SECTION 9.3   Representing Graphs and Graph Isomorphism

Human beings can get a good feeling for a small graph by looking at a picture of it drawn with points in the plane and lines or curves joining pairs of these points. If a graph is at all large (say with more than a dozen vertices or so), then the picture soon becomes too crowded to be useful. A computer has little use for nice pictures, no matter how small the vertex set. Thus people and machines need more precise—more discrete—representations of graphs. In this section we learned about some useful representations. They are for the most part exactly what any intelligent person would come up with, given the assignment to do so.

The only tricky idea in this section is the concept of graph isomorphism. It is a special case of a more general notion of isomorphism, or sameness, of mathematical objects in various settings. Isomorphism tries to capture the idea that all that really matters in a graph is the adjacency structure. If we can find a way to superimpose the graphs so that the adjacency structures match, then the graphs are, for all purposes that matter, the same. In trying to show that two graphs are isomorphic, try moving the vertices around in your mind to see whether you can make the graphs look the same. Of course there are often lots of things to help. For example, in every isomorphism, vertices that correspond must have the same degree.

A good general strategy for determining whether two graphs are isomorphic might go something like this. First check the degrees of the vertices to make sure there are the same number of each degree. See whether vertices of corresponding degrees follow the same adjacency pattern (e.g., if there is a vertex of degree 1 adjacent to a vertex of degree 4 in one of the graphs, then there must be the same pattern in the other, if the graphs are isomorphic). Then look for triangles in the graphs, and see whether they correspond. Sometimes, if the graphs have lots of edges, it is easier to see whether the complements are isomorphic (see Exercise 46). If you cannot find a good reason for the graphs not to be isomorphic (an invariant on which they differ), then try to write down a one-to-one and onto function that shows them to be isomorphic (there may be more than one such function); such a function has to have vertices of like degrees correspond, so often the function practically writes itself. Then check each edge of the first graph to make sure that it corresponds to an edge of the second graph under this correspondence.

Unfortunately, no one has yet discovered a really good algorithm for determining graph isomorphism that works on all pairs of graphs. Research in this subject has been quite active in recent years. See Writing Project 7.

1. Adjacency lists are lists of lists. The adjacency list of an undirected graph is simply a list of the vertices of the given graph, together with a list of the vertices adjacent to each. The list for this graph is as follows. Since, for instance, b is adjacent to a and d, we list a and d in the row for b.

Vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

3. To form the adjacency list of a directed graph, we list, for each vertex in the graph, the terminal vertex of each edge that has the given vertex as its initial vertex. The list for this directed graph is as follows. For example, since there are edges from d to each of b, c, and d, we put those vertices in the row for d.

Initial vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

5. For Exercises 5–8 we assume that the vertices are listed in alphabetical order. The matrix contains a 1 as entry (i, j) if there is an edge from vertex i to vertex j; otherwise that entry is 0.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

7. This is similar to Exercise 5. Note that edges have direction here, so that, for example, the $(1, 2)$ entry is a 1 since there is an edge from a to b, but the $(2, 1)$ entry is a 0 since there is no edge from b to a. Also, the $(1, 1)$ entry is a 1 since there is a loop at a, but the $(2, 2)$ entry is a 0 since there is no loop at b.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

9. We can solve these problems by first drawing the graph, then labeling the vertices, and finally constructing the matrix by putting a 1 in position (i, j) whenever vertices i and j are joined by an edge. It helps to choose a nice order, since then the matrix will have nice patterns in it.

a) The order of the vertices does not matter, since they all play the same role. The matrix has 0's on the diagonal, since there are no loops in the complete graph.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b) We put the vertex in the part by itself first.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c) We put the vertices in the part of size 2 first. Notice the block structure.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d) We put the vertices in the same order in the matrix as they are around the cycle.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

e) We put the center vertex first. Note that the last four columns of the last four rows represent a C_4.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

f) We can label the vertices by the binary numbers from 0 to 7. Thus the first row (also the first column) of this matrix corresponds to the string 000, the second to the string 001, and so on. Since Q_3 has 8 vertices, this is an 8×8 matrix.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

11. This graph has four vertices and is directed, since the matrix is not symmetric. We draw the four vertices as points in the plane, then draw a directed edge from vertex i to vertex j whenever there is a 1 in position (i, j) in the given matrix.
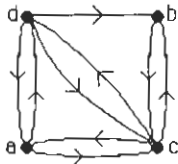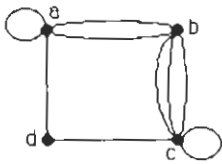

13. We use alphabetical order of the vertices for Exercises 13–15. If there are k parallel edges between vertices i and j, then we put the number k into the $(i, j)^{\text{th}}$ entry of the matrix. In this exercise, there is only one pair of parallel edges.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

15. This is similar to Exercise 13. In this graph there are loops, which are represented by entries on the diagonal. For example, the loop at c is shown by the 1 as the $(3, 3)^{\text{th}}$ entry.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

17. Because of the numbers larger than 1, we need multiple edges in this graph.

19. We use alphabetical order of the vertices. We put a 1 in position (i, j) if there is a directed edge from vertex i to vertex j; otherwise we make that entry a 0. Note that loops are represented by 1's on the diagonal.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

21. This is similar to Exercise 19, except that there are parallel directed edges. If there are k parallel edges from vertex i to vertex j, then we put the number k into the $(i, j)^{\text{th}}$ entry of the matrix. For example, since there are 2 edges from a to c, the $(1, 3)^{\text{th}}$ entry of the adjacency matrix is 2; the loop at c is shown by the 1 as the $(3, 3)^{\text{th}}$ entry.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

23. Since the matrix is not symmetric, we need directed edges; furthermore, it must be a directed multigraph because of the entries larger than 1. For example, the 2 in position $(3, 2)$ means that there are two parallel edges from vertex c to vertex b.
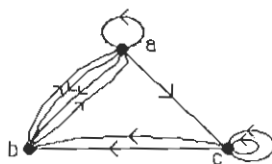
25. Since the matrix is symmetric, it has to be square, so it represents a graph of some sort. In fact, such a matrix does represent a simple graph. The fact that it is a zero-one matrix means that there are no parallel edges. The fact that there are 0's on the diagonal means that there are no loops. The fact that the matrix is symmetric means that the edges can be assumed to be undirected. Note that such a matrix also represents a directed graph in which all the edges happen to appear in antiparallel pairs (see the solution to Exercise 1d in Section 9.1 for a definition), but that is irrelevant to this question; the answer to the question asked is "yes."

27. In an incidence matrix we have one column for each edge. We use alphabetical order of the vertices. Loops are represented by columns with one 1; other edges are represented by columns with two 1's. The order in which the columns are listed is immaterial.

Exercise 13
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise 14
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Exercise 15
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

29. In an undirected graph, each edge incident to a vertex j contributes 1 in the jth column; thus the sum of the entries in that column is just the number of edges incident to j. Another way to state the answer is that the sum of the entries is the degree of j minus the number of loops at j, since each loop counts 2 toward the degree.

In a directed graph, each edge whose terminal vertex is j contributes 1 in the jth column; thus the sum of the entries in that column is just the number of edges that have j as their terminal vertex. Another way to state the answer is that the sum of the entries is the in-degree of j.

31. Since each column represents an edge, the sum of the entries in the column is either 2, if the edge has 2 incident vertices (i.e., is not a loop), or 1 if it has only 1 incident vertex (i.e., is a loop).

33. a) The incidence matrix for K_n has n rows and $C(n,2)$ columns. For each i and j with $1 \le i < j \le n$, there is a column with 1's in rows i and j and 0's elsewhere.

    b) The matrix looks like this, with n rows and n columns.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

    c) The matrix looks like the matrix for C_n, except with an extra row of 0's (which we have put at the end), since the vertex "in the middle" is not involved in the edges "around the outside," and n more columns for the "spokes." We show some extra space between the rim edge columns and the spoke columns; this is for human convenience only and does not have any bearing on the matrix itself.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & \quad & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & \quad & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & \quad & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \quad & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \quad & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \quad & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & \quad & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \quad & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

    d) This matrix has $m + n$ rows and mn columns, one column for each pair (i,j) with $1 \le i \le m$ and $1 \le j \le n$. We have put in some extra spacing for readability of the pattern.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \quad 1 & 1 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 1 & \cdots & 1 \\ & & & & & & & & & & & & \\ 1 & 0 & \cdots & 0 & \quad 1 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \quad 0 & 1 & \cdots & 0 & \quad \cdots & \quad 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \quad 0 & 0 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 1 \end{bmatrix}$$

35. These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$, and $f(u_5) = v_4$.

37. These graphs are isomorphic, since each is the 7-cycle (this is just like Exercise 35).

39. These two graphs are isomorphic. One can see this visually—just imagine "moving" vertices u_1 and u_4 into the inside of the rectangle, thereby obtaining the picture on the right. Formally, one isomorphism is $f(u_1) = v_5$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_6$, $f(u_5) = v_4$, and $f(u_6) = v_1$.

41. These graphs are not isomorphic. In the first graph the vertices of degree 3 are adjacent to a common vertex. This is not true of the second graph.

43. These are isomorphic. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_9$, $f(u_3) = v_4$, $f(u_4) = v_3$, $f(u_5) = v_2$, $f(u_6) = v_8$, $f(u_7) = v_7$, $f(u_8) = v_5$, $f(u_9) = v_{10}$, and $f(u_{10}) = v_6$.

45. We must show that being isomorphic is reflexive, symmetric, and transitive. It is reflexive since the identity function from a graph to itself provides the isomorphism (the one-to-one correspondence)—certainly the identity function preserves adjacency and nonadjacency. It is symmetric, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, then f^{-1} is a one-to-one correspondence that makes G_2 isomorphic to G_1; that is, f^{-1} is a one-to-one and onto function from V_2 to V_1 such that c and d are adjacent in G_2 if and only if $f^{-1}(c)$ and $f^{-1}(d)$ are adjacent in G_1. It is transitive, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, and g is a one-to-one correspondence that makes G_2 isomorphic to G_3, then $g \circ f$ is a one-to-one correspondence that makes G_1 isomorphic to G_3.

47. If a vertex is isolated, then it has no adjacent vertices. Therefore in the adjacency matrix the row and column for that vertex must contain all 0's.

49. Let V_1 and V_2 be the two parts, say of sizes m and n, respectively. We can number the vertices so that all the vertices in V_1 come before all the vertices in V_2. The adjacency matrix has $m + n$ rows and $m + n$ columns. Since there are no edges between two vertices in V_1, the first m columns of the first m rows must all be 0's. Similarly, since there are no edges between two vertices in V_2, the last n columns of the last n rows must all be 0's. This is what we were asked to prove.

51. There are two such graphs, which can be found by trial and error. (We need only look for graphs with 5 vertices and 5 edges, since a self-complementary graph with 5 vertices must have $C(5,2)/2 = 5$ edges. If nothing else, we can draw them all and find the complement of each. See the pictures for the solution of Exercise 45d in Section 9.4.) One such graph is C_5. The other consists of a triangle, together with an edge from one vertex of the triangle to the fourth vertex, and an edge from another vertex of the triangle to the fifth vertex.

53. If C_n is to be self-complementary, then C_n must have the same number of edges as its complement. We know that C_n has n edges. Its complement has the number of edges in K_n minus the number of edges in C_n, namely $C(n,2) - n = [n(n-1)/2] - n$. If we set these two quantities equal we obtain $[n(n-1)/2] - n = n$, which has $n = 5$ as its only solution. Thus C_5 is the only C_n that *might* be self-complementary—our argument just shows that it has the same number of edges as its complement, not that it is indeed isomorphic to its complement. However, it we draw C_5 and then draw its complement, then we see that the complement is again a copy of C_5. Thus $n = 5$ is the answer to the problem.

55. We need to enumerate these graphs carefully to make sure of getting them all—leaving none out and not duplicating any. Let us organize our catalog by the degrees of the vertices. Since there are only 3 edges, the largest the degree could be is 3, and the only graph with 5 vertices, 3 edges, and a vertex of degree 3 is a $K_{1,3}$ together with an isolated vertex. If all the vertices that are not isolated have degree 2, then the graph must consist of a C_3 and 2 isolated vertices. The only way for there to be two vertices of degree 2 (and therefore also 2 of degree 1) is for the graph to be three edges strung end to end, together with an isolated vertex. The only other possibility is for 2 of the edges to be adjacent and the third to be not adjacent to either of the others. All in all, then, we have the 4 possibilities shown below.

57. a) Both graphs consist of 2 sides of a triangle; they are clearly isomorphic.

   b) The graphs are not isomorphic, since the first has 4 edges and the second has 5 edges.

   c) The graphs are not isomorphic, since the first has 4 edges and the second has 3 edges.

59. There are at least two approaches we could take here. One approach is to have a correspondence not only of the vertices but also of the edges, with incidence (and nonincidence) preserved. In detail, we say that two pseudographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there are one-to-one and onto functions $f : V_1 \to V_2$ and $g : E_1 \to E_2$ such that for each vertex $v \in V_1$ and edge $e \in E_1$, v is incident to e if and only if $f(v)$ is incident to $g(e)$.

     Another approach is simply to count the number of edges between pairs of vertices. Thus we can define $G_1 = (V_1, E_1)$ to be isomorphic to $G_2 = (V_2, E_2)$ if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for every pair of (not necessarily distinct) vertices u and v in V_1, there are exactly the same number of edges in E_1 with $\{u, v\}$ as their set of endpoints as there are edges in E_2 with $\{f(u), f(v)\}$ as their set of endpoints.

61. We can tell by looking at the loop, the parallel edges, and the degrees of the vertices that if these directed graphs are to be isomorphic, then the isomorphism has to be $f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_2$, and $f(u_4) = v_1$. We then need to check that each directed edge (u_i, u_j) corresponds to a directed edge $(f(u_i), f(u_j))$. We check that indeed it does for each of the 7 edges (and there are only 7 edges in the second graph). Therefore the two graphs are isomorphic.

63. If there is to be an isomorphism, the vertices with the same in-degree would have to correspond, and the edge between them would have to point in the same direction, so we would need u_1 to correspond to v_3, and u_2 to correspond to v_1. Similarly we would need u_3 to correspond to v_4, and u_4 to correspond to v_2. If we check all 6 edges under this correspondence, then we see that adjacencies are preserved (in the same direction), so the graphs are isomorphic.

65. If f is an isomorphism from a directed graph G to a directed graph H, then f is also an isomorphism from G^c to H^c. This is clear, because (u, v) is an edge of G^c if and only if (v, u) is an edge of G if and only if $(f(v), f(u))$ is an edge of H if and only if $(f(u), f(v))$ is an edge of H^c.

67. A graph with a triangle will not be bipartite, but cycles of even length are bipartite. So we could let one graph be C_6 and the other be the union of two disjoint copies of C_3.

69. Suppose that the graph has v vertices and e edges. Then the incidence matrix is a $v \times e$ matrix, so its transpose is an $e \times v$ matrix. Therefore the product is a $v \times v$ matrix. Suppose that we denote the typical entry of this product by a_{ij}. Let t_{ik} be the typical entry of the incidence matrix; it is either a 0 or a 1. By definition

$$a_{ij} = \sum_{k=1}^{e} t_{ik} t_{jk}.$$

We can now read off the answer from this equation. If $i \neq j$, then a_{ij} is just a count of the number of edges incident to both i and j—in other words, the number of edges between i and j. On the other hand a_{ii} is equal to the number of edges incident to i.

71. Perhaps the simplest example would be to have the graphs have all degrees equaling 2. One way for this to happen is for the graph to be a cycle. But it will also happen if the graph is a disjoint union of cycles. The smallest example occurs when there are six vertices. If G_1 is the 6-cycle and G_2 is the union of two triangles, then the degree sequences are $(2, 2, 2, 2, 2, 2)$ for both, but obviously the graphs are not isomorphic. If we want a connected example, then look at Exercise 41, where the degree sequence is $(3, 3, 2, 2, 1, 1, 1, 1)$ for each graph.