CHAPTER 8
Relations

SECTION 8.1 Relations and Their Properties

This chapter is one of the most important in the book. Many structures in mathematics and computer science are formulated in terms of relations. Not only is the terminology worth learning, but the experience to be gained by working with various relations will prepare the student for the more advanced structures that he or she is sure to encounter in future work.

This section gives the basic terminology, especially the important notions of reflexivity, symmetry, antisymmetry, and transitivity. If we are given a relation as a set of ordered pairs, then reflexivity is easy to check for: we make sure that each element is related to itself. Symmetry is also fairly easy to test for: we make sure that no pair (a,b) is in the relation without its opposite (b,a) being present as well. To check for antisymmetry we make sure that no pair (a,b) with $a \neq b$ and its opposite are both in the relation. In other words, at most one of (a,b) and (b,a) is in the relation if $a \neq b$. Transitivity is much harder to verify, since there are many triples of elements to check. A common mistake to try to avoid is forgetting that a transitive relation that has pairs (a,b) and (b,a) must also include (a,a) and (b,b).

More importantly, we can be given a relation as a rule as to when elements are related. Exercises 4–7 are particularly useful in helping to understand the notions of reflexivity, symmetry, antisymmetry, and transitivity for relations given in this manner. Here you have to ask yourself the appropriate questions in order to determine whether the properties hold. Is every element related to itself? If so, the relation is reflexive. Are the roles of the variables in the definition interchangeable? If so, then the relation is symmetric. Does the definition preclude two different elements from each being related to the other? If so, then the relation is antisymmetric. Does the fact that one element is related to a second, which is in turn related to a third, mean that the first is related to the third? If so, then the relation is transitive.

In general, try to think of a relation in these two ways at the same time: as a set of ordered pairs and as a propositional function describing a relationship among objects.

1. In each case, we need to find all the pairs (a,b) with $a \in A$ and $b \in B$ such that the condition is satisfied. This is straightforward.

a) $\{(0,0),(1,1),(2,2),(3,3)\}$ **b)** $\{(1,3),(2,2),(3,1),(4,0)\}$

c) $\{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$

d) Recall that $a \mid b$ means that b is a multiple of a (a is not allowed to be 0). Thus the answer is $\{(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,0),(3,3),(4,0)\}$.

e) We need to look for pairs whose greatest common divisor is 1—in other words, pairs that are relatively prime. Thus the answer is $\{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,1),(4,3)\}$.

f) There are not very many pairs of numbers (by definition only positive integers are considered) whose least common multiple is 2: only 1 and 2, and 2 and 2. Thus the answer is $\{(1,2),(2,1),(2,2)\}$.

3. **a)** This relation is not reflexive, since it does not include, for instance $(1,1)$. It is not symmetric, since it includes, for instance, $(2,4)$ but not $(4,2)$. It is not antisymmetric since it includes both $(2,3)$ and $(3,2)$, but $2 \neq 3$. It is transitive. To see this we have to check that *whenever* it includes (a,b) and (b,c), then it also includes (a,c). We can ignore the element 1 since it never appears. If (a,b) is in this relation, then by inspection we see that a must be either 2 or 3. But $(2,c)$ and $(3,c)$ are in the relation for all $c \neq 1$; thus (a,c) has to be in this relation whenever (a,b) and (b,c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.

b) This relation is reflexive, since all the pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$ are in it. It is clearly symmetric, the only nontrivial case to note being that both $(1,2)$ and $(2,1)$ are in the relation. It is not antisymmetric because both $(1,2)$ and $(2,1)$ are in the relation. It is transitive; the only nontrivial cases to note are that since both $(1,2)$ and $(2,1)$ are in the relation, we need to have (and do have) both $(1,1)$ and $(2,2)$ included as well.

c) This relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both $(2,4)$ and $(4,2)$ are in the relation. It is not transitive, since although $(2,4)$ and $(4,2)$ are in the relation, $(2,2)$ is not.

d) This relation is clearly not reflexive. It is not symmetric, since, for instance, $(1,2)$ is included but $(2,1)$ is not. It is antisymmetric, since there are no cases of (a,b) and (b,a) both being in the relation. It is not transitive, since although $(1,2)$ and $(2,3)$ are in the relation, $(1,3)$ is not.

e) This relation is clearly reflexive and symmetric. It is trivially antisymmetric since there are no pairs (a,b) in the relation with $a \neq b$. It is trivially transitive, since the only time the hypothesis $(a,b) \in R \wedge (b,c) \in R$ is met is when $a = b = c$.

f) This relation is clearly not reflexive. The presence of $(1,4)$ and absence of $(4,1)$ shows that it is not symmetric. The presence of both $(1,3)$ and $(3,1)$ shows that it is not antisymmetric. It is not transitive; both $(2,3)$ and $(3,1)$ are in the relation, but $(2,1)$ is not, for instance.

5. Recall the definitions: R is reflexive if $(a,a) \in R$ for all a; R is symmetric if $(a,b) \in R$ always implies $(b,a) \in R$; R is antisymmetric if $(a,b) \in R$ and $(b,a) \in R$ always implies $a = b$; and R is transitive if $(a,b) \in R$ and $(b,c) \in R$ always implies $(a,c) \in R$.

a) It is tautological that everyone who has visited Web page a has also visited Web page a, so R is reflexive. It is not symmetric, because there surely are Web pages a and b such that the set of people who visited a is a proper subset of the set of people who visited b (for example, the only link to page a may be on page b). Whether R is antisymmetric in truth is hard to say, but it is certainly conceivable that there are two different Web pages a and b that have had exactly the same set of visitors. In this case, $(a,b) \in R$ and $(b,a) \in R$, so R is not antisymmetric. Finally, R is transitive: if everyone who has visited a has also visited b, and everyone who has visited b has also visited c, then clearly everyone who has visited a has also visited c.

b) This relation is not reflexive, because for any page a that has links on it, $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have no common links found on them. Finally, R is not transitive, because the two Web pages just mentioned, assuming they have links at all, give an example of the failure of the definition: $(a,b) \in R$ and $(b,a) \in R$, but $(a,a) \notin R$.

c) This relation is not reflexive, because for any page a that has no links on it, $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have a common link found on them. Finally, R is surely not transitive. Page a might have only one link (say to this textbook), page c might have only one link different from this (say to the Erdős Number Project), and page b may have only the two links mentioned in this sentence. Then $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$.

d) This relation is probably not reflexive, because there probably exist Web pages out there with no links at all to them (for example, when they are in the process of being written and tested); for any such page a we have $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that are referenced by some third page. Finally, R is surely not transitive. Page a might have only one page that links to it, page c might also have only one page, different from this, that links to it, and page b may be cited on both of these two pages. Then there would be no page that includes links to both pages a and c, so we have $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$.

7. **a)** This relation is not reflexive since it is not the case that $1 \neq 1$, for instance. It is symmetric: if $x \neq y$, then of course $y \neq x$. It is not antisymmetric, since, for instance, $1 \neq 2$ and also $2 \neq 1$. It is not transitive, since $1 \neq 2$ and $2 \neq 1$, for instance, but it is not the case that $1 \neq 1$.
b) This relation is not reflexive, since $(0,0)$ is not included. It is symmetric, because the commutative property of multiplication tells us that $xy = yx$, so that one of these quantities is greater than or equal to 1 if and only if the other is. It is not antisymmetric, since, for instance, $(2,3)$ and $(3,2)$ are both included. It is transitive. To see this, note that the relation holds between x and y if and only if either x and y are both positive or x and y are both negative. So assume that (a,b) and (b,c) are both in the relation. There are two cases, nearly identical. If a is positive, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$. If a is negative, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$.
c) This relation is not reflexive, since $(1,1)$ is not included, for instance. It is symmetric; the equation $x = y - 1$ is equivalent to the equation $y = x + 1$, which is the same as the equation $x = y + 1$ with the roles of x and y reversed. (A more formal proof of symmetry would be by cases. If x is related to y then either $x = y + 1$ or $x = y - 1$. In the former case, $y = x - 1$, so y is related to x; in the latter case $y = x + 1$, so y is related to x.) It is not antisymmetric, since, for instance, both $(1,2)$ and $(2,1)$ are in the relation. It is not transitive, since, for instance, although both $(1,2)$ and $(2,1)$ are in the relation, $(1,1)$ is not.
d) Recall that $x \equiv y \pmod 7$ means that $x - y$ is a multiple of 7, i.e., that $x - y = 7t$ for some integer t. This relation is reflexive, since $x - x = 7 \cdot 0$ for all x. It is symmetric, since if $x \equiv y \pmod 7$, then $x - y = 7t$ for some t; therefore $y - x = 7(-t)$, so $y \equiv x \pmod 7$. It is not antisymmetric, since, for instance, we have both $2 \equiv 9$ and $9 \equiv 2 \pmod 7$. It is transitive. Suppose $x \equiv y$ and $y \equiv z \pmod 7$. This means that $x - y = 7s$ and $y - z = 7t$ for some integers s and t. The trick is to add these two equations and note that the y disappears; we get $x - z = 7s + 7t = 7(s+t)$. By definition, this means that $x \equiv z \pmod 7$, as desired.
e) Every number is a multiple of itself (namely 1 times itself), so this relation is reflexive. (There is one bit of controversy here; we assume that 0 is to be considered a multiple of 0, even though we do not consider that 0 is a divisor of 0.) It is clearly not symmetric, since, for instance, 6 is a multiple of 2, but 2 is not a multiple of 6. The relation is not antisymmetric either; we have that 2 is a multiple of -2, for instance, and -2 is a multiple of 2, but $2 \neq -2$. The relation is transitive, however. If x is a multiple of y (say $x = ty$), and y is a multiple of z (say $y = sz$), then we have $x = t(sz) = (ts)z$, so we know that x is a multiple of z.
f) This relation is reflexive, since a and a are either both negative or both nonnegative. It is clearly symmetric from its form. It is not antisymmetric, since 5 is related to 6 and 6 is related to 5, but $5 \neq 6$. Finally, it is transitive, since if a is related to b and b is related to c, then all three of them must be negative, or all three must be nonnegative.
g) This relation is not reflexive, since, for instance, $17 \neq 17^2$. It is not symmetric, since although $289 = 17^2$, it is not the case that $17 = 289^2$. To see whether it is antisymmetric, suppose that we have both (x,y) and (y,x) in the relation. Then $x = y^2$ and $y = x^2$. To solve this system of equations, plug the second into the first, to obtain $x = x^4$, which is equivalent to $x - x^4 = 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$, so the solutions for x are 0 and 1 (and a pair of irrelevant complex numbers). The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x = y^2$ and $y = x^2$ is when $x = y$; this means that the relation is antisymmetric. It is not transitive, since, for example, $16 = 4^2$ and $4 = 2^2$, but $16 \neq 2^2$.
h) This relation is not reflexive, since, for instance, $17 \not\geq 17^2$. It is not symmetric, since although $289 \geq 17^2$, it is not the case that $17 \geq 289^2$. To see whether it is antisymmetric, we assume that both (x,y) and (y,x)

are in the relation. Then $x \geq y^2$ and $y \geq x^2$. Since both sides of the second inequality are nonnegative, we can square both sides to get $y^2 \geq x^4$. Combining this with the first inequality, we have $x \geq x^4$, which is equivalent to $x - x^4 \geq 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$. The last factor is always positive, so we can divide the original inequality by it to obtain the equivalent inequality $x(1-x) \geq 0$. Now if $x > 1$ or $x < 0$, then the factors have different signs, so the inequality does not hold. Thus the only solutions are $x = 0$ and $x = 1$. The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x \geq y^2$ and $y \geq x^2$ is when $x = y$; this means that the relation is antisymmetric. It is transitive. Suppose $x \geq y^2$ and $y \geq z^2$. Again the second inequality implies that both sides are nonnegative, so we can square both sides to obtain $y^2 \geq z^4$. Combining these inequalities gives $x \geq z^4$. Now we claim that it is always the case that $z^4 \geq z^2$; if so, then we combine this fact with the last inequality to obtain $x \geq z^2$, so x is related to z. To verify the claim, note that since we are working with integers, it is always the case that $z^2 \geq |z|$ (equality for $z = 0$ and $z = 1$, strict inequality for other z). Squaring both sides gives the desired inequality.

9. The relations in parts **(a)**, **(b)**, and **(e)** all have at least one pair of the form (x, x) in them, so they are not irreflexive. The relations in parts **(c)**, **(d)**, and **(f)** do not, so they are irreflexive.

11. According to the preamble to Exercise 9, an irreflexive relation is one for which a is never related to itself; i.e., $\forall a((a,a) \notin R)$.

a) Since we saw in Exercise 5a that $\forall a((a,a) \in R)$, clearly R is not irreflexive.

b) Since there are probably pages a with no links at all, and for such pages it is true that there are no common links found on both page a and page a, this relation is probably not irreflexive.

c) This relation is not irreflexive, because for any page a that has links on it, $(a,a) \in R$.

d) This relation is not irreflexive, because for any page a that has links on it that are ever cited, $(a,a) \in R$.

13. The relation in Exercise 3a is neither reflexive nor irreflexive. It contains some of the pairs (a,a) but not all of them.

15. Of course many answers are possible. The empty relation is always irreflexive (x is never related to y). A less trivial example would be $(a,b) \in R$ if and only if a is taller than b. Since nobody is taller than him/herself, we always have $(a,a) \notin R$.

17. The relation in part **(a)** is asymmetric, since if a is taller than b, then certainly b cannot be taller than a. The relation in part **(b)** is not asymmetric, since there are many instances of a and b born on the same day (both cases in which $a = b$ and cases in which $a \neq b$), and in all such cases, it is also the case that b and a were born on the same day. The relations in part **(c)** and part **(d)** are just like that in part **(b)**, so they, too, are not asymmetric.

19. According to the preamble to Exercise 16, an asymmetric relation is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. Thus R is asymmetric if and only if R is antisymmetric and also irreflexive.

a) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

b) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

c) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

d) not asymmetric since $(0,0) \in R$

e) not asymmetric since $(2,1) \in R$ and $(1,2) \in R$

f) not asymmetric since $(0,1) \in R$ and $(1,0) \in R$

g) not asymmetric since $(1,1) \in R$

h) not asymmetric since $(2,1) \in R$ and $(1,2) \in R$

21. According to the preamble to Exercise 16, an asymmetric relation is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously. In symbols, this is simply $\forall a \forall b \neg((a,b) \in R \wedge (b,a) \in R)$. Alternatively, $\forall a \forall b\,((a,b) \in R \rightarrow (b,a) \notin R)$.

23. There are mn elements of the set $A \times B$, if A is a set with m elements and B is a set with n elements. A relation from A to B is a subset of $A \times B$. Thus the question asks for the number of subsets of the set $A \times B$, which has mn elements. By the product rule, it is 2^{mn}.

25. **a)** By definition the answer is $\{\,(b,a) \mid a \text{ divides } b\,\}$, which, by changing the names of the dummy variables, can also be written $\{\,(a,b) \mid b \text{ divides } a\,\}$. (The universal set is still the set of positive integers.)
b) By definition the answer is $\{\,(a,b) \mid a \text{ does not divide } b\,\}$. (The universal set is still the set of positive integers.)

27. The inverse relation is just the graph of the inverse function. Somewhat more formally, we have $R^{-1} = \{\,(f(a),a) \mid a \in A\,\} = \{\,(b, f^{-1}(b)) \mid b \in B\,\}$, since we can index this collection just as easily by elements of B as by elements of A (using the correspondence $b = f(a)$).

29. This exercise is just a matter of the definitions of the set operations.
a) the set of pairs (a,b) where a is required to read b in a course or has read b
b) the set of pairs (a,b) where a is required to read b in a course and has read b
c) the set of pairs (a,b) where a is required to read b in a course or has read b, but not both; equivalently, the set of pairs (a,b) where a is required to read b in a course but has not done so, or has read b although not required to do so in a course
d) the set of pairs (a,b) where a is required to read b in a course but has not done so
e) the set of pairs (a,b) where a has read b although not required to do so in a course

31. To find $S \circ R$ we want to find the set of pairs (a,c) such that for some person b, a is a parent of b, and b is a sibling of c. Since brothers and sisters have the same parents, this means that a is also the parent of c. Thus $S \circ R$ is contained in the relation R. More specifically, $(a,c) \in S \circ R$ if and only if a is the parent of c, and c has a sibling (who is necessarily also a child of a). To find $R \circ S$ we want to find the set of pairs (a,c) such that for some person b, a is a sibling of b, and b is a parent of c. This is the same as the condition that a is the aunt or uncle of c (by blood, not marriage).

33. **a)** The union of two relations is the union of these sets. Thus $R_2 \cup R_4$ holds between two real numbers if R_2 holds or R_4 holds (or both, it goes without saying). Since it is always true that $a \leq b$ or $b \leq a$, $R_2 \cup R_4$ is all of $\mathbf{R}^2$, i.e., the relation that always holds.
b) For (a,b) to be in $R_3 \cup R_6$, we must have $a < b$ or $a \neq b$. Since this happens precisely when $a \neq b$, we see that the answer is R_6.
c) The intersection of two relations is the intersection of these sets. Thus $R_3 \cap R_6$ holds between two real numbers if R_3 holds and R_6 holds as well. Thus for (a,b) to be in $R_3 \cap R_6$, we must have $a < b$ and $a \neq b$. Since this happens precisely when $a < b$, we see that the answer is R_3.
d) For (a,b) to be in $R_4 \cap R_6$, we must have $a \leq b$ and $a \neq b$. Since this happens precisely when $a < b$, we see that the answer is R_3.
e) Recall that $R_3 - R_6 = R_3 \cap \overline{R_6}$. But $\overline{R_6} = R_5$, so we are asked for $R_3 \cap R_5$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.
f) Reasoning as in part (e), we want $R_6 \cap \overline{R_3} = R_6 \cap R_2$, which is clearly R_1 (since $a \neq b$ and $a \geq b$ precisely when $a > b$).

g) Recall that $R_2 \oplus R_6 = (R_2 \cap \overline{R_6}) \cup (R_6 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_6} = R_2 \cap R_5 = R_5$, and $R_6 \cap \overline{R_2} = R_6 \cap R_3 = R_3$. Thus our answer is $R_5 \cup R_3 = R_4$.
h) Recall that $R_3 \oplus R_5 = (R_3 \cap \overline{R_5}) \cup (R_5 \cap \overline{R_3})$. We see that $R_3 \cap \overline{R_5} = R_3 \cap R_6 = R_3$, and $R_5 \cap \overline{R_3} = R_5 \cap R_2 = R_5$. Thus our answer is $R_3 \cup R_5 = R_4$.

35. Recall that the composition of two relations all defined on a common set is defined as follows: $(a,c) \in S \circ R$ if and only if there is some element b such that $(a,b) \in R$ and $(b,c) \in S$. We have to apply this in each case.
a) For (a,c) to be in $R_2 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_2$. This means that $a > b$ and $b \geq c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_2 \circ R_1 = R_1$.
b) For (a,c) to be in $R_2 \circ R_2$, we must find an element b such that $(a,b) \in R_2$ and $(b,c) \in R_2$. This means that $a \geq b$ and $b \geq c$. Clearly this can be done if and only if $a \geq c$ to begin with. But that is precisely the statement that $(a,c) \in R_2$. Therefore we have $R_2 \circ R_2 = R_2$. In particular, this shows that R_2 is transitive.
c) For (a,c) to be in $R_3 \circ R_5$, we must find an element b such that $(a,b) \in R_5$ and $(b,c) \in R_3$. This means that $a = b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = a$). But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_3 \circ R_5 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).
d) For (a,c) to be in $R_4 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_4$. This means that $a > b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_1 = \mathbf{R}^2$, the relation that always holds.
e) For (a,c) to be in $R_5 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_5$. This means that $a < b$ and $b = c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = c$). But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_5 \circ R_3 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the left—although it is also an identity on the right as well).
f) For (a,c) to be in $R_3 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_3$. This means that $a \neq b$ and $b < c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_3 \circ R_6 = \mathbf{R}^2$, the relation that always holds.
g) For (a,c) to be in $R_4 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_4$. This means that $a \neq b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_6 = \mathbf{R}^2$, the relation that always holds.
h) For (a,c) to be in $R_6 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_6$. This means that $a \neq b$ and $b \neq c$. Clearly this can always be done simply by choosing b to be something other than a or c. Therefore we have $R_6 \circ R_6 = \mathbf{R}^2$, the relation that always holds. Note that since the answer is not R_6 itself, we know that R_6 is not transitive.

37. One earns a doctorate by, among other things, writing a thesis under an advisor, so this relation makes sense. (We ignore anomalies like someone having two advisors or someone being awarded a doctorate without having an advisor.) For (a,b) to be in R^2, we must find a c such that $(a,c) \in R$ and $(c,b) \in R$. In our context, this says that b got his/her doctorate under someone who got his/her doctorate under a. Colloquially, a is the academic grandparent of b, or b is the academic grandchild of a. Generalizing, $(a,b) \in R^n$ precisely when there is a sequence of $n+1$ people, starting with a and ending with b, such that each is the advisor of the next person in the sequence. People with doctorates like to look at these sequences (and trace their ancestry) back as far as they can. There are at least two websites where these relations are listed: one in theoretical computer science (`http://sigact.acm.org/genealogy/`, which seems to be not currently updated or maintained) and one in mathematics (`http://www.genealogy.math.ndsu.nodak.edu/`).

39. a) The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a, b) \in R_1 \cup R_2$ if and only if $a \equiv b \pmod 3$ or $a \equiv b \pmod 4$. There is not a good easier way to state this, other than perhaps to say that $a - b$ is a multiple of either 3 or 4, or to work modulo 12 and write $a - b \equiv 0, 3, 4, 6, 8,$ or $9 \pmod{12}$.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a, b) \in R_1 \cap R_2$ if and only if $a \equiv b \pmod 3$ and $a \equiv b \pmod 4$. Since this means that $a - b$ is a multiple of both 3 and 4, and that happens if and only if $a - b$ is a multiple of 12, we can state this more simply as $a \equiv b \pmod{12}$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a, b) \in R_1 - R_2$ if and only if $a \equiv b \pmod 3$ and $a \not\equiv b \pmod 4$. We could, if we wished, state this working modulo 12: $(a, b) \in R_1 - R_2$ if and only if $a - b \equiv 3, 6,$ or $9 \pmod{12}$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a, b) \in R_2 - R_1$ if and only if $a \equiv b \pmod 4$ and $a \not\equiv b \pmod 3$. We could, if we wished, state this working modulo 12: $(a, b) \in R_2 - R_1$ if and only if $a - b \equiv 4$ or $8 \pmod{12}$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part **(c)** and part **(d)**. Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. We can write this in symbols by saying that $(a, b) \in R_1 \oplus R_2$ if and only if ($a \equiv b \pmod 3$ and $a \not\equiv b \pmod 4$) or ($a \equiv b \pmod 4$ and $a \not\equiv b \pmod 3$). We could, if we wished, state this working modulo 12: $(a, b) \in R_1 \oplus R_2$ if and only if $a - b \equiv 3, 4, 6, 8$ or $9 \pmod{12}$. We could also say that $a - b$ is a multiple of 3 or 4 but not both.

41. A relation is just a subset. A subset can either contain a specified element or not; half of them do and half of them do not. Therefore 8 of the 16 relations on $\{0, 1\}$ contain the pair $(0, 1)$. Alternatively, a relation on $\{0, 1\}$ containing the pair $(0, 1)$ is just a set of the form $\{(0, 1)\} \cup X$, where $X \subseteq \{(0, 0), (1, 0), (1, 1)\}$. Since this latter set has 3 elements, it has $2^3 = 8$ subsets.

43. This is similar to Example 16 in this section.

a) A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, we are asking for the number of subsets of a set with n^2 elements, which is 2^{n^2}. In our case $n = 4$, so the answer is $2^{16} = 65{,}536$.

b) In solving part **(a)**, we had 16 binary choices to make—whether to include a pair (x, y) in the relation or not as x and y ranged over the set $\{a, b, c, d\}$. In this part, one of those choices has been made for us: we *must* include (a, a). We are free to make the other 15 choices. So the answer is $2^{15} = 32{,}768$. See Exercise 45 for more problems of this type.

45. These are combinatorics problems, some harder than others. Let A be the set with n elements on which the relations are defined.

a) To specify a symmetric relation, we need to decide, for each unordered pair $\{a, b\}$ of distinct elements of A, whether to include the pairs (a, b) and (b, a) or leave them out; this can be done in 2 ways for each such unordered pair. Also, for each element $a \in A$, we need to decide whether to include (a, a) or not, again 2 possibilities. We can think of these two parts as one by considering an element to be an unordered pair with repetition allowed. Thus we need to make this 2-fold choice $C(n + 1, 2)$ times, since there are $C(n + 2 - 1, 2)$ ways to choose an unordered pair with repetition allowed. Therefore the answer is $2^{C(n+1,2)} = 2^{n(n+1)/2}$.

b) This is somewhat similar to part **(a)**. For each unordered pair $\{a, b\}$ of distinct elements of A, we have a 3-way choice—either include (a, b) only, include (b, a) only, or include neither. For each element of A we have a 2-way choice. Therefore the answer is $3^{C(n,2)} 2^n = 3^{n(n-1)/2} 2^n$.

c) As in part **(b)** we have a 3-way choice for $a \neq b$. There is no choice about including (a,a) in the relation—the definition prohibits it. Therefore the answer is $3^{C(n,2)} = 3^{n(n-1)/2}$.

d) For each ordered pair (a,b), with $a \neq b$ (and there are $P(n,2)$ such pairs), we can choose to include (a,b) or to leave it out. There is no choice for pairs (a,a). Therefore the answer is $2^{P(n,2)} = 2^{n(n-1)}$.

e) This is just like part **(a)**, except that there is no choice about including (a,a). For each unordered pair of distinct elements of A, we can choose to include neither or both of the corresponding ordered pairs. Therefore the answer is $2^{C(n,2)} = 2^{n(n-1)/2}$.

f) We have complete freedom with the ordered pairs (a,b) with $a \neq b$, so that part of the choice gives us $2^{P(n,2)}$ possibilities, just as in part **(d)**. For the decision as to whether to include (a,a), two of the 2^n possibilities are prohibited: we cannot include all such pairs, and we cannot leave them all out. Therefore the answer is $2^{P(n,2)}(2^n - 2) = 2^{n^2-n}(2^n - 2) = 2^{n^2} - 2^{n^2-n+1}$.

47. The secoud sentence of the proof asks us to "take an element $b \in A$ such that $(a,b) \in R$." There is no guarantee that such an element exists for the taking. This is the only mistake in the proof. If one could be guaranteed that each element in A is related to at least one element, then symmetry and transitivity would indeed imply reflexivity. Without this assumption, however, the proof and the proposition are wrong. As a simple example, take the relation $\emptyset$ on any nonempty set. This relation is vacuously symmetric and transitive, but not reflexive. Here is another counterexample: the relation $\{(1,1),(1,2),(2,1),(2,2)\}$ on the set $\{1,2,3\}$.

49. We need to show two things. First, we need to show that if a relation R is symmetric, then $R = R^{-1}$, which means we must show that $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$. To do this, let $(a,b) \in R$. Since R is symmetric, this implies that $(b,a) \in R$. But since R^{-1} consists of all pairs (a,b) such that $(b,a) \in R$, this means that $(a,b) \in R^{-1}$. Thus we have shown that $R \subseteq R^{-1}$. Next let $(a,b) \in R^{-1}$. By definition this means that $(b,a) \in R$. Since R is symmetric, this implies that $(a,b) \in R$ as well. Thus we have shown that $R^{-1} \subseteq R$.

Second we need to show that $R = R^{-1}$ implies that R is symmetric. To this end we let $(a,b) \in R$ and try to show that (b,a) is also necessarily an element of R. Since $(a,b) \in R$, the definition tells us that $(b,a) \in R^{-1}$. But since we are under the hypothesis that $R = R^{-1}$, this tells us that $(b,a) \in R$, exactly as desired.

51. Suppose that R is reflexive. We must show that R^{-1} is reflexive, i.e., that $(a,a) \in R^{-1}$ for each $a \in A$. Now since R is reflexive, we know that $(a,a) \in R$ for each $a \in R$. By definition, this tells us that $(a,a) \in R^{-1}$, as desired. (Interchanging the two a's in the pair (a,a) leaves it as it was.) Conversely, if R^{-1} is reflexive, then $(a,a) \in R^{-1}$ for each $a \in A$. By definition this means that $(a,a) \in R$ (again we interchanged the two a's).

53. We prove this by induction on n. The case $n = 1$ is trivial, since it is the statement $R = R$. Assume the inductive hypothesis that $R^n = R$. We must show that $R^{n+1} = R$. By definition $R^{n+1} = R^n \circ R$. Thus our task is to show that $R^n \circ R \subseteq R$ and $R \subseteq R^n \circ R$. The first uses the transitivity of R, as follows. Suppose that $(a,c) \in R^n \circ R$. This means that there is an element b such that $(a,b) \in R$ and $(b,c) \in R^n$. By the inductive hypothesis, the latter statement implies that $(b,c) \in R$. Thus by the transitivity of R, we know that $(a,c) \in R$, as desired.

Next assume that $(a,b) \in R$. We must show that $(a,b) \in R^n \circ R$. By the inductive hypothesis, $R^n = R$, and therefore R^n is reflexive by assumption. Thus $(b,b) \in R^n$. Since we have $(a,b) \in R$ and $(b,b) \in R^n$, we have by definition that (a,b) is an element of $R^n \circ R$, exactly as desired. (The first half of this proof was not really necessary, since Theorem 1 in this section already told us that $R^n \subseteq R$ for all n.)

55. We use induction on n, the result being trivially true for $n = 1$. Assume that R^n is reflexive; we must show that R^{n+1} is reflexive. Let $a \in A$, where A is the set on which R is defined. By definition $R^{n+1} = R^n \circ R$. By the inductive hypothesis, R^n is reflexive, so $(a,a) \in R^n$. Also, since R is reflexive by assumption, $(a,a) \in R$. Therefore by the definition of composition, $(a,a) \in R^n \circ R$, as desired.

57. It is not necessarily true that R^2 is irreflexive when R is. We might have pairs (a,b) and (b,a) both in R, with $a \neq b$; then it would follow that $(a,a) \in R^2$, preventing R^2 from being irreflexive. As the simplest example, let $A = \{1,2\}$ and let $R = \{(1,2),(2,1)\}$. Then R is clearly irreflexive. In this case $R^2 = \{(1,1),(2,2)\}$, which is not irreflexive.