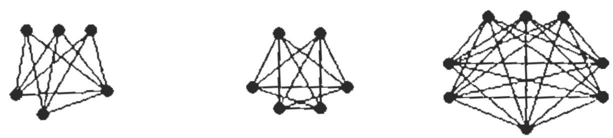
SUPPLEMENTARY EXERCISES FOR CHAPTER 9

2. A graph must be nonempty, so the subgraph can have 1, 2, or 3 vertices. If it has 1 vertex, then it has no edges, so there is clearly just one possibility, K_1. If the subgraph has 2 vertices, then it can have no edges or the one edge joining these two vertices; this gives 2 subgraphs. Finally, if all three vertices are in the subgraph, then the graph can contain no edges, one edge (and we get isomorphic graphs, no matter which edge is used), two edges (ditto), or all three edges. This gives 4 different subgraphs with 3 vertices. Therefore the answer is $1+2+4=7$.

4. Each vertex in the first graph has degree 4. This statement is not true for the second graph. Therefore the graphs cannot be isomorphic. (In fact, the number of edges is different.)

6. We draw these graphs by putting the points in each part close together in clumps, and joining all vertices in different clumps.



(a) (b) (c)

8. a) The statement is true, and we can prove it using the pigeonhole principle. Suppose that the graph has n vertices. The degrees have to be numbers from 0 to $n-1$, inclusive, a total of n possibilities. Now if there is a vertex of degree $n-1$, then it is adjacent to every other vertex, and hence there can be no vertex of degree 0. Thus not all n of the possible degrees can be used. Therefore by the pigeonhole principle, some degree must occur twice.

b) The statement is false for multigraphs. As a simple example, let the multigraph have three vertices a, b, and c. Let there be one edge between a and b, and two edges between b and c. Then it is easy to see that the degrees of the vertices are 1, 3, and 2.

10. Since all the vertices in the subgraph are adjacent in K_n, they are adjacent in the subgraph, i.e., the subgraph is complete.

12. Some staring at the graph convinces us that there are no K_6's. There is one K_5, namely the clique $ceghi$. There are two K_4's not contained in this K_5, which therefore are cliques: $abce$, and $cdeg$. All the K_3's not contained in any of the cliques listed so far are also cliques. We find only aef and efg. All the edges are in at least one of the cliques listed so far (and there are no isolated vertices), so we are done.

14. Since e is adjacent to every other vertex, the (unique) minimum dominating set is $\{e\}$.

16. It is easy to check that the set $\{c, e, j, l\}$ is dominating. We must show that no set with only three vertices is dominating. Suppose that there were such a set. First suppose that the vertex f is to be included. Then at least two more vertices are needed to take care of vertices a and i, unless vertex e is chosen. If vertex e is not chosen, therefore, the dominating set must have more than three vertices, since no pair of vertices covering a and i can cover d, for instance. On the other hand, if e is chosen, then since no single vertex covers c and l, again at least four vertices are required. Thus we may assume that f (and by symmetry g as well) is not in the dominating set with only three elements. This means that we need to find three vertices from the 10-cycle $a, b, c, d, h, l, k, j, i, e, a$ that cover all ten of these vertices. This is impossible, since each vertex covers only three, and $3 \cdot 3 < 10$. Therefore we conclude that there is no dominating set with only three vertices.

18. If G is the graph representing the $n \times n$ chessboard, then a minimum dominating set for G corresponds exactly to a set of squares on which we may place the minimum number of queens to control the board.

20. This isomorphism need not hold. For the simplest counterexample, let G_1, G_2, and H_1 each be the graph consisting of the single vertex v, and let H_2 be the graph consisting of the single vertex w. Then of course G_1 and H_1 are isomorphic, as are G_2 and H_2. But $G_1 \cup G_2$ is a graph with one vertex, and $H_1 \cup H_2$ is a graph with two vertices.

22. Since a 1 in the adjacency matrix indicates the presence of an edge and a 0 the absence of an edge, to obtain the adjacency matrix for $\overline{G}$ we change each 1 in the adjacency matrix for G to a 0, and we change each 0 not on the main diagonal to a 1 (we do not want to introduce loops).

24. a) If no degree is greater than 2, then the graph must consist either of the 5-cycle or a path with no vertices repeated. Therefore there are just two graphs.

b) Certainly every graph besides K_5 that contains K_4 as a subgraph will have chromatic number 4. There are 3 such graphs, since the vertex not in "the" K_4 can be adjacent to one, two or three of the other four vertices. A little further trial and error will convince one that there are no other graphs meeting these conditions, so the answer is 3.

c) Since every proper subgraph of K_5 is planar, there is only one such graph, namely K_5.

26. This follows from the transitivity of the "is isomorphic to" relation and Exercise 65 in Section 9.3. If G is self-converse, then G is isomorphic to G^c. Since H is isomorphic to G, H^c is also isomorphic to G^c. Stringing together these isomorphisms, we see that H is isomorphic to H^c, as desired.

28. This graph is not orientable because of the cut edge $\{c, d\}$, exactly as in Exercise 27.

30. Since we need the city to be strongly connected, we need to find an orientation of the undirected graph representing the city's streets, where the edges represent streets and the vertices represent intersections.

32. There are $C(n, 2) = n(n-1)/2$ edges in a tournament. We must decide how to orient each one, and there are 2 ways to do this for each edge. Therefore the answer is $2^{n(n-1)/2}$. Note that we have not answered the question of how many nonisomorphic tournaments there are—that is much harder.

34. We proceed by induction on n, the number of vertices in the tournament. The base case is $n = 2$, and the single edge is the Hamilton path. Now let G be a tournament with $n+1$ vertices. Delete one vertex, say v, and find (by the inductive hypothesis) a Hamilton path $v_1, v_2, \ldots, v_n$ in the tournament that remains. Now if (v_n, v) is an edge of G, then we have the Hamilton path $v_1, v_2, \ldots, v_n, v$; similarly if (v, v_1) is an edge of G, then we have the Hamilton path $v, v_1, v_2, \ldots, v_n$. Otherwise, there must exist a smallest i such that (v_i, v) and (v, v_{i+1}) are edges of G. We can then splice v into the previous path to obtain the Hamilton path $v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_n$.

36. We follow the hint, arbitrarily pairing the vertices of odd degree and adding an extra edge joining the vertices in each pair. The resulting multigraph has all vertices of even degree, and so it has an Euler circuit. If we delete the new edges, then this circuit is split into k paths. Since no two of the added edges were adjacent, each path is nonempty. The edges and vertices in each of these paths constitute a subgraph, and these subgraphs constitute the desired collection.

38. a) The diameter is clearly 1, since the maximum distance between two vertices is 1. The radius is also 1, with any vertex serving as the center.

b) The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2, with any vertex serving as the center.

c) Vertices at diagonally opposite corners of the cube are a distance 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

d) Vertices at opposite corners of the hexagon are a distance 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

40. Suppose that we follow the given circuit through the multigraph, but instead of using edges more than once, we put in a new parallel edge whenever needed. The result is an Euler circuit through a larger multigraph. If we added new parallel edges in only $m-1$ or fewer places in this process, then we have modified at most $2(m-1)$ vertex degrees. This means that there are at least $2m - 2(m-1) = 2$ vertices of odd degree remaining, which is impossible in a multigraph with an Euler circuit. Therefore we must have added new edges in at least m places, which means the circuit must have used at least m edges more than once.

42. We assume that only simple paths are of interest here. There may be no such path, so no such algorithm is possible. If we want an algorithm that looks for such a path and either finds one or determines that none exists, we can proceed as follows. First we use Dijkstra's algorithm (or some other algorithm) to find a shortest path from a to z (the given vertices). Then for each edge e in that path (one at a time), we delete e from the graph and find a shortest path between a and z in the graph that remains, or determine that no such path exists (again using, say, Dijkstra's algorithm). The second shortest path from a to z is a path of minimum length among all the paths so found, or does not exist if no such paths are found.

44. If we want a shortest path from a to z that passes through m, then clearly we need to find a shortest path from a to m and a shortest path from m to z, and then concatenate them. Each of these paths can be found using Dijkstra's algorithm.

46. a) No two vertices are not adjacent, so the independence number is 1.

b) If n is even, then we can take every other vertex as our independent set, so the independence number is $n/2$. If n is odd, then this does not quite work, but clearly we can take every other vertex except for one vertex. In this case the independence number is $(n-1)/2$. We can state this answer succinctly as $\lfloor n/2 \rfloor$.

c) Since Q_n is a bipartite graph with 2^{n-1} vertices in each part, the independence number is at least 2^{n-1} (take one of the parts as the independent set). We prove that there can be no more than this many independent vertices by induction on n. It is trivial for $n = 1$. Assume the inductive hypothesis, and suppose that there are more than 2^n independent vertices in Q_{n+1}. Recall that Q_{n+1} contains two copies of Q_n in it (with each pair of corresponding points joined by an edge). By the pigeonhole principle, at least one of these Q_n's must contain more than $2^n/2 = 2^{n-1}$ independent vertices. This contradicts the inductive hypothesis. Thus Q_{n+1} has only 2^n independent vertices, as desired.

d) The independence number is clearly the larger of m and n; the independent set to take is the part with this number of vertices.

48. In order to prove this statement it is sufficient to find a coloring with $v-i+1$ colors. We color the graph as follows. Let S be an independent set with i vertices. Color each vertex of S with color $v-i+1$. Color each of the other $v-i$ vertices a different color.

50. a) Obviously adding edges can only help in making the graph connected, so this property is monotone increasing. It is not monotone decreasing, because by removing edges one can disconnect a connected graph.

b) This is dual to part (a); the property is monotone decreasing. To see this, note that removing edges from a nonconnected graph cannot possibly make it connected, while adding edges certainly can.

c) This property is neither monotone increasing nor monotone decreasing. We need to provide examples to verify this. Consider the graph C_4, a square. It has an Euler circuit. However, if we add one edge or remove one edge, then the resulting graph will no longer have an Euler circuit.

d) This property is monotone increasing (since the extra edges do not interfere with the Hamilton circuit already there) but not monotone decreasing (e.g., start with a cycle).

e) This property is monotone decreasing. If a graph can be drawn in the plane, then clearly each of its subgraphs can also be drawn in the plane (just get out your eraser!). The property is not monotone increasing; for example, adding the missing edge to the complete graph on five vertices minus an edge changes the graph from being planar to being nonplanar.

f) This property is neither monotone increasing nor monotone decreasing. It is easy to find examples in which adding edges increases the chromatic number and removing them decreases it (e.g., start with C_5).

g) As in part (f), adding edges can easily decrease the radius and removing them can easily increase it, so this property is neither monotone increasing nor monotone decreasing. For example, C_7 has radius three, but adding enough edges to make K_7 reduces the radius to 1, and removing enough edges to disconnect the graph renders the radius infinite.

h) As in part (g), this is neither monotone increasing nor monotone decreasing.

52. Suppose that G is a graph on n vertices randomly generated using edge probability p, and G' is a graph on n vertices randomly generated using edge probability p', where $p < p'$. Recall that this means that for G we go through all pairs of vertices and independently put an edge between them with probability p; and similarly for G'. We must show that G is no more likely to have property P than G' is. To see this, we will imagine a different way of forming G. First we generate a random graph G' using edge probability p'; then we go through the edges that are present, and independently erase each of them with probability $1 - (p/p')$. Clearly, for an edge to end up in G, it must first get generated and then not get erased, which has probability $p' \cdot (p/p') = p$; therefore this is a valid way to generate G. Now whenever G has property P, then so does G', since P is monotone increasing. Thus the probability that G has property P is no greater than the probability that G' does; in fact it will usually be less, since once a G' having property P is generated, it is possible that it will lose the property as edges are erased.