SECTION 7.2 Solving Linear Recurrence Relations

2. a) linear, homogeneous, with constant coefficients; degree 2
   b) linear with constant coefficients but not homogeneous
   c) not linear
   d) linear, homogeneous, with constant coefficients; degree 3
   e) linear and homogeneous, but not with constant coefficients
   f) linear with constant coefficients, but not homogeneous
   g) linear, homogeneous, with constant coefficients; degree 7

4. For each problem, we first write down the characteristic equation and find its roots. Using this we write down the general solution. We then plug in the initial conditions to obtain a system of linear equations. We solve these equations to determine the arbitrary constants in the general solution, and finally we write down the unique answer.

   a) $r^2 - r - 6 = 0 \qquad r = -2, 3$

$$a_n = \alpha_1(-2)^n + \alpha_2 3^n$$
$$3 = \alpha_1 + \alpha_2$$
$$6 = -2\alpha_1 + 3\alpha_2$$
$$\alpha_1 = 3/5 \qquad \alpha_2 = 12/5$$
$$a_n = (3/5)(-2)^n + (12/5)3^n$$

b) $r^2 - 7r + 10 = 0 \qquad r = 2, 5$

$a_n = \alpha_1 2^n + \alpha_2 5^n$

$2 = \alpha_1 + \alpha_2$

$1 = 2\alpha_1 + 5\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = -1$

$a_n = 3 \cdot 2^n - 5^n$

c) $r^2 - 6r + 8 = 0 \qquad r = 2, 4$

$a_n = \alpha_1 2^n + \alpha_2 4^n$

$4 = \alpha_1 + \alpha_2$

$10 = 2\alpha_1 + 4\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = 1$

$a_n = 3 \cdot 2^n + 4^n$

d) $r^2 - 2r + 1 = 0 \qquad r = 1, 1$

$a_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$

$4 = \alpha_1$

$1 = \alpha_1 + \alpha_2$

$\alpha_1 = 4 \qquad \alpha_2 = -3$

$a_n = 4 - 3n$

e) $r^2 - 1 = 0 \qquad r = -1, 1$

$a_n = \alpha_1(-1)^n + \alpha_2 1^n = \alpha_1(-1)^n + \alpha_2$

$5 = \alpha_1 + \alpha_2$

$-1 = -\alpha_1 + \alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = 2$

$a_n = 3 \cdot (-1)^n + 2$

f) $r^2 + 6r + 9 = 0 \qquad r = -3, -3$

$a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$

$3 = \alpha_1$

$-3 = -3\alpha_1 - 3\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = -2$

$a_n = 3(-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n$

g) $r^2 + 4r - 5 = 0 \qquad r = -5, 1$

$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$

$2 = \alpha_1 + \alpha_2$

$8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1 \qquad \alpha_2 = 3$

$a_n = -(-5)^n + 3$

6. The model is the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-2} = a_{n-1} + 2a_{n-2}$, with $a_0 = a_1 = 1$ (see the technique of Exercise 35 in Section 7.1). To solve this, we use the characteristic equation $r^2 - r - 2 = 0$, which has roots -1 and 2. Therefore the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. Plugging in the initial conditions gives the equations $1 = \alpha_1 + \alpha_2$ and $1 = -\alpha_1 + 2\alpha_2$, which solve to $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$. Therefore in n microseconds $(1/3)(-1)^n + (2/3)2^n$ messages can be transmitted.

8. a) The recurrence relation is, by the definition of average, $L_n = (1/2)L_{n-1} + (1/2)L_{n-2}$.

b) The characteristic equation is $r^2 - (1/2)r - (1/2) = 0$, which gives us $r = -1/2$ and $r = 1$. Therefore the general solution is $L_n = \alpha_1(-1/2)^n + \alpha_2$. Plugging in the initial conditions $L_1 = 100000$ and $L_2 = 300000$ gives $100000 = (-1/2)\alpha_1 + \alpha_2$ and $300000 = (1/4)\alpha_1 + \alpha_2$. Solving these yields $\alpha_1 = 800000/3$ and $\alpha_2 = 700000/3$. Therefore the answer is $L_n = (800000/3)(-1/2)^n + (700000/3)$.

10. The proof may be found in textbooks such as *Introduction to Combinatorial Mathematics* by C. L. Liu (McGraw-Hill, 1968), Chapter 3. It is similar to the proof of Theorem 1.

12. The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are 1, -1, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6$, $\alpha_2 = -2$, and $\alpha_3 = -1$. Therefore the answer is $a_n = 6 - 2(-1)^n - 2^n$.

14. The characteristic equation is $r^4 - 5r^2 + 4 = 0$. This factors as $(r^2-1)(r^2-4) = (r-1)(r+1)(r-2)(r+2) = 0$, so the roots are 1, -1, 2, and -2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n + \alpha_4(-2)^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $2 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$, $6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$, and $8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$. The solution to this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 0$. Therefore the answer is $a_n = 1 + (-1)^n + 2^n$.

16. This requires some linear algebra, but follows the same basic idea as the proof of Theorem 1. See textbooks such as *Introduction to Combinatorial Mathematics* by C. L. Liu (McGraw-Hill, 1968), Chapter 3.

18. This is a third degree recurrence relation. The characteristic equation is $r^3 - 6r^2 + 12r - 8 = 0$. By the rational root test, the possible rational roots are $\pm 1, \pm 2, \pm 4$. We find that $r = 2$ is a root. Dividing $r - 2$ into $r^3 - 6r^2 + 12r - 8$, we find that $r^3 - 6r^2 + 12r - 8 = (r-2)(r^2 - 4r + 4)$. By inspection we factor the rest, obtaining $r^3 - 6r^2 + 12r - 8 = (r-2)^3$. Hence the only root is 2, with multiplicity 3, so the general solution is (by Theorem 4) $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned} -5 &= a_0 = \alpha_1 \\ 4 &= a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \\ 88 &= a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 . \end{aligned}$$

Solving this system of equations, we get $\alpha_1 = -5$, $\alpha_2 = 1/2$, and $\alpha_3 = 13/2$. Therefore the answer is $a_n = -5 \cdot 2^n + (n/2) \cdot 2^n + (13n^2/2) \cdot 2^n = -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot 2^{n-1}$.

20. This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, which factors as $(r^2-4)^2$, which then further factors into $(r-2)^2(r+2)^2$. The roots are 2 and -2, each with multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3(-2)^n + \alpha_4 n \cdot (-2)^n$.

22. This is similar to Example 6. We can immediately write down the general solution using Theorem 4. In this case there are four distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1. So the general solution is $a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n + (\alpha_{3,0} + \alpha_{3,1}n)5^n + \alpha_{4,0}7^n$.

24. a) We compute the right-hand side of the recurrence relation: $2(n-1)2^{n-1} + 2^n = (n-1)2^n + 2^n = n2^n$, which is the left-hand side.

b) The solution of the associated homogeneous equation $a_n = 2a_{n-1}$ is easily found to be $a_n = \alpha 2^n$. Therefore the general solution of the inhomogeneous equation is $a_n = \alpha 2^n + n2^n$.

c) Plugging in $a_0 = 2$, we obtain $\alpha = 2$. Therefore the solution is $a_n = 2 \cdot 2^n + n2^n = (n+2)2^n$.

26. We need to use Theorem 6, and so we need to find the roots of the characteristic polynomial of the associated homogeneous recurrence relation. The characteristic equation is $r^3 - 6r^2 + 12r - 8 = 0$, and as we saw in Exercise 18, $r = 2$ is the only root, and it has multiplicity 3.

a) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_2n^2 + p_1n + p_0$. In the notation of Theorem 6, $s = 1$ here.

b) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3p_0 2^n$.

c) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3(p_1 n + p_0)2^n$.

d) Since -2 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_0(-2)^n$.

e) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3(p_2n^2 + p_1 n + p_0)2^n$.

f) Since -2 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $(p_3n^3 + p_2n^2 + p_1n + p_0)(-2)^n$.

g) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form p_0. In the notation of Theorem 6, $s = 1$ here.

28. **a)** The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = p_2n^2 + p_1n + p_0$. (Note that $s = 1$ here, and 1 is not a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $p_2n^2 + p_1n + p_0 = 2(p_2(n-1)^2 + p_1(n-1) + p_0) + 2n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2 - 2)n^2 + (4p_2 - p_1)n + (-2p_2 + 2p_1 - p_0) = 0$. In order for this equation to be true for all n, we must have $p_2 = -2$, $4p_2 = p_1$, and $-2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -8$ and $p_0 = -12$. Therefore the particular solution we seek is $a_n^{(p)} = -2n^2 - 8n - 12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

b) We plug the initial condition into our solution from part **(a)** to obtain $4 = a_1 = 2\alpha - 2 - 8 - 12$. This tells us that $\alpha = 13$. So the solution is $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$.

30. **a)** The associated homogeneous recurrence relation is $a_n = -5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 + 5r + 6 = 0$, find that $r = -2$ and $r = -3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha(-2)^n + \beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c \cdot 4^n$. We plug this into our recurrence relation and obtain $c \cdot 4^n = -5c \cdot 4^{n-1} - 6c \cdot 4^{n-2} + 42 \cdot 4^n$. We divide through by 4^{n-2}, obtaining $16c = -20c - 6c + 42 \cdot 16$, whence with a little simple algebra $c = 16$. Therefore the particular solution we seek is $a_n^{(p)} = 16 \cdot 4^n = 4^{n+2}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha(-2)^n + \beta(-3)^n + 4^{n+2}$.

b) We plug the initial conditions into our solution from part **(a)** to obtain $56 = a_1 = -2\alpha - 3\beta + 64$ and $278 = a_2 = 4\alpha + 9\beta + 256$. A little algebra yields $\alpha = 1$ and $\beta = 2$. So the solution is $a_n = (-2)^n + 2(-3)^n + 4^{n+2}$.

32. The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = cn \cdot 2^n$. We plug this into our recurrence relation and obtain $cn \cdot 2^n = 2c(n-1)2^{n-1} + 3 \cdot 2^n$. We divide through by 2^{n-1}, obtaining $2cn = 2c(n-1) + 6$, whence with a little simple algebra $c = 3$. Therefore the particular solution we seek is $a_n^{(p)} = 3n \cdot 2^n$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + 3n \cdot 2^n = (3n + \alpha)2^n$.

34. The associated homogeneous recurrence relation is $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$. To solve it we find the characteristic equation $r^3 - 7r^2 + 16r - 12 = 0$. By the rational root test we soon discover that $r = 2$ is a root and factor our equation into $(r-2)^2(r-3) = 0$. Therefore the general solution of the homogeneous relation is $a_n^{(h)} = \alpha 2^n + \beta n \cdot 2^n + \gamma 3^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = (cn+d)4^n$, since the coefficient of 4^n in our given relation is a linear function of n, and 4 is not a root of the characteristic equation. We plug this into our recurrence relation and obtain $(cn+d)4^n = 7(cn-c+d)4^{n-1} - 16(cn-2c+d)4^{n-2} + 12(cn-3c+d)4^{n-3} + n \cdot 4^n$. We divide through by 4^{n-2}, expand and collect terms (a tedious process, to be sure), obtaining $(c-16)n + (5c+d) = 0$. Therefore $c = 16$ and $d = -80$, so the particular solution we seek is $a_n^{(p)} = (16n-80)4^n$. Thus the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + \beta n \cdot 2^n + \gamma 3^n + (16n-80)4^n$. Next we plug in the initial conditions to obtain $-2 = a_0 = \alpha + \gamma - 80$, $0 = a_1 = 2\alpha + 2\beta + 3\gamma - 256$, and $5 = a_2 = 4\alpha + 8\beta + 9\gamma - 768$. We solve this system of three linear equations in three unknowns by standard methods to obtain $\alpha = 17$, $\beta = 39/2$, and $\gamma = 61$. So the solution is $a_n = 17 \cdot 2^n + 39n \cdot 2^{n-1} + 61 \cdot 3^n + (16n-80)4^n$. As a check of our work (it would be too much to hope that we could always get this far without making an algebraic error), we can compute a_3 both from the recurrence and from the solution, and we find that $a_3 = 203$ both ways.

36. Obviously the n^{th} term of the sequence comes from the $(n-1)^{\text{st}}$ term by adding n^2; in symbols, $a_{n-1} + n^2 = \left(\sum_{k=1}^{n-1} k^2\right) + n^2 = \sum_{k=1}^{n} k^2 = a_n$. Also, the sum of the first square is clearly 1. To solve this recurrence relation, we easily see that the homogeneous solution is $a_n = \alpha$, so since the nonhomogeneous term is a second degree polynomial, we need a particular solution of the form $a_n = cn^3 + dn^2 + en$. Plugging this into the recurrence relation gives $cn^3 + dn^2 + en = c(n-1)^3 + d(n-1)^2 + e(n-1) + n^2$. Expanding and collecting terms, we have $(3c-1)n^2 + (-3c+2d)n + (c-d+e) = 0$, whence $c = 1/3$, $d = 1/2$, and $e = 1/6$. Thus $a_n^{(h)} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. So the general solution is $a_n = \alpha + \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. It is now a simple matter to plug in the initial condition to see that $\alpha = 0$. Note that we can find a common denominator and write our solution in the familiar form $a_n = n(n+1)(2n+1)/6$, as was noted in Table 2 of Section 2.4 and proved by mathematical induction in Exercise 3 of Section 4.1.

38. a) The characteristic equation is $r^2 - 2r + 2 = 0$, whose roots are, by the quadratic formula, $1 \pm \sqrt{-1}$, in other words, $1+i$ and $1-i$.

b) The general solution is, by part (a), $a_n = \alpha_1(1+i)^n + \alpha_2(1-i)^n$. Plugging in the initial conditions gives us $1 = \alpha_1 + \alpha_2$ and $2 = (1+i)\alpha_1 + (1-i)\alpha_2$. Solving these linear equations tells us that $\alpha_1 = \frac{1}{2} - \frac{1}{2}i$ and $\alpha_2 = \frac{1}{2} + \frac{1}{2}i$. Therefore the solution is $a_n = (\frac{1}{2} - \frac{1}{2}i)(1+i)^n + (\frac{1}{2} + \frac{1}{2}i)(1-i)^n$.

40. First we reduce this system to a recurrence relation and initial conditions involving only a_n. If we subtract the two equations, we obtain $a_n - b_n = 2a_{n-1}$, which gives us $b_n = a_n - 2a_{n-1}$. We plug this back into the first equation to get $a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2}$, our desired recurrence relation in one variable. Note also that the first of the original equations gives us the necessary second initial condition, namely $a_1 = 3a_0 + 2b_0 = 7$. We now solve this problem for $\{a_n\}$ in the usual way. The roots of the characteristic equation $r^2 - 5r + 4 = 0$ are 1 and 4, and the solution, after solving for the arbitrary constants, is $a_n = -1 + 2 \cdot 4^n$. Finally, we plug this back into the equation $b_n = a_n - 2a_{n-1}$ to find that $b_n = 1 + 4^n$.

42. We can prove this by induction on n. If $n = 1$, then the assertion is $a_1 = s \cdot f_0 + t \cdot f_1 = s \cdot 0 + t \cdot 1 = t$, which is given; and if $n = 2$, then the assertion is $a_2 = s \cdot f_1 + t \cdot f_2 = s \cdot 1 + t \cdot 1 = s + t$, which is true, since $a_2 = a_1 + a_0 = t + s$. Having taken care of the base cases, we assume the inductive hypothesis, that the statement is true for values less than n. Then $a_n = a_{n-1} + a_{n-2} = (sf_{n-2} + tf_{n-1}) + (sf_{n-3} + tf_{n-2}) = s(f_{n-2} + f_{n-3}) + t(f_{n-1} + f_{n-2}) = sf_{n-1} + tf_n$, as desired.

44. We can compute the first few terms by hand. For $n = 1$, the matrix is just the number 2, so $d_1 = 2$. For the initial condition for $\{b_n\}$ is $b_0 = g(1)Q(1)a_0 = g(1)(1/g(1))a_0 = a_0 = C$, since it is conventional to view an empty product as the number 1.

b) Since $\{b_n\}$ satisfies the trivial recurrence relation shown in part (a), we see immediately that

$$\begin{aligned} b_n &= Q(n)h(n) + b_{n-1} = Q(n)h(n) + Q(n-1)h(n-1) + b_{n-2} = \cdots \\ &= \sum_{i=1}^{n} Q(i)h(i) + b_0 = \sum_{i=1}^{n} Q(i)h(i) + C. \end{aligned}$$

The value of a_n follows from the definition of b_n given in part (a).

50. a) We can show this by proving that $nC_n - (n+1)C_{n-1} = 2n$, so let us calculate, using the given recurrence:

$$\begin{aligned} nC_n - (n+1)C_{n-1} &= nC_n - (n-1)C_{n-1} - 2C_{n-1} \\ &= n^2 + n + 2\sum_{k=0}^{n-1} C_k - (n-1)\left(n + \frac{2}{n-1}\sum_{k=0}^{n-2} C_k\right) - 2C_{n-1} \\ &= n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1} - n^2 + n - 2\sum_{k=0}^{n-2} C_k - 2C_{n-1} = 2n. \end{aligned}$$

b) We use the formula given in Exercise 48. Note first that $f(n) = n$, $g(n) = n+1$, and $h(n) = 2n$. Thus $Q(n) = \dfrac{(n-1)!}{(n+1)!} = \dfrac{1}{n(n+1)}$. Plugging this into the formula gives

$$\frac{0 + \sum_{i=1}^{n} \dfrac{2i}{i(i+1)}}{(n+2) \cdot \dfrac{1}{(n+1)(n+2)}} = 2(n+1)\sum_{i=1}^{n} \frac{1}{i+1}.$$

There is no nice closed form way to write this sum (the harmonic series), but we can check that both this formula and the recurrence yield the same values of C_n for small n (namely, $C_1 = 2$, $C_2 = 5$, $C_3 = 26/3$, and so on).

52. A proof of this theorem can be found in textbooks such as *Discrete Mathematics with Applications* by H. E. Mattson, Jr. (Wiley, 1993), Chapter 11.