# CHAPTER 7
# Advanced Counting Techniques

## SECTION 7.1 Recurrence Relations

**2.** In each case we simply plug $n = 0,1,2,3,4,5$, using the initial conditions for the first few and then the recurrence relation.

a) $a_0 = -1$, $a_1 = -2a_0 = 2$, $a_2 = -2a_1 = -4$, $a_3 = -2a_2 = 8$, $a_4 = -2a_3 = -16$, $a_5 = -2a_4 = 32$

b) $a_0 = 2$, $a_1 = -1$, $a_2 = a_1 - a_0 = -3$, $a_3 = a_2 - a_1 = -2$, $a_4 = a_3 - a_2 = 1$, $a_5 = a_4 - a_3 = 3$

c) $a_0 = 1$, $a_1 = 3a_0^2 = 3$, $a_2 = 3a_1^2 = 27 = 3^3$, $a_3 = 3a_2^2 = 2187 = 3^7$, $a_4 = 3a_3^2 = 14348907 = 3^{15}$, $a_5 = 3a_4^2 = 617673396283947 = 3^{31}$

d) $a_0 = -1$, $a_1 = 0$, $a_2 = 2a_1 + a_0^2 = 1$, $a_3 = 3a_2 + a_1^2 = 3$, $a_4 = 4a_3 + a_2^2 = 13$, $a_5 = 5a_4 + a_3^2 = 74$

e) $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = a_2 - a_1 + a_0 = 2$, $a_4 = a_3 - a_2 + a_1 = 1$, $a_5 = a_4 - a_3 + a_2 = 1$

**4.** a) $-3a_{n-1} + 4a_{n-2} = -3 \cdot 0 + 4 \cdot 0 = 0 = a_n$ b) $-3a_{n-1} + 4a_{n-2} = -3 \cdot 1 + 4 \cdot 1 = 1 = a_n$

c) $-3a_{n-1} + 4a_{n-2} = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} = (-4)^{n-2}((-3)(-4) + 4) = (-4)^{n-2} \cdot 16 = (-4)^{n-2}(-4)^2 = (-4)^n = a_n$

d) $-3a_{n-1} + 4a_{n-2} = -3 \cdot (2(-4)^{n-1} + 3) + 4 \cdot (2(-4)^{n-2} + 3) = (-4)^{n-2}((-6)(-4) + 4 \cdot 2) - 9 + 12 = (-4)^{n-2} \cdot 32 + 3 = (-4)^{n-2}(-4)^2 \cdot 2 + 3 = 2 \cdot (-4)^n + 3 = a_n$

**6.** In each case, one possible answer is just the equation as presented (it is a recurrence relation of degree 0). We will give an alternate answer.

a) One possible answer is $a_n = a_{n-1}$.

b) Note that $a_n - a_{n-1} = 2n - (2n-2) = 2$. Therefore we have $a_n = a_{n-1} + 2$ as one possible answer.

c) Just as in part (b), we have $a_n = a_{n-1} + 2$.

d) Probably the simplest answer is $a_n = 5a_{n-1}$.

e) Since $a_n - a_{n-1} = n^2 - (n-1)^2 = 2n - 1$, we have $a_n = a_{n-1} + 2n - 1$.

f) This is similar to part (e). One answer is $a_n = a_{n-1} + 2n$.

g) Note that $a_n - a_{n-1} = n + (-1)^n - (n-1) - (-1)^{n-1} = 1 + 2(-1)^n$. Thus we have $a_n = a_{n-1} + 1 + 2(-1)^n$.

h) $a_n = na_{n-1}$

**8.** In the iterative approach, we write $a_n$ in terms of $a_{n-1}$, then write $a_{n-1}$ in terms of $a_{n-2}$ (using the recurrence relation with $n-1$ plugged in for $n$), and so on. When we reach the end of this procedure, we use the given initial value of $a_0$. This will give us an explicit formula for the answer or it will give us a finite series, which we then sum to obtain an explicit formula for the answer.

a) $a_n = -a_{n-1} = (-1)^2 a_{n-2} = \cdots = (-1)^n a_{n-n} = (-1)^n a_0 = 5 \cdot (-1)^n$

b) $a_n = 3 + a_{n-1} = 3 + 3 + a_{n-2} = 2 \cdot 3 + a_{n-2} = 3 \cdot 3 + a_{n-3} = \cdots = n \cdot 3 + a_{n-n} = n \cdot 3 + a_0 = 3n + 1$

c)
$$\begin{aligned}
a_n &= -n + a_{n-1} \\
&= -n + \big(-(n-1) + a_{n-2}\big) = -\big(n + (n-1)\big) + a_{n-2} \\
&= -\big(n + (n-1)\big) + \big(-(n-2) + a_{n-3}\big) = -\big(n + (n-1) + (n-2)\big) + a_{n-3} \\
&\ \ \vdots \\
&= -\big(n + (n-1) + (n-2) + \cdots + (n-(n-1))\big) + a_{n-n} \\
&= -\big(n + (n-1) + (n-2) + \cdots + 1\big) + a_0 \\
&= -\frac{n(n+1)}{2} + 4 = \frac{-n^2 - n + 8}{2}
\end{aligned}$$

d)
$$\begin{aligned}
a_n &= -3 + 2a_{n-1} \\
&= -3 + 2(-3 + 2a_{n-2}) = -3 + 2(-3) + 4a_{n-2} \\
&= -3 + 2(-3) + 4(-3 + 2a_{n-3}) = -3 + 2(-3) + 4(-3) + 8a_{n-3} \\
&= -3 + 2(-3) + 4(-3) + 8(-3 + 2a_{n-4}) = -3 + 2(-3) + 4(-3) + 8(-3) + 16a_{n-4} \\
&\ \ \vdots \\
&= -3(1 + 2 + 4 + \cdots + 2^{n-1}) + 2^n a_{n-n} = -3(2^n - 1) + 2^n(-1) = -2^{n+2} + 3
\end{aligned}$$

e)
$$\begin{aligned}
a_n &= (n+1)a_{n-1} = (n+1)na_{n-2} \\
&= (n+1)n(n-1)a_{n-3} = (n+1)n(n-1)(n-2)a_{n-4} \\
&\ \ \vdots \\
&= (n+1)n(n-1)(n-2)(n-3)\cdots(n-(n-2))\, a_{n-n} \\
&= (n+1)n(n-1)(n-2)(n-3)\cdots 2 \cdot a_0 \\
&= (n+1)! \cdot 2 = 2(n+1)!
\end{aligned}$$

f)
$$\begin{aligned}
a_n &= 2na_{n-1} \\
&= 2n\big(2(n-1)a_{n-2}\big) = 2^2\big(n(n-1)\big)a_{n-2} \\
&= 2^2\big(n(n-1)\big)\big(2(n-2)a_{n-3}\big) = 2^3\big(n(n-1)(n-2)\big)a_{n-3} \\
&\ \ \vdots \\
&= 2^n n(n-1)(n-2)(n-3)\cdots\big(n-(n-1)\big)a_{n-n} \\
&= 2^n n(n-1)(n-2)(n-3)\cdots 1 \cdot a_0 \\
&= 3 \cdot 2^n n!
\end{aligned}$$

g)
$$\begin{aligned}
a_n &= n - 1 - a_{n-1} \\
&= n - 1 - \big((n-1-1) - a_{n-2}\big) = (n-1) - (n-2) + a_{n-2} \\
&= (n-1) - (n-2) + \big((n-2-1) - a_{n-3}\big) = (n-1) - (n-2) + (n-3) - a_{n-3} \\
&\ \ \vdots \\
&= (n-1) - (n-2) + \cdots + (-1)^{n-1}(n-n) + (-1)^n a_{n-n} \\
&= \frac{2n - 1 + (-1)^n}{4} + (-1)^n \cdot 7
\end{aligned}$$

10. a) The amount after $n-1$ years is multiplied by 1.09 to give the amount after $n$ years, since 9% of the value must be added to account for the interest. Thus we have $a_n = 1.09a_{n-1}$. The initial condition is $a_0 = 1000$.
b) Since we multiply by 1.09 for each year, the solution is $a_n = 1000(1.09)^n$.
c) $a_{100} = 1000(1.09)^{100} \approx \$5{,}529{,}041$

**12.** This is just like Exercise 10. We are letting $a_n$ be the population, in billions of people, $n$ years after 2002.
a) $a_n = 1.013a_{n-1}$, with $a_0 = 6.2$ b) $a_n = 6.2 \cdot (1.013)^n$
c) $a_{20} = 6.2 \cdot (1.013)^{20} \approx 8.0$ billion people

**14.** We let $a_n$ be the salary, in thousands of dollars, $n$ years after 1999.
a) $a_n = 1 + 1.05a_{n-1}$, with $a_0 = 50$
b) Here $n = 8$. We can either iterate the recurrence relation 8 times, or we can use the result of part (c). The answer turns out to be approximately $a_8 = 83.4$, i.e., a salary of approximately \$83,400.
c) We use the iterative approach.

$$\begin{aligned}
a_n &= 1 + 1.05a_{n-1} \\
&= 1 + 1.05(1 + 1.05a_{n-2}) \\
&= 1 + 1.05 + (1.05)^2 a_{n-2} \\
&\;\;\vdots \\
&= 1 + 1.05 + (1.05)^2 + \cdots + (1.05)^{n-1} + (1.05)^n a_0 \\
&= \frac{(1.05)^n - 1}{1.05 - 1} + 50 \cdot (1.05)^n \\
&= 70 \cdot (1.05)^n - 20
\end{aligned}$$

**16.** a) Each month our account accrues some interest that must be paid. Since the balance the previous month is $B(k-1)$, the amount of interest we owe is $(r/12)B(k-1)$. After paying this interest, the rest of the $P$ dollar payment we make each month goes toward reducing the principle. Therefore we have $B(k) = B(k-1) - (P - (r/12)B(k-1))$. This can be simplified to $B(k) = (1 + (r/12))B(k-1) - P$. The initial condition is that $B(0) =$ the amount borrowed.
b) Solving this by iteration yields

$$B(k) = (1 + (r/12))^k (B(0) - 12P/r) + 12P/r\,.$$

Setting $B(k) = 0$ and solving this for $k$ yields the desired value of $T$ after some messy algebra, namely

$$T = \frac{\log(-12P/(B(0)r - 12P))}{\log(1 + (r/12))}\,.$$

**18.** a) A permutation of a set with $n$ elements consists of a choice of a first element (which can be done in $n$ ways), followed by a permutation of a set with $n-1$ elements. Therefore $P_n = nP_{n-1}$. Note that $P_0 = 1$, since there is just one permutation of a set with no objects, namely the empty sequence.
b) $P_n = nP_{n-1} = n(n-1)P_{n-2} = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot P_0 = n!$

**20.** This is similar to Exercise 19 and solved in exactly the same way. The recurrence relation is $a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}$. It would be quite tedious to write down the 100 initial conditions.

**22.** a) Let $s_n$ be the number of such sequences. A string ending in $n$ must consist of a string ending in something less than $n$, followed by an $n$ as the last term. Therefore the recurrence relation is $s_n = s_{n-1} + s_{n-2} + \cdots + s_2 + s_1$. Here is another approach, with a more compact form of the answer. A sequence ending in $n$ is either a sequence ending in $n-1$, followed by $n$ (and there are clearly $s_{n-1}$ of these), or else it does not contain $n-1$ as a term at all, in which case it is *identical* to a sequence ending in $n-1$ in which the $n-1$ has been replaced by an $n$ (and there are clearly $s_{n-1}$ of these as well). Therefore $s_n = 2s_{n-1}$.

Finally we notice that we can derive the second form from the first (or vice versa) algebraically (for example, $s_4 = 2s_3 = s_3 + s_3 = s_3 + s_2 + s_2 = s_3 + s_2 + s_1$).

**b)** We need two initial conditions if we use the second formulation above, $s_1 = 1$ and $s_2 = 1$ (otherwise, our argument is invalid, because the first and last terms are the same). There is one sequence ending in 1, namely the sequence with just this 1 in it, and there is only the sequence 1,2 ending in 2. If we use the first formulation above, then we can get by with just the initial condition $s_1 = 1$.

**c)** Clearly the solution to this recurrence relation and initial condition is $s_n = 2^{n-2}$ for all $n \geq 2$.

**24.** This is very similar to Exercise 23, except that we need to go one level deeper.

**a)** Let $a_n$ be the number of bit strings of length $n$ containing three consecutive 0's. In order to construct a bit string of length $n$ containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

**b)** There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

**c)** We will compute $a_3$ through $a_7$ using the recurrence relation:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1 \\
a_4 &= a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3 \\
a_5 &= a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8 \\
a_6 &= a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20 \\
a_7 &= a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47
\end{aligned}$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

**26.** First let us solve this problem without using recurrence relations at all. It is clear that the only strings that do not contain the string 01 are those that consist of a string of 1's follows by a string of 0's. The string can consist of anywhere from 0 to $n$ 1's, so the number of such strings is $n + 1$. All the rest have at least one occurrence of 01. Therefore the number of bit strings that contain 01 is $2^n - (n+1)$. However, this approach does not meet the instructions of this exercise.

**a)** Let $a_n$ be the number of bit strings of length $n$ that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n - 1$ that contains 01, and there are $a_{n-1}$ of these. Alternatively, for any $k$ from 1 to $n - 1$, we could start with $k$ 0's, follow this by a 1, and then follow this by any $n - k - 1$ bits. For each such $k$ there are $2^{n-k-1}$ such strings, since the final bits are free. Therefore the number of such strings is $2^0 + 2^1 + 2^2 + \cdots + 2^{n-2}$, which equals $2^{n-1} - 1$. Thus our recurrence relation is $a_n = a_{n-1} + 2^{n-1} - 1$. It is valid for all $n \geq 2$.

**b)** The initial conditions are $a_0 = a_1 = 0$, since no string of length less than 2 can have 01 in it.

**c)** We will compute $a_2$ through $a_7$ using the recurrence relation:

$$\begin{aligned}
a_2 &= a_1 + 2^1 - 1 = 0 + 2 - 1 = 1 \\
a_3 &= a_2 + 2^2 - 1 = 1 + 4 - 1 = 4 \\
a_4 &= a_3 + 2^3 - 1 = 4 + 8 - 1 = 11 \\
a_5 &= a_4 + 2^4 - 1 = 11 + 16 - 1 = 26 \\
a_6 &= a_5 + 2^5 - 1 = 26 + 32 - 1 = 57 \\
a_7 &= a_6 + 2^6 - 1 = 57 + 64 - 1 = 120
\end{aligned}$$

Thus there are 120 bit strings of length 7 containing 01. Note that this agrees with our nonrecursive analysis, since $2^7 - (7+1) = 120$.

**28.** This is identical to Exercise 27, one level deeper.

a) Let $a_n$ be the number of ways to climb $n$ stairs. In order to climb $n$ stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in $a_{n-1}$ ways) or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in $a_{n-2}$ ways) or else start with a step of three stairs and then climb $n-3$ stairs (and this can be done in $a_{n-3}$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b) The initial conditions are $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb two stairs (one step twice or two steps at once). Note that the recurrence relation is the same as that for Exercise 25.

c) Each term in our sequence $\{a_n\}$ is the sum of the previous three terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 13$, $a_6 = 24$, $a_7 = 44$, $a_8 = 81$. Thus a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

**30.** a) Let $a_n$ be the number of ternary strings that contain two consecutive 0's. To construct such a string we could start with either a 1 or a 2 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-1}$ ways), or we could start with 01 or 02 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-2}$ ways), we could start with 00 and follow with any ternary string of length $n-2$ (of which there are clearly $3^{n-2}$). Therefore the recurrence relation, valid for all $n \geq 2$, is $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute $a_2$ through $a_6$ using the recurrence relation:

$$\begin{aligned}
a_2 &= 2a_1 + 2a_0 + 3^0 = 2\cdot 0 + 2\cdot 0 + 1 = 1\\
a_3 &= 2a_2 + 2a_1 + 3^1 = 2\cdot 1 + 2\cdot 0 + 3 = 5\\
a_4 &= 2a_3 + 2a_2 + 3^2 = 2\cdot 5 + 2\cdot 1 + 9 = 21\\
a_5 &= 2a_4 + 2a_3 + 3^3 = 2\cdot 21 + 2\cdot 5 + 27 = 79\\
a_6 &= 2a_5 + 2a_4 + 3^4 = 2\cdot 79 + 2\cdot 21 + 81 = 281
\end{aligned}$$

Thus there are 281 bit strings of length 6 containing two consecutive 0's.

**32.** a) Let $a_n$ be the number of ternary strings that contain either two consecutive 0's or two consecutive 1's. To construct such a string we could start with a 2 and follow with a string containing either two consecutive 0's or two consecutive 1's, and this can be done in $a_{n-1}$ ways. There are other possibilities, however. For each $k$ from 0 to $n-2$, the string could start with $n-1-k$ alternating 0's and 1's, followed by a 2, and then be followed by a string of length $k$ containing either two consecutive 0's or two consecutive 1's. The number of such strings is $2a_k$, since there are two ways for the initial part to alternate. The other possibility is that the string has no 2's at all. Then it must consist $n-k-2$ alternating 0's and 1's, followed by a pair of 0's or 1's, followed by any string of length $k$. There are $2\cdot 3^k$ such strings. Now the sum of these quantities as $k$ runs from 0 to $n-2$ is (since this is a geometric progression) $3^{n-1} - 1$. Putting this all together, we have the following recurrence relation, valid for all $n \geq 2$: $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0 + 3^{n-1} - 1$. (By subtracting this recurrence relation from the same relation with $n-1$ substituted for $n$, we can obtain the following closed form recurrence relation for this problem: $a_n = 2a_{n-1} + a_{n-2} + 2\cdot 3^{n-2}$.)

b) Clearly $a_0 = a_1 = 0$.

c) We will compute $a_2$ through $a_6$ using the recurrence relation:

$$\begin{aligned}
a_2 &= a_1 + 2a_0 + 3^1 - 1 = 0 + 2 \cdot 0 + 3 - 1 = 2 \\
a_3 &= a_2 + 2a_1 + 2a_0 + 3^2 - 1 = 2 + 2 \cdot 0 + 2 \cdot 0 + 9 - 1 = 10 \\
a_4 &= a_3 + 2a_2 + 2a_1 + 2a_0 + 3^3 - 1 = 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 27 - 1 = 40 \\
a_5 &= a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^4 - 1 = 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 81 - 1 = 144 \\
a_6 &= a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^5 - 1 \\
&= 144 + 2 \cdot 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 243 - 1 = 490
\end{aligned}$$

Thus there are 490 ternary strings of length 6 containing two consecutive 0's or two consecutive 1's.

**34.** a) Let $a_n$ be the number of ternary strings that contain two consecutive symbols that are the same. We will develop a recurrence relation for $a_n$ by exploiting the symmetry among the three symbols. In particular, it must be the case that $a_n/3$ such strings start with each of the three symbols. Now let us see how we might specify a string of length $n$ satisfying the condition. We can choose the first symbol in any of three ways. We can follow this by a string that starts with a different symbol but has in it a pair of consecutive symbols; by what we have just said, there are $2a_{n-1}/3$ such strings. Alternatively, we can follow the initial symbol by another copy of itself and then any string of length $n-2$; there are clearly $3^{n-2}$ such strings. Thus the recurrence relation is $a_n = 3 \cdot ((2a_{n-1}/3) + 3^{n-2}) = 2a_{n-1} + 3^{n-1}$. It is valid for all $n \geq 2$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute $a_2$ through $a_6$ using the recurrence relation:

$$\begin{aligned}
a_2 &= 2a_1 + 3^1 = 2 \cdot 0 + 3 = 3 \\
a_3 &= 2a_2 + 3^2 = 2 \cdot 3 + 9 = 15 \\
a_4 &= 2a_3 + 3^3 = 2 \cdot 15 + 27 = 57 \\
a_5 &= 2a_4 + 3^4 = 2 \cdot 57 + 81 = 195 \\
a_6 &= 2a_5 + 3^5 = 2 \cdot 195 + 243 = 633
\end{aligned}$$

Thus there are 633 bit strings of length 6 containing two consecutive 0's, 1's, or 2's.

**36.** We let $a_n$ be the number of ways to pay a toll of $5n$ cents. (Obviously there is no way to pay a toll that is not a multiple of 5 cents.)

a) This problem is isomorphic to Exercise 27, so the answer is the same: $a_n = a_{n-1} + a_{n-2}$, with $a_0 = a_1 = 1$.

b) Iterating, we find that $a_9 = 55$.

**38.** a) We start by computing the first few terms to get an idea of what's happening. Clearly $R_1 = 2$, since the equator, say, splits the sphere into two hemispheres. Also, $R_2 = 4$ and $R_3 = 8$. Let's try to analyze what happens when the $n^{\text{th}}$ great circle is added. It must intersect each of the other circles twice (at diametrically opposite points), and each such intersection results in one prior region being split into two regions, as in Exercise 37. There are $n-1$ previous great circles, and therefore $2(n-1)$ new regions. Therefore $R_n = R_{n-1} + 2(n-1)$. If we impose the initial condition $R_1 = 2$, then our values of $R_2$ and $R_3$ found above are consistent with this recurrence. Note that $R_4 = 14$, $R_5 = 22$, and so on.

b) We follow the usual technique, as in Exercise 9. In the last line we use the familiar formula for the sum of the first $n-1$ positive integers. Note that the formula agrees with the values computed above.

$$\begin{aligned}
R_n &= 2(n-1) + R_{n-1} \\
&= 2(n-1) + 2(n-2) + R_{n-2} \\
&= 2(n-1) + 2(n-2) + 2(n-3) + R_{n-3} \\
&\;\;\vdots
\end{aligned}$$

$$= 2(n-1) + 2(n-2) + 2(n-3) + 2 \cdot 1 + R_1$$
$$= n(n-1) + 2 = n^2 - n + 2$$

**40.** Let $e_n$ be the number of bit sequences of length $n$ with an even number of 0's. Note that therefore there are $2^n - e_n$ bit sequences with an odd number of 0's. There are two ways to get a bit string of length $n$ with an even number of 0's. It can begin with a 1 and be followed by a bit string of length $n-1$ with an even number of 0's, and there are $e_{n-1}$ of these; or it can begin with a 0 and be followed by a bit string of length $n-1$ with an odd number of 0's, and there are $2^{n-1} - e_{n-1}$ of these. Therefore $e_n = e_{n-1} + 2^{n-1} - e_{n-1}$, or simply $e_n = 2^{n-1}$. See also Exercise 31 in Section 5.4.

**42.** Let $a_n$ be the number of coverings.

a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the left-most $n-1$ columns, and this can be done in $a_{n-1}$ ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in $a_{n-2}$ ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

b) Clearly $a_1 = 1$ and $a_2 = 2$.

c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ..., so the answer to this part is 2584.

**44.** The initial conditions are of course true. We prove the recurrence relation by induction on $n$, starting with base cases $n = 5$ and $n = 6$, in which cases we find $5f_1 + 3f_0 = 5 = f_5$ and $5f_2 + 3f_1 = 8 = f_6$. Assume the inductive hypothesis. Then we have $5f_{n-4} + 3f_{n-5} = 5(f_{n-5} + f_{n-6}) + 3(f_{n-6} + f_{n-7}) = (5f_{n-5} + 3f_{n-6}) + (5f_{n-6} + 3f_{n-7}) = f_{n-1} + f_{n-2} = f_n$ (we used both the inductive hypothesis and the recursive definition of the Fibonacci numbers). Finally, we prove that $f_{5n}$ is divisible by 5 by induction on $n$. It is true for $n = 1$, since $f_5 = 5$ is divisible by 5. Assume that it is true for $f_{5n}$. Then $f_{5(n+1)} = f_{5n+5} = 5f_{5n+1} + 3f_{5n}$ is divisible by 5, since both summands in this expression are divisible by 5.

**46.** a) We do this systematically, based on the position of the outermost dot, working from left to right:

$$x_0 \cdot (x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$$
$$x_0 \cdot (x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$$
$$x_0 \cdot ((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$$
$$x_0 \cdot ((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$$
$$x_0 \cdot (((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$$
$$(x_0 \cdot x_1) \cdot (x_2 \cdot (x_3 \cdot x_4))$$
$$(x_0 \cdot x_1) \cdot ((x_2 \cdot x_3) \cdot x_4)$$
$$(x_0 \cdot (x_1 \cdot x_2)) \cdot (x_3 \cdot x_4)$$
$$((x_0 \cdot x_1) \cdot x_2) \cdot (x_3 \cdot x_4)$$
$$(x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))) \cdot x_4$$
$$(x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)) \cdot x_4$$
$$((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)) \cdot x_4$$
$$((x_0 \cdot (x_1 \cdot x_2)) \cdot x_3) \cdot x_4$$
$$(((x_0 \cdot x_1) \cdot x_2) \cdot x_3) \cdot x_4$$

b) We know from Example 8 that $C_0 = 1$, $C_1 = 1$, and $C_3 = 5$. It is also easy to see that $C_2 = 2$, since there are only two ways to parenthesize the product of three numbers. Therefore the recurrence relation tells us that $C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$. We have the correct number of solutions listed above.

c) Here $n = 4$, so the formula gives $\frac{1}{5}C(8,4) = \frac{1}{5} \cdot 8 \cdot 7 \cdot 6 \cdot 5/4! = 14$.

**48.** We let $a_n$ be the number of moves required for this puzzle.

a) In order to move the bottom disk off peg 1, we must have transferred the other $n-1$ disks to peg 3 (since we must move the bottom disk to peg 2); this will require $a_{n-1}$ steps. Then we can move the bottom disk to peg 2 (one more step). Our goal, though, was to move it to peg 3, so now we must move the other $n-1$ disks from peg 3 back to peg 1, leaving the bottom disk quietly resting on peg 2. By symmetry, this again takes $a_{n-1}$ steps. One more step lets us move the bottom disk from peg 2 to peg 3. Now it takes $a_{n-1}$ steps to move the remaining disks from peg 1 to peg 3. So our recurrence relation is $a_n = 3a_{n-1} + 2$. The initial condition is of course that $a_0 = 0$.

b) Computing the first few values, we find that $a_1 = 2$, $a_2 = 8$, $a_3 = 26$, and $a_4 = 80$. It appears that $a_n = 3^n - 1$. This is easily verified by induction: The base case is $a_0 = 3^0 - 1 = 1 - 1 = 0$, and $3a_{n-1} + 2 = 3 \cdot (3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1 = a_n$.

c) The only choice in distributing the disks is which peg each disk goes on, since the order of the disks on a given peg is fixed. Since there are three choices for each disk, the answer is $3^n$.

d) The puzzle involves $1 + a_n = 3^n$ arrangements of disks during its solution—the initial arrangement and the arrangement after each move. None of these arrangements can repeat a previous arrangement, since if it did so, there would have been no point in making the moves between the two occurrences of the same arrangement. Therefore these $3^n$ arrangements are all distinct. We saw in part (c) that there are exactly $3^n$ arrangements, so every arrangement was used.

**50.** If we follow the hint, then it certainly looks as if $J(n) = 2k + 1$, where $k$ is the amount left over after the largest possible power of 2 has been subtracted from $n$ (i.e., $n = 2^m + k$ and $k < 2^m$).

**52.** The basis step is trivial, since when $n = 1 = 2^0 + 0$, the conjecture in Exercise 50 states that $J(n) = 2 \cdot 0 + 1 = 1$, which is correct. For the inductive step, we look at two cases, depending on whether there are an even or an odd number of players. If there are $2n$ players, suppose that $2n = 2^m + k$, as in the hint for Exercise 50. Then $k$ must be even and we can write $n = 2^{m-1} + (k/2)$, and $k/2 < 2^{m-1}$. By the inductive hypothesis, $J(n) = 2(k/2) + 1 = k + 1$. Then by the recurrence relation from Exercise 51, $J(2n) = 2J(n) - 1 = 2(k+1) - 1 = 2k + 1$, as desired. For the other case, assume that there are $2n + 1$ players, and again write $2n+1 = 2^m + k$, as in the hint for Exercise 50. Then $k$ must be odd and we can write $n = 2^{m-1} + (k-1)/2$, where $(k-1)/2 < 2^{m-1}$. By the inductive hypothesis, $J(n) = 2((k-1)/2) + 1 = k$. Then by the recurrence relation from Exercise 51, $J(2n+1) = 2J(n) + 1 = 2k + 1$, as desired.

**54.** Since we can only move one disk at a time, we need one move to lift the smallest disk off the middle disk, and another to lift the middle disk off the largest. Similarly, we need two moves to rejoin these disks. And of course we need at least one move to get the largest disk off peg 1. Therefore we can do no better than five moves. To see that this is possible, we just make the obvious moves (disk 1 is the smallest, and $a \xrightarrow{b} c$ means to move disk $b$ from peg $a$ to peg $c$: $1 \xrightarrow{1} 2$, $1 \xrightarrow{2} 3$, $1 \xrightarrow{3} 4$, $3 \xrightarrow{2} 4$, $2 \xrightarrow{1} 4$.

**56.** In our notation, disk 1 is the smallest and disk $n$ is the largest; $a \xrightarrow{b} c$ means to move disk $b$ from peg $a$ to peg $c$.

a) According to the algorithm, we take $k = 3$, since 5 is between the triangular numbers $t_2 = 3$ and $t_3 = 6$. The moves are to first move $5 - 3 = 2$ disks from peg 1 to peg 2 ($1 \xrightarrow{1} 3$, $1 \xrightarrow{2} 2$, $3 \xrightarrow{1} 2$), then working with pegs 1, 3, and 4 move disks 3, 4, and 5 to peg 4 ($1 \xrightarrow{3} 4$, $1 \xrightarrow{4} 3$, $4 \xrightarrow{3} 3$, $1 \xrightarrow{5} 4$, $3 \xrightarrow{3} 1$, $3 \xrightarrow{4} 4$, $1 \xrightarrow{3} 4$), and then move disks 1 and 2 from peg 2 to peg 4 ($2 \xrightarrow{1} 3$, $2 \xrightarrow{2} 4$, $3 \xrightarrow{1} 4$). Note that this took 13 moves in all.

**b)** According to the algorithm, we take $k = 3$, since 6 is between the triangular numbers $t_2 = 3$ and $t_3 = 6$. The moves are to first move $6 - 3 = 3$ disks from peg 1 to peg 2 ($1 \xrightarrow{1} 3$, $1 \xrightarrow{2} 4$, $1 \xrightarrow{3} 2$, $4 \xrightarrow{2} 2$, $3 \xrightarrow{1} 2$), then working with pegs 1, 3, and 4 move disks 4, 5, and 6 to peg 4 ($1 \xrightarrow{4} 4$, $1 \xrightarrow{5} 3$, $4 \xrightarrow{4} 3$, $1 \xrightarrow{6} 4$, $3 \xrightarrow{4} 1$, $3 \xrightarrow{5} 4$, $1 \xrightarrow{4} 4$), and then move disks 1, 2, and 3 from peg 2 to peg 4 ($2 \xrightarrow{1} 3$, $2 \xrightarrow{2} 1$, $2 \xrightarrow{3} 4$, $1 \xrightarrow{2} 4$, $3 \xrightarrow{1} 4$). Note that this took 17 moves in all.

**c)** According to the algorithm, we take $k = 4$, since 7 is between the triangular numbers $t_3 = 6$ and $t_4 = 10$. The moves are to first move $7 - 4 = 3$ disks from peg 1 to peg 2 (five moves, as in part **(b)**), then working with pegs 1, 3, and 4 move disks 4, 5, 6, and 7 to peg 4 (15 moves, using the usual Tower of Hanoi algorithm), and then move disks 1, 2, and 3 from peg 2 to peg 4 (again five moves, as in part **(b)**). Note that this took 25 moves in all.

**d)** According to the algorithm, we take $k = 4$, since 8 is between the triangular numbers $t_3 = 6$ and $t_4 = 10$. The moves are to first move $8 - 4 = 4$ disks from peg 1 to peg 2 (nine moves, as in Exercise 55, with peg 2 playing the role of peg 4), then working with pegs 1, 3, and 4 move disks 5, 6, 7, and 8 to peg 4 (15 moves, using the usual Tower of Hanoi algorithm), and then move disks 1, 2, 3, and 4 from peg 2 to peg 4 (again nine moves, as above). Note that this took 33 moves in all.

**58.** To clarify the problem, we note that $k$ is chosen to be the smallest nonnegative integer such that $n \le k(k+1)/2$. If $n - 1 \ne k(k-1)/2$, then this same value of $k$ applies to $n - 1$ as well; otherwise the value for $n - 1$ is $k - 1$. If $n - 1 \ne k(k-1)/2$, it also follows by subtracting $k$ from both sides of the inequality that the smallest nonnegative integer $m$ such that $n - k \le m(m+1)/2$ is $m = k - 1$, so $k - 1$ is the value selected by the Frame-Stewart algorithm for $n - k$. Now we proceed by induction, the basis steps being trivial. There are two cases for the inductive step. If $n - 1 \ne k(k-1)/2$, then we have from the recurrence relation in Exercise 57 that $R(n) = 2R(n-k) + 2^k - 1$ and $R(n-1) = 2R(n-k-1) + 2^k - 1$. Subtracting yields $R(n) - R(n-1) = 2(R(n-k) - R(n-k-1))$. Since $k - 1$ is the value selected for $n - k$, the inductive hypothesis tells us that this difference is $2 \cdot 2^{k-2} = 2^{k-1}$, as desired. On the other hand, if $n - 1 = k(k-1)/2$, then $R(n) - R(n-1) = 2R(n-k) + 2^k - 1 - (2R(n-1-(k-1)) + 2^{k-1} - 1 = 2^{k-1}$.

**60.** Since the Frame-Stewart algorithm solves the puzzle, the number of moves it uses, $R(n)$, is an upper bound to the number of moves needed to solve the puzzle. By Exercise 59 we have a recurrence or formula for these numbers. The table below shows $n$, the corresponding $k$ and $t_k$, and $R(n)$.

| $n$ | $k$ | $t_k$ | $R(n)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 3 |
| 3 | 2 | 3 | 5 |
| 4 | 3 | 6 | 9 |
| 5 | 3 | 6 | 13 |
| 6 | 3 | 6 | 17 |
| 7 | 4 | 10 | 25 |
| 8 | 4 | 10 | 33 |
| 9 | 4 | 10 | 41 |
| 10 | 4 | 10 | 49 |
| 11 | 5 | 15 | 65 |
| 12 | 5 | 15 | 81 |
| 13 | 5 | 15 | 97 |
| 14 | 5 | 15 | 113 |
| 15 | 5 | 15 | 129 |

| | | | |
|---|---|---|---|
| 16 | 6 | 21 | 161 |
| 17 | 6 | 21 | 193 |
| 18 | 6 | 21 | 225 |
| 19 | 6 | 21 | 257 |
| 20 | 6 | 21 | 289 |
| 21 | 6 | 21 | 321 |
| 22 | 7 | 28 | 353 |
| 23 | 7 | 28 | 417 |
| 24 | 7 | 28 | 481 |
| 25 | 7 | 28 | 545 |

**62.** a) $\nabla a_n = 4 - 4 = 0$ b) $\nabla a_n = 2n - 2(n-1) = 2$
c) $\nabla a_n = n^2 - (n-1)^2 = 2n - 1$ d) $\nabla a_n = 2^n - 2^{n-1} = 2^{n-1}$

**64.** This follows immediately (by algebra) from the definition.

**66.** We prove this by induction on $k$. The case $k = 1$ was Exercise 64. Assume the inductive hypothesis, that $a_{n-k}$ can be expressed in terms of $a_n$, $\nabla a_n$, $\ldots$, $\nabla^k a_n$, for all $n$. We will show that $a_{n-(k+1)}$ can be expressed in terms of $a_n$, $\nabla a_n$, $\ldots$, $\nabla^k a_n$, $\nabla^{k+1} a_n$. Note from the definitions that $a_{n-1} = a_n - \nabla a_n$ and that $\nabla^i a_{n-1} = \nabla^i a_n - \nabla^{i+1} a_n$ for all $i$. By the inductive hypothesis, we know that $a_{(n-1)-k}$ (which is just $a_{n-(k+1)}$ rewritten) can be expressed as $f(a_{n-1}, \nabla a_{n-1}, \ldots, \nabla^k a_{n-1}) = f(a_n - \nabla a_n, \nabla a_n - \nabla^2 a_n, \ldots, \nabla^k a_n - \nabla^{k+1} a_n)$— exactly what we wished to show. Note that in fact all the equations involved are linear.

**68.** By Exercise 66, each $a_{n-i}$ can be so expressed (as a linear function), so the entire recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ can be written as $a_n = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$, where each $f_i$ is a linear expression involving $a_n$, $\nabla a_n$, $\ldots$, $\nabla^k a_n$. This gives us the desired difference equation.