SECTION 7.5 Inclusion–Exclusion

2. $|C \cup D| = |C| + |D| - |C \cap D| = 345 + 212 - 188 = 369$

4. $|M \cap S| = |M| + |S| - |M \cup S| = 650000 + 1250000 - 1450000 = 450{,}000$

6. a) In this case the union is just A_3, so the answer is $|A_3| = 10{,}000$.
   b) The cardinality of the union is the sum of the cardinalities in this case, so the answer is $100+1000+10000 = 11{,}100$.
   c) $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11{,}095$

8. $270 - 64 - 94 - 58 + 26 + 28 + 22 - 14 = 116$

10. $100 - \lfloor 100/5 \rfloor - \lfloor 100/7 \rfloor + \lfloor 100/(5 \cdot 7) \rfloor = 100 - 20 - 14 + 2 = 68$

12. There are $\lfloor \sqrt{1000} \rfloor = 31$ squares and $\lfloor \sqrt[3]{1000} \rfloor = 10$ cubes. Furthermore there are $\lfloor \sqrt[6]{1000} \rfloor = 3$ numbers that are both squares and cubes, i.e., sixth powers. Therefore the answer is $31 + 10 - 3 = 38$.

14. There are $26!$ strings in all. To count the strings that contain *fish*, we glue these four letters together as one and permute it and the 22 other letters, so there are $23!$ such strings. Similarly there are $24!$ strings that contain *rat* and $23!$ strings that contain *bird*. Furthermore, there are $21!$ strings that contain both *fish* and *rat* (glue each of these sets of letters together), but there are no strings that contain both *bird* and another of these strings. Therefore the answer is $26! - 23! - 24! - 23! + 21! \approx 4.0 \times 10^{26}$.

16. $4 \cdot 100 - 6 \cdot 50 + 4 \cdot 25 - 5 = 195$

18. There are $C(10,1) + C(10,2) + \cdots + C(10,10) = 2^{10} - C(10,0) = 1023$ terms on the right-hand side of the equation.

20. $5 \cdot 10000 - 10 \cdot 1000 + 10 \cdot 100 - 5 \cdot 10 + 1 = 40{,}951$

22. The base case is $n = 2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. Look at a situation with $n+1$ sets, and temporarily consider $A_n \cup A_{n+1}$ as one set. Then by the inductive hypothesis we have

$$\begin{aligned}|A_1 \cup \cdots \cup A_{n+1}| &= \sum_{i<n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n} |A_i \cap A_j| \\ &\quad - \sum_{i<n} |A_i \cap (A_n \cup A_{n+1})| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_{n-1} \cap (A_n \cup A_{n+1})|.\end{aligned}$$

Next we apply the distributive law to each term on the right involving $A_n \cup A_{n+1}$, giving us

$$\sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum |(A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1})|.$$

Now we apply the basis step to rewrite each of these terms as

$$\sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}|,$$

which gives us precisely the summation we want.

24. Let E_1, E_2, and E_3 be these three events, in the order given. Then $p(E_1) = C(5,3)/2^5 = 10/32$; $p(E_2) = 2^3/2^5 = 8/32$; and $p(E_3) = 2^3/2^5 = 8/32$. Furthermore $p(E_1 \cap E_2) = C(3,1)/2^5 = 3/32$; $p(E_1 \cap E_3) = 1/32$; and $p(E_2 \cap E_3) = 2/32$. Finally $p(E_1 \cap E_2 \cap E_3) = 1/32$. Therefore the probability that at least one of these events occurs is $(10 + 8 + 8 - 3 - 1 - 2 + 1)/32 = 21/32$.

26. We only need to list the terms that have one or two events in them. Thus we have

$$p(E_1 \cup E_2 \cup E_3 \cup E_4) = \sum_{1 \le i \le 4} p(E_i) - \sum_{1 \le i < j \le 4} p(E_i \cap E_j),$$

or, explicitly, $p(E_1 \cup E_2 \cup E_3 \cup E_4) = p(E_1) + p(E_2) + p(E_3) + p(E_4) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_3 \cap E_4)$.

28. The probability of the union, in this case, is the sum of the probabilities of the events:

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} p(E_i) = p(E_1) + p(E_2) + \cdots + p(E_n)$$