## SECTION 8.3 Representing Relations

*Matrices and directed graphs provide useful ways for computers and humans to represent relations and manipulate them. Become familiar with working with these representations and the operations on them (especially the matrix operation for forming composition) by working these exercises. Some of these exercises explore how properties of a relation can be found from these representations.*

**1.** In each case we use a $3 \times 3$ matrix, putting a 1 in position $(i,j)$ if the pair $(i,j)$ is in the relation and a 0 in position $(i,j)$ if the pair $(i,j)$ is not in the relation. For instance, in part **(a)** there are 1's in the first row, since each of the pairs $(1,1)$, $(1,2)$, and $(1,3)$ are in the relation, and there are 0's elsewhere.

**a)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

**3. a)** Since the $(1,1)^{\text{th}}$ entry is a $1$, $(1,1)$ is in the relation. Since $(1,2)^{\text{th}}$ entry is a $0$, $(1,2)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,3)$, $(2,2)$, $(3,1)$, and $(3,3)$. **b)** $(1,2)$, $(2,2)$, and $(3,2)$ **c)** $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(3,3)$

**5.** An irreflexive relation (see the preamble to Exercise 9 in Section 8.1) is one in which no element is related to itself. In the matrix, this means that there are no 1's on the main diagonal (position $m_{ii}$ for some $i$). Equivalently, the relation is irreflexive if and only if every entry on the main diagonal of the matrix is $0$.

**7.** For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the maiu diagonal); and for transitivity we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

a) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,3)$ and $(3,1)$. Finally, the Boolean square of this matrix is itself, so the relation is transitive.

b) Since there are both 0's and 1's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is not symmetric, the relation is not symmetric (look at positions $(1,2)$ and $(2,1)$, for example). The relation is antisymmetric since there are never two 1's symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is itself, so the relation is transitive.

c) Since there are both 0's and 1's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,3)$ and $(3,1)$, for example. Finally, the Boolean square of this matrix is the matrix with all 1's, so the relation is not transitive ( $1$ is related to $2$, and $2$ is related to $1$, but $2$ is not related to $2$).

**9.** Note that the total number of entries in the matrix is $100^2 = 10{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding $100$, namely in position $(a,b)$ where $a > b$. Thus the answer is the number of subsets of size 2 from a set of 100 elements, i.e., $C(100,2) = 4950$.

b) There is a 1 in the matrix at each position except the 100 positions on the main diagonal. Therefore the answer is $100^2 - 100 = 9900$.

c) There is a 1 in the matrix at each entry just below the main diagonal (i.e., in positions $(2,1)$, $(3,2)$, $\ldots$, $(100,99)$. Therefore the answer is $99$.

d) The entire first row of this matrix corresponds to $a = 1$. Therefore the matrix has 100 nonzero entries.

e) This relation has only the one element $(1,1)$ in it, so the matrix has just one nonzero entry.

**11.** Since the relation $\overline{R}$ is the relation that contains the pair $(a,b)$ (where $a$ and $b$ are elements of the appropriate sets) if and only if $R$ does not contain that pair, we can form the matrix for $\overline{R}$ simply by changing all the 1's to 0's and 0's to 1's in the matrix for $R$.

**13.** Exercise 12 tells us how to do part (a) (we take the transpose of the given matrix $\mathbf{M}_R$, which in this case happens to be the matrix itself). Exercise 11 tells us how to do part (b) (we change 1's to 0's and 0's to 1's in $\mathbf{M}_R$). For part (c) we take the Boolean product of $\mathbf{M}_R$ with itself.

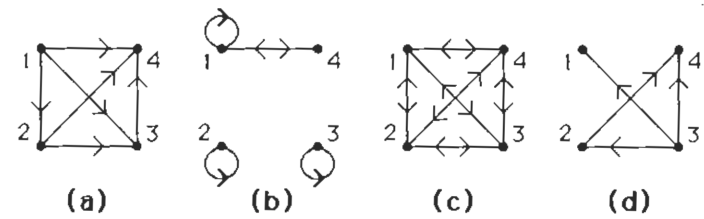
a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**15.** We compute the Boolean powers of $\mathbf{M}_R$; thus $\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R$, $\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \mathbf{M}_R \odot \mathbf{M}_R^{[2]}$, and $\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \mathbf{M}_R \odot \mathbf{M}_R^{[3]}$.

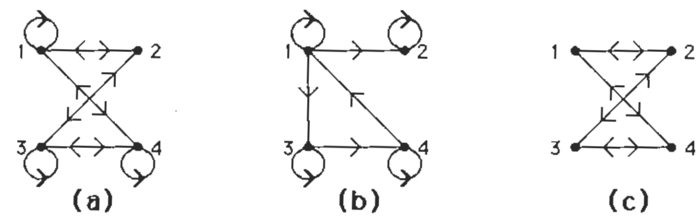
a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

**17.** The matrix for the complement has a 1 wherever the matrix for the relation has a 0, and vice versa. Therefore the number of nonzero entries in $\mathbf{M}_{\overline{R}}$ is $n^2 - k$, since these matrices have $n$ rows and $n$ columns.

**19.** In each case we need a vertex for each of the elements, and we put in a directed edge from $x$ to $y$ if there is a 1 in position $(x,y)$ of the matrix. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.




**21.** In each case we need a vertex for each of the elements, and we put in a directed edge from $x$ to $y$ if there is a 1 in position $(x, y)$ of the matrix. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.




**23.** We list all the pairs $(x, y)$ for which there is an edge from $x$ to $y$ in the directed graph: $\{(a, b), (a, c), (b, c), (c, b)\}$.

**25.** We list all the pairs $(x, y)$ for which there is an edge from $x$ to $y$ in the directed graph: $\{(a, c), (b, a), (c, d), (d, b)\}$.

**27.** We list all the pairs $(x, y)$ for which there is an edge from $x$ to $y$ in the directed graph: $\{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (d, d)\}$.

**29.** An asymmetric relation is one for which it never happens that $a$ is related to $b$ and simultaneously $b$ is related to $a$, even when $a = b$. In terms of the directed graph, this means that we must see no loops and no closed paths of length 2 (i.e., no pairs of edges between two vertices going in opposite directions).

**31.** Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in **antiparallel** pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; and transitive if all paths of length 2 (a pair of edges $(x, y)$ and $(y, z)$) are accompanied by the corresponding path of length 1 (the edge $(x, z)$). The relation drawn in Exercise 23 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge $(a, b)$ is present but not the edge $(b, a)$. It is not antisymmetric, since both edges $(b, c)$ and $(c, b)$ are present. It is not transitive, since the path $(b, c), (c, b)$ from $b$ to $b$ is not accompanied by the edge $(b, b)$. The relation drawn in Exercise 24 is reflexive and not irreflexive since there is a loop at each vertex. It is not symmetric, since, for instance, the edge $(b, a)$ is present but not the edge $(a, b)$. It is antisymmetric, since there are no pairs of antiparallel edges. It is transitive, since the only nontrivial path of length 2 is $bac$, and the edge $(b, c)$ is present. The relation drawn in Exercise 25 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge $(b, a)$ is present but not the edge $(a, b)$. It is antisymmetric, since there are no pairs of antiparallel edges. It is not transitive, since the edges $(a, c)$ and $(c, d)$ are present, but not $(a, d)$.

**33.** Since the inverse relation consists of all pairs $(b, a)$ for which $(a, b)$ is in the original relation, we just have to take the digraph for $R$ and reverse the direction on every edge.

**35.** We prove this statement by induction on $n$. The basis step $n = 1$ is tautologically true, since $\mathbf{M}_R^{[1]} = \mathbf{M}_R$. Assume the inductive hypothesis that $\mathbf{M}_R^{[n]}$ is the matrix representing $R^n$. Now $\mathbf{M}_R^{[n+1]} = \mathbf{M}_R \odot \mathbf{M}_R^{[n]}$. By the inductive hypothesis and the assertion made before Example 5, that $\mathbf{M}_{S\circ R} = \mathbf{M}_R \odot \mathbf{M}_S$, the right-hand side is the matrix representing $R^n \circ R$. But $R^n \circ R = R^{n+1}$, so our proof is complete.