SUPPLEMENTARY EXERCISES FOR CHAPTER 2

2. We are given that $A \subseteq B$. We want to prove that the power set of A is a subset of the power set of B, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 15 in Section 2.1.

4. a) $\mathbf{Z}$ b) $\emptyset$ c) $\mathbf{O}$ d) $\mathbf{E}$

6. If $A \subseteq B$, then every element in A is also in B, so clearly $A \cap B = A$. Conversely, if $A \cap B = A$, then every element of A must also be in $A \cap B$, and hence in B. Therefore $A \subseteq B$.

8. This identity is true, so we must show that every element in the left-hand side is also an element in the right-hand side and conversely. Let $x \in (A - B) - C$. Then $x \in A - B$ but $x \notin C$. This means that $x \in A$, but $x \notin B$ and $x \notin C$. Therefore $x \in A - C$, and therefore $x \in (A - C) - B$. The converse is proved in exactly the same way.

10. The inequality follows from the obvious fact that $A \cap B \subseteq A \cup B$. Equality can hold only if there are no elements in either A or B that are not in both A and B, and this can happen only if $A = B$.

12. Since $\overline{A} \cap \overline{B} = \overline{(A \cup B)}$, we are asked to show that $|\overline{(A \cup B)}| = |U| - (|A| + |B| - |A \cap B|)$. This follows immediately from the facts that $|\overline{X}| = |U| - |X|$ (which is clear from the definitions) and (see the discussion following Example 5 in Section 2.2) that $|A \cup B| = |A| + |B| - |A \cap B|$.

14. We showed in Exercise 36b in Section 2.3 that $f(S \cap T) \subseteq f(S) \cap f(T)$. Thus it remains to show the opposite inclusion, assuming that f is one-to-one. Suppose $y \in f(S) \cap f(T)$. Then $y = f(s)$ for some $s \in S$ and $y = f(t)$ for some $t \in T$. Since f is one-to-one, it must be that $s = t$. Thus f is the image of an element that lies in both S and T, so $y \in f(S \cap T)$.

16. **a)** We are given that f is one-to-one, and we must show that S_f is one-to-one. So suppose that $X_1 \neq X_2$, where these are subsets of A. We have to show that $S_f(X_1) \neq S_f(X_2)$. Without loss of generality there is an element $a \in X_1 - X_2$. This means that $f(a) \in S_f(X_1)$. If $f(a)$ were also an element of $S_f(X_2)$, then we would need an element $a' \in X_2$ such that $f(a') = f(a)$. But since f is one-to-one, this forces $a' = a$, which is impossible, because $a \notin X_2$. Therefore $f(a) \in S_f(X_1) - S_f(X_2)$, so $S_f(X_1) \neq S_f(X_2)$.
b) We are given that f is onto, and we must show that S_f is onto. So suppose that $Y \subseteq B$. We have to find $X \subseteq A$ such that $S_f(X) = Y$. Let $X = \{ x \in A \mid f(x) \in Y \}$. We claim that $S_f(X) = Y$. Clearly $S_f(X) \subseteq Y$. To see that $Y \subseteq S_f(X)$, suppose that $b \in Y$. Then because f is onto, there is some $a \in A$ such that $f(a) = b$. By our definition of X, $a \in X$. Therefore by definition $b \in S_f(X)$.
c) We are given that f is onto, and we must show that $S_{f^{-1}}$ is one-to-one. So suppose that $Y_1 \neq Y_2$, where these are subsets of B. We have to show that $S_{f^{-1}}(Y_1) \neq S_{f^{-1}}(Y_2)$. Without loss of generality there is an element $b \in Y_1 - Y_2$. Because f is onto, there is an $a \in A$ such that $f(a) = b$. Therefore $a \in S_{f^{-1}}(Y_1)$. But we also know that $a \notin S_{f^{-1}}(Y_2)$, because if a were an element of $S_{f^{-1}}(Y_2)$, then we would have $b = f(a) \in Y_2$, contrary to our choice of b. The existence of this a shows that $S_{f^{-1}}(Y_1) \neq S_{f^{-1}}(Y_2)$.
d) We are given that f is one-to-one, and we must show that $S_{f^{-1}}$ is onto. So suppose that $X \subseteq A$. We have to find $Y \subseteq B$ such that $S_{f^{-1}}(Y) = X$. Let $Y = S_f(X)$. In other words, $Y = \{ f(x) \mid x \in X \}$. We must show that $S_{f^{-1}}(Y) = X$, which means that we must show that $\{ u \in A \mid f(u) \in \{ f(x) \mid x \in X \} \} = X$ (we changed the dummy variable to u for clarity). That the right-hand side is a subset of the left-hand side is immediate, because if $u \in X$, then $f(u)$ is an $f(x)$ for some $x \in X$. Conversely, suppose that u is in the left-hand side. Thus $f(u) = f(x_0)$ for some $x_0 \in X$. But because f is one-to-one, we know that $u = x_0$; that is $u \in X$.
e) This follows immediately from the earlier parts, because to be a one-to-one correspondence means to be one-to-one and onto.

18. If n is even , then $n/2$ is an integer, so $\lceil n/2 \rceil + \lfloor n/2 \rfloor = (n/2) + (n/2) = n$. If n is odd, then $\lceil n/2 \rceil = (n+1)/2$ and $\lfloor n/2 \rfloor = (n-1)/2$, so again the sum is n.

20. This is certainly true if either x or y is an integer, since then this equation is equivalent to the identity (4b) in Table 1 of Section 2.3. Otherwise, write x and y in terms of their integer and fractional parts: $x = n + \epsilon$ and $y = m + \delta$, where $n = \lfloor x \rfloor$, $0 < \epsilon < 1$, $m = \lfloor y \rfloor$, and $0 < \delta < 1$. If $\delta + \epsilon > 1$, then the equation is true, since both sides equal $m + n + 2$; if $\delta + \epsilon \leq 1$, then the equation is false, since the left-hand side equals $m + n + 1$, but the right-hand side equals $m + n + 2$. To summarize: the equation is true if and only if either at least one of x and y is an integer or the sum of the fractional parts of x and y exceeds 1.

22. The values of the floor and ceiling function will depend on whether their arguments are integral or not. So there seem to be two cases here. First let us suppose that n is even. Then $n/2$ is an integer, and $n^2/4$ is also an integer, so the equation is a simple algebraic fact. The second case is harder. Suppose that n is odd, say $n = 2k + 1$. Then $n/2 = k + \frac{1}{2}$. Therefore the left-hand side gives us $k(k+1) = k^2 + k$, since we have to round down for the first factor and round up for the second. What about the right-hand side?

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$, so $n^2/4 = k^2 + k + \frac{1}{4}$. Therefore the floor function gives us $k^2 + k$, and the proof is completed.

24. Since we are dividing by 4, let us write $x = 4n - k$, where $0 \le k < 4$. In other words, write x in terms of how much it is less than the smallest multiple of 4 not less than it. There are three cases. If $k = 0$, then x is already a multiple of 4, so both sides equal n. If $0 < k \le 2$, then $\lfloor x/2 \rfloor = 2n - 1$, so the left-hand side is $\lfloor n - \frac{1}{2} \rfloor = n - 1$. Of course the right-hand side is $n - 1$ as well, so again the two sides agree. Finally, suppose that $2 < k < 4$. Then $\lfloor x/2 \rfloor = 2n - 2$, and the left-hand side is $\lfloor n - 1 \rfloor = n - 1$; of course the right-hand side is still $n - 1$, as well. Since we proved that the two sides are equal in all cases, the proof is complete.

26. If x is an integer, then of course the two sides are identical. So suppose that $x = k + \epsilon$, where k is an integer and ϵ is a real number with $0 < \epsilon < 1$. Then the values of the left-hand side, which is $\lfloor (k+n)/m \rfloor$, and the right-hand side, which is $\lfloor (k + n + \epsilon)/m \rfloor$, are the same, since adding a number strictly between 0 and 1 to the numerator of a fraction whose numerator and denominator are integers cannot cause the fraction to reach the next higher integer value (the numerator cannot reach the next multiple of m).

28. a) $1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69$

b) Suppose there were only a finite set of Ulam numbers, say $u_1 < u_2 < \cdots < u_n$. Then it is clear that $u_{n-1} + u_n$ can be written uniquely as the sum of two distinct Ulam numbers, so this is an Ulam number larger than u_n, a contradiction. Therefore there are an infinite number of Ulam numbers.

30. If we work at this long enough, we might notice that each term after the first three is the sum of the previous three terms. With this rule the next four terms will be 169, 311, 572, 1052. One way to use the power of technology here is to submit the given sequence to The On-Line Encyclopedia of Integer Sequences (`http://www.research.att.com/~njas/sequences/`).

32. If either A or B is empty, then $A \times B = \emptyset$, and we are finished. Otherwise, the hypothesis tells us that we can list these sets: $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. (If one or both of these sets are finite, then we can still form these lists by repeating elements infinitely often.) Now we just need a systematic way to put all the elements of $A \times B$, namely pairs of the form (a_i, b_j), into a sequence. We do this by listing first all the pairs in which $i + j = 2$ (there is only one such pair, (a_1, b_1)), then all the elements in which $i + j = 3$ (there are only two such pairs, (a_1, b_2) and (a_2, b_1)), and so on; except that we do not list any element that we have already listed. So, assuming that these elements are distinct, our list starts (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), (a_1, b_4), ... (If any of these terms duplicates a previous term, then it is simply omitted.) The result of this process will be either an infinite sequence or a finite sequence containing all the elements of $A \times B$. Thus $A \times B$ is countable. Compare this to Exercises 41–42 in Section 2.4.