SECTION 5.6 Generating Permutations and Combinations

2. 156423, 165432, 231456, 231465, 234561, 314562, 432561, 435612, 541236, 543216, 654312, 654321

4. These can be done using Algorithm 1 or Example 2. This will be explained in detail for part (a); the others are similar. In the last four parts of this exercise, the next permntation exchanges only the last two elements.
a) The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_2 = 3$ and $a_3 = 4$. The least integer to the right of 3 that is greater than 3 is 4. Hence 4 is placed in the second position. The integers 2 and 3 are then placed in order in the last two positions, giving the permutation 1423.
b) 51234 c) 13254 d) 612354 e) 1623574 f) 23587461

6. The first subset corresponds to the bit string 0000, namely the empty set. The next subset corresponds to the bit string 0001, namely the set $\{4\}$. The next bit string is 0010, corresponding to the set $\{3\}$, and then 0011, which corresponds to the set $\{3,4\}$. We continue in this manner, giving the remaining sets: $\{2\}$, $\{2,4\}$, $\{2,3\}$, $\{2,3,4\}$, $\{1\}$, $\{1,4\}$, $\{1,3\}$, $\{1,3,4\}$, $\{1,2\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{1,2,3,4\}$.

8. Since the new permutation agrees with the old one in positions 1 to $j-1$, and since the new permutation has a_k in position j, whereas the old one had a_j, with $a_k > a_j$, the new permutation succeeds the old one in lexicographic order. Furthermore the new permutation is the first one (in lexicographic order) with a_1, a_2, $\ldots$, a_{j-1}, a_k in positions 1 to j, and the old permutation was the last one with a_1, a_2, $\ldots$, a_{j-1}, a_j in those positions. Since a_k was picked to be the smallest number greater than a_j among a_{j+1}, a_{j+2}, $\ldots$, a_n, there can be no permutation between these two.

10. One algorithm would combine Algorithm 3 and Algorithm 1. Using Algorithm 3, we generate all the r-combinations of the set with n elements. At each stage, after we have found each r-combination, we use Algorithm 1, with $n = r$ (and a different collection to be permuted than $\{1, 2, \ldots, n\}$), to generate all the permutations of the elements in this combination. See the solution to Exercise 11 for an example.

12. a) We find that $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 2$, and $a_5 = 3$. Therefore the number is $1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 2 \cdot 4! + 3 \cdot 5! = 1 + 2 + 12 + 48 + 360 = 423$.

b) Each $a_k = 0$, so the number is 0.

c) We find that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, and $a_5 = 5$. Therefore the number is $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! = 1 + 4 + 18 + 96 + 600 = 719 = 6! - 1$, as expected, since this is the last permutation.

14. a) We find the Cantor expansion of 3 to be $1 \cdot 1! + 1 \cdot 2!$. Therefore we know that $a_4 = 0$, $a_3 = 0$, $a_2 = 1$, and $a_1 = 1$. Following the algorithm given in the solution to Exercise 13, we put 5 in position $5 - 0 = 5$, put 4 in position $4 - 0 = 4$, put 3 in position $3 - 1 = 2$, and put 2 in the position that is 1 from the rightmost available position, namely position 1. Therefore the answer is 23145.

b) We find that $89 = 1 \cdot 1! + 2 \cdot 2! + 2 \cdot 3! + 3 \cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 3, 2, 2, and 1 positions from the right among the available positions, obtaining 35421.

c) We find that $111 = 1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 4 \cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 4, 2, 1, and 1 positions from the right among the available positions, obtaining 52431.