SECTION 9.5 Euler and Hamilton Paths

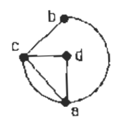
2. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, f, i, h, g, d, e, h, f, e, b, d, a$.

4. This graph has no Euler circuit, since the degree of vertex c (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $f, a, b, c, d, e, f, b, d, a, e, c$.

6. This graph has no Euler circuit, since the degree of vertex b (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $b, c, d, e, f, d, g, i, d, a, h, i, a, b, i, c$.

8. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a$.

10. The graph model for this exercise is as shown here.



Vertices a and b are the banks of the river, and vertices c and d are the islands. Each vertex has even degree, so the graph has an Euler circuit, such as a, c, b, a, d, c, a. Therefore a walk of the type described is possible.

12. The algorithm is essentially the same as Algorithm 1. If there are no vertices of odd degree, then we simply use Algorithm 1, of course. If there are exactly two vertices of odd degree, then we begin constructing the initial path at one such vertex, and it will necessarily end at the other when it cannot be extended any further. Thereafter we follow Algorithm 1 exactly, splicing new circuits into the path we have constructed so far until no unused edges remain.

14. See the comments in the solution to Exercise 13. This graph has exactly two vertices of odd degree; therefore it has an Euler path and can be so traced.

16. First suppose that the directed multigraph has an Euler circuit. Since this circuit provides a path from every vertex to every other vertex, the graph must be strongly connected (and hence also weakly connected). Also, we can count the in-degrees and out-degrees of the vertices by following this circuit; as the circuit passes through a vertex, it adds one to the count of both the in-degree (as it comes in) and the out-degree (as it leaves). Therefore the two degrees are equal for each vertex.

Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem 1 and construct an Euler circuit.

18. For Exercises 18–23 we use the results of Exercises 16 and 17. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from a to d. One such path is $a, b, d, b, c, d, c, a, d$.

20. The conditions of Exercise 16 are met, so there is an Euler circuit, which is perforce also an Euler path. One such path is $a, d, b, d, e, b, e, c, b, a$.

22. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from c to b. One such path is $c, e, b, d, c, b, f, d, e, f, e, a, f, a, b, c, b$. (There is no Euler circuit, however, since the conditions of Exercise 22 are not met.)

24. The algorithm is identical to Algorithm 1.

26. a) The degrees of the vertices ($n-1$) are even if and only if n is odd. Therefore there is an Euler circuit if and only if n is odd (and greater than 1, of course).

b) For all $n \geq 3$, clearly C_n has an Euler circuit, namely itself.

c) Since the degrees of the vertices around the rim are all odd, no wheel has an Euler circuit.

d) The degrees of the vertices are all n. Therefore there is an Euler circuit if and only if n is even (and greater than 0, of course).

28. a) Since the degrees of the vertices are all m and n, this graph has an Euler circuit if and only if both of the positive integers m and n are even.

b) All the graphs listed in part (a) have an Euler circuit, which is also an Euler path. In addition, the graphs $K_{n,2}$ for odd n have exactly 2 vertices of odd degree, so they have an Euler path but not an Euler circuit. Also, $K_{1,1}$ obviously has an Euler path. All other complete bipartite graphs have too many vertices of odd degree.

30. This graph can have no Hamilton circuit because of the cut edge $\{c,f\}$. Every simple circuit must be confined to one of the two components obtained by deleting this edge.

32. As in Exercise 30, the cut edge ($\{e,f\}$ in this case) prevents a Hamilton circuit.

34. This graph has no Hamilton circuit. If it did, then certainly the circuit would have to contain edges $\{d,a\}$ and $\{a,b\}$, since these are the only edges incident to vertex a. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These eight edges already complete a circuit, and this circuit omits the nine vertices on the inside. Therefore there is no Hamilton circuit.

36. It is easy to find a Hamilton circuit here, such as a, d, g, h, i, f, c, e, b, and back to a.

38. This graph has the Hamilton path a,b,c,d,e.

40. This graph has no Hamilton path. There are three vertices of degree 1; each of them would have to be an end vertex of every Hamilton path. Since a path has only 2 ends, this is impossible.

42. It is easy to find the Hamilton path d, c, a, b, e here.

44. a) Obviously K_n has a Hamilton circuit for all $n \geq 3$ but not for $n \leq 2$.

b) Obviously C_n has a Hamilton circuit for all $n \geq 3$.

c) A Hamilton circuit for C_n can easily be extended to one for W_n by replacing one edge along the rim of the wheel by two edges, one going to the center and the other leading from the center. Therefore W_n has a Hamilton circuit for all $n \geq 3$.

d) This is Exercise 49; see the solution given for it.

46. We do the easy part first, showing that the graph obtained by deleting a vertex from the Petersen graph has a Hamilton circuit. By symmetry, it makes no difference which vertex we delete, so assume that it is vertex j. Then a Hamilton circuit in what remains is a,e,d,i,g,b,c,h,f,a. Now we show that the entire graph has no Hamilton circuit. Assume that a Hamilton circuit exists. Not all the edges around the outside can be used, so without loss of generality assume that $\{c,d\}$ is not used. Then $\{e,d\}$, $\{d,i\}$, $\{h,c\}$, and $\{b,c\}$ must all be used. If $\{a,f\}$ is not used, then $\{e,a\}$, $\{a,b\}$, $\{f,i\}$, and $\{f,h\}$ must be used, forming a premature circuit. Therefore $\{a,f\}$ is used. Without loss of generality we may assume that $\{e,a\}$ is also used, and $\{a,b\}$ is not used. Then $\{b,g\}$ is also used, and $\{e,j\}$ is not. But this requires $\{g,j\}$ and $\{h,j\}$ to be used, forming a premature circuit b,c,h,j,g,b. Hence no Hamilton circuit can exist in this graph.

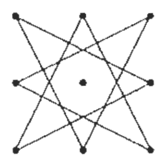
48. We want to look only at odd n, since if n is even, then being at least $(n-1)/2$ is the same as being at least $n/2$, in which case Dirac's Theorem would apply. One way to avoid having a Hamilton circuit is to have a cut vertex—a vertex whose removal disconnects the graph. The simplest example would be the "bow-tie" graph with five vertices (a, b, c, d, and e), where cut vertex c is adjacent to each of the other vertices, and the only other edges are ab and de. Every vertex has degree at least $(5-1)/2 = 2$, but there is no Hamilton circuit.

50. Let us begin at vertex a and walk toward vertex b. Then the circuit begins a, b, c. At this point we must choose among three edges to continue the circuit. If we choose edge $\{c, f\}$, then we will have disconnected the graph that remains, so we must not choose this edge. Suppose instead that the circuit continues with edge $\{c, d\}$. Then the entire circuit is forced to be a, b, c, d, e, c, f, a.

52. This proof is rather hard. See page 63 of *Graph Theory with Applications* by J. A. Bondy and U. S. R. Murty (American Elsevier, 1976).

54. An Euler path will cover every link, so it can be used to test the links. A Hamilton path will cover all the devices, so it can be used to test the devices.

56. We draw one vertex for each of the 9 squares on the board. We then draw an edge from a vertex to each vertex that can be reached by moving 2 units horizontally and 1 unit vertically or vice versa. The result is as shown.



58. a) In a Hamilton path we need to visit each vertex once, moving along the edges. A knight's tour is precisely such a path, since we visit each square once, making legal moves.

b) This is the same as part (a), except that a re-entrant tour must return to its starting point, just as a Hamilton circuit must return to its starting point.

60. In a 3×3 board, the middle vertex is isolated (see solution to Exercise 56). In other words, there is no knight move to or from the middle square. Thus there can clearly be no knight's tour. There is a tour of the rest of the squares, however, as the picture above shows.

62. Each square of the board can be thought of as a pair of integers (x, y). Let A be the set of squares for which $x + y$ is odd, and let B be the set of squares for which $x + y$ is even. This partitions the vertex set of the graph representing the legal moves of a knight on the board into two parts. Now every move of the knight changes $x+y$ by an odd number—either $1+2=3$, $2-1=1$, $1-2=-1$, or $-1-2=-3$. Therefore every edge in this graph joins a vertex in A to a vertex in B. Thus the graph is bipartite.

64. A little trial and error, loosely following the hint, produced the following solution. The numbers show the order in which the squares are to be traversed.

1	28	13	26	3	38	41	16
64	25	2	39	52	15	4	37
29	12	27	14	57	40	17	42
24	63	56	53	60	51	36	5
11	30	49	62	55	58	43	18
48	23	54	59	50	61	6	35
31	10	21	46	33	8	19	44
22	47	32	9	20	45	34	7