GUIDE TO REVIEW QUESTIONS FOR CHAPTER 8

1. **a)** See p. 521 (which refers to the definition on p. 519). **b)** See Example 6 in Section 8.1.

2. **a)** See p. 522. **b)** See p. 523. **c)** See p. 523. **d)** See p. 524.

3. **a)** $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$
b) $\emptyset$ **c)** $\{(1,1),(1,2),(2,2),(2,3),(3,3),(4,4)\}$
d) $\{(1,1),(2,2),(3,3),(4,4)\}$ **e)** $\{(1,1),(2,2),(3,3),(4,4)\}$

4. **a)** See Example 16 in Section 8.1. **b)** See Exercise 45a in Section 8.1.
c) See Exercise 45b in Section 8.1.

5. **a)** See p. 531. **b)** Take the projection $P_{1,4,5}$.
c) First rearrange the order of the fields in the relations, so that the first is in the order address, telephone number, name, major, and the second is in the order name, major, student number, number of credit hours. Then form the join J_2, to get a single relation with the fields in the order address, telephone number, name, major, student number, number of credit hours. Finally, if desired, rearrange the fields to a more natural order.

6. **a)** See p. 538. **b)** See pp. 538–539.

7. **a)** See p. 541. **b)** See pp. 541–542.

8. **a)** See pp. 544–545. **b)** Add all the pairs (a,a).
c) Whenever a pair (a,b) is in the relation, add the pair (b,a).
d) The reflexive closure is $\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,1),(3,3),(4,4)\}$, and the symmetric closure is $\{(1,2),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,2)\}$.

9. **a)** the smallest transitive relation containing R **b)** no **c)** See Algorithms 1 and 2 in Section 8.4.
d) the relation that always holds between two elements of $\{1,2,3,4\}$ (in symbols, $\{1,2,3,4\} \times \{1,2,3,4\}$)

10. a) See p. 555.

b) One equivalence relation is the one with equivalence classes $\{a, b, d\}$ and $\{c\}$. The only other one meeting this condition is the relation that always holds, i.e., in which there is just one equivalence class, $\{a, b, c, d\}$.

11. a) See Example 3 in Section 8.5.

b) It is easy to see that this relation is
$\{(0,0),(1,1),(1,6),(6,1),(6,6),(2,2),(2,5),(5,2),(5,5),(3,3),(3,4),(4,3),(4,4)\}$.

12. a) See p. 558.

b) $[0] = \{\ldots,-5,0,5,\ldots\}$, $[1] = \{\ldots,-4,1,6,\ldots\}$, $[2] = \{\ldots,-3,2,7,\ldots\}$, $[3] = \{\ldots,-2,3,8,\ldots\}$, and $[4] = \{\ldots,-1,4,9,\ldots\}$

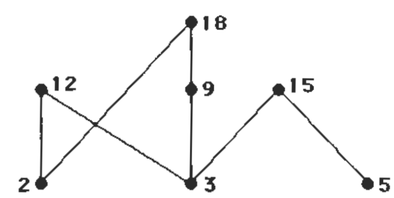
c) $\{0\}$, $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$

13. See Theorem 2 in Section 8.5.

14. a) See p. 566. b) See Example 2 in Section 8.6.

15. See the definition of lexicographic ordering on p. 569.

16. a) See p. 571. b) Here is the Hasse diagram:



17. a) See pp. 572–573. b) $(\{1,2,3,4,5\}, \mid)$ c) $(\{1,2,3,4,5\}, \leq)$

18. a) See p. 574.

b) Extreme examples are the lattice $(\{1,2,3,4,5\}, \leq)$ (every total order is a lattice) and the nonlattice $(\{1,2,3,4,5\}, =)$ (no two distinct elements have an upper bound).

19. a) See Exercise 41 in Section 8.6. b) See Exercise 51 in Section 8.6

20. a) See p. 568. (We assume for the rest of this question that the set is finite, in which case a well-ordered set is the same as a totally ordered set, as defined also on p. 568.) b) See topological sorting on pp. 576–577.

c) See Example 27 in Section 8.6.

SUPPLEMENTARY EXERCISES FOR CHAPTER 8

1. a) This relation is not reflexive, since most strings have many letters in common with themselves. Whether it is irreflexive depends on whether we mean to include the empty string; the empty string is the only string s such that $(s,s) \in R_1$ (the empty string has no letters in common with itself, since it has no letters). Thus if we mean not to include the empty string in the underlying set, then the relation is irreflexive; otherwise it is not. The relation is symmetric by inspection (the roles of a and b in the sentence are symmetric). It is not antisymmetric, since there are many pairs of strings such that $(a,b) \in R_1 \wedge (b,a) \in R_1$; for instance $a =$ *bullfinch* and $b =$ *parrot*. The relation is not transitive, since, for example, although *bullfinch* and *parrot* are related, and *parrot* and *chicken* are related, *bullfinch* and *chicken* do have letters in common and so are not related.

b) This relation is very similar to the relation R_1. For no string is $(a,a) \in R_2$, so the relation is irreflexive and not reflexive. It is symmetric by inspection, and not antisymmetric (same example as above). It is also not transitive, since, for instance, *finch* is related to *parrot*, which is related to *robin*, but *finch* is not related to *robin*, since they are the same length.

c) No string is longer than itself, so R_3 is irreflexive and not reflexive. It is not symmetric, since, for instance, *robin* is longer than *wren*, but *wren* is not longer than *robin*. It is antisymmetric: there is no way for both a to be longer than b and b to be longer than a. Finally, it is clearly transitive, since if a is longer than b which is longer than c, then a is longer than c.

3. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x-y$. Therefore by Exercise 9 in Section 8.5, this is an equivalence relation.

5. Suppose that $(a,b) \in R$. We must show that $(a,b) \in R^2$. By reflexivity, we know that $(b,b) \in R$. Therefore by the definition of R^2, we combine the facts that $(a,b) \in R$ and $(b,b) \in R$ to conclude that $(a,b) \in R^2$.

7. Both of these conclusions are valid. Since each pair (a,a) is in both R_1 and R_2, we can conclude that each pair (a,a) is in $R_1 \cap R_2$ and $R_1 \cup R_2$.

9. Both of these conclusions are valid. For the first, suppose that $(a,b) \in R_1 \cap R_2$. This means that $(a,b) \in R_1$ and $(a,b) \in R_2$. By the symmetry of R_1 and R_2, we conclude that $(b,a) \in R_1$ and $(b,a) \in R_2$. Therefore (b,a) is in their intersection, as desired. For the second part, suppose that $(a,b) \in R_1 \cup R_2$. This means that $(a,b) \in R_1$ or $(a,b) \in R_2$. By the symmetry of R_1 and R_2, we conclude either that $(b,a) \in R_1$ or that $(b,a) \in R_2$. Therefore (b,a) is in their union, as desired.

11. A primary key is one for which there are no two different rows with the same value in this field. If there were two different rows with the same value after projection, then there certainly would have been two different rows with the same value before projection.

13. The key point is that $\Delta^{-1} = \Delta$, where Δ consists of all the pairs (a,a). Thus it does not matter whether we add the pairs in Δ before or after we add the reverse of every pair in the original relation.

15. a) We observed in Exercise 29b in Section 8.4 that we need to take the symmetric closure first in order to insure that the result is symmetric. The relation given in that exercise provides an example. An even simpler one is the relation $\{(0,1),(2,1)\}$; the symmetric closure of the transitive closure is $\{(0,1),(1,0),(1,2),(2,1)\}$, but the transitive closure of the symmetric closure is all of $\{0,1,2\} \times \{0,1,2\}$.

b) Suppose that (a,b) is in the symmetric closure of the transitive closure of R. We must show that (a,b) is also in the transitive closure of the symmetric closure of R. Now either (a,b) or (b,a) is in the transitive closure of R. This means that either there is a path from a to b or a path from b to a in R. In the former case, there is perforce a path from a to b in the symmetric closure of R. In the latter case, the path from b to a can be followed backwards in the symmetric closure of R, since the symmetric closure adds the reverses of all the edges in R. Therefore in either case (a,b) is in the transitive closure of the symmetric closure of R. (See also the related Exercise 23 in Section 8.4.)

17. The closure of S with respect to **P** is a relation S' which contains S as a subset and has property **P**. Since $R \subseteq S$, we conclude that $R \subseteq S'$. By definition of closure, then, the closure of R must be a subset of S', as desired.

19. We use the basic idea of Warshall's algorithm, except that $w_{ij}^{[k]}$ will be a numerical variable (taking values from 0 to ∞, inclusive) representing the length of the longest path from v_i to v_j all of whose interior vertices are labeled less than or equal to k, rather than simply a Boolean variable indicating whether such a path exists. A value of 0 for $w_{ij}^{[k]}$ will mean that there is no path from v_i to v_j all of whose interior vertices are labeled less than or equal to k. To compute $w_{ij}^{[k]}$ from the matrix $\mathbf{W}_{k-1}$, we determine, for each pair (i,j), whether there are paths from v_i to v_k and from v_k to v_j using no interior vertices labeled greater than $k-1$. If either of $w_{ik}^{[k-1]}$ or $w_{kj}^{[k-1]}$ equals 0, then such a pair of paths does not exist, so we set $w_{ij}^{[k]}$ equal to $w_{ij}^{[k-1]}$. Otherwise (if such a pair of paths does exist), then there are two possibilities. If $w_{kk}^{[k-1]} > 0$, then we now know that there are paths of arbitrary length from v_i to v_j, since we can loop around v_k as long as we please; in this case we set $w_{ij}^{[k]}$ to ∞. If $w_{kk}^{[k-1]} = 0$, then we do not yet have such looping, so we set $w_{ij}^{[k]}$ to the larger of $w_{ij}^{[k-1]}$ and $w_{ik}^{[k-1]} + w_{kj}^{[k-1]}$. (Initially we set $\mathbf{W}_0$ equal to the matrix representing the relation.)

21. There are 52 partitions in all, but that is not the question. If there are to be three equivalence classes, then the classes must have sizes $3,1,1$ or $2,2,1$. There are $C(5,3) = 10$ partitions into one set with 3 elements and the other two sets of 1 element each, since the only choice involved is choosing the 3-set. There are $C(5,2)C(3,2)/2 = 15$ ways to partition our set into sets of size 2, 2, and 1; we need to choose the 2 elements for the first set of size 2, then we need to choose the 2 elements from the 3 remaining for the second set of size 2, except that we have overcounted by a factor of 2, since we could choose these two 2-sets in either order. Therefore there are $10 + 15 = 25$ partitions into three classes.

23. There is no question that the collection defined here is a refinement of each of the given partitions, since each set $A_i \cap B_j$ is a subset of A_i and of B_j. We must show that it is actually a partition. By construction, each of the sets in this collection is nonempty. To see that their union is all of S, let $s \in S$. Since P_1 and P_2 are partitions of S, there are sets A_i and B_j such that $s \in A_i$ and $s \in B_j$. Therefore $s \in A_i \cap B_j$, which shows that s is in one of the sets in our collection. Finally, to see that these sets are pairwise disjoint, simply note that unless $i = i'$ and $j = j'$, then $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = (A_i \cap A_{i'}) \cap (B_j \cap B_{j'})$ is empty, since either $(A_i \cap A_{i'})$ or $(B_j \cap B_{j'})$ is empty.

25. The subset relation is a partial order on *every* collection of sets, since it is reflexive, antisymmetric, and transitive. Here the collection of sets happens to be $\mathbf{R}(S)$.

27. We need to find a total order compatible with this partial order. We work from the bottom up, writing down a task (vertex in the diagram) and removing it from the diagram, so that at each stage we choose a vertex with no vertices below it. One such order is: Find recipe $\prec$ Buy seafood $\prec$ Buy groceries $\prec$ Wash shellfish $\prec$ Cut ginger and garlic $\prec$ Clean fish $\prec$ Steam rice $\prec$ Cut fish $\prec$ Wash vegetables $\prec$ Chop water chestnuts $\prec$ Make garnishes $\prec$ Cook in wok $\prec$ Arrange on platter $\prec$ Serve.

29. Since every subset of an antichain is clearly an antichain, we will list only the maximal antichains; the actual answers will be everything we list together with all the subsets of them.

a) Here every two elements are comparable except c and d. Thus the maximal antichains are $\{c,d\}$, $\{a\}$, and $\{b\}$. (There are three more antichains which are subsets of these: $\{c\}$, $\{d\}$, and $\emptyset$.)

b) Here the maximal antichains are $\{a\}$, $\{b,c\}$, $\{c,e\}$, and $\{d,e\}$.

c) In this case there are only three maximal antichains: $\{a,b,c\}$, $\{d,e,f\}$, and $\{g\}$.

31. Let C be a maximal chain. We must show that C contains a minimal element of S. Since C can itself be viewed as a finite poset (being a subset of a poset), it contains a minimal element m. We need to show that m is also a minimal element of S. If it were not, then there would be another element $a \in S$ such that $a \prec m$. Now we claim that $C \cup \{a\}$ is a chain, which will contradict the maximality of C. We need to show that a is comparable to every element of C. We already know that a is comparable to m. Let x be any other element of C. Since m is minimal in C, it cannot be that $x \prec m$; thus since x and m have to be comparable (they are both in C), it must be tbat $m \prec x$. Now by transitivity we have $a \prec x$, and we are done.

33. Consider the relation R on the set of $mn + 1$ people given by $(a, b) \in R$ if and only if a is a descendant of or equal to b. This makes the collection into a poset. In the terminology of Exercise 32, if there is not a subset of $n + 1$ people none of whom is a descendant of any other, then $k \leq n$, since such a subset is certainly an antichain. Therefore the poset can be partitioned into $k \leq n$ chains. Now by the generalized pigeonhole principle, at least one of these chains must contain at least $m + 1$ elements, and this is the desired list of descendants.

35. Recall the definition of well-founded from the preamble to Exercise 55 in Section 8.6—that there is no infinite decreasing sequence. We must show that under this hypothesis, and if $\forall x((\forall y(y \prec x \rightarrow P(y))) \rightarrow P(x))$, then $P(x)$ is true for all $x \in S$. We give a proof by contradiction. If it does not hold that $P(x)$ is true for all $x \in S$, let x_1 be an element of S such that $P(x_1)$ is not true. Then by the conditional statement given above, it must be the case that $\forall y(y \prec x_1 \rightarrow P(y))$ is not true. This means that there is some y with $y \prec x_1$ such that $P(y)$ is not true. Rename this y as x_2. So we know that $P(x_2)$ is not true. Again invoking the conditional statement, we get an $x_3 \prec x_2$ such that $P(x_3)$ is not true. And so on forever. This contradicts the well-foundedness of our poset. Therefore $P(x)$ is true for all $x \in S$.

37. We assume that R is reflexive and transitive on A, and we must show that $R \cap R^{-1}$ is reflexive, symmetric, and transitive. Reflexivity is easy: if $a \in A$, then we know that $(a, a) \in R$, so by the definition of R^{-1} as the reverses of the pairs in R, we know that $(a, a) \in R^{-1}$ as well, whence it follows that $(a, a) \in R \cap R^{-1}$. *Every* relation of the form $R \cap R^{-1}$ is symmetric, no matter what R is, since if $(a, b) \in R$, then $(b, a) \in R^{-1}$ and vice versa. For transitivity, suppose that $(a, b) \in R \cap R^{-1}$ and $(b, c) \in R \cap R^{-1}$. We must show that $(a, c) \in R \cap R^{-1}$. Since $(a, b) \in R$ and $(b, c) \in R$, and since R is transitive, $(a, c) \in R$. Similarly, since $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, $(b, a) \in R$ and $(c, b) \in R$. Again, since R is transitive, $(c, a) \in R$, and hence $(a, c) \in R^{-1}$. Putting these two parts together, we conclude that $(a, c) \in R \cap R^{-1}$, as desired.

39. There is not much to show in this exercise, since the definitions of greatest lower bound and least upper bound exhibit these properties by their very form.

a) The g.l.b. of x and y was defined to be the greatest element that is a lower bound of both x and y. The roles of x and y in this statement are symmetric, so it follows immediately that $x \wedge y = y \wedge x$. Similarly for least upper bound.

b) By definition, $(x \wedge y) \wedge z$ is a lower bound of x, y, and z that is greater than every other common lower bound (this is how we proceeded in Exercise 45 of Section 8.6). Since x, y, and z play interchangeable roles in this statement, grouping does not matter, so $x \wedge (y \wedge z)$ is the same element. Similarly for l.u.b.

c) The two statements are duals, so we will prove just the first one; the proof of the second can be obtained formally simply by exchanging each symbol and word for its dual. To show that $x \wedge (x \vee y) = x$, we must show that x is the greatest lower bound of x and $x \vee y$. Clearly x is a lower bound for x, and since $x \vee y$ is by definition greater than or equal to x, x is a lower bound for it as well. Therefore x is a lower bound. But every other lower bound for x has to be less than x, so x is the greatest lower bound.

d) Obviously x is a lower (upper) bound for itself and itself, and the greatest (least) such.

41. There is nothing very deep going on here—it's just a matter of applying the definitions.

a) Since 1 is the only element greater than or equal to 1, it is the only upper bound for 1 and therefore the only possible value of the least upper bound of x and 1.

b) Clearly x is a lower bound for both x and 1 (since $x \preceq 1$), and clearly no other lower bound can be greater than x, so $x \wedge 1 = x$.

c) This is the dual to part **(b)**. We formed the following proof on the word processor used to produce this solutions manual by copying the words in the solution to part **(b)** and replacing each word and symbol by its dual: Clearly x is an upper bound for both x and 0 (since $0 \preceq x$), and clearly no other upper bound can be smaller than x, so $x \vee 0 = x$.

d) This is the dual of part **(a)**: Since 0 is the only element less than or equal to 0, it is the only lower bound for 0 and therefore the only possible value of the greatest lower bound of x and 0.

43. One way to solve this problem is to play around with some small examples. Here is one counter-example that the author obtained in this way. The lattice has as its elements $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$, and $\{1,2,3\}$, with, as usual, the relation $\subseteq$. (Draw its Hasse diagram!) It is easy to check that every two elements have both a least upper bound and a greatest lower bound (note that $\emptyset$ is a lower bound for the whole lattice, and $\{1,2,3\}$ is an upper bound for the whole lattice). Take $x = \{1\}$, $y = \{2\}$, and $z = \{3\}$, and compute both sides of the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Note that since we do not have the full subset lattice, least upper bounds are not just unions. The left-hand side is x, since $y \wedge z = \emptyset$. The right-hand side is the greatest lower bound of $\{1,2\}$ and $\{1,2,3\}$, which is $\{1,2\}$. Since these are different, the lattice is uot distributive.

45. Yes. First, recall from Example 22 in Section 8.6 that $x \wedge y$ is the greatest common divisor (gcd) of x and y, while $x \vee y$ is their least common multiple (lcm). We can analyze this problem by looking at prime factorizations. The power to which a prime p appears in the gcd of two numbers is the minimum of the powers to which it appears in the two numbers. Similarly, the power to which p appears in the lcm is the maximum of the powers to which it appears in the two numbers. Thus if we let a, b, and c represent the powers to which p appears in x, y, and z, respectively, the first identity we need to prove is

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)).$$

We consider the several cases. If a is the largest of the three numbers, then both sides equal a. If a is the smallest, then both sides equal the smaller of b and c. Otherwise, we can suppose without loss of generality (since the roles of b and c are symmetric) that $b \le a \le c$, in which case we easily compute that both sides equal a. The proof of the other statement is dual to this proof. The result now follows from the Fundamental Theorem of Arithmetic, since numbers are determined by their prime factorizations.

47. As might be expected from the name, the complement of a subset $X \subseteq S$ is its complement $S - X$. To prove this, we need to prove that $X \vee (S - X) = 1$ and $X \wedge (S - X) = 0$, which translated into our particular setting reads: $X \cup (S - X) = S$ and $X \cap (S - X) = \emptyset$. But these are trivially true.

49. Think of the rectangular grid as representing elements in a matrix. Thus we number from top to bottom and within that from left to right. For example, $(2,4)$ is the element in row 2, column 4. The partial order is that $(a,b) \preceq (c,d)$ if $a \le c$ and $b \le d$. Note that $(1,1)$ is the least element under this relation. The rules for Chomp as explained in Chapter 1 coincide with the rules stated in the preamble here. But now we can identify the point (a,b) with the natural number $p^{a-1}q^{b-1}$ for all a and b with $1 \le a \le m$ and $1 \le b \le n$. This identifies the points in the rectangular grid with the set S in this exercise, and the partial order $\preceq$ just described is the same as the divides relation, because $p^{a-1}q^{b-1} \mid p^{c-1}q^{d-1}$ if and only if the exponent on p on the left does not exceed the exponent of p on the right, and similarly for q.

WRITING PROJECTS FOR CHAPTER 8

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. See the same references as suggested for fuzzy logic in Writing Project 2 of Chapter 1, as well as [Zi].

2. There are numerous textbooks on databases. Try to consult one that is fairly recent, because in many areas of computer science, progress is so fast that books soon become out-dated. You will find them in the QA 76.9 area of the library's shelves. Two recommended ones are are [Da1] and [Ma1].

3. Try author or key-word search in an appropriate database (e.g., one provided by *Mathematical Reviews*, which is available on the Web as MathSciNet). Consult the oldest reference you can find that talks about these topics, and it will probably lead you to the original sources. Simultaneous discovery occurs in many branches of intellectual pursuit, not just mathematics and computer science. See Writing Project 15 in Chapter 10 for something along the same lines.

4. The abstraction and difficulty here is part of what makes fractions hard for many children (and some adults) to handle. Be careful to avoid 0 in the denominator! You should be able to figure this out without consulting other sources, and it is a good project to work on with other people.

5. See the hints for Writing Project 3.

6. Entire books have been written on security issues in computer systems ([Pf], for one), and it should not be hard to find a chapter or two on the subject in many more general books (try [De1]).

7. Textbooks on project scheduling should be a good source of information. See [Mo], for example. Scheduling is a topic in a branch of mathematics known as Operations Research. It has its own journals, conferences, subspecialties, software, etc.

8. See the suggestions for Writing Project 7.

9. We have hinted at duality in many of the exercise solutions in this *Guide*. A book on lattice theory (such as [Gr1]) will make the concept more precise.

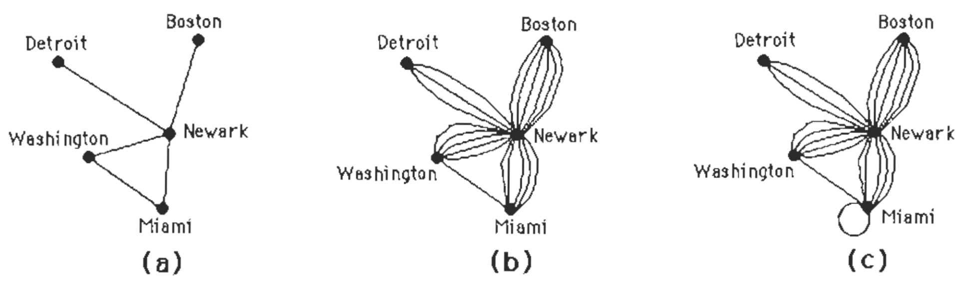
10. As mentioned in the previons suggestion, yon can find entire books on lattice theory. In fact, *Mathematical Reviews* (which is available on the Web as MathSciNet) devotes a whole category (numbered 06) to lattices and other kinds of ordered sets and ordered algebraic structures.

CHAPTER 9
Graphs

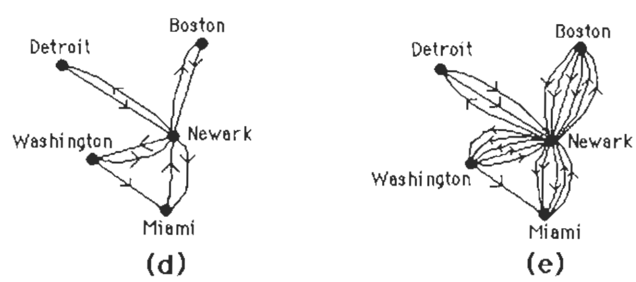
SECTION 9.1 Graphs and Graph Models

The examples and exercises give a good picture of the ways in which graphs can model various real world applications. In constructing graph models you need to determine what the vertices will represent, what the edges will represent, whether the edges will be directed or undirected, whether loops should be allowed, and whether a simple graph or multigraph is more appropriate.

1. In part **(a)** we have a simple graph, with undirected edges, no loops or multiple edges. In part **(b)** we have a multigraph, since there are multiple edges (making the figure somewhat less than ideal visually). In part **(c)** we have the same picture as in part **(b)** except that there is now a loop at one vertex; thus this is a pseudograph.



In part **(d)** we have a directed graph, the directions of the edges telling the directions of the flights; note that the **antiparallel edges** (pairs of the form (u, v) and (v, u)) are not parallel. In part **(e)** we have a directed multigraph, since there are parallel edges.



3. This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

5. This is a pseudograph; the edges are undirected, but there are loops and parallel edges.

7. This is a directed graph; the edges are directed, but there are no parallel edges. (Loops and antiparallel edges—see the solution to Exercise 1d for a definition—are allowed in a directed graph.)

9. This is a directed multigraph; the edges are directed, and there is a set of parallel edges.