CHAPTER 5
Counting

SECTION 5.1 The Basics of Counting

The secret to solving counting problems is to look at the problem the right way and apply the correct rules (usually the product rule or the sum rule), often with some common sense and cleverness thrown in. This is usually easier said than done, but it gets easier the more problems you do. Sometimes you need to count more than you want and then subtract the overcount. (This notion is made more precise in Section 7.5, where the inclusion–exclusion principle is discussed explicitly.) At other times you compensate by dividing; see Exercise 61, for example. Counting problems are sometimes ambiguous, so it is possible that your answer, although different from the answer we obtain, is the correct answer to a different interpretation of the problem; try to figure out whether that is the case.

If you have trouble with a problem, simplify the parameters to make them more manageable (if necessary) and try to list the set in question explicitly. This will often give you an idea of what is going on and suggest a general method of attack that will solve the problem as given. For example, in Exercise 11 you are asked about bit strings of length 10. If you are having difficulty, investigate the analogous question about bit strings of length 2 or 3, where you can write down the entire set, and see if a pattern develops. (Some people define mathematics as the study of patterns.) Sometimes tree diagrams make the analysis in these small cases easier to keep track of.

See the solution to Exercise 41 for a discussion of the powerful tool of **symmetry**, *which you will often find helpful. Another useful trick is* **gluing**; *see the solution to Exercise 53.*

Finally, do not be misled, if you find these exercises easy, into thinking that combinatorial problems are a piece of cake. It is very easy to ask combinatorial questions that look just like the ones asked here but in fact are extremely difficult, if not impossible. For example, try your hand at the following problem: how many strings are there, using 10 A's, 12 B's, 11 C's, and 15 D's, such that no A is followed by a B, and no C is followed by a D?

1. This problem illustrates the difference between the product rule and the sum rule. If we must make one choice *and* then another choice, the product rule applies, as in part (a). If we must make one choice *or* another choice, the sum rule applies, as in part (b). We assume in this problem that there are no double majors.

   a) The product rule applies here, since we want to do two things, one after the other. First, since there are 18 mathematics majors, and we are to choose one of them, there are 18 ways to choose the mathematics major. Then we must choose the computer science major from among the 325 computer science majors, and that can clearly be done in 325 ways. Therefore there are, by the product rule, $18 \cdot 325 = 5850$ ways to pick the two representatives.

   b) The sum rule applies here, since we want to do one of two mutually exclusive things. Either we can choose a mathematics major to be the representative, or we can choose a computer science major. There are 18 ways to choose a mathematics major, and there are 325 ways to choose a computer science major. Since these two actions are mutually exclusive (no one is both a mathematics major and a computer science major), and since we want to do one of them or the other, there are $18 + 325 = 343$ ways to pick the representative.

3. **a)** The product rule applies, since the student will perform each of 10 tasks, one after the other. There are 4 ways to do each task. Therefore there are $4 \cdot 4 \cdots 4 = 4^{10} = 1{,}048{,}576$ ways to answer the questions on the test.

b) This is identical to part **(a)**, except that now there are 5 ways to answer each question—give any of the 4 answers or give no answer at all. Therefore there are $5^{10} = 9{,}765{,}625$ ways to answer the questions on the test.

5. The product rule applies here, since a flight is determined by choosing an airline for the flight from New York to Denver (which can be done in 6 ways) and then choosing an airline for the flight from Denver to San Francisco (which can be done in 7 ways). Therefore there are $6 \cdot 7 = 42$ different possibilities for the entire flight.

7. Three-letter initials are determined by specifying the first initial (26 ways), then the second initial (26 ways), and then the third initial (26 ways). Therefore by the product rule there are $26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$ possible three-letter initials.

9. There is only one way to specify the first initial, but as in Exercise 7, there are 26 ways to specify each of the other initials. Therefore there are, by the product rule, $1 \cdot 26 \cdot 26 = 26^2 = 676$ possible three-letter initials beginning with A.

11. A bit string is determined by choosing the bits in the string, one after another, so the product rule applies. We want to count the number of bit strings of length 10, except that we are not free to choose either the first bit or the last bit (they are mandated to be 1's). Therefore there are 8 choices to make, and each choice can be made in 2 ways (the bit can be either a 1 or a 0). Thus the product rule tells us that there are $2^8 = 256$ such strings.

13. This is a trick question, since it is easier than one might expect. Since the string is given to consist entirely of 1's, there is nothing to choose except the length. Since there are $n+1$ possible lengths not exceeding n (if we include the empty string, of length 0), the answer is simply $n+1$. Note that the empty string consists—vacuously—entirely of 1's.

15. By the sum rule we can count the number of strings of length 4 or less by counting the number of strings of length i, for $0 \le i \le 4$, and then adding the results. Now there are 26 letters to choose from, and a string of length i is specified by choosing its characters, one after another. Therefore, by the product rule there are 26^i strings of length i. The answer to the question is thus $\sum_{i=0}^{4} 26^i = 1 + 26 + 676 + 17576 + 456976 = 475{,}255$.

17. The easiest way to count this is to find the total number of ASCII strings of length five and then subtract off the number of such strings that do not contain the @ character. Since there are 128 characters to choose from in each location in the string, the answer is $128^5 - 127^5 = 34{,}359{,}738{,}368 - 33{,}038{,}369{,}407 = 1{,}321{,}368{,}961$.

19. Because neither 100 nor 50 is divisible by either 7 or 11, whether the ranges are meant to be inclusive or exclusive of their endpoints is moot.

a) There are $\lfloor 100/7 \rfloor = 14$ integers less than 100 that are divisible by 7, and $\lfloor 50/7 \rfloor = 7$ of them are less than 50 as well. This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7. They are 56, 63, 70, 77, 84, 91, and 98.

b) There are $\lfloor 100/11 \rfloor = 9$ integers less than 100 that are divisible by 11, and $\lfloor 50/11 \rfloor = 4$ of them are less than 50 as well. This leaves $9 - 4 = 5$ numbers between 50 and 100 that are divisible by 11. They are 55, 66, 77, 88, and 99.

c) A number is divisible by both 7 and 11 if and only if it is divisible by their least common multiple, which is 77. Obviously there is only one such number between 50 and 100, namely 77. We could also work this out as we did in the previous parts: $\lfloor 100/77 \rfloor - \lfloor 50/77 \rfloor = 1 - 0 = 1$. Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

21. This problem deals with the set of positive integers between 100 and 999, inclusive. Note that there are exactly $999 - 100 + 1 = 900$ such numbers. A second way to see this is to note that to specify a three-digit number, we need to choose the first digit to be nonzero (which can be done in 9 ways) and then the second and third digits (which can each be done in 10 ways), for a total of $9 \cdot 10 \cdot 10 = 900$ ways, by the product rule. A third way to see this (perhaps most relevant for this problem) is to note that a number of the desired form is a number less than or equal to 999 (and there are 999 such numbers) but not less than or equal to 99 (and there are 99 such numbers); therefore there are $999 - 99 = 900$ numbers in the desired range.

a) Every seventh number—7, 14, and so on—is divisible by 7. Therefore the number of positive integers less than or equal to n and divisible by 7 is $\lfloor n/7 \rfloor$ (the floor function is used—we have to round down—because the first six positive integers are not multiples of 7; for example there are only $\lfloor 20/7 \rfloor = 2$ multiples of 7 less than or equal to 20). So we find that there are $\lfloor 999/7 \rfloor = 142$ multiples of 7 not exceeding 999, of which $\lfloor 99/7 \rfloor = 14$ do not exceed 99. Therefore there are exactly $142 - 14 = 128$ numbers in the desired range divisible by 7.

b) This is similar to part **(a)**, with 7 replaced by 2, but with the added twist that we want to count the numbers *not* divisible by 2. Mimicking the analysis in part **(a)**, we see that there are $\lfloor 999/2 \rfloor = 499$ even numbers not exceeding 999, and therefore $999 - 499 = 500$ odd ones; there are similarly $99 - \lfloor 99/2 \rfloor = 50$ odd numbers less than or equal to 99. Therefore there are $500 - 50 = 450$ odd numbers between 100 and 999 inclusive.

c) There are just 9 possible digits that a three-digit number can start with. If all of its digits are to be the same, then there is no choice after the leading digit has been specified. Therefore there are 9 such numbers.

d) This is similar to part **(b)**, except that 2 is replaced by 4. Following the analysis there, we find that there are $999 - \lfloor 999/4 \rfloor = 750$ positive integers less than or equal to 999 not divisible by 4, and $99 - \lfloor 99/4 \rfloor = 75$ such positive integers less than or equal to 99. Therefore there are $750 - 75 = 675$ three-digit integers not divisible by 4.

e) The method is similar to that used in the earlier parts. There are $\lfloor 999/3 \rfloor - \lfloor 99/3 \rfloor = 300$ three-digit numbers divisible by 3, and $\lfloor 999/4 \rfloor - \lfloor 99/4 \rfloor = 225$ three-digit numbers divisible by 4. Moreover there are $\lfloor 999/12 \rfloor - \lfloor 99/12 \rfloor = 75$ numbers divisible by both 3 and 4, i.e., divisible by 12. In order to count each number divisible by 3 or 4 once and only once, we need to add the number of numbers divisible by 3 to the number of numbers divisible by 4, and then subtract the number of numbers divisible by both 3 and 4 so as not to count them twice. Therefore the answer is $300 + 225 - 75 = 450$.

f) In part **(e)** we found that there were 450 three-digit integers that *are* divisible by either 3 or 4. The others, of course, are not. Therefore there are $900 - 450 = 450$ three-digit integers that are not divisible by either 3 or 4.

g) We saw in part **(e)** that there are 300 three-digit numbers divisible by 3 and that 75 of them are also divisible by 4. The remainder of those 300 numbers, therefore, are not divisible by 4. Thus the answer is $300 - 75 = 225$.

h) We saw in part **(e)** that there are 75 three-digit numbers divisible by both 3 and 4.

23. This problem involves 1000 possible strings, since there is a choice of 10 digits for each of the three positions in the string.

a) This is most easily done by subtracting from the total number of strings the number of strings that violate the condition. Clearly there are 10 strings that consist of the same digit three times (000, 111, ..., 999). Therefore there are $1000 - 10 = 990$ strings that do not.

b) If we must begin our string with an odd digit, then we have only 5 choices for this digit. We still have 10 choices for each of the remaining digits. Therefore there are $5 \cdot 10 \cdot 10 = 500$ such strings. Alternatively, we note that by symmetry exactly half the strings begin with an odd digit (there being the same number of odd digits as even ones). Therefore half of the 1000 strings, or 500 of them, begin with an odd digit.

c) Here we need to choose the position of the digit that is not a 4 (3 ways) and choose that digit (9 ways). Therefore there are $3 \cdot 9 = 27$ such strings.

25. There are 50 choices to make, each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator. By the product rule the answer is therefore $3^{50} \approx 7.2 \times 10^{23}$.

27. By the sum rule we need to add the number of license plates of the first type and the number of license plates of the second type. By the product rule there are $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}760{,}000$ license plates consisting of 2 letters followed by 4 digits; and there are $10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 45{,}697{,}600$ license plates consisting of 2 digits followed by 4 letters. Therefore the answer is $6{,}760{,}000 + 45{,}697{,}600 = 52{,}457{,}600$.

29. First let us compute the number of ways to choose the letters. By the sum rule this is the sum of the number of ways to use two letters and the number of ways to use three letters. By the product rule there are 26^2 ways to choose two letters and 26^3 ways to choose three letters. Therefore there are $26^2 + 26^3$ ways to choose the letters. By similar reasoning there are $10^2 + 10^3$ ways to choose the digits. Thus the answer to the question is $(26^2 + 26^3)(10^2 + 10^3) = 18252 \cdot 1100 = 20{,}077{,}200$.

31. We take as known that there are 26 letters including 5 vowels in the English alphabet.

a) There are 8 slots, each of which can be filled with one of the $26 - 5 = 21$ nonvowels (consonants), so by the product rule the answer is $21^8 = 37{,}822{,}859{,}361$.

b) There are 21 choices for the first slot in our string, but only 20 choices for the second slot, 19 for the third, and so on. So the answer is $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = 8{,}204{,}716{,}800$.

c) There are 26 choices for each slot except the first, for which there are 5 choices, so the answer is $5 \cdot 26^7 = 40{,}159{,}050{,}880$.

d) This is similar to (b), except that there are only five choice in the first slot, and we are free to choose from all the letters not used so far, rather than just the consonants. Thus the answer is $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 12{,}113{,}640{,}000$.

e) We subtract from the total number of strings (26^8) the number that do not contain at least one vowel (21^8, the answer to (a)), obtaining the answer $26^8 - 21^8 = 208{,}827{,}064{,}576 - 37{,}822{,}859{,}361 = 171{,}004{,}205{,}215$.

f) The best way to do this is first to decide where the vowel goes (8 choices), then to decide what the vowel is to be (A, E, I, O, or U—5 choices), and then to fill the remaining slots with any consonants (21^7 choices, since one slot has already been filled). Therefore the answer is $8 \cdot 5 \cdot 21^7 = 72{,}043{,}541{,}640$.

g) We can ignore the first slot, since there is no choice. Now the problem is almost identical to (e), except that there are only 7 slots to fill. So the answer is $26^7 - 21^7 = 8{,}031{,}810{,}176 - 1{,}801{,}088{,}541 = 6{,}230{,}721{,}635$.

h) The problem is almost identical to (g), except that there are only 6 slots to fill. So the answer is $26^6 - 21^6 = 308{,}915{,}776 - 85{,}766{,}121 = 223{,}149{,}655$.

33. For each part of this problem, we need to find the number of one-to-one functions from a set with 5 elements to a set with k elements. To specify such a function, we need to make 5 choices, in succession, namely the values of the function at each of the 5 elements in its domain. Therefore the product rule applies. The first choice can be made in k ways, since any element of the codomain can be the image of the first element of the domain. After that choice has been made, there are only $k - 1$ elements of the codomain available to be the image of the second element of the domain, since images must be distinct for the function to be one-to-one. Similarly, for the third element of the domain, there are $k - 2$ possible choices for a function value. Continuing in this way, and applying the product rule, we see that there are $k(k-1)(k-2)(k-3)(k-4)$ one-to-one functions from a set with 5 elements to a set with k elements.

a) By the analysis above the answer is $4\cdot 3\cdot 2\cdot 1\cdot 0 = 0$, what we would expect since there are no one-to-one functions from a set to a strictly smaller set.

b) By the analysis above the answer is $5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.

c) By the analysis above the answer is $6\cdot 5\cdot 4\cdot 3\cdot 2 = 720$.

d) By the analysis above the answer is $7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520$.

35. a) There can clearly be no one-to-one function from $\{1,2,\ldots,n\}$ to $\{0,1\}$ if $n > 2$. If $n = 1$, then there are 2 such functions, the one that sends 1 to 0, and the one that sends 1 to 1. If $n = 2$, then there are again 2 such functions, since once it is determined where 1 is sent, the function must send 2 to the other value in the codomain.

b) If the function assigns 0 to both 1 and n, then there are $n-2$ function values free to be chosen. Each can be chosen in 2 ways. Therefore, by the product rule (since we have to choose values for all the elements of the domain) there are 2^{n-2} such functions, as long as $n > 1$. If $n = 1$, then clearly there is just one such function.

c) If $n = 1$, then there are no such functions, since there are no positive integers less than n. So assume $n > 1$. In order to specify such a function, we have to decide which of the numbers from 1 to $n-1$, inclusive, will get sent to 1. There are $n-1$ ways to make this choice. There is no choice for the remaining numbers from 1 to $n-1$, inclusive, since they all must get sent to 0. Finally, we are free to specify the value of the function at n, and this may be done in 2 ways. Hence, by the product rule the final answer is $2(n-1)$.

37. The easiest way to view a partial function in terms of counting is to add an additional element to the codomain of the function—let's call it u for "undefined"— and then imagine that the function assigns a value to *all* elements of the domain. If the original function f had previously been undefined at x, we now say that $f(x) = u$. Thus all we have done is to increase the size of the codomain from n elements to $n+1$ elements. By Example 6 we conclude that there are $(n+1)^m$ such partial functions.

39. The trick here is to realize that a palindrome of length n is completely determined by its first $\lceil n/2\rceil$ bits. This is true because once these bits are specified, the remaining bits, read from right to left, must be identical to the first $\lfloor n/2\rfloor$ bits, read from left to right. Furthermore, these first $\lceil n/2\rceil$ bits can be specified at will, and by the product rule there are $2^{\lceil n/2\rceil}$ ways to do so.

41. a) Here is a good way (but certainly not the only way) to approach this problem. Since the bride and groom must stand next to each other, let us treat them as one unit. Then the question asks for the number of ways to arrange five units in a row (the bride-and-groom unit and the four other people). We can think of filling five positions one at a time, so by the product rule there are $5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ ways to make these choices. This is not quite the answer, however, since there are also two ways to decide on which side of the groom the bride will stand. Therefore the final answer is $120\cdot 2 = 240$.

b) There are clearly $6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 720$ arrangements in all. We just determined in part **(a)** that 240 of them involve the bride standing next to the groom. Therefore there are $720 - 240 = 480$ ways to arrange the people with the bride not standing next to the groom.

c) Of the 720 arrangements of these people (see part **(b)**), exactly half must have the bride somewhere to the left of the groom. (We are invoking **symmetry** here—a useful tool for solving some combinatorial problems.) Therefore the answer is $720/2 = 360$.

43. There are 2^7 bit strings of length 10 that begin with three 0's, since each of the remaining seven bits has two possible values. Similarly, there are 2^8 bit strings of length 10 that end with two 0's. Furthermore, there are 2^5 bit strings of length 10 that both begin with three 0's and end with two 0's, since each of the five "middle" bits can be specified in two ways. Using the principle of inclusion–exclusion, we conclude that there are $2^7 + 2^8 - 2^5 = 352$ such strings. The idea behind this principle here is that the strings that both begin with three 0's and end with two 0's were counted twice when we added 2^7 and 2^8, so we need to subtract for the overcounting. It is definitely *not* the case that we are subtracting because we do not want to count such strings at all.

45. First let us count the number of 8-bit strings that contain three consecutive 0's. We will organize our count by looking at the leftmost bit that contains a 0 followed by at least two more 0's. If this is the first bit, then the second and third bits are determined (namely, they are both 0), but bits 4 through 8 are free to be specified, so there are $2^5 = 32$ such strings. If it is the second, third, or fourth bits, then the bit preceding it must be a 1 and the two bits after it must be 0's, but the other four bits are free. Therefore there are $2^4 = 16$ such strings in each of these three cases, or 48 in all. Next let us suppose that the substring of three or more 0's starts at bit 5. Then bit 4 must be a 1. Bits 1 through 3 can be anything other than three 0's (if it were three 0's, then we already counted this string); thus there are $2^3 - 1 = 7$ ways to specify them. Bit 8 is free. Therefore there are $7 \cdot 2 = 14$ such strings. Finally, suppose that the substring 000 starts in bit 6, so that bit 5 is a 1. There are $2^4 = 16$ possibilities for the first four bits, but three of them contain three consecutive 0's (0000, 0001, and 1000); therefore there are only 13 such strings. Adding up all the cases we have discussed, we obtain the final answer of $32 + 48 + 14 + 13 = 107$ for the number of 8-bit strings that contain three consecutive 0's.

Next we need to compute the number of 8-bit strings that contain four consecutive 1's. The analysis is very similar to what we have just done. There are $2^4 = 16$ such strings starting with 1111; there are $2^3 = 8$ starting 01111; there are $2^3 = 8$ starting $x01111$ (where x is either 0 or 1); and there are similarly 8 starting each of $xy01111$ and $xyz01111$. This gives us a total of $16 + 4 \cdot 8 = 48$ strings containing four consecutive 1's.

Finally, we need to count the number of strings that contain both three consecutive 0's and four consecutive 1's. It is easy enough to just list them all: 00001111, 00011111, 00011110, 10001111, 11110000, 11111000, 01111000, and 11110001, eight in all. Now applying the principle of inclusion–exclusion to what we have calculated above, we obtain the answer to the entire problem: $107 + 48 - 8 = 147$. There are only 256 bit strings of length eight altogether, so this answer is somewhat surprising, in that more than half of them satisfy the stated condition.

47. We can solve this problem by computing the number of positive integers not exceeding 100 that are divisible by 4, the number of them divisible by 6, and the number divisible by both 4 and 6; and then applying the principle of inclusion–exclusion. There are clearly $100/4 = 25$ multiples of 4 in this range, since every fourth number is divisible by 4. The number of integers in this range divisible by 6 is $\lfloor 100/6 \rfloor = 16$; we needed to round down because the multiples of 6 occur at the end of each consecutive block of 6 integers (6, 12, 18, etc.). Furthermore a number is divisible by both 4 and 6 if and only if it is divisible by their least common multiple, namely 12. Therefore there are $\lfloor 100/12 \rfloor = 8$ numbers divisible by both 4 and 6 in this range. Finally, applying inclusion–exclusion, we see that the number of positive integers not exceeding 100 that are divisible either by 4 or by 6 is $25 + 16 - 8 = 33$.

49. a) We are told that there are $26 + 26 + 10 + 6 = 68$ available characters. A password of length k using these characters can be formed in 68^k ways. Therefore the number of passwords with the specified length restriction is $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} = 9{,}920{,}671{,}339{,}261{,}325{,}541{,}376$, which is about 9.9×10^{21} or about ten sextillion.

b) For a password *not* to contain one of the special characters, it must be constructed from the other 62 characters. There are $62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12} = 3{,}279{,}156{,}377{,}874{,}257{,}103{,}616$ of these. Thus there are 6,641,514,961,387,068,437,760 $\approx 6.6 \times 10^{21}$ (about seven sextillion) passwords that do contain at least one occurrence of one of the special symbols.

c) Assuming no restrictions, it will take one nanosecond (one billionth of a second, or 10^{-9} sec) for each password. We just multiply this by our answer in part **(a)** to find the number of seconds the hacker will require. We can convert this to years by dividing by $60 \cdot 60 \cdot 24 \cdot 365.2425$ (the average number of seconds in a year). It will take about 314,374 years.

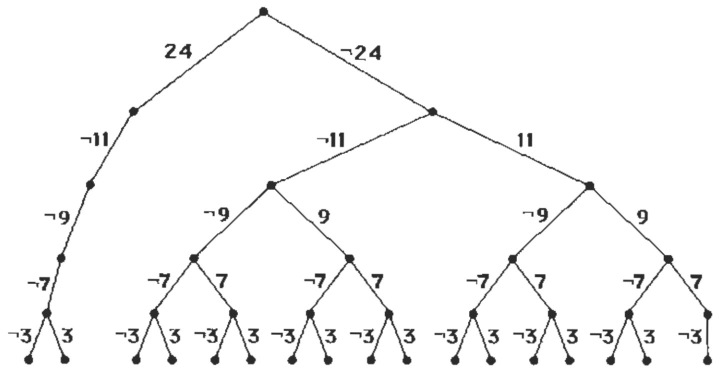
51. By the result of Example 8, there are $C = 6{,}400{,}000{,}000$ possible numbers of the form NXX-NXX-$XXXX$. To determine the number of different telephone numbers worldwide, then, we need to determine how many country codes there are and multiply by C. There are clearly 10 country codes of length 1, 100 country codes of length 2, and 1000 country codes of length 3. Thus there are $10 + 100 + 1000 = 1110$ country codes in all, by the sum rule. Our final answer is $1110 \cdot C = 7{,}104{,}000{,}000{,}000$.

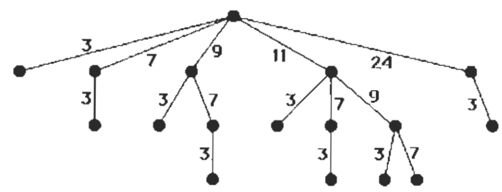
53. We assume that what is intended is that each of the 4 letters is to be used exactly once. There are at least two ways to do this problem. First let us break it into two cases, depending on whether the a comes at the end of the string or not. If a is not at the end, then there are 3 places to put it. After we have placed the a, there are only 2 places to put the b, since it cannot go into the position occupied by the a and it cannot go into the position following the a. Then there are 2 positions in which the c can go, and only 1 position for the d. Therefore, there are by the product rule $3 \cdot 2 \cdot 2 \cdot 1 = 12$ allowable strings in which the a does not come last. Second, if the a comes last, then there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange the letters b, c, and d in the first three positions. The answer, by the sum rule, is therefore $12 + 6 = 18$.

Here is another approach. Ignore for a moment the restriction that a b cannot follow an a. Then we need to choose the letter that comes first (which can be done in 4 ways), then the letter that comes second (which can be done in 3 ways, since one letter has already been used), then the letter that comes third (which can be done in 2 ways, since two of the letters have already been used), and finally the letter that comes last (which can only be done in 1 way, since there is only one unused letter at that point). Therefore there are, by the product rule, $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such strings. Now we need to subtract from this total the number of strings in which the a is immediately followed by the b. To count these, let us imagine the a and b glued together into one superletter, ab. (This **gluing** technique often comes in handy.) Now there are 3 things to arrange. We can choose any of them (the letters c or d or the superletter ab) to come first, and that can be done in 3 ways. We can choose either of the other two to come second (which can be done in 2 ways), and we are forced to choose the remaining one to come third. By the product rule there are $3 \cdot 2 \cdot 1 = 6$ ways to make these choices. Therefore our final answer is $24 - 6 = 18$.

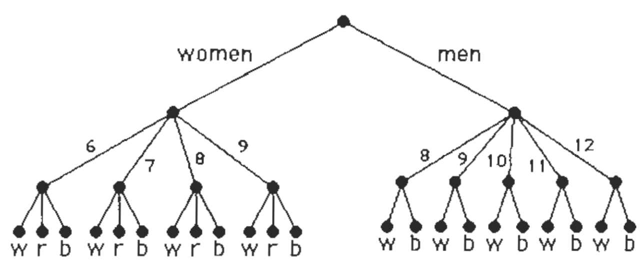
55. There are at least two approaches that are effective here. In our first tree, we let each branching point represent a decision as to whether to include the next element in the set (starting with the largest element). At the top of the tree, for example, we can either choose to include 24 or to exclude it (denoted $\neg 24$). We branch one way for each possibility. In the first figure below, the entire subtree to the right represents those sets that do not include 24, and the subtree to the left represents those that do. At the point below and to the left of the 24, we have only one branch, $\neg 11$, since after we have included 24 in our set, we cannot include 11, because the sum would not be less than 28 if we did. At the point below and to the right of the $\neg 24$, however, we again branch twice, since we can choose either to include 11 or to exclude it. To answer the question, we look at the points in the last row of the tree. Each represents a set whose sum is less than 28. For example, the sixth point from the right represents the set $\{11, 3\}$. Since there are 17 such points, the answer to the problem is 17.

Our other solution is more compact. In the tree below we show branches from a point only for the inclusion of new numbers in the set. The set formed by including no more numbers is represented by the point itself. This time *every* point represents a set. For example, the point at the top represents the empty set, the point below and to the right of the number 11 represents the set $\{11\}$, and the left-most point on the bottom row represents the set $\{3,7,9\}$. In general the set that a point represents is the set of numbers found on the path up the tree from that point to the root of the tree (the point at the top). We only included branches when the sum would be less than 28. Since there are 17 points altogether in this figure, the answer to the problem is 17.





57. a) The tree shown here enumerates the possible outcomes. First we branch on gender, then on size, and finally on color. There are 22 ends, so the answer to the question is 22.



b) First we apply the sum rule: the number of shoes is the sum of the number of men's shoes and the number of women's shoes. Next we apply the product rule. For a woman's shoe we need to specify size (4 choices) and then for each choice of size, we need to specify color (3 choices). Therefore there are $4 \cdot 3 = 12$ possible women's models. Similarly, there are $5 \cdot 2 = 10$ men's models. Therefore the answer is $12 + 10 = 22$.

59. We want to prove $P(m)$, the sum rule for m tasks, which says that if tasks T_1, T_2, $\ldots$, T_m can be done in n_1, n_2, $\ldots$, n_m ways, respectively, and no two of them can be done at the same time, then there are $n_1 + n_2 + \cdots + n_m$ ways to do one of the tasks. The basis step of our proof by mathematical induction is $m = 2$, and that has already been given. Now assume that $P(m)$ is true, and we want to prove $P(m+1)$. There are $m+1$ tasks, no two of which can be done at the same time; we want to do one of them. Either we choose one from among the first m, or we choose the task T_{m+1}. By the sum rule for two tasks, the number of ways we can do this is $n + n_{m+1}$, where n is the number of ways we can do one of the tasks among the first m. But by the inductive hypothesis $n = n_1 + n_2 + \cdots + n_m$. Therefore the number of ways we can do one of the $m+1$ tasks is $n_1 + n_2 + \cdots + n_m + n_{m+1}$, as desired.

61. A diagonal joins two vertices of the polygon, but they must be vertices that are not already joined by a side of the polygon. Thus there are $n-3$ diagonals emanating from each vertex of the polygon (we've excluded two of the $n-1$ other vertices as possible targets for diagonals). If we multiply $n-3$ by n, the number of vertices, we will have counted each diagonal exactly twice—once for each endpoint. We compensate for this overcounting by dividing by 2. Therefore there are $n(n-3)/2$ diagonals. (Note that the convexity of the polygon had nothing to do with the problem—we were counting the diagonals, whether or not we could be sure that they all lay inside the polygon.)