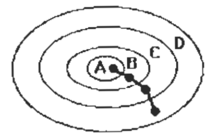
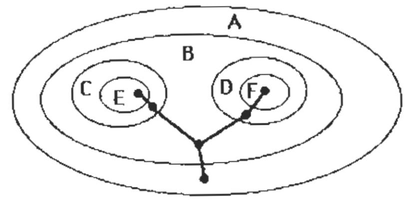
SECTION 9.8 Graph Coloring

2. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A and C, and the other for B and D.

4. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A, C, and D, and the other for B, E, and F.

6. Since there is a triangle, at least 3 colors are needed. To show that 3 colors suffice, notice that we can color the vertices around the outside alternately using red and blue, and color vertex g green.

8. Since there is a triangle, at least 3 colors are needed. The coloring in which b and c are blue, a and f are red, and d and e are green shows that 3 colors suffice.

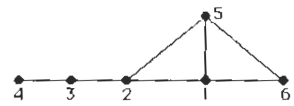
10. Since vertices b, c, h, and i form a K_4, at least 4 colors are required. A coloring using only 4 colors (and we can get this by trial and error, without much difficulty) is to let a and c be red; b, d, and f, blue; g and i, green; and e and h, yellow.

12. In Exercise 5 the chromatic number is 3, but if we remove vertex a, then the chromatic number will fall to 2. In Exercise 6 the chromatic number is 3, but if we remove vertex g, then the chromatic number will fall to 2. In Exercise 7 the chromatic number is 3, but if we remove vertex b, then the chromatic number will fall to 2. In Exercise 8 the chromatic number was shown to be 3. Even if we remove a vertex, at least one of the two triangles ace and bdf must remain, since they share no vertices. Therefore the smaller graph will still have chromatic number 3. In Exercise 9 the chromatic number is 2. Obviously it is not possible to reduce it to 1 by removing one vertex, since at least one edge will remain. In Exercise 10 the chromatic number was shown to be 4, and a coloring was provided. If we remove vertex h and recolor vertex e red, then we can eliminate color yellow from that solution. Therefore we will have reduced the chromatic number to 3. Finally, the graph in Exercise 11 will still have a triangle, no matter what vertex is removed, so we cannot lower its chromatic number below 3 by removing a vertex.

14. Since the map is planar, we know that four colors suffice. That four colors are necessary can be seen by looking at Kentucky. It is surrounded by Tennessee, Missouri, Illinois, Indiana, Ohio, West Virginia, and Virginia; furthermore the states in this list form a C_7, each one adjacent to the next. Therefore at least three colors are needed to color these seven states (see Exercise 16), and then a fourth is necessary for Kentucky.

16. Let the circuit be $v_1, v_2, \ldots, v_n, v_1$, where n is odd. Suppose that two colors (red and blue) sufficed to color the graph containing this circuit. Without loss of generality let the color of v_1 be red. Then v_2 must be blue, v_3 must be red, and so on, until finally v_n must be red (since n is odd). But this is a contradiction, since v_n is adjacent to v_1. Therefore at least three colors are needed.

18. We draw the graph in which two vertices (representing locations) are adjacent if the locations are within 150 miles of each other.



Clearly three colors are necessary and sufficient to color this graph, say red for vertices 4, 2, and 6; blue for 3 and 5; and yellow for 1. Thus three channels are necessary and sufficient.

20. We let the vertices of a graph be the animals, and we draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits. A coloring of this graph gives an assignment of habitats (the colors are the habitats).

22. **a)** See Section 20 of *Introduction to Graph Theory*, second edition, by Robin J. Wilson (Academic Press, 1979), for a proof that the edge chromatic number of K_n is $n-1$ if n is even, and n if n is odd. The proof is nontrivial.

b) Let t be the larger of m and n. Since all the vertices in one part have degree t, it is clear that at least t colors are needed. That t colors suffice is less clear. See the reference given in part **(a)** for a proof that they do.

c) If n is even, then clearly we can alternate 2 colors around the cycle, so the edge chromatic number is 2. If n is odd, then by the same reasoning the edge chromatic number cannot be 2 and yet is equal to 3.

d) The edges incident to the middle of the wheel must be painted different colors, so n colors are necessary. Furthermore, it is easy to find a free color for each edge along the rim, so n is sufficient as well.

24. Since each of the n vertices in this subgraph must have a different color, the chromatic number must be at least n.

26. Our pseudocode is as follows. The comments should explain how it implements the algorithm.

```
procedure coloring(G : simple graph)
{assume that the vertices are labeled 1, 2, ..., n so that
  deg(1) ≥ deg(2) ≥ ··· ≥ deg(n) }
for i := 1 to n
        c(i) := 0 {originally no vertices are colored}
count := 0 {no vertices colored yet}
color := 1 {try the first color}
while count < n {there are still vertices to be colored}
begin
        for i := 1 to n {try to color vertex i with color color }
                if c(i) = 0 {vertex i is not yet colored} then
                begin
                        c(i) := color {assume we can do it until we find out otherwise}
                        for j := 1 to n
                                if {i, j} is an edge and c(j) = color
                                        then c(i) := 0 {we found out otherwise}
                        if c(i) = color
                                then count := count + 1 {the new coloring of i worked}
                end
        color := color + 1 {we have to go on to the next color}
end {the coloring is complete}
```

28. We know that the chromatic number of an odd cycle is 3 (see Example 4). If we remove one edge, then we get a path, which clearly can be colored with two colors. This shows that the cycle is chromatically 3-critical.

30. Although the chromatic number of W_4 is 3, if we remove one edge then the graph still contains a triangle, so its chromatic number remains 3. Therefore W_4 is not chromatically 3-critical.

32. First let us prove some general results. In a complete graph, each vertex is adjacent to every other vertex, so each vertex must get its own set of k different colors. Therefore if there are n vertices, kn colors are clearly necessary and sufficient. Thus $\chi_k(K_n) = kn$. In a bipartite graph, every vertex in one part can get the same set of k colors, and every vertex in the other part can get the same set of k colors (a disjoint set from the colors assigned to the vertices in the first part). Therefore $2k$ colors are sufficient, and clearly $2k$ colors are required if there is at least one edge. Let us now look at the specific graphs.

a) For this complete graph situation we have $k = 2$ and $n = 3$, so $2 \cdot 3 = 6$ colors are necessary and sufficient.

b) As in part (a), the answer is kn, which here is $2 \cdot 4 = 8$.

c) Call the vertex in the middle of the wheel m, and call the vertices around the rim, in order, a, b, c, and d. Since m, a, and b form a triangle, we need at least 6 colors. Assign colors 1 and 2 to m, 3 and 4 to a, and 5 and 6 to b. Then we can also assign 3 and 4 to c, and 5 and 6 to d, completing a 2-tuple coloring with 6 colors. Therefore $\chi_2(W_4) = 6$.

d) First we show that 4 colors are not sufficient. If we had only colors 1 through 4, then as we went around the cycle we would have to assign, say, 1 and 2 to the first vertex, 3 and 4 to the second, 1 and 2 to the third, and 3 and 4 to the fourth. This gives us no colors for the final vertex. To see that 5 colors are sufficient, we simply give the coloring: In order around the cycle the colors are $\{1,2\}$, $\{3,4\}$, $\{1,5\}$, $\{2,4\}$, and $\{3,5\}$. Therefore $\chi_2(C_5) = 5$.

e) By our general result on bipartite graphs, the answer is $2k = 2 \cdot 2 = 4$.

f) By our general result on complete graphs, the answer is $kn = 3 \cdot 5 = 15$.

g) We claim that the answer is 8. To see that eight colors suffice, we can color the vertices as follows in order around the cycle: $\{1,2,3\}$, $\{4,5,6\}$, $\{1,2,7\}$, $\{3,6,8\}$, and $\{4,5,7\}$. Showing that seven colors are not sufficient is harder. Assume that a coloring with seven colors exists. Without loss of generality, color the first vertex $\{1,2,3\}$ and color the second vertex $\{4,5,6\}$. If the third vertex is colored $\{1,2,3\}$, then the fourth and fifth vertices would need to use six colors different from 1, 2, and 3, for a total of nine colors. Therefore without loss of generality, assume that the third vertex is colored $\{1,2,7\}$. But now the other two vertices cannot have colors 1 or 2, and they must have six different colors, so eight colors would be required in all. This is a contradiction, so there is in fact no coloring with just seven colors.

h) By our general result on bipartite graphs, the answer is $2k = 2 \cdot 3 = 6$.

34. As we observed in the solution to Exercise 34, the answer is $2k$ if G has at least one edge (and it is clearly k if G has no edges, since every vertex can get the same colors).

36. We use induction on the number of vertices of the graph. Every graph with six or fewer vertices can be colored with six or fewer colors, since each vertex can get a different color. That takes care of the basis case(s). So we assume that all graphs with k vertices can be 6-colored and consider a graph G with $k + 1$ vertices. By Corollary 2 in Section 8.7, G has a vertex v with degree at most 5. Remove v to form the graph G'. Since G' has only k vertices, we 6-color it by the inductive hypothesis. Now we can 6-color G by assigning to v a color not used by any of its five or fewer neighbors. This completes the inductive step, and the theorem is proved.

38. Clearly any convex polygon can be guarded by one guard, because every vertex sees all points on or inside the polygon. This takes care of triangles and convex quadrilaterals ($n = 3$ and some of $n = 4$). It is also clear that for a nonconvex quadrilateral, a guard placed at the vertex with the reflex angle can see all points on or inside the polygon. This completes the proof that $g(3) = g(4) = 1$.

40. By Lemma 1 in Section 4.2 every hexagon has an interior diagonal, which will divide the hexagon into two polygons, each with fewer than six sides (either two quadrilaterals or one triangle and one pentagon). By Exercises 38 and 39, one guard suffices for each, so $g(6) \leq 2$. By Exercise 41, we also know that $g(6) \geq 2$. Therefore $g(6) = 2$.

42. By Theorem 1 in Section 4.2, we can triangulate the polygon. We claim that it is possible to color the vertices of the triangulated polygon using three colors so that no two adjacent vertices have the same color. We prove this by induction. The basis step ($n = 3$) is trivial. Assume the inductive hypothesis that every triangulated polygon with k vertices can be 3-colored, and consider a triangulated polygon with $k+1$ vertices. By Exercise 23 in Section 4.2, one of the triangles in the triangulation has two sides that were sides of the original polygon. If we remove those two sides and their common vertex, the result is a triangulated polygon with k vertices. By the inductive hypothesis, we can 3-color its vertices. Now put the removed edges and vertex back. The vertex is adjacent to only two other vertices, so we can extend the coloring to it by assigning it the color not used by those vertices. This completes the proof of our claim. Now some color must be used no more than $n/3$ times; if not, then every color would be used more than $n/3$ times, and that would account for more than $3 \cdot n/3 = n$ vertices. (This argument is in the spirit of the pigeonhole principle.) Say that red is the color used least in our coloring. Then there are at most $n/3$ vertices colored red, and since this is an integer, there are at most $\lfloor n/3 \rfloor$ vertices colored red. Put guards at all these vertices. Since each triangle must have its vertices colored with three different colors, there is a guard who can see all points on or in the interior of each triangle in the triangulation. But this is all the points on or in the interior of the polygon, and our proof is complete. Combining this with Exercise 41, we have proved that $g(n) = \lfloor n/3 \rfloor$.