SECTION 9.2 Graph Terminology and Special Types of Graphs

2. In this pseudograph there are 5 vertices and 13 edges. The degree of vertex a is 6, since in addition to the 4 nonloops incident to a, there is a loop contributing 2 to the degree. The degrees of the other vertices are $\deg(b) = 6$, $\deg(c) = 6$, $\deg(d) = 5$, and $\deg(e) = 3$. There are no pendant or isolated vertices in this pseudograph.

4. For the graph in Exercise 1, the sum is $2+4+1+0+2+3 = 12 = 2\cdot 6$; there are 6 edges. For the pseudograph in Exercise 2, the sum is $6+6+6+5+3 = 26 = 2\cdot 13$; there are 13 edges. For the pseudograph in Exercise 3, the sum is $3+2+4+0+6+0+4+2+3 = 24 = 2\cdot 12$; there are 12 edges.

6. Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1 the sum of the degrees is even (it is $2e$).

8. In this directed multigraph there are 4 vertices and 8 edges. The degrees are $\deg^-(a) = 2$, $\deg^+(a) = 2$, $\deg^-(b) = 3$, $\deg^+(b) = 4$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 1$.

10. For Exercise 7 the sum of the in-degrees is $3+1+2+1 = 7$, and the sum of the out-degrees is $1+2+1+3 = 7$; there are 7 edges. For Exercise 8 the sum of the in-degrees is $2+3+2+1 = 8$, and the sum of the out-degrees is $2+4+1+1 = 8$; there are 8 edges. For Exercise 9 the sum of the in-degrees is $6+1+2+4+0 = 13$, and the sum of the out-degrees is $1+5+5+2+0 = 13$; there are 13 edges.

12. Since there is an edge from a person to each of his or her acquaintances, the degree of v is the number of people v knows. An isolated vertex would be a person who knew on one, and a pendant vertex would be a person who knew just one other person (it is doubtful that there are many, if any, isolated or pendant vertices). If the average degree is 1000, then the average person knows 1000 other people.

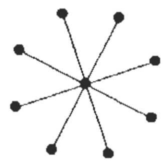
14. Since there is an edge from a person to each of the other actors that person has appeared with in a movie, the degree of v is the number of other actors that person has appeared with. An isolated vertex would be a person who has appeared only in movies in which he or she was the only actor, and a pendant vertex would be a person who has appeared with only one other actor in any movie (it is doubtful that there are many, if any, isolated or pendant vertices).

16. Since there is an edge from a page to each page that it links to, the outdegree of a vertex is the number of links on that page, and the in-degree of a vertex is the number of other pages that have a link to it.

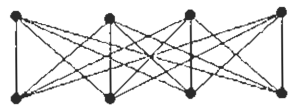
18. This is essentially the same as Exercise 36 in Section 5.2, where the graph models the "know each other" relation on the people at the party. See the solution given for that exercise. The number of people a person knows is the degree of the corresponding vertex in the graph.

20. a) This graph has 7 vertices, with an edge joining each pair of distinct vertices.

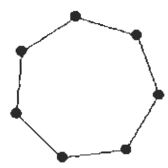
b) This graph is the complete bipartite graph on parts of size 1 and 8; we have put the part of size 1 in the middle.



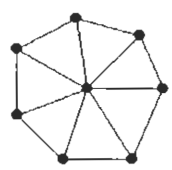
c) This is the complete bipartite graph with 4 vertices in each part.



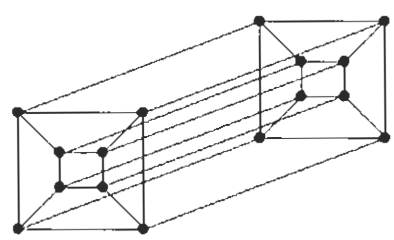
d) This is the 7-cycle.



e) The 7-wheel is the 7-cycle with an extra vertex joined to the other 7 vertices.



f) We take two copies of Q_3 and join corresponding vertices.



22. This graph is bipartite, with bipartition $\{a, c\}$ and $\{b, d, e\}$. In fact this is the complete bipartite graph $K_{2,3}$. If this graph were missing the edge between a and d, then it would still be bipartite on the same sets, but not a complete bipartite graph.

24. This is just like Exercise 22, but here we have the complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

26. **a)** By the definition given in the text, K_1 does not have enough vertices to be bipartite. Clearly K_2 is bipartite. There is a triangle in K_n for $n > 2$, so those complete graphs are not bipartite. (See Exercise 23.)

b) First we need $n \geq 3$ for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

c) Every wheel contains triangles, so no W_n is bipartite.

d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

28. a) The partite sets are the set of women ({Anna, Barbara, Carol, Diane, Elizabeth}) and the set of men ({Jason, Kevin, Larry, Matt, Nick, Oscar}). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, and EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.

b) By trial and error we easily find a matching (it's not unique), such as AL, BK, CJ, DN, and EM.

30. We just have to count the number of edges at each vertex, and then arrange these counts in nonincreasing order. For #21, we have $4,1,1,1,1$. For #22, we have $3,3,2,2,2$. For #23, we have $4,3,3,2,2,2$. For #24, we have $4,4,2,2,2,2$. For #25, we have $3,3,3,3,2,2$.

32. Assume that $m \geq n$. Then each of the n vertices in one part has degree m, and each of the m vertices in other part has degree n. Thus the degree sequence is $m, m, \ldots, m, n, n, \ldots, n$, where the sequence contains n copies of m and m copies of n. We put the m's first because we assumed that $m \geq n$. If $n \geq m$, then of course we would put the m copies of n first. If $m = n$, this would mean a total of $2n$ copies of n.

34. The 4-wheel (see Figure 5) with one edge along the rim deleted is such a graph. It has $(4+3+3+2+2)/2 = 7$ edges.

36. a) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

c) A 6-cycle is such a graph. (See picture below.)

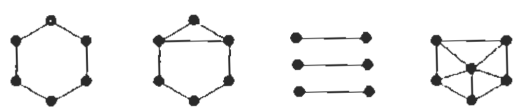
d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

e) A 6-cycle with one of its diagonals added is such a graph. (See picture below.)

f) A graph consisting of three edges with no common vertices is such a graph. (See picture below.)

g) The 5-wheel is such a graph. (See picture below.)

h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.



(c) (e) (f) (g)

38. Since isolated vertices play no essential role, we can assume that $d_n > 0$. The sequence is graphic, so there is some simple graph G such that the degrees of the vertices are d_1, d_2, $\ldots$, d_n. Without loss of generality, we can label the vertices of our graph so that $d(v_i) = d_i$. Among all such graphs, choose G to be one in which v_1 is adjacent to as many of v_2, v_3, $\ldots$, v_{d_1+1} as possible. (The worst case might be that v_1 is not adjacent to any of these vertices.) If v_1 is adjacent to all of them, then we are done. We will show that if there is a vertex among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is not adjacent to, then we can find another graph with $d(v_i) = d_i$ and having v_1 adjacent to one more of the vertices v_2, v_3, $\ldots$, v_{d_1+1} than is true for G. This is a contradiction to the choice of G, and hence we will have shown that G satisfies the desired condition.

Under this assumption, then, let u be a vertex among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is not adjacent to, and let w be a vertex not among v_2, v_3, $\ldots$, v_{d_1+1} that v_1 is adjacent to; such a vertex w has to exist because $d(v_1) = d_1$. Because the degree sequence is listed in nonincreasing order, we have $d(u) \geq d(w)$. Consider all the vertices that are adjacent to u. It cannot be the case that w is adjacent to each of them, because then w would have a higher degree than u (because w is adjacent to v_1 as well, but u is not). Therefore there is some vertex x such that edge ux is present but edge xw is not present. Note also that edge v_1w is present but edge v_1u is not present. Now construct the graph G' to be the same as G except that edges ux and v_1w are removed and edges xw and v_1u are added. The degrees of all vertices are unchanged, but this graph has v_1 adjacent to more of the vertices among $v_2, v_3, \ldots, v_{d_1+1}$ than is the case in G. That gives the desired contradiction, and our proof is complete.

40. Given a sequence $d_1, d_2, \ldots, d_n$, if $n = 2$, then the sequence is graphic if and only if $d_1 = d_2 = 1$ (the graph consists of one edge)—this is one base case. Otherwise, if $n < d_1 + 1$, then the sequence is not graphic—this is the other base case. Otherwise (this is the recursive step), form a new sequence by deleting d_1, subtracting 1 from each of $d_2, d_3, \ldots, d_{d_1+1}$, deleting all 0's, and rearranging the terms into nonincreasing order. The original sequence is graphic if and only if the resulting sequence (with $n-1$ terms) is graphic.

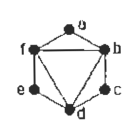
42. We list the subgraphs: the subgraph consisting of K_2 itself, the subgraph consisting of two vertices and no edges, and two subgraphs with 1 vertex each. Therefore the answer is 4.

44. We need to count this in an organized manner. First note that W_3 is the same as K_4, and it will be easier if we think of it as K_4. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 4 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(4,2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \cdot 2 = 12$ subgraphs with two vertices. If a subgraph is to have three vertices, then there are $C(4,3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways in each case to decide whether or not to include each of the edges joining pairs of them. This gives us $4 \cdot 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide which edges to include. Thus our answer is $4 + 12 + 32 + 64 = 112$.

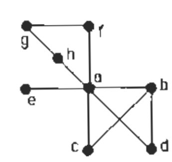
46. a) We want to show that $2e \geq vm$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot be less than the sum of m for each vertex, since each degree is no less than m.
b) We want to show that $2e \leq vM$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot exceed the sum of M for each vertex, since each degree is no greater than M.

48. Since the vertices in one part have degree m, and vertices in the other part have degree n, we conclude that $K_{m,n}$ is regular if and only if $m = n$.

50. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



52. The union is shown here. The only common vertex is a, so we have reoriented the drawing so that the pieces will not overlap.



54. The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n-1)/2$ edges, we want to solve $n(n-1)/2 = 28$. Thus $n = 8$.

56. Following the ideas given in the solution to Exercise 57, we see that the degree sequence is obtained by subtracting each of these numbers from 4 (the number of vertices) and reversing the order. We obtain $2, 2, 1, 1, 0$.

58. Suppose the parts are of sizes k and $v-k$. Then the maximum number of edges the graph may have is $k(v-k)$ (an edge between each pair of vertices in different parts). By algebra or calculus, we know that the function $f(k) = k(v-k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

60. We start by coloring any vertex red. Then we color all the vertices adjacent to this vertex blue. Then we color all the vertices adjacent to blue vertices red, then color all the vertices adjacent to red vertices blue, and so on. If we ever are in the position of trying to color a vertex with the color opposite to the color it already has, then we stop and know that the graph is not bipartite. If the process terminates (successfully) before all the vertices have been colored, then we color some uncolored vertex red (it will necessarily not be adjacent to any vertices we have already colored) and begin the process again. Eventually we will have either colored all the vertices (producing the bipartition) or stopped and decided that the graph is not bipartite.

62. Obviously $(G^c)^c$ and G have the same vertex set, so we need only show that they have the same directed edges. But this is clear, since an edge (u,v) is in $(G^c)^c$ if and only if the edge (v,u) is in G^c if and only if the edge (u,v) is in G.

64. Let $|V_1| = n_1$ and $|V_2| = n_2$. Then the number of endpoints of edges in V_1 is $n \cdot n_1$, and the number of endpoints of edges in V_2 is $n \cdot n_2$. Since every edge must have one endpoint in each part, these two expressions must be equal, and it follows (because $n \neq 0$) that $n_1 = n_2$, as desired.

66. In addition to the connections shown in Figure 13, we need to make connections between $P(i,3)$ and $P(i,0)$ for each i, and between $P(3,j)$ and $P(0,j)$ for each j. The complete network is shown here. We can imagine this drawn on a torus.