SECTION 3.8 Matrices

In addition to routine exercises with matrix calculations, there are several exercises here asking for proofs of various properties of matrix operations. In most cases the proofs follow immediately from the definitions of the matrix operations and properties of operations on the set from which the entries in the matrices are drawn. Also, the important notion of the (multiplicative) **inverse** *of a matrix is examined in Exercises 18–21. Keep in mind that some matrix operations are performed "entrywise," whereas others operate on whole rows or columns at a time. The general problem of efficient calculation of multiple matrix products, suggested by Exercises 23–25, is interesting and nontrivial. Exercise 31 foreshadows material in Section 8.4.*

1. a) Since $\mathbf{A}$ has 3 rows and 4 columns, its size is 3×4.

b) The third column of $\mathbf{A}$ is the 3×1 matrix $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

c) The second row of $\mathbf{A}$ is the 1×4 matrix $[2\ 0\ 4\ 6]$.

d) This is the element in the third row, second column, namely 1.

e) The transpose of $\mathbf{A}$ is the 4×3 matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$.

3. a) We use the definition of matrix multiplication to obtain the four entries in the product $\mathbf{AB}$. The $(1,1)^{\text{th}}$ entry is the sum $a_{11}b_{11} + a_{12}b_{21} = 2 \cdot 0 + 1 \cdot 1 = 1$. Similarly, the $(1,2)^{\text{th}}$ entry is the sum $a_{11}b_{12} + a_{12}b_{22} = 2 \cdot 4 + 1 \cdot 3 = 11$; $(2,1)^{\text{th}}$ entry is the sum $a_{21}b_{11} + a_{22}b_{21} = 3 \cdot 0 + 2 \cdot 1 = 2$; and $(2,2)^{\text{th}}$ entry is the sum $a_{21}b_{12} + a_{22}b_{22} = 3 \cdot 4 + 2 \cdot 3 = 18$. Therefore the answer is $\begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$.

b) The calculation is similar. Again, to get the $(i,j)^{\text{th}}$ entry of the product, we need to add up all the products $a_{ik}b_{kj}$. You can visualize "lifting" the i^{th} row from the first factor ($\mathbf{A}$) and placing it on top of the j^{th} column from the second factor ($\mathbf{B}$), multiplying the pairs of numbers that lie on top of each other, and taking the sum. Here we have

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + (-1) \cdot 1 & 1 \cdot (-2) + (-1) \cdot 0 & 1 \cdot (-1) + (-1) \cdot 2 \\ 0 \cdot 3 + 1 \cdot 1 & 0 \cdot (-2) + 1 \cdot 0 & 0 \cdot (-1) + 1 \cdot 2 \\ 2 \cdot 3 + 3 \cdot 1 & 2 \cdot (-2) + 3 \cdot 0 & 2 \cdot (-1) + 3 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}.$$

c) The calculation is similar to the previous parts:

$$\begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4\cdot(-1)+(-3)\cdot 0 & 4\cdot 3+(-3)\cdot(-1) & 4\cdot 2+(-3)\cdot 4 & 4\cdot(-2)+(-3)\cdot(-3) \\ 3\cdot(-1)+(-1)\cdot 0 & 3\cdot 3+(-1)\cdot(-1) & 3\cdot 2+(-1)\cdot 4 & 3\cdot(-2)+(-1)\cdot(-3) \\ 0\cdot(-1)+(-2)\cdot 0 & 0\cdot 3+(-2)\cdot(-1) & 0\cdot 2+(-2)\cdot 4 & 0\cdot(-2)+(-2)\cdot(-3) \\ (-1)\cdot(-1)+5\cdot 0 & (-1)\cdot 3+5\cdot(-1) & (-1)\cdot 2+5\cdot 4 & (-1)\cdot(-2)+5\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 15 & -4 & 1 \\ -3 & 10 & 2 & -3 \\ 0 & 2 & -8 & 6 \\ 1 & -8 & 18 & -13 \end{bmatrix}$$

5. First we need to observe that $\mathbf{A} = [a_{ij}]$ must be a 2×2 matrix; it must have two rows since the matrix it is being multiplied by on the left has two columns, and it must have two columns since the answer obtained has two columns. If we write out what the matrix multiplication means, then we obtain the following system of linear equations:

$$2a_{11} + 3a_{21} = 3$$
$$2a_{12} + 3a_{22} = 0$$
$$1a_{11} + 4a_{21} = 1$$
$$1a_{12} + 4a_{22} = 2$$

Solving these equations by elimination of variables (or other means—it's really two systems of two equations each in two unknowns), we obtain $a_{11} = 9/5$, $a_{12} = -6/5$, $a_{21} = -1/5$, $a_{22} = 4/5$. As a check we compute that, indeed,

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 9/5 & -6/5 \\ -1/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

7. Since the $(i,j)^{\text{th}}$ entry of $\mathbf{0} + \mathbf{A}$ is the sum of the $(i,j)^{\text{th}}$ entry of $\mathbf{0}$ (namely 0) and the $(i,j)^{\text{th}}$ entry of $\mathbf{A}$, this entry is the same as the $(i,j)^{\text{th}}$ entry of $\mathbf{A}$. Therefore by the definition of matrix equality, $\mathbf{0} + \mathbf{A} = \mathbf{A}$. A similar argument shows that $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

9. We simply look at the $(i,j)^{\text{th}}$ entries of each side. The $(i,j)^{\text{th}}$ entry of the left-hand side is $a_{ij} + (b_{ij} + c_{ij})$. The $(i,j)^{\text{th}}$ entry of the right-hand side is $(a_{ij} + b_{ij}) + c_{ij}$. By the associativity law for real number addition, these are equal. The conclusion follows.

11. In order for $\mathbf{AB}$ to be defined, the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$. In order for $\mathbf{BA}$ to be defined, the number of columns of $\mathbf{B}$ must equal the number of rows of $\mathbf{A}$. Thus for some positive integers m and n, it must be the case that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix. Another way to say this is to say that $\mathbf{A}$ must have the same size as $\mathbf{B}^t$ (and/or vice versa).

13. Let us begin with the left-hand side and find its $(i,j)^{\text{th}}$ entry. First we need to find the entries of $\mathbf{BC}$. By definition, the $(q,j)^{\text{th}}$ entry of $\mathbf{BC}$ is $\sum_{r=1}^{k} b_{qr}c_{rj}$. (See Section 2.4 for the meaning of summation notation. This symbolism is a shorthand way of writing $b_{q1}c_{1j} + b_{q2}c_{2j} + \cdots + b_{qk}c_{kj}$.) Therefore the $(i,j)^{\text{th}}$ entry of $\mathbf{A(BC)}$ is $\sum_{q=1}^{p} a_{iq}(\sum_{r=1}^{k} b_{qr}c_{rj})$. By distributing multiplication over addition (for real numbers), we can move the term a_{iq} inside the inner summation, to obtain $\sum_{q=1}^{p}\sum_{r=1}^{k} a_{iq}b_{qr}c_{rj}$. (We are also implicitly using associativity of multiplication of real numbers here, to avoid putting parentheses in the product $a_{iq}b_{qr}c_{rj}$.)

A similar analysis with the right-hand side shows that the $(i,j)^{\text{th}}$ entry there is equal to $\sum_{r=1}^{k}(\sum_{q=1}^{p} a_{iq}b_{qr})c_{rj}$ $= \sum_{r=1}^{k}\sum_{q=1}^{p} a_{iq}b_{qr}c_{rj}$. Now by the commutativity of addition, the order of summation (whether we sum over r first and then q, or over q first and then r) does not matter, so these two expressions are equal, and the proof is complete.

15. Let us begin by computing $\mathbf{A}^n$ for the first few values of n.

$$\mathbf{A}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

It seems clear from this pattern, then, that $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. (A proof of this fact could be given using mathematical induction, discussed in Section 4.1.)

17. **a)** The $(i,j)^{\text{th}}$ entry of $(\mathbf{A}+\mathbf{B})^t$ is the $(j,i)^{\text{th}}$ entry of $\mathbf{A}+\mathbf{B}$, namely $a_{ji}+b_{ji}$. On the other hand, the $(i,j)^{\text{th}}$ entry of $\mathbf{A}^t+\mathbf{B}^t$ is the sum of the $(i,j)^{\text{th}}$ entries of $\mathbf{A}^t$ and $\mathbf{B}^t$, which are the $(j,i)^{\text{th}}$ entries of $\mathbf{A}$ and $\mathbf{B}$, again $a_{ji}+b_{ji}$. Hence $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t+\mathbf{B}^t$.

b) The $(i,j)^{\text{th}}$ entry of $(\mathbf{AB})^t$ is the $(j,i)^{\text{th}}$ entry of $\mathbf{AB}$, namely $\sum_{k=1}^{n} a_{jk}b_{ki}$. (See Section 2.4 for the meaning of summation notation. This symbolism is a shorthand way of writing $ba_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}$.) On the other hand, the $(i,j)^{\text{th}}$ entry of $\mathbf{B}^t\mathbf{A}^t$ is $\sum_{k=1}^{n} b_{ki}a_{jk}$ (since the $(i,k)^{\text{th}}$ entry of $\mathbf{B}^t$ is b_{ki} and the $(k,j)^{\text{th}}$ entry of $\mathbf{A}^t$ is a_{jk}). By the commutativity of multiplication of real numbers, these two values are the same, so the matrices are equal.

19. All we have to do is form the products $\mathbf{AA}^{-1}$ and $\mathbf{A}^{-1}\mathbf{A}$, using the purported $\mathbf{A}^{-1}$, and see that both of them are the 2×2 identity matrix. It is easy to see that the upper left and lower right entries in each case are $(ad-bc)/(ad-bc)=1$, and the upper right and lower left entries are all 0.

21. We must show that $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{I}$, where $\mathbf{I}$ is the $n\times n$ identity matrix. Since matrix multiplication is associative, we can write this product as

$$\mathbf{A}^n\left((\mathbf{A}^{-1})^n\right) = \mathbf{A}(\mathbf{A}\ldots(\mathbf{A}(\mathbf{AA}^{-1})\mathbf{A}^{-1})\ldots\mathbf{A}^{-1})\mathbf{A}^{-1}.$$

By dropping each $\mathbf{AA}^{-1} = \mathbf{I}$ from the center as it is obtained, this product reduces to $\mathbf{I}$. Similarly $\left((\mathbf{A}^{-1})^n\right)\mathbf{A}^n = \mathbf{I}$. Therefore by definition $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$. (A more formal proof requires mathematical induction; see Section 4.1.)

23. In order to compute the $(i,j)^{\text{th}}$ entry of the product $\mathbf{AB}$, we need to compute the product $a_{ik}b_{kj}$ for each k from 1 to m_2, requiring m_2 multiplications. Since there are m_1m_3 such pairs (i,j), we need a total of $m_1m_2m_3$ multiplications.

25. There are five different ways to perform this multiplication:

$$(\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_3\mathbf{A}_4), \quad ((\mathbf{A}_1\mathbf{A}_2)\mathbf{A}_3)\mathbf{A}_4, \quad \mathbf{A}_1(\mathbf{A}_2(\mathbf{A}_3\mathbf{A}_4)), \quad (\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3))\mathbf{A}_4, \quad \mathbf{A}_1((\mathbf{A}_2\mathbf{A}_3)\mathbf{A}_4).$$

We can use the result of Exercise 23 to find the numbers of multiplications needed in these five cases. For example, in the first case we need $10\cdot 2\cdot 5 = 100$ multiplications to compute the 10×5 matrix $\mathbf{A}_1\mathbf{A}_2$, $5\cdot 20\cdot 3 = 300$ multiplications to compute the 5×3 matrix $\mathbf{A}_3\mathbf{A}_4$, and then $10\cdot 5\cdot 3 = 150$ multiplications to multiply these two matrices together to obtain the final answer. This gives a total of $100+300+150 = 550$ multiplications. Similar calculations for the other four cases yield $10\cdot 2\cdot 5 + 10\cdot 5\cdot 20 + 10\cdot 20\cdot 3 = 1700$, $5\cdot 20\cdot 3 + 2\cdot 5\cdot 3 + 10\cdot 2\cdot 3 = 390$, $2\cdot 5\cdot 20 + 10\cdot 2\cdot 20 + 10\cdot 20\cdot 3 = 1200$, and $2\cdot 5\cdot 20 + 2\cdot 20\cdot 3 + 10\cdot 2\cdot 3 = 380$, respectively. The winner is therefore $\mathbf{A}_1((\mathbf{A}_2\mathbf{A}_3)\mathbf{A}_4)$, requiring 380 multiplications. Note that the worst arrangement requires 1700 multiplications: it will take over four times as long.

27. Using the idea in Exercise 26, we see that the given system can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is the coefficient matrix, $\mathbf{X}$ is an $n \times 1$ matrix with x_i the entry in its i^{th} row, and $\mathbf{B}$ is the $n \times 1$ matrix of right-hand sides. Specifically we have

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}.$$

If we can find the inverse $\mathbf{A}^{-1}$, then we can find $\mathbf{X}$ simply by computing $\mathbf{A}^{-1}\mathbf{B}$. But Exercise 18 tells us that $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$. Therefore

$$\mathbf{X} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

We should plug in $x_1 = 1$, $x_2 = -1$, and $x_3 = -2$ to see that these do indeed form the solution.

29. These routine exercises simply require application of the appropriate definitions. Parts **(a)** and **(b)** are entrywise operations, whereas the operation $\odot$ in part **(c)** is similar to matrix multiplication (the $(i,j)^{\text{th}}$ entry of $\mathbf{A} \odot \mathbf{B}$ depends on the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$).

a) $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ b) $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

31. Note that $\mathbf{A}^{[2]}$ means $\mathbf{A} \odot \mathbf{A}$, and $\mathbf{A}^{[3]}$ means $\mathbf{A} \odot \mathbf{A} \odot \mathbf{A}$. We just apply the definition.

a) $\mathbf{A}^{[2]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b) $\mathbf{A}^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ c) $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

33. These are immediate from the commutativity of the corresponding logical operations on variables.
a) $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}] = [b_{ij} \vee a_{ij}] = \mathbf{B} \vee \mathbf{A}$
b) $\mathbf{B} \wedge \mathbf{A} = [b_{ij} \wedge a_{ij}] = [a_{ij} \wedge b_{ij}] = \mathbf{A} \wedge \mathbf{B}$

35. These are immediate from the distributivity of the corresponding logical operations on variables.
a) $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = [a_{ij} \vee (b_{ij} \wedge c_{ij})] = [(a_{ij} \vee b_{ij}) \wedge (a_{ij} \vee c_{ij})] = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$
b) $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = [a_{ij} \wedge (b_{ij} \vee c_{ij})] = [(a_{ij} \wedge b_{ij}) \vee (a_{ij} \wedge c_{ij})] = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$

37. The proof is identical to the proof in Exercise 13, except that real number multiplication is replaced by $\wedge$, and real number addition is replaced by $\vee$. Briefly, in symbols, $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = \left[\bigvee_{q=1}^{p} a_{iq} \wedge \left(\bigvee_{r=1}^{k} b_{qr} \wedge c_{rj}\right)\right] = \left[\bigvee_{q=1}^{p} \bigvee_{r=1}^{k} a_{iq} \wedge b_{qr} \wedge c_{rj}\right] = \left[\bigvee_{r=1}^{k} \bigvee_{q=1}^{p} a_{iq} \wedge b_{qr} \wedge c_{rj}\right] = \left[\bigvee_{r=1}^{k} \left(\bigvee_{q=1}^{p} a_{iq} \wedge b_{qr}\right) \wedge c_{rj}\right] = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$.