CHAPTER 6
Discrete Probability

SECTION 6.1   An Introduction to Discrete Probability

2. The probability is $1/6 \approx 0.17$, since there are six equally likely outcomes.

4. Since April has 30 days, the answer is $30/366 = 5/61 \approx 0.082$.

6. There are 16 cards that qualify as being an ace or a heart, so the answer is $16/52 = 4/13 \approx 0.31$. We could also compute this from Theorem 2 as $4/52 + 13/52 - 1/52$.

8. We saw in Example 11 of Section 5.3 that there are $C(52,5)$ possible poker hands, and we assume by symmetry that they are all equally likely. In order to solve this problem, we need to compute the number of poker hands that contain the ace of hearts. There is no choice about choosing the ace of hearts. To form the rest of the hand, we need to choose 4 cards from the 51 remaining cards, so there are $C(51,4)$ hands containing the ace of hearts. Therefore the answer to the question is the ratio

$$\frac{C(51,4)}{C(52,5)} = \frac{5}{52} \approx 9.6\%\,.$$

The problem can also be done by subtracting from 1 the answer to Exercise 9, since a hand contains the ace of hearts if and only if it is not the case that it does not contain the ace of hearts.

10. This is similar to Exercise 8. We need to compute the number of poker hands that contain the two of diamonds and the three of spades. There is no choice about choosing these two cards. To form the rest of the hand, we need to choose 3 cards from the 50 remaining cards, so there are $C(50,3)$ hands containing these two specific cards. Therefore the answer to the question is the ratio

$$\frac{C(50,3)}{C(52,5)} = \frac{5}{663} \approx 0.0075\,.$$

12. There are 4 ways to specify the ace. Once the ace is chosen for the hand, there are $C(48,4)$ ways to choose nonaces for the remaining four cards. Therefore there are $4C(48,4)$ hands with exactly one ace. Since there are $C(52,5)$ equally likely hands, the answer is the ratio

$$\frac{4C(48,4)}{C(52,5)} \approx 0.30\,.$$

14. We saw in Example 11 of Section 5.3 that there are $C(52,5) = 2{,}598{,}960$ different hands, and we assume by symmetry that they are all equally likely. We need to count the number of hands that have 5 different kinds (ranks). There are $C(13,5)$ ways to choose the kinds. For each card, there are then 4 ways to choose the suit. Therefore there are $C(13,5) \cdot 4^5 = 1{,}317{,}888$ ways to choose the hand. Thus the probability is $1317888/2598960 = 2112/4165 \approx 0.51$.

16. Of the $C(52,5) = 2,598,960$ hands, $4 \cdot C(13,5) = 5148$ are flushes, since we can specify a flush by choosing a suit and then choosing 5 cards from that suit. Therefore the answer is $5148/2598960 = 33/16660 \approx 0.0020$.

18. There are clearly only $10 \cdot 4 = 40$ straight flushes, since all we get to specify for a straight flush is the starting (lowest) kind in the straight (anything from ace up to ten) and the suit. Therefore the answer is $40/C(52,5) = 40/2598960 = 1/64974$.

20. There are 4 royal flushes, one in each suit. Therefore the answer is $4/C(52,5) = 4/2598960 = 1/649740$.

22. There are $\lfloor 100/3 \rfloor = 33$ multiples of 3 among the integers from 1 to 100 (inclusive), so the answer is $33/100 = 0.33$.

24. In each case, if the numbers are chosen from the integers from 1 to n, then there are $C(n,6)$ possible entries, only one of which is the winning one, so the answer is $1/C(n,6)$.
  a) $1/C(30,6) = 1/593775 \approx 1.7 \times 10^{-6}$ b) $1/C(36,6) = 1/1947792 \approx 5.1 \times 10^{-7}$
  c) $1/C(42,6) = 1/5245786 \approx 1.9 \times 10^{-7}$ d) $1/C(48,6) = 1/12271512 \approx 8.1 \times 10^{-8}$

26. In each case, if the numbers are chosen from the integers from 1 to n, then there are $C(n,6)$ possible entries. If we wish to avoid all the winning numbers, then we must make our choice from the $n - 6$ nonwinning numbers, and this can be done in $C(n - 6, 6)$ ways. Therefore, since the winning numbers are picked at random, the probability is $C(n - 6, 6)/C(n,6)$.
  a) $C(34,6)/C(40,6) = 1344904/3838380 \approx 0.35$ b) $C(42,6)/C(48,6) = 5245786/12271512 \approx 0.43$
  c) $C(50,6)/C(56,6) = 15890700/32468436 \approx 0.49$ d) $C(58,6)/C(64,6) = 40475358/74974368 \approx 0.54$

28. We need to find the number of ways for the computer to select its 11 numbers, and we need to find the number of ways for it to select its 11 numbers so as to contain the 7 numbers that we chose. For the former, the number is clearly $C(80,11)$. For the latter, the computer must select four more numbers besides the ones we chose, from the $80 - 7 = 73$ other numbers, so there are $C(73,4)$ ways to do this. Therefore the probability that we win is the ratio $C(73,4)/C(80,11)$, which works out to $3/28879240$, or about one chance in ten million (1.04×10^{-7}). The same answer can be obtained by counting things in the other direction: the number of ways for us to choose 7 of the computer's predestined 11 numbers divided by the number of ways for us to pick 7 numbers. This gives $C(11,7)/C(80,7)$, which has the same value as before.

30. In order to specify a winning ticket, we must choose five of the six numbers to match ($C(6,5) = 6$ ways to do so) and one number from among the remaining 34 numbers not to match ($C(34,1) = 34$ ways to do so). Therefore there are $6 \cdot 34 = 204$ winning tickets. Since there are $C(40,6) = 3,838,380$ tickets in all, the answer is $204/3838380 = 17/319865 \approx 5.3 \times 10^{-5}$, or about 1 chance in 19,000.

32. The number of ways for the drawing to turn out is $100 \cdot 99 \cdot 98$. The number of ways of ways for the drawing to cause Kumar, Janice, and Pedro each to win a prize is $3 \cdot 2 \cdot 1$ (three ways for one of these to be picked to win first prize, two ways for one of the others to win second prize, one way for the third to win third prize). Therefore the probability we seek is $(3 \cdot 2 \cdot 1)/(100 \cdot 99 \cdot 98) = 1/161700$.

34. a) There are $50 \cdot 49 \cdot 48 \cdot 47$ equally likely outcomes of the drawings. In only one of these do Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively. Therefore the probability is $1/(50 \cdot 49 \cdot 48 \cdot 47) = 1/5527200$.
  b) There are $50 \cdot 50 \cdot 50 \cdot 50$ equally likely outcomes of the drawings. In only one of these do Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively. Therefore the probability is $1/50^4 = 1/6250000$.

36. Reasoning as in Example 2, we see that there are 5 ways to get a total of 8 when two dice are rolled: $(6,2)$, $(5,3)$, $(4,4)$, $(3,5)$, and $(2,6)$. There are $6^2 = 36$ equally likely possible outcomes of the roll of two dice, so the probability of getting a total of 8 when two dice are rolled is $5/36 \approx 0.139$. For three dice, there are $6^3 = 216$ equally likely possible outcomes, which we can represent as ordered triples (a,b,c). We need to enumerate the possibilities that give a total of 8. This is done in a more systematic way in Section 5.5, but we will do it here by brute force. The first die could turn out to be a 6, giving rise to the 1 triple $(6,1,1)$. The first die could be a 5, giving rise to the 2 triples $(5,2,1)$, and $(5,1,2)$. Continuing in this way, we see that there are 3 triples giving a total of 8 when the first die shows a 4, 4 triples when it shows a 3, 5 triples when it shows a 2, and 6 triples when it shows a 1 (namely $(1,6,1)$, $(1,5,2)$, $(1,4,3)$, $(1,3,4)$, $(1,2,5)$, and $(1,1,6)$). Therefore there are $1+2+3+4+5+6 = 21$ possible outcomes giving a total of 8. This tells us that the probability of rolling a 9 when three dice are thrown is $21/216 \approx 0.097$, smaller than the corresponding value for two dice. Thus rolling a total of 9 is more likely when using two dice than when using three.

38. a) Intuitively, these should be independent, since the first event seems to have no influence on the second. In fact we can compute as follows. First $p(E_1) = 1/2$ and $p(E_2) = 1/2$ by the symmetry of coin tossing. Furthermore, $E_1 \cap E_2$ is the event that the first two coins come up tails and heads, respectively. Since there are four equally likely outcomes for the first two coins (HH, HT, TH, and TT), $p(E_1 \cap E_2) = 1/4$. Therefore $p(E_1 \cap E_2) = 1/4 = (1/2) \cdot (1/2) = p(E_1)p(E_2)$, so the events are indeed independent.
b) Again $p(E_1) = 1/2$. For E_2, note that there are 8 equally likely outcomes for the three coins, and in 2 of these cases E_2 occurs (namely HHT and THH); therefore $p(E_2) = 2/8 = 1/4$. Thus $p(E_1)p(E_2) = (1/2) \cdot (1/4) = 1/8$. Now $E_1 \cap E_2$ is the event that the first coin comes up tails, and two but not three heads come up in a row. This occurs precisely when the outcome is THH, so the probability is $1/8$. This is the same as $p(E_1)p(E_2)$, so the events are independent.
c) As in part **(b)**, $p(E_1) = 1/2$ and $p(E_2) = 1/4$. This time $p(E_1 \cap E_2) = 0$, since there is no way to get two heads in a row if the second coin comes up tails. Since $p(E_1)p(E_2) \neq p(E_1 \cap E_2)$, the events are not independent.

40. You had a 1/4 chance of winning with your original selection. Just as in the original problem, the host's action did not change this, since he would act the same way regardless of whether your selection was a winner or a loser. Therefore you have a 1/4 chance of winning if you do not change. This implies that there is a 3/4 chance of the prize's being behind one of the other doors. Since there are two such doors and by symmetry the probabilities for each of them must be the same, your chance of winning after switching is half of 3/4, or 3/8.