One.III Reduced Echelon Form

Linear Algebra

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Gauss-Jordan reduction

Pivoting

Example Here is an extension of Gauss's Method with some advantages.

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\begin{array}{rcrcrcr}
x & + & y & - & z & = & 2 \\
2x & - & y & & & = & -1 \\
x & - & 2y & + & 2z & = & -1
\end{array}
\xrightarrow[-1\rho_1+\rho_3]{-2\rho_1+\rho_2}
\begin{array}{rcrcrcr}
x & + & y & - & z & = & 2 \\
 & - & 3y & & 2z & = & -5 \\
 & - & 3y & + & 3z & = & -3
\end{array}
$$

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Instead of doing back substitution, we continue using the row operations. First make all the leading entries one.

$$\xrightarrow{(-1/3)\rho_2} \quad \begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & & y & - & (2/3)z & = & 5/3 \\ & & & & z & = & 2 \end{array}$$

Finish by eliminating upwards with the leading entries.

$$\begin{array}{rcl} x+y-z &=& 2 \\ y-(2/3)z &=& 5/3 \\ z &=& 2 \end{array} \;\xrightarrow[(2/3)\rho_3+\rho_2]{\rho_3+\rho_1}\; \begin{array}{rcl} x+y &=& 4 \\ y &=& 3 \\ z &=& 2 \end{array} \;\xrightarrow{-\rho_2+\rho_1}\; \begin{array}{rcl} x &=& 1 \\ y &=& 3 \\ z &=& 2 \end{array}$$

Now we can read off the solution.

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$$

Now we can read off the solution.

Using one entry to clear out the rest of a column is *pivoting* on that entry.

Example With this system

$$\begin{aligned} x - y - 2w &= 2 \\ x + y + 3z + w &= 1 \\ -y + z - w &= 0 \end{aligned}$$

we can rewrite in matrix notation and do Gauss's Method.

$$\xrightarrow{-1\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & 0 & 5/2 & 1/2 & -1/2 \end{array}\right)$$

We can combine the operations making the leading entries to one.

$$\xrightarrow[(2/5)\rho_3]{(1/2)\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 3/2 & 3/2 & -1/2 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

Now eliminate upwards.

$$\xrightarrow{-(3/2)\rho_3+\rho_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 2 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right) \xrightarrow{\rho_2+\rho_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

The final augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -4/5 & 9/5 \\ 0 & 1 & 0 & 6/5 & -1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 \end{array}\right)$$

leads to this description of the solution set.

$$\{\begin{pmatrix} 9/5 \\ -1/5 \\ -1/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4/5 \\ -6/5 \\ -1/5 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R}\}$$

Gauss-Jordan reduction

This extension of Gauss's Method is the *Gauss-Jordan Method* or *Gauss-Jordan reduction*.

1.3 *Definition* A matrix or linear system is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a one and is the only nonzero entry in its column.

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The cost of using Gauss-Jordan reduction to solve a system is the additional arithmetic. But it has the benefit that we can just read off the solution set from the reduced echelon form.

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Proof For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i + \rho_j} \xrightarrow{-k\rho_i + \rho_j} A$$

(The third clause is false if $i = j$.) QED

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We say that matrices that reduce to each other are equivalent with respect to the relationship of row reducibility. The next result justifies this, using the definition of an equivalence.

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For transitivity, suppose that A reduces to B and that B reduces to C. Following the reduction steps from $A \to \cdots \to B$ with those from $B \to \cdots \to C$ gives a reduction from A to C. QED

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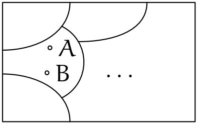
The diagram below shows the collection of all matrices as a box. Inside that box each matrix lies in a class. Matrices are in the same class if and only if they are interreducible. The classes are disjoint — no matrix is in two distinct classes. We have partitioned the collection of matrices into *row equivalence classes*.



Linear Combination Lemma

How Gauss's method acts

Example Consider this reduction.

$$\left(\begin{array}{cc|c}1 & 3 & 5\\ 2 & 4 & 8\end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|c}1 & 3 & 5\\ 0 & -2 & -2\end{array}\right) \xrightarrow{-(1/2)\rho_2} \left(\begin{array}{cc|c}1 & 3 & 5\\ 0 & 1 & 1\end{array}\right)$$
$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & 1\end{array}\right)$$

Denote the matrices as $A \to D \to G \to B$ The steps take us through these combinations.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} \delta_1 = \alpha_1 \\ \delta_2 = -2\alpha_1 + \alpha_2 \end{pmatrix}$$
$$\xrightarrow{-(1/2)\rho_2} \begin{pmatrix} \gamma_1 = \alpha_1 \\ \gamma_2 = -1\alpha_1 + (1/2)\alpha_2 \end{pmatrix}$$
$$\xrightarrow{-3\rho_2+\rho_1} \begin{pmatrix} \beta_1 = 4\alpha_1 - (3/2)\alpha_2 \\ \beta_2 = -1\alpha_1 + (1/2)\alpha_2 \end{pmatrix}$$

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Gauss's method systemmatically develops linear combinations of rows.

Linear Combination Lemma

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Proof Given the set $c_{1,1}x_1 + \cdots + c_{1,n}x_n$ through $c_{m,1}x_1 + \cdots + c_{m,n}x_n$ of linear combinations of the x's, consider a combination of those

$$d_1(c_{1,1}x_1 + \cdots + c_{1,n}x_n) + \cdots + d_m(c_{m,1}x_1 + \cdots + c_{m,n}x_n)$$

where the d's are scalars along with the c's. Distributing those d's and regrouping gives

$$= (d_1c_{1,1} + \cdots + d_mc_{m,1})x_1 + \cdots + (d_1c_{1,n} + \cdots + d_mc_{m,n})x_n$$

which is also a linear combination of the x's. QED

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Proof For any two interreducible matrices A and B there is some minimum number of row operations that will take one to the other. We proceed by induction on that number.

In the base step, that we can go from the first to the second using zero reduction operations, the two matrices are equal. Then each row of B is trivially a combination of A's rows $\vec{\beta}_i = 0 \cdot \vec{\alpha}_1 + \cdots + 1 \cdot \vec{\alpha}_i + \cdots + 0 \cdot \vec{\alpha}_m$.

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For the inductive step assume the inductive hypothesis: with $k \geqslant 0$, any matrix that can be derived from A in k or fewer operations has rows that are linear combinations of A's rows. Consider a matrix B such that reducing A to B requires $k+1$ operations. In that reduction there is a next-to-last matrix G, so that $A \longrightarrow \cdots \longrightarrow G \longrightarrow B$. The inductive hypothesis applies to this G because it is only k steps away from A. That is, each row of G is a linear combination of the rows of A.

We will verify that the rows of B are linear combinations of the rows of G. Then the Linear Combination Lemma, Lemma 2.3 , applies to show that the rows of B are linear combinations of the rows of A.

If the row operation taking G to B is a swap then the rows of B are just the rows of G reordered and each row of B is a linear combination of the rows of G. If the operation taking G to B is multiplication of a row by a scalar $c\rho_i$ then $\vec{\beta}_i = c\vec{\gamma}_i$ and the other rows are unchanged. Finally, if the row operation is adding a multiple of one row to another $r\rho_i + \rho_j$ then only row j of B differs from the matching row of G, and $\vec{\beta}_j = r\gamma_i + \gamma_j$, which is indeed a linear combinations of the rows of G. QED

2.5 *Lemma* In an echelon form matrix, no nonzero row is a linear combination of the other nonzero rows.

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Proof Let R be an echelon form matrix and consider its non-$\vec{0}$ rows. First observe that if we have a row written as a combination of the others $\vec{\rho}_i = c_1\vec{\rho}_1 + \cdots + c_{i-1}\vec{\rho}_{i-1} + c_{i+1}\vec{\rho}_{i+1} + \cdots + c_m\vec{\rho}_m$ then we can rewrite that equation as

$$\vec{0} = c_1\vec{\rho}_1 + \cdots + c_{i-1}\vec{\rho}_{i-1} + c_i\vec{\rho}_i + c_{i+1}\vec{\rho}_{i+1} + \cdots + c_m\vec{\rho}_m \qquad (*)$$

where not all the coefficients are zero; specifically, $c_i = -1$. The converse holds also: given equation $(*)$ where some $c_i \neq 0$ we could express $\vec{\rho}_i$ as a combination of the other rows by moving $c_i\vec{\rho}_i$ to the left and dividing by $-c_i$. Therefore we will have proved the theorem if we show that in $(*)$ all of the coefficients are 0. For that we use induction on the row number i.

The base case is the first row $i = 1$ (if there is no such nonzero row, so that R is the zero matrix, then the lemma holds vacuously). Let ℓ_i be the column number of the leading entry in row i. Consider the entry of each row that is in column ℓ_1. Equation $(*)$ gives this.

$$0 = c_1 r_{1,\ell_1} + c_2 r_{2,\ell_1} + \cdots + c_m r_{m,\ell_1} \qquad (**)$$

The matrix is in echelon form so every row after the first has a zero entry in that column $r_{2,\ell_1} = \cdots = r_{m,\ell_1} = 0$. Thus equation $(**)$ shows that $c_1 = 0$, because $r_{1,\ell_1} \neq 0$ as it leads the row.

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The inductive step is much the same as the base step. Again consider equation $(*)$. We will prove that if the coefficient c_i is 0 for each row index $i \in \{1, \ldots, k\}$ then c_{k+1} is also 0. We focus on the entries from column ℓ_{k+1}.

$$0 = c_1 r_{1,\ell_{k+1}} + \cdots + c_{k+1} r_{k+1,\ell_{k+1}} + \cdots + c_m r_{m,\ell_{k+1}}$$

By the inductive hypothesis $c_1, \ldots c_k$ are all 0 so this reduces to the equation $0 = c_{k+1} r_{k+1,\ell_{k+1}} + \cdots + c_m r_{m,\ell_{k+1}}$. The matrix is in echelon form so the entries $r_{k+2,\ell_{k+1}}, \ldots, r_{m,\ell_{k+1}}$ are all 0. Thus $c_{k+1} = 0$, because $r_{k+1,\ell_{k+1}} \neq 0$ as it is the leading entry. QED

Example In this non-echelon form matrix the third row is the sum of the first and second.

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But in the matrix that we get from Gauss's Method

$$\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

the only linear relationship among the nonzero rows

$$\vec{0} = c_1 \begin{pmatrix} 1 & -1 & 3 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 2 & -2 \end{pmatrix}$$

is the trivial relationship, since the equation of first entries $0 = c_1 \cdot 1$ gives that $c_1 = 0$ and then the equation of second entries $0 = 0 \cdot (-1) + c_2 \cdot 2$ gives $c_2 = 0$.

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Proof Fix a number of rows m. We will proceed by induction on the number of columns n.

The base case is that the matrix has $n = 1$ column. If this is the zero matrix then its echelon form is the zero matrix. If instead it has any nonzero entries then when the matrix is brought to reduced echelon form it must have at least one nonzero entry, which must be a 1 in the first row. Either way, its reduced echelon form is unique.

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For the inductive step we assume that $n > 1$ and that all m row matrices having fewer than n columns have a unique reduced echelon form. Consider an $m \times n$ matrix A and suppose that B and C are two reduced echelon form matrices derived from A. We will show that these two must be equal.

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Let $\hat{A}$ be the matrix consisting of the first $n - 1$ columns of A. Observe that any sequence of row operations that bring A to reduced echelon form will also bring $\hat{A}$ to reduced echelon form. By the inductive hypothesis this reduced echelon form of $\hat{A}$ is unique, so if B and C differ then the difference must occur in column n.

Consider a homogeneous system of equations for which A is the matrix of coefficients.

$$\begin{aligned} a_{1,1}x_1 + \ a_{1,2}x_2 + \cdots + \ a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + \ a_{2,2}x_2 + \cdots + \ a_{2,n}x_n &= 0 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= 0 \end{aligned} \tag{$*$}$$

By Theorem One.I.1.5 the set of solutions to that system is the same as the set of solutions to B's system

$$\begin{aligned} b_{1,1}x_1 + \ b_{1,2}x_2 + \cdots + \ b_{1,n}x_n &= 0 \\ b_{2,1}x_1 + \ b_{2,2}x_2 + \cdots + \ b_{2,n}x_n &= 0 \\ &\vdots \\ b_{m,1}x_1 + b_{m,2}x_2 + \cdots + b_{m,n}x_n &= 0 \end{aligned} \tag{$**$}$$

and to C's.

$$\begin{aligned} c_{1,1}x_1 + \ c_{1,2}x_2 + \cdots + \ c_{1,n}x_n &= 0 \\ c_{2,1}x_1 + \ c_{2,2}x_2 + \cdots + \ c_{2,n}x_n &= 0 \\ &\vdots \\ c_{m,1}x_1 + c_{m,2}x_2 + \cdots + c_{m,n}x_n &= 0 \end{aligned} \tag{$***$}$$

With B and C different only in column n, suppose that they differ in row i. Subtract row i of $(***)$ from row i of $(**)$ to get the equation $(b_{i,n} - c_{i,n}) \cdot x_n = 0$. We've assumed that $b_{i,n} \neq c_{i,n}$ so $x_n = 0$. Thus in $(**)$ and $(***)$ the n-th column contains a leading entry, or else the variable x_n would be free. That's a contradiction because with B and C equal on the first $n-1$ columns, the leading entries in the n-th column would have to be in the same row, and with both matrices in reduced echelon form, both leading entries would have to be 1, and would have to be the only nonzero entries in that column. So $B = C$. QED

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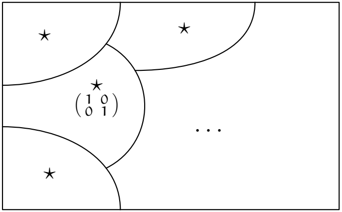
So the reduced echelon form is a canonical form for row equivalence: the reduced echelon form matrices are representatives of the classes.



Example To decide if these two are row equivalent

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 & -2 \\ 6 & 2 & -4 \\ 1 & 0 & 2 \end{pmatrix}$$

use Gauss-Jordan elimination to bring each to reduced echelon form and see if they are equal.

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use Gauss-Jordan elimination to bring each to reduced echelon form and see if they are equal. The results are

$$\xrightarrow[-(4/3)\rho_1+\rho_3]{-(1/3)\rho_1+\rho_2} \xrightarrow{-1\rho_2+\rho_3} \xrightarrow[-(3/5)\rho_2]{(1/3)\rho_1} \xrightarrow{-(2/3)\rho_2+\rho_1} \begin{pmatrix}1 & 0 & 4/5\\ 0 & 1 & -6/5\\ 0 & 0 & 0\end{pmatrix}$$

and

$$\xrightarrow[-(1/3)\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{\rho_2\leftrightarrow\rho_3} \xrightarrow[-3\rho_2]{(1/3)\rho_1} \xrightarrow{-(1/3)\rho_2+\rho_1} \begin{pmatrix}1 & 0 & 2\\ 0 & 1 & -8\\ 0 & 0 & 0\end{pmatrix}$$

so the original matrices are not row equivalent.