SECTION 9.4 Connectivity

2. a) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

b) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

c) This is not a path, since there is no edge from d to b.

d) This is not a path, since there is no edge from b to d.

4. This graph is connected—it is easy to see that there is a path from every vertex to every other vertex.

6. The graph in Exercise 3 has three components: the piece that looks like a $\wedge$, the piece that looks like a $\vee$, and the isolated vertex. The graph in Exercise 4 is connected, with just one component. The graph in Exercise 5 has two components, each a triangle.

8. A connected component of a collaboration graph represent a maximal set of people with the property that for any two of them, we can find a string of joint works that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

10. An actor is in the same connected component as Kevin Bacon if there is a path from that person to Bacon. This means that the actor was in a movie with someone who was in a movie with someone who ... who was in a movie with Kevin Bacon. This includes Kevin Bacon, all actors who appeared in a movie with Kevin Bacon, all actors who appeared in movies with those people, and so on.

12. a) Notice that there is no path from f to a, so the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

b) Notice that the sequence a, b, c, d, e, f, a provides a path from every vertex to every other vertex, so this graph is strongly connected.

c) The underlying undirected graph is clearly not connected (one component consists of the triangle), so this graph is neither strongly nor weakly connected.

14. a) The cycle *baeb* guarantees that these three vertices are in one strongly connected component. Since there is no path from c to any other vertex, and there is no path from any other vertex to d, these two vertices are in strong components by themselves. Therefore the strongly connected components are $\{a, b, e\}$, $\{c\}$, and $\{d\}$.

b) The cycle *cdec* guarantees that these three vertices are in one strongly connected component. The vertices a, b, and f are in strong components by themselves, since there are no paths both to and from each of these to every other vertex. Therefore the strongly connected components are $\{a\}$, $\{b\}$ $\{c, d, e\}$, and $\{f\}$.

c) The cycle *abcdfghia* guarantees that these eight vertices are in one strongly connected component. Since there is no path from e to any other vertex, this vertex is in a strong component by itself. Therefore the strongly connected components are $\{a, b, c, d, f, g, h, i\}$ and $\{e\}$.

16. Let $a, b, c, \ldots, z$ be the directed path. Since z and a are in the same strongly connected component, there is a directed path from z to a. This path appended to the given path gives us a circuit. We can reach any vertex on the original path from any other vertex on that path by going around this circuit.

18. The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

20. We notice that there are two vertices in each graph that are not in cycles of size 4. So let us try to construct an isomorphism that matches them, say $u_1 \leftrightarrow v_2$ and $u_8 \leftrightarrow v_6$. Now u_1 is adjacent to u_2 and u_3, and v_2 is adjacent to v_1 and v_3, so we try $u_2 \leftrightarrow v_1$ and $u_3 \leftrightarrow v_3$. Then since u_4 is the other vertex adjacent to u_3 and v_4 is the other vertex adjacent to v_3 (and we already matched u_3 and v_3), we must have $u_4 \leftrightarrow v_4$. Proceeding along similar lines, we then complete the bijection with $u_5 \leftrightarrow v_8$, $u_6 \leftrightarrow v_7$, and $u_7 \leftrightarrow v_5$. Having thus been led to the only possible isomorphism, we check that the 12 edges of G exactly correspond to the 12 edges of H, and we have proved that the two graphs are isomorphic.

22. **a)** Adjacent vertices are in different parts, so every path between them must have odd length. Therefore there are no paths of length 2.

b) A path of length 3 is specified by choosing a vertex in one part for the second vertex in the path and a vertex in the other part for the third vertex in the path (the first and fourth vertices are the given adjacent vertices). Therefore there are $3 \cdot 3 = 9$ paths.

c) As in part (a), the answer is 0.

d) This is similar to part (b); therefore the answer is $3^4 = 81$.

24. Probably the best way to do this is to write down the adjacency matrix for this graph and then compute its powers. The matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

a) To find the number of paths of length 2, we need to look at $\mathbf{A}^2$, which is

$$\begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 3 & 2 & 2 \\ 2 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 & 4 & 1 \\ 2 & 2 & 1 & 2 & 1 & 3 \end{bmatrix}.$$

Since the $(3,4)^{\text{th}}$ entry is 0, so there are no paths of length 2.

b) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^3$ turns out to be 8, so there are 8 paths of length 3.

c) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^4$ turns out to be 10, so there are 10 paths of length 4.

d) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^5$ turns out to be 73, so there are 73 paths of length 5.

e) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^6$ turns out to be 160, so there are 160 paths of length 6.

f) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^7$ turns out to be 739, so there are 739 paths of length 7.

26. We show this by induction on n. For $n = 1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

28. Let v be a vertex of odd degree, and let H be the component of G containing v. Then H is a graph itself, so it has an even number of vertices of odd degree. In particular, there is another vertex w in H with odd degree. By definition of connectivity, there is a path from v to w.

30. Vertices c and d are the cut vertices. The removal of either one creates a graph with two components. The removal of any other vertex does not disconnect the graph.

32. The graph in Exercise 29 has no cut edges; any edge can be removed, and the result is still connected. For the graph in Exercise 30, $\{c,d\}$ is the only cut edge. There are several cut edges for the graph in Exercise 31: $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{c,e\}$, $\{e,i\}$, and $\{h,i\}$.

34. First we show that if c is a cut vertex, then there exist vertices u and v such that every path between them passes through c. Since the removal of c increases the number of components, there must be two vertices in G that are in different components after the removal of c. Then every path between these two vertices has to pass through c. Conversely, if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

36. First suppose that $e=\{u,v\}$ is a cut edge. Every circuit containing e must contain a path from u to v in addition to just the edge e. Since there are no such paths if e is removed from the graph, every such path must contain e. Thus e appears twice in the circuit, so the circuit is not simple. Conversely, suppose that e is not a cut edge. Then in the graph with e deleted u and v are still in the same component. Therefore there is a simple path P from u to v in this deleted graph. The circuit consisting of P followed by e is a simple circuit containing e.

38. (The answers given here are not unique.) In the directed graph in Exercise 7, there is a path from b to each of the other three vertices, so $\{b\}$ is a vertex basis (and a smallest one). For the directed graph in Exercise 8, there is a path from b to each of a and c; on the other hand, d must clearly be in every vertex basis. Thus $\{b,d\}$ is a smallest vertex basis. Every vertex basis for the directed graph in Exercise 9 must contain vertex e, since it has no incoming edges. On the other hand, from any other vertex we can reach all the other vertices, so e together with any one of the other four vertices will form a vertex basis.

40. By definition of graph, both G_1 and G_2 are nonempty. If they have no common vertex, then there clearly can be no paths from $v_1\in G_1$ to $v_2\in G_2$. In that case G would not be connected, contradicting the hypothesis.

42. First we obtain the inequality given in the hint. We claim that the maximum value of $\sum n_i^2$, subject to the constraint that $\sum n_i = n$, is obtained when one of the n_i's is as large as possible, namely $n-k+1$, and the remaining n_i's (there are $k-1$ of them) are all equal to 1. To justify this claim, suppose instead that two of the n_i's were a and b, with $a\ge b\ge 2$. If we replace a by $a+1$ and b by $b-1$, then the constraint is still satisfied, and the sum of the squares has changed by $(a+1)^2+(b-1)^2-a^2-b^2=2(a-b)+2\ge 2$. Therefore the maximum cannot be attained unless the n_i's are as we claimed. Since there are only a finite number of possibilities for the distribution of the n_i's, the arrangement we give must in fact yield the maximum. Therefore $\sum n_i^2 \le (n-k+1)^2+(k-1)\cdot 1^2 = n^2-(k-1)(2n-k)$, as desired.

Now by Exercise 41, the number of edges of the given graph does not exceed $\sum C(n_i,2)=\sum(n_i^2+n_i)/2 = ((\sum n_i^2)+n)/2$. Applying the inequality obtained above, we see that this does not exceed $(n^2-(k-1)(2n-k)+n)/2$, which after a little algebra is seen to equal $(n-k)(n-k+1)/2$. The upshot of all this is that the most edges are obtained if there is one component as large as possible, with all the other components consisting of isolated vertices.

44. Under these conditions, the matrix has a block structure, with all the 1's confined to small squares (of various sizes) along the main diagonal. The reason for this is that there are no edges between different components. See the picture for a schematic view. The only 1's occur inside the small submatrices (but not all the entries in these squares are 1's, of course).

$$\begin{bmatrix} [\ ] & & & 0 \\ & [\ ] & & \\ & & \ddots & \\ 0 & & & [\ ] \end{bmatrix}$$

46. The length of a shortest path is the smallest l such that there is at least one path of length l from v to w. Therefore we can find the length by computing successively $\mathbf{A}^1$, $\mathbf{A}^2$, $\mathbf{A}^3$, ..., until we find the first l such that the $(i,j)^{\text{th}}$ entry of $\mathbf{A}^l$ is not 0, where v is the i^{th} vertex and w is the j^{th}.

48. First we write down the adjacency matrix for this graph, namely

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then we compute $\mathbf{A}^2$ and $\mathbf{A}^3$, and look at the $(1,3)^{\text{th}}$ entry of each. We find that these entries are 0 and 1, respectively. By the reasoning given in Exercise 47, we conclude that a shortest path has length 3.

50. Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say $u_1, u_2, \ldots, u_k, u_1$, then we claim that $f(u_1), f(v_2), \ldots, f(u_k), f(v_1)$ is a simple circuit in H. Certainly this is a circuit, since each edge u_iu_{i+1} (and u_ku_1) in G corresponds to an edge $f(u_i)f(u_{i+1})$ (and $f(u_k)f(u_1)$) in H. Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

52. The adjacency matrix of G is as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We compute $\mathbf{A}^2$ and $\mathbf{A}^3$, obtaining

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^3 = \begin{bmatrix} 2 & 3 & 5 & 2 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 & 1 & 2 & 1 \\ 5 & 5 & 4 & 6 & 1 & 6 & 1 \\ 2 & 2 & 6 & 2 & 3 & 5 & 1 \\ 1 & 1 & 1 & 3 & 0 & 1 & 1 \\ 2 & 2 & 6 & 5 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 3 & 0 \end{bmatrix}.$$

Already every off-diagonal entry in $\mathbf{A}^3$ is nonzero, so we know that there is a path of length 3 between every pair of distinct vertices in this graph. Therefore the graph G is connected.

On the other hand, the adjacency matrix of H is as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

We compute $\mathbf{A}^2$ through $\mathbf{A}^5$, obtaining the following matrices:

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 3 & 3 & 2 \end{bmatrix}$$

$$\mathbf{A}^4 = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 5 & 5 \\ 0 & 0 & 0 & 5 & 6 & 5 \\ 0 & 0 & 0 & 5 & 5 & 6 \end{bmatrix} \qquad \mathbf{A}^5 = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 11 & 11 \\ 0 & 0 & 0 & 11 & 10 & 11 \\ 0 & 0 & 0 & 11 & 11 & 10 \end{bmatrix}$$

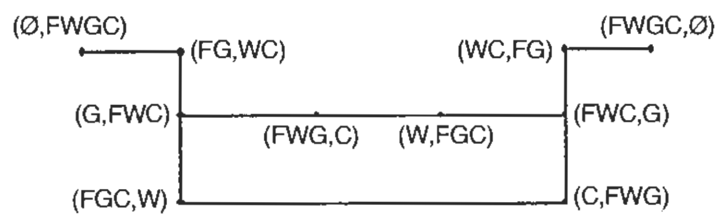
If we compute the sum $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \mathbf{A}^4 + \mathbf{A}^5$ we obtain

$$\begin{bmatrix} 6 & 7 & 7 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 7 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 21 & 21 \\ 0 & 0 & 0 & 21 & 20 & 21 \\ 0 & 0 & 0 & 21 & 21 & 20 \end{bmatrix}.$$

There is a 0 in the $(1,4)$ position, telling us that there is no path of length at most 5 from vertex a to vertex d. Since the graph only has six vertices, this tells us that there is no path at all from a to d. Thus the fact that there was a 0 as an off-diagonal entry in the sum told us that the graph was not connected.

54. a) To proceed systematically, we list the states in order of decreasing population on the left shore. The allowable states are then $(FWGC, \emptyset)$, (FWG, C), (FWC, G), (FGC, W), (FG, WC), (WC, FG) (C, FWG), (G, FWC), (W, FGC), and $(\emptyset, FWGC)$. Notice that, for example, (GC, FW) and (WGC, F) are not allowed by the rules.

b) The graph is as shown here. Notice that the boat can carry only the farmer and one other object, so the transitions are rather restricted.



c) The path in the graph corresponds to the moves in the solution.

d) There are two simple paths from $(FWGC, \emptyset)$ to $(\emptyset, FWGC)$ that can be easily seen in the graph. One is $(FWGC, \emptyset)$, (WC, FG), (FWC, G), (W, FGC), (FWG, C), (G, FWC), (FG, WC), $(\emptyset, FWGC)$. The other is $(FWGC, \emptyset)$, (WC, FG), (FWC, G), (C, FWG), (FGC, W), (G, FWC), (FG, WC), $(\emptyset, FWGC)$.

e) Both solutions cost \$4.

56. If we use the ordered pair (a, b) to indicate that the three-gallon jug has a gallons in it and the five-gallon jug has b gallons in it, then we start with $(0,0)$ and can do the following things: fill a jug that is empty or partially empty (so that, for example, we can go from $(0,3)$ to $(3,3)$); empty a jug; or transfer some or all of the contents of a jug to the other jug , as long as we either completely empty the donor jug or completely fill the receiving jug. A simple solution to the puzzle uses this directed path: $(0,0) \to (3,0) \to (0,3) \to (3,3) \to (1,5)$.