SECTION 8.6 Partial Orderings

2. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) This relation is not reflexive because 1 is not related to itself. Therefore R is not a partial ordering. The relation is antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c \neq 1$ so a is related to c.

b) This is a partial ordering, because it is reflexive and the pairs $(2,0)$ and $(2,3)$ will not introduce any violations of antisymmetry or transitivity.

c) This is not a partial ordering, because it is not transitive: $3\,R\,1$ and $1\,R\,2$, but 3 is not related to 2. It is reflexive and the pairs $(1,2)$ and $(3,1)$ will not introduce any violations of antisymmetry.

d) This is not a partial ordering, because it is not transitive: $1\,R\,2$ and $2\,R\,0$, but 1 is not related to 0. It is reflexive and the nonreflexive pairs will not introduce any violations of antisymmetry.

e) The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

4. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S, R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S, R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b, then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b, and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

6. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs (a,a).)

b) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs (a,a).

c) This is a poset, very similar to Example 1.

d) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

8. a) This relation is $\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$. It is clearly reflexive and antisymmetric. The only pairs that might present problems with transitivity are the nondiagonal pairs, $(2,1)$ and $(1,3)$. If the relation were to be transitive, then we would also need the pair $(2,3)$ in the relation. Since it is not there, the relation is not a partial order.

b) Reasoning as in part (a), we see that this relation is a partial order, since the pair $(3,1)$ can cause no problem with transitivity.

c) A little trial and error shows that this relation is not transitive ($(1,3)$ and $(3,4)$ are present, but not $(1,4)$) and therefore not a partial order.

10. This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

12. This follows immediately from the definition. Clearly R^{-1} is reflexive if R is. For antisymmetry, suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, whence $(b,a) \notin R^{-1}$. Finally, if $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, so $(c,a) \in R$ (since R is transitive), and therefore $(a,c) \in R^{-1}$; thus R^{-1} is transitive.

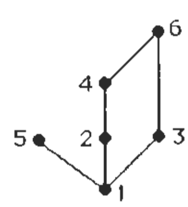
14. **a)** These are comparable, since $5 \mid 15$.
b) These are not comparable since neither divides the other.
c) These are comparable, since $8 \mid 16$.
d) These are comparable, since $7 \mid 7$.

16. **a)** We need either a number less than 2 in the first coordinate, or a 2 in the first coordinate and a number less than 3 in the second coordinate. Therefore the answer is $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, and $(2,2)$.
b) We need either a number greater than 3 in the first coordinate, or a 3 in the first coordinate and a number greater than 1 in the second coordinate. Therefore the answer is $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(3,2)$, $(3,3)$, and $(3,4)$.
c) The Hasse diagram is a straight line with 16 points on it, since this is a total order. The pair $(4,4)$ is at the top, $(4,3)$ beneath it, $(4,2)$ beneath that, and so on, with $(1,1)$ at the bottom. To save space, we will not actually draw this picture.

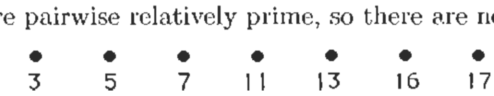
18. **a)** The string *quack* comes first, since it is an initial substring of *quacking*, which comes next (since the other three strings all begin *qui*, not *qua*). Similarly, these last three strings are in the order *quick*, *quicksand*, *quicksilver*.
b) The order is *open*, *opened*, *opener*, *opera*, *operand*.
c) The order is *zero*, *zeo*, *zoological*, *zoology*, *zoom*.

20. The Hasse diagram for this total order is a straight line, as shown, with 0 at the top (it is the "largest" element under the "is greater than or equal to" relation) and 5 at the bottom.

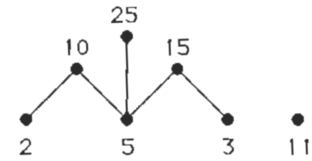
22. In each case we put a above b and draw a line between them if $b \mid a$ but there is no element c other than a and b such that $b \mid c$ and $c \mid a$.
a) Note that 1 divides all numbers, so the numbers on the second level from the bottom are the primes.



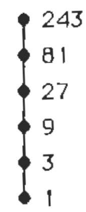
b) In this case these numbers are pairwise relatively prime, so there are no lines in the Hasse diagram.



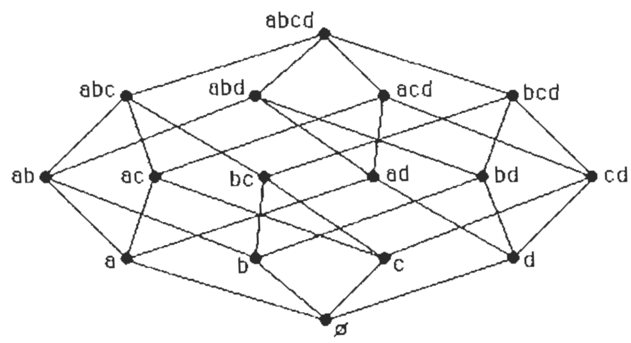
c) Note that we can place the points as we wish, as long as a is above b when $b \mid a$.



d) In this case these numbers each divide the next, so the Hasse diagram is a straight line.



24. This picture is a four-dimensional cube. We draw the sets with k elements at level k: the empty set at level 0 (the bottom), the entire set at level 4 (the top).



26. The procedure is the same as in Exercise 25: $\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,d),(d,d),(e,e)\}$

28. In this problem $a \preceq b$ when $a \mid b$. For (a,b) to be in the covering relation, we need a to be a proper divisor of b but we also must have no element in our set $\{1, 2, 3, 4, 6, 12\}$ being a proper multiple of a and a proper divisor of b. For example, $(2,12)$ is not in the covering relation, since $2 \mid 6$ and $6 \mid 12$. With this understanding it is easy to list the pairs in the covering relation: $(1,2)$, $(1,3)$, $(2,4)$, $(2,6)$, $(3,6)$, $(4,12)$,and $(6,12)$.

30. In the Hasse diagram, if x is lower than y and there is an edge joining x and y, then $x \prec y$. We claim that in this case (x,y) is also in the covering relation. Indeed, the definition of the Hasse diagram stated that there is to be no edge from x to y if there is some z with $x \prec z \prec y$ (such edges forced by transitivity were to be removed in the construction). This is equivalent to having (x,y) in the covering relation. Conversely, if (x,y) is in the covering relation, the $x \prec y$, and there is no z that would have caused the edge between x and y to be removed in the construction of the Hasse diagram. This completes the proof.

32. a) The maximal elements are the ones with no other elements above them, namely l and m.

b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

c) There is no greatest element, since neither l nor m is greater than the other.

d) There is no least element, since neither a nor b is less than the other.

e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l, and m are the elements fitting this description.

f) Since k is less than both l and m, it is the least upper bound of a, b, and c.

g) No element is less than both f and h, so there are no lower bounds.

h) Since there are no lower bounds, there can be no greatest lower bound.

34. The reader should draw the Hasse diagram to aid in answering these questions.

a) Clearly the numbers 27, 48, 60, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.

b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.

c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

d) There is no least element, since there is no number in the set that divides both 2 and 9.

e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.

f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

g) We need to find numbers in the list that are divisors of both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.

h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

36. a) One example is the natural numbers under "is less than or equal to." Here 1 is the (only) minimal element, and there are no maximal elements.

b) Dual to part (a), the answer is the natural numbers under "is greater than or equal to."

c) Combining the answers for the first two parts, we look at the set of integers under "is less than or equal to." Clearly there are no maximal or minimal elements.

38. Reflexivity is clear from the definition. To show antisymmetry, suppose that $a_1 \ldots a_m < b_1 \ldots b_n$, and let $t = \min(m,n)$. This means that either $a_1 \ldots a_t = b_1 \ldots b_t$ and $m < n$, so that $b_1 \ldots b_n \not< a_1 \ldots a_m$, or else $a_1 \ldots a_t < b_1 \ldots b_t$, so that $b_1 \ldots b_t \not< a_1 \ldots a_t$ and hence again $b_1 \ldots b_n \not< a_1 \ldots a_m$. Finally for transitivity, suppose that $a_1 \ldots a_m < b_1 \ldots b_n < c_1 \ldots c_p$. Let $t = \min(m,n)$, $r = \min(n,p)$, $s = \min(m,p)$, and $l = \min(m,n,p)$. Now if $a_1 \ldots a_l < b_1 \ldots b_l < c_1 \ldots c_l$, then clearly $a_1 \ldots a_m < c_1 \ldots c_p$. Otherwise, without loss of generality we may assume that $a_1 \ldots a_l = b_1 \ldots b_l$. If $l = t$, then $m < n$ and $m \le p$. Furthermore, either $b_1 \ldots b_r < c_1 \ldots c_r$, or $b_1 \ldots b_r = c_1 \ldots c_r$ and $n < p$. In the former case, if $r > l$, then since $p > m$ we have $a_1 \ldots a_m < c_1 \ldots c_p$, whereas if $r = l$, then $a_1 \ldots a_l < c_1 \ldots c_l$. In the latter case, $a_1 \ldots a_s = c_1 \ldots c_s$ and $m < p$, so again $a_1 \ldots a_m < c_1 \ldots c_p$. If $l < t$, then we must have $b_1 \ldots b_l < c_1 \ldots c_l$, whence $a_1 \ldots a_l < c_1 \ldots c_l$.

40. a) If x and y are both greatest elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both least elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

42. a) If x and y are both least upper bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both greatest lower bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

46. By the duality in the definitions, the greatest lower bound of two elements of S under R is their least upper bound under R^{-1}, and their least upper bound under R is their greatest lower bound under R^{-1}. Therefore, if (S, R) is a lattice (i.e., all the l.u.b.'s and g.l.b.'s exist), then so is (S, R^{-1}).

48. We need to verify the various defining properties of a lattice. First, we need to show that S is a poset under the given $\preceq$ relation. Clearly $(A, C) \preceq (A, C)$, since $A \leq A$ and $C \subseteq C$; thus we have established reflexivity. For antisymmetry, suppose that $(A_1, C_1) \preceq (A_2, C_2)$ and $(A_2, C_2) \preceq (A_1, C_1)$. This means that $A_1 \leq A_2$, $C_1 \subseteq C_2$, $A_2 \leq A_1$, and $C_2 \subseteq C_1$. By the properties of $\leq$ and $\subseteq$ it immediately follows that $A_1 = A_2$ and $C_1 = C_2$, so $(A_1, C_1) = (A_2, C_2)$. Transitivity is proved in a similar way, using the transitivity of $\leq$ and $\subseteq$. Second, we need to show that greatest lower bounds and least upper bounds exist. Suppose that (A_1, C_1) and (A_2, C_2) are two elements of S; we claim that $(\min(A_1, A_2), C_1 \cap C_2)$ is their greatest lower bound. Clearly $\min(A_1, A_2) \leq A_1$ and $\min(A_1, A_2) \leq A_2$; and $C_1 \cap C_2 \subseteq C_1$ and $C_1 \cap C_2 \subseteq C_2$. Therefore $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_1, C_1)$ and $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_2, C_2)$, so this is a lower bound. On the other hand, if (A, C) is any lower bound, then $A \leq A_1$, $A \leq A_2$, $C \subseteq C_1$, and $C \subseteq C_2$. It follows from the properties of $\leq$ and $\subseteq$ that $A \leq \min(A_1, A_2)$ and $C \subseteq C_1 \cap C_2$. Therefore $(A, C) \preceq (\min(A_1, A_2), C_1 \cap C_2)$. This means that $(\min(A_1, A_2), C_1 \cap C_2)$ is the greatest lower bound. The proof that $(\max(A_1, A_2), C_1 \cup C_2)$ is the least upper bound is exactly dual to this argument.

50. This issue was already dealt with in our solution to Exercise 44, parts (b) and (c). If $(S, \leq)$ is a total (linear) order, then the least upper bound of two elements is the larger one, and their greatest lower bound is the smaller.

52. By Exercise 50, we can try to choose our examples from among total orders, such as subsets of $\mathbf{Z}$ under $\leq$.
a) $(\mathbf{Z}, \leq)$ b) $(\mathbf{Z}^+, \leq)$ c) $(\mathbf{Z}^-, \leq)$, where $\mathbf{Z}^-$ is the set of negative integers d) $(\{1\}, \leq)$

54. In each case, the issue is whether every subset contains a least element.

a) The is well-ordered, since the minimum element in each set is its smallest element.

b) This is not well-ordered. For example, the set $\{\frac{1}{n} \mid n \in \mathbf{N}\}$ contains no minimum element.

c) This is a finite totally-ordered set, so it is well-ordered.

d) This is well-ordered, since has the same structure as the positive integers under $\leq$, because $x \geq y$ if and only if $-x \leq -y$.

56. Let x_0 and x_1 be two elements in the dense poset, with $x_0 \prec x_1$ (guaranteed by the conditions stated). By density, there is an element x_2 between x_0 and x_1, i.e., with $x_0 \prec x_2 \prec x_1$. Again by density, there is an element x_3 between x_0 and x_2, i.e., with $x_0 \prec x_3 \prec x_2$. We continue in this manner and have produced an infinite decreasing sequence: $\cdots \prec x_4 \prec x_3 \prec x_2 \prec x_1$. Thus the poset is not well-founded.

58. It is not well-founded because of the infinite decreasing sequence $\cdots \prec aaab \prec aab \prec ab \prec b$. It is not dense, because there is no element between a and aa in this order.

60. This is dual to Lemma 1. We can simply copy the proof, changing every "minimal" to "maximal" and reversing each inequality.

62. Since a larger number can never divide a smaller one, the "is less than or equal to" relation on any set is a compatible total order for the divisibility relation. This gives $1 \prec_t 2 \prec_t 3 \prec_t 6 \prec_t 8 \prec_t 12 \prec_t 24 \prec_t 36$.

64. There are many compatible total orders here. We just need to work from the bottom up. One answer is to take Foundation $\prec$ Framing $\prec$ Roof $\prec$ Exterior siding $\prec$ Wiring $\prec$ Plumbing $\prec$ Flooring $\prec$ Wall – board $\prec$ Exterior painting $\prec$ Interior painting $\prec$ Carpeting $\prec$ Interior fixtures $\prec$ Exterior fixtures $\prec$ Completion.