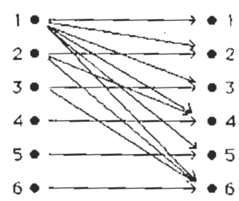
CHAPTER 8
Relations

SECTION 8.1 Relations and Their Properties

2. a) $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,2)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,6)$, $(4,4)$, $(5,5)$, $(6,6)$

b) We draw a line from a to b whenever a divides b, using separate sets of points; an alternate form of this graph would have just one set of points.



c) We put an $\times$ in the i^{th} row and j^{th} column if and only if i divides j.

R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

4. a) Being taller than is not reflexive (I am not taller than myself), nor symmetric (I am taller than my daughter, but she is not taller than I). It is antisymmetric (vacuously, since we never have A taller than B, and B taller than A, even if $A = B$). It is clearly transitive.

b) This is clearly reflexive, symmetric, and transitive (it is an equivalence relation—see Section 8.5). It is not antisymmetric, since twins, for example, are unequal people born on the same day.

c) This has exactly the same answers as part (b), since having the same first name is just like having the same birthday.

d) This is clearly reflexive and symmetric. It is not antisymmetric, since my cousin and I have a common grandparent, and I and my cousin have a common grandparent, but I am not equal to my cousin. This relation is not transitive. My cousin and I have a common grandparent; my cousin and her cousin on the other side of her family have a common grandparent. My cousin's cousin and I do not have a common grandparent.

6. a) Since $1 + 1 \neq 0$, this relation is not reflexive. Since $x + y = y + x$, it follows that $x + y = 0$ if and only if $y + x = 0$, so the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.

b) Since $x = \pm x$ (choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1.

c) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x - y) = y - x$. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, not that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$.

d) Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2,1) \in R$, but $(1,2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x,y) and (y,x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4,2) \in R$ and $(2,1) \in R$, but $(4,1) \notin R$.

e) This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,3)$ and $(3,2)$ are both in R. It is not transitive; for example, $(1,0) \in R$ and $(0,-2) \in R$, but $(1,-2) \notin R$.

f) This is not reflexive, since $(1,1) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,0)$ and $(0,2)$ are both in R. It is not transitive; for example, $(1,0) \in R$ and $(0,-2) \in R$, but $(1,-2) \notin R$.

g) This is not reflexive, since $(2,2) \notin R$. It is not symmetric, since $(1,2) \in R$ but $(2,1) \notin R$. It is antisymmetric, because if $(x,y) \in R$ and $(y,x) \in R$, then $x = 1$ and $y = 1$, so $x = y$. It is transitive, because if $(x,y) \in R$ and $(y,z) \in R$, then $x = 1$ (and $y = 1$, although that doesn't matter), so $(x,z) \in R$.

h) This is not reflexive, since $(2,2) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2,1)$ and $(1,2)$ are both in R. It is not transitive; for example, $(3,1) \in R$ and $(1,7) \in R$, but $(3,7) \notin R$.

8. We give the simplest example in each case.

a) the empty set on $\{a\}$ (vacuously symmetric and antisymmetric)

b) $\{(a,b),(b,a),(a,c)\}$ on $\{a,b,c\}$

10. Only the relation in part **(a)** is irreflexive (the others are all reflexive).

12. a) not irreflexive, since $(0,0) \in R$. b) not irreflexive, since $(0,0) \in R$.
c) not irreflexive, since $(0,0) \in R$. d) not irreflexive, since $(0,0) \in R$.
e) not irreflexive, since $(0,0) \in R$. f) not irreflexive, since $(0,0) \in R$.
g) not irreflexive, since $(1,1) \in R$. h) not irreflexive, since $(1,1) \in R$.

14. $\forall x\,((x,x) \notin R)$

16. The relations in parts **(a)**, **(b)**, and **(e)** are not asymmetric since they contain pairs of the form (x,x). Clearly the relation in part **(c)** is not asymmetric. The relation in part **(f)** is not asymmetric (both $(1,3)$ and $(3,1)$ are in the relation). It is easy to see that the relation in part **(d)** is asymmetric.

18. According to the preamble to Exercise 16, an asymmetric relation is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. Thus R is asymmetric if and only if R is antisymmetric and also irreflexive.

a) This is not asymmetric, since in fact (a,a) is always in R.

b) For any page a with no links, $(a,a) \in R$, so this is not asymmetric.

c) For any page a with links, $(a,a) \in R$, so this is not asymmetric.

d) For any page a that is linked to, $(a,a) \in R$, so this is not asymmetric.

20. An asymmetric relation must be antisymmetric, since the hypothesis of the condition for antisymmetry is false if the relation is asymmetric. The relation $\{(a,a)\}$ on $\{a\}$ is antisymmetric but not asymmetric, however, so the answer to the second question is no. In fact, it is easy to see that R is asymmetric if and only if R is antisymmetric and irreflexive.

22. Of course many answers are possible. The empty relation is always asymmetric (x is never related to y). A less trivial example would be $(a,b) \in R$ if and only if a is taller than b. Clearly it is impossible that both a is taller than b and b is taller than a at the same time.

24. a) $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(a,b) \mid a > b\}$
b) $\overline{R} = \{(a,b) \mid (a,b) \notin R\} = \{(a,b) \mid a \not< b\} = \{(a,b) \mid a \geq b\}$

26. a) Since this relation is symmetric, $R^{-1} = R$.
b) This relation consists of all pairs (a,b) in which state a does not border state b.

28. These are merely routine exercises in set theory. Note that $R_1 \subseteq R_2$.
a) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\} = R_2$ b) $\{(1,2),(2,3),(3,4)\} = R_1$
c) $\emptyset$ d) $\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

30. Since $(1,2) \in R$ and $(2,1) \in S$, we have $(1,1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1,1),(1,2),(2,1),(2,2)\}$.

32. a) The union of two relations is the union of these sets. Thus $R_1 \cup R_3$ holds between two real numbers if R_1 holds or R_3 holds (or both, it goes without saying). Here this means that the first number is greater than the second or vice versa—in other words, that the two numbers are not equal. This is just relation R_6.
b) For (a,b) to be in $R_3 \cup R_6$, we must have $a > b$ or $a = b$. Since this happens precisely when $a \geq b$, we see that the answer is R_2.
c) The intersection of two relations is the intersection of these sets. Thus $R_2 \cap R_4$ holds between two real numbers if R_2 holds and R_4 holds as well. Thus for (a,b) to be in $R_2 \cap R_4$, we must have $a \geq b$ and $a \leq b$. Since this happens precisely when $a = b$, we see that the answer is R_5.
d) For (a,b) to be in $R_3 \cap R_5$, we must have $a < b$ and $a = b$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.
e) Recall that $R_1 - R_2 = R_1 \cap \overline{R_2}$. But $\overline{R_2} = R_3$, so we are asked for $R_1 \cap R_3$. It is impossible for $a > b$ and $a < b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.
f) Reasoning as in part (f), we want $R_2 \cap \overline{R_1} = R_2 \cap R_4$, which is R_5 (this was part (c)).
g) Recall that $R_1 \oplus R_3 = (R_1 \cap \overline{R_3}) \cup (R_3 \cap \overline{R_1})$. We see that $R_1 \cap \overline{R_3} = R_1 \cap R_2 = R_1$, and $R_3 \cap \overline{R_1} = R_3 \cap R_4 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).
h) Recall that $R_2 \oplus R_4 = (R_2 \cap \overline{R_4}) \cup (R_4 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_4} = R_2 \cap R_1 = R_1$, and $R_4 \cap \overline{R_2} = R_4 \cap R_3 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).

34. Recall that the composition of two relations all defined on a common set is defined as follows: $(a,c) \in S \circ R$ if and only if there is some element b such that $(a,b) \in R$ and $(b,c) \in S$. We have to apply this in each case.
a) For (a,c) to be in $R_1 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_1$. This means that $a > b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_1 = R_1$. We can interpret (part of) this as showing that R_1 is transitive.
b) For (a,c) to be in $R_1 \circ R_2$, we must find an element b such that $(a,b) \in R_2$ and $(b,c) \in R_1$. This means that $a \geq b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_2 = R_1$.
c) For (a,c) to be in $R_1 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_1$. This means that $a < b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

d) For (a,c) to be in $R_1 \circ R_4$, we must find an element b such that $(a,b) \in R_4$ and $(b,c) \in R_1$. This means that $a \leq b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_4 = \mathbf{R}^2$, the relation that always holds.

e) For (a,c) to be in $R_1 \circ R_5$, we must find an element b such that $(a,b) \in R_5$ and $(b,c) \in R_1$. This means that $a = b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with (choose $b = a$). But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_5 = R_1$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).

f) For (a,c) to be in $R_1 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_1$. This means that $a \neq b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

g) For (a,c) to be in $R_2 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_2$. This means that $a < b$ and $b \geq c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_2 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

h) For (a,c) to be in $R_3 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_3$. This means that $a < b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with. But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_3 \circ R_3 = R_3$. We can interpret (part of) this as showing that R_3 is transitive.

36. For (a,b) to be an element of R^3, we must find people c and d such that $(a,c) \in R$, $(c,d) \in R$, and $(d,b) \in R$. In words, this says that a is the parent of someone who is the parent of someone who is the parent of b. More simply, a is a great-grandparent of b.

38. Note that these two relations are inverses of each other, since a is a multiple of b if and only if b divides a (see the preamble to Exercise 24).

a) The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a,b) \in R_1 \cup R_2$ if and only if $a \mid b$ or $b \mid a$. There is not a good easier way to state this.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a,b) \in R_1 \cap R_2$ if and only if $a \mid b$ and $b \mid a$. This happens if and only if $a = \pm b$ and $a \neq 0$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a,b) \in R_1 - R_2$ if and only if $a \mid b$ and $b \nmid a$. This is equivalent to saying that $a \mid b$ and $a \neq \pm b$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a,b) \in R_2 - R_1$ if and only if $b \mid a$ and $a \nmid b$. This is equivalent to saying that $b \mid a$ and $a \neq \pm b$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part (c) and part (d). Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. This happens if and only if $a \mid b$ or $b \mid a$, but $a \neq \pm b$.

40. These are just the 16 different subsets of $\{(0,0),(0,1),(1,0),(1,1)\}$.

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$

6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

42. We list the relations by number as given in the solution above.
a) 8, 13, 14, 16 **b)** 1, 3, 4, 9 **c)** 1, 2, 5, 8, 9, 12, 15, 16
d) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14 **e)** 1, 3, 4 **f)** 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16

44. This is similar to Example 16 in this section. A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, so there are 2^{n^2} relations on S if no restrictions are imposed. One might observe here that the condition that $a \neq b$ is not relevant.
a) Half of these relations contain (a,b) and half do not, so the answer is $2^{n^2}/2 = 2^{n^2-1}$. Looking at it another way, we see that there are $n^2 - 1$ choices involved in specifying such a relation, since we have no choice about (a,b).
b) The analysis and answer are exactly the same as in part **(a)**.
c) Of the n^2 possible pairs to put in S, exactly n of them have a as their first element. We must use none of these, so there are $n^2 - n$ pairs that we are free to work with. Therefore there are 2^{n^2-n} possible choices for S.
d) By part **(c)** we know that there are 2^{n^2-n} relations that do not contain at least one ordered pair with a as its first element, so all the other relations, namely $2^{n^2} - 2^{n^2-n}$ of them, do contain at least one ordered pair with a as its first element.
e) We reason as in part **(c)**. There are n ordered pairs that have a as their first element, and n more that have b as their second element, although this counts (a,b) twice, so there are a total of $2n-1$ pairs that violated the condition. This means that there are $n^2 - 2n + 1 = (n-1)^2$ pairs that we are free to choose for S. Thus the answer is $2^{(n-1)^2}$. Another way to look at this is to visualize the matrix representing S. The a^{th} row must be all 0's, as must the b^{th} column. If we cross out that row and column we have in effect an $n-1$ by $n-1$ matrix, with $(n-1)^2$ entries. Since we can fill each entry with either a 0 or a 1, there are $2^{(n-1)^2}$ choices for specifying S.
f) This is the opposite condition from part **(e)**. Therefore reasoning as in part **(d)**, we have $2^{n^2} - 2^{(n-1)^2}$ possible relations.

46. **a)** There are two relations on a set with only one element, and they are both transitive.
b) There are 16 relations on a set with two elements, and we saw in Exercise 42f that 13 of them are transitive.
c) For $n = 3$ there are $2^{3^2} = 512$ relations. One way to find out how many of them are transitive is to use a computer to generate them all and check each one for transitivity. If we do this, then we find that 171 of them are transitive. Doing this by hand is not pleasant, since there are many cases to consider.

48. **a)** Since R contains all the pairs (x,x), so does $R \cup S$. Therefore $R \cup S$ is reflexive.
b) Since R and S each contain all the pairs (x,x), so does $R \cap S$. Therefore $R \cap S$ is reflexive.

c) Since R and S each contain all the pairs (x,x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x,x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

e) Since R and S each contain all the pairs (x,x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

50. By definition, to say that R is antisymmetric is to say that $R \cap R^{-1}$ contains only pairs of the form (a,a). The statement we are asked to prove is just a rephrasing of this.

52. This is immediate from the definition, since R is reflexive if and only if it contains all the pairs (x,x), which in turn happens if and only if $\overline{R}$ contains none of these pairs, i.e., $\overline{R}$ is irreflexive.

54. We just apply the definition each time. We find that R^2 contains all the pairs in $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$ except $(2,3)$ and $(4,5)$; and R^3, R^4, and R^5 contain all the pairs.

56. We prove this by induction on n. There is nothing to prove in the basis step ($n = 1$). Assume the inductive hypothesis that R^n is symmetric, and let $(a,c) \in R^{n+1} = R^n \circ R$. Then there is a $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R^n$. Since R^n and R are symmetric, $(b,a) \in R$ and $(c,b) \in R^n$. Thus by definition $(c,a) \in R \circ R^n$. We will have completed the proof if we can show that $R \circ R^n = R^{n+1}$. This we do in two steps. First, composition of relations is associative, that is, $(R \circ S) \circ T = R \circ (S \circ T)$ for all relations with appropriate domains and codomains. (The proof of this is straightforward applications of the definition.) Second we show that $R \circ R^n = R^{n+1}$ by induction on n. Again the basis step is trivial. Under the inductive hypothesis, then, $R \circ R^{n+1} = R \circ (R^n \circ R) = (R \circ R^n) \circ R = R^{n+1} \circ R = R^{n+2}$, as desired.