SUPPLEMENTARY EXERCISES FOR CHAPTER 4

2. The proposition is true for $n = 1$, since $1^3 + 3^3 = 28 = 1(1+1)^2(2\cdot 1^2 + 4\cdot 1 + 1)$. Assume the inductive hypothesis. Then

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n+1)^3 + (2n+3)^3 &= (n+1)^2(2n^2+4n+1) + (2n+3)^3 \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 36n^2 + 54n + 27 \\
&= 2n^4 + 16n^3 + 47n^2 + 60n + 28 \\
&= (n+2)^2(2n^2+8n+7) \\
&= (n+2)^2(2(n+1)^2 + 4(n+1) + 1).
\end{aligned}$$

4. Our proof is by induction, it being trivial for $n = 1$, since $1/3 = 1/3$. Under the inductive hypothesis

$$\begin{aligned}
\frac{1}{1\cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{1}{2n+1}\left(n + \frac{1}{2n+3}\right) \\
&= \frac{1}{2n+1}\left(\frac{2n^2+3n+1}{2n+3}\right) \\
&= \frac{1}{2n+1}\left(\frac{(2n+1)(n+1)}{2n+3}\right) = \frac{n+1}{2n+3},
\end{aligned}$$

as desired.

6. We prove this statement by induction. The base case is $n = 5$, and indeed $5^2 + 5 = 30 < 32 = 2^5$. Assuming the inductive hypothesis, we have $(n+1)^2 + (n+1) = n^2 + 3n + 2 < n^2 + 4n < n^2 + n^2 = 2n^2 < 2(n^2+n)$, which is less than $2\cdot 2^n$ by the inductive hypothesis, and this equals 2^{n+1}, as desired.

8. We can let $N = 16$. We prove that $n^4 < 2^n$ for all $n > N$. The base case is $n = 17$, when $17^4 = 83521 < 131072 = 2^{17}$. Assuming the inductive hypothesis, we have $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 < n^4 + 4n^3 + 6n^3 + 4n^3 + 2n^3 = n^4 + 16n^3 < n^4 + n^4 = 2n^4$, which is less than $2\cdot 2^n$ by the inductive hypothesis, and this equals 2^{n+1}, as desired.

10. If $n = 0$ (base case), then the expression equals $0 + 1 + 8 = 9$, which is divisible by 9. Assume that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. We must show that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is also divisible by 9. The difference of these two expressions is $(n+3)^3 - n^3 = 9n^2 + 27n + 27 = 9(n^2+3n+3)$, a multiple of 9. Therefore since the first expression is divisible by 9, so is the second.

12. The two parts are nearly identical, so we do only part **(a)**. Part **(b)** is proved in the same way, substituting multiplication for addition throughout. The basis step is the tautology that if $a_1 \equiv b_1 \pmod{m}$, then $a_1 \equiv b_1 \pmod{m}$. Assume the inductive hypothesis. This tells us that $\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \pmod{m}$. Combining this fact with the fact that $a_{n+1} \equiv b_{n+1} \pmod{m}$, we obtain the desired congruence, $\sum_{j=1}^{n+1} a_j \equiv \sum_{j=1}^{n+1} b_j \pmod{m}$ from Theorem 5 in Section 3.4.

14. After some computation we conjecture that $n + 6 < (n^2 - 8n)/16$ for all $n \geq 28$. (We find that it is not true for smaller values of n.) For the basis step we have $28 + 6 = 34$ and $(28^2 - 8\cdot 28)/16 = 35$, so the statement is true. Assume that the statement is true for $n = k$. Then since $k > 27$ we have

$$\begin{aligned}
\frac{(k+1)^2 - 8(k+1)}{16} = \frac{k^2-8k}{16} + \frac{2k-7}{16} &> k + 6 + \frac{2k-7}{16} \quad \text{by the inductive hypothesis} \\
&> k + 6 + \frac{2\cdot 27 - 7}{16} > k + 6 + 2.9 > (k+1) + 6,
\end{aligned}$$

as desired.

16. When $n = 1$, we are looking for the derivative of $g(x) = e^{cx}$, which is ce^{cx} by the chain rule, so the statement is true for $n = 1$. Assume that the statement is true for $n = k$, that is, the kth derivative is given by $g^{(k)} = c^k e^{cx}$. Differentiating by the chain rule again (and remembering that c^k is constant) gives us the $(k+1)$st derivative: $g^{(k+1)} = c \cdot c^k e^{cx} = c^{k+1}e^{cx}$, as desired.

18. We look at the first few Fibonacci numbers to see if there is a pattern (all congruences are modulo 3): $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3 \equiv 0$, $f_5 = 5 \equiv 2$, $f_6 = 8 \equiv 2$, $f_7 = 13 \equiv 1$, $f_8 = 21 \equiv 0$, $f_9 = 34 \equiv 1$. We may not see a pattern yet, but note that f_8 and f_9 are the same, modulo 3, as f_0 and f_1. Therefore the sequence must continue to repeat from this point, since the recursive definition gives f_n just in terms of n_{n-1} and f_{n-2}. In particular, $f_{10} \equiv f_2 = 1$, $f_{11} \equiv f_3 = 2$, and so on. Since the pattern has period 8, we can formulate our conjecture as follows:

$$f_n \equiv 0 \pmod 3 \text{ if } n \equiv 0 \text{ or } 4 \pmod 8$$
$$f_n \equiv 1 \pmod 3 \text{ if } n \equiv 1,\ 2, \text{ or } 7 \pmod 8$$
$$f_n \equiv 2 \pmod 3 \text{ if } n \equiv 3,\ 5, \text{ or } 6 \pmod 8$$

To prove this by mathematical induction is tedious. There are two base cases, $n = 0$ and $n = 1$. The conjecture is certainly true in each of them, since $0 \equiv 0 \pmod 8$ and $f_0 \equiv 0 \pmod 3$, and $1 \equiv 1 \pmod 8$ and $f_0 \equiv 1 \pmod 3$. So we assume the inductive hypothesis and consider a given $n+1$. There are eight cases to consider, depending on the value of $(n+1)$ **mod** 8. We will carry out one of them; the other seven cases are similar. If $n+1 \equiv 5 \pmod 8$, for example, then $n-1$ and n are congruent to 3 and 4 modulo 8, respectively. By the inductive hypothesis, $f_{n-1} \equiv 2 \pmod 3$ and $f_n \equiv 0 \pmod 3$. Therefore f_{n+1}, which is the sum of these two numbers, is equivalent to $2+0$, or 2, modulo 3, as desired.

20. There are two base cases: for $n = 0$ we have $f_0 + f_2 = 0 + 1 = 1 = l_1$, and $f_1 + f_3 = 1 + 2 = 3 = l_2$, as desired. Assume the inductive hypothesis, that $f_k + f_{k+2} = l_{k+1}$ for all $k \le n$ (we are using strong induction here). Then $f_{n+1} + f_{n+3} = f_n + f_{n-1} + f_{n+2} + f_{n+1} = (f_n + f_{n+2}) + (f_{n-1} + f_{n+1}) = l_{n+1} + l_n$ by the inductive hypothesis (with $k = n$ and $k = n-1$). This last expression equals $l_{n+2} = l_{(n+1)+1}$, however, by the definition of the Lucas numbers, as desired.

22. We follow the hint. Starting with the trivial identity

$$\frac{m+n-1}{n} = \frac{m-1}{n} + 1$$

and multiplying both sides by

$$\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$$

we obtain the identity given in the hint:

$$\frac{m(m+1)\cdots(m+n-1)}{n!} = \frac{(m-1)m(m+1)\cdots(m+n-2)}{n!} + \frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$$

Now we want to show that the product of any n consecutive positive integers is divisible by $n!$. We prove this by induction on n. The case $n = 1$ is clear, since every integer is divisible by $1!$. Assume the inductive hypothesis, that the statement is true for $n-1$. To prove the statement for n, now, we will give a proof using induction on the starting point of the sequence of n consecutive positive integers. Call this starting point m. The basis step, $m = 1$, is again clear, since the product of the first n positive integers *is* $n!$. Assume the inductive hypothesis that the statement is true for $m-1$. Note that we have two inductive hypotheses active here: the statement is true for $n-1$, and the statement is true also for $m-1$ and n. We are trying to prove the statement true for m and n. At this point we simply stare at the identity given above. The first term on the right-hand side is an integer by the inductive hypothesis about $m-1$ and n. The second term on the right-hand side is an integer by the inductive hypothesis about $n-1$. Therefore the expression is an integer. But the statement that the left-hand side is an integer is precisely what we wanted—that the product of the n positive integers starting with m is divisible by $n!$.

24. The algebra gets very messy here, but the ideas are not advanced. We will use the following standard trigonometric identity, which is proved using the standard formulae for the sine and cosine of sums and differences:

$$\cos A \sin B = \frac{\sin(A+B) - \sin(A-B)}{2}$$

The proof of the identity in this exercise is by induction, of course. The basis step ($n=1$) is the true statement that

$$\cos x = \frac{\cos x \sin(x/2)}{\sin(x/2)}.$$

Assume the inductive hypothesis:

$$\sum_{j=1}^{n} \cos jx = \frac{\cos((n+1)x/2)\sin(nx/2)}{\sin(x/2)}$$

Now it is clear that the inductive step is equivalent to showing that adding the $(n+1)^{\text{th}}$ term in the sum to the expression on the right-hand side of the last displayed equation yields the same expression with $n+1$ substituted for n. In other words, we must show that

$$\cos(n+1)x + \frac{\cos((n+1)x/2)\sin(nx/2)}{\sin(x/2)} = \frac{\cos((n+2)x/2)\sin((n+1)x/2)}{\sin(x/2)},$$

which can be rewritten without fractions as

$$\sin(x/2)\cos(n+1)x + \cos((n+1)x/2)\sin(nx/2) = \cos((n+2)x/2)\sin((n+1)x/2).$$

But this follows after a little calculation using the trigonometric identity displayed at the beginning of this solution, since both sides equal

$$\frac{\sin((2n+3)x/2) - \sin(x/2)}{2}.$$

26. We compute a few terms to get a feel for what is going on: $x_1 = \sqrt{6} \approx 2.45$, $x_2 = \sqrt{\sqrt{6}+6} \approx 2.91$, $x_3 \approx 2.98$, and so on. The values seem to be approaching 3 from below in an increasing manner.

a) Clearly $x_0 < x_1$. Assume that $x_{k-1} < x_k$. Then $x_k = \sqrt{x_{k-1}+6} < \sqrt{x_k+6} = x_{k+1}$, and the inductive step is proved.

b) Since $\sqrt{6} < \sqrt{9} = 3$, the basis step is proved. Assume that $x_k < 3$. Then $x_{k+1} = \sqrt{x_k+6} < \sqrt{3+6} = 3$, and the inductive step is proved.

c) By a result from mathematical analysis, an increasing bounded sequence converges to a limit. If we call this limit L, then we must have $L = \sqrt{L+6}$, by letting $n \to \infty$ in the defining equation. Solving this equation for L yields $L = 3$. (The root $L = -2$ is extraneous, since L is positive.)

28. We first prove that such an expression exists. The basis step will handle all $n < b$. These cases are clear, because we can take $k = 0$ and $a_0 = n$. Assume the inductive hypothesis, that we can express all nonnegative integers less than n in this way, and consider an arbitrary $n \geq b$. By the Division Algorithm (Theorem 2 in Section 3.4), we can write n as $q \cdot b + r$, where $0 \leq r < b$. By the inductive hypothesis, we can write q as $a_k b^k + a_{k-1}b^{k-1} + \cdots + a_1 b + a_0$. This means that $n = (a_k b^k + a_{k-1}b^{k-1} + \cdots + a_1 b + a_0) \cdot b + r = a_k b^{k+1} + a_{k-1}b^k + \cdots + a_1 b^2 + a_0 b + r$, and this is in the desired form.

For uniqueness, suppose that $a_kb^k + a_{k-1}b^{k-1} + \cdots + a_1b + a_0 = c_kb^k + c_{k-1}b^{k-1} + \cdots + c_1b + c_0$, where we have added initial terms with zero coefficients if necessary so that each side has the same number of terms; thus we have $0 \le a_i < b$ and $0 \le c_i < b$ for all i. Subtracting the second expansion from both sides gives us $(a_k - c_k)b^k + (a_{k-1} - c_{k-1})b^{k-1} + \cdots + (a_1 - c_1)b + (a_0 - c_0) = 0$. If the two expressions are different, then there is a smallest integer j such that $a_j \ne c_j$; that means that $a_i = c_i$ for $i = 0, 1, \ldots, j-1$. Hence

$$b^j\big((a_k - c_k)b^{k-j} + (a_{k-1} - c_{k-1})b^{k-j-1} + \cdots + (a_{j+1} - c_{j+1})b + (a_j - c_j)\big) = 0,$$

so

$$(a_k - c_k)b^{k-j} + (a_{k-1} - c_{k-1})b^{k-j-1} + \cdots + (a_{j+1} - c_{j+1})b + (a_j - c_j) = 0.$$

Solving for $a_j - c_j$ we have

$$\begin{aligned} a_j - c_j &= (c_k - a_k)b^{k-j} + (c_{k-1} - a_{k-1})b^{k-j-1} + \cdots + (c_{j+1} - a_{j+1})b \\ &= b\big((c_k - a_k)b^{k-j-1} + (c_{k-1} - a_{k-1})b^{k-j-2} + \cdots + (c_{j+1} - a_{j+1})\big). \end{aligned}$$

But this means that b divides $a_j - c_j$. Because both a_j and c_j are between 0 and $b-1$, inclusive, this is possible only if $a_j = b_j$, a contradiction. Thus the expression is unique.

30. For simplicity we will suppress the arguments ("(x)")and just write f' for the derivative of f. We also assume, of course, that denominators are not zero. If $n = 1$ there is nothing to prove, and the $n = 2$ case is just an application of the product rule:

$$\frac{(f_1f_2)'}{f_1f_2} = \frac{f_1'f_2 + f_1f_2'}{f_1f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

Assume the inductive hypothesis and consider the situation for $n+1$:

$$\begin{aligned} \frac{(f_1f_2\cdots f_nf_{n+1})'}{f_1f_2\cdots f_nf_{n+1}} &= \frac{(f_1f_2\cdots f_n)'f_{n+1} + (f_1f_2\cdots f_n)f_{n+1}'}{(f_1f_2\cdots f_n)f_{n+1}} \\ &= \frac{(f_1f_2\cdots f_n)'}{(f_1f_2\cdots f_n)} + \frac{f_{n+1}'}{f_{n+1}} \\ &= \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n} + \frac{f_{n+1}'}{f_{n+1}}. \end{aligned}$$

The first line followed from the product rule, the second line was algebra, and the third line followed from the inductive hypothesis.

32. Call a coloring proper if no two regions that have an edge in common have a common color. For the basis step we can produce a proper coloring if there is only one line by coloring the half of the plane on one side of the line red and the other half blue. Assume that a proper coloring is possible with k lines. If we have $k+1$ lines, remove one of the lines, properly color the configuration produced by the remaining lines, and then put the last line back. Reverse all the colors on one side of the last line. The resulting coloring will be proper.

34. It will be convenient to clear fractions by multiplying both sides by the product of all the x_s's; this makes the desired inequality

$$(x_1^2 + 1)(x_2^2 + 1)\cdots(x_n^2 + 1) \ge (x_1x_2 + 1)(x_2x_3 + 1)\cdots(x_{n-1}x_n + 1)(x_nx_1 + 1).$$

The basis step is

$$(x_1^2 + 1)(x_2^2 + 1) \ge (x_1x_2 + 1)(x_2x_1 + 1).$$

which after algebraic simplification and factoring becomes $(x_1 - x_2)^2 \ge 0$ and therefore is correct. For the inductive step, we assume that the inequality is true for n and hope to prove

$$(x_1^2 + 1)(x_2^2 + 1)\cdots(x_n^2 + 1)(x_{n+1}^2 + 1) \ge (x_1x_2 + 1)(x_2x_3 + 1)\cdots(x_{n-1}x_n + 1)(x_nx_{n+1} + 1)(x_{n+1}x_1 + 1).$$

Because of the cyclic form of this inequality, we can without loss of generality assume that x_{n+1} is the largest (or tied for the largest) of all the given numbers. By the inductive hypothesis we have

$$(x_1^2+1)(x_2^2+1)\cdots(x_n^2+1)(x_{n+1}^2+1) \ge (x_1x_2+1)(x_2x_3+1)\cdots(x_{n-1}x_n+1)(x_nx_1+1)(x_{n+1}^2+1),$$

so it suffices to show that

$$(x_nx_1+1)(x_{n+1}^2+1) \ge (x_nx_{n+1}+1)(x_{n+1}x_1+1).$$

But after algebraic simplification and factoring, this becomes $(x_{n+1}-x_1)(x_{n+1}-x_n) \ge 0$, which is true by our assumption that x_{n+1} is the largest number in the list.

36. (It will be helpful for the reader to draw a diagram to help in following this proof.) We use induction on n, the number of cities, the result being trivial if $n=1$ or $n=2$. Assume the inductive hypothesis and suppose that we have a country with $k+1$ cities, labeled c_1 through c_{k+1}. Remove c_{k+1} and apply the inductive hypothesis to find a city c that can be reached either directly or with one intermediate stop from each of the other cities among c_1 through c_k. If the one-way road leads from c_{k+1} to c, then we are done, so we can assume that the road leads from c to c_{k+1}. If there are any one-way roads from c_{k+1} to a city with a one-way road to c, then we are also done, so we can assume that each road between c_{k+1} and a city with a one-way road to c leads from such a city to c_{k+1}. Thus c and all the cities with a one-way road to c have a direct road to c_{k+1}. All the remaining cities must have a one-way road from them to a city with a one-way road to c (that was part of the definition of c), and so they have paths of length 2 to c_{k+1}, via some such city. Therefore c_{k+1} satisfies the conditions of the problem, and the proof is complete.

38. We have to assume from the statement of the problem that all the cars get are equally efficient in terms of miles per gallon. We proceed by induction on n, the number of cars in the group. If $n=1$, then the one car has enough fuel to complete the lap. Assume the inductive hypothesis that the statement is true for a group of k cars, and suppose we have a group of $k+1$ cars. It helps to think of the cars as stationary, not moving yet. We claim that at least one car c in the group has enough fuel to reach the next car in the group. If this were not so, then the total amount of fuel in all the cars combined would not cover the full lap (think of each car as traveling as far as it can on its own fuel). So now pretend that the car d just ahead of car c is not present, and instead the fuel in that car is in c's tank. By the inductive hypothesis (we still have the same total amount of fuel), some car in this situation can complete a lap by obtaining fuel from other cars as it travels around the track. Then this same car can complete the lap in the actual situation, because if and when it needs to move from the location of car c to the location of the car d, the amount of fuel it has available without d's fuel that we are pretending c already has will be sufficient for it to reach d, at which time this extra fuel becomes available (because this car made it to c's location and car c has enough fuel to reach d's location).

40. **a)** The basis step is to prove the statement that this algorithm terminates for all fractions of the form $1/q$. Since this fraction is already a unit fraction, there is nothing more to prove.

b) For the inductive step, assume that the algorithm terminates for all proper positive fractions with numerators smaller than p, suppose that we are starting with the proper positive fraction p/q, and suppose that the algorithm selects $1/n$ as the first step in the algorithm. Note that necessarily $n>1$. Therefore we can write $p/q = p'/q' + 1/n$. If $p/q = 1/n$, we are done, so assume that $p/q > 1/n$. By finding a common denominator and subtracting, we see that we can take $p' = np - q$ and $q' = nq$. We claim that $p' < p$, which algebraically is easily seen to be equivalent to $p/q < 1/(n-1)$, and this is true by the choice of n such that $1/n$ is the largest unit fraction not exceeding p/q. Therefore by the inductive hypothesis we can write p'/q' as the sum of distinct unit fractions with increasing denominators, and thereby have written p/q as the sum of unit fractions. The only thing left to check is that $p'/q' < 1/n$, so that the algorithm will not try to choose $1/n$ again for p'/q'. But if this were not the case, then $p/q \ge 2/n$, and combining this with the inequality $p/q < 1/(n-1)$ given above, we would have $2/n < 1/(n-1)$, which would mean that $n=1$, a contradiction.

42. What we really need to show is that the definition "terminates" for every n. It is conceivable that trying to apply the definition gets us into some kind of infinite loop, using the second line; we need to show that this is not the case. We will give a very strange kind of proof by mathematical induction. First, following the hint, we will show that the definition tells us that $M(n) = 91$ for all positive integers $n \leq 101$. We do this by backwards induction, starting with $n = 101$ and going down toward $n = 1$. There are 11 base cases: $n = 101, 100, 99, \ldots, 91$. The first line of the definition tells us immediately that $M(101) = 101 - 10 = 91$. To compute $M(100)$ we have

$$\begin{aligned} M(100) &= M(M(100+11)) = M(M(111)) \\ &= M(111-10) = M(101) = 91\,. \end{aligned}$$

The last equality came from the fact that we had already computed $M(101)$. Similarly,

$$\begin{aligned} M(99) &= M(M(99+11)) = M(M(110)) \\ &= M(110-10) = M(100) = 91\,, \end{aligned}$$

and so on down to

$$\begin{aligned} M(91) &= M(M(91+11)) = M(M(102)) \\ &= M(102-10) = M(92) = 91\,. \end{aligned}$$

In each case the final equality comes from the previously computed value. Now assume the inductive hypothesis, that $M(k) = 91$ for all k from $n+1$ through 101 (i.e., if $n+1 \leq k \leq 101$); we mnst prove that $M(n) = 91$, where n is some fixed positive integer less than 91. To compute $M(n)$, we have

$$M(n) = M(M(n+11)) = M(91) = 91$$

where the next to last equality comes from the fact that $n+11$ is between $n+1$ and 101. Thns we have proved that $M(n) = 91$ for all $n \leq 101$. The first line of the definition takes care of values of n greater than 101, so the entire function is well-defined.

44. We proceed by induction on n. The case $n = 2$ is just the definition of symmetric difference. Assume that the statement is true for $n-1$; we must show that it is true for n. By definition $R_n = R_{n-1} \oplus A_n$. We must show that an element x is in R_n if and only if it belongs to an odd number of the sets $A_1, A_2, \ldots, A_n$. The inductive hypothesis tells us that x is in R_{n-1} if and only if x belongs to an odd number of the sets $A_1, A_2, \ldots, A_{n-1}$. There are four cases. Suppose first that $x \in R_{n-1}$ and $x \in A_n$. Then x belongs to an odd number of the sets $A_1, A_2, \ldots, A_{n-1}$ and therefore belongs to an even number of the sets $A_1, A_2, \ldots, A_n$; thus $x \notin R_n$, which is correct by the definition of $\oplus$. Next suppose that $x \in R_{n-1}$ and $x \notin A_n$. Then x belongs to an odd number of the sets $A_1, A_2, \ldots, A_{n-1}$ and therefore belongs to an odd number of the sets $A_1, A_2, \ldots, A_n$; thus $x \in R_n$, which is again correct by the definition of $\oplus$. For the third case, suppose that $x \notin R_{n-1}$ and $x \in A_n$. Then x belongs to an even number of the sets $A_1, A_2, \ldots, A_{n-1}$ and therefore belongs to an odd number of the sets $A_1, A_2, \ldots, A_n$; thus $x \in R_n$, which is again correct by the definition of $\oplus$. The last case ($x \notin R_{n-1}$ and $x \notin A_n$) is similar.

46. This problem is similar to and uses the result of Exercise 58 in Section 4.1. The lemma we need is that if there are n planes meeting the stated conditions, then adding one more plane, which intersects the original figure in the manner described, results in the addition of $(n^2+n+2)/2$ new regions. The reason for this is that the pattern formed on the new plane by all the lines of intersection of this plane with the planes already present has, by Exercise 58 in Section 4.1, $(n^2+n+2)/2$ regions; and each of these two-dimensional regions separates the three-dimensional region through which it passes into two three-dimensional regions. Therefore the proof by induction of the present exercise reduces to noting that one plane separates space into $(1^3+5\cdot 1+6)/6 = 2$ regions, and verifying the algebraic identity

$$\frac{n^3+5n+6}{6} + \frac{n^2+n+2}{2} = \frac{(n+1)^3+5(n+1)+6}{6}\,.$$

48. a) This set is not well ordered, since the set itself has no least element (the negative integers get smaller and smaller).

b) This set is well ordered—the problem inherent in part (a) is not present here because the entire set has -99 as its least element. Every subset also has a least element.

c) This set is not well ordered. The entire set, for example, has no least element, since the numbers of the form $1/n$ for n a positive integer get smaller and smaller.

d) This set is well ordered. The situation is analogous to part (b).

50. In the preamble to Exercise 48 in Section 3.7 an algorithm was described for writing the greatest common divisor of two positive integers as a linear combination of these two integer (see also Theorem 1 in that section). We can use that algorithm, together with the result of Exercise 49, to solve this problem. For $n = 1$ there is nothing to do, since $a_1 = a_1$, and we already have an algorithm for $n = 2$. For $n > 2$, we can write $\gcd(a_{n-1}, a_n)$ as a linear combination of a_{n-1} and a_n, say as

$$\gcd(a_{n-1}, a_n) = c_{n-1}a_{n-1} + c_n a_n \,.$$

Then we apply the algorithm recursively to the numbers a_1, a_2, ..., a_{n-2}, $\gcd(a_{n-1}, a_n)$. This gives us the following equation:

$$\gcd(a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)) = c_1a_1 + c_2a_2 + \cdots + c_{n-2}a_{n-2} + Q \cdot \gcd(a_{n-1}, a_n)$$

Plugging in from the previous display, we have the desired linear combination:

$$\begin{aligned} \gcd(a_1, a_2, \ldots, a_n) &= \gcd(a_1, a_2, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)) \\ &= c_1a_1 + c_2a_2 + \cdots + c_{n-2}a_{n-2} + Q(c_{n-1}a_{n-1} + c_n a_n) \\ &= c_1a_1 + c_2a_2 + \cdots + c_{n-2}a_{n-2} + Qc_{n-1}a_{n-1} + Qc_n a_n \end{aligned}$$

52. The following definition works. The empty string is in the set, and if x and y are in the set, then so are xy, $1x00$, $00x1$, and $0x1y0$. One way to see this is to think of graphing, for a string in this set, the quantity (number of 0's) $- 2 \cdot$ (number of 1's) as a function of the position in the string. This graph must start and end at the horizontal axis. If it contains another point on the axis, then we can split the string into xy where x and y are both in the set. If the graph stays above the axis, then the string must be of the form $00x1$, and if it stays below the axis, then it must be of the form $1x00$. The only other case is that in which the graph crosses the axis at a 1 in the string, without landing on the axis. In this case, the string must look like $0x1y0$.

54. a) The set contains three strings of length 3, and each of them gives us four more strings of length 6, using the fourth through seventh rules, except that there is a bit of overlap, so that in fact there are only 13 strings in all. The strings are *abc*, *bac*, *acb*, *abcabc*, *ababcc*, *aabcbc*, *abcbac*, *abbacc*, *abacbc*, *bacabc*, *abcacb*, *aacbbc*, and *acbabc*.

b) We prove this by induction on the length of the string. The basis step is vacuously true, since there are no strings in the set of length 0 (and it is trivially true anyway, since 0 is a multiple of 3). Assume the inductive hypothesis that the statement is true for shorter strings, and let y be a string in S. If $y \in S$ by one of the first three rules, then y has length 3. If $y \in S$ by one of the last four rules, then the length of y is equal to 3 plus the length of x. By the inductive hypothesis, the length of x is a multiple of 3, so the length of y is also a multiple of 3.

56. By applying the recursive rules we get the following list: ((())), (()()), ()()(), ()(()), (())().

58. We use induction on the length of the string x of balanced parentheses. If $x = \lambda$, then the statement is true since $0 = 0$. Otherwise $x = (a)$ or $x = ab$, where a and b are shorter balanced strings of parentheses. In the first case, the number of parentheses of each type in x is one more than the corresponding number in a, so by the inductive hypothesis these numbers are equal. In the second case, the number of parentheses of each type in x is the sum of the corresponding numbers in a and b, so again by the inductive hypothesis these numbers are equal.

60. We prove the "only if" part by induction on the length of the balanced string w. If $w = \lambda$, then there is nothing to prove. If $w = (x)$, then we have by the inductive hypothesis that $N(x) = 0$ and that $N(a) \geq 0$ if a is a prefix of x. Then $N(w) = 1 + 0 + (-1) = 0$; and $N(b) \geq 1 \geq 0$ if b is a nonempty prefix of w, since $b = (a$. If $w = xy$, then we have by the inductive hypothesis that $N(x) = N(y) = 0$; and $N(a) \geq 0$ if a is a prefix of x or y. Then $N(w) = 0 + 0 = 0$; and $N(b) \geq 0$ if b is a prefix of w, since either b is a prefix of x or $b = xa$ where a is a prefix of y.

We also prove the "if" part by induction on the length of the string w. Suppose that w satisfies the condition. If $w = \lambda$, then $w \in B$. Otherwise w must begin with a parenthesis, and it must be a left parenthesis, since otherwise the prefix of length 1 would give us $N()) = -1$. Now there are two cases: either $w = ab$, where $N(a) = N(b) = 0$ and $a \neq \lambda \neq b$, or not. If so, then a and b are balanced strings of parentheses by the inductive hypothesis (noting that prefixes of a are prefixes of w, and prefixes of b are a followed by prefixes of w), so w is balanced by the recursive definition of the set of balanced strings. In the other case, $N(u) \geq 1$ for all nonempty prefixes u of w, other than w itself. Thus w must end with a right parenthesis to make $N(w) = 0$. So $w = (x)$, and $N(x) = 0$. Furthermore $N(u) \geq 0$ for every prefix u of x, since if $N(u)$ dipped to -1, then $N((u) = 0$ and we would be in the first case. Therefore by the inductive hypothesis x is balanced, and so by the definition of balanced strings w is balanced, as desired.

62. We copy the definition into an algorithm.

procedure $gcd(a, b:$ nonnegative integers, not both zero$)$
if $a > b$ **then** $gcd(a,b) := gcd(b,a)$
else if $a = 0$ **then** $gcd(a,b) := b$
else if a and b are even **then** $gcd(a,b) := 2 \cdot gcd(a/2, b/2)$
else if a is even and b is odd **then** $gcd(a,b) := gcd(a/2, b)$
else $gcd(a,b) := gcd(a, b-a)$

64. To prove that a recursive program is correct, we need to check that it works correctly for the base case, and that it works correctly for the inductive step under the inductive assumption that it works correctly on its recursive call. To apply this rule of inference to Algorithm 1 in Section 4.4, we reason as follows. The base case is $n = 1$. In that case the **then** clause is executed, and not the **else** clause, and so the procedure gives the correct value, namely 1. Now assume that the procedure works correctly for $n-1$, and we want to show that it gives the correct value for the input n, where $n > 1$. In this case, the **else** clause is executed, and not the **then** clause, so the procedure gives us n times whatever the procedure gives for input $n-1$. By the inductive hypothesis, we know that this latter value is $(n-1)!$. Therefore the procedure gives $n \cdot (n-1)!$, which by definition is equal to $n!$, exactly as we wished.

66. We apply the definition:

$$\begin{aligned}
a(0) &= 0\\
a(1) &= 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1\\
a(2) &= 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1\\
a(3) &= 3 - a(a(2)) = 3 - a(1) = 3 - 1 = 2\\
a(4) &= 4 - a(a(3)) = 4 - a(2) = 4 - 1 = 3\\
a(5) &= 5 - a(a(4)) = 5 - a(3) = 5 - 2 = 3\\
a(6) &= 6 - a(a(5)) = 6 - a(3) = 6 - 2 = 4\\
a(7) &= 7 - a(a(6)) = 7 - a(4) = 7 - 3 = 4\\
a(8) &= 8 - a(a(7)) = 8 - a(4) = 8 - 3 = 5\\
a(9) &= 9 - a(a(8)) = 9 - a(5) = 9 - 3 = 6
\end{aligned}$$

68. We follow the hint. First note that by algebra, $\mu^2 = 1 - \mu$, and that $\mu \approx 0.618$. Therefore we have $(\mu n - \lfloor \mu n \rfloor) + (\mu^2 n - \lfloor \mu^2 n \rfloor) = \mu n - \lfloor \mu n \rfloor + (1 - \mu)n - \lfloor (1 - \mu)n \rfloor = \mu n - \lfloor \mu n \rfloor + n - \mu n - \lfloor n - \mu n \rfloor = \mu n - \lfloor \mu n \rfloor + n - \mu n - n - \lfloor -\mu n \rfloor = -\lfloor \mu n \rfloor - (-\lceil \mu n \rceil) = -\lfloor \mu n \rfloor + \lceil \mu n \rceil = 1$, since μn is irrational and therefore not an integer. (We used here some of the properties of the floor and ceiling function from Table 1 in Section 2.3.) Next, continuing with the hint, suppose that $0 \le \alpha < 1 - \mu$, and consider $\lfloor (1 + \mu)(1 - \alpha) \rfloor + \lfloor \alpha + \mu \rfloor$. The second floor term is 0, since $\alpha < 1 - \mu$. The product $(1 + \mu)(1 - \alpha)$ is greater than $(1 + \mu)\mu = \mu + \mu^2 = 1$ and less than $(1 + 1 - \alpha)(1 - \alpha) < 2 \cdot 1 = 2$, so the whole sum equals 1, as desired. For the other case, suppose that $1 - \mu < \alpha < 1$, and again consider $\lfloor (1 + \mu)(1 - \alpha) \rfloor + \lfloor \alpha + \mu \rfloor$. Here $\alpha + \mu$ is between 1 and 2, and $(1 + \mu)(1 - \alpha) < 1$, so again the sum is 1.

The rest of the proof is pretty messy algebra. Since we already know from Exercise 67 that the function $a(n)$ is well-defined by the recurrence $a(n) = n - a(a(n - 1))$ for all $n \ge 1$ and initial condition $a(0) = 0$, it suffices to prove that $\lfloor (n + 1)\mu \rfloor$ satisfies these equations. It clearly satisfies the second, since $0 < \mu < 1$. Thus we must show that $\lfloor (n + 1)\mu \rfloor = n - \lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor$ for all $n \ge 1$. Let $\alpha = n\mu - \lfloor n\mu \rfloor$; then $0 \le \alpha < 1$, and $\alpha \ne 1 - \mu$, since μ is irrational. First consider $\lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor$. It equals $\lfloor \mu(1 + \mu n - \alpha) \rfloor = \lfloor \mu + \mu^2 n - \alpha\mu \rfloor = \lfloor \mu + 1 - \alpha + \lfloor \mu^2 n \rfloor - \alpha\mu \rfloor$ by the first fact proved above. Since $\lfloor \mu^2 n \rfloor$ is an integer, this equals $\lfloor \mu^2 n \rfloor + \lfloor \mu + 1 - \alpha - \alpha\mu \rfloor = \lfloor \mu^2 n \rfloor + \lfloor (1 + \mu)(1 - \alpha) \rfloor = \mu^2 n - 1 + \alpha + \lfloor (1 + \mu)(1 - \alpha) \rfloor$. Next consider $\lfloor (n + 1)\mu \rfloor$. It equals $\lfloor \mu n + \mu \rfloor = \lfloor \lfloor \mu n \rfloor + \alpha + \mu \rfloor = \lfloor \mu n \rfloor + \lfloor \alpha + \mu \rfloor = \mu n - \alpha + \lfloor \alpha + \mu \rfloor$. Putting these together we have $\lfloor (\lfloor n\mu \rfloor + 1)\mu \rfloor + \lfloor (n + 1)\mu \rfloor - n = \mu^2 n - 1 + \alpha + \lfloor (1 + \mu)(1 - \alpha) \rfloor + \mu n - \alpha + \lfloor \alpha + \mu \rfloor - n = (\mu^2 + \mu - 1)n - 1 + \lfloor (1 + \mu)(1 - \alpha) \rfloor + \lfloor \alpha + \mu \rfloor$, which equals $0 - 1 + 1 = 0$ by the definition of μ and the second fact proved above. This is equivalent to what we wanted.

70. a) We apply the definition:

$$\begin{aligned}
a(0) &= 0\\
a(1) &= 1 - a(a(a(0))) = 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1\\
a(2) &= 2 - a(a(a(1))) = 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1\\
a(3) &= 3 - a(a(a(2))) = 3 - a(a(1)) = 3 - a(1) = 3 - 1 = 2\\
a(4) &= 4 - a(a(a(3))) = 4 - a(a(2)) = 4 - a(1) = 4 - 1 = 3\\
a(5) &= 5 - a(a(a(4))) = 5 - a(a(3)) = 5 - a(2) = 5 - 1 = 4\\
a(6) &= 6 - a(a(a(5))) = 6 - a(a(4)) = 6 - a(3) = 6 - 2 = 4\\
a(7) &= 7 - a(a(a(6))) = 7 - a(a(4)) = 7 - a(3) = 7 - 2 = 5\\
a(8) &= 8 - a(a(a(7))) = 8 - a(a(5)) = 8 - a(4) = 8 - 3 = 5\\
a(9) &= 9 - a(a(a(8))) = 9 - a(a(5)) = 9 - a(4) = 9 - 3 = 6
\end{aligned}$$

b) We apply the definition:

$$\begin{aligned}
a(0) &= 0\\
a(1) &= 1 - a(a(a(a(0)))) = 1 - a(a(a(0))) = 1 - a(a(0)) = 1 - a(0) = 1 - 0 = 1\\
a(2) &= 2 - a(a(a(a(1)))) = 2 - a(a(a(1))) = 2 - a(a(1)) = 2 - a(1) = 2 - 1 = 1\\
a(3) &= 3 - a(a(a(a(2)))) = 3 - a(a(a(1))) = 3 - a(a(1)) = 3 - a(1) = 3 - 1 = 2\\
a(4) &= 4 - a(a(a(a(3)))) = 4 - a(a(a(2))) = 4 - a(a(1)) = 4 - a(1) = 4 - 1 = 3\\
a(5) &= 5 - a(a(a(a(4)))) = 5 - a(a(a(3))) = 5 - a(a(2)) = 5 - a(1) = 5 - 1 = 4\\
a(6) &= 6 - a(a(a(a(5)))) = 6 - a(a(a(4))) = 6 - a(a(3)) = 6 - a(2) = 6 - 1 = 5\\
a(7) &= 7 - a(a(a(a(6)))) = 7 - a(a(a(5))) = 7 - a(a(4)) = 7 - a(3) = 7 - 2 = 5\\
a(8) &= 8 - a(a(a(a(7)))) = 8 - a(a(a(5))) = 8 - a(a(4)) = 8 - a(3) = 8 - 2 = 6\\
a(9) &= 9 - a(a(a(a(8)))) = 9 - a(a(a(6))) = 9 - a(a(5)) = 9 - a(4) = 9 - 3 = 6
\end{aligned}$$

c) We apply the definition:

$$\begin{aligned}
a(1) &= 1\\
a(2) &= 1\\
a(3) &= a(3 - a(2)) + a(3 - a(1)) = a(3 - 1) + a(3 - 1) = a(2) + a(2) = 1 + 1 = 2\\
a(4) &= a(4 - a(3)) + a(4 - a(2)) = a(4 - 2) + a(4 - 1) = a(2) + a(3) = 1 + 2 = 3\\
a(5) &= a(5 - a(4)) + a(5 - a(3)) = a(5 - 3) + a(5 - 2) = a(2) + a(3) = 1 + 2 = 3\\
a(6) &= a(6 - a(5)) + a(6 - a(4)) = a(6 - 3) + a(6 - 3) = a(3) + a(3) = 2 + 2 = 4\\
a(7) &= a(7 - a(6)) + a(7 - a(5)) = a(7 - 4) + a(7 - 3) = a(3) + a(4) = 2 + 3 = 5\\
a(8) &= a(8 - a(7)) + a(8 - a(6)) = a(8 - 5) + a(8 - 4) = a(3) + a(4) = 2 + 3 = 5\\
a(9) &= a(9 - a(8)) + a(9 - a(7)) = a(9 - 5) + a(9 - 5) = a(4) + a(4) = 3 + 3 = 6\\
a(10) &= a(10 - a(9)) + a(10 - a(8)) = a(10 - 6) + a(10 - 5) = a(4) + a(5) = 3 + 3 = 6
\end{aligned}$$

72. The first term a_1 tells how many 1's there are. If $a_1 \geq 2$, then the sequence would not be nondecreasing, since a 1 would follow this 2. Therefore $a_1 = 1$. This tells us that there is one 1, so the next term must be at least 2. By the same reasoning as before, a_2 can't be 3 or larger, so $a_2 = 2$. This tells us that there are two 2's, and they must all come together since the sequence is nondecreasing. So $a_3 = 2$ as well. But now we know that there are two 3's, and of course they must come next. We continue in this way and obtain the first 20 terms:

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8$$