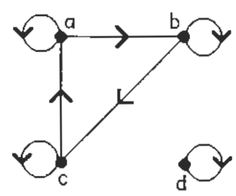
SECTION 8.4 Closures of Relations

2. When we add all the pairs (x,x) to the given relation we have all of $\mathbf{Z} \times \mathbf{Z}$; in other words, we have the relation that always holds.

4. To form the reflexive closure, we simply need to add a loop at each vertex that does not already have one.

6. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



8. To form the digraph of the symmetric closure, we simply need to add an edge from x to y whenever this edge is not already in the directed graph but the edge from y to x is.

10. The symmetric closure was found in Example 2 to be the "is not equal to" relation. If we now make this relation reflexive as well, we will have the relation that always holds.

12. $\mathbf{M}_R \vee \mathbf{I}_n$ is by definition the same as $\mathbf{M}_R$ except that it has all 1's on the main diagonal. This must represent the reflexive closure of R, since this closure is the same as R except for the addition of all the pairs (x,x) that were not already present.

14. Suppose that the closure C exists. We must show that C is the intersection I of all the relations S that have property $\mathbf{P}$ and contain R. Certainly $I \subseteq C$, since C is one of the sets in the intersection. Conversely, by definition of closure, C is a subset of every relation S that has property $\mathbf{P}$ and contains R; therefore C is contained in their intersection.

16. In each case, the sequence is a path if and only if there is an edge from each vertex in the sequence to the vertex following it.
a) This is a path. **b)** This is not a path (there is no edge from e to c). **c)** This is a path.
d) This is not a path (there is no edge from d to a). **e)** This is a path.
f) This is not a path (there is no loop at b).

18. In the language of Chapter 9, this digraph is strongly connected, so there will be a path from every vertex to every other vertex.
a) One path is a, b. **b)** One path is b, e, a. **c)** One path is b, c, b; a shorter one is just b.
d) One path is a, b, e. **e)** One path is b, e, d. **f)** One path is c, e, d.
g) One path is d, e, d. Another is the path of length 0 from d to itself.
h) One path is e, a. Another is $e, a, b, e, a, b, e, a, b, e, a$. **i)** One path is e, a, b, c.

20. a) The pair (a,b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b—in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.
b) The pair (a,b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b—in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.
c) The pair (a,b) is in R^* precisely when it is possible to fly from a to b.

22. Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$.

24. It is certainly possibly for R^2 to contain some pairs (a, a). For example, let $R = \{(1,2),(2,1)\}$.

26. a) We show the various matrices that are involved. First,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}^{[2]} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{A}^{[3]} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{A}.$$

It follows that $\mathbf{A}^{[4]} = \mathbf{A}^{[2]}$ and $\mathbf{A}^{[5]} = \mathbf{A}^{[3]}$. Therefore the answer $\mathbf{B}$, the meet of all the $\mathbf{A}$'s, is $\mathbf{A} \vee \mathbf{A}^{[2]}$, namely

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

b) For this and the remaining parts we just exhibit the matrices that arise.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \mathbf{A}^{[5]} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

d)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \mathbf{B}$$

28. We compute the matrices $\mathbf{W}_i$ for $i = 0, 1, 2, 3, 4, 5$, and then $\mathbf{W}_5$ is the answer.

a)
$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \mathbf{W}_5$$

b)
$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \mathbf{W}_1 \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \mathbf{W}_3 = \mathbf{W}_4$$

$$\mathbf{W}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

c)
$$\mathbf{W}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{W}_3$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d)
$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

30. Let m be the length of the shortest path from a to b, and let $a = x_0, x_1, \ldots, x_{m-1}, x_m = b$ be such a path. If $m > n - 1$, then $m \geq n$, so $m + 1 \geq n + 1$, which means that not all of the vertices x_0, x_1, x_2, $\ldots$, x_m are distinct. Thus $x_i = x_j$ for some i and j with $0 \leq i < j \leq m$ (but not both $i = 0$ and $j = m$, since $a \neq b$). We can then excise the circuit from x_i to x_j, leaving a shorter path from a to b, namely $x_0, \ldots, x_i, x_{j+1}, \ldots, x_m$. This contradicts the choice of m. Therefore $m \leq n - 1$, as desired.

32. Warshall's algorithm determines the existence of paths. If instead we keep track of the lengths of paths, then we can get the desired information. Thus we make the following changes in Algorithm 2. First, instead of initializing $\mathbf{W}$ to be $\mathbf{M}_R$, we initialize it to be $\mathbf{M}_R$ with each 0 replaced by ∞. Second, the computational step becomes $w_{ij} := \min(w_{ij}, w_{ik} + w_{kj})$.

34. All we need to do is make sure that all the pairs (x, x) are included. An easy way to accomplish this is to add them at the end, by setting $\mathbf{W} := \mathbf{W} \vee \mathbf{I}_n$.