SUPPLEMENTARY EXERCISES FOR CHAPTER 7

2. a) Let a_n be the amount that remains after n hours. Then $a_n = 0.99a_{n-1}$.

b) By iteration we find the solution $a_n = (0.99)^n a_0$, where a_0 is the original amount of the isotope.

4. a) Let B_n be the number of bacteria after n hours. The initial conditions are $B_0 = 100$ and $B_1 = 300$. Thereafter, $B_n = B_{n-1} + 2B_{n-1} - B_{n-2} = 3B_{n-1} - B_{n-2}$.

b) The characteristic equation is $r^2 - 3r + 1 = 0$, which has roots $(3 \pm \sqrt{5})/2$. Therefore the general solution is $B_n = \alpha_1((3+\sqrt{5})/2)^n + \alpha_2((3-\sqrt{5})/2)^n$. Plugging in the initial conditions we determine that $\alpha_1 = 50+30\sqrt{5}$ and $\alpha_2 = 50 - 30\sqrt{5}$. Therefore the solution is $B_n = (50 + 30\sqrt{5})((3+\sqrt{5})/2)^n + (50 - 30\sqrt{5})((3-\sqrt{5})/2)^n$.

c) Plugging in small values of n, we find that $B_9 = 676{,}500$ and $B_{10} = 1{,}771{,}100$. Therefore the colony will contain more than one million bacteria after 10 hours.

6. We can put any of the stamps on first, leaving a problem with a smaller number of cents to solve. Thus the recurrence relation is $a_n = a_{n-4} + a_{n-6} + a_{n-10}$. We need 10 initial conditions, and it is easy to see that $a_0 = 1$, $a_1 = a_2 = a_3 = a_5 = a_7 = a_9 = 0$, and $a_4 = a_6 = a_8 = 1$.

8. If we add the equations, we obtain $a_n + b_n = 2a_{n-1}$, which means that $b_n = 2a_{n-1} - a_n$. If we now substitute this back into the first equation, we have $a_n = a_{n-1} + (2a_{n-2} - a_{n-1}) = 2a_{n-2}$. The initial conditions are $a_0 = 1$ (given) and $a_1 = 3$ (follows from the first recurrence relation and the given initial conditions). We can solve this using the characteristic equation $r^2 - 2 = 0$, but a simpler approach, that avoids irrational numbers, is as follows. It is clear that $a_{2n} = 2^n a_0 = 2^n$, and $a_{2n+1} = 2^n a_1 = 3 \cdot 2^n$. This is a nice explicit formula, which is all that "solution" really means. We also need a formula for b_n, of course. From $b_n = 2a_{n-1} - a_n$ (obtained above), we have $b_{2n} = 3 \cdot 2^n - 2^n = 2^{n+1}$, and $b_{2n+1} = 2 \cdot 2^n - 3 \cdot 2^n = -2^n$.

10. Following the hint, we let $b_n = \log a_n$. Then the recurrence relation becomes $b_n = 3b_{n-1} + 2b_{n-2}$, with initial conditions $b_0 = b_1 = 1$. This is solved in the usual manner. The characteristic equation is $r^2 - 3r - 2 = 0$, which gives roots $(3 \pm \sqrt{17})/2$. Plugging the initial conditions into the general solution and doing some messy algebra gives

$$b_n = \frac{17 - \sqrt{17}}{34}\left(\frac{3+\sqrt{17}}{2}\right)^n + \frac{17+\sqrt{17}}{34}\left(\frac{3-\sqrt{17}}{2}\right)^n.$$

The solution to the original problem is then $a_n = 2^{b_n}$.

12. The characteristic equation is $r^3 - 3r^2 + 3r - 1 = 0$. This factors as $(r-1)^3 = 0$, so there is only one root, 1, and its multiplicity is 3. Therefore the general solution is $a_n = \alpha_1 + \alpha_2 n + \alpha_3 n^2$. Plugging in the initial conditions gives us $2 = \alpha_1$, $2 = \alpha_1 + \alpha_2 + \alpha_3$, and $4 = \alpha_1 + 2\alpha_2 + 4\alpha_3$. Solving yields $\alpha_1 = 2$, $\alpha_2 = -1$, and $\alpha_3 = 1$. Therefore the solution is $a_n = 2 - n + n^2$.

14. We use the result of Exercise 31 in Section 7.3, with $a = 3$, $b = 5$, $c = 2$, and $d = 4$. Thus the solution is $f(n) = 625n^4/311 - 314n^{\log_5 3}/311$.

16. The algorithm compares the largest elements of the two halves (this is one comparison), and then it compares the smaller largest element with the second largest element of the other half (one more comparison). This is sufficient to determine the largest and second largest elements of the list. (If the list has only one element in it, then the second largest element is declared to be $-\infty$.) Let $f(n)$ be the number of comparisons used by this algorithm on a list of size n. The list is split into two lists, of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively. Thus our recurrence relation is $f(n) = f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil) + 2$, with initial condition $f(1) = 0$. (This algorithm could be made slightly more efficient by having the base cases be $n = 2$ and $n = 3$, rather than $n = 1$.)

18. a) $\Delta a_n = 3 - 3 = 0$ b) $\Delta a_n = 4(n+1) + 7 - (4n+7) = 4$
c) $\Delta a_n = \big((n+1)^2 + (n+1) + 1\big) - (n^2 + n + 1) = 2n + 2$

20. We prove something a bit stronger. If $a_n = P(n)$ is a polynomial of degree at most d, then Δa_n is a polynomial of degree at most $d-1$. To see this, let $P(n) = c_d n^d + (\text{lower order terms})$. Then

$$\begin{aligned}\Delta P(n) &= c_d(n+1)^d + (\text{lower order terms}) - c_d n^d + (\text{lower order terms})\\ &= c_d n^d + (\text{lower order terms}) - c_d n^d + (\text{lower order terms})\\ &= (\text{lower order terms}).\end{aligned}$$

If we apply this result $d+1$ times, then we get that $\Delta^{d+1} a_n$ has degree at most -1, i.e., is identically 0.

22. Since it is valid to use the commutative, associative, and distributive laws for absolutely convergent infinite series, we simply write

$$(cF + dG)(x) = cF(x) + dG(x) = c\sum_{k=0}^{\infty} a_k x^k + d\sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty}(ca_k + db_k)x^k .$$

24. $14 + 18 - 22 = 10$

26. If the queries are correct, then by inclusion–exclusion the number of students who are freshmen and have not taken courses in either subject must equal $2175 - 1675 - 1074 - 444 + 607 + 350 + 201 - 143 = -3$. Since a negative number here is not possible, we conclude that the responses cannot all be accurate.

28. There will be $C(7,i)$ terms involving combinations of i of the sets at a time. Therefore the answer is $C(7,1) + C(7,2) + C(7,3) + C(7,4) + C(7,5) = 119$.

30. For a more compact notation, let us write 1,000,000 as M.
a) $\lfloor M/2 \rfloor + \lfloor M/3 \rfloor + \lfloor M/5 \rfloor - \lfloor M/(2\cdot 3) \rfloor - \lfloor M/(2\cdot 5) \rfloor - \lfloor M/(3\cdot 5) \rfloor + \lfloor M/(2\cdot 3\cdot 5) \rfloor = 733{,}334$
b) $M - \lfloor M/7 \rfloor - \lfloor M/11 \rfloor - \lfloor M/13 \rfloor + \lfloor M/(7\cdot 11) \rfloor + \lfloor M/(7\cdot 13) \rfloor + \lfloor M/(11\cdot 13) \rfloor - \lfloor M/(7\cdot 11\cdot 13) \rfloor = 719{,}281$
c) This is asking for numbers divisible by 3 but not by 21. Since the set of numbers divisible by 21 is a subset of the set of numbers divisible by 3, this is simply $\lfloor M/3 \rfloor - \lfloor M/21 \rfloor = 285{,}714$.

32. After the assignments of the hardest and easiest job have been made, there are 4 different jobs to assign to 3 different employees. No restrictions are stated, so we assume that there are none. Therefore we are just looking for the number of functions from a set with 4 elements to a set with 3 elements, and there are $3^4 = 81$ such functions. (If we impose the restriction that every employee must get at least one job, then it is a little harder. In particular, we must rule out all the assignments in which the jobs go only to the two employees that already have jobs. There are $2^4 = 16$ such assignments, so the answer would be $81 - 16 = 65$ in this case.)

34. We will count the number of bit strings that do contain four consecutive 1's. Bits 1 through 4 could be 1's, or bits 2 through 5, or bits 3 through 6, and in each case there are 4 strings meeting those conditions (since the other two bits are free). This gives a total of 12. However we overcounted, since there are ways in which more than one of these can happen. There are 2 strings in which bits 1 through 4 and bits 2 through 5 are 1's, 2 strings in which bits 2 through 5 and bits 3 through 6 are 1's, and 1 string in which bits 1 through 4 and bits 3 through 6 are 1's. Finally, there is 1 string in which all three substrings are 1's. Thus the number of bit strings with 4 consecutive 1's is $12 - 2 - 2 - 1 + 1 = 8$. Therefore the answer to the exercise is $2^6 - 8 = 56$.