SUPPLEMENTARY EXERCISES FOR CHAPTER 1

2. The truth table is as follows.

p	q	r	$p \vee q$	$p \wedge \neg r$	$(p \vee q) \to (p \wedge \neg r)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

4. **a)** The converse is "If I drive to work today, then it will rain." The contrapositive is "If I do not drive to work today, then it will not rain." The inverse is "If it does not rain today, then I will not drive to work."
b) The converse is "If $x \geq 0$ then $|x| = x$." The contrapositive is "If $x < 0$ then $|x| \neq x$." The inverse is "If $|x| \neq x$, then $x < 0$."
c) The converse is "If n^2 is greater than 9, then n is greater than 3." The contrapositive is "If n^2 is not greater than 9, then n is not greater than 3." The inverse is "If n is not greater than 3, then n^2 is not greater than 9."

6. The inverse of $p \to q$ is $\neg p \to \neg q$. Therefore the inverse of the inverse is $\neg\neg p \to \neg\neg q$, which is equivalent to $p \to q$ (the original proposition). The converse of $p \to q$ is $q \to p$. Therefore the inverse of the converse is $\neg q \to \neg p$, which is the contrapositive of the original proposition. The inverse of the contrapositive is $q \to p$, which is the same as the converse of the original statement.

8. Let t be "Sergei takes the job offer"; let b be "Sergei gets a signing bonus"; and let h be "Sergei will receive a higher salary." The given statements are $t \to b$, $t \to h$, $b \to \neg h$, and t. By modus ponens we can conclude b and h from the first two conditional statements, and therefore we can conclude $\neg h$ from the third conditional statement. We now have the contradiction $h \wedge \neg h$, so these statements are inconsistent.

10. Since both knights and knaves claim that they are knights (the former truthfully and the latter deceivingly), we know that A is a knave. But since A's statement must be false, and the first part of the conjunction is true, the second part must be false, so we know that B must be a knave as well. If C were a knight, then B's statement would be true, and knaves must lie, so C must also be a knave. Thus all three are knaves.

12. If S is a proposition, then it is either true or false. If S is false, then the statement "If S is true, then unicorns live" is vacuously true; but this statement *is* S, so we would have a contradiction. Therefore S is true, so the statement "If S is true, then unicorns live" is true and has a true hypothesis. Hence it has a true conclusion (modus ponens), and so unicorns live. But we know that unicorns do not live. It follows that S cannot be a proposition.

14. **a)** The answer is $\exists x P(x)$ if we do not read any significance into the use of the plural, and $\exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$ if we do.
b) $\neg \forall x P(x)$, or, equivalently, $\exists x \neg P(x)$ **c)** $\forall y Q(y)$
d) $\forall x P(x)$ (the class has nothing to do with it) **e)** $\exists y \neg Q(y)$

16. The given statement tells us that there are exactly two elements in the domain. Therefore the statement will be true as long as we choose the domain to be anything with size 2, such as the United States presidents named Bush.

18. We want to say that for every y, there do not exist four different people each of whom is the grandmother of y. Thus we have $\forall x \neg \exists a \exists b \exists c \exists d (a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge G(a,y) \wedge G(b,y) \wedge G(c,y) \wedge G(d,y))$.

20. **a)** Since there is no real number whose square is -1, it is true that there exist exactly 0 values of x such that $x^2 = -1$.
b) This is true, because 0 is the one and only value of x such that $|x| = 0$.
c) This is true, because $\sqrt{2}$ and $-\sqrt{2}$ are the only values of x such that $x^2 = 2$.
d) This is false, because there are more than three values of x such that $x = |x|$, namely all positive real numbers.

22. Let us assume the hypothesis. This means that there is some x_0 such that $P(x_0, y)$ holds for all y. Then it is certainly true that for all y there exists an x such that $P(x,y)$ is true, since in each case we can take $x = x_0$. Note that the converse is not always a tautology, since the x in $\forall y \exists x P(x,y)$ can depend on y.

24. No. Here is an example. Let $P(x,y)$ be $x > y$, where we are talking about integers. Then for every y there does exist an x such that $x > y$; we could take $x = y + 1$, for example. However, there does not exist an x such that for *every* y, $x > y$; in other words, there is no superlarge integer (if for no other reason than that no integer can be larger than itself).

26. a) It will snow today, but I will not go skiing tomorrow.
b) Some person in this class does not understand mathematical induction.
c) All students in this class like discrete mathematics.
d) There is some mathematics class in which all the students stay awake during lectures.

28. Let $W(r)$ means that room r is painted white. Let $I(r,b)$ mean that room r is in building b. Let $L(b,u)$ mean that building b is on the campus of United States university u. Then the statement is that there is some university u and some building on the campus of u such that every room in b is painted white. In symbols this is $\exists u \exists b(L(b,u) \wedge \forall r(I(r,b) \rightarrow W(r)))$.

30. To say that there are exactly two elements that make the statement true is to say that two elements exist that make the statement true, and that every element that makes the statement true is one of these two elements. More compactly, we can phrase the last part by saying that an element makes the statement true if and only if it is one of these two elements. In symbols this is $\exists x \exists y(x \neq y \wedge \forall z(P(z) \leftrightarrow (z = x \vee z = y)))$. In English we might express the rule as follows. The hypotheses are that $P(x)$ and $P(y)$ are both true, that $x \neq y$, and that every z that satisfies $P(z)$ must be either x or y. The conclusion is that there are exactly two elements that make P true.

32. We give a proof by contraposition. If x is rational, then $x = p/q$ for some integers p and q with $q \neq 0$. Then $x^3 = p^3/q^3$, and we have expressed x^3 as the quotient of two integers, the second of which is not zero. This by definition means that x^3 is rational, and that completes the proof of the contrapositive of the original statement.

34. Let m be the square root of n, rounded down if it is not a whole number. (In the notation to be introduced in Section 2.3, we are letting $m = \lfloor\sqrt{n}\rfloor$.) We can see that this is the unique solution in a couple of ways. First, clearly the different choices of m correspond to a partition of $\mathbf{N}$, namely into $\{0\}$, $\{1,2,3\}$, $\{4,5,6,7,8\}$, $\{9,10,11,12,13,14,15\}$, ... So every n is in exactly one of these sets. Alternatively, take the square root of the given inequalities to give $m \leq \sqrt{n} < m+1$. That m is then the floor of $\sqrt{n}$ (and that m is unique) follows from statement (1a) of Table 1 in Section 2.3.

36. A constructive proof seems indicated. We can look for examples by hand or with a computer program. The smallest ones to be found are $50 = 5^2 + 5^2 = 1^2 + 7^2$ and $65 = 4^2 + 7^2 = 1^2 + 8^2$.

38. We claim that the number 7 is not the sum of at most two squares and a cube. The first two positive squares are 1 and 4, and the first positive cube is 1, and these are the only numbers that could be used in forming the sum. Clearly no sum of three or fewer of these is 7. This counterexample disproves the statement.

40. We give a proof by contradiction. If $\sqrt{2} + \sqrt{3}$ were rational, then so would be its square, which is $5 + 2\sqrt{6}$. Subtracting 5 and dividing by 2 then shows that $\sqrt{6}$ is rational, but this contradicts the theorem we are told to assume.