SECTION 8.5 Equivalence Relations

2. a) This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).

b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).

c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

d) This is not an equivalence relation since it is clearly not transitive.

e) Again, just as in part (c), this is not transitive.

4. One relation is that a and b are related if they were born in the same U.S. state (with "not in a state of the U.S." counting as one state). Here the equivalence classes are the nonempty sets of students from each state. Another example is for a to be related to b if a and b have lived the same number of complete decades. The equivalence classes are the set of all 10-to-19 year-olds, the set of all 20-to-29 year-olds, and so on (the sets among these that are nonempty, that is). A third example is for a to be related to b if 10 is a divisor of the difference between a's age and b's age, where "age" means the whole number of years since birth, as of the first day of class. For each $i = 0, 1, \ldots, 9$, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit i.

6. One way to partition the classes would be by level. At many schools, classes have three-digit numbers, the first digit of which is approximately the level of the course, so that courses numbered 100–199 are taken by freshman, 200–299 by sophomores, and so on. Formally, two classes are related if their numbers have the same digit in the hundreds column; the equivalence classes are the set of all 100-level classes, the set of all 200-level classes, and so on. A second example would focus on department. Two classes are equivalent if they are offered by the same department; for example, MATH 154 is equivalent to MATH 372, but not to EGR 141. The equivalence classes are the sets of classes offered by each department (the set of math classes, the set of engineering classes, and so on). A third—and more egocentric—classification would be to have one equivalence class be the set of classes that you have completed successfully and the other equivalence class to be all the other classes. Formally, two classes are equivalent if they have the same answer to the question, "Have I completed this class successfully?"

8. Recall (Definition 4 in Section 2.4) that two sets have the same cardinality if there is a bijection (one-to-one and onto function) from one set to the other. We must show that R is reflexive, symmetric, and transitive. Every set has the same cardinality as itself because of the identity function. If f is a bijection from S to T, then f^{-1} is a bijection from T to S, so R is symmetric. Finally, if f is a bijection from S to T and g is a bijection from T to U, then $g \circ f$ is a bijection from T to U, so R is transitive (see Exercise 29 in Section 2.3).

The equivalence class of $\{1,2,3\}$ is the set of all three-element sets of real numbers, including such sets as $\{4,25,1948\}$ and $\{e,\pi,\sqrt{2}\}$. Similarly, $[\mathbf{Z}]$ is the set of all infinite countable sets of real numbers (see Section 2.4), such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1,2,3\}$ (it's too small) or the set of all real numbers (it's too big). See Section 2.4 for more on countable sets.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

12. This follows from Exercise 9, where f is the function that takes a bit string of length $n \geq 3$ to its last $n-3$ bits.

14. This follows from Exercise 9, where f is the function that takes a string of uppercase and lowercase English letters and changes all the lower case letters to their uppercase equivalents (and leaves the uppercase letters unchanged).

16. This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since clearly $ad=bc$ if and only if $a/b=c/d$.

If we want an explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad=bc$, which also means that $cb=da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad=bc$ and $cf=de$. Multiplying these equations gives $acdf=bcde$, and since all these numbers are nonzero, we have $af=be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

18. **a)** This follows from Exercise 9, where the function f from the set of polynomials to the set of polynomials is the operator that takes the derivative n times—i.e., f of a function g is the function $g^{(n)}$. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "n^{th} derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) The third derivative of x^4 is $24x$. Since the third derivative of a polynomial of degree 2 or less is 0, the polynomials of the form x^4+ax^2+bx+c have the same third derivative. Thus these are the functions in the same equivalence class as f.

20. This follows from Exercise 9, where the function f from the set of people to the set of Web-traversing behaviors starting at the given particular Web page takes the person to the behavior that person exhibited.

22. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is an equivalence relation, satisfying all three properties. The equivalence classes are $\{a,d\}$ and $\{b,c\}$.

24. **a)** This is not an equivalence relation, since it is not symmetric.

b) This is an equivalence relation; one equivalence class consists of the first and third elements, and the other consists of the second and fourth elements.

c) This is an equivalence relation; one equivalence class consists of the first, second, and third elements, and the other consists of the fourth element.

26. Only part (a) and part (c) are equivalence relations. In part (a) each element is in an equivalence class by itself. In part (c) the elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

28. Only part (a) and part (d) are equivalence relations. In part (a) there is one equivalence class for each $n \in \mathbf{Z}$, and it contains all those functions whose value at 1 is n. In part (d) there really is no good way to describe the equivalence classes. For one thing, the set of equivalence classes is uncountable. For each function $f : \mathbf{Z} \to \mathbf{Z}$, there is the equivalence class consisting of all those functions g for which there is a constant C such that $g(n) = f(n) + C$ for all $n \in \mathbf{Z}$.

30. a) all the strings whose first three bits are 010 b) all the strings whose first three bits are 101
c) all the strings whose first three bits are 111 d) all the strings whose first three bits are 010

32. Since two strings are related if they agree beyond their first 3 bits, the equivalence class of a bit string $xyzt$, where x, y, and z are bits, and t is a bit string, is the set of all bit strings of the form $x'y'z't$, where x', y', and z' are any bits.
a) the set of all bit strings of length 3 (take $t = \lambda$ in the formulation given above)
b) the set of all bit strings of length 4 that end with a 1
c) the set of all bit strings of length 5 that end 11
d) the set of all bit strings of length 8 that end 10101

34. a) Since this string has length less than 5, its equivalence class consists only of itself.
b) This is similar to part (a): $[1011]_{R_5} = \{1011\}$.
c) Since this string has length 5, its equivalence class consists of all strings that start 11111.
d) This is similar to part (c): $[01010101]_{R_5} = \{01010s \mid s \text{ is any bit string}\}$.

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo m.
a) $\{4 + 2n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 0, 2, 4, \ldots\}$ b) $\{4 + 3n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 1, 4, 7, \ldots\}$
c) $\{4 + 6n \mid n \in \mathbf{Z}\} = \{\ldots, -2, 4, 10, 16, \ldots\}$ d) $\{4 + 8n \mid n \in \mathbf{Z}\} = \{\ldots, -4, 4, 12, 20, \ldots\}$

38. In each case we need to allow all strings that agree with the given string if we ignore the case in which the letters occur.
a) $\{NO, No, nO, no\}$
b) $\{YES, YEs, YeS, Yes, yES, yEs, yeS, yes\}$
c) $\{HELP, HELp, HElP, HElp, HeLP, HeLp, HelP, Help, hELP, hELp, hElP, hElp, heLP, heLp, helP, help\}$

40. a) By our observation in the solution to Exercise 16, the equivalence class of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.
b) Again by our observation, the equivalence classes are the positive rational numbers. (Indeed, this is the way one can rigorously define what a rational number is, and this is why fractions are so difficult for children to understand.)

42. a) This is a partition, since it satisfies the definition.
b) This is not a partition, since the subsets are not disjoint.
c) This is a partition, since it satisfies the definition.
d) This is not a partition, since the union of the subsets leaves out 0.

44. a) This is clearly a partition. b) This is not a partition, since 0 is in neither set.
c) This is a partition by the division algorithm.
d) This is a partition, since the second set mentioned is the set of all number between -100 and 100, inclusive.
e) The first two sets are not disjoint (4 is in both), so this is not a partition.

46. a) This is a partition, since it satisfies the definition.

b) This is a partition, since it satisfies the definition.

c) This is not a partition, since the intervals are not disjoint (they share endpoints).

d) This is not a partition, since the union of the subsets leaves out the integers.

e) This is a partition, since it satisfies the definition.

f) This is a partition, since it satisfies the definition. Each equivalence class consists of all real numbers with a fixed fractional part.

48. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

b) $\{(a,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(f,e),(f,f),(g,g)\}$

c) $\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),$ $(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

d) $\{(a,a),(a,c),(a,e),(a,g),(c,a),(c,c),(c,e),(c,g),(e,a),(e,c),(e,e),(e,g),(g,a),(g,c),(g,e),(g,g),$ $(b,b),(b,d),(d,b),(d,d),(f,f)\}$

50. We need to show that every equivalence class consisting of people living in the same county (or parish) and same state is contained in an equivalence class of all people living in the same state. This is clear. The equivalence class of all people living in county c in state s is a subset of the set of people living in state s.

52. We are asked to show that every equivalence class for R_4 is a subset of some equivalence class for R_3. Let $[y]_{R_4}$ be an arbitrary equivalence class for R_4. We claim that $[y]_{R_4} \subseteq [y]_{R_3}$; proving this claim finishes the proof. To show that one set is a subset of another set, we choose an arbitrary bit string x in the first set and show that it is also an element of the second set. In this case since $y \in [x]_{R_4}$, we know that y is equivalent to x under R_4, that is, that either $y = x$ or y and x are each at least 4 bits long and agree on their first 4 bits. Because strings that are at least 4 bits long and agree on their first 4 bits perforce are at least 3 bits long and agree on their first 3 bits, we know that either $y = x$ or y and x are each at least 3 bits long and agree on their first 3 bits. This means that y is equivalent to x under R_3, that is, that $y \in [x]_{R_3}$.

54. First, suppose that $R_1 \subseteq R_2$. We must show that P_1 is a refinement of P_2. Let $[a]_{R_1}$ be an equivalence class in P_1. We must show that $[a]_{R_1}$ is contained in an equivalence class in P_2. In fact, we will show that $[a]_{R_1} \subseteq [a]_{R_2}$. To this end, let $b \in [a]_{R_1}$. Then $(a,b) \in R_1 \subseteq R_2$. Therefore $b \in [a]_{R_2}$, as desired.

Conversely, suppose that P_1 is a refinement of P_2. Since $a \in [a]_{R_2}$, the definition of "refinement" forces $[a]_{R_1} \subseteq [a]_{R_2}$ for all $a \in A$. This means that for all $b \in A$ we have $(a,b) \in R_1 \rightarrow (a,b) \in R_2$; in other words, $R_1 \subseteq R_2$.

56. a) This need not be an equivalence relation, since it need not be transitive.

b) Since the intersection of reflexive, symmetric, and transitive relations also have these properties (see Section 8.1), the intersection of equivalence relations is an equivalence relation.

c) This will never be an equivalence relation on a nonempty set, since it is not reflexive.

58. This exercise is very similar to Exercise 59, and the reader should look at the solution there for details.

a) As in Exercise 59, the motions of the bracelet form a dihedral group, in this case consisting of six motions: rotations of 0°, 120°, and 240°, and three reflections, each keeping one bead fixed and interchanging the other two. The composition of any two of these operations is again one of these operations. The 0° rotation plays the role of the identity, which says that the relation is reflexive. Each operation has an inverse (reflections are their own inverses, the 0° rotation is its own inverse, and the 120° and 240° rotations are inverses of each other); this proves symmetry. And transitivity follows from the group table.

b) The equivalence classes are the indistinguishable bracelets. If we denote a bracelet by the colors of its beads, then these classes can be described as RRR, WWW, BBB, RRW, RRB, WWR, WWB, BBR, BBW, and RWB. Note that once we specify the colors, then every two bracelets with those colors are equivalent. This would not be the case if there were four or more beads, however. For example, in a 4-bead bracelet with two reds and two whites, the bracelet in which the red beads are adjacent is not equivalent to the one in which they are not.

60. **a)** In Exercise 25 of Section 3.2, we showed that $f(x)$ is $\Theta(g(x))$ if and only if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. To show that R is reflexive, we need to show that $f(x)$ is $O(f(x))$, which is clear by taking $C = 1$ and $k = 1$ in the definition. Symmetry is immediate from the definition, since if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, then $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$. Finally, transitivity follows immediately from the transitive of the "is big-O of" relation, which was proved in Exercise 17 of Section 3.2.

b) This is the class of all functions that asymptotically (i.e., as $n \to \infty$) grow just as fast as a multiple of $f(n) = n^2$. So, for example, functions such as $g(n) = 5n^2 + \log n$, or $g(n) = (n^3 - 17)/(100n + 10^{10})$ belong to this class, but $g(n) = n^{2.01}$ does not (it grows too fast), and $g(n) = n^2/\log n$ does not (it grows too slowly). Another way to express this class is to say that it is the set of all functions g such that there exist constants positive C_1 and C_2 such that the ratio $f(n)/g(n)$ always lies between C_1 and C_2.

62. We will count partitions instead, since equivalence relations are in one-to-one correspondence with partitions. Without loss of generality let the set be $\{1, 2, 3, 4\}$. There is 1 partition in which all the elements are in the same set, namely $\{\{1, 2, 3, 4\}\}$. There are 4 partitions in which the sizes of the sets are 1 and 3, namely $\{\{1\}, \{2, 3, 4\}\}$ and three more like it. There are 3 partitions in which the sizes of the sets are 2 and 2, namely $\{\{1, 2\}, \{3, 4\}\}$ and two more like it. There are 6 partitions in which the sizes of the sets are 2, 1, and 1, namely $\{\{1, 2\}, \{3\}, \{4\}\}$ and five more like it. Finally, there is 1 partition in which all the elements are in separate sets. This gives a total of 15. To actually list the 15 relations would be tedious.

64. No. Here is a counterexample. Start with $\{(1, 2), (3, 2)\}$ on the set $\{1, 2, 3\}$. Its transitive closure is itself. The reflexive closure of that is $\{(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)\}$. The symmetric closure of that is $\{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$. The result is not transitive; for example, $(1, 3)$ is missing. Therefore this is not an equivalence relation.

66. We end up with the original partition P.

68. We will develop this recurrence relation in the context of partitions of the set $\{1, 2, \ldots, n\}$. Note that $p(0) = 1$, since there is only one way to partition the empty set (namely, into the empty collection of subsets). For warm-up, we also note that $p(1) = 1$, since $\{\{1\}\}$ is the only partition of $\{1\}$; that $p(2) = 2$, since we can partition $\{1, 2\}$ either as $\{\{1, 2\}\}$ or as $\{\{1\}, \{2\}\}$; and that $p(3) = 5$, since there are the following partitions: $\{\{1, 2, 3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$, $\{\{1\}, \{2\}, \{3\}\}$. Now to partition $\{1, 2, \ldots, n\}$, we first decide how many other elements of this set will go into the same subset as n goes into. Call this number j, and note that j can take any value from 0 through $n - 1$. Once we have determined j, we can specify the partition by deciding on the subset of j elements from $\{1, 2, \ldots, n - 1\}$ that will go into the same subset as n (and this can be done in $C(n - 1, j)$ ways), and then we need to decide how to partition the remaining $n - 1 - j$ elements (and this can be done in $p(n - j - 1)$ ways). The given recurrence relation now follows.