1, n-2$, and so on. The largest available number is then used for the label of the next vertex on the spine. We continue in this manner until we have labeled the entire caterpillar. It is clear that the labeling is graceful. See the example below.



40. By Exercise 48 in Section 10.4, we can number the vertices while doing depth-first search in order of their finishing. It follows from the solution given there that this order corresponds to postorder in the spanning tree. We claim that the opposite order of these numbers gives a topological sort of the vertices in the graph. We must show that there is no directed edge uv such that u's number in this process is less than v's number (prior to reversing the order). Clearly this is true if uv is a tree edge, since the numbers of all of a vertex's descendants are less than the number of that vertex. By Exercise 56 in Section 10.4, there are no back edges in our acyclic digraph. By Exercise 47 in Section 10.4, if uv is a forward edge, then it connects a vertex to a descendant, so the number of u exceeds the number of v, and that is consistent with our given partial order. And if uv is a cross edge, then v is in a previously visited subtree, so the number on v is less than the number on u, again consistent with the given partial order.

42. We form a graph whose vertices are the allowable positions of the people and boat. Each vertex, then, contains the information as to which of the six people and the boat are on, say, the near bank (the remaining people and/or boat are on the far bank). If we label the people X, Y, Z, x, y, z (the husbands in upper case letters and the wives in the corresponding lower case letters) and the boat B, then the initial position is $XYZxyzB$ and the desired final position is the empty set. Two vertices are joined by an edge if it is possible to obtain one position from the other with one legal boat ride (where "legal" means of course that the rules of the puzzle are not violated—that no man is left alone with a woman other than his wife, and that the boat crosses the river only with one or two people in it). For example, the vertex $YZyz$ is adjacent to the vertex $XYZxyzB$, since the married couple Xx can travel to the opposite bank in the boat. Our task is to find a path in this graph from the initial position to the desired final position. Dijkstra's algorithm could be used to find such a path. The graph is too large to draw here, but with this notation (and arrows for readability), one path is $XYZxyzB \to YZyz \to YZxyzB \to YZy \to YZyzB \to Zz \to ZyzB \to Z \to ZzB \to \emptyset$.

44. We assume that what is being asked for here is not "a minimum spanning tree of the graph that also happens to satisfy the degree constraint" but rather "a tree of minimum weight among all spanning trees that satisfy the degree constraint."
a) Since b is a cut vertex we must include at least one of the two edges $\{b, c\}$ and $\{b, d\}$, and one of the other three edges incident to b. Thus the best we can do is to include edges $\{b, c\}$ and $\{a, b\}$. It is then easy to see

that the unique minimum spanning tree with degrees constrained to be at most 2 consists of these two edges, together with $\{c,d\}$, $\{a,f\}$, and $\{e,f\}$.

b) Obviously we must include edge $\{a,b\}$. We cannot include edge $\{b,g\}$, because this would force some vertex to have degree greater than 2 in the spanning tree. For a similar reason we cannot include edge $\{b,d\}$. A little more thought shows that the minimum spanning tree under these constraints consists of edge $\{a,b\}$, together with edges $\{b,c\}$, $\{c,d\}$, $\{d,g\}$, $\{f,g\}$, and $\{e,f\}$.