GUIDE TO REVIEW QUESTIONS FOR CHAPTER 4

1. a) no

b) Sometimes yes. If the given formula is correct, then it is often possible to prove it using the principle of mathematical induction (although it would be wishful thinking to believe that *every* such true formnla could be so proved). If the formnla is incorrect, then induction would not work, of course; thus an incorrect formula could not be shown to be incorrect nsing the principle.

c) See Exercise 9 in Section 4.1.

2. a) $n \geq 7$

b) For the basis step we just check that $11 \cdot 7 + 17 \leq 2^7$. Fix $n \geq 7$, and assume the inductive hypothesis, that $11n+17 \leq 2^n$. Then $11(n+1)+17 = (11n+17)+11 \leq 2^n + 11 < 2^n + 2^n = 2^{n+1}$. The strict inequality here follows from tbe fact that $n \geq 4$.

3. a) Carefully considering all the possibilities shows that the amounts of postage less than 32 cents that can be achieved are 0, 5, 9, 10, 14, 15, 18, 19, 20, 23, 24, 25, 27, 28, 29, and 30. All amonnts greater than or equal to 32 cents can be achieved.

b) To prove tbis latter statement, we check the basis step by noting that $32 = 9+9+9+5$. Assume that we can achieve n cents, and consider $n+1$ cents, where $n \geq 32$. If the stamps used for n cents included a 9-cent stamp, then replacing it by two 5-cent stamps gives us $n+1$ cents, as desired. Otherwise only 5-cent stamps were used to achieve n cents, and since $n > 30$, there must be at least seven such stamps. Replace seven of the 5-cent stamps by four 9-cent stamps; this increases the amount of postage by $4 \cdot 9 - 7 \cdot 5 = 1$ cent, again as desired.

c) We check the base cases $32 = 3 \cdot 9 + 5$, $33 = 2 \cdot 9 + 3 \cdot 5$, $34 = 9 + 5 \cdot 5$, $35 = 7 \cdot 5$, and $36 = 4 \cdot 9$. Fix $n \geq 37$ and assume that all amounts from 32 to $n-1$ can be achieved. To achieve n cents postage, take the stamps used for $n-5$ cents (since $n \geq 37$, $n-5 \geq 32$, so the inductive hypothesis applies) and adjoin a 5-cent stamp.

d) Let n be an integer greater than or equal to 32. We want to express n as a sum of a nonnegative multiple of 5 and a nonnegative multiple of 9. Divide n by 5 to obtain a quotient q and remainder r such that $n = 5q + r$ and $0 \leq r \leq 4$. Note that since $n \geq 32$, $q \geq 6$. If $r = 0$, then we already have n expressed in the desired form. If $r = 1$, then $n \geq 36$, so $q \geq 7$; thus we can write $n = 5q + 1 = 5(q-7) + 4 \cdot 9$ to get the desired decomposition. If $r = 2$, then we rewrite $n = 5q + 2 = 5(q-5) + 3 \cdot 9$. If $r = 3$, then we rewrite $n = 5q + 3 = 5(q-3) + 2 \cdot 9$. And if $r = 4$, then we rewrite $n = 5q + 4 = 5(q-1) + 9$. In each case we have the desired sum.

4. See Examples 2 and 3 in Section 4.2.

5. a) See p. 278 and Appendix 1 (Axiom 4 for the positive integers).

b) Let S be the set of positive integers that cannot be written as the product of primes. If $S \neq \emptyset$, then S has a least element, c. Clearly $c \neq 1$, since 1 is the product of no primes. Thus c is greater than 1. Now c cannot be prime, since as such it would already be written as the product of primes (namely itself). Therefore c is a composite number, say $c = ab$, where a and b are both positive integers less than c. Since c is the smallest element of S, neither a nor b is in S. Therefore both a and b *can* be written as the product of primes. But multiplying these products together patently shows that c is the product of primes. This is a contradiction to the choice of c. Therefore our assumption that $S \neq \emptyset$ was wrong, and the theorem is proved.

6. a) See Exercise 56 in Section 4.3. b) $f(1) = 2$, and $f(n) = (n+1)f(n-1)$ for all $n \geq 2$

7. a) See p. 297. b) See Example 6 in Section 4.3.

8. a) See Exercise 57 in Section 4.3. b) $a_n = 3 \cdot 2^{n-3}$

9. See Examples 10 and 11 in Section 4.3.

10. a) See Example 9 in Section 4.3. b) See Example 14 in Section 4.3.

11. a) See p. 311.

b) Call the sequence a_1, a_2, $\ldots$, a_n. If $n = 1$, then the $sum(a_1) = a_1$. Otherwise $sum(a_1, a_2, \ldots, a_n) = a_n + sum(a_1, a_2, \ldots, a_{n-1})$.

12. See Example 4 in Section 4.4.

13. a) See p. 317.

b) We split the list into the two halves: $4, 10, 1, 5, 3$ and $8, 7, 2, 6, 9$. We then merge sort each half by applying this algorithm recursively and merging the results. For the first half, for example, this means splitting $4, 10, 1, 5, 3$ into the two halves $4, 10, 1$ and $5, 3$, recursively sorting each half, and merging. For the second half of this, for example, it means splitting into 5 and 3, recursively sorting each half, and merging. Since these two halves are already sorted, we just merge, into the sorted list $3, 5$. Similarly, we will get $1, 4, 10$ for the result of merge sort applied to $4, 10, 1$. When we merge $1, 4, 10$ and $3, 5$, we get $1, 3, 4, 5, 10$. Finally, we merge this with the sorted second half, $2, 6, 7, 8, 9$, to obtain the completely sorted list $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

c) $O(n \log n)$; see pp. 319–321

14. a) no

b) No—you also need to show that it halts for all inputs, and the initial and final assertions for which you provide a proof of partial correctness need to be appropriate ones (i.e., relevant to the question of whether the program produces the correct output).

15. See the rules displayed in Section 4.5.

16. See p. 326.

SUPPLEMENTARY EXERCISES FOR CHAPTER 4

1. Let $P(n)$ be the statement that this equation holds. The basis step consists of verifying that $P(1)$ is true, which is trivial because $2/3 = 1 - (1/3^1)$. For the inductive step we assume that $P(k)$ is true and try to prove $P(k+1)$. We have

$$\begin{aligned}\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^n}+\frac{2}{3^{n+1}} &= 1-\frac{1}{3^n}+\frac{2}{3^{n+1}} \quad \text{(by the inductive hypothesis)}\\ &= 1-\frac{3}{3^{n+1}}+\frac{2}{3^{n+1}}\\ &= 1-\frac{1}{3^{n+1}},\end{aligned}$$

as desired.

3. We prove this by induction on n. If $n = 1$ (basis step), then the equation reads $1\cdot 2^0 = (1-1)\cdot 2^1 + 1$, which is the true statement $1 = 1$. Assume that the statement is true for n:

$$1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \cdots + n\cdot 2^{n-1} = (n-1)\cdot 2^n + 1$$

We must show that it is true for $n+1$. Thus we have

$$\begin{aligned}&1\cdot 2^0 + 2\cdot 2^1 + 3\cdot 2^2 + \cdots + n\cdot 2^{n-1} + (n+1)\cdot 2^n\\ &\quad = (n-1)\cdot 2^n + 1 + (n+1)\cdot 2^n \quad \text{(by the inductive hypothesis)}\\ &\quad = (2n)\cdot 2^n + 1\\ &\quad = n\cdot 2^{n+1} + 1\\ &\quad = \big((n+1)-1\big)\cdot 2^{n+1} + 1,\end{aligned}$$

exactly as desired.

5. We prove this by induction on n. If $n = 1$ (basis step), then the equation reads $1/(1\cdot 4) = 1/4$, which is true. Assume that the statement is true for n:

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

We must show that it is true for $n+1$. Thus we have

$$\begin{aligned}&\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\cdots+\frac{1}{(3n-2)(3n+1)}+\frac{1}{\big(3(n+1)-2\big)\big(3(n+1)+1\big)}\\ &\quad = \frac{n}{3n+1}+\frac{1}{\big(3(n+1)-2\big)\big(3(n+1)+1\big)} \quad \text{(by the inductive hypothesis)}\\ &\quad = \frac{n}{3n+1}+\frac{1}{(3n+1)(3n+4)}\\ &\quad = \frac{1}{3n+1}\left(n+\frac{1}{3n+4}\right)\\ &\quad = \frac{1}{3n+1}\left(\frac{3n^2+4n+1}{3n+4}\right)\\ &\quad = \frac{1}{3n+1}\left(\frac{(3n+1)(n+1)}{3n+4}\right)\\ &\quad = \frac{n+1}{3n+4}\\ &\quad = \frac{n+1}{3(n+1)+1},\end{aligned}$$

exactly as desired.

7. Let $P(n)$ be the statement $2^n > n^3$. We want to prove that $P(n)$ is true for all $n > 9$. The basis step is $n = 10$, in which we have $2^{10} = 1024 > 1000 = 10^3$. Assume $P(n)$; we want to show $P(n+1)$. Then we have

$$\begin{aligned}(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\ &\le n^3 + 3n^2 + 3n^2 + 3n^2 \quad (\text{since } n \ge 1) \\ &= n^3 + 9n^2 \\ &< n^3 + n^3 \quad (\text{since } n > 9) \\ &= 2n^3 < 2 \cdot 2^n \quad (\text{by the inductive hypothesis}) \\ &= 2^{n+1},\end{aligned}$$

as desired.

9. This problem deals with factors in algebra. We have to be just a little clever. Let $P(n)$ be the statement that $a - b$ is a factor of $a^n - b^n$. We want to show that $P(n)$ is true for all positive integers n, and of course we will do so by induction. If $n = 1$, then we have the trivial statement that $a - b$ is a factor of $a - b$. Next assume the inductive hypothesis, that $P(n)$ is true. We want to show $P(n+1)$, that $a - b$ is a factor of $a^{n+1} - b^{n+1}$. The trick is to rewrite $a^{n+1} - b^{n+1}$ by subtracting and adding ab^n (and hence not changing its value). We obtain $a^{n+1} - b^{n+1} = a^{n+1} - ab^n + ab^n - b^{n+1} = a(a^n - b^n) + b^n(a - b)$. Now this expression contains two terms. By the inductive hypothesis, $a - b$ is a factor of the first term. Obviously $a - b$ is a factor of the second. Therefore $a - b$ is a factor of the entire expression, and we are done.

11. Let $P(n)$ be the given equation. It is certainly true for $n = 0$, since it reads $a = a$ in that case. Assume that $P(n)$ is true:

$$a + (a+d) + \cdots + (a+nd) = \frac{(n+1)(2a+nd)}{2}$$

Then

$$\begin{aligned}&a + (a+d) + \cdots + (a+nd) + \big(a + (n+1)d\big) \\ &\quad = \frac{(n+1)(2a+nd)}{2} + \big(a + (n+1)d\big) \quad (\text{by the inductive hypothesis}) \\ &\quad = \frac{(n+1)(2a+nd) + 2\big(a + (n+1)d\big)}{2} \\ &\quad = \frac{(n+1)(2a+nd) + 2a + nd + nd + 2d}{2} \\ &\quad = \frac{(n+2)(2a+nd) + (n+2)d}{2} \\ &\quad = \frac{(n+2)\big(2a + (n+1)d\big)}{2},\end{aligned}$$

which is exactly $P(n+1)$.

13. We use induction. If $n = 1$, then the left-hand side has just one term, namely $5/6$, and the right-hand side is $10/12$, which is the same number. Assume that the equation holds for $n = k$, and consider $n = k + 1$. Then we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i+4}{i(i+1)(i+2)} &= \sum_{i=1}^{k} \frac{i+4}{i(i+1)(i+2)} + \frac{k+5}{(k+1)(k+2)(k+3)} \\
&= \frac{k(3k+7)}{2(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)} \qquad \text{(by the inductive hypothesis)} \\
&= \frac{1}{(k+1)(k+2)} \cdot \left(\frac{k(3k+7)}{2} + \frac{k+5}{k+3} \right) \\
&= \frac{1}{2(k+1)(k+2)(k+3)} \cdot (k(3k+7)(k+3) + 2(k+5)) \\
&= \frac{1}{2(k+1)(k+2)(k+3)} \cdot (3k^3 + 16k^2 + 23k + 10) \\
&= \frac{1}{2(k+1)(k+2)(k+3)} \cdot (3k+10)(k+1)^2 \\
&= \frac{1}{2(k+2)(k+3)} \cdot (3k+10)(k+1) \\
&= \frac{(k+1)(3(k+1)+7)}{2((k+1)+1)((k+1)+2)},
\end{aligned}$$

as desired.

15. When $n = 1$, we are looking for the derivative of $g(x) = xe^x$, which, by the product rule, is $x \cdot e^x + e^x = (x+1)e^x$, so the statement is true for $n = 1$. Assume that the statement is true for $n = k$, that is, the k^{th} derivative is given by $g^{(k)} = (x+k)e^x$. Differentiating by the product rule gives us the $(k+1)^{\text{st}}$ derivative: $g^{(k+1)} = (x+k)e^x + e^x = (x+(k+1))e^x$, as desired.

17. We look at the first few Fibonacci numbers to see if there is a pattern: $f_0 = 0$ (even), $f_1 = 1$ (odd), $f_2 = 1$ (odd), $f_3 = 2$ (even), $f_4 = 3$ (odd), $f_5 = 5$ (odd), ... The pattern seems to be even-odd-odd, repeated forever. Since the pattern has period 3, we can formulate our conjecture as follows: f_n is even if $n \equiv 0 \pmod 3$, and is odd in the other two cases. Let us prove this by mathematical induction. There are two base cases, $n = 0$ and $n = 1$. The conjecture is certainly true in each of them, since $0 \equiv 0 \pmod 3$ and f_0 is even, and $1 \not\equiv 0 \pmod 3$ and f_0 is odd. So we assume the inductive hypothesis and consider a given $n+1$. There are three cases to consider, depending on the value of $(n+1) \textbf{ mod } 3$. If $n+1 \equiv 0 \pmod 3$, then $n-1$ and n are congruent to 1 and 2 modulo 3, respectively. By the inductive hypothesis, both f_{n-1} and f_n are odd. Therefore f_{n+1}, which is the sum of these two numbers, is even, as desired. The other two cases are similar. If $n+1 \equiv 1 \pmod 3$, then $n-1$ and n are congruent to 2 and 0 modulo 3, respectively. By the inductive hypothesis, f_{n-1} is odd and f_n is even. Therefore f_{n+1}, which is the sum of these two numbers, is odd, as desired. On the other hand, if $n+1 \equiv 2 \pmod 3$, then $n-1$ and n are congruent to 0 and 1 modulo 3, respectively. By the inductive hypothesis, f_{n-1} is even and f_n is odd. Therefore f_{n+1}, which is the sum of these two numbers, is odd, as desired.

19. The important point to note here is that k can be thought of as a universally quantified variable for each n. Thus the statement we wish to prove is $P(n)$: for every k, $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$. We use mathematical induction. If $n = 0$ (the first base case), then we want to prove $P(0)$: for every k, $f_k f_0 + f_{k+1} f_1 = f_{0+k+1}$, which reduces to the identity $f_{k+1} = f_{k+1}$, since $f_0 = 0$ and $f_1 = 1$. If $n = 1$ (the second base case), then we want to prove $P(1)$: for every k, $f_k f_1 + f_{k+1} f_2 = f_{1+k+1}$, which reduces to the defining recurrence $f_k + f_{k+1} = f_{k+2}$, since $f_1 = 1$ and $f_2 = 1$. Now we assume the inductive hypothesis $P(n)$ and try to prove $P(n+1)$. It is a straightforward calculation, using the inductive hypothesis and the recursive definition of the

Fibonacci numbers:

$$\begin{aligned} f_k f_{n+1} + f_{k+1} f_{n+2} &= f_k(f_{n-1} + f_n) + f_{k+1}(f_n + f_{n+1}) \\ &= f_k f_{n-1} + f_k f_n + f_{k+1} f_n + f_{k+1} f_{n+1} \\ &= (f_k f_{n-1} + f_{k+1} f_n) + (f_k f_n + f_{k+1} f_{n+1}) \\ &= f_{n-1+k+1} + f_{n+k+1} = f_{n+k+2}, \end{aligned}$$

as desired.

21. Let $P(n)$ be the statement $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$. We easily verify the two base cases, $P(0)$ and $P(1)$, since $2^2 = 2 \cdot 1 + 2$ and $2^2 + 1^2 = 1 \cdot 3 + 2$. Next assume the inductive hypothesis and consider $P(n+1)$. We have

$$\begin{aligned} l_0^2 + l_1^2 + \cdots + l_n^2 + l_{n+1}^2 &= l_n l_{n+1} + 2 + l_{n+1}^2 \\ &= l_{n+1}(l_n + l_{n+1}) + 2 \\ &= l_{n+1} l_{n+2} + 2, \end{aligned}$$

which is exactly what we wanted.

23. The identity is clearly true for $n = 1$. Let us expand the right-hand side for $n+1$, invoking the inductive hypothesis at the appropriate point (and using the suggested trigonometric identities as well as the fact that $i^2 = -1$):

$$\begin{aligned} \cos(n+1)x + i\sin(n+1)x &= \cos(nx + x) + i\sin(nx + x) \\ &= \cos nx \cos x - \sin nx \sin x + i(\sin nx \cos x + \cos nx \sin x) \\ &= \cos x(\cos nx + i \sin nx) + \sin x(-\sin nx + i\cos nx) \\ &= \cos x(\cos nx + i \sin nx) + i \sin x(i \sin nx + \cos nx) \\ &= (\cos nx + i \sin nx)(\cos x + i \sin x) \\ &= (\cos x + i \sin x)^n (\cos x + i \sin x) \\ &= (\cos x + i \sin x)^{n+1} \end{aligned}$$

25. First let's rewrite the right-hand side to make it simpler to work with, namely as $2^{n+1}(n^2 - 2n + 3) - 6$. We use induction. If $n = 1$, then the left-hand side has just one term, namely 2, and the right-hand side is $4 \cdot 2 - 6 = 2$ as well. Assume that the equation holds for $n = k$, and consider $n = k+1$. Then we have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 2^j &= \sum_{j=1}^{k} j^2 2^j + (k+1)^2 2^{k+1} \\ &= 2^{k+1}(k^2 - 2k + 3) - 6 + (k^2 + 2k + 1)2^{k+1} \quad \text{(by the inductive hypothesis)} \\ &= 2^{k+1}(2k^2 + 4) - 6 \\ &= 2^{k+2}(k^2 + 2) - 6 \\ &= 2^{k+2}((k+1)^2 - 2(k+1) + 3) - 6, \end{aligned}$$

as desired.

27. One solution here is to use partial fractions and telescoping. First note that

$$\frac{1}{j^2 - 1} = \frac{1}{2}\left(\frac{1}{j-1} - \frac{1}{j+1}\right).$$

Therefore when summing from 1 to n, the terms being added and the terms being subtracted all cancel out except for $1/(j-1)$ when $j = 2$ and 3, and $1/(j+1)$ when $j = n-1$ and n. Thus the sum is just

$$\frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right).$$

which simplifies, with a little algebra, to the expression given on the right-hand side of the formula in the exercise.

In the spirit of this chapter, however, we also give a proof by mathematical induction. Let $P(n)$ be the formula in the exercise. The basis step is for $n = 2$, in which case both sides reduce to $1/3$. For the inductive step assume that the equation holds for $n = k$, and consider $n = k + 1$. Then we have

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j^2-1} &= \sum_{j=1}^{k} \frac{1}{j^2-1} + \frac{1}{(k+1)^2-1} \\
&= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{(k+1)^2-1} \quad \text{(by the inductive hypothesis)} \\
&= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{k^2+2k} = \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{k(k+2)} \\
&= \frac{(k-1)(3k+2)(k+2)+4(k+1)}{4k(k+1)(k+2)} \\
&= \frac{3k^3+5k^2}{4k(k+1)(k+2)} = \frac{3k^2+5k}{4(k+1)(k+2)} \\
&= \frac{k(3k+5)}{4(k+1)(k+2)} = \frac{((k+1)-1)(3(k+1)+2)}{4(k+1)(k+2)},
\end{aligned}$$

which is exactly what $P(k+1)$ asserts.

29. Let $P(n)$ be the assertion that at least $n+1$ lines are needed to cover the lattice points in the given triangular region. Clearly $P(0)$ is true, because we need at least one line to cover the one point at $(0,0)$. Assume the inductive hypothesis, that at least $k+1$ lines are needed to cover the lattice points with $x \geq 0$, $y \geq 0$, and $x + y \leq k$. Consider the triangle of lattice points defined by $x \geq 0$, $y \geq 0$, and $x + y \leq k+1$. Because this set includes the previous set, at least $k+1$ lines are required just to cover the smaller set (by the inductive hypothesis). By way of contradiction, assume that $k+1$ lines could cover this larger set as well. Then these lines must also cover the $k+2$ points on the line $x + y = k+1$, namely $(0, k+1)$, $(1, k)$, $(2, k-1)$, $\ldots$, $(k, 1)$, $(k+1, 0)$. But only the line $x + y = k+1$ itself can cover more than one of these points, because two distinct lines intersect in at most one point, and this line does nothing toward covering the lattice points in the smaller triangle. Therefore none of the $k+1$ lines that are needed to cover the lattice points in the smaller triangle can cover more than one of the points on the line $x + y = k+1$, and this leaves at least one point uncovered. Therefore our assumption that $k+1$ line could cover the larger set is wrong, and our proof is complete.

31. The basis step is the given statement defining $\mathbf{B}$. Assume the inductive hypothesis, that $\mathbf{B}^k = \mathbf{M}\mathbf{A}^k\mathbf{M}^{-1}$. We want to prove that $\mathbf{B}^{k+1} = \mathbf{M}\mathbf{A}^{k+1}\mathbf{M}^{-1}$. By definition $\mathbf{B}^{k+1} = \mathbf{B}\mathbf{B}^k = \mathbf{M}\mathbf{A}\mathbf{M}^{-1}\mathbf{B}^k = \mathbf{M}\mathbf{A}\mathbf{M}^{-1}\mathbf{M}\mathbf{A}^k\mathbf{M}^{-1}$ by the inductive hypothesis. But this simplifies, using rules for matrices, to $\mathbf{M}\mathbf{A}\mathbf{I}\mathbf{A}^k\mathbf{M}^{-1} = \mathbf{M}\mathbf{A}\mathbf{A}^k\mathbf{M}^{-1} = \mathbf{M}\mathbf{A}^{k+1}\mathbf{M}^{-1}$, as desired.

33. It takes some luck to be led to the solution here. We see that we can write $3! = 3+2+1$. We also have a recursive definition of factorial, that $(n+1)! = (n+1)n!$, so we might hope to multiply each of the divisors we got at the previous stage by $n+1$ to get divisors at this stage. Thus we would have $4! = 4 \cdot 3! = 4(3+2+1) = 12+8+4$, but that gives us only three divisors in the sum, and we want four. That last divisor, which is $n+1$ can, however, be rewritten as the sum of n and 1, so our sum for $4!$ is $12+8+3+1$. Let's see if we can continue this. We have $5! = 5 \cdot 4! = 5(12+8+3+1) = 50+40+15+5 = 50+40+15+4+1$. It seems to be working. The basis step $n = 3$ is already done, so let's see if we can prove the inductive step. Assume that we can write $k!$ as a sum of the desired form, say $k! = a_1 + a_2 + \cdots + a_k$, where each a_i is a divisor of $k!$ and the divisors are listed in strictly decreasing order, and consider $(k+1)!$. Then we have $(k+1)! = (k+1)k! =$

$(k+1)(a_1+a_2+\cdots+a_k) = (k+1)a_1+(k+1)a_2+\cdots+(k+1)a_k = (k+1)a_1+(k+1)a_2+\cdots+k\cdot a_k+a_k$. Because each a_i was a divisor of $k!$, each $(k+1)a_i$ is a divisor of $(k+1)!$, but what about those last two terms? We don't seem to have any way to know that $k\cdot a_k$ is a factor of $(k+1)!$. Hold on, in our exploration we always had the last divisor in our sum being 1. If so, then $k\cdot a_k = k$, which is a divisor of $(k+1)!$, and $a_k = 1$, so the new last summand is again 1. (Notice also that our list of summands is still in strictly decreasing order.) So our proof by mathematical induction needs to be of the following stronger result: For every $n \geq 3$, we can write $n!$ as the sum of n of its distinct positive divisors, one of which is 1. The argument we have just given proves this by mathematical induction.

35. When $n = 1$ the statement is vacuously true. If $n = 2$ there must be a woman first and a man second, so the statement is true. Assume that the statement is true for $n = k$, where $k \geq 2$, and consider $k+1$ people standing in a line, with a woman first and a man last. If the k^{th} person is a woman, then we have that woman standing in front of the man at the end, and we are done. If the k^{th} person is a man, then the first k people in line satisfy the conditions of the inductive hypothesis for the first k people in line, so again we can conclnde that there is a woman directly in front of a man somewhere in the line.

37. (It will be helpful for the reader to draw a diagram to help in following this proof.) When $n = 1$ there is one circle, and we can color the inside blue and the outside red to satisfy the conditions. Assume the indnctive hypothesis that if there are k circles, then the regions can be 2-colored such that no regions with a common boundary have the same color, and consider a situation with $k+1$ circles. Remove one of the circles, producing a picture with k circles, and invoke the inductive hypothesis to color it in the prescribed manner. Then replace the removed circle and change the color of every region inside this circle (from red to blue, and from blue to red). It is clear that the resulting figure satisfies the condition, since if two regions have a common boundary, then either that boundary was an arc of the new circle, in which case the regions on either side used to be the same region and now the inside portion is colored differently from the outside, or else the boundary did not involve the new circle, in which case the regions are colored differently because they were colored differently before the new circle was restored.

39. We use induction. If $n = 1$ then the equation reads $1\cdot 1 = 1\cdot 2/2$, which is trne. Assume that the equation is true for n and consider it for $n+1$. (We use the letter n rather than k, because k is used for something else here.) Then we have, with some messy algebra,

$$\begin{aligned}
\sum_{j=1}^{n+1}(2j-1)\left(\sum_{k=j}^{n+1}\frac{1}{k}\right) &= \sum_{j=1}^{n}(2j-1)\left(\sum_{k=j}^{n+1}\frac{1}{k}\right) + (2(n+1)-1)\cdot\frac{1}{n+1} \\
&= \sum_{j=1}^{n}(2j-1)\left(\frac{1}{n+1}+\sum_{k=j}^{n}\frac{1}{k}\right) + \frac{2n+1}{n+1} \\
&= \left(\frac{1}{n+1}\sum_{j=1}^{n}(2j-1)\right) + \left(\sum_{j=1}^{n}(2j-1)\sum_{k=j}^{n}\frac{1}{k}\right) + \frac{2n+1}{n+1} \\
&= \left(\frac{1}{n+1}\cdot n^2\right) + \frac{n(n+1)}{2} + \frac{2n+1}{n+1} \quad \text{(by the inductive hypothesis)} \\
&= \frac{2n^2+n(n+1)^2+(4n+2)}{2(n+1)} \\
&= \frac{2(n+1)^2+n(n+1)^2}{2(n+1)} \\
&= \frac{(n+1)(n+2)}{2},
\end{aligned}$$

as desired.

41. a) $M(102) = 102 - 10 = 92$ b) $M(101) = 101 - 10 = 91$
c) $M(99) = M(M(99+11)) = M(M(110)) = M(100) = M(M(111)) = M(101) = 91$
d) $M(97) = M(M(108)) = M(98) = M(M(109)) = M(99) = 91$ (using part **(c)**)
e) This one is too long to show in its entirety here, but here is what is involved. First, $M(87) = M(M(98)) = M(91)$, using part **(d)**. Then $M(91) = M(M(102)) = M(92)$ from part **(a)**. In a similar way, we find that $M(92) = M(93)$, and so on, until it equals $M(97)$, which we found in part **(d)** to be 91. Hence the answer is 91.
f) Using what we learned from part **(e)**, we have $M(76) = M(M(87)) = M(91) = 91$.

43. The basis step is wrong. The statement makes no sense for $n = 1$, since the last term on the left-hand side would then be $1/(0 \cdot 1)$, which is undefined. The first n for which it makes sense is $n = 2$, when it reads
$$\frac{1}{1 \cdot 2} = \frac{3}{2} - \frac{1}{2}.$$
Of course this statement is false, since $\frac{1}{2} \neq 1$. Therefore the basis step fails, and so the "theorem" is not true.

45. We will prove by induction that n circles divide the plane into $n^2 - n + 2$ regions. One circle certainly divides the plane into two regions (the inside and the outside), and $1^2 - 1 + 2 = 2$. Thus the statement is correct for $n = 1$. We assume that the statement is true for n circles, and consider it for $n+1$ circles. Let us imagine an arrangement of $n+1$ circles in the plane, each pair intersecting in exactly two points, no point common to three circles. If we remove one circle, then we are left with n circles, and by the inductive hypothesis they divide the plane into $n^2 - n + 2$ regions. Now let us draw the circle that we removed, starting at a point at which it intersects another circle. As we proceed around the circle, every time we encounter a point on one of the circles that was already there, we cut off a new region (in other words, we divide one old region into two). Therefore the number of regions that are added on account of this circle is equal to the number of points of intersection of this circle with the other n circles. We are told that each other circle intersects this one in exactly two points. Therefore there are a total of $2n$ points of intersection, and hence $2n$ new regions. Therefore the number of regions determined by $n+1$ circles is $n^2 - n + 2 + 2n = n^2 + n + 2 = (n+1)^2 - (n+1) + 2$ (the last equality is just algebra). Thus we have derived that the statement is also true for $n+1$, and our proof is complete.

47. We will give a proof by contradiction. Let us consider the set $B = \{\, b\sqrt{2} \mid b \text{ and } b\sqrt{2} \text{ are positive integers} \,\}$. Clearly B is a subset of the set of positive integers. Now if $\sqrt{2}$ is rational, say $\sqrt{2} = p/q$, then $B \neq \emptyset$, since $q\sqrt{2} = p \in B$. Therefore by the well-ordering property, B contains a smallest element, say $a = b\sqrt{2}$. Then $a\sqrt{2} - a = a\sqrt{2} - b\sqrt{2} = (a-b)\sqrt{2}$. Since $a\sqrt{2} = 2b$ and a are both integers, so is this quantity. Furthermore, it is a positive integer, since it equals $a(\sqrt{2}-1)$ and $\sqrt{2} - 1 > 0$. Therefore $a\sqrt{2} - a \in B$. But clearly $a\sqrt{2} - a < a$, since $\sqrt{2} < 2$. This contradicts our choice of a to be the smallest element of B. Therefore our original assumption that $\sqrt{2}$ is rational is false.

49. a) We use the following lemma: A positive integer d is a common divisor of $a_1, a_2, \ldots, a_n$ if and only if d is a divisor of $\gcd(a_1, a_2, \ldots, a_n)$. [Proof: The prime factorization of $\gcd(a_1, a_2, \ldots, a_n)$ is $\prod p_i^{e_i}$, where e_i is the minimum exponent of p_i among $a_1, a_2, \ldots, a_n$. Clearly d divides every a_j if and only if the exponent of p_i in the prime factorization of d is less than or equal to e_i for every i, which happens if and only if $d \mid \gcd(a_1, a_2, \ldots, a_n)$.] Now let $d = \gcd(a_1, a_2, \ldots, a_n)$. Then d must be a divisor of each a_i, and hence must be a divisor of $\gcd(a_{n-1}, a_n)$ as well. Therefore d is a common divisor of $a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)$. To show that it is the greatest common divisor of these numbers, suppose that e is any common divisor of these numbers. Then e is a divisor of each a_i for $1 \le i \le n-2$, and, being a divisor of $\gcd(a_{n-1}, a_n)$, it is also a divisor of a_{n-1} and a_n. Therefore e is a common divisor of *all* the a_i and hence a divisor of their common divisor, d. This shows that d is the greatest common divisor of $a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)$.

b) If $n = 2$, then we just apply the Euclidean algorithm to a_1 and a_2. Otherwise, we apply the Euclidean algorithm to a_{n-1} and a_n, obtaining an answer d, and then apply this algorithm recursively to $a_1, a_2, \ldots, a_{n-2}, d$. Note that this last sequence has only $n-1$ numbers in it.

51. We begin by computing $f(n)$ for the first few values of n, using the recursive definition. Thus we have $f(1) = 1$, $f(2) = f(1)+4-1 = 1+4-1 = 4$, $f(3) = f(2)+6-1 = 4+6-1 = 9$, $f(4) = f(3)+8-1 = 9+8-1 = 16$. The pattern seems clear, so we conjecture that $f(n) = n^2$. Now we prove this by induction. The base case we have already verified. So assume that $f(n) = n^2$. All we have to do is show that $f(n+1) = (n+1)^2$. By the recursive definition we have $f(n+1) = f(n) + 2(n+1) - 1$. This equals $n^2 + 2(n+1) - 1$ by the inductive hypothesis, and by algebra we have $n^2 + 2(n+1) - 1 = (n+1)^2$, as desired.

53. The recursive definition says that we can "grow" strings in S by appending 0's on the left and 1's on the right, as many as we wish.

a) The only string of length 0 in S is λ. There are two strings of length 1 in S, obtained either by appending a 0 to the front of λ or a 1 to the end of λ, namely the strings 0 and 1. The strings of length 2 in S come from the strings of length 1 by appending either a 0 to the front or a 1 to the end; they are 00, 01, and 11. Similarly, we can append a 0 to the front or a 1 to the end of any of these strings to get the strings of length 3 in S, namely 000, 001, 011, and 111. Continuing in this manner, we see that the other strings in S of length less than or equal to 5 are 0000, 0001, 0011, 0111, 1111, 00000, 00001, 00011, 00111, 01111, and 11111.

b) The simplest way to describe these strings is $\{\, 0^m 1^n \mid m \text{ and } n \text{ are nonnegative integers} \,\}$.

55. Applying the first recursive step once to λ tells us that $() \in B$. Then applying the second recursive step to this string tells us that $()() \in B$. Finally, we apply the first recursive step once more to get $(()()) \in B$. To see that $(()))$ is not in B, we invoke Exercise 58. Since the number of left parentheses does not equal the number of right parentheses, this string is not balanced.

57. There is of course the empty string, with 0 symbols. By the first recursive rule, we get the string $()$. If we apply the first recursive rule to this string, then we get $(())$, and if we apply the second recursive rule, then we get $()()$. These are the only strings in B with four or fewer symbols.

59. The definition simply says that N of a string is a count of the parentheses, with each left parenthesis counting $+1$ and each right parenthesis counting -1.

a) There is one left parenthesis and one right parenthesis, so $N\big(()\big) = 1 - 1 = 0$.

b) There are 3 left parentheses and 5 right parentheses, so $N\big())))())(()\big) = 3 - 5 = -2$.

c) There are 4 left parentheses and 2 right parentheses, so $N\big((()(()\big) = 4 - 2 = 2$.

d) There are 6 left parentheses and 6 right parentheses, so $N\big(()((()))(())\big) = 6 - 6 = 0$.

61. The basic idea, of course, is to turn the definition into a procedure. The recursive part of the definition tells us how to find elements of B from shorter elements of B. The naive approach, however, is not very good, because we end up adding to B strings that already are there. For example, the string $()()()$ occurs in two different ways from the rule "$xy \in B$ if $x, y \in B$": by letting $x = ()()$ and $y = ()$, and by letting $x = ()$ and $y = ()()$.

To avoid this problem, we will keep two lists of strings, whose union is the set $B(n)$ of balanced strings of parentheses of length not exceeding n. The set $S(n)$ will be those balanced strings w of length at most n snch that $w = uv$, where $u, v \neq \lambda$ and u and v are balanced. The set $T(n)$ will be all other balanced strings of length at most n. Note that, for example, $\lambda \in T$, $() \in T$, $(()) \in T$, but $()() \in S$. Since all the strings in B are of even length, we really only need to work with even values of n, dragging the odd values along for the ride.

procedure *generate*(n : nonnegative integer)
if n is odd **then**
begin
    $S := S(n-1)$ {the S constructed by *generate*($n-1$)}
    $T := T(n-1)$ {the T constructed by *generate*($n-1$)}
end
else if $n = 0$ **then**
begin
    $S := \emptyset$
    $T := \{\lambda\}$
end
else
begin
    $S' := S(n-2)$ {the S constructed by *generate*($n-2$)}
    $T' := T(n-2)$ {the T constructed by *generate*($n-2$)}
    $T := T' \cup \{\, (x) \mid x \in T' \cup S' \wedge \text{length}(x) = n-2 \,\}$
    $S := S' \cup \{\, xy \mid x \in T' \wedge y \in T' \cup S' \wedge \text{length}(xy) = n \,\}$
end
{$T \cup S$ is the set of balanced strings of length at most n}

63. There are two cases. If $x \leq y$ initially, then the statement $x := y$ is not executed, so x and y remain unchanged and $x \leq y$ is a true final assertion. If $x > y$ initially, then the statement $x := y$ is executed, so $x = y$ at the end, and thus $x \leq y$ is again a true final assertion. These are the only two possibilities associated with the initial condition **T** (true), so our proof is complete.

65. If the list has just one element in it, then the number of 0's is 1 if the element is 0 and is 0 otherwise. That forms the basis step of our algorithm. For the recursive step, the number of occurrences of 0 in a list L is the same as the number of occurrences in the list L without its last term if that last term is not a 0, and is one more than this value if it is. We can write this in pseudocode as follows.

procedure *zerocount*($a_1, a_2, \ldots, a_n$: list of integers)
if $n = 1$ **then**
    if $a_1 = 0$ **then** *zerocount*$(a_1, a_2, \ldots, a_n) := 1$
    else *zerocount*$(a_1, a_2, \ldots, a_n) := 0$
else
    if $a_n = 0$ **then** *zeracount*$(a_1, a_2, \ldots, a_n) :=$ *zerocount*$(a_1, a_2, \ldots, a_{n-1}) + 1$
    else *zerocount*$(a_1, a_2, \ldots, a_n) :=$ *zerocount*$(a_1, a_2, \ldots, a_{n-1})$

67. From the numerical evidence in Exercise 66, it appears that $a(n)$ is a natural number and $a(n) \leq n$ for all n. We prove that a is well-defined by showing that this observation is in fact true. Obviously the proof is by mathematical induction. The basis step is $n = 0$, for which the statement is obviously true, since $a(0) = 0$. Now assume that $a(n-1)$ is a natural number and $a(n-1) \leq n-1$. Then $a(a(n-1))$ is a applied to some natural number less than or equal to $n-1$; by the inductive hypothesis this value is some natural number less than or equal to $n-1$. Therefore $a(a(n-1))$ is also some natural number less than or equal to $n-1$ (again by the inductive hypothesis). Therefore $n - a(a(n-1))$ is n minus some natural number less than or equal to $n-1$, which is some natural number less than or equal to n, and we are done.

69. From Exercise 68 we know that $a(n) = \lfloor (n+1)\mu \rfloor$ and that $a(n-1) = \lfloor n\mu \rfloor$. Since μ is less than 1, these two values are either equal or they differ by 1. First suppose that $\mu n - \lfloor \mu n \rfloor < 1 - \mu$. This is equivalent to $\mu(n+1) < 1 + \lfloor \mu n \rfloor$. If this is true, then clearly $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor$. On the other hand, if $\mu n - \lfloor \mu n \rfloor \geq 1 - \mu$, then $\mu(n+1) \geq 1 + \lfloor \mu n \rfloor$, so $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor + 1$, as desired.

71. We apply the definition:

$$\begin{array}{rl}
m(0) = 0 & f(0) = 1 \\
m(1) = 1 - f(m(0)) = 1 - f(0) = 1 - 1 = 0 & f(1) = 1 - m(f(0)) = 1 - m(1) = 1 - 0 = 1 \\
m(2) = 2 - f(m(1)) = 2 - f(0) = 2 - 1 = 1 & f(2) = 2 - m(f(1)) = 2 - m(1) = 2 - 0 = 2 \\
m(3) = 3 - f(m(2)) = 3 - f(1) = 3 - 1 = 2 & f(3) = 3 - m(f(2)) = 3 - m(2) = 3 - 1 = 2 \\
m(4) = 4 - f(m(3)) = 4 - f(2) = 4 - 2 = 2 & f(4) = 4 - m(f(3)) = 4 - m(2) = 4 - 1 = 3 \\
m(5) = 5 - f(m(4)) = 5 - f(2) = 5 - 2 = 3 & f(5) = 5 - m(f(4)) = 5 - m(3) = 5 - 2 = 3 \\
m(6) = 6 - f(m(5)) = 6 - f(3) = 6 - 2 = 4 & f(6) = 6 - m(f(5)) = 6 - m(3) = 6 - 2 = 4 \\
m(7) = 7 - f(m(6)) = 7 - f(4) = 7 - 3 = 4 & f(7) = 7 - m(f(6)) = 7 - m(4) = 7 - 2 = 5 \\
m(8) = 8 - f(m(7)) = 8 - f(4) = 8 - 3 = 5 & f(8) = 8 - m(f(7)) = 8 - m(5) = 8 - 3 = 5 \\
m(9) = 9 - f(m(8)) = 9 - f(5) = 9 - 3 = 6 & f(9) = 9 - m(f(8)) = 9 - m(5) = 9 - 3 = 6
\end{array}$$

73. By Exercise 72 the sequence starts out 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, ..., and we see that $f(1) = 1$ (since the last occurrence of 1 is in position 1), $f(2) = 3$ (since the last occurrence of 2 is in position 3), $f(3) = 5$ (since the last occurrence of 3 is in position 5), $f(4) = 8$ (since the last occurrence of 4 is in position 8), and so on. Since the sequence is nondecreasing, the last occurrence of n must be in the position for which the total number of 1s, 2s, 3s, ..., n's all together is that position number. But since a_k gives the number of occurrences of k, this is just $\sum_{k=1}^{n} a_k$, as desired. For example,

$$\sum_{k=1}^{6} a_k = 1 + 2 + 2 + 3 + 3 + 4 = 15 = f(6),$$

the position where the last 6 occurs.

Since we just saw that $f(n)$ is the sum of the first n terms of the sequence, $f(f(n))$ must be the sum of the first $f(n)$ terms of the sequence. But since $f(n)$ is the last term whose value is n, this means the sum of all the terms of the sequence whose value is at most n. Since there are a_k terms of the sequence whose value is k, this sum must be $\sum_{k=1}^{n} k \cdot a_k$, as desired. For example,

$$\sum_{k=1}^{3} k \cdot a_k = 1 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 = 11 = f(f(3)) = f(5),$$

the position where the last 5 occurs.

WRITING PROJECTS FOR CHAPTER 4

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. Start with the historical footnote in the text. The standard history of mathematics references, such as [Bo4] and [Ev3], might have something or might provide a hint of where to look next.

2. There is a nice chapter on this in [He]. A Web search should also turn up useful pages, such as this one: `http://www-cgrl.cs.mcgill.ca/~godfried/teaching/cg-projects/97/Octavian/compgeom.html`

3. There are several textbooks on computational geometry, such as [O'R]. A comprehensive website on the subject can be found here: `http://compgeom.cs.uiuc.edu/~jeffe/compgeom/`

4. The ratio of successive Fibonacci numbers approaches a value known as the "golden ratio," so it would be useful to search for this topic as well. A recent and somewhat controversial book on the subject [Li] debunks some of the more outrageous claimed applications. This website seems to have a wealth of information on applications of Fibonacci numbers: `http://www.cs.rit.edu/~pga/Fibo/fact_sheet.html`

5. You can find some references, as well as an historical discussion of the Ackermann function and an iterative algorithm for computing it, in [GrZe].

6. Try searching your library's on-line catalog or the Web under keywords like *program correctness* or *verification*. Or look at [Ba1], [Di], or [Ho1].

7. As in Writing Project 6, a key-word search might turn up something. One book to look at is [De1].