SECTION 5.4 Binomial Coefficients

In this section we usually write the binomial coefficients using the $\binom{n}{r}$ *notation, rather than the* $C(n,r)$ *notation. These numbers tend to come up in many parts of discrete mathematics.*

1. a) When $(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, and a term in the fourth sum are added. Terms of the form x^4, x^3y, x^2y^2, xy^3 and y^4 arise. To obtain a term of the form x^4, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^4 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{4}{4}$.) To obtain a term of the form x^3y, an x must be chosen in three of the four sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 3-combinations of four objects, namely $\binom{4}{3} = 4$. Similarly, the number of terms of the form x^2y^2 is the number of ways to pick two of the four sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{4}{2} = 6$ ways. By the same reasoning there are $\binom{4}{1} = 4$ ways to obtain the xy^3 terms, and only one way (which we can think of as $\binom{4}{0}$) to obtain a y^4 term. Consequently, the product is $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

b) This is explained in Example 2. The expansion is $\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. Note that it does not matter whether we think of the bottom of the binomial coefficient expression as corresponding to the exponent on x, as we did in part **(a)**, or the exponent on y, as we do here.

3. The coefficients are the binomial coefficients $\binom{6}{i}$, as i runs from 0 to 6, namely 1, 6, 15, 20, 15, 6, 1. Therefore $(x+y)^6 = \sum_{i=0}^{6} \binom{6}{i} x^{6-i}y^i = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

5. There is one term for each i from 0 to 100, so there are 101 terms.

7. By the Binomial Theorem the term involving x^9 in the expansion of $(2+(-x))^{19}$ is $\binom{19}{9}2^{10}(-x)^9$. Therefore the coefficient is $\binom{19}{9}2^{10}(-1)^9 = -2^{10}\binom{19}{9} = -94{,}595{,}072$.

9. Using the Binomial Theorem, we see that the term involving x^{101} in the expansion of $((2x)+(-3y))^{200}$ is $\binom{200}{99}(2x)^{101}(-3y)^{99}$. Therefore the coefficient is $\binom{200}{99}2^{101}(-3)^{99} = -2^{101}3^{99}C(200,99)$.

11. Let us apply the Binomial Theorem to the given binomial:

$$\begin{aligned}(x^2 - x^{-1})^{100} &= \sum_{j=0}^{100}\binom{100}{j}(x^2)^{100-j}(-x^{-1})^j \\ &= \sum_{j=0}^{100}\binom{100}{j}(-1)^j x^{200-2j-j} = \sum_{j=0}^{100}\binom{100}{j}(-1)^j x^{200-3j}\end{aligned}$$

Thus the only nonzero coefficients are those of the form $200-3j$ where j is an integer between 0 and 100, inclusive, namely $200,\ 197,\ 194,\ \ldots,\ 2,\ -1,\ -4,\ \ldots,\ -100$. If we denote $200-3j$ by k, then we have $j=(200-k)/3$. This gives us our answer. The coefficient of x^k is zero for k not in the list just given (namely those vales of k between -100 and 200, inclusive, that are congruent to 2 modulo 3), and for those values of k in the list, the coefficient is $(-1)^{(200-k)/3}\binom{100}{(200-k)/3}$.

13. We are asked simply to display these binomial coefficients. Each can be computed from the formula in Theorem 2 in Section 5.3. Alternatively, we can apply Pascal's Identity to the last row of Figure 1(b), adding successive numbers in that row to produce the desired row. We thus obtain

$$1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 .$$

15. There are many ways to see why this is true. By Corollary 1 the sum of *all* the positive numbers $\binom{n}{k}$, as k runs from 0 to n, is 2^n, so certainly each one of them is no bigger than this sum. Another way to see this is to note that $\binom{n}{k}$ counts the number of subsets of an n-set having k elements, and 2^n counts even more—the number of subsets of an n-set with no restriction as to size; so certainly the former is smaller than the latter.

17. We know that

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2}.$$

Now if we make the numerator of the right-hand side larger by raising each factor up to n, and make the denominator smaller by lowering each factor to 2, then we have certainly not decreased the value, so the left-hand side is less than or equal to this altered expression. But the result is precisely $n^k/2^{k-1}$, as desired.

19. Using the formula (Theorem 2 in Section 5.3) we have

$$\begin{aligned}\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} \quad \text{(having found a common denominator)} \\ &= \frac{(n+1)n!}{k!((n+1)-k)!} = \frac{(n+1)!}{k!((n+1)-k)!} = \binom{n+1}{k}.\end{aligned}$$

21. **a)** We show that each side counts the number of ways to choose from a set with n elements a subset with k elements and a distinguished element of that set. For the left-hand side, first choose the k-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways). For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining $k-1$ elements of the subset from the remaining $n-1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways).

b) This is straightforward algebra:

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

23. This identity can be proved algebraically or combinatorially. Algebraically, we compute as follows, starting with the right-hand side (we use twice the fact that $(x+1)x! = (x+1)!$):

$$\begin{aligned}\frac{(n+1)\binom{n}{k-1}}{k} &= \frac{(n+1)n!}{(k-1)!(n-(k-1))!k}\\ &= \frac{(n+1)!}{k!(n-(k-1))!}\\ &= \frac{(n+1)!}{k!((n+1)-k)!}\\ &= \binom{n+1}{k}\end{aligned}$$

For a combinatorial argument, we need to construct a situation in which both sides count the same thing. Suppose that we have a set of $n+1$ people, and we wish to choose k of them. Clearly there are $\binom{n+1}{k}$ ways to do this. On the other hand, we can choose our set of k people by first choosing one person to be in the set (there are $n+1$ choices), and then choosing $k-1$ additional people to be in the set, from the n people remaining. This can be done in $\binom{n}{k-1}$ ways. Therefore apparently there are $(n+1)\binom{n}{k-1}$ ways to choose the set of k people. However, we have overcounted: there are k ways that every such set can be chosen, since once we have the set, we realize that any of the k people could have been chosen first. Thus we have overcounted by a factor of k, and the real answer is $(n+1)\binom{n}{k-1}/k$ (we correct for the overcounting by dividing by k). Comparing our two approaches, one yielding the answer $\binom{n+1}{k}$, and the other yielding the answer $(n+1)\binom{n}{k-1}/k$, we conclude that $\binom{n+1}{k} = (n+1)\binom{n}{k-1}/k$.

Finally, we are asked to use this identity to give a recursive definition of the $\binom{n}{k}$'s. Note that this identity expresses $\binom{n}{k}$ in terms of $\binom{i}{j}$ for values of i and j less than n and k, respectively (namely $i = n-1$ and $j = k-1$). Thus the identity will be the recursive part of the definition. We need the base cases to handle $n = 0$ or $k = 0$. Our full definition becomes

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0\\ 0 & \text{if } k > 0 \text{ and } n = 0\\ n\binom{n-1}{k-1}/k & \text{if } n > 0 \text{ and } k > 0.\end{cases}$$

Actually, if we assume (as we usually do) that $k \le n$, then we do not need the second line of the definition. Note that $\binom{n}{k} = 0$ for $n < k$ under the definition given here, which is consistent with the combinatorial definition, since there are no ways to choose k different elements from a set with fewer than k elements.

25. We use Pascal's Identity twice (Theorem 2 of this section) and Corollary 1 of the previous section:

$$\begin{aligned}\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} &= \frac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right)\\ &= \frac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n}\right) = \frac{1}{2}\left(\binom{2n+2}{n+1}\right)\end{aligned}$$

27. a) We need to find something to count so that the left-hand side of the equation counts it in one way and the right-hand side counts it in a different way. After much thought, we might try the following. We will count the number of bit strings of length $n+r+1$ containing exactly r 0's and $n+1$ 1's. There are $\binom{n+r+1}{r}$ such strings, since a string is completely specified by deciding which r of the bits are to be the 0's. To see that the left-hand side of the identity counts the same thing, let $l+1$ be the position of the last 1 in the string. Since there are $n+1$ 1's, we know that l cannot be less than n. Thus there are disjoint cases for each l from n to $n+r$. For each such l, we completely determine the string by deciding which of the l positions in the string before the last 1 are to be 0's. Since there are n 1's in this range, there are $l-n$ 0's. Thus there are $\binom{l}{l-n}$ ways to choose the positions of the 0's. Now by the sum rule the total number of bit strings will be $\sum_{l=n}^{n+r}\binom{l}{l-n}$. By making the change of variable $k=l-n$, this transforms into the left-hand side, and we are finished.

b) We need to prove this by induction on r; Pascal's Identity will enter at the crucial step. We let $P(r)$ be the statement to be proved. The basis step is clear, since the equation reduces to $\binom{n}{0}=\binom{n+1}{0}$, which is the true proposition $1=1$. Assuming the inductive hypothesis, we derive $P(r+1)$ in the usual way:

$$\begin{aligned}\sum_{k=0}^{r+1}\binom{n+k}{k} &= \left(\sum_{k=0}^{r}\binom{n+k}{k}\right)+\binom{n+r+1}{r+1}\\ &= \binom{n+r+1}{r}+\binom{n+r+1}{r+1} \quad \text{(by the inductive hypothesis)}\\ &= \binom{n+(r+1)+1}{r+1} \quad \text{(by Pascal's Identity)}\end{aligned}$$

29. We will follow the hint and count the number of ways to choose a committee with leader from a set of n people. Note that the size of the committee is not specified, although it clearly needs to have at least one person (its leader). On the one hand, we can choose the leader first, in any of n ways. We can then choose the rest of the committee, which can be any subset of the remaining $n-1$ people; this can be done in 2^{n-1} ways since there are this many subsets. Therefore the right-hand side of the proposed identity counts this. On the other hand, we can organize our count by the size of the committee. Let k be the number of people who will serve on the committee. The number of ways to select a committee with k people is clearly $\binom{n}{k}$, and once we have chosen the committee, there are clearly k ways in which to choose its leader. By the sum rule the left-hand side of the proposed identity therefore also counts the number of such committees. Since the two sides count the same quantity, they must be equal.

31. Corollary 2 says that $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots\pm\binom{n}{n}=0$. If we put all the negative terms on the other side, we obtain $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\cdots$ (one side ends at $\binom{n}{n}$ and the other side ends at $\binom{n}{n-1}$—which is which depends on whether n is even or odd). Now the left-hand side counts the number of subsets with even cardinality of a set with n elements, and the right-hand side counts the number of subsets with odd cardinality of the same set. That these two quantities are equal is precisely what we wanted to prove.

33. a) Clearly a path of the desired type must consist of m moves to the right and n moves up. Therefore each such path can be represented by a bit string consisting of m 0's and n 1's, with the 0's representing moves to the right and the 1's representing moves up. Note that the total length of this bit string is $m+n$.

b) We know from this section that the number of bit strings of length $m+n$ containing exactly n 1's is $\binom{m+n}{n}$, since one need only specify the positions of the 1's. Note that this is the same as $\binom{m+n}{m}$.

35. We saw in Exercise 33 that the number of paths of length n was the same as the number of bit strings of length n, which we know to be 2^n, the right-hand side of the identity. On the other hand, a path of length n must end up at some point the sum of whose coordinates is n, say at $(n-k,k)$ for some k from 0 to n. We saw in Exercise 33 that the number of paths ending up at $(n-k,k)$ was equal to $\binom{n-k+k}{k} = \binom{n}{k}$. Therefore the left-hand side of the identity counts the number of such paths, too.

37. The right-hand side of the identity we are asked to prove counts, by Exercise 33, the number of paths from $(0,0)$ to $(n+1,r)$. Now let us count these paths by breaking them down into $r+1$ cases, depending on how many steps upward they begin with. Let k be the number of steps upward they begin with before taking a step to the right. Then k can take any value from 0 to r. The number of paths from $(0,0)$ to $(n+1,r)$ that begin with exactly k steps upward before turning to the right is clearly the same as the number of paths from $(1,k)$ to $(n+1,r)$, since after these k upward steps and the move to the right we have reached $(1,k)$. This latter quantity is the same as the number of paths from $(0,0)$ to $(n+1-1,r-k)=(n,r-k)$, since we can relabel our diagram to make $(1,k)$ the origin. From Exercise 33, this latter quantity is $\binom{n+r-k}{r-k}$. Therefore the total number of paths is the desired sum

$$\sum_{k=0}^{r}\binom{n+r-k}{r-k} = \sum_{k=0}^{r}\binom{n+k}{k},$$

where the equality comes from changing the dummy variable from k to $r-k$. Since both sides count the same thing, they are equal.

39. **a)** This looks like the third negatively sloping diagonal of Pascal's triangle, starting with the leftmost entry in the second row and reading down and to the right. In other words, the n^{th} term of this sequence is $\binom{n+1}{n-1} = \binom{n+1}{2}$.
b) This looks like the fourth negatively sloping diagonal of Pascal's triangle, starting with the leftmost entry in the third row and reading down and to the right. In other words, the n^{th} term of this sequence is $\binom{n+2}{n-1} = \binom{n+2}{3}$.
c) These seem to be the entries reading straight down the middle of the Pascal's triangle. Only every other row has a middle element. The first entry in the sequence is $\binom{0}{0}$, the second is $\binom{2}{1}$, the third is $\binom{4}{2}$, the fourth is $\binom{6}{3}$, and so on. In general, then, the n^{th} term is $\binom{2n-2}{n-1}$.
d) These seem to be the entries reading dowu the middle of the Pascal's triangle. Only every other row has an exact middle element, but in the other rows, there are two elements sharing the middle. The first entry in the sequence is $\binom{0}{0}$, the second is $\binom{1}{0}$, the third is $\binom{2}{1}$, the fourth is $\binom{3}{1}$, the fifth is $\binom{4}{2}$, the fourth is $\binom{5}{2}$, and so on. In general, then, the n^{th} term is $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.
e) One pattern here is a wandering through Pascal's triangle according to the following rule. The n^{th} term is in the n^{th} row of the triangle. We increase the position in that row as we go down, but as soon as we are about to reach the middle of the row, we jump back to the start of the next row. For example, the fifth term is the first entry in row 5; the sixth term is the second entry in row 6; the seventh term is the third entry in row 7; the eighth term is the fourth entry in row 8. The next entry following that pattern would take us to the middle of the ninth row, so instead we jump back to the beginning, and the ninth term is the first entry of row 9. To come up with a formula here, we see that the n^{th} entry is $\binom{n-1}{k-1}$ for a particular k; let us determine k as a function of n. A little playing around with the pattern reveals that k is n minus the largest power of 2 less than n (where for this purpose we consider 0 to be the largest power of 2 less than 1). For example, the 14^{th} term has $k = 14-8 = 6$, so it is $\binom{13}{5} = 1287$.
f) The terms seem to come from every third row, so the n^{th} term is $\binom{3n-3}{k}$ for some k. A little observation indicates that in fact these terms are $\binom{0}{0}$, $\binom{3}{1}$, $\binom{6}{2}$, $\binom{9}{3}$, $\binom{12}{4}$, and so on. Thus the n^{th} term is $\binom{3n-3}{n-1}$.