Three.VI Projection

Linear Algebra
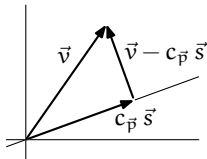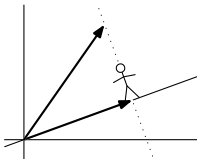Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Orthogonal Projection Into a Line

Project a vector into a line

This shows a figure walking out on the line until they are at a point $\vec{p}$ such that the tip of $\vec{v}$ is directly above them, where "above" does not mean parallel to the y-axis but instead means orthogonal to the line.
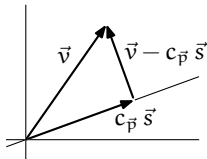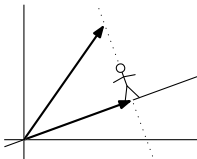
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Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to $c_{\vec{p}}\vec{s}$.
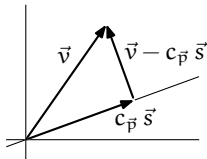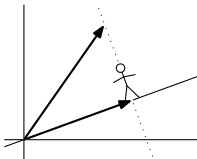
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   To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$, it must be orthogonal to $\vec{s}$ itself. Then $(\vec{v} - c_{\vec{p}}\vec{s}) \cdot \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \cdot \vec{s} / \vec{s} \cdot \vec{s}$.

Note two things about orthogonal projection.

1) We have an idea of 'angle' in $\mathbb{R}^n$ but we have not given such a definition in some other spaces. This section's results will stick to spaces in which we have covered 'orthogonal,' namely the real spaces.

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2) We have decomposed $\vec{v}$ into two parts.
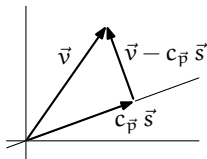
Intuitively, some of $\vec{v}$ lies with the line and that gives the first part $c_{\vec{p}}\vec{s}$. The part of $\vec{v}$ that lies with a line orthogonal to the given line is $\vec{v} - c_{\vec{p}}\vec{s}$. What's compelling about pairing these two parts is that they don't interact in that the projection of one into the line spanned by the other is the zero vector.

1.1 Definition The *orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$* is this vector.

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

Example The projection of this $\mathbb{R}^3$ vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

This picture of that projection
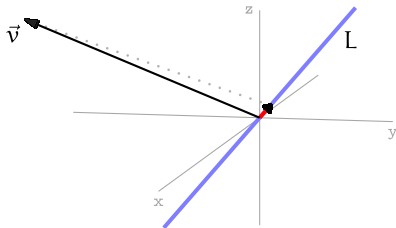
brings out that the projection vector is quite short:
$\|\vec{v}\| = \sqrt{(6)} \approx 2.45$ while $\|\text{proj}_{[\vec{s}]}(\vec{v})\| = \sqrt{1/6} \approx 0.41$. Only a small part of the vector $\vec{v}$ lies in the direction of the line L.

Gram-Schmidt Orthogonalization

Mutually orthogonal vectors

The prior subsection suggests that projecting a vector $\vec{v}$ into the line spanned by $\vec{s}$ decomposes $\vec{v}$ into two parts
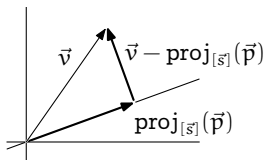
$$\vec{v} = \text{proj}_{[\vec{s}]}(\vec{v}) + \left(\vec{v} - \text{proj}_{[\vec{s}]}(\vec{v})\right)$$

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that are orthogonal and so are not-interacting. We will now develop that suggestion.

2.1 *Definition*   Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are *mutually orthogonal* when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

Example The vectors of the standard basis $\mathcal{E}_3 \subset \mathbb{R}^3$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
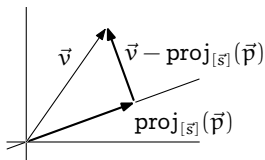
This remains true if we rotate this basis.

Example These two vectors in $\mathbb{R}^2$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
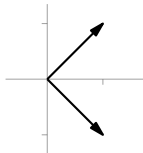
2.2 *Theorem*   If the vectors in a set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^n$ are mutually orthogonal and nonzero then that set is linearly independent.

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Proof   Consider $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. For $i \in \{1, .., k\}$, taking the dot product of $\vec{v}_i$ with both sides of the equation $\vec{v}_i \bullet (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = \vec{v}_i \bullet \vec{0}$, which gives $c_i \cdot (\vec{v}_i \bullet \vec{v}_i) = 0$, shows that $c_i = 0$ since $\vec{v}_i \neq \vec{0}$.   QED

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2.5 *Definition*   An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

2.7 *Theorem*   If $\langle \vec{\beta}_1, \ldots \vec{\beta}_k \rangle$ is a basis for a subspace of $\mathbb{R}^n$ then the vectors

$$\vec{\kappa}_1 = \vec{\beta}_1$$
$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$$
$$\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$$
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Proof   We will use induction to check that each $\vec{\kappa}_i$ is nonzero, is in the span of $\langle \vec{\beta}_1, \dots \vec{\beta}_i \rangle$, and is orthogonal to all preceding vectors $\vec{\kappa}_1 \bullet \vec{\kappa}_i = \dots = \vec{\kappa}_{i-1} \bullet \vec{\kappa}_i = 0$. Then Corollary 2.3 gives that $\langle \vec{\kappa}_1, \dots \vec{\kappa}_k \rangle$ is a basis for the same space as is the starting basis.

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We shall only cover the cases up to $i = 3$, to give the sense of the argument. The full argument is Exercise 25 .

The $i = 1$ case is trivial; taking $\vec{\kappa}_1$ to be $\vec{\beta}_1$ makes it a nonzero vector since $\vec{\beta}_1$ is a member of a basis, it is obviously in the span of $\langle \vec{\beta}_1 \rangle$, and the 'orthogonal to all preceding vectors' condition is satisfied vacuously.

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In the $i = 2$ case the expansion

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1$$

shows that $\vec{\kappa}_2 \neq \vec{0}$ or else this would be a non-trivial linear dependence among the $\vec{\beta}$'s (it is nontrivial because the coefficient of $\vec{\beta}_2$ is 1). It also shows that $\vec{\kappa}_2$ is in the span of $\langle \vec{\beta}_1, \vec{\beta}_2 \rangle$. And, $\vec{\kappa}_2$ is orthogonal to the only preceding vector

$$\vec{\kappa}_1 \cdot \vec{\kappa}_2 = \vec{\kappa}_1 \cdot (\vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)) = 0$$

because this projection is orthogonal.

The $i = 3$ case is the same as the $i = 2$ case except for one detail. As in the $i = 2$ case, expand the definition.

$$\vec{\kappa}_3 = \vec{\beta}_3 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\kappa}_1 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_2}{\vec{\kappa}_2 \bullet \vec{\kappa}_2} \cdot \vec{\kappa}_2$$

$$= \vec{\beta}_3 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\beta}_1 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_2}{\vec{\kappa}_2 \bullet \vec{\kappa}_2} \cdot \left(\vec{\beta}_2 - \frac{\vec{\beta}_2 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\beta}_1 \right)$$

By the first line $\vec{\kappa}_3 \neq \vec{0}$, since $\vec{\beta}_3$ isn't in the span $[\vec{\beta}_1, \vec{\beta}_2]$ and therefore by the inductive hypothesis it isn't in the span $[\vec{\kappa}_1, \vec{\kappa}_2]$. By the second line $\vec{\kappa}_3$ is in the span of the first three $\vec{\beta}$'s. Finally, the calculation below shows that $\vec{\kappa}_3$ is orthogonal to $\vec{\kappa}_1$.

$$\vec{\kappa}_1 \cdot \vec{\kappa}_3 = \vec{\kappa}_1 \cdot \big(\ \vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)\ \big)$$
$$= \vec{\kappa}_1 \cdot \big(\vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3)\big) - \vec{\kappa}_1 \cdot \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$$
$$= 0$$

(Here is the difference with the $i = 2$ case: as happened for $i = 2$ the first term is 0 because this projection is orthogonal, but here the second term in the second line is 0 because $\vec{\kappa}_1$ is orthogonal to $\vec{\kappa}_2$ and so is orthogonal to any vector in the line spanned by $\vec{\kappa}_2$.) A similar check shows that $\vec{\kappa}_3$ is also orthogonal to $\vec{\kappa}_2$. QED

Example This is a basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \rangle$$

We produce the new basis by starting with $\vec{\beta}_1$.

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The next step is $\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$.

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.
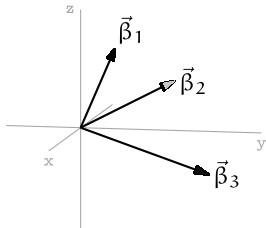
$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

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The members of this basis are mutually orthogonal.

$$K = \langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix} \rangle$$
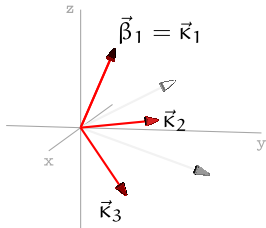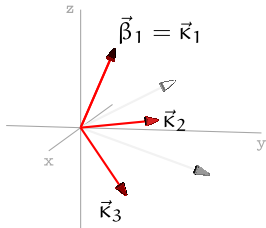
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We could go on to make this basis even more like $\mathcal{E}_3$ by normalizing all of its members to have length 1, making an *orthonormal* basis.