SECTION 5.5 Generalized Permutations and Combinations

As in Section 5.3, we have formulae that give us the answers to some combinatorial problems, if we can figure out which formula applies to which problem, and in what way it applies. Here, even more than in previous sections, the ability to see a problem from the right perspective is the key to solving it. Expect to spend several minutes staring at each problem before any insight comes. Reread the examples in the section and try to imagine yourself going through the thought processes explained there. Gradually your mind will begin to think in the same terms. In particular, ask yourself what is being selected from what, whether ordered or unordered selections are to be made, and whether repetition is allowed. In most cases, after you have answered these questions, you can find the appropriate formula from Table 1.

1. Since order is important here, and since repetition is allowed, this is a simple application of the product rule. There are 3 ways in which the first element can be selected, 3 ways in which the second element can be selected, and so on, with finally 3 ways in which the fifth element can be selected, so there are $3^5 = 243$ ways in which the 5 elements can be selected. The general formula is that there are n^k ways to select k elements from a set of n elements, in order, with unlimited repetition allowed.

3. Since we are considering strings, clearly order matters. The choice for each position in the string is from the set of 26 letters. Therefore, using the same reasoning as in Exercise 1, we see that there are $26^6 = 308{,}915{,}776$ strings.

5. We assume that the jobs and the employees are distinguishable. For each job, we have to decide which employee gets that job. Thus there are 5 ways in which the first job can be assigned, 5 ways in which the second job can be assigned, and 5 ways in which the third job can be assigned. Therefore, by the multiplication principle (just as in Exercise 1) there are $5^3 = 125$ ways in which the assignments can be made. (Note that we do not require that every employee get at least one job.)

7. Since the selection is to be an unordered one, Theorem 2 applies. We want to choose $r = 3$ items from a set with $n = 5$ elements. Theorem 2 tells us that there are $C(5+3-1,3) = C(7,3) = 7 \cdot 6 \cdot 5/(3 \cdot 2) = 35$ ways to do so. (Equivalently, this problem is asking us to count the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 3$, where x_i represents the number of times that the i^{th} element of the 5-element set gets selected.)

9. Let $b_1, b_2, \ldots, b_8$ be the number of bagels of the 8 types listed (in the order listed) that are selected. Order does not matter: we are presumably putting the bagels into a bag to take home, and the order in which we put them there is irrelevant.

a) If we want to choose 6 bagels, then we are asking for the number of nonnegative solutions to the equation $b_1 + b_2 + \cdots + b_8 = 6$. Theorem 2 applies, with $n = 8$ and $r = 6$, giving us the answer $C(8+6-1,6) = C(13,6) = 1716$.

b) This is the same as part (a), except that $r = 12$ rather than 6. Thus there are $C(8+12-1,12) = C(19,12) = C(19,7) = 50{,}388$ ways to make the selection. (Note that $C(19,7)$ was easier to compute than $C(19,12)$, and since they are equal, we chose the latter form.)

c) This is the same as part (a), except that $r = 24$ rather than 6. Thus there are $C(8+24-1,24) = C(31,24) = C(31,7) = 2{,}629{,}575$ ways to make the selection.

d) This one is more complicated. Here we want to solve the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the constraint that each $b_i \geq 1$. We reduce this problem to the form in which Theorem 2 is applicable with the following trick. Let $b_i' = b_i - 1$; then b_i' represents the number of bagels of type i, in excess of the required 1, that are selected. If we substitute $b_i = b_i' + 1$ into the original equation, we obtain $(b_1'+1) + (b_2'+1) + \cdots + (b_8'+1) = 12$, which reduces to $b_1' + b_2' + \cdots + b_8' = 4$. In other words, we are asking how many ways are there to choose the 4 extra bagels (in excess of the required 1 of each type) from among the 8 types, repetitions allowed. By Theorem 2 the number of solutions is $C(8+4-1,4) = C(11,4) = 330$.

e) This final part is even trickier. First let us ignore the restriction that there can be no more than 2 salty bagels (i.e., that $b_4 \leq 2$). We will take into account, however, the restriction that there must be at least 3 egg bagels (i.e., that $b_3 \geq 3$). Thus we want to count the number of solutions to the equation $b_1+b_2+\cdots+b_8 = 12$, subject to the condition that $b_i \geq 0$ for all i and $b_3 \geq 3$. As in part **(d)**, we use the trick of choosing the 3 egg bagels at the outset, leaving only 9 bagels free to be chosen; equivalently, we set $b_3' = b_3 - 3$, to represent the extra egg bagels, above the reqnired 3, that are chosen. Now Theorem 2 applies to the number of solutions of $b_1 + b_2 + b_3' + b_4 + \cdots + b_8 = 9$, so there are $C(8+9-1,9) = C(16,9) = C(16,7) = 11{,}440$ ways to make this selection.

Next we need to worry about the restriction that $b_4 \leq 2$. We will impose this restriction by subtracting from our answer so far the number of ways to violate this restriction (while still obeying the restriction that $b_3 \geq 3$). The difference will be the desired answer. To violate the restriction means to have $b_4 \geq 3$. Thus we want to count the number of solutions to $b_1 + b_2 + \cdots + b_8 = 12$, with $b_3 \geq 3$ and $b_4 \geq 3$. Using the same technique as we have just used, this is equal to the number of nonnegative solutions to the equation $b_1 + b_2 + b_3' + b_4' + b_5 + \cdots + b_8 = 6$ (the 6 on the right being $12 - 3 - 3$). By Theorem 2 there are $C(8+6-1,6) = C(13,6) = 1716$ ways to make this selection. Therefore our final answer is $11440 - 1716 = 9724$.

11. This can be solved by common sense. Since the pennies are all identical and the nickels are all identical, all that matters is the nnmber of each type of coin selected. We can select anywhere from 0 to 8 pennies (and the rest nickels); since there are nine numbers in this range, the answer is 9. (The number of pennies and nickels is irrelevant, as long as each is at least eight.) If we wanted to use a high-powered theorem for this problem, we could observe that Theorem 2 applies, with $n = 2$ (there are two types of coins) and $r = 8$. The formula gives $C(2+8-1,8) = C(9,8) = 9$.

13. Assuming that the warehouses are distinguishable, let w_i be the number of books stored in warehouse i. Then we are asked for the number of solutions to the equation $w_1 + w_2 + w_3 = 3000$. By Theorem 2 there are $C(3+3000-1,3000) = C(3002,3000) = C(3002,2) = 4{,}504{,}501$ of them.

15. **a)** Let $x_1 = x_1' + 1$; thus x_1' is the value that x_1 has in excess of its required 1. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2 + x_3 + x_4 + x_5 = 20$. By Theorem 2 there are $C(5+20-1,20) = C(24,20) = C(24,4) = 10{,}626$ of them.

b) Substitute $x_i = x_i' + 2$ into the equation for each i; thus x_i' is the valne that x_i has in excess of its required 2. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2' + x_3' + x_4' + x_5' = 11$. By Theorem 2 there are $C(5+11-1,11) = C(15,11) = C(15,4) = 1365$ of them.

c) There are $C(5+21-1,21) = C(25,21) = C(25,4) = 12650$ solutions with no restriction on x_1. The restriction on x_1 will be violated if $x_1 \geq 11$. Following the procedure in part **(a)**, we find that there are $C(5+10-1,10) = C(14,10) = C(14,4) = 1001$ solntions in which the restriction is violated. Therefore there are $12650 - 1001 = 11{,}649$ solutions of the equation with its restriction.

d) First let us impose the restrictions that $x_3 \geq 15$ and $x_2 \geq 1$. Then the problem is equivalent to counting the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subject to the constraints that $x_1 \leq 3$ and $x_2' \leq 2$ (the latter coming from the original restriction that $x_2 < 4$). Note that these two restrictions cannot be violated simultaneously. Thus if we count the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subtract the number of its solntions in which $x_1 \geq 4$, and subtract the numbers of its solutions in which $x_2' \geq 3$, then we will have the answer. By Theorem 2 there are $C(5+5-1,5) = C(9,5) = 126$ solutions of the unrestricted eqnation. Applying the first restriction reduces the equation to $x_1' + x_2' + x_3' + x_4 + x_5 = 1$, which has $C(5+1-1,1) = C(5,1) = 5$ solutions. Applying the second restriction reduces the equation to $x_1 + x_2'' + x_3' + x_4 + x_5 = 2$, which has $C(5+2-1,2) = C(6,2) = 15$ solutions. Therefore the answer is $126 - 5 - 15 = 106$.

17. Theorem 3 applies here, with $n = 10$ and $k = 3$. The answer is therefore

$$\frac{10!}{2!3!5!} = 2520\,.$$

19. Theorem 3 applies here, with $n = 14$, $n_1 = n_2 = 3$ (the triplets), $n_3 = n_4 = n_5 = 2$ (the twins), and $n_6 = n_7 = 1$. The answer is therefore

$$\frac{14!}{3!3!2!2!2!1!1!} = 302{,}702{,}400\,.$$

21. If we think of the balls as doing the choosing, then this is asking for the number of ways to choose six bins from the nine given bins, with repetition allowed. (The number of times each bin is chosen is the number of balls in that bin.) By Theorem 2 with $n = 9$ and $r = 6$, this choice can be made in $C(9+6-1,6) = C(14,6) = 3003$ ways.

23. There are several ways to count this. We can first choose the two objects to go into box #1 ($C(12,2)$ ways), then choose the two objects to go into box #2 ($C(10,2)$ ways, since only 10 objects remain), then choose the two objects to go into box #3 ($C(8,2)$ ways), and so on. So the answer is $C(12,2)\cdot C(10,2)\cdot C(8,2)\cdot C(6,2)\cdot C(4,2)\cdot C(2,2) = (12\cdot 11/2)(10\cdot 9/2)(8\cdot 7/2)(6\cdot 5/2)(4\cdot 3/2)(2\cdot 1/2) = 12!/2^6 = 7{,}484{,}400$. Alternatively, just line up the 12 objects in a row (12! ways to do that), and put the first two into box #1, the next two into box #2, and so on. This overcounts by a factor of 2^6, since there are that many ways to swap objects in the permutation without affecting the result (swap the first and second objects or not, and swap the third and fourth objects or not, and so on). So this results in the same answer. Here is a third way to get this answer. First think of pairing the objects. Think of the objects as ordered (a first, a second, and so on). There are 11 ways to choose a mate for the first object, then 9 ways to choose a mate for the first unused object, then 7 ways to choose a mate for the first still unused object, and so on. This gives $11\cdot 9\cdot 7\cdot 5\cdot 3$ ways to do the pairing. Then there are 6! ways to choose the boxes for the pairs. So the answer is the product of these two quantities, which is again 7,484,400.

25. Let d_1, d_2, ..., d_6 be the digits of a natural number less than 1,000,000; they can each be anything from 0 to 9 (in particular, we may as well assume that there are leading 0's if necessary to make the number exactly 6 digits long). If we want the sum of the digits to equal 19, then we are asking for the number of solutions to the equation $d_1 + d_2 + \cdots + d_6 = 19$ with $0 \le d_i \le 9$ for each i. Ignoring the upper bound restriction, there are, by Theorem 2, $C(6+19-1,19) = C(24,19) = C(24,5) = 42504$ of them. We must subtract the number of solutions in which the restriction is violated. If the digits are to add up to 19 and one or more of them is to exceed 9, then exactly one of them will have to exceed 9, since $10 + 10 > 19$. There are 6 ways to choose the digit that will exceed 9. Once we have made that choice (without loss of generality assume it is d_1 that is to be made greater than or equal to 10), then we count the number of solutions to the equation by counting the number of solutions to $d_1' + d_2 + \cdots + d_6 = 19 - 10 = 9$; by Theorem 2 there are $C(6+9-1,9) = C(14,9) = C(14,5) = 2002$ of them. Thus there are $6\cdot 2002 = 12012$ solutions that violate the restriction. Subtracting this from the 42504 solutions altogether, we find that $42504 - 12012 = 30{,}492$ is the answer to the problem.

27. We assume that each problem is worth a whole number of points. Then we want to find the number of integer solutions to $x_1 + x_2 + \cdots + x_{10} = 100$, subject to the constraint that each $x_i \geq 5$. Letting x_i' be the number of points assigned to problem i in excess of its required 5, and substituting $x_i = x_i' + 5$ into the equation, we obtain the equivalent equation $x_1' + x_2' + \cdots + x_{10}' = 50$. By Theorem 2 the number of solutions is given by $C(10 + 50 - 1, 50) = C(59, 50) = C(59, 9) = 12{,}565{,}671{,}261$.

29. There are at least two good ways to do this problem. First we present a solution in the spirit of this section. Let us place the 1's and some gaps in a row. A 1 will come first, followed by a gap, followed by another 1, another gap, a third 1, a third gap, a fourth 1, and a fourth gap. Into the gaps we must place the 12 0's that are in this string. Let g_1, g_2, g_3, and g_4 be the numbers of 0's placed in gaps 1 through 4, respectively. The only restriction is that each $g_i \geq 2$. Thus we want to count the number of solutions to the equation $g_1+g_2+g_3+g_4 = 12$, with $g_i \geq 2$ for each i. Letting $g_i = g_i'+2$, we want to count, equivalently, the number of nonnegative solutions to $g_1'+g_2'+g_3'+g_4' = 4$. By Theorem 2 there are $C(4+4-1, 4) = C(7, 4) = C(7, 3) = 35$ solutions. Thus our answer is 35.

Here is another way to solve the problem. Since each 1 must be followed by two 0's, suppose we glue 00 to the right end of each 1. This uses up 8 of the 0's, leaving 4 unused 0's. Now we have 8 objects, namely 4 0's and 4 100's. We want to find the number of strings we can form with these 8 objects, starting with a 100. After placing the 100 first, there are 7 places left for objects, 3 of which have to be 100's. Clearly there are $C(7, 3) = 35$ ways to choose the positions for the 100's, so our answer is 35.

31. This is a direct application of Theorem 3, with $n = 11$, $n_1 = 5$, $n_2 = 2$, $n_3 = n_4 = 1$, and $n_5 = 2$ (where n_1 represents the number of A's, etc.). Thus the answer is $11!/(5!2!1!1!2!) = 83{,}160$.

33. We need to use the sum rule at the outermost level here, adding the number of strings using each subset of letters. There are quite a few cases. First, there are 3 strings of length 1, namely O, R, and N. There are several strings of length 2. If the string uses no O's, then there are 2; if it uses 1 O, then there are 2 ways to choose the other letter, and 2 ways to permute the letters in the string, so there are 4; and of course there is just 1 string of length 2 using 2 O's. Strings of length 3 can use 1, 2, or 3 O's. A little thought shows that the number of such strings is $3! = 6$, $2 \cdot 3 = 6$, and 1, respectively. There are 3 possibilities of the choice of letters for strings of length 4. If we omit an O, then there are $4!/2! = 12$ strings; if we omit either of the other letters (2 ways to choose the letter), then there are 4 strings. Finally, there are $5!/3! = 20$ strings of length 5. This gives a total of $3 + 2 + 4 + 1 + 6 + 6 + 1 + 12 + 2 \cdot 4 + 20 = 63$ strings using some or all of the letters.

35. We need to consider the three cases determined by the number of characters used in the string: 7, 8, or 9. If all nine letters are to be used, then Theorem 3 applies and we get

$$\frac{9!}{4!2!1!1!1!} = 7560$$

strings. If only eight letters are used, then we need to consider which letter is left out. In each of the cases in which the V, G, or N is omitted, Theorem 3 tells us that there are

$$\frac{8!}{4!2!1!1!} = 840$$

strings, for a total of 2520 for these cases. If an R is left out, then Theorem 3 tells us that there are

$$\frac{8!}{4!1!1!1!} = 1680$$

strings, and if an E is left out, then Theorem 3 tells us that there are

$$\frac{8!}{3!2!1!1!} = 3360$$

strings. This gives a total of $2520 + 1680 + 3360 = 7560$ strings of length 8. (It was not an accident that there are as many strings of length 8 as there are of length 9, since there is a one-to-one correspondence between these two sets, given by associating with a string of length 9 its first 8 characters.) For strings of length 7 there are even more cases. We tabulate them here:

omitting VG	$7!/(4!2!1!) = 105$ strings
omitting VN	$7!/(4!2!1!) = 105$ strings
omitting GN	$7!/(4!2!1!) = 105$ strings
omitting VR	$7!/(4!1!1!1!) = 210$ strings
omitting GR	$7!/(4!1!1!1!) = 210$ strings
omitting NR	$7!/(4!1!1!1!) = 210$ strings
omitting RR	$7!/(4!1!1!1!) = 210$ strings
omitting EV	$7!/(3!2!1!1!) = 420$ strings
omitting EG	$7!/(3!2!1!1!) = 420$ strings
omitting EN	$7!/(3!2!1!1!) = 420$ strings
omitting ER	$7!/(3!1!1!1!1!) = 840$ strings
omitting EE	$7!/(2!2!1!1!1!) = 1260$ strings

Adding up these numbers we see that there are 4515 strings of length 7. Thus the answer is $7560 + 7560 + 4515 = 19{,}635$.

37. We assume that all the fruit is to be eaten; in other words, this process ends after 7 days. This is a permutation problem since the order in which the fruit is consumed matters (indeed, there is nothing else that matters here). Theorem 3 applies, with $n = 7$, $n_1 = 3$, $n_2 = 2$, and $n_3 = 2$. The answer is therefore $7!/(3!2!2!) = 210$.

39. We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, and 5 z's. By Theorem 3 there are

$$\frac{12!}{4!3!5!} = 27720$$

such sequences.

41. This is like Example 9. If we approach it as is done there, we see that the answer is

$$C(52,7)C(45,7)C(38,7)C(31,7)C(24,7) = \frac{52!}{7!45!} \cdot \frac{45!}{7!38!} \cdot \frac{38!}{7!31!} \cdot \frac{31!}{7!24!} \cdot \frac{24!}{7!17!} = \frac{52!}{7!7!7!7!7!17!} \approx 7.0 \times 10^{34}.$$

Applying Theorem 4 will yield the same answer; in this approach we think of the five players and the undealt cards as the six distinguishable boxes.

43. We assume that we are to care about which player gets which cards. For example, a deal in which Laurel gets a royal flush in spades and Blaine gets a royal flush in hearts will be counted as different from a deal in which Laurel gets a royal flush in hearts and Blaine gets a royal flush in spades (and the other four players get the same cards each time). The order in which a player receives his or her cards is not relevant, however, so we are dealing with combinations. We can look at one player at a time. There are $C(48,5)$ ways to choose the cards for the first player, then $C(43,5)$ ways to choose the cards for the second player (because five of the cards are gone), and so on. So the answer, by the multiplication principle, is $C(48,5) \cdot C(43,5) \cdot C(38,5) \cdot C(33,5) \cdot C(28,5) \cdot C(23,5) = 649{,}352{,}163{,}073{,}816{,}339{,}512{,}038{,}979{,}194{,}880 \approx 6.5 \times 10^{32}$.

45. **a)** All that matters is how many copies of the book get placed on each shelf. Letting x_i be the number of copies of the book placed on shelf i, we are asking for the number of solutions to the equation $x_1 + x_2 + \cdots + x_k = n$, with each x_i a nonnegative integer. By Theorem 2 this is $C(k + n - 1, n)$.

b) No generality is lost if we number the books b_1, b_2, ..., b_n and think of placing book b_1, then placing b_2, and so on. There are clearly k ways to place b_1, since we can put it as the first book (for now) on any of the shelves. After b_1 is placed, there are $k+1$ ways to place b_2, since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are $k+2$ ways to place b_3 (to the right of b_1, to the right of b_2, or as the first book on some shelf), $k+3$ ways to place b_4, ..., $k+n-1$ ways to place b_n. Therefore the answer is the product of these numbers, which can more easily be expressed as $(k+n-1)!/(k-1)!$.

Another, perhaps easier, way to obtain this answer is to think of first choosing the locations for the books, which is what we counted in part **(a)**, and then choose a permutation of the n books to put into those locations (shelf by shelf, from the top down, and from left to right on each shelf). Thus the answer is $C(k+n-1,n)\cdot n!$, which evaluates to the same thing we obtained with our other analysis.

47. The first box holds n_1 objects, and there are $C(n,n_1)$ ways to choose those objects from among the n objects in the collection. Once these objects are chosen, we can choose the objects to be placed in the second box in $C(n-n_1,n_2)$ ways, since there are $n-n_1$ objects not yet placed, and we need to put n_2 of them into the second box. Similarly, there are then $C(n-n_1-n_2,n_3)$ ways to choose objects for the third box. We continue in this way, until finally there are $C(n-n_1-n_2-\cdots-n_{k-1},n_k)$ ways to choose the objects to put in the last (k^{th}) box. Note that this last expression equals $C(n_k,n_k)=1$, since $n_1+n_2+\cdots+n_k=n$. Now by the product rule the number of ways to make the entire assignment is

$$C(n,n_1)\cdot C(n-n_1,n_2)\cdot C(n-n_1-n_2,n_3)\cdots C(n-n_1-n_2-\cdots-n_{k-1},n_k).$$

We use the formula for combinations to write this as

$$\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!(n-n_1-n_2-\cdots-n_{k-1}-n_k)!},$$

which simplifies after the telescoping cancellation to

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

(we use the fact that $n-n_1-n_2-\cdots-n_{k-1}=n_k$, since $n_1+n_2+\cdots+n_k=n$), as desired.

49. **a)** The sequence was nondecreasing to begin with. By adding $k-1$ to the k^{th} term, we are adding more to each term than was added to the previous term. Hence even if two successive terms in the sequence were originally equal, the second term must strictly exceed the first after this addition is completed. Therefore the sequence is made up of distinct numbers. The smallest can be no smaller than $1+(1-1)=1$, and the largest can be no larger than $n+(r-1)=n+r-1$; therefore the terms all come from T.
b) If we are given an increasing sequence of r terms from T, then by subtracting $k-1$ from the k^{th} term we have a nondecreasing sequence of r terms from S, repetitions allowed. (The k^{th} term in the original sequence must be between k and $n+r-1-(r-k)=n+(k-1)$, so subtracting $k-1$ leaves a number between 1 and n, inclusive. Furthermore, only 1 more is subtracted from a term than is subtracted from the previous term; thus no term can become strictly smaller than its predecessor, since it exceeded it by at least 1 to start with.) This operation exactly inverts the operation described in part **(a)**, so the correspondence is one-to-one.
c) The first two parts show that there are exactly as many r-combinations with repetition allowed from S as there are r-combinations (without repetition) from T. Since T has $n+r-1$ elements, this latter quantity is clearly $C(n+r-1,r)$.

51. We use the formula given on page 378 for the number of ways to distribute n distinguishable objects into j indistinguishable boxes with no box empty:

$$S(n,j)=\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$

In this case, $n = 6$ and $j = 4$, so we have

$$S(6,4) = \frac{1}{4!}\left(\binom{4}{0}4^6 - \binom{4}{1}3^6 + \binom{4}{2}2^6 - \binom{4}{3}1^6\right) = \frac{1}{4!}(4096 - 2916 + 384 - 4) = 65\,.$$

If we want to work this out from scratch, we can argue as follows. There are two patterns possible. We can put three of the objects into one box and each of the remaining objects into a separate box; there are $C(6,3) = 20$ ways to choose the objects for the crowded box. Alternatively, we can choose a pair of objects for one box ($C(6,2) = 15$ ways) and a pair of remaining objects for the second box ($C(4,2) = 6$ ways) and put the other two objects into separate boxes, but divide by 2 because of the overcounting caused by the indistinguishability of the first two boxes, for a total of 45 ways. Therefore the answer is $20 + 45 = 65$.

53. We assume that people are distinguishable, so this problem is identical to Exercise 51. There are 65 ways to place the employees.

55. Since each box has to contain at least one object, we might as well put one object into each box to begin with. This leaves us with just two more objects, and there are only two choices: we can put them both into the same box (so that the partition we end up with is $6 = 3 + 1 + 1 + 1$), or we can put them into different boxes (so that the partition we end up with is $6 = 2 + 2 + 1 + 1$). So the answer is 2.

57. Since each box has to contain at least two DVDs, we might as well put two DVDs into each box to begin with. This leaves us with just three more DVDs, and there are only three choices: we can put them all into the same box (so that the partition we end up with is $9 = 5 + 2 + 2$), or we can put two into one box and one into another (so that the partition we end up with is $9 = 4 + 3 + 2$), or we can put them all into different boxes (so that the partition we end up with is $9 = 3 + 3 + 3$). So the answer is 3.

59. To begin, notice that because each box must have at least one ball, there are only two basic arrangements: to put three balls into one box and one ball into each of the other two boxes (denoted 3-1-1), or to put one ball into one box and two balls into each of the other two boxes (denoted 1-2-2).

a) For the 3-1-1 arrangement, there are 3 ways to choose the crowded box, $C(5,3) = 10$ ways to choose the balls to be put there, and 2 ways to decide where the other balls go, for a total of $3 \cdot 10 \cdot 2 = 60$ possibilities. For the 1-2-2 arrangement, there are 3 ways to choose the box that will have just one ball, 5 ways to choose which ball goes there, and $C(4,2) = 6$ ways to decide which two balls go into the lower-numbered remaining box, for a total of $3 \cdot 5 \cdot 6 = 90$ possibilities. Thus the answer is $60 + 90 = 150$.

b) There are $C(5,3) = 10$ ways to choose the balls for the crowded box in the 3-1-1 arrangement. For the 1-2-2 arrangement there are 5 ways to choose the lonely ball and 3 ways to choose the partner of the lowest-numbered remaining ball. Therefore the answer is $10 + 5 \cdot 3 = 25$.

c) There are 3 ways to choose the crowded box for the 3-1-1 arrangement, and there are 3 ways to choose the solo box for the 1-2-2 arrangement. Therefore the answer is $3 + 3 = 6$.

d) There are just the 2 possibilities we have been discussing: 3-1-1 and 1-2-2.

61. Without the restriction on site X, we are simply asking for the number of ways to order the ten symbols V, V, W, W, X, X, Y, Y, Z, Z (the ordering will give us the visiting schedule). By Theorem 3 this can be done in $10!/(2!)^5 = 113{,}400$ ways. If the inspector visits site X on consecutive days, then in effect we are ordering nine symbols (including only one X), where now the X means to visit site X twice in a row. There are $9!/(2!)^4 = 22{,}680$ ways to do this. Therefore the answer is $113{,}400 - 22{,}680 = 90{,}720$.

63. When $(x_1 + x_2 + \cdots + x_m)^n$ is expanded, each term will clearly be of the form $Cx_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, for some constants C that depend on the exponents, where the exponents sum to n. Thus the form of the given formula is correct, and the only question is whether the constants are correct. We need to count the number of ways in which a product of one term from each of the n factors can be $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$. In order for this to happen, we must choose n_1 x_1's, n_2 x_2's, ..., n_m x_m's. By Theorem 3 this can be done in

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!n_2!\cdots n_m!}$$

ways.

65. By the Multinomial Theorem, given in Exercise 63, the coefficient is

$$C(10; 3, 2, 5) = \frac{10!}{3!2!5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{12} = 2520\,.$$