SECTION 1.2 Propositional Equivalences

2. There are two cases. If p is true, then $\neg(\neg p)$ is the negation of a false proposition, hence true. Similarly, if p is false, then $\neg(\neg p)$ is also false. Therefore the two propositions are logically equivalent.

4. a) We construct the relevant truth table and note that the fifth and seventh columns are identical.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

b) Again we construct the relevant truth table and note that the fifth and seventh columns are identical.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

6. We see that the fourth and seventh columns are identical.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

8. We need to negate each part and swap "and" with "or."

a) Kwame will not take a job in industry and will not go to graduate school.

b) Yoshiko does not know Java or does not know calculus.

c) James is not young, or he is not strong.

d) Rita will not move to Oregon and will not move to Washington.

10. We construct a truth table for each conditional statement and note that the relevant column contains only T's. For part **(a)** we have the following table.

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

For part **(b)** we have the following table. We omit the columns showing $p \rightarrow q$ and $q \rightarrow r$ so that the table will fit on the page.

p	q	r	$(p \rightarrow q) \rightarrow (q \rightarrow r)$	$q \rightarrow r$	$[(p \rightarrow q) \rightarrow (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	T	F
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	T	F
F	F	T	T	T	F
F	F	F	T	T	T

For part **(c)** we have the following table.

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

For part **(d)** we have the following table. We have omitted some of the intermediate steps to make the table fit.

p	q	r	$(p \vee q) \wedge (p \to r) \wedge (p \to r)$	$[(p \vee q) \wedge (p \to r) \wedge (p \to r)] \to r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

12. We argue directly by showing that if the hypothesis is true, then so is the conclusion. An alternative approach, which we show only for part (a), is to use the equivalences listed in the section and work symbolically.

a) Assume the hypothesis is true. Then p is false. Since $p \vee q$ is true, we conclude that q must be true. Here is a more "algebraic" solution: $[\neg p \wedge (p \vee q)] \to q \equiv \neg[\neg p \wedge (p \vee q)] \vee q \equiv \neg\neg p \vee \neg(p \vee q)] \vee q \equiv p \vee \neg(p \vee q) \vee q \equiv (p \vee q) \vee \neg(p \vee q) \equiv \mathbf{T}$. The reasons for these logical equivalences are, respectively, Table 7, line 1; De Morgan's law; double negation; commutative and associative laws; negation law.

b) We want to show that if the entire hypothesis is true, then the conclusion $p \to r$ is true. To do this, we need only show that if p is true, then r is true. Suppose p is true. Then by the first part of the hypothesis, we conclude that q is true. It now follows from the second part of the hypothesis that r is true, as desired.

c) Assume the hypothesis is true. Then p is true, and since the second part of the hypothesis is true, we conclude that q is also true, as desired.

d) Assume the hypothesis is true. Since the first part of the hypothesis is true, we know that either p or q is true. If p is true, then the second part of the hypothesis tells us that r is true; similarly, if q is true, then the third part of the hypothesis tells us that r is true. Thus in either case we conclude that r is true.

14. This is not a tautology. It is saying that knowing that the hypothesis of an conditional statement is false allows us to conclude that the conclusion is also false, and we know that this is not valid reasoning. To show that it is not a tautology, we need to find truth assignments for p and q that make the entire proposition false. Since this is possible only if the conclusion if false, we want to let q be true; and since we want the hypothesis to be true, we must also let p be false. It is easy to check that if, indeed, p is false and q is true, then the conditional statement is false. Therefore it is not a tautology.

16. The first of these propositions is true if and only if p and q have the same truth value. The second is true if and only if either p and q are both true, or p and q are both false. Clearly these two conditions are saying the same thing.

18. It is easy to see from the definitions of conditional statement and negation that each of these propositions is false in the case in which p is true and q is false, and true in the other three cases. Therefore the two propositions are logically equivalent.

20. It is easy to see from the definitions of the logical operations involved here that each of these propositions is true in the cases in which p and q have the same truth value, and false in the cases in which p and q have opposite truth values. Therefore the two propositions are logically equivalent.

22. Suppose that $(p \to q) \wedge (p \to r)$ is true. We want to show that $p \to (q \wedge r)$ is true, which means that we want to show that $q \wedge r$ is true whenever p is true. If p is true, since we know that both $p \to q$ and $p \to r$ are true from our assumption, we can conclude that q is true and that r is true. Therefore $q \wedge r$ is true, as desired. Conversely, suppose that $p \to (q \wedge r)$ is true. We need to show that $p \to q$ is true and that $p \to r$ is true, which means that if p is true, then so are q and r. But this follows from $p \to (q \wedge r)$.

24. We determine exactly which rows of the truth table will have T as their entries. Now $(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, i.e., when $q \vee r$ is true, which is precisely when $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situations, they are logically equivalent.

26. Applying the third and first equivalences in Table 7, we have $\neg p \to (q \to r) \equiv p \vee (q \to r) \equiv p \vee \neg q \vee r$. Applying the first equivalence in Table 7 to $q \to (p \vee r)$ shows that $\neg q \vee p \vee r$ is equivalent to it. But these are equivalent by the commutative and associative laws.

28. We know that $p \leftrightarrow q$ is true precisely when p and q have the same truth value. But this happens precisely when $\neg p$ and $\neg q$ have the same truth value, that is, $\neg p \leftrightarrow \neg q$.

30. The conclusion $q \vee r$ will be true in every case except when q and r are both false. But if q and r are both false, then one of $p \vee q$ or $\neg p \vee r$ is false, because one of p or $\neg p$ is false. Thus in this case the hypothesis $(p \vee q) \wedge (\neg p \vee r)$ is false. An conditional statement in which the conclusion is true or the hypothesis is false is true, and that completes the argument.

32. We just need to find an assignment of truth values that makes one of these propositions true and the other false. We can let p be true and the other two variables be false. Then the first statement will be $\mathbf{F} \to \mathbf{F}$, which is true, but the second will be $\mathbf{F} \wedge \mathbf{T}$, which is false.

34. We apply the rules stated in the preamble.
a) $p \wedge \neg q$ **b)** $p \vee (q \wedge (r \vee \mathbf{F}))$ **c)** $(p \vee \neg q) \wedge (q \vee \mathbf{T})$

36. If s has any occurrences of $\wedge$, $\vee$, $\mathbf{T}$, or $\mathbf{F}$, then the process of forming the dual will change it. Therefore $s^* = s$ if and only if s is simply one propositional variable (like p). A more difficult question is to determine when s^* will be logically equivalent to s. For example, $p \vee \mathbf{F}$ is logically equivalent to its dual $p \wedge \mathbf{T}$, because both are logically equivalent to p.

38. The table is in fact displayed so as to exhibit the duality. The two identity laws are duals of each other, the two domination laws are duals of each other, etc. The only law not listed with another, the double negation law, is its own dual, since there are no occurrences of $\wedge$, $\vee$, $\mathbf{T}$, or $\mathbf{F}$ to replace.

40. Following the hint, we easily see that the answer is $p \wedge q \wedge \neg r$.

42. The statement of the problem is really the solution. Each line of the truth table corresponds to exactly one combination of truth values for the n atomic propositions involved. We can write down a conjunction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false. If we do this for *each* line of the truth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form.

44. Given a compound proposition p, we can, by Exercise 43, write down a proposition q that is logically equivalent to p and uses only $\neg$, $\wedge$, and $\vee$. Now by De Morgan's law we can get rid of all the $\vee$'s by replacing each occurrence of $p_1 \vee p_2 \vee \cdots \vee p_n$ with $\neg(\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$.

46. We write down the truth table corresponding to the definition.

p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

48. We write down the truth table corresponding to the definition.

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

50. **a)** From the definition (or as seen in the truth table constructed in Exercise 48), $p \downarrow p$ is false when p is true and true when p is false, exactly as $\neg p$ is; thus the two are logically equivalent.

b) The proposition $(p \downarrow q) \downarrow (p \downarrow q)$ is equivalent, by part **(a)**, to $\neg(p \downarrow q)$, which from the definition (or truth table or Exercise 49) is clearly equivalent to $p \vee q$.

c) By Exercise 45, every compound proposition is logically equivalent to one that uses only $\neg$ and $\vee$. But by parts **(a)** and **(b)** of the present exercise, we can get rid of all the negations and disjunctions by using *NOR*'s. Thus every compound proposition can be converted into a logically equivalent compound proposition involving only *NOR*'s.

52. This exercise is similar to Exercise 50. First we can see from the truth tables that $(p \mid p) \equiv (\neg p)$ and that $((p \mid p) \mid (q \mid q)) \equiv (p \vee q)$. Then we argue exactly as in part **(c)** of Exercise 50: by Exercise 45, every compound proposition is logically equivalent to one that uses only $\neg$ and $\vee$. But by our observations at the beginning of the present exercise, we can get rid of all the negations and disjunctions by using *NAND*'s. Thus every compound proposition can be converted into a logically equivalent compound proposition involving only *NAND*'s.

54. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, and r for which the truth values of $p \mid (q \mid r)$ and $(p \mid q) \mid r$ differ. One such assignment is T for p and F for q and r. Then computing from the truth tables (or definitions), we see that $p \mid (q \mid r)$ is false and $(p \mid q) \mid r$ is true.

56. To say that p and q are logically equivalent is to say that the truth tables for p and q are identical; similarly, to say that q and r are logically equivalent is to say that the truth tables for q and r are identical. Clearly if the truth tables for p and q are identical, and the truth tables for q and r are identical, then the truth tables for p and r are identical (this is a fundamental axiom of the notion of equality). Therefore p and r are logically equivalent. (We are assuming—and there is no loss of generality in doing so—that the same atomic variables appear in all three propositions.)

58. If we want the first two of these to be true, then p and q must have the same truth value. If q is true, then the third and fourth expressions will be true, and if r is false, the last expression will be true. So all five of these disjunctions will be true if we set p and q to be true, and r to be false.

60. In each case we hunt for truth assignments that make all the disjunctions true.

a) Since p occurs in four of the five disjunctions, we can make p true, and then make q false (and make r and s anything we please). Thus this proposition is satisfiable.

b) This is satisfiable by, for example, setting p to be false (that takes care of the first, second, and fourth disjunctions), s to be false (for the third and sixth disjunctions), q to be true (for the fifth disjunction), and r to be anything.

c) It is not hard to find a satisfying truth assignment, such as p, q, and s true, and r false.