SECTION 4.2 Strong Induction and Well-Ordering

In this section we extend the technique of proof by mathematical induction by using a stronger inductive hypothesis. The inductive step is now to prove the proposition for $k+1$, assuming the inductive hypothesis that the proposition is true for all values less than or equal to k. Aside from that, the two methods are the same.

1. Let $P(n)$ be the statement that you can run n miles. We want to prove that $P(n)$ is true for all positive integers n. For the basis step we note that the given conditions tell us that $P(1)$ and $P(2)$ are true. For the inductive step, fix $k \geq 2$ and assume that $P(j)$ is true for all $j \leq k$. We want to show that $P(k+1)$ is true. Since $k \geq 2$, $k-1$ is a positive integer less than or equal to k, so by the inductive hypothesis, we know that $P(k-1)$ is true. That is, we know that you can run $k-1$ miles. We were told that "you can always run two more miles once you have run a specified number of miles," so we know that you can run $(k-1)+2=k+1$ miles. This is $P(k+1)$.

Note that we didn't use strong induction exactly as stated on page 284. Instead, we considered both $n=1$ and $n=2$ as part of the basis step. We could have more formally included $n=2$ in the inductive step as a special case. Writing our proof this way, the basis step is just to note that we are told we can run one mile, so $P(1)$ is true. For the inductive step, if $k=1$ then we are already told that we can run two miles. If $k>1$, then the inductive hypothesis tells us that we can run $k-1$ miles, so we can run $(k-1)+2=k+1$ miles.

3. **a)** $P(8)$ is true, because we can form 8 cents of postage with one 3-cent stamp and one 5-cent stamp. $P(9)$ is true, because we can form 9 cents of postage with three 3-cent stamps. $P(10)$ is true, because we can form 10 cents of postage with two 5-cent stamps.
b) The inductive hypothesis is the statement that using just 3-cent and 5-cent stamps we can form j cents postage for all j with $8 \leq j \leq k$, where we assume that $k \geq 10$.
c) In the indnctive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 3-cent and 5-cent stamps.
d) We want to form $k+1$ cents of postage. Since $k \geq 10$, we know that $P(k-2)$ is true, that is, that we can form $k-2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.
e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 8.

5. **a)** We can form the following amounts of postage as indicated: $4=4$, $8=4+4$, $11=11$, $12=4+4+4$, $15=11+4$, $16=4+4+4+4$, $19=11+4+4$, $20=4+4+4+4+4$, $22=11+11$, $23=11+4+4+4$, $24=4+4+4+4+4+4$, $26=11+11+4$, $27=11+4+4+4+4$, $28=4+4+4+4+4+4+4$, $30=11+11+4+4$, $31=11+4+4+4+4+4$, $32=4+4+4+4+4+4+4+4$, $33=11+11+11$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 30 cents using just 4-cent and 11-cent stamps.
b) Let $P(n)$ be the statement that we can form n cents of postage using just 4-cent and 11-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 30$. The basis step, $n=30$, is handled above. Assume that we can form k cents of postage (the inductive hypothesis); we will show how to form $k+1$ cents of postage. If the k cents included an 11-cent stamp, then replace it by three 4-cent stamps ($3 \cdot 4 = 11+1$). Otherwise,

k cents was formed from just 4-cent stamps. Because $k \geq 30$, there must be at least eight 4-cent stamps involved. Replace eight 4-cent stamps by three 11-cent stamps, and we have formed $k+1$ cents in postage $(3 \cdot 11 = 8 \cdot 4 + 1)$.

c) $P(n)$ is the same as in part **(b)**. To prove that $P(n)$ is true for all $n \geq 30$, we note for the basis step that from part **(a)**, $P(n)$ is true for $n = 30, 31, 32, 33$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $30 \leq j \leq k$, where k is a fixed integer greater than or equal to 33. We want to show that $P(k+1)$ is true. Because $k - 3 \geq 30$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 30 and k inclusive, and that enabled us to jump back four steps to a value for which we knew how to form the desired postage.

7. We can form the following amounts of money as indicated: $2 = 2$, $4 = 2+2$, $5 = 5$, $6 = 2+2+2$. By having considered all the combinations, we know that the gaps in this list (\$1 and \$3) cannot be filled. We claim that we can form all amounts of money greater than or equal to 5 dollars. Let $P(n)$ be the statement that we can form n dollars using just 2-dollar and 5-dollar bills. We want to prove that $P(n)$ is true for all $n \geq 5$. We already observed that the basis step is true for $n = 5$ and 6. Assume the inductive hypothesis, that $P(j)$ is true for all j with $5 \leq j \leq k$, where k is a fixed integer greater than or equal to 6. We want to show that $P(k+1)$ is true. Because $k-1 \geq 5$, we know that $P(k-1)$ is true, that is, that we can form $k-1$ dollars. Add another 2-dollar bill, and we have formed $k+1$ dollars, as desired.

9. Following the hint, we let $P(n)$ be the statement that there is no positive integer b such that $\sqrt{2} = n/b$. For the basis step, $P(1)$ is true because $\sqrt{2} > 1 \geq 1/b$ for all positive integers b. For the inductive step, assume that $P(j)$ is true for all $j \leq k$, where k is an arbitrary positive integer; we must prove that $P(k+1)$ is true. So assume the contrary, that $\sqrt{2} = (k+1)/b$ for some positive integer b. Squaring both sides and clearing fractions, we have $2b^2 = (k+1)^2$. This tells us that $(k+1)^2$ is even, and so $k+1$ is even as well (the square of an odd number is odd, by Example 1 in Section 1.6). Therefore we can write $k+1 = 2t$ for some positive integer t. Substituting, we have $2b^2 = 4t^2$, so $b^2 = 2t^2$. By the same reasoning as before, b is even, so $b = 2s$ for some positive integer s. Then we have $\sqrt{2} = (k+1)/b = (2t)/(2s) = t/s$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete.

11. There are four base cases. If $n = 1 = 4 \cdot 0 + 1$, then clearly the first player is doomed, so the second player wins. If there are two, three, or four matches ($n = 4 \cdot 0 + 2$, $n = 4 \cdot 0 + 3$, or $n = 4 \cdot 1$), then the first player can win by removing all but one match. Now assume the strong inductive hypothesis, that in games with k or fewer matches, the first player can win if $k \equiv 0$, 2 or, 3 (mod 4) and the second player can win if $k \equiv 1 \pmod 4$. Suppose we have a game with $k+1$ matches, with $k \geq 4$. If $k+1 \equiv 0 \pmod 4$, then the first player can remove three matches, leaving $k-2$ matches for the other player. Since $k-2 \equiv 1 \pmod 4$, by the inductive hypothesis, this is a game that the second player at that point (who is the first player in our game) can win. Similarly, if $k+1 \equiv 2 \pmod 4$, then the first player can remove one match, leaving k matches for the other player. Since $k \equiv 1 \pmod 4$, by the inductive hypothesis, this is a game that the second player at that point (who is the first player in our game) can win. And if $k+1 \equiv 3 \pmod 4$, then the first player can remove two matches, leaving $k-1$ matches for the other player. Since $k-1 \equiv 1 \pmod 4$, by the inductive hypothesis, this is again a game that the second player at that point (who is the first player in our game) can win. Finally, if $k+1 \equiv 1 \pmod 4$, then the first player must leave k, $k-1$, or $k-2$ matches for the other player. Since $k \equiv 0 \pmod 4$, $k-1 \equiv 3 \pmod 4$, and $k-2 \equiv 2 \pmod 4$, by the inductive hypothesis, this is a game that the first player at that point (who is the second player in our game) can win. Thus the first player in our game is doomed, and the proof is complete.

13. Let $P(n)$ be the statement that exactly $n-1$ moves are required to assemble a puzzle with n pieces. Now $P(1)$ is trivially true. Assume that $P(j)$ is true for all $j < n$, and consider a puzzle with n pieces. The final move must be the joining of two blocks, of size k and $n-k$ for some integer k, $1 \le k \le n-1$. By the inductive hypothesis, it required $k-1$ moves to construct the one block, and $n-k-1$ moves to construct the other. Therefore $1+(k-1)+(n-k-1)=n-1$ moves are required in all, so $P(n)$ is true. Notice that for variety here we proved $P(n)$ under the assumption that $P(j)$ was true for $j<n$; so n played the role that $k+1$ plays in the statement of strong induction given in the text. It is worthwhile to understand how all of these forms are saying the same thing and to be comfortable moving between them.

15. Let the Chomp board have n rows and n columns. We claim that the first player can win the game by making the first move to leave just the top row and left-most column. (He does this by selecting the cookie in the second column of the second row.) Let $P(n)$ be the statement that if a player has presented his opponent with a Chomp configuration consisting of just n cookies in the top row and n cookies in the left-most column (both of these including the poisoned cookie in the upper left corner), then he can win the game. We will prove $\forall n P(n)$ by strong induction. We know that $P(1)$ is true, because the opponent is forced to take the poisoned cookie at his first turn. Fix $k \ge 1$ and assume that $P(j)$ is true for all $j \le k$. We claim that $P(k+1)$ is true. It is the opponent's turn to move. If she picks the poisoned cookie, then the game is over and she loses. Otherwise, assume that she picks the cookie in the top row in column j, or the cookie in the left column in row j, for some j with $2 \le j \le k+1$. The first player now picks the cookie in the left column in row j, or the cookie in the top row in column j, respectively. This leaves the position covered by $P(j-1)$ for his opponent, so by the inductive hypothesis, he can win.

17. Let $P(n)$ be the statement that if a simple polygon with n sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon. We will prove $\forall n \ge 4\, P(n)$. The statement is clearly true for $n=4$, because there is only one diagonal, leaving two triangles with the desired property. Fix $k \ge 4$ and assume that $P(j)$ is true for all j with $4 \le j \le k$. Consider a polygon with $k+1$ sides, and some triangulation of it. Pick one of the diagonals in this triangulation. First suppose that this diagonal divides the polygon into one triangle and one polygon with k sides. Then the triangle has two sides that border the exterior. Furthermore, the k-gon has, by the inductive hypothesis, two triangles that have two sides that border the exterior of that k-gon, and only one of these triangles can fail to be a triangle that has two sides that border the exterior of the original polygon. The only other case is that this diagonal divides the polygon into two polygons with j sides and $k+3-j$ sides for some j with $4 \le j \le k-1$. By the inductive hypothesis, each of these two polygons has two triangles that have two sides that border their exterior, and in each case only one of these triangles can fail to be a triangle that has two sides that border the exterior of the original polygon.

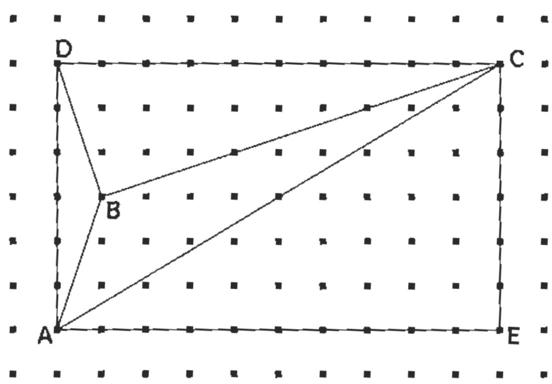
19. Let $P(n)$ be the statement that the area of a simple polygon with n sides and vertices all at lattice points is given by $I+B/2-1$, where I and B are as defined in the exercise. We will prove $\forall n \ge 3\, P(n)$. We begin by proving an additivity lemma. If P is a simple polygon with all vertices at the lattice points, divided into polygons P_1 and P_2 by a diagonal, then $I(P)+B(P)/2-1=(I(P_1)+B(P_1)/2-1)+(I(P_2)+B(P_2)/2-1)$. To see this, suppose there are k lattice points on the diagonal, not counting its endpoints. Then $I(P)=I(P_1)+I(P_2)+k$ and $B(P)=B(P_1)+B(P_2)-2k-2$; and the result follows by simple algebra. What this says in particular is that if Pick's formula gives the correct area for P_1 and P_2, then it must give the correct formula for P, whose area is the sum of the areas for P_1 and P_2; and similarly if Pick's formula gives the correct area for P and one of the P_i's, then it must give the correct formula for the other P_i.

Next we prove the theorem for rectangles whose sides are parallel to the coordinate axes. Such a rectangle necessarily has vertices at (a,b), (a,c), (d,b), and (d,c), where a, b, c, and d are integers with $b<c$ and $a<d$. Its area is clearly $(c-b)(d-a)$. By looking at the perimeter, we see that it has $B=2(c-b+d-a)$, and we see also that it has $I=(c-b-1)(d-a-1)=(c-b)(d-a)-(c-b)-(d-a)+1$. Therefore

$I + B/2 - 1 = (c-b)(d-a) - (c-b) - (d-a) + 1 + (c-b+d-a) - 1 = (c-b)(d-a)$, which is the desired area.

Next consider a right triangle whose legs are parallel to the coordinate axes. This triangle is half a rectangle of the type just considered, for which Pick's formula holds, so by the additivity lemma, it holds for the triangle as well. (The values of B and I are the same for each of the two triangles, so if Pick's formula gave an answer that was either too small or too large, then it would give a correspondingly wrong answer for the rectangle.)

For the next step, consider an arbitrary triangle with vertices at the lattice points that is not of the type already considered. Embed it in as small a rectangle as possible. There are several possible ways this can happen, but in any case (and adding one more edge in one case), the rectangle will have been partitioned into the given triangle and two or three right triangles with sides parallel to the coordinate axes. See the figure for a typical case. Again by the additivity lemma, we are guaranteed that Pick's formula gives the correct area for the central triangle.



Note that we have now proved $P(3)$, the basis step in our strong induction proof. For the inductive step, given an arbitrary polygon, use Lemma 1 in the text to split it into two polygons. Then by the additivity lemma above and the inductive hypothesis, we know that Pick's formula gives the correct area for this polygon.

Here are some good websites for more details:

`http://planetmath.org/encyclopedia/ProofOfPicksTheorem.html`
`http://www.cut-the-knot.org/ctk/Pick_proof.shtml`
`http://mathforum.org/library/drmath/view/65670.html`

21. **a)** Use the left figure. Angle abp is smallest for p, but the segment bp is not an interior diagonal.
b) Use the right figure. The vertex other than b with smallest x-coordinate is d, but the segment bd is not an interior diagonal.
c) Use the right figure. The vertex closest to b is d, but the segment bd is not an interior diagonal.

23. **a)** When we try to prove the inductive step and find a triangle in each subpolygon with at least two sides bordering the exterior, it may happen in each case that the triangle we are guaranteed in fact borders the diagonal (which is part of the boundary of that polygon). This leaves us with no triangles guaranteed to touch the boundary of the *original* polygon.
b) We proved $\forall n \geq 4\, T(n)$ in Exercise 17. Since we can always find two triangles that satisfy the property, perforce, at least one triangle does. Thus we have proved $\forall n \geq 4\, E(n)$.

25. **a)** The inductive step here allows us to conclude that $P(3), P(5), \ldots$ are all true, but we can conclude nothing about $P(2), P(4), \ldots$.
b) We can conclude that $P(n)$ is true for all positive integers n, using strong induction.

c) The inductive step here allows us to conclude that $P(2)$, $P(4)$, $P(8)$, $P(16)$, ... are all true, but we can conclude nothing about $P(n)$ when n is not a power of 2.

d) This is mathematical induction; we can conclude that $P(n)$ is true for all positive integers n.

27. Suppose, for a proof by contradiction, that there is some positive integer n such that $P(n)$ is not true. Let m be the smallest positive integer greater than n for which $P(m)$ *is* true; we know that such an m exists because $P(m)$ is true for infinitely many values of m, and therefore true for more than just $1, 2, \ldots, n-1$. But we are given that $P(m) \rightarrow P(m-1)$, so $P(m-1)$ is true. Thus $m-1$ cannot be greater than n, so $m-1=n$ and $P(n)$ is in fact true. This contradiction shows that $P(n)$ is true for all n.

29. The error is in going from the basis step $n=0$ to the next value, $n=1$. We cannot write 1 as the sum of two smaller natural numbers, so we cannot invoke the inductive hypothesis. In the notation of the "proof," when $k=0$, we cannot write $0+1=i+j$ where $0 \leq i \leq 0$ and $0 \leq j \leq 0$.

31. To show that strong induction is valid, let us suppose that we have a proposition $\forall n P(n)$ which has been proved using it. We must show that in fact $\forall n P(n)$ is true (to say that a principle of proof is valid means that it proves only true propositions). Let S be the set of counterexamples, i.e., $S = \{ n \mid \neg P(n) \}$. We want to show that $S = \emptyset$. We argue by contradiction. Assume that $S \neq \emptyset$. Then by the well-ordering property, S has a smallest element. Since part of the method of strong induction is to show that $P(1)$ is true, this smallest counterexample must be greater than 1. Let us call it $k+1$. Since $k+1$ is the smallest element of S, it must be the case that $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ is true. But the rest of the proof using strong induction involved showing that $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ implied $P(k+1)$; therefore since the hypothesis is true, the conclusion must be true as well, i.e., $P(k+1)$ is true. This contradicts our assumption that $k+1 \in S$. Therefore we conclude that $S = \emptyset$, so $\forall n P(n)$ is true.

33. In each case we will argue on the basis of a "smallest counterexample."

a) Suppose that there is a counterexample, that is, that there are values of n and k such that $P(n,k)$ is not true. Choose a counterexample with $n+k$ as small as possible. We cannot have $n=1$ and $k=1$, because we are given that $P(1,1)$ is true. Therefore either $n>1$ or $k>1$. In the former case, by our choice of counterexample, we know that $P(n-1,k)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction. The latter case is similar. So our supposition that there is a counterexample must be wrong, and $P(n,k)$ is true in all cases.

b) Suppose that there is a counterexample, that is, that there are values of n and k such that $P(n,k)$ is not true. Choose a counterexample with n as small as possible. We cannot have $n=1$, because we are given that $P(1,k)$ is true for all k. Therefore $n>1$. By our choice of counterexample, we know that $P(n-1,k)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction. So our supposition that there is a counterexample must be wrong, and $P(n,k)$ is true in all cases.

c) Suppose that there is a counterexample, that is, that there are values of n and k such that $P(n,k)$ is not true. Choose a counterexample with k as small as possible. We cannot have $k=1$, because we are given that $P(n,1)$ is true for all n. Therefore $k>1$. By our choice of counterexample, we know that $P(n,k-1)$ is true. But the inductive step then forces $P(n,k)$ to be true, a contradiction. So our supposition that there is a counterexample must be wrong, and $P(n,k)$ is true in all cases.

35. We want to calculate the product $a_1 a_2 \cdots a_n$ by inserting parentheses to express the calculation as a sequence of multiplications of two quantities. For example, we can insert parentheses into $a_1 a_2 a_3 a_4 a_5$ to render it $(a_1(a_2 a_3))(a_4 a_5)$, and then the four multiplications are $a_2 \cdot a_3$, $a_4 \cdot a_5$, $a_1 \cdot (a_2 a_3)$, and finally the product of these last two quantities. We must show that no matter how the parentheses are inserted, $n-1$ multiplications will be required. If $n=1$, then clearly 0 multiplications are required, so the basis step is trivial. Now assume the strong inductive hypothesis, that for all $k < n$, no matter how parentheses are inserted into the product of k numbers, $k - 1$ multiplications are required to compute the answer. Consider a parenthesized product of a_1 through a_n, and look at the last multiplication. Thus we have $(a_1 a_2 \cdots a_r) \cdot (a_{r+1} \cdots a_n)$, where we do not care how the parentheses are distributed within the pairs shown here. By the inductive hypothesis, it requires $r - 1$ multiplications to obtain the first product in parentheses and $n - r - 1$ to obtain the second (that second product has $n - r$ factors). Furthermore, 1 additional multiplication is needed to multiply these two answers together. This gives a total of $(r - 1) + (n - r - 1) + 1 = n - 1$ multiplications for the given problem, exactly what we needed to show.

37. Suppose that we have two such pairs, say (q, r) and (q', r'), so that $a = dq + r = dq' + r'$, with $0 \le r, r' < d$. We will show that the pairs are really the same, that is, that $q = q'$ and $r = r'$. From $dq + r = dq' + r'$ we obtain $d(q - q') = r' - r$. Therefore $d \mid (r' - r)$. But $|r' - r| < d$ (since both r' and r are nonnegative integers less than d). The only multiple of d in that range is 0, so we are forced to conclude that $r' = r$. Then it easily follows that $q = q'$ as well, since $q = (a - r)/d = (a - r')/d = q'$.

39. This problem deals with a paradox caused by self-reference. First of all, the answer to the question is clearly "no," because there are a finite number of English words, and so only a finite number of strings of fifteen words or fewer; therefore only a finite number of positive integers can be so described, not all of them. On the other hand, we might offer the following "proof" that every positive integer *can* be so expressed. Clearly 1 can be so expressed (e.g., "one" or "the cardinality of the power set of the empty set"). By the well-ordering principle, if there is a positive integer that cannot be expressed in fifteen words or fewer, then there is a smallest such, say s. Then the phrase "the smallest positive integer that cannot be described using no more than fifteen English words" is a description of s using no more than fifteen English words, a contradiction. Therefore no such s exists, and we seem to have proved that every positive integer can be so expressed, in obvious violation to common sense (and the argument presented above). Paradoxes like this are likely to arise whenever we try to use language to talk about itself; the use of language in this way, while seeming to be meaningful, is in fact nonsense.

41. We will prove this by contradiction. Suppose that the well-ordering property were false. Let S be a counterexample: a nonempty set of nonnegative integers that contains no smallest element. Let $P(n)$ be the statement "$i \notin S$ for all $i \le n$." We will show that $P(n)$ is true for all n (which will contradict the assertion that S is nonempty). Now $P(0)$ must be true, because if $0 \in S$ then clearly S would have a smallest element, namely 0. Suppose now that $P(n)$ is true, so that $i \notin S$ for $i = 0, 1, \ldots, n$. We must show that $P(n + 1)$ is true, which amounts to showing that $n + 1 \notin S$. If $n + 1 \in S$, then $n + 1$ would be the smallest element of S, and this would contradict our assumption. Therefore $n + 1 \notin S$. Thus we have shown by the principle of mathematical induction that $P(n)$ is true for all n, which means that there can be no elements of S. This contradicts our assumption that $S \neq \emptyset$, and our proof by contradiction is complete.

43. Strong induction clearly implies the principle of mathematical induction, for if one has shown that $P(k) \to P(k + 1)$ is true, then one has also shown that $[P(1) \wedge \cdots \wedge P(k)] \to P(k + 1)$ is true. By Exercise 41, the principle of mathematical induction implies the well-ordering property. Therefore by assuming either the principle of mathematical induction or strong induction as an axiom, we can prove the well-ordering property.