## SECTION 9.6 Shortest-Path Problems

**2.** In the solution to Exercise 5 we find a shortest path. Its length is 7.

**4.** In the solution to Exercise 5 we find a shortest path. Its length is 16.

**6.** The solution to this problem is given in the solution to Exercise 7, where the paths themselves are found.

**8.** In theory, we can use Dijkstra's algorithm. In practice with graphs of this size and shape, we can tell by observation what the conceivable answers will be and find the one that produces the minimum total length by inspection.
a) The direct path is the shortest.
b) The path via Chicago only is the shortest.
c) The path via Atlanta and Chicago is the shortest.
d) The path via Atlanta, Chicago and Denver is the shortest.

**10.** The comments for Exercise 8 apply.
a) The direct flight is the cheapest.
b) The path via New York is the cheapest.
c) The path via New York and Chicago is the cheapest.
d) The path via New York is the cheapest.

**12.** The comments for Exercise 8 apply.
a) The path through Chicago is the fastest.
b) The path via Chicago is the fastest.
c) The path via Denver (or the path via Los Angeles) is the fastest.
d) The path via Dallas (or the path via Chicago) is the fastest.

**14.** Here we simply assign the weight of 1 to each edge.

**16.** We need to keep track of the vertex from which a shortest path known so far comes, as well as the length of that path. Thus we add an array $P$ to the algorithm, where $P(v)$ is the previous vertex in the best known path to $v$. We modify Algorithm 1 so that when $L$ is updated by the statement $L(v) := L(u) + w(u, v)$, we also set $P(v) := u$. Once the **while** loop has terminated, we can obtain a shortest path from $a$ to $z$ *in reverse* by starting with $z$ and following the pointers in $P$. Thus the path in reverse is $z$, $P(z)$, $P(P(z))$, $\ldots$, $P(P(\cdots P(z)\cdots)) = a$.

**18.** The shortest path need not be unique. For example, we could have a graph with vertices $a$, $b$, $c$, and $d$, with edges $\{a,b\}$ of weight 3, $\{b,c\}$ of weight 7, $\{a,d\}$ of weight 4, and $\{d,c\}$ of weight 6. There are two shortest paths from $a$ to $c$.

**20.** We give an ad hoc analysis. Recall that a simple path cannot use any edge more than once. Furthermore, since the path must use an odd number of edges incident to $a$ and an odd number of edges incident to $z$, the path must omit at least two edges, one at each end. The best we could hope for, then, in trying for a path of maximum length, is that the path leaves ont the shortest such edges—$\{a,c\}$ and $\{e,z\}$. If the path leaves out these two edges, then it must also leave out one more edge incident to $c$, since the path must use an even number of the three remaining edges incident to $c$. The best we could hope for is that the path omits the two aforementioned edges and edge $\{b,c\}$. Since $2 + 1 < 4$, this is better than the other possibility, namely omitting edge $\{a,b\}$ instead of edge $\{a,c\}$. Finally, we find a simple path omitting only these three edges, namely $a, b, d, c, e, d, z$, with length 35, and thus we conclude that it is a longest simple path from $a$ to $z$.

A similar argument shows that the longest simple path from $c$ to $z$ is $c, a, b, d, c, e, d, z$

**22.** It follows by induction on $i$ that after the $i^{\text{th}}$ pass through the triply nested **for** loop in the pseudocode, $d(v_j, v_k)$ gives, for each $j$ and $k$, the shortest distance between $v_j$ and $v_k$ using only intermediate vertices $v_m$ for $m \le i$. Therefore after the final path, we have obtained the shortest distance.

**24.** Consider the graph with vertices $a$, $b$, and $z$, where the weight of $\{a, z\}$ is 2, the weight of $\{a, b\}$ is 3, and the weight of $\{b, z\}$ is $-2$. Then Dijkstra's algorithm will decide that $L(z) = 2$ and stop, whereas the path $a, b, z$ is shorter (has length 1).

**26.** The following table shows the twelve different Hamilton circuits and their weights:

| Circuit | Weight |
|---|---|
| $a$-$b$-$c$-$d$-$e$-$a$ | $3 + 10 + 6 + 1 + 7 = 27$ |
| $a$-$b$-$c$-$e$-$d$-$a$ | $3 + 10 + 5 + 1 + 4 = 23$ |
| $a$-$b$-$d$-$c$-$e$-$a$ | $3 + 9 + 6 + 5 + 7 = 30$ |
| $a$-$b$-$d$-$e$-$c$-$a$ | $3 + 9 + 1 + 5 + 8 = 26$ |
| $a$-$b$-$e$-$c$-$d$-$a$ | $3 + 2 + 5 + 6 + 4 = 20$ |
| $a$-$b$-$e$-$d$-$c$-$a$ | $3 + 2 + 1 + 6 + 8 = 20$ |
| $a$-$c$-$b$-$d$-$e$-$a$ | $8 + 10 + 9 + 1 + 7 = 35$ |
| $a$-$c$-$b$-$e$-$d$-$a$ | $8 + 10 + 2 + 1 + 4 = 25$ |
| $a$-$c$-$d$-$b$-$e$-$a$ | $8 + 6 + 9 + 2 + 7 = 32$ |
| $a$-$c$-$e$-$b$-$d$-$a$ | $8 + 5 + 2 + 9 + 4 = 28$ |
| $a$-$d$-$b$-$c$-$e$-$a$ | $4 + 9 + 10 + 5 + 7 = 35$ |
| $a$-$d$-$c$-$b$-$e$-$a$ | $4 + 6 + 10 + 2 + 7 = 29$ |

Thus we see that the circuits $a$-$b$-$e$-$c$-$d$-$a$ and $a$-$b$-$e$-$d$-$c$-$a$ (or the same circuits starting at some other point but traversing the vertices in the same or exactly opposite order) are the ones with minimum total weight.

**28.** The following table shows the twelve different Hamilton circuits and their weights, where we abbreviate the cities with the beginning letter of their name, except that New Orleans is $O$:

| Circuit | Weight |
|---|---|
| $S$-$B$-$N$-$O$-$P$-$S$ | $409 + 109 + 229 + 309 + 119 = 1175$ |
| $S$-$B$-$N$-$P$-$O$-$S$ | $409 + 109 + 319 + 309 + 429 = 1575$ |
| $S$-$B$-$O$-$N$-$P$-$S$ | $409 + 239 + 229 + 319 + 119 = 1315$ |
| $S$-$B$-$O$-$P$-$N$-$S$ | $409 + 239 + 309 + 319 + 389 = 1665$ |
| $S$-$B$-$P$-$N$-$O$-$S$ | $409 + 379 + 319 + 229 + 429 = 1765$ |
| $S$-$B$-$P$-$O$-$N$-$S$ | $409 + 379 + 309 + 229 + 389 = 1715$ |
| $S$-$N$-$B$-$O$-$P$-$S$ | $389 + 109 + 239 + 309 + 119 = 1165$ |
| $S$-$N$-$B$-$P$-$O$-$S$ | $389 + 109 + 379 + 309 + 429 = 1615$ |
| $S$-$N$-$O$-$B$-$P$-$S$ | $389 + 229 + 239 + 379 + 119 = 1355$ |
| $S$-$N$-$P$-$B$-$O$-$S$ | $389 + 319 + 379 + 239 + 429 = 1755$ |
| $S$-$O$-$B$-$N$-$P$-$S$ | $429 + 239 + 109 + 319 + 119 = 1215$ |
| $S$-$O$-$N$-$B$-$P$-$S$ | $429 + 229 + 109 + 379 + 119 = 1265$ |

As a check of our arithmetic, we can compute the total weight (price) of all the trips (it comes to 17580) and check that it is equal to 6 times the sum of the weights (which here is 2930), since each edge appears in six paths (and sure enough, $17580 = 6 \cdot 2930$). We see that the circuit $S$-$N$-$B$-$O$-$P$-$S$ (or the same circuit starting at some other point but traversing the vertices in the same or exactly opposite order) is the one with minimum total weight, 1165.

**30.** We follow the hint. Let $G$ be our original weighted graph, and construct a new graph $G'$ as follows. The vertices and edges of $G'$ are the same as the vertices and edges of $G$. For each pair of vertices $u$ and $v$ in $G$, use an algorithm such as Dijkstra's algorithm to find a shortest path (i.e., one of minimum total weight) between $u$ and $v$. Record this path in a table, and assign to the edge $\{u, v\}$ in $G'$ the weight of this path. It is now clear that finding the circuit of minimum total weight in $G'$ that visits each vertex exactly once is equivalent to finding the circuit of minimum total weight in $G$ that visits each vertex at least once.