SECTION 5.3 Permutations and Combinations

2. $P(7,7) = 7! = 5040$

4. There are 10 combinations and 60 permutations. We list them in the following way. Each combination is listed, without punctuation, in increasing order, followed by the five other permutations involving the same numbers, in parentheses, without punctuation.

123 (132 213 231 312 321) 124 (142 214 241 412 421) 125 (152 215 251 512 521)

134 (143 314 341 413 431) 135 (153 315 351 513 531) 145 (154 415 451 514 541)

234 (243 324 342 423 432) 235 (253 325 352 523 532)

245 (254 425 452 524 542) 345 (354 435 453 534 543)

6. a) $C(5,1) = 5$ b) $C(5,3) = C(5,2) = 5 \cdot 4/2 = 10$ c) $C(8,4) = 8 \cdot 7 \cdot 6 \cdot 5/(4 \cdot 3 \cdot 2) = 70$
d) $C(8,8) = 1$ e) $C(8,0) = 1$ f) $C(12,6) = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7/(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 924$

8. $P(5,5) = 5! = 120$

10. $P(6,6) = 6! = 720$

12. a) To specify a bit string of length 12 that contains exactly three 1's, we simply need to choose the three positions that contain the 1's. There are $C(12,3) = 220$ ways to do that.

b) To contain at most three 1's means to contain three 1's, two 1's, one 1, or no 1's. Reasoning as in part (a), we see that there are $C(12,3)+C(12,2)+C(12,1)+C(12,0) = 220+66+12+1 = 299$ such strings.

c) To contain at least three 1's means to contain three 1's, four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, 10 1's, 11 1's, or 12 1's. We could reason as in part (b), but we would have too many numbers to add. A simpler approach would be to figure out the number of ways not to have at least three 1's (i.e., to have two 1's, one 1, or no 1's) and then subtract that from 2^{12}, the total number of bit strings of length 12. This way we get $4096 - (66 + 12 + 1) = 4017$.

d) To have an equal number of 0's and 1's in this case means to have six 1's. Therefore the answer is $C(12,6) = 924$.

14. $C(99,2) = 99 \cdot 98/2 = 4851$

16. We need to compute $C(10,1) + C(10,3) + C(10,5) + C(10,7) + C(10,9) = 10 + 120 + 252 + 120 + 10 = 512$. (In the next section we will see that there are just as many subsets with an odd number of elements as there are subsets with an even number of elements (Exercise 31 in Section 5.4). Since there are $2^{10} = 1024$ subsets in all, the answer is $1024/2 = 512$, in agreement with our computation.)

18. a) Each flip can be either heads or tails, so there are $2^8 = 256$ possible outcomes.

b) To specify an outcome that has exactly three heads, we simply need to choose the three flips that came up heads. There are $C(8,3) = 56$ such outcomes.

c) To contain at least three heads means to contain three heads, four heads, five heads, six heads, seven heads, or eight heads. Reasoning as in part (b), we see that there are $C(8,3)+C(8,4)+C(8,5)+C(8,6)+C(8,7)+C(8,8) = 56 + 70 + 56 + 28 + 8 + 1 = 219$ such outcomes. We could also subtract from 256 the number of ways to get two of fewer heads, namely $28 + 8 + 1 = 37$. Since $256 - 37 = 219$, we obtain the same answer using this alternative method.

d) To have an equal number of heads and tails in this case means to have four heads. Therefore the answer is $C(8,4) = 70$.

20. a) There are $C(10,3)$ ways to choose the positions for the 0's, and that is the only choice to be made, so the answer is $C(10,3) = 120$.

b) There are more 0's than 1's if there are fewer than five 1's. Using the same reasoning as in part (a), together with the sum rule, we obtain the answer $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 210 = 386$. Alternatively, by symmetry, half of all cases in which there are not five 0's have more 0's than 1's; therefore the answer is $(2^{10} - C(10,5)/2 = (1024 - 252)/2 = 386$.

c) We want the number of bit strings with 7, 8, 9, or 10 1's. By the same reasoning as above, there are $C(10,7) + C(10,8) + C(10,9) + C(10,10) = 120 + 45 + 10 + 1 = 176$ such strings.

d) If a string does not have at least three 1's, then it has 0, 1, or 2 1's. There are $C(10,0) + C(10,1) + C(10,2) = 1 + 10 + 45 = 56$ such strings. There are $2^{10} = 1024$ strings in all. Therefore there are $1024 - 56 = 968$ strings with at least three 1's.

22. a) If ED is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of seven items—the letters A, B, C, F, G, and H, and the superletter ED. Therefore the answer is $P(7,7) = 7! = 5040$.

b) Reasoning as in part (a), we see that the answer is $P(6,6) = 6! = 720$.

c) As in part (a), we glue BA into one item and glue FGH into one item. Therefore we need to permute five items, and there are $P(5,5) = 5! = 120$ ways to do it.

d) This is similar to part (c). Glue AB into one item, glue DE into one item, and glue GH into one item, producing five items, so the answer is $P(5,5) = 5! = 120$.

e) If both CAB and BED are substrings, then $CABED$ has to be a substring. So we are really just permuting four items: $CABED$, F, G, and H. Therefore the answer is $P(4,4) = 4! = 24$.

f) There are no permutations with both of these substrings, since B cannot be followed by both C and F at the same time.

24. First position the women relative to each other. Since there are 10 women, there are $P(10,10)$ ways to do this. This creates 11 slots where a man (but not more than one man) may stand: in front of the first woman, between the first and second women, $\ldots$, between the ninth and tenth women, and behind the tenth woman. We need to choose six of these positions, in order, for the first through six man to occupy (order matters, because the men are distinct people). This can be done is $P(11,6)$ ways. Therefore the answer is $P(10,10) \cdot P(11,6) = 10! \cdot 11!/5! = 1{,}207{,}084{,}032{,}000$.

26. a) This is just a matter of choosing 10 players from the group of 13, since we are not told to worry about what positions they play; therefore the answer is $C(13,10) = 286$.

b) This is the same as part (a), except that we need to worry about the order in which the choices are made, since there are 10 distinct positions to be filled. Therefore the answer is $P(13,10) = 13!/3! = 1{,}037{,}836{,}800$.

c) There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men. Therefore (using part (a)) there are $286 - 1 = 285$ ways to choose the players if at least one of them must be a woman.

28. We are just being asked for the number of strings of T's and F's of length 40 with exactly 17 T's. The only choice is which 17 of the 40 positions are to have the T's, so the answer is $C(40,17) \approx 8.9 \times 10^{10}$.

30. a) There are $C(16,5)$ ways to select a committee if there are no restrictions. There are $C(9,5)$ ways to select a committee from just the 9 men. Therefore there are $C(16,5) - C(9,5) = 4368 - 126 = 4242$ committees with at least one woman.

b) There are $C(16,5)$ ways to select a committee if there are no restrictions. There are $C(9,5)$ ways to select a committee from just the 9 men. There are $C(7,5)$ ways to select a committee from just the 7 men. These two possibilities do not overlap, since there are no ways to select a committee containing neither men nor women. Therefore there are $C(16,5) - C(9,5) - C(7,5) = 4368 - 126 - 21 = 4221$ committees with at least one woman and at least one man.

32. a) The only reasonable way to do this is by subtracting from the number of strings with no restrictions the number of strings that do not contain the letter a. The answer is $26^6 - 25^6 = 308915776 - 244140625 = 64{,}775{,}151$.

b) If our string is to contain both of these letters, then we need to subtract from the total number of strings the number that fail to contain one or the other (or both) of these letters. As in part (a), 25^6 strings fail to contain an a; similarly 25^6 fail to contain a b. This is overcounting, however, since 24^6 fail to contain both of these letters. Therefore there are $25^6 + 25^6 - 24^6$ strings that fail to contain at least one of these letters. Therefore the answer is $26^6 - (25^6 + 25^6 - 24^6) = 308915776 - (244140625 + 244140625 - 191102976) = 11{,}737{,}502$.

c) First choose the position for the a; this can be done in 5 ways, since the b must follow it. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $5P(24,4) = 1{,}275{,}120$.

d) First choose the positions for the a and b; this can be done in $C(6,2)$ ways, since once we pick two positions, we put the a in the left-most and the b in the other. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $C(6,2)P(24,4) = 3{,}825{,}360$.

34. Probably the best way to do this is just to break it down into the three cases by sex. There are $C(15,6)$ ways to choose the committee to be composed only of women, $C(15,5)C(10,1)$ ways if there are to be five women and one man, and $C(15,4)C(10,2)$ ways if there are to be four women and two men. Therefore the answer is $C(15,6) + C(15,5)C(10,1) + C(15,4)C(10,2) = 5005 + 30030 + 61425 = 96{,}460$.

36. Glue two 1's to the right of each 0, giving us a collection of nine tokens: five 011's and four 1's. We are asked for the number of strings consisting of these tokens. All that is involved is choosing the positions for the 1's among the nine positions in the string, so the answer is $C(9,4) = 126$.

38. $C(45,3) \cdot C(57,4) \cdot C(69,5) = 14190 \cdot 395010 \cdot 11238513 \approx 6.3 \times 10^{16}$

40. We might as well assume that the first person sits in the northernmost seat. Then there are $P(5,5)$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is $5! = 120$.

42. We can solve this problem by breaking it down into cases depending on the number of ties. There are five cases. (1) If there are no ties, then there are clearly $P(4,4) = 24$ possible ways for the horses to finish. (2) Assume that there are two horses that tie, but the others have distinct finishes. There are $C(4,2) = 6$ ways to choose the horses to be tied; then there are $P(3,3) = 6$ ways to determine the order of finish for the three groups (the pair and the two single horses). Thus there are $6 \cdot 6 = 36$ ways for this to happen. (3) There might be two groups of two horses that are tied. There are $C(4,2) = 6$ ways to choose the winners (and the other two horses are the losers). (4) There might be a group of three horses all tied. There are $C(4,3) = 4$ ways to choose which these horses will be, and then two ways for the race to end (the tied horses win or they lose), so there are $4 \cdot 2 = 8$ possibilities. (5) There is only one way for all the horses to tie. Putting this all together, the answer is $24 + 36 + 6 + 8 + 1 = 75$.

44. a) The complicating factor here is the rule that the penalty kick round (or "group") is over once one team has clinched a victory. For example, if the first team to shoot has missed all of its first four shots and the other team has made two of its first three shots, then the round is over after only seven kicks. There are $2^{10} = 1024$ possible scenarios without this rule (and without worrying yet about whether the score is tied at the end of this round), but it seems rather tedious and dangerous (in the sense of your being likely to make a mistake and leave something out) to try to analyze the more complicated situation by writing out all the possibilities by hand. (This is not impossible, though, and the author has obtained the correct answer in this way.) Rather than do this, one can write a computer program to simulate the situation and do the counting. The result is that there are 672 possible scoring scenarios for a round of penalty kicks, including the possibility that the score is still tied at the end of that round.

Next we need to count the number of ways for the score to end up tied at the end of the round. For this to happen, both teams must score p points, where p is some integer between 0 and 5, inclusive. The scoring scenario is determined by the positions of the kickers who did the scoring. There are $C(5,p)$ ways to choose these positions for each team, or $C(5,p)^2$ ways in all. We need to sum this over the values of p from 0 to 5.

The sum is 252. So there are 252 ways for the score to end up tied. We already noted in the paragraph above that there are 672 different scoring scenarios, so there are $672 - 252 = 420$ scenarios in which the score is not tied. This answers the question for this part of the exercise.

b) This is easy after what we've found above. There are 252 ways for the score to be tied at the end of the first group of penalty kicks, and there are 420 ways for the game to be settled in the second group. So there are $252 \cdot 420 = 105{,}840$ ways for the game to end during the second round.

c) We have already seen that there are 420 ways for the game to end in the first round, and 105,840 more ways for it to end in the second round. In order for it to go into a sudden death period, the first two rounds must have ended tied, which can happen in $420 \cdot 420 = 176{,}400$ ways. Thereafter, the game can end after two more kicks in 2 ways (either team can make their kick and have the other team miss theirs), after four more kicks in $2 \cdot 2 = 4$ ways (the first pair of kicks must have the same result, either both made or both missed, and then either team can win), after six more kicks in $2^2 \cdot 2 = 8$ ways (the first two pairs of kicks must have the same results, and then either team can win), after eight more kicks in 16 ways, and after ten more kicks in 32 ways. Thus there are $2 + 4 + 8 + 16 + 32 = 62$ ways for the sudden death round to end within ten kicks. This needs to be multiplied by the 176,400 ways we can reach sudden death, for a total of 10,936,800 scoring scenarios. So the answer to this last question is $420 + 105840 + 10936800 = 11{,}043{,}060$.