SECTION 2.3 Functions

The importance of understanding what a function is cannot be overemphasized—functions permeate all of mathematics and computer science. This exercise set enables you to make sure you understand functions and their properties. Exercise 29 is a particularly good benchmark to test your full comprehension of the abstractions involved. The definitions play a crucial role in doing proofs about functions. To prove that a function $f : A \to B$ is one-to-one, you need to show that $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in A$. To prove that such a function is onto, you need to show that $\forall y \in B\, \exists x \in A\, (f(x) = y)$.

1. a) The expression $1/x$ is meaningless for $x = 0$, which is one of the elements in the domain; thus the "rule" is no rule at all. In other words, $f(0)$ is not defined.

b) Things like $\sqrt{-3}$ are undefined (or, at best, are complex numbers).

c) The "rnle" for f is ambiguous. We must have $f(x)$ defined uniquely, but here there are two values associated with every x, the positive square root and the negative square root of $x^2 + 1$.

3. a) This is not a function, because there may be no zero bit in S, or there may be more than one zero bit in S. Thus there may be no value for $f(S)$ or more than one. In either case this violates the definition of a function, since $f(S)$ must have a unique value.

b) This is a function from the set of bit strings to the set of integers, since the number of 1 bits is always a clearly defined nonnegative integer.

c) This definition does not tell what to do with a nonempty string consisting of all 0's. Thus, for example, $f(000)$ is undefined. Therefore this is not a function.

5. In each case we want to find the domain (the set on which the function operates, which is implicitly stated in the problem) and the range (the set of possible output values).

a) Clearly the domaiu is the set of all bit strings. The range is $\mathbf{Z}$; the function evaluated at a string with n 1's and no 0's is n, and the function evaluated at a string with n 0's and no 1's is $-n$.

b) Again the domain is clearly the set of all bit strings. Since there can be any natural number of 0's in a bit string, the value of the function can be $0, 2, 4, \ldots$. Therefore the range is the set of even natural numbers.

c) Again the domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7 (if it were more, then we could form another hyte), the range is $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

d) As the problem states, the domain is the set of positive iutegers. Only perfect squares can be function values, and clearly every positive perfect square is possible. Therefore the range is $\{1, 4, 9, 16, \ldots\}$.

7. In each case, the domain is the set of possible inputs for which the fuuction is defined, and the range is the set of all possible outputs on these inputs.

a) The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$, since we are told that the function operates on pairs of positive integers (the word "pair" in mathematics is usnally understood to mean ordered pair). Since the maximum is again a positive integer, and all positive integers are possible maximums (by letting the two elements of the pair be the sanie), the range is $\mathbf{Z}^+$.

b) We are told that the domain is $\mathbf{Z}^+$. Since the decimal representation of an integer has to have at least one digit, at most nine digits do not appear, and of course the number of missing digits could be any number less than 9. Thus the range is $\{0,1,2,3,4,5,6,7,8,9\}$.

c) We are told that the domain is the set of bit strings. The block 11 could appear no times, or it could appear any positive number of times, so the range is $\mathbf{N}$.

d) We are told that the domain is the set of bit strings. Since the first 1 can be anywhere in the string, its position can be $1,2,3,\ldots$. If the bit string contains no 1's, the value is 0 by definition. Therefore the range is $\mathbf{N}$.

9. The floor function rounds down and the ceiling function rounds up.

a) 1 b) 0 c) 0 d) -1 e) 3 f) -1 g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rfloor = \lfloor 2\frac{1}{2} \rfloor = 2$

h) $\lfloor \frac{1}{2} \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot 2 \rfloor = \lfloor 1 \rfloor = 1$

11. We need to determine whether the range is all of $\{a,b,c,d\}$. It is for the function in part (a), but not for the other two functions.

13. a) This function is onto, since every integer is 1 less than some integer. In particular, $f(x+1) = x$.

b) This function is not onto. Since $n^2 + 1$ is always positive, the range cannot include any negative integers.

c) This function is not onto, since the integer 2, for example, is not in the range. In other words, 2 is not the cube of any integer.

d) This function is onto. If we want to obtain the value x, then we simply need to start with $2x$, since $f(2x) = \lceil 2x/2 \rceil = \lceil x \rceil = x$ for all $x \in \mathbf{Z}$.

15. An onto function is one whose range is the entire codomain. Thus we must determine whether we can write every integer in the form given by the rule for f in each case.

a) Given any integer n, we have $f(0,n) = n$, so the function is onto.

b) Clearly the range contains no negative integers, so the function is not onto.

c) Given any integer m, we have $f(m, 25) = m$, so the function is onto. (We could have used any constant in place of 25 in this argument.)

d) Clearly the range contains no negative integers, so the function is not onto.

e) Given any integer m, we have $f(m,0) = m$, so the function is onto.

17. Obviously there are an infinite number of correct answers to each part. The problem asked for a "formula." Parts (a) and (c) seem harder here, since we somehow have to fold the negative integers into the positive ones without overlap. Therefore we probably want to treat the negative integers differently from the positive integers. One way to do this with a formula is to make it a two-part formula. If one objects that this is not "a formula," we can counter as follows. Consider the function $g(x) = \lfloor 2^x \rfloor / 2^x$. Clearly if $x \geq 0$, then 2^x is a positive integer, so $g(x) = 2^x/2^x = 1$. If $x < 0$, then 2^x is a number between 0 and 1, so $g(x) = 0/2^x = 0$. If we want to define a function that has the value $f_1(x)$ when $x \geq 0$ and $f_2(x)$ when $x < 0$, then we can use the formula $g(x) \cdot f_1(x) + (1 - g(x)) \cdot f_2(x)$.

a) We could map the positive integers (and 0) into the positive multiples of 3, say, and the negative integers into numbers that are 1 greater than a multiple of 3, in a one-to-one manner. This will give us a function that leaves some elements out of the range. So let us define our function as follows:

$$f(x) = \begin{cases} 3x+3 & \text{if } x \geq 0 \\ 3|x|+1 & \text{if } x < 0 \end{cases}$$

The values of f on the inputs 0 through 4 are then $3, 6, 9, 12, 15$; and the values on the inputs -1 to -4 are $4, 7, 10, 13$. Clearly this function is one-to-one, but it is not onto since, for example, 2 is not in the range.

b) This is easier. We can just take $f(x) = |x| + 1$. It is clearly onto, but $f(n)$ and $f(-n)$ have the same value for every positive integer n, so f is not one-to-one.

c) This is similar to part (a), except that we have to be careful to hit all values. Mapping the nonnegative integers to the odds and the negative integers to the evens will do the trick:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ 2|x| & \text{if } x < 0 \end{cases}$$

d) Here we can use a trivial example like $f(x) = 17$ or a simple nontrivial one like $f(x) = x^2 + 1$. Clearly these are neither one-to-one nor onto.

19. a) One way to determine whether a function is a bijection is to try to construct its inverse. This function is a bijection, since its inverse (obtained by solving $y = 2x + 1$ for x) is the function $g(y) = (y-1)/2$. Alternatively, we can argue directly. To show that the function is one-to-one, note that if $2x + 1 = 2x' + 1$, then $x = x'$. To show that the function is onto, note that $2((y-1)/2) + 1 = y$, so every number is in the range.

b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written $[1, \infty)$), not all of $\mathbf{R}$. (It is not injective either.)

c) This function is a bijection, since it has an inverse function, namely the function $f(y) = y^{1/3}$ (obtained by solving $y = x^3$ for x).

d) This function is not a bijection. It is easy to see that it is not injective, since x and $-x$ have the same image, for all real numbers x. A little work shows that the range is only $\{\, y \mid 0.5 \leq y < 1 \,\} = [0.5, 1)$.

21. The key here is that larger denominators make smaller fractions, and smaller denominators make larger fractions. We have two things to prove, since this is an "if and only if" statement. First, suppose that f is strictly decreasing. This means that $f(x) > f(y)$ whenever $x < y$. To show that g is strictly increasing, suppose that $x < y$. Then $g(x) = 1/f(x) < 1/f(y) = g(y)$. Conversely, suppose that g is strictly increasing. This means that $g(x) < g(y)$ whenever $x < y$. To show that f is strictly decreasing, suppose that $x < y$. Then $f(x) = 1/g(x) > 1/g(y) = f(y)$.

23. We need to make the function decreasing, but not *strictly* decreasing, so, for example, we could take the trivial function $f(x) = 17$. If we want the range to be all of $\mathbf{R}$, we could define f in parts this way: $f(x) = -x - 1$ for $x < -1$; $f(x) = 0$ for $-1 \leq x \leq 1$; and $f(x) = -x + 1$ for $x > 1$.

25. The function is not one-to-one (for example, $f(2) = 2 = f(-2)$), so it is not invertible. On the restricted domain, the function is the identity function from the set of nonnegative real numbers to itself, $f(x) = x$, so it is one-to-one and onto and therefore invertible; in fact, it is its own inverse.

27. In each case, we need to compute the values of $f(x)$ for each $x \in S$.

a) Note that $f(\pm 2) = \lfloor (\pm 2)^2/3 \rfloor = \lfloor 4/3 \rfloor = 1$, $f(\pm 1) = \lfloor (\pm 1)^2/3 \rfloor = \lfloor 1/3 \rfloor = 0$, $f(0) = \lfloor 0^2/3 \rfloor = \lfloor 0 \rfloor = 0$, and $f(3) = \lfloor 3^2/3 \rfloor = \lfloor 3 \rfloor = 3$. Therefore $f(S) = \{0, 1, 3\}$.

b) In addition to the values we computed above, we note that $f(4) = 5$ and $f(5) = 8$. Therefore $f(S) = \{0, 1, 3, 5, 8\}$.

c) Note this time also that $f(7) = 16$ and $f(11) = 40$, so $f(S) = \{0, 8, 16, 40\}$.

d) $\{f(2), f(6), f(10), f(14)\} = \{1, 12, 33, 65\}$

29. In both cases, we can argue directly from the definitions.

a) Assume that both f and g are one-to-one. We need to show that $f \circ g$ is one-to-one. This means that we need to show that if x and y are two distinct elements of A, then $f(g(x)) \neq f(g(y))$. First, since g is one-to-one, the definition tells us that $g(x) \neq g(y)$. Second, since now $g(x)$ and $g(y)$ are distinct elements of B, and since f is one-to-one, we conclude that $f(g(x)) \neq f(g(y))$, as desired.

b) Assume that both f and g are onto. We need to show that $f \circ g$ is onto. This means that we need to show that if z is any element of C, then there is some element $x \in A$ such that $f(g(x)) = z$. First, since f is onto, we can conclude that there is an element $y \in B$ such that $f(y) = z$. Second, since g is onto and $y \in B$, we can conclude that there is an element $x \in A$ such that $g(x) = y$. Putting these together, we have $z = f(y) = f(g(x))$, as desired.

31. To establish the setting here, let us suppose that $g : A \to B$ and $f : B \to C$. Then $f \circ g: A \to C$. We are told that f and $f \circ g$ are onto. Thus all of C gets "hit" by the images of elements of B; in fact, each element in C gets hit by an element from A under the composition $f \circ g$. But this does not seem to tell us anything about the elements of B getting hit by the images of elements of A. Indeed, there is no reason that they must. For a simple counterexample, suppose that $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Let $g(a) = b_1$, and let $f(b_1) = c$ and $f(b_2) = c$. Then clearly f and $f \circ g$ are onto, but g is not, since b_2 is not in its range.

33. We just perform the indicated operations on the defining expressions. Thus $f + g$ is the function whose value at x is $(x^2 + 1) + (x + 2)$, or, more simply, $(f + g)(x) = x^2 + x + 3$. Similarly fg is the function whose value at x is $(x^2 + 1)(x + 2)$; in other words, $(fg)(x) = x^3 + 2x^2 + x + 2$.

35. We simply solve the equation $y = ax + b$ for x. This gives $x = (y - b)/a$, which is well-defined since $a \neq 0$. Thus the inverse is $f^{-1}(y) = (y-b)/a$. To check that our work is correct, we must show that $f \circ f^{-1}(y) = y$ for all $y \in \mathbf{R}$ and that $f^{-1} \circ f(x) = x$ for all $x \in \mathbf{R}$. Both of these are straightforward algebraic manipulations. For the first, we have $f \circ f^{-1}(y) = f(f^{-1}(y)) = f((y - b)/a) = a\big((y - b)/a\big) + b = y$. The second is similar.

37. Let us arrange for S and T to be nonempty sets that have empty intersection. Then the left-hand side will be $f(\emptyset)$, which is the empty set. If we can make the right-hand side nonempty, then we will be done. We can make the right-hand side nonempty by making the codomain consist of just one element, so that $f(S)$ and $f(T)$ will both be the set consisting of that one element. The simplest example is as follows. Let $A = \{1, 2\}$ and $B = \{3\}$. Let f be the unique function from A to B (namely $f(1) = f(2) = 3$). Let $S = \{1\}$ and $T = \{2\}$. Then $f(S \cap T) = f(\emptyset) = \emptyset$, which is a proper subset of $f(S) \cap f(T) = \{3\} \cap \{3\} = \{3\}$.

39. **a)** We want to find the set of all numbers whose floor is 0. Since all numbers from 0 to 1 (including 0 but not 1) round down to 0, we conclude that $g^{-1}(\{0\}) = \{\, x \mid 0 \leq x < 1 \,\} = [0, 1)$.

b) This is similar to part (**a**). All numbers from -1 to 2 (including -1 but not 2) round down to -1, 0, or 1; we conclude that $g^{-1}(\{-1, 0, 1\}) = \{\, x \mid -1 \leq x < 2 \,\} = [-1, 2)$.

c) Since $g(x)$ is always an integer, there are no values of x such that $g(x)$ is strictly between 0 and 1. Thus the inverse image in this case is the empty set.

41. Note that the complementation here is with respect to the relevant universal set. Thus $\overline{S} = B - S$ and $\overline{f^{-1}(S)} = A - f^{-1}(S)$. There are two things to prove in order to show that these two sets are equal: that the left-hand side of the equation is a subset of the right-hand side, and that the right-hand side is a subset of the left-hand side. First let $x \in f^{-1}(\overline{S})$. This means that $f(x) \in \overline{S}$, or equivalently that $f(x) \notin S$. Therefore by definition of inverse image, $x \notin f^{-1}(S)$, so $x \in \overline{f^{-1}(S)}$. For the other direction, assume that $x \in \overline{f^{-1}(S)}$. Then $x \notin f^{-1}(S)$. By definition this means that $f(x) \notin S$, which means that $f(x) \in \overline{S}$. Therefore by definition, $x \in f^{-1}(\overline{S})$.

43. There are three cases. Define the "fractional part" of x to be $f(x) = x - \lfloor x \rfloor$. Clearly $f(x)$ is always between 0 and 1 (inclusive at 0, exclusive at 1), and $x = \lfloor x \rfloor + f(x)$. If $f(x)$ is less than $\frac{1}{2}$, then $x - \frac{1}{2}$ will have a value slightly less than $\lfloor x \rfloor$, so when we round up, we get $\lfloor x \rfloor$. In other words, in this case $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$, and indeed that is the integer closest to x. If $f(x)$ is greater than $\frac{1}{2}$, then $x - \frac{1}{2}$ will have a value slightly greater than $\lfloor x \rfloor$, so when we round up, we get $\lfloor x \rfloor + 1$. In other words, in this case $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor + 1$, and indeed that is the integer closest to x in this case. Finally, if the fractional part is exactly $\frac{1}{2}$, then x is midway between two integers, and $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$, which is the smaller of these two integers.

45. We can write the real number x as $\lfloor x \rfloor + \epsilon$, where ϵ is a real number satisfying $0 \le \epsilon < 1$. Since $\epsilon = x - \lfloor x \rfloor$, we have $0 \le x - \lfloor x \rfloor < 1$. The first two inequalities, $x - 1 < \lfloor x \rfloor$ and $\lfloor x \rfloor \le x$, follow algebraically. For the other two inequalities, we can write $x = \lceil x \rceil - \epsilon$, where again $0 \le \epsilon < 1$. Then $0 \le \lceil x \rceil - x < 1$, and again the desired inequalities follow by easy algebra.

47. **a)** One direction (the "only if" part) is obvious: If $x < n$, then since $\lfloor x \rfloor \le x$ it follows that $\lfloor x \rfloor < n$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $x \ge n$. Then "the greatest integer not exceeding x" must be at least n, since n is an integer not exceeding x. That is, $\lfloor x \rfloor \ge n$.
b) One direction (the "only if" part) is obvious: If $n < x$, then since $x \le \lceil x \rceil$ it follows that $n < \lceil x \rceil$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $n \ge x$. Then "the smallest integer not less than x" must be no greater than n, since n is an integer not less than x. That is, $\lceil x \rceil \le n$.

49. If n is even, then $n = 2k$ for some integer k. Thus $\lfloor n/2 \rfloor = \lfloor k \rfloor = k = n/2$. If n is odd, then $n = 2k + 1$ for some integer k. Thus $\lfloor n/2 \rfloor = \lfloor k + \frac{1}{2} \rfloor = k = (n-1)/2$.

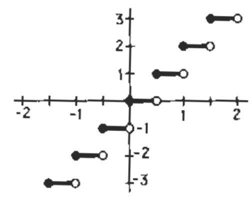
51. Without loss of generality we can assume that $x \ge 0$, since the equation to be proved is equivalent to the same equation with $-x$ substituted for x. Then the left-hand side is $\lceil -x \rceil$ by definition, and the right-hand side is $-\lfloor x \rfloor$. Thus this problem reduces to Exercise 50. Its proof is straightforward. Write x as $n + \epsilon$, where n is a natural number and ϵ is a real number satisfying $0 \le \epsilon < 1$. Then clearly $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n$ and $-\lfloor x \rfloor = -\lfloor n + \epsilon \rfloor = -n$ as well.

53. In some sense this question is its own answer—the number of integers strictly between a and b is the number of integers strictly between a and b. Presumably we seek an expression involving a, b, and the floor and/or ceiling function to answer this question. If we round a down and round b up to integers, then we will be looking at the smallest and largest integers just outside the range of integers we want to count, respectively. These values are of course $\lfloor a \rfloor$ and $\lceil b \rceil$, respectively. Then the answer is $\lceil b \rceil - \lfloor a \rfloor - 1$ (just think of counting all the integers between these two values, excluding both ends—if a row of fenceposts one foot apart extends for k feet, then there are $k - 1$ fenceposts not counting the end posts). Note that this even works when, for example, $a = 0.3$ and $b = 0.7$.

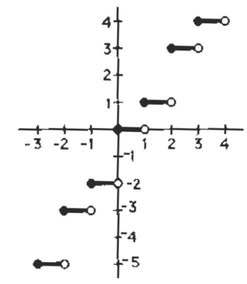
55. Since a byte is eight bits, all we are asking for in each case is $\lceil n/8 \rceil$, where n is the number of bits.
a) $\lceil 7/8 \rceil = 1$ **b)** $\lceil 17/8 \rceil = 3$ **c)** $\lceil 1001/8 \rceil = 126$ **d)** $\lceil 28800/8 \rceil = 3600$

57. In each case we need to divide the number of bytes (octets) by 1500 and round up. In other words, the answer is $\lceil n/1500 \rceil$, where n is the number of bytes.
a) $\lceil 150{,}000/1500 \rceil = 100$ **b)** $\lceil 384{,}000/1500 \rceil = 256$ **c)** $\lceil 1{,}544{,}000/1500 \rceil = 1030$
d) $\lceil 45{,}300{,}000/1500 \rceil = 30{,}200$

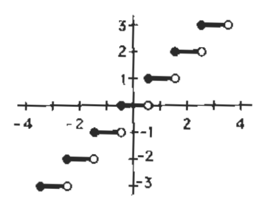
59. The graph will look exactly like the graph of the function $f(x) = \lfloor x \rfloor$, shown in Figure 10a, except that the picture will be compressed by a factor of 2 in the horizontal direction, since x has been replaced by $2x$.



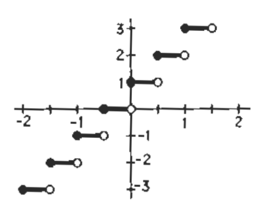
61. This is a step function, with values changing only at the integers. We note the pattern that $f(x)$ jumps by 1 when x passes through an odd integer (because of the $\lfloor x \rfloor$ term), and by 2 when x passes through an even integer (an additional jump caused by the $\lfloor x/2 \rfloor$ term). The result is as shown.



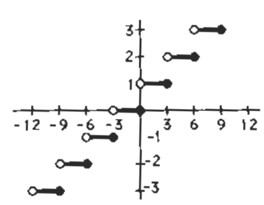
63. **a)** The graph will look exactly like the graph of the function $f(x) = \lfloor x \rfloor$, shown in Figure 10a, except that the picture will be shifted to the left by $\frac{1}{2}$ unit, since x has been replaced by $x + \frac{1}{2} = x - (-\frac{1}{2})$.



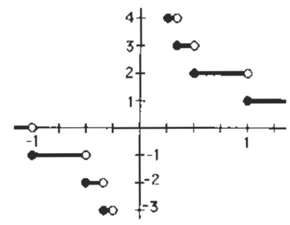
b) The graph will look exactly like the graph of the function $f(x) = \lfloor 2x \rfloor$, shown in the solution to Exercise 59, except that the picture will be shifted to the left by $\frac{1}{2}$ unit, since x has been replaced by $x + \frac{1}{2}$. Alternatively, we can note that $f(x)$ can be rewritten as $\lfloor 2x \rfloor + 1$, so the graph is the graph shown in the solution to Exercise 59 shifted up one unit.



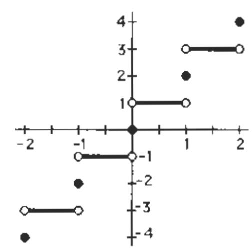
c) The graph will look exactly like the graph of the function $f(x) = \lceil x \rceil$, shown in Figure 10b, except that the x-axis is stretched by a factor of 3. Thus we can use the same picture and just relabel the x-axis.



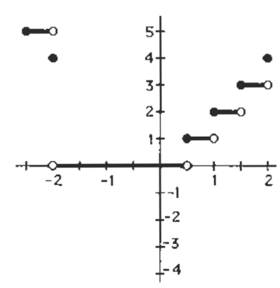
d) The graph is a step version of the usual hyperbola $y = 1/x$. Note that $x = 0$ is not in the domain. The graph can be drawn by first plotting the points at which $1/x$ is an integer ($x = 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$) and then filling in the horizontal segments, making sure to note that they go to the right (for example, if x is a little bigger than $\frac{1}{2}$, then $1/x$ is a little less than 2, so $f(x) = 2$, since we are rouuding up here). Note that $f(x) = 1$ for $x \geq 1$, and $f(x) = 0$ for $x < -1$.



e) The key thing to note is that since we can pull integers outside the floor and ceiling function (identity (4) in Table 1), we can write $f(x)$ more simply as $\lfloor x \rfloor + \lceil x \rceil$. When x is an integer, this is just $2x$. When x is between two integers, however, this has the value of the integer between the two integers $2\lfloor x \rfloor$ and $2\lceil x \rceil$. The graph is therefore as shown here.



f) The basic shape is the parabola, $y = x^2$. In particular, for x an even integer, $f(x) = x^2$, since the terms inside the floor and ceiling function symbols are integers. However, because of these step functions, the curve is broken into steps. At even integers other than $x = 0$ there are isolated points in the graph. Also, the graph takes jumps at all the integer and half-integer values outside the range $-2 < x < \frac{1}{2}$ (where in fact $f(x) = 0$). The portion of the graph near the origin is shown here.



g) Despite the complicated-looking formula, this is really quite similar to part **(a)**; in fact, we'll see that it's identical! First note that the expression inside the outer ceiling function symbols is always going to be a half-integer; therefore we can tell exactly what its rounded-up value will be, namely $\lfloor x - \frac{1}{2} \rfloor + 1$. Furthermore, since identity (4) of Table 1 allows us to move the 1 inside the floor function symbols, we have $f(x) = \lfloor x + \frac{1}{2} \rfloor$. Therefore this is the same function as in part **(a)**.

65. We simply need to solve the equation $y = x^3 + 1$ for x. This is easily done by algebra: $x = (y-1)^{1/3}$. Therefore the inverse function is given by the rule $f^{-1}(y) = (y-1)^{1/3}$ (or, equivalently, by the rule $f^{-1}(x) = (x-1)^{1/3}$, since the variable in the definition is just a dummy variable).

67. We can prove all of these identities by showiug that the left-hand side is equal to the right-hand side for all possible values of x. Iu each instance (except part **(c)**, in which there are only two cases), there are four cases to consider, depending on whether x is in A and/or B.

a) If x is in both A and B, then $f_{A\cap B}(x) = 1$; and the right-hand side is $1 \cdot 1 = 1$ as well. Otherwise $x \notin A \cap B$, so the left-hand side is 0, and the right-hand side is either $0 \cdot 1$ or $1 \cdot 0$ or $0 \cdot 0$, all of which are also 0.

b) If x is in both A and B, then $f_{A\cup B}(x) = 1$; and the right-hand side is $1 + 1 - 1 \cdot 1 = 1$ as well. If x is in A but not B, then $x \in A \cup B$, so the left-hand side is still 1, and the right-hand side is $1 + 0 - 1 \cdot 0 = 1$, as desired. The case in which x is in B but not A is similar. Finally, if x is in neither A nor B, then the left-hand side is 0, and the right-hand side is $0 + 0 - 0 \cdot 0 = 0$ as well.

c) If $x \in A$, then $x \notin \overline{A}$, so $f_{\overline{A}}(x) = 0$. The right-hand side equals $1 - 1 = 0$ in this case, as well. On the other hand, if $x \notin A$, then $x \in \overline{A}$, so the left-hand side is 1, and the right-hand side is $1 - 0 = 1$ as well.

d) If x is in both A and B, then $x \notin A \oplus B$, so $f_{A\oplus B}(x) = 0$. The right-hand side is $1 + 1 - 2 \cdot 1 \cdot 1 = 0$ as well. Next, if $x \in A$ but $x \notin B$, then $x \in A \oplus B$, so the left-hand side is 1. The right-hand side is $1 + 0 - 2 \cdot 1 \cdot 0 = 1$ as well. The case $x \in B \wedge x \notin A$ is similar. Finally, if x is in neither A nor B, then $x \notin A \oplus B$, so the left-hand side is 0; and the right-hand side is also $0 + 0 - 2 \cdot 0 \cdot 0 = 0$.

69. a) This is true. Since $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = \frac{1}{2}$. In that case the left-hand side is $\lfloor 1 \rfloor = 1$, while the right-hand side is $2 \cdot 0 = 0$.

c) This is true. We prove it by cases. If x is an integer, then by identity (4b) in Table 1, we know that $\lceil x + y \rceil = x + \lceil y \rceil$, and it follows that the difference is 0. Similarly, if y is an integer. The remaining case is that $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $x + y$ will be greater than $m + n$ but less than $m + n + 2$, so $\lceil x + y \rceil$ will be either $m + n + 1$ or $m + n + 2$. Therefore the given expression will be either $(n + 1) + (m + 1) - (m + n + 1) = 1$ or $(n + 1) + (m + 1) - (m + n + 2) = 0$, as desired.

d) This is clearly false, as we can find with a little experimentation. Take, for example, $x = 1/10$ and $y = 3$. Then the left-hand side is $\lceil 3/10 \rceil = 1$, but the right-hand side is $1 \cdot 3 = 3$.

e) Again a little trial and error will produce a counterexample. Take $x = 1/2$. Then the left-hand side is 1 while the right-hand side is 0.

71. a) If x is a positive integer, then the two sides are identical. So suppose that $x = n^2 + m + \epsilon$, where n is the largest perfect square integer less than x, m is a nonnegative integer, and $0 < \epsilon < 1$. For example, if $x = 13.2$, then $n = 3$, $m = 4$, and $\epsilon = 0.2$. Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n + 1$. Therefore both sides of the equation equal n.

b) If x is a positive integer, then the two sides are identical. So suppose that $x = n^2 - m - \epsilon$, where n is the smallest perfect square integer greater than x, m is a nonnegative integer, and ϵ is a real number with $0 < \epsilon < 1$. For example, if $x = 13.2$, then $n = 4$, $m = 2$, and $\epsilon = 0.8$. Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n - 1$ and n. Therefore both sides of the equation equal n.

73. In each case we easily read the domain and codomain from the notation. The domain of definition is obtained by determining for which values the definition makes sense. The function is total if the domain of definition is the entire domain (so that there are no values for which the partial function is undefined).

a) The domain is $\mathbf{Z}$ and the codomain is $\mathbf{R}$. Since division is possible by every nonzero number, the domain of definition is all the nonzero integers; $\{0\}$ is the set of values for which f is undefined. (It is not total.)

b) The domain and codomain are both given to be $\mathbf{Z}$. Since the definition makes sense for all integers, this is a total function, whose domain of definition is also $\mathbf{Z}$; the set of values for which f is undefined is $\emptyset$.

c) By inspection, the domain is the Cartesian product $\mathbf{Z} \times \mathbf{Z}$, and the codomain is $\mathbf{Q}$. Since fractions cannot have a 0 in the denominator, we must exclude the "slice" of $\mathbf{Z} \times \mathbf{Z}$ in which the second coordinate is 0. Thus the domain of definition is $\mathbf{Z} \times (\mathbf{Z} - \{0\})$, and the function is undefined for all values in $\mathbf{Z} \times \{0\}$. It is not a total function.

d) The domain is given to be $\mathbf{Z} \times \mathbf{Z}$ and the codomain is given to be $\mathbf{Z}$. Since the definition makes sense for all pairs of integers, this is a total function, whose domain of definition is also $\mathbf{Z} \times \mathbf{Z}$; the set of values for which f is undefined is $\emptyset$.

e) Again the domain and codomain are $\mathbf{Z} \times \mathbf{Z}$ and $\mathbf{Z}$, respectively. Since the definition is only stated for those pairs in which the first coordinate exceeds the second, the domain of definition is $\{\,(m,n) \mid m > n\,\}$, and therefore the set of values for which the function is undefined is $\{\,(m,n) \mid m \le n\,\}$. It is not a total function.

75. **a)** By definition, to say that S has cardinality m is to say that S has exactly m distinct elements. Therefore we can imagine enumerating them, as a child counts objects: assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence.

b) By part (**a**), there is a bijection f from S to $\{1, 2, \ldots, m\}$ and a bijection g from T to $\{1, 2, \ldots, m\}$. This tells us that g^{-1} is a bijection from $\{1, 2, \ldots, m\}$ to T. Then the composition $g^{-1} \circ f$ is the desired bijection from S to T.

77. A little experimentation with this function shows the pattern:

$f(1,1) = 1$	$f(2,1) = 3$	$f(3,1) = 6$	$f(4,1) = 10$	$f(5,1) = 15$	$f(6,1) = 21$
$f(1,2) = 2$	$f(2,2) = 5$	$f(3,2) = 9$	$f(4,2) = 14$	$f(5,2) = 20$	$f(6,2) = 27$
$f(1,3) = 4$	$f(2,3) = 8$	$f(3,3) = 13$	$f(4,3) = 19$	$f(5,3) = 26$	
$f(1,4) = 7$	$f(2,4) = 12$	$f(3,4) = 18$	$f(4,4) = 25$		
$f(1,5) = 11$	$f(2,5) = 17$	$f(3,5) = 24$			
$f(1,6) = 16$	$f(2,6) = 23$				
$f(1,7) = 22$					

We see by looking at the diagonals of this table that the function takes on successive values as $m+n$ increases. When $m+n=2$, $f(m,n)=1$. When $m+n=3$, $f(m,n)$ takes on the values 2 and 3. When $m+n=4$, $f(m,n)$ takes on the values 4, 5, and 6. And so on. It is clear from the formula that the range of values the function takes on for a fixed value of $m+n$, say $m+n=x$, is $\frac{(x-2)(x-1)}{2}+1$ through $\frac{(x-2)(x-1)}{2}+(x-1)$, since m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m+n$ is fixed. To show that this function is oue-to-one and onto, we merely need to show that the range of values for $x+1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1,1)+1 = f(1,x)$. We compute:

$$f(x-1,1)+1 = \frac{(x-2)(x-1)}{2} + (x-1) + 1 = \frac{x^2-x+2}{2} = \frac{(x-1)x}{2} + 1 = f(1,x)$$