SUPPLEMENTARY EXERCISES FOR CHAPTER 6

2. There are $C(52,13)$ possible hands. A hand with no pairs mnst contain exactly one card of each kind. The only choice involved, therefore, is the suit for each of the 13 cards. There are 4 ways to specify the suit, and there are 13 tasks to be performed. Therefore there are 4^{13} hands with no pairs. The probability of drawing such a hand is thus $4^{13}/C(52,13) = 67108864/635013559600 = 4194304/39688347475 \approx 0.000106$.

4. The denominator of each probability is the number of 7-card poker hands, namely $C(52,7) = 133784560$.

a) The number of such hands is $13 \cdot 12 \cdot 4$, since there are 13 ways to choose the kind for the four, then 12 ways to choose another kind for the three, then $C(4,3) = 4$ ways to choose which three cards of that second kind to use. Therefore the probability is $624/133784560 \approx 4.7 \times 10^{-6}$.

b) The number of such hands is $13 \cdot 4 \cdot 66 \cdot 6^2$, since there are 13 ways to choose the kind for the three, $C(4,3) = 4$ ways to choose which three cards of that kind to use, then $C(12,2) = 66$ ways to choose two more kinds for the pairs, then $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use. Therefore the probability is $123552/133784560 \approx 9.2 \times 10^{-4}$.

c) The number of such hands is $286 \cdot 6^3 \cdot 10 \cdot 4$, since there are $C(13,3) = 286$ ways to choose the kinds for the pairs, $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use, 10 ways to choose the kind for the singleton, and 4 ways to choose which card of that kind to use. Therefore the probability is $2471040/133784560 \approx 0.018$.

d) The number of such hands is $78 \cdot 6^2 \cdot 165 \cdot 4^3$, since there are $C(13,2) = 78$ ways to choose the kinds for the pairs, $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use, $C(11,3) = 165$ ways to choose the kinds for the singletons, and 4 ways to choose which card of each of those kinds to use. Therefore the probability is $29652480/133784560 \approx 0.22$.

e) The number of such hands is $1716 \cdot 4^7$, since there are $C(13,7) = 1716$ ways to choose the kinds and 4 ways to choose which card of each of kind to use. Therefore the probability is $28114944/133784560 \approx 0.21$.

f) The number of such hands is $4 \cdot 1716$, since there are 4 ways to choose the suit for the flush and $C(13,7) = 1716$ ways to choose the kinds in that suit. Therefore the probability is $6864/133784560 \approx 5.1 \times 10^{-5}$.

g) The number of such hands is $8 \cdot 4^7$, since there are 8 ways to choose the kind for the straight to start at (A, 2, 3, 4, 5, 6, 7, or 8) and 4 ways to choose the suit for each kind. Therefore the probability is $131072/133784560 \approx 9.8 \times 10^{-4}$.

h) There are only $4 \cdot 8$ straight flushes, since the only choice is the suit and the starting kind (see part **(g)**). Therefore the probability is $32/133784560 \approx 2.4 \times 10^{-7}$.

6. **a)** Each of the outcomes 1 through 12 occurs with probability $1/12$, so the expectation is $(1/12)(1+2+3+\cdots+12) = 13/2$.

b) We compute $V(X) = E(X^2) - E(X)^2 = (1/12)(1^2+2^2+3^2+\cdots+12^2) - (13/2)^2 = (325/6) - (169/4) = 143/12$.

8. **a)** Since expected value is linear, the expected value of the sum is the sum of the expected values, each of which is $13/2$ by Exercise 6a. Therefore the answer is 13.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, each of which is $143/12$ by Exercise 6b. Therefore the answer is $143/6$.

10. a) Since expected value is linear, the expected value of the sum is the sum of the expected values, which are $9/2$ by Exercise 5a and $13/2$ by Exercise 6a. Therefore the answer is $(9/2) + (13/2) = 11$.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, which are $21/4$ by Exercise 5b and $143/12$ by Exercise 6b. Therefore the answer is $(21/4) + (143/12) = 103/6$.

12. We need to determine how many positive integers less than $n = pq$ are divisible by either p or q. Certainly the numbers p, $2p$, $3p$, ..., $(q-1)p$ are all divisible by p. This gives $q-1$ numbers. Similarly, $p-1$ numbers are divisible by q. None of these numbers is divisible by both p and q since $lcm(p,q) = pq/\gcd(p,q) = pq/1 = pq = n$. Therefore $p+q-2$ numbers in this range are divisible by p or q, so the remaining $pq-1-(p+q-2) = pq-p-q+1 = (p-1)(q-1)$ are not. Therefore the probability that a randomly chosen integer in this range is not divisible by either p or q is $(p-1)(q-1)/(pq-1)$.

14. Technically a proof by mathematical induction is required, but we will give a somewhat less formal version. We just apply the definition of conditional probability to the right-hand side and observe that practically everything cancels (each denominator with the numerator of the previous term):

$$\begin{aligned}
&p(E_1)p(E_2|E_1)p(E_3|E_1\cap E_2)\cdots p(E_n|E_1\cap E_2\cap\cdots\cap E_{n-1})\\
&\quad = p(E_1)\cdot\frac{p(E_1\cap E_2)}{p(E_1)}\cdot\frac{p(E_1\cap E_2\cap E_3)}{p(E_1\cap E_2)}\cdots\frac{p(E_1\cap E_2\cap\cdots\cap E_n)}{p(E_1\cap E_2\cap\cdots\cap E_{n-1})}\\
&\quad = p(E_1\cap E_2\cap\cdots\cap E_n)
\end{aligned}$$

16. If n is odd, then it is impossible, so the probability is 0. If n is even, then there are $C(n,n/2)$ ways that an equal number of heads and tails can appear (choose the flips that will be heads), and 2^n outcomes in all, so the probability is $C(n,n/2)/2^n$.

18. There are 2^{11} bit strings. There are 2^6 palindromic bit strings, since once the first six bits are specified arbitrarily, the remaining five bits are forced. If a bit string is picked at random, then, the probability that it is a palindrome is $2^6/2^{11} = 1/32$.

20. a) Since there are b bins, each equally likely to receive the ball, the answer is $1/b$.

b) By linearity of expectation, the fact that n balls are tossed, and the answer to part (a), the answer is n/b.

c) In order for this part to make sense, we ignore n, and assume that the ball supply is unlimited and we keep tossing until the bin contains a ball. The number of tosses then has a geometric distribution with $p = 1/b$ from part (a). The expectation is therefore b.

d) Again we have to assume that the ball supply is unlimited and we keep tossing until every bin contains at least one ball. The analysis is identical to that of Exercise 29 in this set, with b here playing the role of n there. By the solution given there, the answer is $b\sum_{j=1}^{b} 1/j$.

22. a) The intersection of two sets is a subset of each of them, so the largest $p(A\cap B)$ could be would occur when the smaller is a subset of the larger. In this case, that would mean that we want $B\subseteq A$, in which case $A\cap B = B$, so $p(A\cap B) = p(B) = 1/2$. To construct an example, we find a common denominator of the fractions involved, namely 6, and let the sample space consist of 6 equally likely outcomes, say numbered 1 through 6. We let $B = \{1,2,3\}$ and $A = \{1,2,3,4\}$. The smallest intersection would occur when $A\cup B$ is as large as possible, since $p(A\cup B) = p(A)+p(B)-p(A\cap B)$. The largest $A\cup B$ could ever be is the entire sample space, whose probability is 1, and that certainly can occur here. So we have $1 = (2/3)+(1/2)-p(A\cap B)$, which gives $p(A\cap B) = 1/6$. To construct an example, again we find a common denominator of these fractions, namely 6, and let the sample space consist of 6 equally likely outcomes, say numbered 1 through 6. We let $A = \{1,2,3,4\}$ and $B = \{4,5,6\}$. Then $A \cap B = \{4\}$, and $p(A \cap B) = 1/6$.

b) The largest $p(A \cup B)$ could ever be is 1, which occurs when $A \cup B$ is the entire sample space. As we saw in part (a), that is possible here, using the second example above. The union of two sets is a subset of each of them, so the smallest $p(A \cup B)$ could be would occur when the smaller is a subset of the larger. In this case, that would mean that we want $B \subseteq A$, in which case $A \cup B = A$, so $p(A \cup B) = p(A) = 2/3$. This occurs in the first example given above.

24. From $p(B \mid A) < p(B)$ it follows that $p(A \cap B)/p(A) < p(B)$, which is equivalent to $p(A \cap B) < p(A)p(B)$. Dividing both sides by $p(B)$ and using the fact that then $p(A \mid B) = p(A \cap B)/p(B)$ yields the desired result.

26. By Example 6 in Section 6.4, the expected value of X, the number of people who get their own hat back, is 1. By Exercise 37 in that section, the variance of X is also 1. If we apply Chebyshev's Inequality (Theorem 8 in Section 6.4) with $r = 10$, we find that the probability that X is greater than or equal to 11 is at most $1/10^2 = 1/100$.

28. In order for the stated outcome to occur, the first $m+n$ trials must result in exactly m successes and n failures, and the $(m+n)^{\text{th}}$ trial must be a success. There are many ways in which this can occur; specifically, there are $C(n+m-1,n)$ ways to choose which n of the first $n+m-1$ trials are to be the failures. Each particular sequence has probability $q^n p^m$ of occurring, since the successes occur with probability p and the failures occur with probability q. The answer follows.

30. **a)** Clearly each assignment has a probability $1/2^n$.

b) The probability that the random assignment of truth values made the first of the two literals in the clause false is $1/2$, and similarly for the second. Since the coin tosses were independent, the probability that both are false is therefore $(1/2)(1/2) = 1/4$, so the probability that the disjunction is true is $1-(1/4) = 3/4$.

c) By linearity of expectation, the answer is $(3/4)D$.

d) By part (c), averaged over all possible outcomes of the coin flips, $3/4$ of the clauses are true. Since the average cannot be greater than all the numbers being averaged, at least $3/4$ of the clauses must be true for at least one outcome of the coin tosses.

32. Rather than following the hint, we will give a direct argument. The protocol given here has $n!$ possible outcomes, each equally likely, because there are n possible choices for $r(n)$, $n-1$ possible choices for $r(n-1)$, and so on. Therefore if we can argue that each outcome gives rise to exactly one permutation, then each permutation will be equally likely. But this is clear. Suppose $(a_1, a_2, a_3, \ldots, a_n)$ is a permutation of $(1,2,3,\ldots,n)$. In order for this permutation to be generated by the protocol, it must be the case that $r(n) = a_n$, because it is only on round one of the protocol that anything gets moved into the n^{th} position. Next, $r(n-1)$ must the unique value that picks out a_{n-1} to put in the $(n-1)^{\text{st}}$ position (this is not necessarily a_{n-1}, because it might happen that $a_{n-1} = n$, and n could have been put into one of the other positions as a result of round one). And so on. Thus each permutation corresponds to exactly one sequence of choices of the random numbers.