## SECTION 7.5 Inclusion–Exclusion

*Inclusion–exclusion is not a nice compact formula in practice, but it is often the best that is available. In Exercise 19, for example, the answer contains over 30 terms. The applications in this section are somewhat contrived, but much more interesting applications are presented in Section 7.6. The inclusion–exclusion principle in some sense gives a methodical way to apply common sense. Presumably anyone could solve a problem such as Exercise 9 by trial and error or other ad hoc techniques, given enough time; the inclusion–exclusion principle makes the solution straightforward. Be careful when using the inclusion–exclusion principle to get the signs right—some terms need to be subtracted and others need to be added. In general the sign changes when the size of the expression changes.*

**1.** In all cases we use the fact that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 12 + 18 - |A_1 \cap A_2| = 30 - |A_1 \cap A_2|$.
**a)** Here $|A_1 \cap A_2| = 0$, so the answer is $30 - 0 = 30$.
**b)** This time we are told that $|A_1 \cap A_2| = 1$, so the answer is $30 - 1 = 29$.
**c)** This time we are told that $|A_1 \cap A_2| = 6$, so the answer is $30 - 6 = 24$.
**d)** If $A_1 \subseteq A_2$, then $A_1 \cap A_2 = A_1$, so $|A_1 \cap A_2| = |A_1| = 12$. Therefore the answer is $30 - 12 = 18$.

**3.** We may as well treat percentages as if they were cardinalities—as if the population were exactly 100. Let $V$ be the set of households with television sets, and let $P$ be the set of households with phones. Then we are given $|V| = 96$, $|P| = 98$, and $|V \cap P| = 95$. Therefore $|V \cup P| = 96 + 98 - 95 = 99$, so only 1% of the households have neither telephones nor televisions.

**5.** For all parts we need to use the formula $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.
**a)** If the sets are pairwise disjoint, then the cardinality of the union is the sum of the cardinalities, namely 300, since all but the first three terms on the right-hand side of the formula are equal to 0.
**b)** Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 0 = 150$.
**c)** Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 25 = 175$.
**d)** In this case the answer is obviously 100. By the formula, the cardinality of each set on the right-hand side is 100, so we can arrive at this answer through the computation $100 + 100 + 100 - 100 - 100 - 100 + 100 = 100$.

**7.** We need to use the formula $|P \cup F \cup C| = |P| + |F| + |C| - |P \cap F| - |P \cap C| - |F \cap C| + |P \cap F \cap C|$, where, for example, $P$ is the set of students who have taken a course in Pascal. Thus we have $|P \cup F \cup C| = 1876 + 999 + 345 - 876 - 290 - 231 + 189 = 2012$. Therefore, since there are 2504 students altogether, we know that $2504 - 2012 = 492$ have taken none of these courses.

**9.** We need to use the inclusion–exclusion formula for four sets, $C$ (the students taking calculus), $D$ (the students taking discrete mathematics), $S$ (those taking data structures), and $L$ (those taking programming languages). The formula says $|C \cup D \cup S \cup L| = |C| + |D| + |S| + |L| - |C \cap D| - |C \cap S| - |C \cap L| - |D \cap S| - |D \cap L| - |S \cap L| + |C \cap D \cap S| + |C \cap D \cap L| + |C \cap S \cap L| + |D \cap S \cap L| - |C \cap D \cap S \cap L|$. Plugging the given information into this formula gives us a total of $507 + 292 + 312 + 344 - 0 - 14 - 213 - 211 - 43 - 0 + 0 + 0 + 0 + 0 - 0 = 974$.

**11.** There are clearly 50 odd positive integers not exceeding 100 (half of these 100 numbers are odd), and there are 10 squares (from $1^2$ to $10^2$). Furthermore, half of these squares are odd. Thus we compute the cardinality of the set in question to be $50 + 10 - 5 = 55$.

**13.** Let us count the strings that have 6 or more consecutive 0's. There are 4 strings that have 0's in the first six places, since there are $2 \cdot 2 = 4$ ways to specify the last two bits. Similarly, there are 4 strings that have 0's in bits 2 through 7, and there are 4 strings that have 0's in bits 3 through 8. We have overcounted, though. There are 2 strings that have 0's in bits 1 through 7 (the intersection of the first two sets mentioned above); 2 strings that have 0's in bits 2 through 8 (the intersection of the last two sets mentioned above); and 1 string that has 0's in all bits (the intersection of the first and last sets mentioned above). Moreover, there is 1 string with 0's in bits 1 through 8, the intersection of all three sets mentioned above. Putting this all together, we know that the number of strings with 6 consecutive 0's is $4+4+4-2-2-1+1 = 8$. Since there are $2^8 = 256$ strings in all, there must be $256 - 8 = 248$ that do not contain 6 consecutive 0's.

**15.** We need to use inclusion–exclusion with three sets. There are 7! permutations that begin 987, since there are 7 digits free to be permuted among the last 7 spaces (we are assuming that it is meant that the permutations are to start with 987 *in that order*, not with 897, for instance). Similarly, there are 8! permutations that have 45 in the fifth and sixth positions, and there are 7! that end with 123. (We assume that the intent is that these digits are to appear in the order given.) There are 5! permutations that begin with 987 and have 45 in the fifth and sixth positions; 4! that begin with 987 and end with 123; and 5! that have 45 in the fifth and sixth positions and end with 123. Finally, there are 2! permutations that begin with 987, have 45 in the fifth and sixth positions, and end with 123 (since only the 0 and the 6 are left to place). Therefore the total number of permutations meeting any of these conditions is $7! + 8! + 7! - 5! - 4! - 5! + 2! = 50{,}138$.

**17.** By inclusion–exclusion, the answer is $50+60+70+80-6\cdot 5+4\cdot 1-0 = 234$. Note that there were $C(4,2) = 6$ pairs to worry about (each with 5 elements in common) and $C(4,1) = 4$ triples to worry about (each with 1 element in common).

**19.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_2 \cap A_5| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_5| + |A_1 \cap A_4 \cap A_5| + |A_2 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_4 \cap A_5| + |A_3 \cap A_4 \cap A_5| - |A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - |A_1 \cap A_2 \cap A_4 \cap A_5| - |A_1 \cap A_3 \cap A_4 \cap A_5| - |A_2 \cap A_3 \cap A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$

**21.** Since no three of the sets have a common intersection, we need only carry our expression out as far as pairs. Thus we have $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_1 \cap A_6| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_2 \cap A_6| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_3 \cap A_6| - |A_4 \cap A_5| - |A_4 \cap A_6| - |A_5 \cap A_6|$.

**23.** Since the probability of an event (i.e., a set) $E$ is proportional to the number of elements in the set $E$, this problem is just asking about cardinalities, and so inclusion–exclusion gives us the answer. Thus $p(E_1 \cup E_2 \cup E_3) = p(E_1) + p(E_2) + p(E_3) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_3)$.

**25.** We can do this problem either by working directly with probabilities or by counting ways to satisfy the condition. We choose to do the former. First we need to determine the probability that all the numbers are odd. There are $C(100,4)$ ways to choose the numbers, and there are $C(50,4)$ ways to choose them all to be odd (since there are 50 odd numbers in the given interval). Therefore the probability that they are all odd is $C(50,4)/C(100,4)$. Similarly, since there are 33 multiples of 3 in the given interval, the probability of having all four numbers divisible by 3 is $C(33,4)/C(100,4)$. Finally, the probability that all four are divisible by 5 is $C(20,4)/C(100,4)$.

Next we need to know the probabilities that two of these events occur simultaneously. A number is both odd and divisible by 3 if and only if it is divisible by 3 but not by 6; therefore, since there are $\lfloor 100/6 \rfloor = 16$ multiples of 6 in the given interval, there are $33 - 16 = 17$ numbers that are both odd and divisible by 3. Thus the probability is $C(17,4)/C(100,4)$. Similarly there are 10 odd numbers divisible by 5, so the probability that all four numbers meet those conditions is $C(10,4)/C(100,4)$. Finally, the probability that all four numbers are divisible by both 3 and 5 is $C(6,4)/C(100,4)$, since there are only $\lfloor 100/15 \rfloor = 6$ such numbers.

Finally, the only numbers satisfying all three conditions are the odd multiples of 15, namely 15, 45, and 75. Since there are only 3 such numbers, it is impossible that all chosen four numbers are divisible by 2, 3, and 5; in other words, the probability of that event is 0. We are now ready to apply the result of Exercise 23 (i.e., inclusion–exclusion viewed in terms of probabilities). We get

$$\begin{aligned}
&\frac{C(50,4)}{C(100,4)}+\frac{C(33,4)}{C(100,4)}+\frac{C(20,4)}{C(100,4)}-\frac{C(17,4)}{C(100,4)}-\frac{C(10,4)}{C(100,4)}-\frac{C(6,4)}{C(100,4)}+0\\
&=\frac{230300+40920+4845-2380-210-15}{3921225}\\
&=\frac{273460}{3921225}=\frac{4972}{71295}\approx 0.0697\,.
\end{aligned}$$

**27.** We are asked to write down inclusion–exclusion for five sets, just as in Exercise 19, except that intersections of more than three sets can be omitted. Furthermore, we are to use event notation, rather than set notation. Thus we have $p(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_1 \cap E_5) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_2 \cap E_5) - p(E_3 \cap E_4) - p(E_3 \cap E_5) - p(E_4 \cap E_5) + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_1 \cap E_2 \cap E_5) + p(E_1 \cap E_3 \cap E_4) + p(E_1 \cap E_3 \cap E_5) + p(E_1 \cap E_4 \cap E_5) + p(E_2 \cap E_3 \cap E_4) + p(E_2 \cap E_3 \cap E_5) + p(E_2 \cap E_4 \cap E_5) + p(E_3 \cap E_4 \cap E_5)$.

**29.** We are simply asked to rephrase Theorem 1 in terms of probabilities of events. Thus we have

$$\begin{aligned}
p(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{1\le i\le n} p(E_i) - \sum_{1\le i<j\le n} p(E_i \cap E_j) + \sum_{1\le i<j<k\le n} p(E_i \cap E_j \cap E_k)\\
\cdots + (-1)^{n+1} p(E_1 \cap E_2 \cap \cdots \cap E_n)\,.
\end{aligned}$$

**1.** We want to find the number of apples that have neither of the properties of having worms or of having bruises. By inclusion-exclusion, we know that this is equal to the number of apples, minus the numbers with each of the properties, plus the number with both properties. In this case, this is $100-20-15+10=75$.

**3.** We need first to find the number of solutions with no restrictions. By the results of Section 5.5, there are $C(3+13-1,13)=C(15,13)=C(15,2)=105$. Next we need to find the number of solutions in which each restriction is violated. There are three variables that can fail to be less than 6, and the situation is symmetric, so the total number of solutions in which each restriction is violated is 3 times the number of solutions in which $x_1 \geq 6$. By the trick we used in Section 5.5, this is the same as the number of nonnegative integer solutions to $x_1'+x_2+x_3=7$, where $x_1=x_1'+6$. This of course is $C(3+7-1,7)=C(9,7)=C(9,2)=36$. Therefore there are $3\cdot 36=108$ solutions in which at least one of the restrictions is violated (with some of these counted more than once).

Next we need to find the number of solutions with at least two of the restrictions violated. There are $C(3,2)=3$ ways to choose the pair to be violated, so the number we are seeking is 3 times the number of solutions in which $x_1 \geq 6$ and $x_2 \geq 6$. Again by the trick we used in Section 5.5, this is the same as the number of nonnegative integer solutions to $x_1'+x_2'+x_3=1$, where $x_1=x_1'+6$ and $x_2=x_2'+6$. This of course is $C(3+1-1,1)=C(3,1)=3$. Therefore there are $3\cdot 3=9$ solutions in which two of the restrictions are violated. Finally, we note that there are no solutions in which all three of the solutions are violated, since if each of the variables is at least 6, then their sum is at least 18, and hence cannot equal 13.

Thus by inclusion–exclusion, we see that there are $105-108+9=6$ solutions to the original problem. (We can check this on an ad hoc basis. The only way the sum of three numbers, not as big as 6, can be 13, is to have either two 5's and one 3, or else one 5 and two 4's. There are three variables that can be the "odd man out" in each case, for a total of 6 solutions.)

**5.** We follow the procedure described in the text. There are 198 positive integers less than 200 and greater than 1. The ones that are not prime are divisible by at least one of the primes in the set $\{2,3,5,7,11,13\}$. The number of integers in the given range divisible by the prime $p$ is given by $\lfloor 199/p \rfloor$. Therefore we apply inclusion exclusion and obtain the following number of integers from 2 to 199 that are not divisible by at least one of the primes in our set. (We have only listed those terms that contribute to the result, deleting all those that equal 0.)

$$\begin{aligned}
198 &- \left\lfloor\frac{199}{2}\right\rfloor - \left\lfloor\frac{199}{3}\right\rfloor - \left\lfloor\frac{199}{5}\right\rfloor - \left\lfloor\frac{199}{7}\right\rfloor - \left\lfloor\frac{199}{11}\right\rfloor - \left\lfloor\frac{199}{13}\right\rfloor + \left\lfloor\frac{199}{2\cdot 3}\right\rfloor \\
&+ \left\lfloor\frac{199}{2\cdot 5}\right\rfloor + \left\lfloor\frac{199}{2\cdot 7}\right\rfloor + \left\lfloor\frac{199}{2\cdot 11}\right\rfloor + \left\lfloor\frac{199}{2\cdot 13}\right\rfloor + \left\lfloor\frac{199}{3\cdot 5}\right\rfloor + \left\lfloor\frac{199}{3\cdot 7}\right\rfloor + \left\lfloor\frac{199}{3\cdot 11}\right\rfloor \\
&+ \left\lfloor\frac{199}{3\cdot 13}\right\rfloor + \left\lfloor\frac{199}{5\cdot 7}\right\rfloor + \left\lfloor\frac{199}{5\cdot 11}\right\rfloor + \left\lfloor\frac{199}{5\cdot 13}\right\rfloor + \left\lfloor\frac{199}{7\cdot 11}\right\rfloor + \left\lfloor\frac{199}{7\cdot 13}\right\rfloor + \left\lfloor\frac{199}{11\cdot 13}\right\rfloor \\
&- \left\lfloor\frac{199}{2\cdot 3\cdot 5}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 7}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 11}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 13}\right\rfloor - \left\lfloor\frac{199}{2\cdot 5\cdot 7}\right\rfloor - \left\lfloor\frac{199}{2\cdot 5\cdot 11}\right\rfloor \\
&- \left\lfloor\frac{199}{2\cdot 5\cdot 13}\right\rfloor - \left\lfloor\frac{199}{2\cdot 7\cdot 11}\right\rfloor - \left\lfloor\frac{199}{2\cdot 7\cdot 13}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 7}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 11}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 13}\right\rfloor \\
&= 198-99-66-39-28-18-15+33+19+14+9+7+13+9+6+5 \\
&\quad +5+3+3+2+2+1-6-4-3-2-2-1-1-1-1-1-1-1=40
\end{aligned}$$

These 40 numbers are therefore all prime, as are the 6 numbers in our set. Therefore there are exactly 46 prime numbers less than 200.

**7.** We can apply inclusion-exclusion if we reason as follows. First, we restrict ourselves to numbers greater than 1. If the number $N$ is the power of an integer, then it is certainly the prime power of an integer, since if