GUIDE TO REVIEW QUESTIONS FOR CHAPTER 7

1. **a)** See pp. 449–450. **b)** $\$1{,}000{,}000 \cdot 1.09^n$

2. See Example 4 in Section 7.1.

3. See Example 5 in Section 7.1.

4. **a)** See Example 6 in Section 7.1 (interchange the roles of 0 and 1). **b)** See Exercise 27 in Section 7.1.

5. an equation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$

6. **a)** See Theorem 1 and Example 3 in Section 7.2 if the roots of the characteristic equation are distinct; otherwise see Theorem 2 and Example 5.
b) The characteristic equation is $r^2 - 13r + 22 = 0$, leading to roots 2 and 11. This gives the general solution $a_n = \alpha_1 2^n + \alpha_2 11^n$. Substituting in the initial conditions gives $\alpha_1 = 2$ and $\alpha_2 = 1$. Therefore the solution is $a_n = 2^{n+1} + 11^n$.
c) The characteristic equation is $r^2 - 14r + 49 = 0$, leading to the repeated root 7. This gives the general solution $a_n = \alpha_1 7^n + \alpha_2 n 7^n$. Substituting in the initial conditions gives $\alpha_1 = 3$ and $\alpha_2 = 2$. Therefore the solution is $a_n = (3 + 2n)7^n$.

7. a) See p. 476. The exact solution is $f(b^k) = a^k f(1) + \sum_{j=0}^{k-1} a^j g(b^{k-j})$. b) 1442

8. a) See Example 1 in Section 7.3. b) $O(\log n)$

9. a) See p. 502. b) See pp. 501–502.
c) Note that the number of integers not exceeding 1000 that are divisible by a and b is $\lfloor 1000/\text{lcm}(a,b) \rfloor$. Thus the answer is

$$\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,10)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,15)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(10,15)} \right\rfloor + \left\lfloor \frac{1000}{\text{lcm}(6,10,15)} \right\rfloor$$
$$= 166 + 100 + 66 - 33 - 33 - 33 + 33 = 266\,.$$

This is similar to (but slightly harder than) the discussion on p. 507.
d) For a similar problem, see Example 1 in Section 7.6. The solution is $C(4+22-1,22) - C(4+14-1,14) - C(4+16-1,16) - C(4+17-1,17) + C(4+8-1,8) + C(4+9-1,9) + C(4+11-1,11) - C(4+3-1,3)$.

10. a) See Example 5 in Section 7.5. b) $4 \cdot 25 - 6 \cdot 5 + 4 \cdot 2 - 1 = 77$

11. See Theorem 1 in Section 7.5.

12. See p. 509.

13. a) Count the number of onto functions from an m-set to an n-set, using Theorem 1 in Section 7.6; see Example 3 in Section 7.6.
b) $3^7 - 3 \cdot 2^7 + 3 \cdot 1^7 = 1806$

14. See the discussion of the Sieve of Eratosthenes on p. 507.

15. a) See p. 510.
b) Think of the hats permuted among the heads (which are the positions for the objects being permuted).
c) See Theorem 2 in Section 7.6.

SUPPLEMENTARY EXERCISES FOR CHAPTER 7

1. Let L_n be the number of chain letters sent at the n^{th} stage.
a) Since each person receiving a letter sends it to 4 new people, there will be 4 times as many letters sent at the n^{th} stage as were sent at the $(n-1)^{\text{st}}$ stage. Therefore the recurrence relation is $L_n = 4L_{n-1}$.
b) The initial condition is that at the first stage 40 letters are sent (each of the original 10 people sent it to 4 others), i.e., $L_1 = 40$.
c) We need to solve this recurrence relation. We do so easily by iteration, since $L_n = 4L_{n-1} = 4^2 L_{n-2} = \cdots = 4^{n-1} L_1 = 4^{n-1} \cdot 40$, or more simply $L_n = 10 \cdot 4^n$.

3. Let M_n be the amount of money (in dollars) that the government prints in the n^{th} hour.
a) According to the given information, the amount of money printed in the n^{th} hour is \$10,000 in \$1 bills, \$20,000 in \$5 bills, \$30,000 in \$10 bills, \$50,000 in \$20 bills, and \$50,000 in \$50 bills, for a total of \$160,000. Therefore our recurrence relation is $M_n = M_{n-1} + 160000$.
b) Since 1000 of each bill was produced in the first hour, we know that $M_1 = 1000(1+5+10+20+50+100) = 186000$.

c) We solve the recurrence relation by iteration:

$$\begin{aligned} M_n &= 160000 + M_{n-1} \\ &= 160000 + 160000 + M_{n-2} = 2 \cdot 160000 + M_{n-2} \\ &\vdots \\ &= (n-1) \cdot 160000 + M_1 \\ &= 160000(n-1) + 186000 = 160000n + 26000 \end{aligned}$$

d) Let T_n be the total amount of money produced in the first n hours. Then $T_n = T_{n-1} + M_n$, since the total amount of money produced in the first n hours is the same as the total amount of money produced in the first $n-1$ hours, plus the amount of money produced in the n^{th} hour. Thus, from our result in part **(c)**, the recurrence relation is $T_n = T_{n-1} + 160000n + 26000$, with initial condition $T_0 = 0$ (no money is produced in 0 hours).

e) We solve the recurrence relation from part **(d)** by iteration:

$$\begin{aligned} T_n &= 26000 + 160000n + T_{n-1} \\ &= 26000 + 160000n + 26000 + 160000(n-1) + T_{n-2} \\ &= 2 \cdot 26000 + \big(n + (n-1)\big) \cdot 160000 + T_{n-2} \\ &\vdots \\ &= n \cdot 26000 + 160000 \cdot \big(n + (n-1) + \cdots + 1\big) + T_0 \\ &= 26000n + 160000 \cdot \frac{n(n+1)}{2} = 80000n^2 + 106000n \end{aligned}$$

5. This problem is similar to Exercise 35 in Section 7.1. Let m_n be the number of messages that can be sent in n microseconds.

a) A message must begin with either the two-microsecond signal or the three-microsecond signal. If it begins with the two-microsecond signal, then the rest of the message is of length $n-2$; if it begins with the three-microsecond signal, then the message continues as a message of length $n-3$. Therefore the recurrence relation is $m_n = m_{n-2} + m_{n-3}$.

b) We need initial conditions for $n = 0$, 1, and 2, since the recurrence relation has degree 3. Clearly $m_0 = 1$, since the empty message is the one and only message of length 0. Also $m_1 = 0$, since every nonempty message contains at least one signal, and the shortest signal has length 2. Finally $m_2 = 1$, since there is only one message of length 2, namely the one that uses one of the shorter signals and none of the longer signals.

c) There are two approaches here. One is to solve the recurrence relation, using the methods of Section 7.2. Unfortunately, the characteristic equation is $r^3 - r - 1 = 0$, and it has no rational roots. It is possible to find real roots, but the formula for solving third degree equations is messy, and the algebra in completing the solution this way would not be pleasant. (Alternatively, one could get approximations to the roots, then get approximations to the coefficients in the solution, plug in $n = 12$, and round to the nearest integer; again the calculation involved would be unpleasant.)

The other approach is simply to use the recurrence relation to compute m_3, m_4, $\ldots$, m_{12}. First $m_3 = m_1 + m_0 = 0 + 1 = 1$; then $m_4 = m_2 + m_1 = 1 + 0 = 1$, then $m_5 = m_3 + m_2 = 1 + 1 = 2$, and so on. Starting with m_6, the sequence continues 2, 3, 4, 5, 7, 9, 12. Therefore there are 12 different messages that can be sent in exactly 12 microseconds. (If we wanted to find the number of nonempty messages that could be sent in at most 12 microseconds—which is certainly one interpretation of the question—then we would add m_1 through m_{12}, obtaining 47 as our answer.)

7. The recurrence relation found in Exercise 6 is of degree 10, namely $a_n = a_{n-4} + a_{n-6} + a_{n-10}$. It needs 10 initial conditions, namely $a_0 = 1$, $a_1 = a_2 = a_3 = a_5 = a_7 = a_9 = 0$, and $a_4 = a_6 = a_8 = 1$.

a) $a_{12} = a_8 + a_6 + a_2 = 1 + 1 + 0 = 2$ (Indeed, the 2 ways to affix 12 cents postage is either to use 3 4-cent stamps or to use 2 6-cent stamps.)

b) First we need to compute $a_{10} = a_6 + a_4 + a_0 = 1 + 1 + 1 = 3$. Then $a_{14} = a_{10} + a_8 + a_4 = 3 + 1 + 1 = 5$.

c) We use the results of previous parts here: $a_{18} = a_{14} + a_{12} + a_8 = 5 + 2 + 1 = 8$.

d) First we need to compute $a_{16} = a_{12} + a_{10} + a_6 = 2 + 3 + 1 = 6$. Using this (and previous parts), we have $a_{22} = a_{18} + a_{16} + a_{12} = 8 + 6 + 2 = 16$.

9. Following the hint, let $b_n = \log a_n$ (remember that we mean log base 2). Then using the property that the log of a quotient is the difference of the logs and the log of a power is the multiple of the log, we take the logarithm of both sides of the recurrence relation for a_n to obtain $b_n = 2b_{n-1} - b_{n-2}$. The initial conditions translate into $b_0 = \log a_0 = \log 1 = 0$ and $b_1 = \log a_1 = \log 2 = 1$. Thus we have transformed our problem into a linear, homogeneous, second degree recurrence relation with constant coefficients.

To solve $b_n = 2b_{n-1} - b_{n-2}$, we form the characteristic equation $r^2 - 2r + 1 = 0$, which has the repeated root $r = 1$. By Theorem 2 in Section 7.2, the general solution is $b_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$. Plugging in the initial conditions gives the equations $\alpha_1 = 0$ and $\alpha_1 + \alpha_2 = 1$, whence $\alpha_2 = 1$. Therefore the solution is $b_n = n$. Finally, $b_n = \log a_n$ implies that $a_n = 2^{b_n}$. Therefore our solution to the original problem is $a_n = 2^n$.

11. The characteristic equation of the associated homogeneous equation is $r^3 - 3r^2 + 3r - 1 = 0$. This factors as $(r-1)^3 = 0$, so there is only one root, 1, and its multiplicity is 3. Therefore the general solution is $a_n^{(h)} = \alpha + \beta n + \gamma n^2$. Since the nonhomogeneous term is 1, Theorem 6 in Section 7.2 tells us to look for a particular solution of the form $a_n = c \cdot n^3$. Plugging this into the recurrence gives $c \cdot n^3 = 3c(n-1)^3 - 3c(n-2)^3 + (n-3)^3 + 1$. Simplifying this by multiplying it out and collecting like powers of n gives us $6c = 1$ (all the other terms cancel out), so $c = 1/6$. Thus $a_n^{(p)} = n^3/6$. Plugging in the initial conditions to the general solution $a_n = \alpha + \beta n + \gamma n^2 + n^3/6$ gives us $2 = \alpha$, $4 = \alpha + \beta + \gamma + 1/6$, and $8 = \alpha + 2\beta + 4\gamma + 4/3$. Solving yields $\alpha = 2$, $\beta = 4/3$, and $\gamma_3 = 1/2$. Therefore the solution is $a_n = 2 + 4n/3 + n^2/2 + n^3/6$. As a check we can compute a_3 both from the recurrence and from the formula, and we get 15 in both cases.

13. One way to approach this problem is by temporarily using three variables. We assume that rabbits are born at the beginning of the month. Let a_n be the number of pairs of $\frac{1}{2}$-month-old rabbits present in the middle of the n^{th} month, let b_n be the number of pairs of $1\frac{1}{2}$-month-old rabbits present in the middle of the n^{th} month, and let c_n be the number of pairs of $2\frac{1}{2}$-month-old rabbits present in the middle of the n^{th} month. All the older rabbits have left the island, by the conditions of the exercise. Let us see how each of these depends on previous values. First note that $b_n = a_{n-1}$, since these rabbits are one month older. Similarly $c_n = b_{n-1}$. Combining these two equations gives $c_n = a_{n-2}$. Finally, $a_n = b_{n-1} + c_{n-1}$, since newborns come from these two groups of rabbits. Writing this last equation totally in terms of a_n (using the previous equations) gives $a_n = a_{n-2} + a_{n-3}$.

Now we are interested in $T_n = a_n + b_n + c_n$, the total number of pairs of rabbits in the middle of the n^{th} month. Since we have seen that the sequences $\{b_n\}$ and $\{c_n\}$ are the same as the sequence $\{a_n\}$, just shifted by one or two months, they must satisfy the same recurrence relation, so we have $b_n = b_{n-2} + b_{n-3}$ and $c_n = c_{n-2} + c_{n-3}$. If we add these three recurrence relations, we obtain $T_n = T_{n-2} + T_{n-3}$. We can take as the initial conditions $T_1 = T_2 = 1$ and $T_3 = 2$.

(We are not asked to solve this recurrence relation, and fortunately so. The characteristic equation, $r^3 - r - 1 = 0$ has no nice roots—one is irrational and two are complex. The roots are distinct, however, so let us call them r_1, r_2, and r_3. Then the general solution to the recurrence relation is $T_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$.

We could in principle determine the values of the α's by plugging in the initial conditions, thereby obtaining an explicit solution. We will not do this.)

15. We use Theorem 2 in Section 7.3, with $a = 3$, $b = 5$, $c = 2$ and $d = 4$. Since $a < b^d$, we have that $f(n)$ is $O(n^d) = O(n^4)$.

17. In the algorithm in Exercise 16, we need 2 comparisons to determine the largest and second largest elements of the sequence, knowing the largest and second largest elements of the first half and the second half. Thus letting $f(n)$ be the number of comparisons needed for a list with n elements, and assuming that n is even, we have $f(n) = 2f(n/2) + 2$. Now by Theorem 2 in Section 7.3, with $a = 2$, $b = 2$, $c = 2$ and $d = 0$, we know that $f(n)$ is $O(n^{\log_b a}) = O(n^1) = O(n)$.

19. First we have to find Δa_n. By definition we have $\Delta a_n = a_{n+1} - a_n = 3(n+1)^3 + (n+1) + 2 - (3n^3 + n + 2) = 9n^2 + 9n + 4$.
a) By definition $\Delta^2 a_n = \Delta a_{n+1} - \Delta a_n = 9(n+1)^2 + 9(n+1) + 4 - (9n^2 + 9n + 4) = 18n + 18$.
b) By definition $\Delta^3 a_n = \Delta^2 a_{n+1} - \Delta^2 a_n = 18(n+1) + 18 - (18n + 18) = 18$.
c) By definition $\Delta^4 a_n = \Delta^3 a_{n+1} - \Delta^3 a_n = 18 - 18 = 0$.

21. We apply the definition, starting with the right-hand side:
$$\begin{aligned} a_{n+1}(\Delta b_n) + b_n(\Delta a_n) &= a_{n+1}(b_{n+1} - b_n) + b_n(a_{n+1} - a_n) \\ &= a_{n+1}b_{n+1} - a_n b_n \quad \text{(by algebra)} \\ &= \Delta(a_n b_n) \quad \text{(by definition)} \end{aligned}$$

23. **a)** Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$. Therefore
$$G'(x) - G(x) = \sum_{n=0}^{\infty} ((n+1)a_{n+1} - a_n)x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x ,$$
as desired. That $G(0) = a_0 = 1$ is given.
b) We compute the indicated derivative:
$$(e^{-x}G(x))' = e^{-x}G'(x) - e^{-x}G(x) = e^{-x}(G'(x) - G(x)) = e^{-x} \cdot e^x = 1$$
This means that $e^{-x}G(x)$ is x plus a constant, say $x + c$. So $G(x) = xe^x + ce^x$. Plugging in the initial condition shows that $c = 1$, and we are done.
c) We work with the generating function for the exponential function:
$$G(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
Therefore $a_n = 1/(n-1)! + 1/n!$ for all $n \geq 1$ (and $a_0 = 1$). As a check we can compute the first few terms of the sequence both from this solution and from the recurrence, and in each case we find the sequence $a_0 = 1$, $a_1 = 2$, $a_2 = 3/2$, $a_3 = 2/3$, $a_4 = 5/24$, ...

25. Let H, C, and S stand for the sets of farms that have horses, cows, and sheep, respectively. We are told that $|H \cup C \cup S| = 323$, $|H| = 224$, $|C| = 85$, $|S| = 57$, and $|H \cap C \cap S| = 18$. We are asked to find $|H \cap C| + |H \cap S| + |C \cap S| - 3|H \cap C \cap S|$ (the reason for the subtraction is that the indicated sum counts the farms with all three animals 3 times, and we wish to count it no times). By the principle of inclusion–exclusion we know that $|H \cup C \cup S| = |H| + |C| + |S| - |H \cap C| - |H \cap S| - |C \cap S| + |H \cap C \cap S|$. Solving for the expression we are interested in, we get $|H \cap C| + |H \cap S| + |C \cap S| - 3|H \cap C \cap S| = |H| + |C| + |S| - |H \cup C \cup S| - 2|H \cap C \cap S| = 224 + 85 + 57 - 323 - 2 \cdot 18 = 7$. Thus 7 farms have exactly two of the three types of animals.

27. We apply the principle of inclusion–exclusion: $|AM \cup PM \cup OR \cup CS| = 23 + 17 + 44 + 63 - 5 - 8 - 4 - 6 - 5 - 14 + 2 + 2 + 1 + 1 - 1 = 110$.

29. Since the largest possible value for $x_1 + x_2 + x_3$ under these constraints is $5+9+4 = 18$, there are no solutions to the given equation.

31. **a)** We solve this problem in the same manner as we solved Exercise 7 in Section 7.6. As explained in our solution there, we need only look at prime powers. Let us restrict ourselves to integers greater than 1, and add 1 at the end. There are $\lfloor\sqrt{199}\rfloor - 1 = 13$ perfect second powers in the given range, namely 2^2 through 14^2. There are $\lfloor\sqrt[3]{199}\rfloor - 1 = 4$ perfect third powers, $\lfloor\sqrt[5]{199}\rfloor - 1 = 1$ perfect fifth power, and $\lfloor\sqrt[7]{199}\rfloor - 1 = 1$ perfect seventh power. Furthermore, there is $\lfloor\sqrt[6]{199}\rfloor - 1 = 1$ perfect sixth power, which is both a perfect square and a perfect cube. Therefore by inclusion–exclusion, the number of numbers between 2 and 199 inclusive that are powers greater than the first power of an integer is $13 + 4 + 1 + 1 - 1 = 18$; adding on the number 1 itself (since $1 = 1^2$), we get the answer 19.

b) We saw in Exercise 5 in Section 7.6 that there are 46 primes less than 200, and we just saw above that there are 19 powers. Since these two sets are disjoint, we just add the cardinalities, obtaining $19 + 46 = 65$.

c) Solving this problem is like counting prime numbers, except that the squares of primes play the role of the primes themselves. The squares of primes relevant to the problem are 4, 9, 25, 49, 121, and 169. The number of positive integers less than 200 divisible by p^2 is $\lfloor 199/p^2\rfloor$. There is overcounting, however, since a number divisible by a number like $36 = 2^2 \cdot 3^2$ is counted in both $\lfloor 199/2^2\rfloor$ and $\lfloor 199/3^2\rfloor$; hence we need to subtract $\lfloor 199/6^2\rfloor$. The number of numbers divisible by squares of primes is therefore

$$\left\lfloor\frac{199}{2^2}\right\rfloor + \left\lfloor\frac{199}{3^2}\right\rfloor + \left\lfloor\frac{199}{5^2}\right\rfloor + \left\lfloor\frac{199}{7^2}\right\rfloor + \left\lfloor\frac{199}{11^2}\right\rfloor + \left\lfloor\frac{199}{13^2}\right\rfloor - \left\lfloor\frac{199}{6^2}\right\rfloor - \left\lfloor\frac{199}{10^2}\right\rfloor - \left\lfloor\frac{199}{14^2}\right\rfloor,$$

which is just $49 + 22 + 7 + 4 + 1 + 1 - 5 - 1 - 1 = 77$. Therefore there are $199 - 77 = 122$ positive integers less than 200 that are not divisible by the square of an integer greater than 1.

d) This is similar to part (c), with cubes in place of squares. Reasoning the same way, we get

$$199 - \left(\left\lfloor\frac{199}{2^3}\right\rfloor + \left\lfloor\frac{199}{3^3}\right\rfloor + \left\lfloor\frac{199}{5^3}\right\rfloor\right) = 199 - (24 + 7 + 1) = 167.$$

e) For each set of three prime numbers $\{p, q, r\}$, the number of positive integers less than 200 divisible by p, q, and r is given by $\lfloor 200/(pqr)\rfloor$. There is no overcounting to worry about in this problem, since no number less than 200 is divisible by four primes (the smallest such number is $2\cdot3\cdot5\cdot7 = 210$). Therefore the number of positive integers less than 200 divisible by three primes is the sum of $\lfloor 200/(pqr)\rfloor$ over all triples of distinct primes whose product is at most 200. A tedious listing shows that there are 19 such triples, and when we form the sum we get 31. Therefore there are $199 - 31 = 168$ positive integers less than 200 that are not divisible by three or more primes.

33. There are n ways to choose which person is to receive the correct hat, and there are D_{n-1} ways to have the remaining hats returned totally incorrectly (where D_{n-1} is the number of derangements of $n-1$ objects). On the other hand there are $n!$ possible ways to return the hats. Therefore the probability is $nD_{n-1}/n! = D_{n-1}/(n-1)!$. Note that this happens to be the same as the probability that none of $n-1$ people is given the correct hat; therefore it is approximately $1/e \approx 0.368$ for large n.

35. There are $2^6 = 64$ bit strings of length 6. We need to find the number that contain at least 4 1's. The number that contain exactly i 1's is $C(6,i)$, since such a string is determined by choosing i of the 6 positions to contain the 1's. Therefore there are $C(6,4)+C(6,5)+C(6,6) = C(6,2)+C(6,1)+C(6,0) = 15+6+1 = 22$ strings with at least 4 1's. Hence the probability in question is $22/64 = 11/32$.

WRITING PROJECTS FOR CHAPTER 7

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. Obviously you will need to find a translated version if you want to read what Fibonacci actually said. The search technique of gradually working your way backwards usually works: If you can't find what you want in the place you start (here, for example, maybe with a standard mathematics history textbook), then search the references provided by that work, then check the references in the references, and so on backwards.

2. Articles and books at all levels have dealt with this subject. You might find something in, say, *Scientific American* (which is indexed in hard-copy and electronic versions of *Readers' Guide*); you might find some articles in materials for high-school students (see, for example, *Mathematics Teacher*, a magazine for high school teachers); and just browsing through the mathematics section of a public library or popular bookstore might yield something on this topic. Talk to someone who teaches a "math for poets" course at your school (i.e., a course with almost no mathematical prerequisite that deals with appreciating the beauty or applications of mathematics); some of the textbooks for that kind of course have material on this topic, as well as references to other sources of information.

3. Paul Stockmeyer, a professor at the College of William and Mary, describes the Tower of Hanoi problem and its variations as his "main professional hobby." Consult his website (`http://www.cs.wm.edu/~pkstoc`) for some of his papers on the subject, as well as some great links.

4. There are many articles about the Catalan numbers, as well as treatments in textbooks. One article to start with might be [HiPe2].

5. Obviously, consult the reference mentioned in that exercise.

6. Andrzej Pelc has written several papers on this topic; search for his Web page.

7. Numerous websites discuss this, and several of them have working demos, as well. Search for the key words.

8. See [Ro3] for a brief introduction. An extensive discussion, as well as a long list of references, can be found in [Gu].

9. One book on sieve methods is [HaRi].

10. See the article [Da2]. There is also relevant material in the chapter on arrangements with forbidden positions in [MiRo].

11. A wonderful book on generating functions is [Wi2], and [GrKn] also has a lot of relevant material. There are sections on generating functions in [Gr2] as well as any of the advanced combinatorics books mentioned in the general suggestions at the back of this *Guide*.

12. See advanced combinatorics texts, such as [Ro1] or [Tu1]. For another writing project, find out about George Polyá, a fascinating figure in 20th century mathematics and mathematics education.

13. See [BoDo], which should also have pointers to historical sources.

14. Advanced combinatorics texts, such as [Br2] or [Ro1], discuss this topic.