SECTION 7.4 Generating Functions

2. The generating function is $f(x) = 1 + 4x + 16x^2 + 64x^3 + 256x^4$. Since the i^{th} term in this sequence (the coefficient of x^i) is 4^i for $0 \le i \le 4$, we can also write the generating function as

$$f(x) = \sum_{i=0}^{4} (4x)^i = \frac{1-(4x)^5}{1-4x}.$$

4. We will use Table 1 in much of this solution.

a) Apparently all the terms are 0 except for the seven -1's shown. Thus $f(x) = -1-x-x^2-x^3-x^4-x^5-x^6$. This is already in closed form, but we can also write it more compactly as $f(x) = -(1-x^7)/(1-x)$, making use of the identity from Example 2.

b) This sequence fits the pattern in Table 1 for $1/(1-ax)$ with $a = 3$. Therefore the generating function is $1/(1-3x)$.

c) We can factor out $3x^2$ and write the generating function as $3x^2(1-x+x^2-x^3+\cdots) = 3x^2/(1+x)$, again using the identity in Table 1.

d) Except for the extra x (the coefficient of x is 2 rather than 1), the generating function is just $1/(1-x)$. Therefore the answer is $x + (1/(1-x))$.

e) From Table 1, we see that the Binomial Theorem applies and we can write this as $(1+2x)^7$.

f) We can factor out -3 and write the generating function as $-3(1-x+x^2-x^3+\cdots) = -3/(1+x)$, using the identity in Table 1.

g) We can factor out x and write the generating function as $x(1-2x+4x^2-8x^3+\cdots) = x/(1+2x)$, using the sixth identity in Table 1 with $a = -2$.

h) From Table 1 we see that the generating function here is $1/(1-x^2)$.

6. a) Since the sequence with $a_n = 1$ for all n has generating function $1/(1-x)$, this sequence has generating function $-1/(1-x)$.

b) By Table 1, the generating function for the sequence in which $a_n = 2^n$ for *all* n is $1/(1-2x)$. Here we can either think of subtracting out the missing constant term (since $a_0 = 0$) or factoring out $2x$. Therefore the answer can be written as either $1/(1-2x) - 1$ or $2x/(1-2x)$, which are of course algebraically equivalent.

c) We need to split this into two parts. Since we know that the generating function for the sequence $\{n+1\}$ is $1/(1-x)^2$, we write $n - 1 = (n+1) - 2$. Therefore the generating function is $(1/(1-x)^2) - (2/(1-x))$. We can combine terms and write this function as $(2x-1)/(1-x)^2$, but there is no particular reason to prefer that form in general.

d) The power series for the function e^x is $\sum_{n=0}^{\infty} x^n/n!$. That is almost what we have here; the difference is that the denominator is $(n+1)!$ instead of $n!$. So we have

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

by a change of variable. This last sum is $e^x - 1$ (only the first term is missing), so our answer is $(e^x - 1)/x$.

e) Let $f(x)$ be the generating function we seek. From Table 1 we know that $1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, and that is almost what we have here. To transform this to $f(x)$ need to factor out x^2 and change the variable of summation:

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} C(n+2,2)x^n = \frac{1}{x^2}\sum_{n=0}^{\infty} C(n+2,2)x^{n+2} = \frac{1}{x^2}\sum_{n=2}^{\infty} C(n,2)x^n = \frac{1}{x^2}\cdot(f(x) - f(0) - f(1))$$

Noting that $f(0) = f(1) = 0$ by definition, we have $f(x) = x^2/(1-x)^3$.

f) We again use Table 1:

$$\sum_{n=0}^{\infty} C(10,n+1)x^n = \sum_{n=1}^{\infty} C(10,n)x^{n-1} = \frac{1}{x}\sum_{n=1}^{\infty} C(10,n)x^n = \frac{1}{x}((1+x)^{10}-1)$$

8. a) By the Binomial Theorem (the third line of Table 1) we get $a_{2n} = C(3,n)$ for $n = 0, 1, 2, 3$, and the other coefficients are all 0. Alternatively, we could just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = 1$, $a_2 = 3$, $a_4 = 3$, $a_6 = 1$.

b) This is like part (a). First we need to factor out -1 and write this as $-(1-3x)^3$. Then by the Binomial Theorem (the second line of Table 1) we get $a_n = -C(3,n)(-3)^n$ for $n = 0, 1, 2, 3$, and the other coefficients are all 0. Alternatively, we could (by hand or with *Maple*) just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = -1$, $a_1 = 9$, $a_2 = -27$, $a_3 = 27$.

c) This problem requires a combination of the results of the sixth and seventh identities in Table 1. The coefficient of x^{2n} is 2^n, and the odd coefficients are all 0.

d) We know that $x^2/(1-x)^3 = x^2\sum_{n=0}^{\infty} C(n+2,2)x^n = \sum_{n=0}^{\infty} C(n+2,2)x^{n+2} = \sum_{n=2}^{\infty} C(n,2)x^n$. Therefore $a_n = C(n,2) = n(n-1)/2$ for $n \geq 2$ and $a_0 = a_1 = 0$. (Actually, since $C(0,2) = C(1,2) = 0$, we really don't need to make a special statement for $n < 2$.)

e) The last term gives us, from Table 1, $a_n = 3^n$. We need to adjust this for $n = 0$ and $n = 1$ because of the first two terms. Thus $a_0 = -1 + 3^0 = 0$, and $a_1 = 1 + 3^1 = 4$.

f) We split this into two parts and proceed as in part (d):

$$\begin{aligned}\frac{1}{(1+x)^3} + \frac{x^3}{(1+x)^3} &= \sum_{n=0}^{\infty}(-1)^n C(n+2,2)x^n + x^3\sum_{n=0}^{\infty}(-1)^n C(n+2,2)x^n \\ &= \sum_{n=0}^{\infty}(-1)^n C(n+2,2)x^n + \sum_{n=0}^{\infty}(-1)^n C(n+2,2)x^{n+3} \\ &= \sum_{n=0}^{\infty}(-1)^n C(n+2,2)x^n + \sum_{n=3}^{\infty}(-1)^{n-3} C(n-1,2)x^n\end{aligned}$$

Note that n and $n-3$ have opposite parities. Therefore $a_n = (-1)^n C(n+2,2) + (-1)^{n-3}C(n-1,2) = (-1)^n(C(n+2,2) - C(n-1,2)) = (-1)^n 3n$ for $n \geq 3$ and $a_n = (-1)^n C(n+2,2) = (-1)^n(n+2)(n+1)/2$ for $n < 3$. This answer can be confirmed using the `series` command in *Maple*.

g) The key here is to recall the algebraic identity $1 - x^3 = (1-x)(1+x+x^2)$. Therefore the given function can be rewritten as $x(1-x)/(1-x^3)$, which can then be split into $x/(1-x^3)$ plus $-x^2/(1-x^3)$. From Table 1 we know that $1/(1-x^3) = 1 + x^3 + x^6 + x^9 + \cdots$. Therefore $x/(1-x^3) = x + x^4 + x^7 + x^{10} + \cdots$, and $-x^2/(1-x^3) = -x^2 - x^5 - x^8 - x^{11} - \cdots$. Thus we see that a_n is 0 when n is a multiple of 3, it is 1 when n is 1 greater than a multiple of 3, and it is -1 when n is 2 greater than a multiple of 3. One can check this answer with *Maple*.

h) From Table 1 we know that $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$. It follows that

$$e^{3x^2} = 1 + 3x^2 + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!} + \cdots.$$

We can therefore read off the coefficients of the generating function for $e^{3x^2} - 1$. First, clearly $a_0 = 0$. Second, $a_n = 0$ when n is odd. Finally, when n is even, we have $a_{2m} = 3^m/m!$.

10. Different approaches are possible for obtaining these answers. One can use brute force algebra and just multiply everything out, either by hand or with computer algebra software such as *Maple*. One can view the problem as asking for the solution to a particular combinatorial problem and solve the problem by other means (e.g., listing all the possibilities). Or one can get a closed form expression for the coefficients, using the generating function theory developed in this section.

a) First we view this combinatorially. By brute force we can list the ten ways to obtain x^9 when this product is multiplied out (where "ijk" means choose an x^i term from the first factor, an x^j term from the second factor, and an x^k term from the third factor): 009, 036, 063, 090, 306, 333, 360, 603, 630, 900. Second, it is clear that we can view this problem as asking for the coefficient of x^3 in $(1+x+x^2+x^3+\cdots)^3$, since each x^3 in the original is playing the role of x here. Since $(1+x+x^2+x^3+\cdots)^3 = 1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, the answer is clearly $C(3+2,2) = C(5,2) = 10$. A third way to get the answer is to ask *Maple* to expand $(1+x^3+x^6+x^9)^3$ and look at the coefficient of x^9, which will turn out to be 10. Note that we don't have to go beyond x^9 in each factor, because the higher terms can't contribute to an x^9 term in the answer.

b) If we factor out x^2 from each factor, we can write this as $x^6(1+x+x^2+\cdots)^3$. Thus we are seeking the coefficient of x^3 in $(1+x+x^2+\cdots)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, so the answer is $C(3+2,2) = 10$. The other two methods explained in part (a) work here as well.

c) If we factor out as high a power of x from each factor as we can, then we can write this as

$$x^7(1+x^2+x^3)(1+x)(1+x+x^2+x^3+\cdots),$$

and so we seek the coefficient of x^2 in $(1+x^2+x^3)(1+x)(1+x+x^2+x^3+\cdots)$. We could do this by brute force, but let's try it more analytically. We write our expression in closed form as

$$\frac{(1+x^2+x^3)(1+x)}{1-x} = \frac{1+x+x^2+\text{higher order terms}}{1-x} = \frac{1}{1-x} + x\cdot\frac{1}{1-x} + x^2\cdot\frac{1}{1-x} + \text{irrelevant terms.}$$

The coefficient of x^2 in this power series comes either from the coefficient of x^2 in the first term in the final expression displayed above, or from the coefficient of x^1 in the second factor of the second term of that expression, or from the coefficient of x^0 in the second factor of the third term. Each of these coefficients is 1, so our answer is 3. This could also be confirmed by having *Maple* multiply out ("**expand**") the original expression (truncating the last factor at x^3).

d) The easiest approach here is simply to note that there are only two combinations of terms that will give us an x^9 term in the product: $x \cdot x^8$ and $x^7 \cdot x^2$. So the answer is 2.

e) The highest power of x appearing in this expression when multiplied out is x^6. Therefore the answer is 0.

12. These can all be checked by using the `series` command in *Maple*.

a) By Table 1, the coefficient of x^n in this power series is $(-3)^n$. Therefore the answer is $(-3)^{12} = 531{,}441$.

b) By Table 1, the coefficient of x^n in this power series is $2^nC(n+1,1)$. Thus the answer is $2^{12}C(12+1,1) = 53{,}248$.

c) By Table 1, the coefficient of x^n in this power series is $(-1)^nC(n+7,7)$. Therefore the answer is $(-1)^{12}C(12+7,7) = 50{,}388$.

d) By Table 1, the coefficient of x^n in this power series is $4^nC(n+2,2)$. Thus the answer is $4^{12}C(12+2,2) = 1{,}526{,}726{,}656$.

e) This is really asking for the coefficient of x^9 in $1/(1+4x)^2$. Following the same idea as in part (d), we see that the answer is $(-4)^9C(9+1,1) = -2{,}621{,}440$.

14. Each child will correspond to a factor in our generating function. We can give 0, 1, 2, or 3 figures to the child; therefore the generating function for each child is $1+x+x^2+x^3$. We want to find the coefficient of x^{12} in the expansion of $(1+x+x^2+x^3)^5$. We can multiply this out (preferably with a computer algebra package such as *Maple*), and the coefficient of x^{12} turns out to be 35. To solve it analytically, we write our generating function as

$$\left(\frac{1-x^4}{1-x}\right)^5 = \frac{1-5x^4+10x^8-10x^{12}+\text{higher order terms}}{(1-x)^5}.$$

There are four contributions to the coefficient of x^{12}, one for each term in the numerator, from the power series for $1/(1-x)^5$. Since the coefficient of x^n in $1/(1-x)^5$ is $C(n+4,4)$, our answer is $C(12+4,4) - 5C(8+4,4) + 10C(4+4,4) - 10C(0+4,4) = 1820 - 2475 + 700 - 10 = 35$.

16. The factors in the generating function for choosing the egg and plain bagels are both $x^2 + x^3 + x^4 + \cdots$. The factor for choosing the salty bagels is $x^2 + x^3$. Therefore the generating function for this problem is $(x^2 + x^3 + x^4 + \cdots)^2(x^2 + x^3)$. We want to find the coefficient of x^{12}, since we want 12 bagels. This is equivalent to finding the coefficient of x^6 in $(1 + x + x^2 + \cdots)^2(1 + x)$ This function is $(1 + x)/(1 - x)^2$, so we want the coefficient of x^6 in $1/(1 - x)^2$, which is 7, plus the coefficient of x^5 in $1/(1 - x)^2$, which is 6. Thus the answer is 13.

18. Without changing the answer, we can assume that the jar has an infinite number of balls of each color; this will make the algebra easier. For the red and green balls the generating function is $1 + x + x^2 + \cdots$, but for the blue balls the generating function is $x^3 + x^4 + \cdots + x^{10}$, so the generating function for the whole problem is $(1 + x + x^2 + \cdots)^2(x^3 + x^4 + \cdots + x^{10})$. We seek the coefficient of x^{14}. This is the same as the coefficient of x^{11} in

$$(1 + x + x^2 + \cdots)^2(1 + x + \cdots + x^7) = \frac{1 - x^8}{(1 - x)^3}.$$

Since the coefficient of x^n in $1/(1 - x)^3$ is $C(n + 2, 2)$, and we have two contributing terms determined by the numerator, our answer is $C(11 + 2, 2) - C(3 + 2, 2) = 68$.

20. We want the coefficient of x^k to be the number of ways to make change for k pesos. Ten-peso bills contribute 10 each to the exponent of x. Thus we can model the choice of the number of 10-peso bills by the choice of a term from $1 + x^{10} + x^{20} + x^{30} + \cdots$. Twenty-peso bills contribute 20 each to the exponent of x. Thus we can model the choice of the number of 20-peso bills by the choice of a term from $1 + x^{20} + x^{40} + x^{60} + \cdots$. Similarly, 50-peso bills contribute 50 each to the exponent of x, so we can model the choice of the number of 50-peso bills by the choice of a term from $1 + x^{50} + x^{100} + x^{150} + \cdots$. Similar reasoning applies to 100-peso bills. Thus the generating function is $f(x) = (1 + x^{10} + x^{20} + x^{30} + \cdots)(1 + x^{20} + x^{40} + x^{60} + \cdots)(1 + x^{50} + x^{100} + x^{150} + \cdots)(1 + x^{100} + x^{200} + x^{300} + \cdots)$, which can also be written as

$$f(x) = \frac{1}{(1 - x^{10})(1 - x^{20})(1 - x^{50})(1 - x^{100})}$$

by Table 1. Note that $c_k = 0$ unless k is a multiple of 10, and the power series has no terms whose exponents are not powers of 10.

22. Let e_i, for $i = 1, 2, \ldots, n$, be the exponent of x taken from the i^{th} factor in forming a term x^6 in the expansion. Thus $e_1 + e_2 + \cdots + e_n = 6$. The coefficient of x^6 is therefore the number of ways to solve this equation with nonnegative integers, which, from Section 5.5, is $C(n + 6 - 1, 6) = C(n + 5, 6)$. Its value, of course, depends on n.

24. a) The restriction on x_1 gives us the factor $x^3 + x^4 + x^5 + \cdots$. The restriction on x_2 gives us the factor $x + x^2 + x^3 + x^4 + x^5$. The restriction on x_3 gives us the factor $1 + x + x^2 + x^3 + x^4$. And the restriction on x_4 gives us the factor $x + x^2 + x^3 + \cdots$. Thus the answer is the product of these:

$$(x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots)$$

We can use algebra to rewrite this in closed form as $x^5(1 + x + x^2 + x^3 + x^4)^2/(1 - x)^2$.

b) We want the coefficient of x^7 in this series, which is the same as the coefficient of x^2 in the series for

$$\frac{(1 + x + x^2 + x^3 + x^4)^2}{(1 - x)^2} = \frac{1 + 2x + 3x^2 + \text{higher order terms}}{(1 - x)^2}.$$

Since the coefficient of x^n in $1/(1 - x)^2$ is $n + 1$, our answer is $1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 10$.

26. a) On each roll, we can get a total of one pip, two pips, ..., six pips. So the generating function for each roll is $x + x^2 + x^3 + x^4 + x^5 + x^6$. The exponent on x gives the number of pips. If we want to achieve a total of k pips in n rolls, then we need the coefficient of x^k in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$. Since n is free to vary here, we must add these generating functions for all possible values of n. Therefore the generating function for this problem is $\sum_{n=0}^{\infty}(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$. By the formula for summing a geometric series, this is the same as $1/(1 - (x + x^2 + x^3 + x^4 + x^5 + x^6)) = 1/(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)$.
b) We seek the coefficient of x^8 in the power series for our answer to part (a). The best way to get the answer is probably asking *Maple* or another computer algebra package to find this power series, which it will probably do using calculus. If we do so, the answer turns out to be 125 (the series starts out $1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 32x^6 + 63x^7 + 125x^8 + 248x^9$).

28. In each case, the generating function for the choice of pennies is $1 + x + x^2 + \cdots = 1/(1 - x)$ or some portion of this to account for restrictions on the number of pennies used. Similarly, the generating function for the choice of nickels is $1 + x^5 + x^{10} + \cdots = 1/(1 - x^5)$ (or some portion); and similarly for the dimes and quarters. For each part we will write down the generating function (a product of the generating functions for each coin) and then invoke a computer algebra system to get the answer.
a) The generating function for the pennies is $1 + x + x^2 + \cdots + x^{10} = (1 - x^{11})/(1 - x)$. Thus our entire generating function is

$$\frac{1 - x^{11}}{1 - x} \cdot \frac{1}{1 - x^5} \cdot \frac{1}{1 - x^{10}} \cdot \frac{1}{1 - x^{25}}.$$

Maple says that the coefficient of x^{100} in this is 79.
b) This is just like part (a), except that now the generating function is

$$\frac{1 - x^{11}}{1 - x} \cdot \frac{1 - (x^5)^{11}}{1 - x^5} \cdot \frac{1}{1 - x^{10}} \cdot \frac{1}{1 - x^{25}}.$$

This time *Maple* reports that the answer is 58.
c) This problem can be solved by using a generating function with two variables, one for the number of coins (say y) and one for the values (say x). Then the generating function for nickels, for instance, is

$$1 + x^5y + x^{10}y^2 + \cdots = \frac{1}{1 - x^5y}.$$

We multiply the four generating functions together, for the four different denominations, and get a function of x and y. Then we ask *Maple* to expand this as a power series and get the coefficient of x^{100}. This coefficient is a polynomial in y. We ask *Maple* to extract and simplify this polynomial and it turns out to be $y^4 + y^6 + 2y^7 + 2y^8 + 2y^9 + 4y^{10}$ plus higher order terms that we don't want, since we need the number of coins (which is what the exponent on y tells us) to be less than 11. Since the total of these coefficients is 12, the answer is 12, which can be confirmed by brute force enumeration.

30. a) Multiplication distributes over addition, even when we are talking about infinite sums, so the generating function is just $2G(x)$.
b) What used to be the coefficient of x^0 is now the coefficient of x^1, and similarly for the other terms. The way that happened is that the whole series got multiplied by x. Therefore the generating function for this series is $xG(x)$. In symbols,

$$a_0x + a_1x^2 + a_2x^3 + \cdots = x(a_0 + a_1x + a_2x^2 + \cdots) = xG(x).$$

c) The terms involving a_0 and a_1 are missing; $G(x) - a_0 - a_1x = a_2x^2 + a_3x^3 + \cdots$. Here, however, we want a_2 to be the coefficient of x^4, not x^2 (and similarly for the other powers), so we must throw in an extra factor. Thus the answer is $x^2(G(x) - a_0 - a_1x)$.
d) This is just like part (c), except that we slide the powers down. Thus the answer is $(G(x) - a_0 - a_1x)/x^2$.

e) Following the hint, we differentiate $G(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain $G'(x) = \sum_{n=0}^{\infty} n\, a_n x^{n-1}$. By a change of variable this becomes $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, which is the generating function for precisely the sequence we are given. Thus $G'(x)$ is the generating function for this sequence.

f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x) \cdot G(x)$.

32. This problem is like Example 16. First let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by changing the name of the variable from k to $k+1$). Thus

$$G(x) - 7xG(x) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 7a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 7a_{k-1})x^k = a_0 + 0 = 5\,,$$

because of the given recurrence relation and initial condition. Thus $G(x)(1-7x) = 5$, so $G(x) = 5/(1-7x)$. From Table 1 we know then that $a_k = 5 \cdot 7^k$.

34. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$). Thus

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k = 1 + \sum_{k=1}^{\infty} 4^{k-1}x^k \\ &= 1 + x\sum_{k=1}^{\infty} 4^{k-1}x^{k-1} = 1 + x\sum_{k=0}^{\infty} 4^k x^k = 1 + x \cdot \frac{1}{1-4x} = \frac{1-3x}{1-4x}\,. \end{aligned}$$

Thus $G(x)(1-3x) = (1-3x)/(1-4x)$, so $G(x) = 1/(1-4x)$. Therefore $a_k = 4^k$, from Table 1.

36. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and $x^2G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$\begin{aligned} G(x) - xG(x) - 2x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1}x^k - \sum_{k=2}^{\infty} 2a_{k-2}x^k = a_0 + a_1 x - a_0 x + \sum_{k=2}^{\infty} 2^k \cdot x^k \\ &= 4 + 8x + \frac{1}{1-2x} - 1 - 2x = \frac{4-12x^2}{1-2x}\,, \end{aligned}$$

because of the given recurrence relation, the initial conditions, Table 1, and algebra. Since the left-hand side of this equation factors as $G(x)(1-2x)(1+x)$, we have $G(x) = (4-12x^2)/((1+x)(1-2x)^2)$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}\,,$$

multiplying through by the common denominator, and equating coefficients, we find that $A = -8/9$, $B = 38/9$, and $C = 2/3$. Thus

$$G(x) = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2} = \sum_{k=0}^{\infty}\left(-\frac{8}{9}(-1)^k + \frac{38}{9}\cdot 2^k + \frac{2}{3}(k+1)2^k\right)x^k$$

(from Table 1). Therefore $a_k = (-8/9)(-1)^k + (38/9)2^k + (2/3)(k+1)2^k$. Incidentally, it would be wise to check our answers, either with a computer algebra package, or by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 24$ both ways).

38. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and similarly $x^2G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$\begin{aligned} G(x) - 2xG(x) - 3x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 2a_{k-1}x^k - \sum_{k=2}^{\infty} 3a_{k-2}x^k = a_0 + a_1 x - 2a_0 x + \sum_{k=2}^{\infty}(4^k+6)\cdot x^k \\ &= 20 + 20x + \frac{1}{1-4x} + \frac{6}{1-x} - 7 - 10x = 13 + 10x + \frac{1}{1-4x} + \frac{6}{1-x} \\ &= \frac{20 - 80x + 2x^2 + 40x^3}{(1-4x)(1-x)}, \end{aligned}$$

because of the given recurrence relation, the initial conditions, and Table 1. Since the left-hand side of this equation factors as $G(x)(1-3x)(1+x)$, we know that

$$G(x) = \frac{20 - 80x + 2x^2 + 40x^3}{(1-4x)(1-x)(1+x)(1-3x)}.$$

At this point we must use partial fractions to break up the denominator. Setting this last expression equal to

$$\frac{A}{1-4x} + \frac{B}{1-x} + \frac{C}{1+x} + \frac{D}{1-3x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A = 16/5$, $B = -3/2$, $C = 31/20$, and $D = 67/4$. Thus

$$G(x) = \frac{16/5}{1-4x} + \frac{-3/2}{1-x} + \frac{31/20}{1+x} + \frac{67/4}{1-3x} = \sum_{k=0}^{\infty}\left(\frac{16}{5}\cdot 4^k - \frac{3}{2} + \frac{31}{20}(-1)^k + \frac{67}{4}\cdot 3^k\right)x^k$$

(from Table 1). Therefore $a_k = (16/5)4^k - (3/2) + (31/20)(-1)^k + (67/4)3^k$. We check our answer by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 202$ both ways). Alternatively, we ask *Maple* for the solution:

```
rsolve({a(k) = 2 * a(k - 1) + 3 * a(k - 2) + 4^k + 6, a(0) = 20, a(1) = 60}, a(k));
```

40. a) By definition,

$$\begin{aligned} \binom{-1/2}{n} &= \frac{(-1/2)(-3/2)(-5/2)\cdots(-(2n-1)/2)}{n!} \\ &= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} \\ &= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}\cdot\frac{2\cdot 4\cdot 6\cdot(2n)}{2^n\, n!} \\ &= (-1)^n \frac{(2n)!}{n!\, n!\, 4^n} \\ &= (-1)^n \binom{2n}{n}\frac{1}{4^n} = \binom{2n}{n}\frac{1}{(-4)^n} \end{aligned}$$

b) By the extended Binomial Theorem (Theorem 2), with $-4x$ in place of x and $u = -1/2$, we have

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4x)^n = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(-4)^n}(-4x)^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

42. First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the Binomial Theorem, we have

$$\begin{aligned} \sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\ &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r. \end{aligned}$$

Thus

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n.$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r)+C(n-1,r-1)$ for $1\le r\le n-1$, as desired.

44. Let $G(x)=\sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}$, where $a_n = 1^2+2^2+3^2+\cdots+n^2$.
a) We use the method of generating functions to solve the recurrence relation and initial condition that our sequence satisfies: $a_n = a_{n-1}+n^2$ with $a_0=0$ (as in, for example, Exercise 34):

$$G(x)-xG(x)=\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=0}^{\infty} n^2 x^n .$$

By Exercise 37, the generating function for $\{n^2\}$ is

$$\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{1}{1-x}=\frac{x^2+x}{(1-x)^3},$$

so $(1-x)G(x)=(x^2+x)/(1-x)^3$. Dividing both sides by $1-x$ gives the desired expression for $G(x)$.
b) We split the generating function we found for $G(x)=\sum_{n=0}^{\infty} a_n x^n$ into two pieces and use Table 1:

$$\begin{aligned}\frac{x^2}{(1-x)^4}+\frac{x}{(1-x)^4} &= \sum_{n=0}^{\infty} C(n+3,3)x^{n+2}+\sum_{n=0}^{\infty} C(n+3,3)x^{n+1}\\ &= \sum_{n=0}^{\infty} C(n+1,3)x^n + \sum_{n=0}^{\infty} C(n+2,3)x^n\\ &= \sum_{n=0}^{\infty} \frac{(n+1)n(n-1)+(n+2)(n+1)n}{6}x^n\\ &= \sum_{n=0}^{\infty} \frac{n(n+1)(2n+1)}{6}x^n ,\end{aligned}$$

as desired. (Note that we did not need to change the limits of summation in line 3 because $C(1,3)=C(2,3)=0$.)

46. We will make heavy use of the identity $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$.

a) $\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n = 2\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = e^{-2x}$
b) $\sum_{n=0}^{\infty}\frac{-1}{n!}x^n = -\sum_{n=0}^{\infty}\frac{1}{n!}x^n = -e^x$
c) $\sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = xe^x$, by a change of variable (This could also be done using calculus.)
d) This generating function can be obtained either with calculus or without. To do it without calculus, write $\sum_{n=0}^{\infty} n(n-1)\frac{x^n}{n!} = \sum_{n=2}^{\infty}\frac{x^n}{(n-2)!} = x^2\sum_{n=0}^{\infty}\frac{x^n}{n!} = x^2e^x$, by a change of variable. To do it with calculus, start with $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and differentiate both sides twice to obtain $e^x=\sum_{n=0}^{\infty}\frac{n(n-1)}{n!}x^{n-2}=\frac{1}{x^2}\sum_{n=0}^{\infty} n(n-1)\frac{x^n}{n!}$.
Therefore $\sum_{n=0}^{\infty} n(n-1)\frac{x^n}{n!} = x^2e^x$.

e) This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} = \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{x^n}{n!} = \frac{1}{x^2}(e^x - x - 1).$$

To do it with calculus, integrate $e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!}$ from 0 to t to obtain

$$e^t - 1 = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \cdot \frac{1}{n!}.$$

Then differentiate again, from 0 to x, to obtain

$$e^x - x - 1 = \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)(n+1)n!} = x^2 \sum_{n=0}^{\infty} \frac{x^n}{(n+2)(n+1)n!}.$$

Thus $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \cdot \frac{x^n}{n!} = (e^x - x - 1)/x^2$.

48. In many of these cases, it's a matter of plugging the exponent of e into the generating function for e^x. We let a_n denote the n^{th} term of the sequence whose generating function is given.

a) The generating function is $e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} 3^n \frac{x^n}{n!}$, so the sequence is $a_n = 3^n$.

b) The generating function is $2e^{-3x+1} = (2e)e^{-3x} = 2e \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} (2e(-3)^n) \frac{x^n}{n!}$, so the sequence is $a_n = 2e(-3)^n$.

c) The generating function is $e^{4x} + e^{-4x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \sum_{n=0}^{\infty} (4^n + (-4)^n) \frac{x^n}{n!}$, so the sequence is $a_n = 4^n + (-4)^n$.

d) The sequence whose exponential generating function is e^{3x} is clearly $\{3^n\}$, as in part (a). Since

$$1 + 2x = \frac{1}{0!}x^0 + \frac{2}{1!}x^1 + \sum_{n=2}^{\infty} \frac{0}{n!}x^n,$$

we know that $a_n = 3^n$ for $n \geq 2$, with $a_1 = 3^1 + 2 = 5$ and $a_0 = 3^0 + 1 = 2$.

e) We know that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} x^n,$$

so the sequence for which $1/(1+x)$ is the exponential generating function is $\{(-1)^n n!\}$. Combining this with the rest of the function (where the generating function is just $\{1\}$), we have $a_n = 1 - (-1)^n n!$.

f) Note that

$$xe^x = \sum_{n=0}^{\infty} x \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = \sum_{n=1}^{\infty} n \cdot \frac{x^n}{n!} = \sum_{n=0}^{\infty} n \cdot \frac{x^n}{n!}.$$

(We changed variable in the middle.) Therefore $a_n = n$, as in Exercise 46c.

g) First we note that

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!} = 1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots$$
$$= \frac{x^0}{0!} \cdot \frac{0!}{0!} + \frac{x^3}{3!} \cdot \frac{3!}{1!} + \frac{x^6}{6!} \cdot \frac{6!}{2!} + \frac{x^9}{9!} \cdot \frac{9!}{3!} + \cdots.$$

Therefore we see that $a_n = 0$ if n is not a multiple of 3, and $a_n = n!/(n/3)!$ if n is a multiple of 3.

50. **a)** Since all 4^n base-four strings of length n fall into one of the four categories counted by a_n, b_n, c_n, and d_n, obviously $d_n = 4^n - a_n - b_n - c_n$. Next let's see how a string of various types of length $n+1$ can be obtained from a string of length n by adding one digit. To get a string of length $n+1$ with an even number of 0s and an even number of 1s, we can take a string of length n with these same parities and append a 2 or a 3 (thus there are $2a_n$ such strings of this type), or we can take a string of length n with an even number of 0s and an odd number of 1s and append a 1 (thus there are b_n such strings of this type), or we can take a string of length n with an odd number of 0s and an even number of 1s and append a 0 (thus there are c_n such strings of this type). Therefore we have $a_{n+1} = 2a_n + b_n + c_n$. In the same way we find that $b_{n+1} = 2b_n + a_n + d_n$, which equals $b_n - c_n + 4^n$ after substituting the identity with which we began this solution. Similarly, $c_{n+1} = 2c_n + a_n + d_n = c_n - b_n + 4^n$.

b) The strings of length 1 are 0, 1, 2, and 3. So clearly $a_1 = 2$, $b_1 = c_1 = 1$, and $d_1 = 0$. (Note that 0 is an even number.) In fact we can also say that $a_0 = 1$ (the empty string) and $b_0 = c_0 = d_0 = 0$.

c) We apply the recurrences from part (a) twice:

$$\begin{aligned} a_2 &= 2\cdot 2 + 1 + 1 = 6 & a_3 &= 2\cdot 6 + 4 + 4 = 20 \\ b_2 &= 1 - 1 + 4 = 4 & b_3 &= 4 + 16 - 4 = 16 \\ c_2 &= 1 - 1 + 4 = 4 & c_3 &= 4 + 16 - 4 = 16 \\ d_2 &= 16 - 6 - 4 - 4 = 2 & d_3 &= 64 - 20 - 16 - 16 = 12 \end{aligned}$$

d) Before proceeding as the problem asks, we note a shortcut. By symmetry, b_n must be the same as c_n. Substituting this into our recurrences, we find immediately that $b_n = c_n = 4^{n-1}$ for $n \ge 1$. Therefore $a_n = 2a_{n-1} + 2\cdot 4^{n-2}$. This recurrence with the initial condition $a_1 = 2$ can easily be solved by the methods of either this section or Section 7.2 to give $a_n = 2^{n-1} + 4^{n-1}$. But let's proceed as instructed.

Let $A(x)$, $B(x)$, and $C(x)$ be the desired generating functions. Then $xA(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$ and similarly for B and C, so we have

$$A(x) - xB(x) - xC(x) - 2xA(x) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} b_{n-1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = a_0 = 1.$$

Similarly,

$$\begin{aligned} B(x) - xB(x) + xC(x) &= \sum_{n=0}^{\infty} b_n x^n - \sum_{n=1}^{\infty} b_{n-1}x^n + \sum_{n=1}^{\infty} c_{n-1}x^n \\ &= b_0 + \sum_{n=1}^{\infty} 4^{n-1}x^n = 0 + x\sum_{n=0}^{\infty} 4^n x^n = \frac{x}{1-4x}. \end{aligned}$$

Obviously C satisfies the same equation. Therefore our system of three equations (suppressing the arguments on A, B, and C) is

$$\begin{aligned} (1-2x)A - xB - xC &= 1 \\ (1-x)B + xC &= \frac{x}{1-4x} \\ xB + (1-x)C &= \frac{x}{1-4x}. \end{aligned}$$

e) Subtracting the third equation in part (d) from the second shows that $B = C$, and then plugging that back into the second equation immediately gives

$$B(x) = C(x) = \frac{x}{1-4x}.$$

Plugging these into the first equation yields

$$(1-2x)A - 2x\cdot\frac{x}{1-4x} = 1,$$

and solving for A gives us

$$A(x) = \frac{1-4x+2x^2}{(1-2x)(1-4x)}.$$

Now that we know the generating functions, we can recover the coefficients. For B and C (using Table 1) we immediately get a coefficient of 4^{n-1} for all $n \geq 1$, with $b_0 = c_0 = 0$. We rewrite $A(x)$ using partial fractions as

$$A(x) = \frac{1}{4} + \frac{1/2}{1-2x} + \frac{1/4}{1-4x},$$

so we have $a_n = \frac{1}{2} \cdot 2^n + \frac{1}{4} \cdot 4^n = 2^{n-1} + 4^{n-1}$ for $n \geq 1$, with $a_0 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$.

52. To form a partition of n using only odd-sized parts, we must choose some 1s, some 3s, some 5s, and so on. The generating function for choosing 1s is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

(the exponent gives the number so obtained). Similarly, the generating function for choosing 3s is

$$1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}$$

(again the exponent gives the number so obtained). The other choices have analogous generating functions. Therefore the generating function for the entire problem, so that the coefficient of x^n will give $p_o(n)$, the number of partitions of n into odd-sized part, is the infinite product

$$\frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots .$$

54. We need to carefully organize our work so as not to miss any of the partitions. We start with largest-sized parts first in all cases. For $n = 1$, we have $1 = 1$ as the only partition of either type, and so $p_o(1) = p_d(1) = 1$. For $n = 2$, we have $2 = 2$ as the only partition into distinct parts, and $2 = 1 + 1$ as the only partition into odd parts, so $p_o(1) = p_d(1) = 1$. For $n = 3$, we have $3 = 3$ and $3 = 2 + 1$ as the only partitions into distinct parts, and $3 = 3$ and $3 = 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 2$. For $n = 4$, we have $4 = 4$ and $4 = 3 + 1$ as the only partitions into distinct parts, and $4 = 3 + 1$ and $4 = 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 2$. For $n = 5$, we have $5 = 5$, $5 = 4 + 1$, and $5 = 3 + 2$ as the only partitions into distinct parts, and $5 = 5$, $5 = 3 + 1 + 1$, and $5 = 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 3$. For $n = 6$, we have $6 = 6$, $6 = 5 + 1$, $6 = 4 + 2$, and $6 = 3 + 2 + 1$ as the only partitions into distinct parts, and $6 = 5 + 1$, $6 = 3 + 3$, $6 = 3 + 1 + 1 + 1$, and $6 = 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 4$. For $n = 7$, we have $7 = 7$, $7 = 6 + 1$, $7 = 5 + 2$, $7 = 4 + 3$, and $7 = 4 + 2 + 1$ as the only partitions into distinct parts, and $7 = 7$, $7 = 5 + 1 + 1$, $7 = 3 + 3 + 1$, $7 = 3 + 1 + 1 + 1 + 1$, and $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 5$. Finally, for $n = 8$, we have $8 = 8$, $8 = 7 + 1$, $8 = 6 + 2$, $8 = 5 + 3$, $8 = 5 + 2 + 1$, and $8 = 4 + 3 + 1$ as the only partitions into distinct parts, and $8 = 7 + 1$, $8 = 5 + 3$ $8 = 5 + 1 + 1 + 1$, $8 = 3 + 3 + 1 + 1$, $8 = 3 + 1 + 1 + 1 + 1 + 1$, and $8 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 6$. As we will prove in Exercise 55, it is no coincidence that these numbers all agree.

56. This is a very difficult problem. A solution can be found in *The Theory of Partitions* by George Andrews (Addison-Wesley, 1976), Chapter 6.

58. a) In order to have the first success on the n^{th} trial, where $n \geq 1$, we must have $n - 1$ failures followed by a success. Therefore $p(X = n) = q^{n-1}p$, where p is the probability of success and $q = 1 - p$ is the probability of failure. Therefore the probability generating function is

$$G(x) = \sum_{n=1}^{\infty} q^{n-1}px^n = px\sum_{n=1}^{\infty}(qx)^{n-1} = px\sum_{n=0}^{\infty}(qx)^n = \frac{px}{1-qx}.$$

b) By Exercise 57, $E(X)$ is the derivative of $G(x)$ at $x = 1$. Here we have

$$G'(x) = \frac{p}{(1-qx)^2}, \quad \text{so} \quad G'(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

From the same exercise, we know that the variance is $G''(1) + G'(1) - G'(1)^2$; so we compute:

$$G''(x) = \frac{2pq}{(1-qx)^3}, \quad \text{so} \quad G''(1) = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2},$$

and therefore

$$V(X) = G''(1) + G'(1) - G'(1)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}.$$

60. We start with the definition and then use the fact that the only way for the sum of two nonnegative integers to be k is for one of them to be i and the other to be $k-i$, for some i between 0 and k, inclusive. We then invoke independence, and finally the definition of multiplication of infinite series:

$$\begin{aligned}
G_{X+Y}(x) &= \sum_{k=0}^{\infty} p(X+Y=k)x^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} p(X=i \text{ and } Y=k-i)\right)x^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} p(X=i)\cdot p(Y=k-i)\right)x^k \\
&= G_X(x)\cdot G_Y(x)
\end{aligned}$$