SECTION 2.3 Functions

2. a) This is not a function because the rule is not well-defined. We do not know whether $f(3)=3$ or $f(3)=-3$. For a function, it cannot be both at the same time.

b) This is a function. For all integers n, $\sqrt{n^2+1}$ is a well-defined real number.

c) This is not a function with domain $\mathbf{Z}$, since for $n=2$ (and also for $n=-2$) the value of $f(n)$ is not defined by the given rule. In other words, $f(2)$ and $f(-2)$ are not specified since division by 0 makes no sense.

4. a) The domain is the set of nonnegative integers, and the range is the set of digits (0 through 9).

b) The domain is the set of positive integers, and the range is the set of integers greater than 1.

c) The domain is the set of all bit strings, and the range is the set of nonnegative integers.

d) The domain is the set of all bit strings, and the range is the set of nonnegative integers (a bit string can have length 0).

6. a) The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$ and the range is $\mathbf{Z}^+$.

b) Since the largest decimal digit of a strictly positive integer cannot be 0, we have domain $\mathbf{Z}^+$ and range $\{1,2,3,4,5,6,7,8,9\}$.

c) The domain is the set of all bit strings. The number of 1's minus number of 0's can be any positive or negative integer or 0, so the range is $\mathbf{Z}$.

d) The domain is given as $\mathbf{Z}^+$. Clearly the range is $\mathbf{Z}^+$ as well.

e) The domain is the set of bit strings. The range is the set of strings of 1's, i.e., $\{\lambda, 1, 11, 111, \ldots\}$, where λ is the empty string (containing no symbols).

8. We simply round up or down in each case.
a) 1 b) 2 c) -1 d) 0 e) 3 f) -2 g) $\lfloor \frac{1}{2} + 1 \rfloor = \lfloor \frac{3}{2} \rfloor = 1$
h) $\lceil 0 + 1 + \frac{1}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$

10. a) This is one-to-one. b) This is not one-to-one, since b is the image of both a and b.
c) This is not one-to-one, since d is the image of both a and d.

12. a) This is one-to-one, since if $n_1 - 1 = n_2 - 1$, then $n_1 = n_2$.
b) This is not one-to-one, since, for example, $f(3) = f(-3) = 10$.
c) This is one-to-one, since if $n_1^3 = n_2^3$, then $n_1 = n_2$ (take the cube root of each side).
d) This is not one-to-one, since, for example, $f(3) = f(4) = 2$.

14. a) This is clearly onto, since $f(0, -n) = n$ for every integer n.
b) This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m-n)(m+n) = 2$, then m and n must have same parity (both even or both odd). In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4 and hence cannot equal 2.
c) This is clearly onto, since $f(0, n-1) = n$ for every integer n.
d) This is onto. To achieve negative values we set $m = 0$, and to achieve nonnegative values we set $n = 0$.
e) This is not onto, for the same reason as in part (b). In fact, the range here is clearly a subset of the range in that part.

16. a) $f(n) = n + 17$ b) $f(n) = \lceil n/2 \rceil$
c) We let $f(n) = n - 1$ for even values of n, and $f(n) = n + 1$ for odd values of n. Thus we have $f(1) = 2$, $f(2) = 1$, $f(3) = 4$, $f(4) = 3$, and so on. Note that this is just one function, even though its definition used two formulae, depending on the the parity of n.
d) $f(n) = 17$

18. If we can find an inverse, the function is a bijection. Otherwise we must explain why the function is not on-to-one or not onto.
a) This is a bijection since the inverse function is $f^{-1}(x) = (4 - x)/3$.
b) This is not one-to-one since $f(17) = f(-17)$, for instance. It is also not onto, since the range is the interval $(-\infty, 7]$. For example, 42548 is not in the range.
c) This function is a bijection, but not from $\mathbf{R}$ to $\mathbf{R}$. To see that the domain and range are not $\mathbf{R}$, note that $x = -2$ is not in the domain, and $x = 1$ is not in the range. On the other hand, f is a bijection from $\mathbf{R} - \{-2\}$ to $\mathbf{R} - \{1\}$, since its inverse is $f^{-1}(x) = (1 - 2x)/(x - 1)$.
d) It is clear that this continuous function is increasing throughout its entire domain ($\mathbf{R}$) and it takes on both arbitrarily large values and arbitrarily small (large negative) ones. So it is a bijection. Its inverse is clearly $f^{-1}(x) = \sqrt[5]{x - 1}$.

20. The key here is that larger denominators make smaller fractions, and smaller denominators make larger fractions. We have two things to prove, since this is an "if and only if" statement. First, suppose that f is strictly increasing. This means that $f(x) < f(y)$ whenever $x < y$. To show that g is strictly decreasing, suppose that $x < y$. Then $g(x) = 1/f(x) > 1/f(y) = g(y)$. Conversely, suppose that g is strictly decreasing. This means that $g(x) > g(y)$ whenever $x < y$. To show that f is strictly increasing, suppose that $x < y$. Then $f(x) = 1/g(x) < 1/g(y) = f(y)$.

22. We need to make the function increasing, but not *strictly* increasing, so, for example, we could take the trivial function $f(x) = 17$. If we want the range to be all of $\mathbf{R}$, we could define f in parts this way: $f(x) = x$ for $x < 0$; $f(x) = 0$ for $0 \le x \le 1$; and $f(x) = x - 1$ for $x > 1$.

24. For the function to be invertible, it must be a one-to-one correspondence. This means that it has to be one-to-one, which it is, and onto, which it is not, because, its range is the set of *positive* real numbers, rather than the set of *all* real numbers. When we restrict the codomain to be the set of positive real numbers, we get an invertible function. In fact, there is a well-known name for the inverse function in this case—the natural logarithm function ($g(x) = \ln x$).

26. In all parts, we simply need to compute the values $f(-1)$, $f(0)$, $f(2)$, $f(4)$, and $f(7)$ and collect the values into a set.
a) $\{1\}$ (all five values are the same) b) $\{-1, 1, 5, 8, 15\}$ c) $\{0, 1, 2\}$ d) $\{0, 1, 5, 16\}$

28. a) the set of even integers b) the set of positive even integers c) the set of real numbers

30. To clarify the setting, suppose that $g : A \to B$ and $f : B \to C$, so that $f \circ g : A \to C$. We will prove that if $f \circ g$ is one-to-one, then g is also one-to-one, so not only is the answer to the question "yes," but part of the hypothesis is not even needed. Suppose that g were not one-to-one. By definition this means that there are distinct elements a_1 and a_2 in A such that $g(a_1) = g(a_2)$. Then certainly $f(g(a_1)) = f(g(a_2))$, which is the same statement as $(f \circ g)(a_1) = (f \circ g)(a_2)$. By definition this means that $f \circ g$ is not one-to-one, and our proof is complete.

32. We have $(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 5$, whereas $(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 2 = x^2 + 3$. Note that they are not equal.

34. Forming the compositions we have $(f \circ g)(x) = acx + ad + b$ and $(g \circ f)(x) = cax + cb + d$. These are equal if and only if $ad + b = cb + d$. In other words, equality holds for all 4-tuples (a, b, c, d) for which $ad + b = cb + d$.

36. a) This really has two parts. First suppose that b is in $f(S \cup T)$. Thus $b = f(a)$ for some $a \in S \cup T$. Either $a \in S$, in which case $b \in f(S)$, or $a \in T$, in which case $b \in f(T)$. Thus in either case $b \in f(S) \cup f(T)$. This shows that $f(S \cup T) \subseteq f(S) \cup f(T)$. Conversely, suppose $b \in f(S) \cup f(T)$. Then either $b \in f(S)$ or $b \in f(T)$. This means either that $b = f(a)$ for some $a \in S$ or that $b = f(a)$ for some $a \in T$. In either case, $b = f(a)$ for some $a \in S \cup T$, so $b \in f(S \cup T)$. This shows that $f(S) \cup f(T) \subseteq f(S \cup T)$, and our proof is complete.
b) Suppose $b \in f(S \cap T)$. Then $b = f(a)$ for some $a \in S \cap T$. This implies that $a \in S$ and $a \in T$, so we have $b \in f(S)$ and $b \in f(T)$. Therefore $b \in f(S) \cap f(T)$, as desired.

38. a) The answer is the set of all solutions to $x^2 = 1$, namely $\{1, -1\}$.
b) In order for x^2 to be strictly between 0 and 1, we need x to be either strictly between 0 and 1 or strictly between -1 and 0. Therefore the answer is $\{ x \mid -1 < x < 0 \vee 0 < x < 1 \}$.
c) In order for x^2 to be greater than 4, we need either $x > 2$ or $x < -2$. Therefore the answer is $\{ x \mid x > 2 \vee x < -2 \}$.

40. a) We need to prove two things. First suppose $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore either $f(x) \in S$ or $f(x) \in T$. In the first case $x \in f^{-1}(S)$, and in the second case $x \in f^{-1}(T)$. In either case, then, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have shown that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. Conversely, suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then either $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, so either $f(x) \in S$ or $f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$, as desired.
b) This is similar to part (a). We have $x \in f^{-1}(S \cap T)$ if and only if $f(x) \in S \cap T$, if and only if $f(x) \in S$ and $f(x) \in T$, if and only if $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, if and only if $x \in f^{-1}(S) \cap f^{-1}(T)$.

42. There are three cases. Define the "fractional part" of x to be $f(x) = x - \lfloor x \rfloor$. Clearly $f(x)$ is always between 0 and 1 (inclusive at 0, exclusive at 1), and $x = \lfloor x \rfloor + f(x)$. If $f(x)$ is less than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly less than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$, and indeed that is the integer closest to x. If $f(x)$ is greater than $\frac{1}{2}$, then $x + \frac{1}{2}$ will have a value slightly greater than $\lfloor x \rfloor + 1$, so when we round down, we get $\lfloor x \rfloor + 1$. In other words, in this case $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, and indeed that is the integer closest to x in this case. Finally, if the fractional part is exactly $\frac{1}{2}$, then x is midway between two integers, and $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor + 1$, which is the larger of these two integers.

44. If x is not an integer, then $\lceil x \rceil$ is the integer just larger than x, and $\lfloor x \rfloor$ is the integer just smaller than x. Clearly they differ by 1. If x is an integer, then $\lceil x \rceil - \lfloor x \rfloor = x - x = 0$.

46. Write $x = n - \epsilon$, where n is an integer and $0 \le \epsilon < 1$; thus $\lceil x \rceil = n$. Then $\lceil x + m \rceil = \lceil n - \epsilon + m \rceil = n + m = \lceil x \rceil + m$. Alternatively, we could proceed along the lines of the proof of property 4a of Table 1, shown in the text.

48. a) The "if" direction is trivial, since $x \le \lceil x \rceil$. For the other direction, suppose that $x \le n$. Since n is an integer no smaller than x, and $\lceil x \rceil$ is by definition the smallest such integer, clearly $\lceil x \rceil \le n$.
b) The "if" direction is trivial, since $\lfloor x \rfloor \le x$. For the other direction, suppose that $n \le x$. Since n is an integer not exceeding x, and $\lfloor x \rfloor$ is by definition the largest such integer, clearly $n \le \lfloor x \rfloor$.

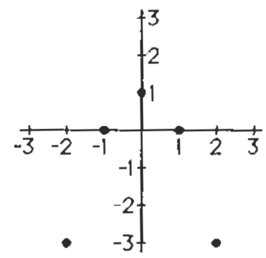
50. To prove the first equality, write $x = n - \epsilon$, where n is an integer and $0 \le \epsilon < 1$; thus $\lceil x \rceil = n$. Therefore, $\lfloor -x \rfloor = \lfloor -n + \epsilon \rfloor = -n = -\lceil x \rceil$. The second equality is proved in the same manner, writing $x = n + \epsilon$, where n is an integer and $0 \le \epsilon < 1$. This time $\lfloor x \rfloor = n$, and $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n = -\lfloor x \rfloor$.

52. In some sense this question is its own answer—the number of integers between a and b, inclusive, is the number of integers between a and b, inclusive. Presumably we seek an expression involving a, b, and the floor and/or ceiling function to answer this question. If we round a up and round b down to integers, then we will be looking at the smallest and largest integers just inside the range of integers we want to count, respectively. These values are of course $\lceil a \rceil$ and $\lfloor b \rfloor$, respectively. Then the answer is $\lfloor b \rfloor - \lceil a \rceil + 1$ (just think of counting all the integers between these two values, including both ends—if a row of fenceposts one foot apart extends for k feet, then there are $k+1$ fenceposts). Note that this even works when, for example, $a = 0.3$ and $b = 0.7$.

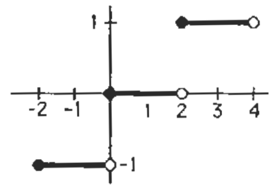
54. Since a byte is eight bits, all we are asking for in each case is $\lceil n/8 \rceil$, where n is the number of bits.
a) $\lceil 4/8 \rceil = 1$ b) $\lceil 10/8 \rceil = 2$ c) $\lceil 500/8 \rceil = 63$ d) $\lceil 3000/8 \rceil = 375$

56. From Example 26 we know that one ATM cell is 53 bytes, or $53 \cdot 8 = 424$ bits long. Thus in each case we need to divide the number of bits transmitted in 10 seconds by 424 and round down.
a) In 10 seconds, this link can transmit $128{,}000 \cdot 10 = 1{,}280{,}000$ bits. Therefore the answer is $\lfloor 1{,}280{,}000/424 \rfloor = 3018$.
b) In 10 seconds, this link can transmit $300{,}000 \cdot 10 = 3{,}000{,}000$ bits. So the answer is $\lfloor 3{,}000{,}000/424 \rfloor = 7075$.
c) In 10 seconds, this link can transmit $1{,}000{,}000 \cdot 10 = 10{,}000{,}000$ bits. So the answer is $\lfloor 10{,}000{,}000/424 \rfloor = 23{,}584$.

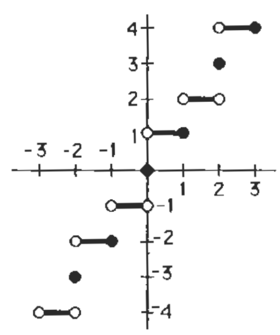
58. The graph consists of the points $(n, 1 - n^2)$ for all $n \in \mathbf{Z}$. The picture shows part of the graph on the usual coordinate axes.



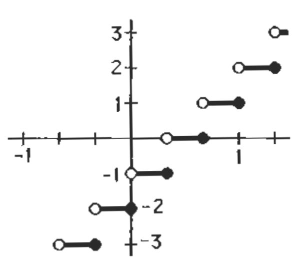
60. The graph is similar to the graph of $f(x) = \lfloor x \rfloor$; the only difference is a change in the scale of the x-axis.



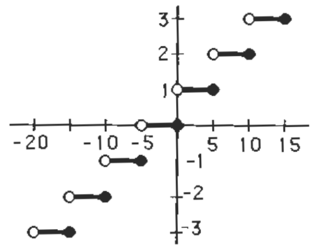
62. The function values for this step function change only at integer values of x, and different things happen for odd x and for even x because of the $x/2$ term. Whatever jump pattern is established on the closed interval $[0, 2]$ must repeat indefinitely in both directions. A thoughtful analysis then yields the following graph.



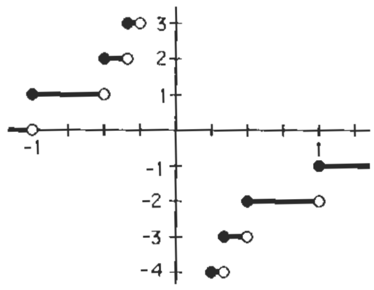
64. **a)** We can rewrite this as $f(x) = \lceil 3(x - \frac{2}{3}) \rceil$. The graph will therefore look look exactly like the graph of the function $f(x) = \lceil 3x \rceil$, except that the picture will be shifted to the right by $\frac{2}{3}$ unit, since x has been replaced by $x - \frac{2}{3}$. The graph of $f(x) = \lceil 3x \rceil$ is just like the graph shown in Figure 10b, except that the x-axis needs to be rescaled by a factor of 3 (the first jump on the positive x-axis occurs at $x = \frac{1}{3}$ here). Putting this all together yields the following picture. (Alternatively, we can think of this as the graph of $f(x) = \lceil 3x \rceil$ shifted down 2 units, since $\lceil 3x - 2 \rceil = \lceil 3x \rceil - 2$.)



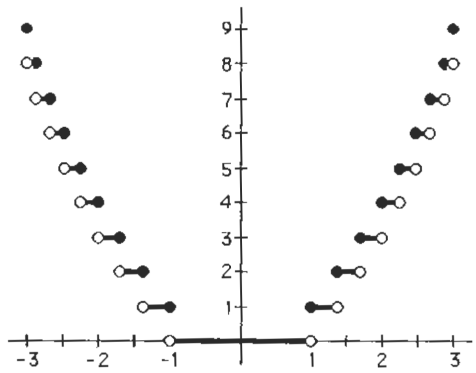
b) The graph will look exactly like the graph shown in Figure 10b, except that the x-axis needs to be rescaled by a factor of 5 (the first jump on the positive x-axis occurs at $x = 5$ here).



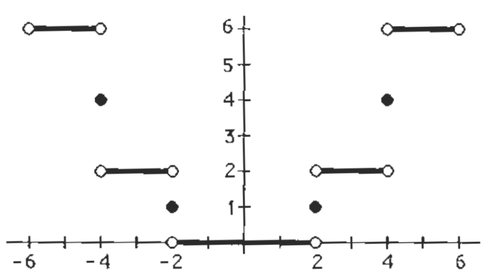
c) Since $\lfloor -1/x \rfloor = -\lceil 1/x \rceil$ (see Exercise 50), the picture is just the picture for Exercise 63d flipped upside down.



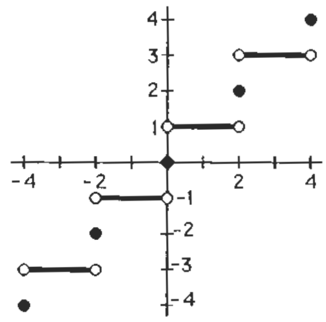
d) The basic shape is the parabola, $y = x^2$. However, because of the greatest integer function, the curve is broken into steps, with jumps at $x = \pm 1,\ \pm\sqrt{2},\ \pm\sqrt{3}, \ldots$. Note the symmetry around the y-axis.



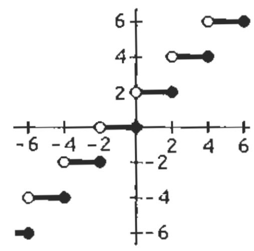
e) The basic shape is the parabola, $y = x^2/4$. However, because of the step functions, the curve is broken into steps. For x an even integer, $f(x) = x^4/4$, since the terms inside the floor and ceiling function symbols are integers. Note how these are isolated point, as in Exercise 63f.



f) When x is an even integer, this is just x. When x is between two even integers, however, this has the value of the odd integer between them. The graph is therefore as shown here.



g) Despite the complicated-looking formula, this is not too hard. Note that the expression inside the outer floor function symbols is always going to be an integer plus $\frac{1}{2}$; therefore we can tell exactly what its rounded-down value will be, namely $2\lceil x/2\rceil$. This is just the graph in Figure 10b, rescaled on both axes.



66. This follows immediately from the definition. We want to show that $((f\circ g)\circ(g^{-1}\circ f^{-1}))(z) = z$ for all $z\in Z$ and that $((g^{-1}\circ f^{-1})\circ(f\circ g))(x) = x$ for all $x\in X$. For the first we have

$$\begin{aligned}((f\circ g)\circ(g^{-1}\circ f^{-1}))(z) &= (f\circ g)((g^{-1}\circ f^{-1})(z))\\ &= (f\circ g)(g^{-1}(f^{-1}(z)))\\ &= f(g(g^{-1}(f^{-1}(z))))\\ &= f(f^{-1}(z)) = z\,.\end{aligned}$$

The second equality is similar.

68. If f is one-to-one, then every element of A gets sent to a different element of B. If in addition to the range of A there were another element in B, then $|B|$ would be at least one greater than $|A|$. This cannot happen, so we conclude that f is onto. Conversely, suppose that f is onto, so that every element of B is the image of some element of A. In particular, there is an element of A for each element of B. If two or more elements of A were sent to the same element of B, then $|A|$ would be at least one greater than the $|B|$. This cannot happen, so we conclude that f is one-to-one.

70. a) This is true. Since $\lceil x\rceil$ is already an integer, $\lfloor\lceil x\rceil\rfloor = \lceil x\rceil$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = y = \frac{3}{4}$. In this case the left-hand side is $\lfloor 3/2\rfloor = 1$, while the right-hand side is $0+0=0$.

c) A little trial and error fails to produce a counterexample, so maybe this is true. We look for a proof. Since we are dividing by 4, let us write $x = 4n+k$, where $0\le k<4$. In other words, write x in terms of how much it exceeds the largest multiple of 4 not exceeding it. There are three cases. If $k=0$, then x is already a multiple of 4, so both sides equal n. If $0<k\le 2$, then $\lceil x/2\rceil = 2n+1$, so the left-hand side is $\lceil n+\frac{1}{2}\rceil = n+1$. Of course the right-hand side is $n+1$ as well, so again the two sides agree. Finally, suppose that $2<k<4$. Then $\lceil x/2\rceil = 2n+2$, and the left-hand side is $\lceil n+1\rceil = n+1$; of course the right-hand side is still $n+1$, as well. Since we proved that the two sides are equal in all cases, the proof is complete.

d) For $x = 8.5$, the left-hand side is 3, whereas the right-hand side is 2.

e) This is true. Write $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are nonnegative real numbers less than 1. The left-hand side is $n+m+(n+m)$ or $n+m+(n+m+1)$, the latter occurring if and only if $\epsilon + \delta \geq 1$. The right-hand side is the sum of two quantities. The first is either $2n$ (if $\epsilon < \frac{1}{2}$) or $2n+1$ (if $\epsilon \geq \frac{1}{2}$). The second is either $2m$ (if $\delta < \frac{1}{2}$) or $2m+1$ (if $\delta \geq \frac{1}{2}$). The only way, then, for the left-hand side to exceed the right-hand side is to have the left-hand side be $2n+2m+1$ and the right-hand side be $2n+2m$. This can occur only if $\epsilon + \delta \geq 1$ while $\epsilon < \frac{1}{2}$ and $\delta < \frac{1}{2}$. But that is an impossibility, since the sum of two numbers less than $\frac{1}{2}$ cannot be as large as 1. Therefore the right-hand side is always at least as large as the left-hand side.

72. A straightforward way to do this problem is to consider the three cases determined by where in the interval between two consecutive integers the real number x lies. Certainly every real number x lies in an interval $[n, n+1)$ for some integer n; indeed, $n = \lfloor x \rfloor$. (Recall that $[s,t)$ is the notation for the set of real numbers greater than or equal to s and less than t.) If $x \in [n, n+\frac{1}{3})$, then $3x$ lies in the interval $[3n, 3n+1)$, so $\lfloor 3x \rfloor = 3n$. Moreover in this case $x + \frac{1}{3}$ is still less than $n+1$, and $x + \frac{2}{3}$ is still less than $n+1$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + n = 3n$ as well. For the second case, we assume that $x \in [n+\frac{1}{3}, n+\frac{2}{3})$. This time $3x \in [3n+1, 3n+2)$, so $\lfloor 3x \rfloor = 3n+1$. Moreover in this case $x + \frac{1}{3}$ is in $[n+\frac{2}{3}, n+1)$, and $x + \frac{2}{3}$ is in $[n+1, n+\frac{4}{3})$, so $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = n + n + (n+1) = 3n+1$ as well. The third case, $x \in [n+\frac{2}{3}, n+1)$, is similar, with both sides equaling $3n+2$.

74. a) We merely have to remark that f^* is well-defined by the rule given here. For each $a \in A$, either a is in the domain of definition of f or it is not. If it is, then $f^*(a)$ is the well-defined element $f(a) \in B$, and otherwise $f^*(a) = u$. In either case $f^*(a)$ is a well-defined element of $B \cup \{u\}$.

b) We simply need to set $f^*(a) = u$ for each a not in the domain of definition of f. In part (a), then, $f^*(n) = 1/n$ for $n \neq 0$, and $f^*(0) = u$. In part (b) we have a total function already, so $f^*(n) = \lceil n/2 \rceil$ for all $n \in \mathbf{Z}$. In part (c) $f^*(m,n) = m/n$ if $n \neq 0$, and $f^*(m,0) = u$ for all $m \in \mathbf{Z}$. In part (d) we have a total function already, so $f^*(m,n) = mn$ for all values of m and n. In part (e) the rule only applies if $m > n$, so $f^*(m,n) = m - n$ if $m > n$, and $f^*(m,n) = u$ if $m \leq n$.

76. For the "if" direction, we simply need to note that if S is a finite set, with cardinality m, then every proper subset of S has cardinality strictly smaller than m, so there is no possible one-to-one correspondence between the elements of S and the elements of the proper subset. (This is essentially the pigeonhole principle, to be discussed in Section 5.2.)

The "only if" direction is much deeper. Let S be the given infinite set. Clearly S is not empty, because by definition, the empty set has cardinality 0, a nonnegative integer. Let a_0 be one element of S, and let $A = S - \{a_0\}$. Clearly A is also infinite (because if it were finite, then we would have $|S| = |A| + 1$, making S finite). We will now construct a one-to-one correspondence between S and A; think of this as a one-to-one and onto function f from S to A. (This construction is an infinite process; technically we are using something called the Axiom of Choice.) In order to define $f(a_0)$, we choose an arbitrary element a_1 in A (which is possible because A is infinite) and set $f(a_0) = a_1$. Next we define f at a_1. To do so, we choose an arbitrary element a_2 in $A - \{a_1\}$ (which is possible because $A - \{a_1\}$ is necessarily infinite) and set $f(a_1) = a_2$. Next we define f at a_2. To do so, we choose an arbitrary element a_3 in $A - \{a_1, a_2\}$ (which is possible because $A - \{a_1, a_2\}$ is necessarily infinite) and set $f(a_2) = a_3$. We continue this process forever. Finally, we let f be the identity function on $S - \{a_0, a_1, a_2, \ldots\}$. The function thus defined has $f(a_i) = a_{i+1}$ for all natural numbers i and $f(x) = x$ for all $x \in S - \{a_0, a_1, a_2, \ldots\}$. Our construction forced f to be one-to-one and onto.

78. We saw in Exercise 77 that

$$f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$$

is a one-to-one function with domain $\mathbf{Z}^+ \times \mathbf{Z}^+$. We want to expand the domain to be $\mathbf{Z} \times \mathbf{Z}$, so things need to be spread out a little if we are to keep it one-to-one. If we can find a one-to-one function g from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}^+ \times \mathbf{Z}^+$, then composing these two functions will be our desired one-to-one function from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}$ (we know from Exercise 29 that the composition of one-to-one functions is one-to-one). The function suggested here is $g(m,n) = ((3m+1)^2, (3n+1)^2)$, so that the composed function is $(f \circ g)(m,n) = ((3m+1)^2 + (3n+1)^2 - 2)((3m+1)^2 + (3n+1)^2 - 1)/2 + (3m+1)^2$. To see that g is one-to-one, first note that it is enough to show that the behavior in each coordinate is one-to-one; that is, the function that sends integer k to positive integer $(3k+1)^2$ is one-to-one. To see this, first note that if $k_1 \neq k_2$ and k_1 and k_2 are both positive or both negative, then $(3k_1+1)^2 \neq (3k_2+1)^2$. And if one is nonnegative and the other is negative, then they cannot have the same images under this function because the nonnegative integers are sent to squares of numbers that leave a remainder of 1 when divided by 3 ($0 \to 1^2$, $1 \to 4^2$, $2 \to 7^2$, $\ldots$), but negative integers are sent to squares of numbers that leave a remainder of 2 when divided by 3 ($-1 \to 2^2$, $-2 \to 5^2$, $-3 \to 8^2, \ldots$).