GUIDE TO REVIEW QUESTIONS FOR CHAPTER 5

1. $1 + 2 + 2 \cdot 2 + 2 \cdot 2 \cdot 2 + \cdots + 2 \cdot 2 \cdots 2 = 2047$

2. Subtract 11 from the answer to the previous review question, since λ, 1, 11, ..., $11\ldots1$ are the bit strings that do not have at least one 0 bit.

3. **a)** See Example 6 in Section 5.1. **b)** 10^5 **c)** See Example 7 in Section 5.1. **d)** $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ **e)** 0

4. with a tree diagram; see Example 20 in Section 5.1 (extended to larger tree)

5. Using the inclusion–exclusion principle, we get $2^7 + 2^7 - 2^4$; see Example 17 in Section 5.1.

6. **a)** See p. 347. **b)** 11 pigeons, 10 holes (digits)

7. **a)** See p. 349. **b)** $N = 91$, $k = 10$

8. **a)** Permutations are ordered arrangements; combinations are unordered (or just arbitrarily ordered for convenience) selections.
b) $P(n,r) = C(n,r) \cdot r!$ (see the proof of Theorem 2 in Section 5.3) **c)** $C(25,6)$ **d)** $P(25,6)$

9. **a)** See pp. 366–367. **b)** by adding the two numbers above each number in the new row

10. A combinatorial proof is a proof of an algebraic identity that shows that both sides count the same thing (in some application). An algebraic proof is totally different—it shows that the two sides are equal by doing formal manipulations with the unknowns, with no reference to what the expressions might mean in an application.

11. See p. 366.

12. a) See p. 363. b) See p. 363. c) $2^{100}5^{101}C(201, 101)$

13. a) See Theorem 2 in Section 5.5. b) $C(5+12-1, 12)$ c) $C(5+9-1, 9)$
d) $C(5+12-1, 12) - C(5+7-1, 7)$ e) $C(5+10-1, 10) - C(5+6-1, 6)$

14. a) See Example 5 in Section 5.5. b) $C(4+17-1, 17)$
c) $C(4+13-1, 13)$ (see Exercise 15a in Section 5.5)

15. a) See Theorem 3 in Section 5.5. b) $14!/(2!2!1!3!1!1!3!1!)$

16. See pp. 382–384.

17. a) $C(52, 5) \cdot C(47, 5) \cdot C(42, 5) \cdot C(37, 5) \cdot C(32, 5) \cdot C(27, 5)$ b) See Theorem 4 in Section 5.5.

18. See p. 385.

SUPPLEMENTARY EXERCISES FOR CHAPTER 5

1. In each part of this problem we have $n = 10$ and $r = 6$.

a) If the items are to be chosen in order with no repetition allowed, then we need a simple permutation. Therefore the answer is $P(10, 6) = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151{,}200$.

b) If repetition is allowed, then this is just a simple application of the product rule, with 6 tasks, each of which can be done in 10 ways. Therefore the answer is $10^6 = 1{,}000{,}000$.

c) If the items are to be chosen without regard to order but with no repetition allowed, then we need a simple combination. Therefore the answer is $C(10, 6) = C(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 \cdot /(4 \cdot 3 \cdot 2) = 210$.

d) Unordered choices with repetition allowed are counted by $C(n+r-1, r)$, which in this case is $C(15, 6) = 5005$.

3. The student has 3 choices for each question: true, false, and no answer. There are 100 questions, so by the product rule there are $3^{100} \approx 5.2 \times 10^{47}$ ways to answer the test.

5. We will apply the inclusion–exclusion principle from Section 5.1. First let us calculate the number of these strings with exactly three a's. To specify such a string we need to choose the positions for the a's, which can be done in $C(10, 3)$ ways. Then we need to choose either a b or a c to fill each of the other 7 positions in the string, which can be done in 2^7 ways. Therefore there are $C(10, 3) \cdot 2^7 = 15360$ such strings. Similarly, there are $C(10, 4) \cdot 2^6 = 13440$ strings with exactly four b's. Next we need to compute the number of strings satisfying both of these conditions. To specify a string with exactly three a's and exactly four b's, we need to choose the positions for the a's, which can be done in $C(10, 3)$ ways, and then choose the positions for the b's, which can be done in $C(7, 4)$ ways (only seven slots remain after the a's are placed). Therefore there are $C(10, 3) \cdot C(7, 4) = 4200$ such strings. Finally, by the inclusion–exclusion principle the number of strings having either exactly three a's or exactly four b's is $15360 + 13440 - 4200 = 24{,}600$.

7. **a)** We want a combination with repetition allowed, with $n = 28$ and $r = 3$. By Theorem 2 of Section 5.5, there are $C(28+3-1,3) = C(30,3) = 4060$ possibilities.

b) This is just a simple application of the product rule. There are 28 ways to choose the ice cream, 8 ways to choose the sauce, and 12 ways to choose the topping, so there are $28 \cdot 8 \cdot 12 = 2688$ possible small sundaes.

c) By the product rule we have to multiply together the number of ways to choose the ice cream, the number of ways to choose the sauce, and the number of ways to choose the topping. There are $C(28+3-1,3)$ ways to choose the ice cream, just as in part **(a)**. There are $C(8,2)$ ways to choose the sauce, since repetition is not allowed. There are similarly $C(12,3)$ ways to choose the toppings. Multiplying these numbers together, we find that the answer is $4060 \cdot 28 \cdot 220 = 25{,}009{,}600$ different large sundaes.

9. We can solve this problem by counting the number of numbers that have the given digit in 1, 2, or 3 places.

a) The digit 0 appears in 1 place in some two-digit numbers and in some three-digit numbers. There are clearly 9 two-digit numbers in which 0 appears, namely $10, 20, \ldots, 90$. We can count the number of three-digit numbers in which 0 appears exactly once as follows: first choose the place in which it is to appear (2 ways, since it cannot be the leading digit), then choose the left-most of the remaining digits (9 ways, since it cannot be a 0), then choose the final digit (also 9 ways). Therefore there are $9 + 2 \cdot 9 \cdot 9 = 171$ numbers in which the 0 appears exactly once, accounting for 171 appearances of the digit 0. Finally there are another 9 numbers in which the digit 0 appears twice, namely $100, 200, \ldots, 900$. This accounts for 18 more 0's. And of course the number 1000 contributes 3 0's. Therefore our final answer is $171 + 18 + 3 = 192$.

b) The analysis for the digit 1 is not the same as for the digit 0, since we can have leading 1's but not leading 0's. One 1 appears in the one-digit numbers. Two-digit numbers can have a 1 in the ones place (and there are 9 of these, namely $11, 21, \ldots, 91$), or in the tens place (and there are 10 of these, namely 10 through 19). Of course the number 11 is counted in both places, but that is proper, since we want to count each appearance of a 1. Therefore there are $10 + 9 = 19$ 1's appearing in two-digit numbers. Similarly, the three-digit numbers have 90 1's appearing in the ones place (every tenth number, and there are 900 numbers), 90 1's in the tens place (10 per decade, and there are 9 decades), and 100 1's in the hundreds place (100 through 199); therefore there are 280 ones appearing in three-digit numbers. Finally there is a 1 in 1000, so the final answer is $1 + 19 + 280 + 1 = 301$.

c) The analysis for the digit 2 is exactly the same as for the digit 1, with the exception that we do not get any 2's in 1000. Therefore the answer is $301 - 1 = 300$.

d) The analysis for the digit 9 is exactly the same as for the digit 2, so the answer is again 300.

Let us check all of the answers to this problem simultaneously. There are 300 each of the digits 2 through 9, for a total of 2400 digits. There are 192 0's and 301 1's. Therefore $2400 + 192 + 301 = 2893$ digits are used altogether. Let us count this another way. There are 9 one-digit numbers, 90 two-digit numbers, 900 three-digit numbers, and 1 four-digit number, so the total number of digits is $9 \cdot 1 + 90 \cdot 2 + 900 \cdot 3 + 1 \cdot 4 = 2893$. This agreement tends to confirm our analysis.

11. This is a negative instance of the generalized pigeonhole principle. The worst case would be if the student gets each fortune 3 times, for a total of $3 \times 213 = 639$ meals. If the student ate 640 or more meals, then the student will get the same fortune at least $\lceil 640/213 \rceil = 4$ times.

13. We have no guarantee ahead of time that this will work, but we will try applying the pigeonhole principle. Let us count the number of different possible sums. If the numbers in the set do not exceed 50, then the largest possible sum of a 5-element subset will be $50 + 49 + 48 + 47 + 46 = 240$. The smallest possible sum will be $1 + 2 + 3 + 4 + 5 = 15$. Therefore the sum has to be a number between 15 and 240, inclusive, and there are $240 - 15 + 1 = 226$ such numbers. Now let us count the number of different subsets. That is of course $C(10,5) = 252$. Since there are more subsets (pigeons) than sums (pigeonholes), we know that there must be two subsets with the same sum.

15. We assume that the drawings of the cards is done without replacement (i.e., no repetition allowed).

a) The worst case would be that we drew 1 ace and the 48 cards that are not aces, a total of 49 cards. Therefore we need to draw 50 cards to guarantee at least 2 aces (and it is clear that 50 is sufficient, since at worst 2 of the 4 aces would theu be left in the deck).

b) The same analysis as in part (a) applies, so again the answer is 50.

c) In this problem we can use the pigeonhole principle. If we drew 13 cards, then they might all be of different kinds (ranks). If we drew 14 cards, however, then since there are only 13 kinds we would be assured of having at least two of the same kind. (The drawn cards are the pigeons and the kinds are the pigeonholes.)

d) If we drew 4 cards, they might all be of the same kind. However, if we draw 5 cards, then since there are only 4 of one kind, we are assured of seeing at least two different kinds.

17. This problem can be solved using the pigeonhole principle if we look at it correctly. Let s_i be the sum of the first i of these numbers, where $1 \le i \le m$. Now if $s_i \equiv 0 \pmod{m}$ for some i, then we have our desired consecutive terms whose sum is divisible by m. Otherwise the numbers s_1 **mod** m, s_2 **mod** m, $\ldots$, s_m **mod** m are all integers in the set $\{1, 2, \ldots, m-1\}$. By the pigeonhole principle we know that two of them are the same, say s_i **mod** $m = s_j$ **mod** m with $i < j$. Then $s_j - s_i$ is divisible by m. But $s_j - s_i$ is just the sum of the $(i+1)^{\text{th}}$ through j^{th} terms in the sequence, and we are done.

19. The decimal expansion of a rational number a/b (we can assume that a and b are positive integers) can be obtained by long division of the number b into the number a, where a is written with a decimal point and an arbitrarily long string of 0's following it. The basic step in long division is finding the next digit of the qnotient, which is just $\lfloor r/b \rfloor$, where r is the current remainder with the next digit of the dividend brought down. Now in our case, eventually the dividend has nothing but 0's to bring down. Furthermore there are only b possible remainders, namely the numbers from 0 to $b-1$. Thus at some point in our calculation after we have passed the decimal point, we will, by the pigeonhole principle, be looking at exactly the same situation as we had previously. From that point onward, then, the calcnlation must follow the same pattern as it did previously. In particular, the digits of the quotient will repeat.

For example, to compute the decimal expansion of the rational number $349/11$, we divide 11 into $349.00\ldots$. The first digit of the quotient is 3, and the remainder is 1. The next digit of the quotient is 1 and the remainder is 8. At this point there are only 0's left to bring down. The next digit of the quotient is a 7 with a remainder of 3, and then a quotient digit of 2 with a remainder of 8. We are now in exactly the same situation as at the previous appearance of a remainder of 8, so the quotient digits 72 repeat forever. Thus $349/11 = 31.\overline{72}$.

21. a) This is a simple combination, so the answer is $C(20, 12) = 125{,}970$.

b) The only choice is the choice of a variety, so the answer is 20.

c) We assume that order does not matter (all the donuts will go into a bag). Therefore, since repetitions are allowed, Theorem 2 of Section 5.5 applies, and the answer is $C(20 + 12 - 1, 12) = C(31, 12) = 141{,}120{,}525$.

d) We can simply subtract from our answer to part (c) our answer to part (b), which asks for the number of ways this restriction can be violated. Therefore the answer is 141,120,505.

e) We put the 6 blueberry filled donuts into our bag, and the problem becomes one of choosing 6 donuts with no restrictions. In analogy with part (c), we obtain the answer $C(20 + 6 - 1, 6) = C(25, 6) = 177{,}100$.

f) There are $C(20 + 5 - 1, 5) = C(24, 5) = 42504$ ways to choose at least 7 blueberry donuts among our dozen (the calculation is essentially the same as that in part (e)). Our answer is therefore 42504 less than our unrestricted answer to part (c): $141120525 - 42504 = 141{,}078{,}021$.

23. a) The given equation is equivalent to $n(n-1)/2 = 45$, which reduces to $n^2 - n - 90 = 0$. The quadratic formula (or factoring) tells us that the roots are $n = 10$ and $n = -9$. Since n is assumed to be nonnegative, the only relevant solution is $n = 10$.

b) The given equation is equivalent to $n(n-1)(n-2)/6 = n(n-1)$. Since $P(n,2)$ is not defined for $n < 2$, we know that neither n nor $n-1$ is 0, so we can divide both sides by these factors, obtaining $n-2=6$, whence $n=8$.

c) Recall the identity $C(n,k) = C(n,n-k)$. The given equation fits that model if $n=7$ and $k=5$. Hence $n=7$ is a solution. That there are no more solutions follows from the fact that $C(n,k)$ is an increasing function in k for $0 \le k \le n/2$, and hence there are no other numbers i and j for which $C(n,i) = C(n,j)$.

25. Following the hint, we see that each element of S falls into exactly one of three categories: either it is an element of A, or else it is not an element of A but is an element of B (in other words, is an element of $B-A$), or else it is not an element of B either (in other words, is an element of $S-B$). So the number of ways to choose sets A and B to satisfy these conditions is the same as the number of ways to place each element of S into one of these three categories. Therefore the answer is 3^n. For example, if $n=2$ and $S=\{x,y\}$, then there are 9 pairs: $(\emptyset,\emptyset)$, $(\emptyset,\{x\})$, $(\emptyset,\{y\})$, $(\emptyset,\{x,y\})$, $(\{x\},\{x\})$, $(\{x\},\{x,y\})$, $(\{y\},\{y\})$, $(\{y\},\{x,y\})$, $(\{x,y\},\{x,y\})$.

27. We start with the right-hand side and use Pascal's Identity three times to obtain the left-hand side:

$$\begin{aligned}
C(n+2,r+1) &- 2C(n+1,r+1) + C(n,r+1) \\
&= C(n+1,r+1) + C(n+1,r) - 2C(n+1,r+1) + C(n,r+1) \\
&= C(n+1,r) - C(n+1,r+1) + C(n,r+1) \\
&= [C(n,r) + C(n,r-1)] - [C(n,r+1) + C(n,r)] + C(n,r+1) \\
&= C(n,r-1)
\end{aligned}$$

29. Substitute $x=1$ and $y=3$ into the Binomial Theorem (Theorem 1 in Section 5.4) and we obtain exactly this identity.

31. The trick to the analysis here is to imagine what such a string has to look like. Every string of 0's and 1's can be thought of as consisting of alternating blocks—a block of 1's (possibly empty) followed by a block of 0's followed by a block of 1's followed by a block of 0's, and so on, ending with a block of 0's (again, possibly empty). If we want there to be exactly two occurrences of 01, then in fact there must be exactly six such blocks, the middle four all being nonempty (the transitions from 0's to 1's create the 01's) and the outer two possibly being empty. In other words, the string must look like this:

$$x_1 \text{ 1's} \;—\; x_2 \text{ 0's} \;—\; x_3 \text{ 1's} \;—\; x_4 \text{ 0's} \;—\; x_5 \text{ 1's} \;—\; x_6 \text{ 0's},$$

where $x_1 + x_2 + \cdots + x_6 = n$ and $x_1 \ge 0$, $x_6 \ge 0$, and $x_i \ge 1$ for $i = 2,3,4,5$. Clearly such a string is totally specified by the values of the x_i's. Therefore we are simply asking for the number of solutions to the equation $x_1 + x_2 + \cdots + x_6 = n$ subject to the stated constraints. This kind of problem is solved in Section 5.5 (Example 5 and several exercises). The stated problem is equivalent to finding the number of solutions to $x_1 + x_2' + x_3' + x_4' + x_5' + x_6 = n-4$ where each variable here is nonnegative (we let $x_i = x_i' + 1$ for $i = 2,3,4,5$ in order to insure that these x_i's are strictly positive). The number of such solutions is, by the results just cited, $C(6+n-4-1, n-4)$, which simplifies to $C(n+1,n-4)$ or $C(n+1,5)$.

33. An answer key is just a permutation of 8 a's, 3 b's, 4 c's, and 5 d's. We know from Theorem 3 in Section 5.5 that there are

$$\frac{20!}{8!3!4!5!} = 3{,}491{,}888{,}400$$

such permutations.

35. We assume that each student is to get one advisor, that there are no other restrictions, and that the students and advisors are to be considered distinct. Then there are 5 ways to assign each student, so by the product rule there are $5^{24} \approx 6.0 \times 10^{16}$ ways to assign all of them.

37. For all parts of this problem, Theorem 2 in Section 5.5 is used.

a) We let $x_1 = x_1' + 2$, $x_2 = x_2' + 3$, and $x_3 = x_3' + 4$. Then the restrictions are equivalent to requiring that each of the x_i''s be nonnegative. Therefore we want the number of nonnegative integer solutions to the equation $x_1' + x_2' + x_3' = 8$. There are $C(3+8-1,8) = C(10,8) = C(10,2) = 45$ of them.

b) The number of solutions with $x_3 > 5$ is the same as the number of solutions to $x_1 + x_2 + x_3' = 11$, where $x_3 = x_3' + 6$. There are $C(3+11-1,11) = C(13,11) = C(13,2) = 78$ of these. Now we want to subtract the number of solutions for which also $x_1 \geq 6$. This is equivalent to the number of solutions to $x_1' + x_2 + x_3' = 5$, where $x_1 = x_1' + 6$. There are $C(3+5-1,5) = C(7,5) = C(7,2) = 21$ of these. Therefore the answer to the problem is $78 - 21 = 57$.

c) Arguing as in part (b), we know that there are 78 solutions to the equation $x_1 + x_2 + x_3' = 11$, which is equivalent to the number of solutions to $x_1 + x_2 + x_3 = 17$ with $x_3 > 5$. We now need to subtract the number of these solutions that violate one or both of the restrictions $x_1 < 4$ and $x_2 < 3$. The number of solntions with $x_1 \geq 4$ is the number of solutions to $x_1' + x_2 + x_3' = 7$, namely $C(3+7-1,7) = C(9,7) = C(9,2) = 36$. The number of solutions with $x_2 \geq 3$ is the nnmber of solutions to $x_1 + x_2' + x_3' = 8$, namely $C(3+8-1,7) = C(10,8) = C(10,2) = 45$. However, there are also solutions in which both restrictions are violated, namely the solutions to $x_1' + x_2' + x_3' = 4$. There are $C(3+4-1,4) = C(6,4) = C(6,2) = 15$ of these. Therefore the number of solutions in which one or both conditions are violated is $36 + 45 - 15 = 66$; we needed to subtract the 15 so as not to count these solutions twice. Putting this all together, we see that there are $78 - 66 = 12$ solutions of the given problem.

39. a) We want to find the number of r-element subsets for $r = 0,1,2,3,4$ and add. Therefore the answer is $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 210 = 386$.

b) This time we want $C(10,8) + C(10,9) + C(10,10) = C(10,2) + C(10,1) + C(10,0) = 45 + 10 + 1 = 56$.

c) This time we want $C(10,1) + C(10,3) + C(10,5) + C(10,7) + C(10,9) = C(10,1) + C(10,3) + C(10,5) + C(10,3) + C(10,1) = 10 + 120 + 252 + 120 + 10 = 512$. We can also solve this problem by using the fact from Exercise 31 in Section 5.4 that a set has the same number of subsets with an even number of elements as it has subsets with an odd number of elements. Since the set has $2^{10} = 1024$ subsets altogether, half of these—512 of them—must have an odd number of elements.

41. Since the objects are identical, all that matters is the number of objects put into each container. If we let x_i be the number of objects put into the i^{th} container, then we are asking for the number of solutions to the equation $x_1 + x_2 + \cdots + x_m = n$ with the restriction that each $x_i \geq 1$. By the usual trick this is equivalent to asking for the number of nonnegative integer solutions to $x_1' + x_2' + \cdots + x_m' = n - m$, where we have set $x_i = x_i' + 1$ to insure that each container gets at least one object. By Theorem 2 in Section 5.5, there are $C(m + (n-m) - 1, n-m) = C(n-1, n-m)$ solutions. This can also be written as $C(n-1, m-1)$, since $(n-1) - (n-m) = m - 1$. (Of course if $n < m$, then there are no solutions, since it would be impossible to put at least one object in each container. Our answer is consistent with this observation if we think of $C(x,y)$ as being 0 if $y > x$.)

43. a) This can be done with the multiplication principle. There are five choices for each ball, so the answer is $5^6 = 15{,}625$.

b) This is like Example 10 in Section 5.5, and we can use the formula on page 378:

$$\sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

with $n = 6$ and $k = 5$. We get

$$\frac{1}{1!}1^6+\frac{1}{2!}(1\cdot2^6-2\cdot1^6)+\frac{1}{3!}(1\cdot3^6-3\cdot2^6+3\cdot1^6)+\frac{1}{4!}(1\cdot4^6-4\cdot3^6+6\cdot2^6-4\cdot1^6)+\frac{1}{5!}(1\cdot5^6-5\cdot4^6+10\cdot3^6-10\cdot2^6+5\cdot1^6),$$

which is $1 + 31 + 90 + 65 + 15 = 202$. The command in *Maple* for this, using the "combinat" package, is `sum(stirling2(6,j),j=1..5);`.

c) We saw in the discussion surrounding Example 9 in Section 5.5 that the number of ways to distribute n unlabeled objects into k labeled boxes is $C(n+k-1, k-1)$, because this is really the same as the problem of choosing an n-combination from the set of k boxes, with repetitions allowed. In this case we have $n = 6$ and $k = 5$, so the answer is $C(10, 4) = 210$.

d) Since both the boxes and the objects are indistinguishable, what is really being asked is how many different ways there are to write 6 as the sum of five nonnegative integers, with order ignored. We will just enumerate the possibilities and count them. We have $6 = 6+0+0+0+0$; $6 = 5+1+0+0+0$; $6 = 4+2+0+0+0$; $6 = 4+1+1+0+0$; $6 = 3+3+0+0+0$; $6 = 3+2+1+0+0$; $6 = 3+1+1+1+0$; $6 = 2+2+2+0+0$; $6 = 2+2+1+1+0$; and $6 = 2+1+1+1+1$. There are ten ways in all. Notice that we are allowing some of the boxes to be empty.

45. For convenience let us assume that the finite set is $\{1, 2, \ldots, n\}$. If we call a permutation $a_1a_2\ldots a_r$, then we simply need to allow each of the variables a_i to take on all n of the values from 1 to n. This is essentially just counting in base n, so our algorithm will be similar to Algorithm 2 in Section 5.6. The procedure shown here generates the next permutation. To get all the permutations, we just start with $11\ldots1$ and call this procedure $r^n - 1$ times.

procedure *next permutation*(n : positive integer, $a_1, a_2, \ldots, a_r$: positive integers $\le n$)
{this procedure replaces the input with the next permutation, repetitions allowed,
in lexicographic order; assume that there is a next permutation, i.e., $a_1a_2\ldots a_r \ne nn\ldots n$}
$i := r$
while $a_i = n$
begin
$a_i := 1$
$i := i - 1$
end
$a_i := a_i + 1$
{$a_1a_2\ldots a_r$ is the next permutation in lexicographic order}

47. We must show that if there are $R(m, n-1) + R(m-1, n)$ people at a party, then there must be at least m mutual friends or n mutual enemies. Consider one person; let's call him Jerry. Then there are $R(m-1, n) + R(m, n-1) - 1$ other people at the party, and by the pigeonhole principle there must be at least $R(m-1, n)$ friends of Jerry or $R(m, n-1)$ enemies of Jerry among these people. First let's suppose there are $R(m-1, n)$ friends of Jerry. By the definition of R, among these people we are guaranteed to find either $m-1$ mutual friends or n mutual enemies. In the former case these $m-1$ mutual friends together with Jerry are a set of m mutual friends; and in the latter case we have the desired set of n mutual enemies. The other situation is similar: Suppose there are $R(m, n-1)$ enemies of Jerry; we are guaranteed to find among them either m mutual friends or $n-1$ mutual enemies. In the former case we have the desired set of m mutual friends, and in the latter case these $n-1$ mutual enemies together with Jerry are a set of n mutual enemies.

WRITING PROJECTS FOR CHAPTER 5

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. You might start with the standard history of mathematics books, such as [Bo4] or [Ev3].

2. To learn about telephone numbers in North America, refer to books on telecommunications, such [Fr]. The term to look for in an index is the North American Numbering Plan.

3. A lot of progress has been made recently by research mathematicians such as Herbert Wilf in finding general methods of proving essentially *all* true combinatorial identities, more or less mechanically. See whether you can find some of this work by looking in *Mathematical Reviews* (MathSciNet on the Web) or the book [PeWi]. There is also some discussion of this in [Wi2], a book on generating functions. Also, a classical book on combinatorial identities is [Ri2].

4. Students who have had an advanced physics course will be at an advantage here. Maybe you have a friend who is a physics major! In any case, it should not be hard to find a fairly elementary textbook on this subject.

5. More advanced combinatorics textbooks usually deal with Stirling numbers, at least in the exercises. See [Ro1], for instance. Other sources here are a chapter in [MiRo] and the amazing [GrKn].

6. See the comments for Writing Project 5.

7. There are entire books devoted to Ramsey theory, dealing not only with the classical Ramsey numbers, but also with applications to number theory, graph theory, geometry, linear algebra, etc. For a fairly advanced such book, see [GrRo]; for a gentler introduction, see the relevant sections of [Ro1] or the chapter in [MiRo]. Up-to-the-minute results can be found with a Web search.

8. Try books with titles such as "combinatorial algorithms"—that's what methods of generating permutations are, after all. See [Ev1] or [ReNi], for example. Another fascinating source (which deals with combinatorial algorithms as well as many other topics relevant to this text) is [GrKn]. Volume 2 of Knuth's classic [Kn] should have some relevant material. There is also an older article you might want to check ont, [Le1]. An interesting related problem is to generate a *random* permutation; this is needed, for example, when using a computer to simulate the shuffling of a deck of cards for playing card games.

9. See the comments for Writing Project 8.