SECTION 5.5 Generalized Permutations and Combinations

2. There are 5 choices each of 5 times, so the answer is $5^5 = 3125$.

4. There are 6 choices each of 7 times, so the answer is $6^7 = 279{,}936$.

6. By Theorem 2 the answer is $C(3+5-1,5) = C(7,5) = C(7,2) = 21$.

8. By Theorem 2 the answer is $C(21+12-1,12) = C(32,12) = 225{,}792{,}840$.

10. a) $C(6+12-1,12) = C(17,12) = 6188$ b) $C(6+36-1,36) = C(41,36) = 749{,}398$

c) If we first pick the two of each kind, then we have picked $2 \cdot 6 = 12$ croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part **(a)**, namely $C(6+12-1,12) = C(17,12) = 6188$.

d) We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1,21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1,24) = C(29,24) = 118755$ ways to pick 24 croissants without any restriction, there must be $118755 - 65780 = 52{,}975$ ways to choose two dozen croissants with no more than two broccoli.

e) Eight croissants are specified, so this problem is the same as choosing $24 - 8 = 16$ croissants without restriction, which can be done in $C(6+16-1,16) = C(21,16) = 20{,}349$ ways.

f) First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are $24 - 9 = 15$ left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1,15) = C(20,15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1,11) = C(16,11) = 4368$ choices. Therefore the number of ways to make the selection without violating the restriction is $15504 - 4368 = 11{,}136$.

12. There are 5 things to choose from, repetitions allowed, and we want to choose 20 things, order not important. Therefore by Theorem 2 the answer is $C(5+20-1,20) = C(24,20) = C(24,4) = 10{,}626$.

14. By Theorem 2 the answer is $C(4+17-1,17) = C(20,17) = C(20,3) = 1140$.

16. a) We require each $x_i \geq 2$. This uses up 12 of the 29 total required, so the problem is the same as finding the number of solutions to $x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = 17$ with each x'_i a nonnegative integer. The number of solutions is therefore $C(6+17-1, 17) = C(22, 17) = 26{,}334$.

b) The restrictions use up 22 of the total, leaving a free total of 7. Therefore the answer is $C(6+7-1, 7) = C(12, 7) = 792$.

c) The number of solutions without restriction is $C(6+29-1, 29) = C(34, 29) = 278256$. The number of solution violating the restriction by having $x_1 \geq 6$ is $C(6+23-1, 23) = C(28, 23) = 98280$. Therefore the answer is $278256 - 98280 = 179{,}976$.

d) The number of solutions with $x_2 \geq 9$ (as required) but without the restriction on x_1 is $C(6+20-1, 20) = C(25, 20) = 53130$. The number of solution violating the additional restriction by having $x_1 \geq 8$ is $C(6+12-1, 12) = C(17, 12) = 6188$. Therefore the answer is $53130 - 6188 = 46{,}942$.

18. It follows directly from Theorem 3 that the answer is

$$\frac{20!}{2!4!3!1!2!3!2!3!} \approx 5.9 \times 10^{13}.$$

20. We introduce the nonnegative slack variable x_4, and our problem becomes the same as the problem of counting the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 11$. By Theorem 2 the answer is $C(4+11-1, 11) = C(14, 11) = C(14, 3) = 364$.

22. If we think of the balls as doing the choosing, then this is asking for the number of ways to choose 12 bins from the six given bins, with repetition allowed. (The number of times each bin is chosen is the number of balls in that bin.) By Theorem 2 with $n = 6$ and $r = 12$, this choice can be made in $C(6+12-1, 12) = C(17, 12) = 6188$ ways.

24. We assume that this problem leaves us free to pick which boxes get which numbers of balls. There are several ways to count this. Here is one. Line up the 15 objects in a row ($15!$ ways to do that), and line up the five boxes in a row ($5!$ ways to do that). Now put the first object into the first box, the next two into the second box, the next three into the third box, and so on. This overcounts by a factor of $1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!$, since there are that many ways to swap objects in the permutation without affecting the result. Therefore the answer is $15! \cdot 5!/(1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!) = 4{,}540{,}536{,}000$.

26. We can model this problem by letting x_i be the i^{th} digit of the number for $i = 1, 2, 3, 4, 5, 6$, and asking for the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 13$, where each x_i is between 0 and 8, inclusive, except that one of them equals 9. First, there are 6 ways to decide which of the digits is 9. Without loss of generality assume that $x_6 = 9$. Then the number of ways to choose the remaining digits is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 4$ (note that the restriction that each $x_i \leq 8$ was moot, since the sum was only 4). By Theorem 2 there are $C(5+4-1, 4) = C(8, 4) = 70$ solutions. Therefore the answer is $6 \cdot 70 = 420$.

28. (Note that the roles of the letters n and r here are reversed from the usual roles, as, for example, in Theorem 2.) We can choose the required objects first, and there are $q_1 + q_2 + \cdots + q_r$ of these. Then $n - (q_1 + q_2 + \cdots + q_r) = n - q_1 - q_2 - \cdots - q_r$ objects remain to be chosen. There are still r types. Therefore by Theorem 2, the number of ways to make this choice is $C(r + (n - q_1 - q_2 - \cdots - q_r) - 1, (n - q_1 - q_2 - \cdots - q_r)) = C(n + r - q_1 - q_2 - \cdots - q_r - 1, n - q_1 - q_2 - \cdots - q_r)$.

30. By Theorem 3 the answer is $11!/(4!4!2!) = 34{,}650$.

32. We can treat the 3 consecutive A's as one letter. Thus we have 6 letters, of which 2 are the same (the two R's), so by Theorem 3 the answer is $6!/2! = 360$.

34. We need to calculate separately, using Theorem 3, the number of strings of length 5, 6, and 7. There are $7!/(3!3!1!) = 140$ strings of length 7. For strings of length 6, we can omit the R and form $6!/(3!3!) = 20$ string; omit an E and form $6!/(3!2!1!) = 60$ strings, or omit an S and also form 60 strings. This gives a total of 140 strings of length 6. For strings of length 5, we can omit two E's or two S's, each giving $5!/(3!1!1!) = 20$ strings; we can omit one E and one S ($5!/(2!2!1!) = 30$ strings); or we can omit the R and either an E or an S ($5!/(3!2!) = 10$ strings each). This gives a total of 90 strings of length 5, for a grand total of 370 strings of length 5 or greater.

36. We simply need to choose the 6 positions, out of the 14 available, to make 1's. There are $C(14,6) = 3003$ ways to do so.

38. We assume that the forty issues are distinguishable.
a) Theorem 4 says that the answer is $40!/10!^4 \approx 4.7 \times 10^{21}$.
b) Each distribution into identical boxes gives rise to $4! = 24$ distributions into labeled boxes, since once we have made the distribution into unlabeled boxes we can arbitrarily label the boxes. Therefore the answer is the same as the answer in part (a) divided by 24, namely $(40!/10!^4)/4! \approx 2.0 \times 10^{20}$.

40. We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, 5 z's, and 4 w's. By Theorem 3, there are
$$\frac{16!}{4!3!5!4!} = 50{,}450{,}400$$
such sequences.

42. Theorem 4 says that the answer is $52!/13!^4 \approx 5.4 \times 10^{28}$, since each player gets 13 cards.

44. a) All that matters is the number of books on each shelf, so the answer is the number of solutions to $x_1 + x_2 + x_3 + x_4 = 12$, where x_i is being viewed as the number of books on shelf i. The answer is therefore $C(4+12-1,12) = C(15,12) = 455$.
b) No generality is lost if we number the books $b_1, b_2, \ldots, b_{12}$ and think of placing book b_1, then placing b_2, and so on. There are clearly 4 ways to place b_1, since we can put it as the first book (for now) on any of the shelves. After b_1 is placed, there are 5 ways to place b_2, since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are 6 ways to place b_3 (to the right of b_1, to the right of b_2, or as the first book on any of the shelves), 7 ways to place $b_4, \ldots,$ 15 ways to place b_{12}. Therefore the answer is the product of these numbers $4 \cdot 5 \cdots 15 = 217{,}945{,}728{,}000$.

46. We follow the hint. There are 5 bars (chosen books), and therefore there are 6 places where the 7 stars (nonchosen books) can fit (before the first bar, between the first and second bars, ..., after the fifth bar). Each of the second through fifth of these slots must have at least one star in it, so that adjacent books are not chosen. Once we have placed these 4 stars, there are 3 stars left to be placed in 6 slots. The number of ways to do this is therefore $C(6+3-1,3) = C(8,3) = 56$.

48. We can think of the n distinguishable objects to be distributed into boxes as numbered from 1 to n. Since such a distribution is completely determined by assigning a box number (from 1 to k) to each object, we can think of a distribution simply as a sequence of box numbers $a_1, a_2, \ldots, a_n$, where a_i is the box into which object i goes. Furthermore, since we want n_i objects to go into box i, this sequence must contain n_i copies of the number i (for each i from 1 to k). But this is precisely a permutation of n objects (namely, numbers) with n_i indistinguishable objects of type i (namely, n_i copies of the number i). Thus we have established the desired one-to-one correspondence. Since Theorem 3 tells us that there are $n!/(n_1!n_2!\cdots n_k!)$ permutations, there must also be this many ways to do the distribution into boxes, and the proof of Theorem 4 is complete.

50. This is actually a problem about partitions of sets. Let us call the set of 5 objects $\{a,b,c,d,e\}$. We want to partition this set into three pairwise disjoint subsets (some possibly empty). We count in a fairly ad hoc way. First, we could put all five objects into one subset (i.e., all five objects go into one box, with the other two boxes empty). Second, we could put four of the objects into one subset and one into another, such as $\{a,b,c,d\}$ together with $\{e\}$. There are 5 ways to do this, since each of the five objects can be the singleton. Third, we could put three of the objects into one set (box) and two into another; there are $C(5,2)=10$ ways to do this, since there are that many ways to choose which objects are to be the doubleton. Similarly, there are 10 ways to distribute the elements so that three go into one set and one each into the other two sets (for example, $\{a,b,c\}$, $\{d\}$, and $\{e\}$). Finally, we could put two items into one set, two into another, and one into the third (for example, $\{a,b\}$, $\{c,d\}$, and $\{e\}$). Here we need to choose the singleton (5 ways), and then we need to choose one of the 3 ways to separate the remaining four elements into pairs; this gives a total of 15 partitions. In all we have 41 different partitions.

This can also be solved by using the formulae on page 378:

$$S(5,1)=\frac{1}{1!}\left(\binom{1}{0}1^5\right)=\frac{1}{1!}(1)=1$$

$$S(5,2)=\frac{1}{2!}\left(\binom{2}{0}2^5-\binom{2}{1}1^5\right)=\frac{1}{2!}(32-2)=15$$

$$S(5,3)=\frac{1}{3!}\left(\binom{3}{0}3^5-\binom{3}{1}2^5+\binom{3}{2}1^5\right)=\frac{1}{3!}(243-96+3)=25$$

$$\sum_{j=1}^{3}S(5,j)=1+15+25=41$$

52. This is similar to Exercise 50, with 3 replaced by 4. We compute this using the formulae:

$$S(5,1)=\frac{1}{1!}\left(\binom{1}{0}1^5\right)=\frac{1}{1!}(1)=1$$

$$S(5,2)=\frac{1}{2!}\left(\binom{2}{0}2^5-\binom{2}{1}1^5\right)=\frac{1}{2!}(32-2)=15$$

$$S(5,3)=\frac{1}{3!}\left(\binom{3}{0}3^5-\binom{3}{1}2^5+\binom{3}{2}1^5\right)=\frac{1}{3!}(243-96+3)=25$$

$$S(5,4)=\frac{1}{4!}\left(\binom{4}{0}4^5-\binom{4}{1}3^5+\binom{4}{2}2^5-\binom{4}{3}1^5\right)=\frac{1}{4!}(1024-972+192-4)=10$$

$$\sum_{j=1}^{4}S(5,j)=1+15+25+10=51$$

54. We are asked for the partitions of 5 into at most 3 parts; notice that we are not required to use all three boxes. We can easily list these partitions explicitly: $5=5$, $5=4+1$, $5=3+2$, $5=3+1+1$, and $5=2+2+1$. Therefore the answer is 5.

56. This is similar to Exercise 55. Since each box has to contain at least one object, we might as well put one object into each box to begin with. This leaves us with just three more objects, and there are only three choices: we can put them all into the same box (so that the partition we end up with is $8 = 4+1+1+1+1$), or we can put them into three different boxes (so that the partition we end up with is $8 = 2+2+2+1+1$), or we can put two into one box and the last into another (so that the partition we end up with is $8 = 3+2+1+1+1$). So the answer is 3.

58. **a)** This is a straightforward application of the product rule: There are 7 choices for the first ball, 6 choices for the second ball, and so on, for an answer of $7\cdot 6\cdot 5\cdot 4\cdot 3 = 2520$.

b) Since each ball must be in a separate box and the boxes are unlabeled, there is only one way to do this.

c) This is just a matter of choosing which five boxes to put balls into, so the answer is $C(7,5) = 21$.

d) As noted in part **(b)**, there is only one way to do this.

60. There are 31 other teams to play, and we can denote these with the symbols x_1, x_2, ..., x_{31}. We are asked for a list of $4\cdot 4 + 11\cdot 3 + 16\cdot 2 = 81$ of these symbols that contains exactly 4 copies of each of x_1 through x_4, exactly 3 copies of each of x_5 through x_{15}, and exactly 2 copies of each of x_{16} through x_{31}. Theorem 3 tells us that the number of possible lists is

$$\frac{81!}{(4!)^4\cdot(3!)^{11}\cdot(2!)^{16}} \approx 7.35\times 10^{101}\,.$$

(The arithmetic was done with *Maple*.)

62. Each term must be of the form $Cx_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, where the n_i's are nonnegative integers whose sum is n. The number of ways to specify a term, then, is the number of nonnegative integer solutions to $n_1+n_2+\cdots+n_m = n$, which by Theorem 2 is $C(m+n-1,n)$. Note that the coefficients C for these terms can be computed using Theorem 3—see Exercise 63.

64. From Exercise 62, we know that there are $C(3+4-1,4) = C(6,4) = 15$ terms, and the coefficients come from Exercise 63. The answer is $x^4+y^4+z^4+4x^3y+4xy^3+4x^3z+4xz^3+4y^3z+4yz^3+6x^2y^2+6x^2z^2+6y^2z^2+12x^2yz+12xy^2z+12xyz^2$.

66. By Exercise 62, the answer is $C(3+100-1,100) = C(102,100) = C(102,2) = 5151$.