SECTION 7.2 Solving Linear Recurrence Relations

In many ways this section is extremely straightforward. Theorems 1–6 give an algorithm for solving linear homogeneous recurrence relations with constant coefficients. The only difficulty that sometimes occurs is that the algebra involved becomes messy or impossible. (Although the Fundamental Theorem of Algebra says that every n^{th} degree polynomial equation has exactly n roots (counting multiplicities), there is in general no way to find their exact values. For example, there is nothing analogous to the quadratic formula for equations of degree 5. Also, the roots may be irrational, as we saw in Example 4, or complex, as is discussed in Exercises 38 and 39. Patience is required with the algebra in such cases.) Many other techniques are available in other special cases, in analogy to the situation with differential equations; see Exercises 48–50, for example. If you have access to a computer algebra package, you should investigate how good it is at solving recurrences. See the solution to Exercise 49 for the kind of command to use in Maple.

1. **a)** This is linear (the terms a_i all appear to the first power), has constant coefficients (3, 4, and 5), and is homogeneous (no terms are functions of just n). It has degree 3, since a_n is expressed in terms of a_{n-1}, a_{n-2}, and a_{n-3}.

b) This does not have constant coefficients, since the coefficient of a_{n-1} is the nonconstant $2n$.

c) This is linear, homogeneous, with constant coefficients. It has degree 4, since a_n is expressed in terms of a_{n-1}, a_{n-2}, a_{n-3} and a_{n-4} (the fact that the coefficient of a_{n-2}, for example, is 0 is irrelevant—the degree is the largest k such that a_{n-k} is present).

d) This is not homogeneous because of the 2.

e) This is not linear, since the term a_{n-1}^2 appears.

f) This is linear, homogeneous, with constant coefficients. It has degree 2.

g) This is linear but not homogeneous because of the n.

3. **a)** We can solve this problem by iteration (or even by inspection), but let us use the techniques in this section instead. The characteristic equation is $r-2=0$, so the only root is $r=2$. Therefore the general solution to the recurrence relation, by Theorem 3 (with $k=1$), is $a_n=\alpha 2^n$ for some constant α. We plug in the initial condition to solve for α. Since $a_0=3$ we have $3=\alpha 2^0$, whence $\alpha=3$. Therefore the solution is $a_n=3\cdot 2^n$.

b) Again this is trivial to solve by inspection, but let us use the algorithm. The characteristic equation is $r-1=0$, so the only root is $r=1$. Therefore the general solution to the recurrence relation, by Theorem 3 (with $k=1$), is $a_n=\alpha 1^n=\alpha$ for some constant α. In other words, the sequence is constant. We plug in the initial condition to solve for α. Since $a_0=2$ we have $\alpha=2$. Therefore the solution is $a_n=2$ for all n.

c) The characteristic equation is $r^2-5r+6=0$, which factors as $(r-2)(r-3)=0$, so the roots are $r=2$ and $r=3$. Therefore by Theorem 1 the general solution to the recurrence relation is $a_n=\alpha_1 2^n+\alpha_2 3^n$ for some constants α_1 and α_2. We plug in the initial condition to solve for the α's. Since $a_0=1$ we have $1=\alpha_1+\alpha_2$, and since $a_1=0$ we have $0=2\alpha_1+3\alpha_2$. These linear equations are easily solved to yield $\alpha_1=3$ and $\alpha_2=-2$. Therefore the solution is $a_n=3\cdot 2^n-2\cdot 3^n$.

d) The characteristic equation is $r^2-4r+4=0$, which factors as $(r-2)^2=0$, so there is only one root, $r=2$, which occurs with multiplicity 2. Therefore by Theorem 2 the general solution to the recurrence relation is $a_n=\alpha_1 2^n+\alpha_2 n2^n$ for some constants α_1 and α_2. We plug in the initial conditions to solve for the α's. Since $a_0=6$ we have $6=\alpha_1$, and since $a_1=8$ we have $8=2\alpha_1+2\alpha_2$. These linear equations are easily solved to yield $\alpha_1=6$ and $\alpha_2=-2$. Therefore the solution is $a_n=6\cdot 2^n-2\cdot n2^n=(6-2n)2^n$. Incidentally, there is a good way to check a solution to a recurrence relation problem, namely by calculating the next term in two ways. In this exercise, the recurrence relation tells us that $a_2=4a_1-4a_0=4\cdot 8-4\cdot 6=8$, whereas the solution tells us that $a_2=(6-2\cdot 2)2^2=8$. Since these answers agree, we are somewhat confident that our solution is correct. We could calculate a_3 in two ways for another confirmation.

e) This time the characteristic equation is $r^2+4r+4=0$, which factors as $(r+2)^2=0$, so again there is only one root, $r=-2$, which occurs with multiplicity 2. Therefore by Theorem 2 the general solution to the recurrence relation is $a_n = \alpha_1(-2)^n + \alpha_2 n(-2)^n$ for some constants α_1 and α_2. We plug in the initial conditions to solve for the α's. Since $a_0 = 0$ we have $0 = \alpha_1$, and since $a_1 = 1$ we have $1 = -2\alpha_1 - 2\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 0$ and $\alpha_2 = -1/2$. Therefore the solution is $a_n = (-1/2)n(-2)^n = n(-2)^{n-1}$.

f) The characteristic equation is $r^2 - 4 = 0$, so the roots are $r = 2$ and $r = -2$. Therefore the solution is $a_n = \alpha_1 2^n + \alpha_2(-2)^n$ for some constants α_1 and α_2. We plug in the initial conditions to solve for the α's. We have $0 = \alpha_1 + \alpha_2$, and $4 = 2\alpha_1 - 2\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 1$ and $\alpha_2 = -1$. Therefore the solution is $a_n = 2^n - (-2)^n$.

g) The characteristic equation is $r^2 - 1/4 = 0$, so the roots are $r = 1/2$ and $r = -1/2$. Therefore the solution is $a_n = \alpha_1(1/2)^n + \alpha_2(-1/2)^n$ for some constants α_1 and α_2. We plug in the initial conditions to solve for the α's. We have $1 = \alpha_1 + \alpha_2$, and $0 = \alpha_1/2 - \alpha_2/2$. These linear equations are easily solved to yield $\alpha_1 = \alpha_2 = 1/2$. Therefore the solution is $a_n = (1/2)(1/2)^n + (1/2)(-1/2)^n = (1/2)^{n+1} - (-1/2)^{n+1}$.

5. The recurrence relation found in Exercise 35 of Section 7.1 was $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_0 = a_1 = 1$. To solve this, we look at the characteristic equation $r^2 - r - 1 = 0$ (exactly as in Example 4) and obtain, by the quadratic formula, the roots $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Therefore from Theorem 1 we know that the solution is given by

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2. The initial conditions $a_0 = 1$ and $a_1 = 1$ allow us to determine these constants. We plug them into the equation displayed above and obtain

$$\begin{aligned} 1 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right). \end{aligned}$$

By algebra we solve these equations (one way is to solve the first for α_2 in terms of α_1, and plug that into the second equation to get one equation in α_1, which can then be solved—the fact that these coefficients are messy irrational numbers involving $\sqrt{5}$ does not change the rules of algebra, of course). The solutions are $\alpha_1 = (5+\sqrt{5})/10$ and $\alpha_2 = (5-\sqrt{5})/10$. Therefore the specific solution is given by

$$a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Alternatively, by not rationalizing the denominators when we solve for α_1 and α_2, we get $\alpha_1 = (1+\sqrt{5})/(2\sqrt{5})$ and $\alpha_2 = -(1-\sqrt{5})/(2\sqrt{5})$. With these expressions, we can write our solution as

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

7. First we need to find a recurrence relation and initial conditions for the problem. Let t_n be the number of ways to tile a $2 \times n$ board with 1×2 and 2×2 pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a 2×2 tile there, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in t_{n-2} ways. We can place a 1×2 tile at the edge, oriented vertically, leaving a $2 \times (n-1)$ board to be tiled, which of course can be done in t_{n-1} ways. Finally, we can place two 1×2 tiles horizontally, one above the other, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in t_{n-2} ways. These three possibilities are disjoint. Therefore our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$. The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile a 2×0 board (the way that uses no tiles) and only one way to tile a 2×1 board. This recurrence relation is the same one that appeared in Example 3; it has characteristic roots 2 and -1, so the general solution is

$$t_n = \alpha_1 2^n + \alpha_2(-1)^n .$$

To determine the coefficients we plug in the initial conditions, giving us the equations

$$1 = t_0 = \alpha_1 + \alpha_2$$
$$1 = t_1 = 2\alpha_1 - \alpha_2 .$$

Solving these yields $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$, so our final solution is $t_n = 2^{n+1}/3 + (-1)^n/3$.

9. **a)** The amount P_n in the account at the end of the n^{th} year is equal to the amount at the end of the previous year (P_{n-1}), plus the 20% dividend on that amount ($0.2P_{n-1}$) plus the 45% dividend on the amount at the end of the year before that ($0.45P_{n-2}$). Thus we have $P_n = 1.2P_{n-1} + 0.45P_{n-2}$. We need two initial conditions, since the equation has degree 2. Clearly $P_0 = 100000$. The other initial condition is that $P_1 = 120000$, since there is only one dividend at the end of the first year.

b) Solving this recurrence relation requires looking at the characteristic equation $r^2 - 1.2r - 0.45 = 0$. By the quadratic formula, the roots are $r_1 = 1.5$ and $r_2 = -0.3$. Therefore the general solution of the recurrence relation is $P_n = \alpha_1(1.5)^n + \alpha_2(-0.3)^n$. Plugging in the initial conditions gives us the equations $100000 = \alpha_1 + \alpha_2$ and $120000 = 1.5\alpha_1 - 0.3\alpha_2$. These are easily solved to give $\alpha_1 = 250000/3$ and $\alpha_2 = 50000/3$. Therefore the solution of our problem is

$$P_n = \frac{250000}{3}(1.5)^n + \frac{50000}{3}(-0.3)^n .$$

11. **a)** We prove this by induction on n. We need to verify two base cases. For $n = 1$ we have $L_1 = 1 = 0 + 1 = f_0 + f_2$; and for $n = 2$ we have $L_2 = 3 = 1 + 2 = f_1 + f_3$. Assume the inductive hypothesis that $L_k = f_{k-1} + f_{k+1}$ for $k < n$. We must show that $L_n = f_{n-1} + f_{n+1}$. To do this, we let $k = n - 1$ and $k = n - 2$:

$$L_{n-1} = f_{n-2} + f_n$$
$$L_{n-2} = f_{n-3} + f_{n-1} .$$

If we add these two equations, we obtain

$$L_{n-1} + L_{n-2} = (f_{n-2} + f_{n-3}) + (f_n + f_{n-1}) ,$$

which is the same as

$$L_n = f_{n-1} + f_{n+1}$$

as desired, using the recurrence relations for the Lucas and Fibonacci numbers.

b) To find an explicit formula for the Lucas numbers, we need to solve the recurrence relation and initial conditions. Since the recurrence relation is the same as that of the Fibonacci numbers, we get the same general solution as in Example 4, namely

$$L_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

for some constants α_1 and α_2. The initial conditions are different, though. When we plug them in we get the system

$$2 = L_0 = \alpha_1 + \alpha_2$$
$$1 = L_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) .$$

By algebra we solve these equations, yielding $\alpha_1 = \alpha_1 = 1$. Therefore the specific solution is given by

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

13. This is a third degree equation. The characteristic equation is $r^3 - 7r - 6 = 0$. Assuming the composer of the problem has arranged that the roots are nice numbers, we use the rational root test, which says that rational roots must be of the form $\pm p/q$, where p is a factor of the constant term (6 in this case) and q is a factor of the coefficient of the leading term (the coefficient of r^3 is 1 in this case). Hence the possible rational roots are $\pm1, \pm2, \pm3, \pm6$. We find that $r = -1$ is a root, so one factor of $r^3 - 7r + 6$ is $r + 1$. Dividing $r + 1$ into $r^3 - 7r - 6$ by long (or synthetic) division, we find that $r^3 - 7r - 6 = (r + 1)(r^2 - r - 6)$. By inspection we factor the rest, obtaining $r^3 - 7r - 6 = (r+1)(r-3)(r+2)$. Hence the roots are -1, 3, and -2, so the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 3^n + \alpha_3(-2)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned} 9 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3 \\ 10 &= a_1 = -\alpha_1 + 3\alpha_2 - 2\alpha_3 \\ 32 &= a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3 . \end{aligned}$$

Solving this system of equations (by elimination, for instance), we get $\alpha_1 = 8$, $\alpha_2 = 4$, and $\alpha_3 = -3$. Therefore the specific solution is $a_n = 8(-1)^n + 4 \cdot 3^n - 3(-2)^n$.

15. This is a third degree recurrence relation. The characteristic equation is $r^3 - 2r^2 - 5r + 6 = 0$. By the rational root test, the possible rational roots are $\pm1, \pm2, \pm3, \pm6$. We find that $r = 1$ is a root. Dividing $r - 1$ into $r^3 - 2r^2 - 5r + 6$, we find that $r^3 - 2r^2 - 5r + 6 = (r - 1)(r^2 - r - 6)$. By inspection we factor the rest, obtaining $r^3 - 2r^2 - 5r + 6 = (r - 1)(r - 3)(r + 2)$. Hence the roots are 1, 3, and -2, so the general solution is $a_n = \alpha_1 1^n + \alpha_2 3^n + \alpha_3(-2)^n$, or more simply $a_n = \alpha_1 + \alpha_2 3^n + \alpha_3(-2)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned} 7 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3 \\ -4 &= a_1 = \alpha_1 + 3\alpha_2 - 2\alpha_3 \\ 8 &= a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3 . \end{aligned}$$

Solving this system of equations, we get $\alpha_1 = 5$, $\alpha_2 = -1$, and $\alpha_3 = 3$. Therefore the specific solution is $a_n = 5 - 3^n + 3(-2)^n$.

17. We almost follow the hint and let a_{n+1} be the right-hand side of the stated identity. Clearly $a_1 = C(0,0) = 1$ and $a_2 = C(1,0) = 1$. Thus $a_1 = f_1$ and $a_2 = f_2$. Now if we can show that the sequence $\{a_n\}$ satisfies the same recurrence relation that the Fibonacci numbers do, namely $a_{n+1} = a_n + a_{n-1}$, then we will know that $a_n = f_n$ for all $n \ge 1$ (precisely what we want to show), since the solution of a second degree recurrence relation with two initial conditions is unique.

To show that $a_{n+1} = a_n + a_{n-1}$, we start with the right-hand side, which is, by our definition, $C(n-1,0) + C(n-2,1) + \cdots + C(n-1-k,k) + C(n-2,0) + C(n-3,1) + \cdots + C(n-2-l,l)$, where $k = \lfloor (n-1)/2 \rfloor$ and $l = \lfloor (n-2)/2 \rfloor$. Note that $k = l$ if n is even, and $k = l + 1$ if n is odd. Let us first take the case in which $k = l = (n-2)/2$. By Pascal's Identity, we regroup the sum above and rewrite it as

$$\begin{aligned} &C(n-1,0) + [C(n-2,0) + C(n-2,1)] + [C(n-3,1) + C(n-3,2)] + \cdots \\ &\qquad + [C(n-2-((n-2)/2-1),(n-2)/2-1) + C(n-1-(n-2)/2,(n-2)/2)] \\ &\qquad + C(n-2-(n-2)/2,(n-2)/2) \\ &= C(n-1,0) + C(n-1,1) + C(n-2,2) + \cdots \\ &\qquad + C(n-(n-2)/2,(n-2)/2) + C(n-2-(n-2)/2,(n-2)/2) \\ &= 1 + C(n-1,1) + C(n-2,2) + \cdots + C(n-(n-2)/2,(n-2)/2) + 1 \\ &= C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n-(n-2)/2,(n-2)/2) + C(n-n/2,n/2) \\ &= C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n-j,j), \end{aligned}$$

where $j = n/2 = \lfloor n/2 \rfloor$. This is precisely a_{n+1}, as desired. In case n is odd, so that $k = (n-1)/2$ and $l = (n-3)/2$, we have a similar calculation (in this case the sum involving k has one more term than the sum involving l):

$$\begin{aligned}
&C(n-1,0)+[C(n-2,0)+C(n-2,1)]+[C(n-3,1)+C(n-3,2)]+\cdots \\
&\qquad +[C(n-2-(n-3)/2,(n-3)/2)+C(n-1-(n-1)/2,(n-1)/2)] \\
&\quad = C(n-1,0)+C(n-1,1)+C(n-2,2)+\cdots+C(n-(n-1)/2,(n-1)/2) \\
&\quad = 1+C(n-1,1)+C(n-2,2)+\cdots+C(n-(n-1)/2,(n-1)/2) \\
&\quad = C(n,0)+C(n-1,1)+C(n-2,2)+\cdots+C(n-j,j)\,,
\end{aligned}$$

where $j=(n-1)/2=\lfloor n/2\rfloor$. Again, this is precisely a_{n+1}, as desired.

19. This is a third degree recurrence relation. The characteristic equation is $r^3+3r^2+3r+1=0$. We easily recognize this polynomial as $(r+1)^3$. Hence the only root is -1, with multiplicity 3, so the general solution is (by Theorem 4) $a_n=\alpha_1(-1)^n+\alpha_2 n(-1)^n+\alpha_3 n^2(-1)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned}
5 &= a_0 = \alpha_1 \\
-9 &= a_1 = -\alpha_1-\alpha_2-\alpha_3 \\
15 &= a_2 = \alpha_1+2\alpha_2+4\alpha_3
\end{aligned}$$

Solving this system of equations, we get $\alpha_1=5$, $\alpha_2=3$, and $\alpha_3=1$. Therefore the answer is $a_n=5(-1)^n+3n(-1)^n+n^2(-1)^n$. We could also write this in factored form, of course, as $a_n=(n^2+3n+5)(-1)^n$. As a check of our answer, we can calculate a_3 both from the recurrence and from our formula, and we find that it comes out to be -23 in both cases.

21. This is similar to Example 6. We can immediately write down the general solution using Theorem 4. In this case there are four distinct roots, so $t=4$. The multiplicities are 4, 3, 2, and 1. So the general solution is $a_n=(\alpha_{1,0}+\alpha_{1,1}n+\alpha_{1,2}n^2+\alpha_{1,3}n^3)+(\alpha_{2,0}+\alpha_{2,1}n+\alpha_{2,2}n^2)(-2)^n+(\alpha_{3,0}+\alpha_{3,1}n)3^n+\alpha_{4,0}(-4)^n$.

23. Theorem 5 tells us that the general solution to the inhomogeneous linear recurrence relation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}+F(n)$$

can be found by finding one particular solution of this recurrence relation and adding it to the general solution of the corresponding homogeneous recurrence relation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}\,.$$

If we let f_n be the particular solution to the inhomogeneous recurrence relation and g_n be the general solution to the homogeneous recurrence relation (which will have some unspecified parameters α_1, α_2, ..., α_k), then the general solution to the inhomogeneous recurrence relation is f_n+g_n (so it, too, will have some unspecified parameters α_1, α_2, ..., α_k).

a) To show that $a_n=-2^{n+1}$ is a solution to $a_n=3a_{n-1}+2^n$, we simply substitute it in and see if we get a true statement. Upon substituting into the right-hand side we get $3a_{n-1}+2^n=3(-2^n)+2^n=2^n(-3+1)=-2^{n+1}$, which is precisely the left-hand side.

b) By Theorem 5 and the comments above, we need to find the general solution to the corresponding homogeneous recurrence relation $a_n=3a_{n-1}$. This is easily seen to be $a_n=\alpha 3^n$ (either by the iterative method or by the method of this section with a linear characteristic equation). Putting these together as discussed above, we find the general solution to the given recurrence relation: $a_n=\alpha 3^n-2^{n+1}$.

c) To find the solution with $a_0=1$, we need to plug this initial condition (where $n=0$) into our answer to part **(b)**. Doing so gives the equation $1=\alpha-2$, whence $\alpha=3$. Therefore the solution to the given recurrence relation and initial condition is $a_n=3\cdot 3^n-2^{n+1}=3^{n+1}-2^{n+1}$.

25. See the introductory remarks to Exercise 23, which apply here as well.

a) We solve this problem by wishful thinking. Suppose that $a_n = An + B$, and substitute into the given recurrence relation. This gives us $An+B = 2\big(A(n-1)+B\big)+n+5$, which simplifies to $(A+1)n+(-2A+B+5) = 0$. Now if this is going to be true for all n, then both of the quantities in parentheses will have to be 0. In other words, we need to solve the simultaneous equations $A+1=0$ and $-2A+B+5=0$. The solution is $A=-1$ and $B=-7$. Therefore a solution to the recurrence relation is $a_n = -n-7$.

b) By Theorem 5 and the comments at the beginning of Exercise 23, we need to find the general solution to the corresponding homogeneous recurrence relation $a_n = 2a_{n-1}$. This is easily seen to be $a_n = \alpha 2^n$ (either by the iterative method or by the method of this section with a linear characteristic equation). Putting these together as discussed above, we find the general solution to the given recurrence relation: $a_n = \alpha 2^n - n - 7$.

c) To find the solution with $a_0 = 4$, we need to plug this initial condition (where $n=0$) into our answer to part (b). Doing so gives the equation $4 = \alpha - 7$, whence $\alpha = 11$. Therefore the solution to the given recurrence relation and initial condition is $a_n = 11\cdot 2^n - n - 7$.

27. We need to use Theorem 6, and so we need to find the roots of the characteristic polynomial of the associated homogeneous recurrence relation. The characteristic equation is $r^4 - 8r^2 + 16 = 0$, and as we saw in Exercise 20, $r = \pm 2$ are the only roots, each with multiplicity 2.

a) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_3n^3 + p_2n^2 + p_1n + p_0$. Note that $s=1$ here, in the notation of Theorem 6.

b) Since -2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2p_0(-2)^n$.

c) Since 2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_1n+p_0)2^n$.

d) Since 4 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $(p_2n^2 + p_1n + p_0)4^n$.

e) Since -2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_2n^2+p_1n+p_0)(-2)^n$. Note that we needed a second degree polynomial inside the parenthetical expression because the polynomial in $F(n)$ was second degree.

f) Since 2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_4n^4+p_3n^3+p_2n^2+p_1n+p_0)2^n$.

g) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form p_0. Note that $s=1$ here, in the notation of Theorem 6.

29. a) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c\cdot 3^n$. We plug this into our recurrence relation and obtain $c\cdot 3^n = 2c\cdot 3^{n-1} + 3^n$. We divide through by 3^{n-1} and simplify, to find easily that $c=3$. Therefore the particular solution we seek is $a_n^{(p)} = 3\cdot 3^n = 3^{n+1}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + 3^{n+1}$.

b) We plug the initial condition into our solution from part (a) to obtain $5 = a_1 = 2\alpha + 9$. This tells us that $\alpha = -2$. So the solution is $a_n = -2\cdot 2^n + 3^{n+1} = -2^{n+1} + 3^{n+1}$. At this point it would be very useful to check our answer. One method is to let a computer do the work; a computer algebra package such as *Maple* will solve equations of this type (see Exercise 49 for the syntax of the command). Alternatively, we can compute the next term of the sequence in two ways and verify that we obtain the same answer in each case. From the recurrence relation, we expect that $a_2 = 2 \cdot a_1 + 3^2 = 2 \cdot 5 + 9 = 19$. On the other hand, our solution tells us that $a_2 = -2^{2+1} + 3^{2+1} = -8 + 27 = 19$. Since the values agree, we can be fairly confident that our solution is correct.

31. The associated homogeneous recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 - 5r + 6 = 0$, find that $r = 2$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha 2^n + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice (or following the hint), we want to look for a function of the form $a_n = cn \cdot 2^n + dn + e$. (The reason for the factor n in front of 2^n is that 2^n was already a solution of the homogeneous equation. The reason for the term $dn + e$ is the first degree polynomial $3n$.) We plug this into our recurrence relation and obtain $cn \cdot 2^n + dn + e = 5c(n-1) \cdot 2^{n-1} + 5d(n-1) + 5e - 6c(n-2) \cdot 2^{n-2} - 6d(n-2) - 6e + 2^n + 3n$. In order for this equation to be true, the exponential parts must be equal, and the polynomial parts must be equal. Therefore we have $c \cdot 2^n = 5c(n-1) \cdot 2^{n-1} - 6c(n-2) \cdot 2^{n-2} + 2^n$ and $dn + e = 5d(n-1) + 5e - 6d(n-2) - 6e + 3n$. To solve the first of these equations, we divide through by 2^{n-1}, obtaining $2c = 5c(n-1) - 3c(n-2) + 2$, whence a little algebra yields $c = -2$. To solve the second equation, we note that the coefficients of n as well as the constant terms must be equal on each side, so we know that $d = 5d - 6d + 3$ and $e = -5d + 5e + 12d - 6e$. This tells us that $d = 3/2$ and $e = 21/4$. Therefore the particular solution we seek is $a_n^{(p)} = -2n \cdot 2^n + 3n/2 + 21/4$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + \beta 3^n - 2n \cdot 2^n + 3n/2 + 21/4 = \alpha 2^n + \beta 3^n - n \cdot 2^{n+1} + 3n/2 + 21/4$.

33. The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 4a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 4 = 0$, find that $r = 2$ is a repeated root, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha 2^n + \beta n \cdot 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = n^2(cn + d)2^n$. (The reason for the factor $cn + d$ is that there is a linear polynomial factor in front of 2^n in the nonhomogeneous term; the reason for the factor n^2 is that the root $r = 2$ already appears twice in the associated homogeneous relation.) We plug this into our recurrence relation and obtain $n^2(cn+d)2^n = 4(n-1)^2(cn-c+d)2^{n-1} - 4(n-2)^2(cn-2c+d)2^{n-2} + (n+1)2^n$. We divide through by 2^n, obtaining $n^2(cn + d) = 2(n-1)^2(cn - c + d) - (n-2)^2(cn - 2c + d) + (n + 1)$. Some algebra transforms this into $cn^3 + dn^2 = cn^3 + dn^2 + (-6c + 1)n + (6c - 2d + 1)$. Equating like powers of n tells us that $c = 1/6$ and $d = 1$. Therefore the particular solution we seek is $a_n^{(p)} = n^2(n/6 + 1)2^n$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = (\alpha + \beta n + n^2 + n^3/6)2^n$.

35. The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 3a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 3 = 0$, find that $r = 1$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice, we want to look for a function of the form $a_n = c \cdot 2^n + n(dn + e) = c \cdot 2^n + dn^2 + en$. (The factor n in front of $(dn + e)$ is needed since 1 is already a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $c \cdot 2^n + dn^2 + en = 4c \cdot 2^{n-1} + 4d(n-1)^2 + 4e(n-1) - 3c \cdot 2^{n-2} - 3d(n-2)^2 - 3e(n-2) + 2^n + n + 3$. A lot of messy algebra transforms this into the following equation, where we group by function of n: $2^{n-2}(-c - 4) + n^2 \cdot 0 + n(-4d - 1) + (8d - 2e - 3) = 0$. The coefficients must therefore all be 0, whence $c = -4$, $d = -1/4$, and $e = -5/2$. Therefore the particular solution we seek is $a_n^{(p)} = -4 \cdot 2^n - n^2/4 - 5n/2$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + \alpha + \beta 3^n$. Next we plug in the initial conditions to obtain $1 = a_0 = -4 + \alpha + \beta$ and $4 = a_1 = -8 - 11/4 + \alpha + 3\beta$. We solve this system of equations to obtain $\alpha = 1/8$ and $\beta = 39/8$. So the final solution is $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$. As a check of our work (it would be too much to hope that we could always get this far without making an algebraic error), we can compute a_2 both from the recurrence and from the solution, and we find that $a_2 = 22$ both ways.

37. Obviously the n^{th} term of the sequence comes from the $(n-1)^{\text{st}}$ term by adding the n^{th} triangular number; in symbols, $a_{n-1} + n(n+1)/2 = \left(\sum_{k=1}^{n-1} k(k+1)/2\right) + n(n+1)/2 = \sum_{k=1}^{n} k(k+1)/2 = a_n$. Also, the sum of the first triangular number is clearly 1. To solve this recurrence relation, we easily see that the homogeneous solution is $a_n^{(h)} = \alpha$, so since the nonhomogeneous term is a second degree polynomial, we need a particular solution of the form $a_n = cn^3 + dn^2 + en$. Plugging this into the recurrence relation gives $cn^3 + dn^2 + en = c(n-1)^3 + d(n-1)^2 + e(n-1) + n(n+1)/2$. Expanding and collecting terms, we have $(3c-\frac{1}{2})n^2 + (-3c+2d-\frac{1}{2})n + (c-d+e) = 0$, whence $c = \frac{1}{6}$, $d = \frac{1}{2}$, and $e = \frac{1}{3}$. Thus $a_n^{(p)} = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$. So the general solution is $a_n = \alpha + \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$. It is now a simple matter to plug in the initial condition $a_1 = 1$ to see that $\alpha = 0$. Note that we can find a common denominator and write our solution in the nice form $a_n = n(n+1)(n+2)/6$, which is the binomial coefficient $C(n+2,3)$.

39. Nothing in the discussion of solving recurrence relations by the methods of this section relies on the roots of the characteristic equation being real numbers. Sometimes the roots are complex numbers (involving $i = \sqrt{-1}$). The situation is analogous to the fact that we sometimes get irrational numbers when solving the characteristic equation (for example, for the Fibonacci numbers), even though the coefficients are all integers and the terms in the sequence are all integers. It is just that we need irrational numbers in order to write down an algebraic solution. Here we need complex numbers in order to write down an algebraic solution, even though all the terms in the sequence are real.

a) The characteristic equation is $r^4 - 1 = 0$. This factors as $(r-1)(r+1)(r^2+1) = 0$, so the roots are $r = 1$ and $r = -1$ (from the first two factors) and $r = i$ and $r = -i$ (from the third factor).

b) By our work in part **(a)**, the general solution to the recurrence relation is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 i^n + \alpha_4(-i)^n$. In order to figure out the α's we plug in the initial conditions, yielding the following system of linear equations:

$$\begin{aligned} 1 = a_0 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ 0 = a_1 &= \alpha_1 - \alpha_2 + i\alpha_3 - i\alpha_4 \\ -1 = a_2 &= \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 \\ 1 = a_3 &= \alpha_1 - \alpha_2 - i\alpha_3 + i\alpha_4 \end{aligned}$$

Remembering that i is just a constant, we solve this system by elimination or other means. For instance, we could begin by subtracting the third equation from the first, to give $2 = 2\alpha_3 + 2\alpha_4$ and subtracting the fourth from the second to give $-1 = 2i\alpha_3 - 2i\alpha_4$. This gives us two equation in two unknowns. Solving them yields $\alpha_3 = (2i-1)/(4i)$ which can be put into nicer form by multiplying by i/i, so $\alpha_3 = (2+i)/4$; and then $\alpha_4 = 1 - \alpha_3 = (2-i)/4$. We plug these values back into the first and fourth equations, obtaining $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 1/2$. These tell us that $\alpha_1 = 1/4$ and $\alpha_2 = -1/4$. Therefore the answer to the problem is

$$a_n = \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{2+i}{4}i^n + \frac{2-i}{4}(-i)^n .$$

41. a) To say that f_n is the integer closest to $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ is to say that the absolute difference between these two numbers is less than $\dfrac{1}{2}$. But the difference is just $\left|\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right|$. Thus we are asked to show that this latter number is less than $\frac{1}{2}$. The value within the parentheses is about -0.62. When raised to the n^{th} power, for $n \geq 0$, we get a number of absolute value less than or equal to 1. When we then divide by $\sqrt{5}$ (which is greater than 2), we get a number less than $\frac{1}{2}$, as desired.

b) Clearly the second term in the formula for f_n alternates sign as n increases: a positive number is being subtracted for n even, and a negative number is being subtracted for n odd. Therefore f_n is less than $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ for even n and greater than $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ for odd n.

43. We follow the hint and let $b_n = a_n + 1$, or, equivalently, $a_n = b_n - 1$. Then the recurrence relation becomes $b_n - 1 = b_{n-1} - 1 + b_{n-2} - 1 + 1$, or $b_n = b_{n-1} + b_{n-2}$; and the initial conditions become $b_0 = a_0 + 1 = 0 + 1 = 1$ and $b_1 = a_1 + 1 = 1 + 1 = 2$. We now apply the result of Exercise 42, with b playing the role of a, and $s = 1$ and $t = 2$, to get $b_n = f_{n-1} + 2f_n$. Therefore $a_n = f_{n-1} + 2f_n - 1$. We can check this with a few small values of n: for $n = 2$, our solution predicts that $a_2 = f_1 + 2f_2 - 1 = 1 + 2 \cdot 1 - 1 = 2$; similarly, $a_3 = f_2 + 2f_3 - 1 = 1 + 2 \cdot 2 - 1 = 4$ and $a_4 = 7$. These are precisely the values we would get by applying directly the recurrence relation defining a_n in this problem. A reality check like this is a good way to increase the chances that we haven't made a mistake.

An alternative answer is $a_n = f_{n+2} - 1$. We can prove this as follows:

$$f_{n-1} + 2f_n - 1 = f_{n-1} + f_n + f_n - 1 = f_{n+1} + f_n - 1 = f_{n+2} - 1$$

45. Let a_n be the desired quantity, the number of pairs of rabbits on the island after n months. So $a_0 = 1$, since one pair is there initially. We need to read the problem carefully and decide how we will interpret what it says. Since a pair produces two new pairs "at the age of one month" and six new pairs "at the age of two months" and every month thereafter, the original pair has already produced two new pairs at the end of one month, so $a_1 = 3$ (the original pair plus two new pairs), and $a_2 = 3 + 6 + 4 = 13$ (the three pairs that were already there, six new pairs produced by the original inhabitants, and two new pairs produced by each of the two pairs born at the end of the first month). If you interpret the wording to imply that births do not occur until after the month has finished, then naturally you will get different answers from those we are about to find.

a) We already have stated the initial conditions $a_0 = 1$ and $a_1 = 3$. To obtain a recurrence relation for a_n, the number of pairs of rabbits present at the end of the n^{th} month, we observe (as was the case in analyzing Fibonacci's example) that all the rabbit pairs who were present at the end of the $(n-2)^{\text{nd}}$ month will give rise to six new ones, giving us $6a_{n-2}$ new pairs; and all the rabbit pairs who were present at the end of the $(n-1)^{\text{st}}$ month but not at the end of the $(n-2)^{\text{nd}}$ month will give rise to two new ones, namely $2(a_{n-1} - a_{n-2})$ new pairs. Of course, the a_{n-1} pairs who were there stay around as well. Thus our recurrence relation is $a_n = 6a_{n-2} + 2(a_{n-1} - a_{n-2}) + a_{n-1}$, or, more simply, $a_n = 3a_{n-1} + 4a_{n-2}$. As a check, we compute that $a_2 = 3 \cdot 3 + 4 \cdot 1 = 13$, which is the number we got above.

b) We proceed by the method of this section, as we did in, say, Exercise 3. The characteristic equation is $r^2 - 3r - 4 = 0$, which factors as $(r - 4)(r + 1) = 0$, so we get roots 4 and -1. Thus the general solution is $a_n = \alpha_1 4^n + \alpha_2(-1)^n$. Plugging in the initial conditions $a_0 = 1$ and $a_1 = 3$, we find $1 = \alpha_1 + \alpha_2$ and $3 = 4\alpha_1 - \alpha_2$, which are easily solved to yield $\alpha_1 = 4/5$ and $\alpha_2 = 1/5$. Therefore the number of pairs of rabbits on the island after n months is $a_n = 4 \cdot 4^n/5 + (-1)^n/5 = \left(4^{n+1} + (-1)^n\right)/5$. As a check, we see that $a_2 = (4^3 + 1)/5 = 65/5 = 13$, the number we found above.

47. Let a_n be the employee's salary for the n^{th} year of employment, in tens of thousands of dollars (this makes the numbers easier to work with). Thus we are told that $a_1 = 5$, and applying the given rule for raises, we have $a_2 = 2 \cdot 5 + 1 = 11$, $a_3 = 2 \cdot 11 + 2 = 24$, and so on.

a) For her n^{th} year of employment, she has $n - 1$ years of experience, so the raise rule says that $a_n = 2a_{n-1} + (n - 1)$. (Remember that we are using \$10,000 as the unit of pay here.)

b) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$, which clearly has the solution $a_n^{(h)} = \alpha 2^n$. For the particular solution of the given relation, we note that the nonhomogeneous term is a linear function of n and try $a_n = cn + d$. Plugging into the relation yields $cn + d = 2c(n - 1) + 2d + n - 1$, which, upon grouping like terms, becomes $(-c - 1)n + (2c - d + 1) = 0$. Therefore $c = -1$ and $d = -1$, so $a_n^{(p)} = -n - 1$. Therefore the general solution is $a_n = \alpha 2^n - n - 1$. Plugging in the initial condition gives $5 = a_1 = 2\alpha - 2$, whence $\alpha = 7/2$. Our solution is therefore $a_n = 7 \cdot 2^{n-1} - n - 1$. We can check that this gives the correct salary for the first few years, as computed above.

49. Using the notation of Exercise 48 we have $f(n) = n+1$, $g(n) = n+3$, $h(n) = n$, and $C = 1$. Therefore

$$Q(n) \cdot n = \frac{(2 \cdot 3 \cdot 4 \cdots n) \cdot n}{4 \cdot 5 \cdot 6 \cdots (n+3)} = \frac{6n}{(n+1)(n+2)(n+3)} = \frac{-3}{n+1} + \frac{12}{n+2} + \frac{-9}{n+3}.$$

The last decomposition was by standard partial fractions techniques from calculus (write the fraction as $A/(n+1) + B/(n+2) + C/(n+3)$ and solve for A, B, and C by multiplying it out and equating like powers of n with the origiual fraction). Now we can give a closed form for $\sum_{i=1}^{n} Q(i)i$, since almost all the terms cancel out in a telescoping manner:

$$\begin{aligned}\sum_{i=1}^{n} Q(i)i &= \sum_{i=1}^{n} \frac{-3}{i+1} + \frac{12}{i+2} + \frac{-9}{i+3} \\ &= -\frac{3}{2} + \frac{12}{3} - \frac{9}{4} - \frac{3}{3} + \frac{12}{4} - \frac{9}{5} - \frac{3}{4} + \frac{12}{5} - \frac{9}{6} - \frac{3}{5} + \frac{12}{6} - \frac{9}{7} + \cdots \\ &\quad - \frac{3}{n-1} + \frac{12}{n} - \frac{9}{n+1} - \frac{3}{n} + \frac{12}{n+1} - \frac{9}{n+2} - \frac{3}{n+1} + \frac{12}{n+2} - \frac{9}{n+3} \\ &= -\frac{3}{2} + \frac{12}{3} - \frac{3}{3} - \frac{9}{n+2} + \frac{12}{n+2} - \frac{9}{n+3} = \frac{3}{2} - \frac{6n+9}{(n+2)(n+3)}\end{aligned}$$

This plus 1 gives us the numerator for a_n, according to the formula given in Exercise 48. For the denominator, we need

$$g(n+1)Q(n+1) = \frac{(n+4) \cdot 2 \cdot 3 \cdots (n+1)}{4 \cdot 5 \cdots (n+4)} = \frac{6}{(n+2)(n+3)}.$$

Putting this all together algebraically, we obtain $a_n = (5n^2 + 13n + 12)/12$. We can (and should!) check that this conforms to the recurrence when we calculate a_1, a_2, and so on. Indeed, we get $a_1 = 5/2$ and $a_2 = 29/6$ both ways. It is interesting to note that asking *Maple* to do this with the command

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rsolve({a(n) = ((n + 3) * a(n - 1) + n)/(n + 1), a(0) = 1}, a(n));
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produces the correct answer.

51. A proof of this theorem can be fonnd in textbooks such as *Discrete Mathematics with Applications* by H. E. Mattson, Jr. (Wiley, 1993), Chapter 11.

53. We follow the hint, letting $n = 2^k$ and $a_k = \log T(n) = \log T(2^k)$. We take the log (base 2, of course) of both sides of the given recurrence relation and use the properties of logarithms to obtain

$$\log T(n) = \log n + 2 \log T(n/2),$$

so we have

$$\log T(2^k) = k + 2 \log T(2^{k-1})$$

or

$$a_k = k + 2a_{k-1}.$$

The initial condition becomes $a_0 = \log 6$. Using the techniques in this section, we find that the general solution of the recurrence relation is $a_k = c \cdot 2^k - k - 2$. Plugging in the initial condition leads to $c = 2 + \log 6$. Now we have to translate this back into terms involving T. Since $T(n) = 2^{a_k}$ and $n = 2^k$ we have

$$T(n) = 2^{(2+\log 6) \cdot 2^k - k - 2} = (2^{\log 6})^{2^k} (2^{2 \cdot 2^k - 2})(2^{-k}) = \frac{6^n \cdot 4^{n-1}}{n}.$$