Four.II Geometry of Determinants

Linear Algebra

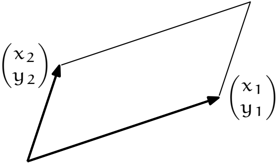
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Determinants as size functions

Box

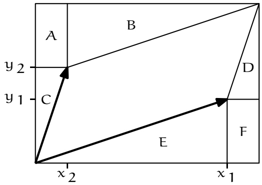
This parallelogram is defined by the two vectors.



1.1 *Definition* In $\mathbb{R}^n$ the *box* (or *parallelepiped*) formed by $\langle \vec{v}_1, \ldots, \vec{v}_n \rangle$ is the set $\{t_1\vec{v}_1 + \cdots + t_n\vec{v}_n \mid t_1, \ldots, t_n \in [0..1]\}$.

The box is like the span, except that the scalars are from the unit interval.

Area of a two dimensional box



$$
\begin{aligned}
\text{box area} &= \text{rectangle area} - \text{area of } A - \cdots - \text{area of F} \\
&= (x_1 + x_2)(y_1 + y_2) - x_2 y_1 - x_1 y_1 / 2 \\
&\quad - x_2 y_2 / 2 - x_2 y_2 / 2 - x_1 y_1 / 2 - x_2 y_1 \\
&= x_1 y_2 - x_2 y_1
\end{aligned}
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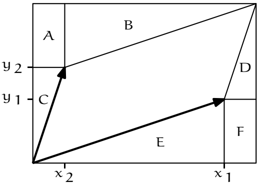

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&= x_1 y_2 - x_2 y_1
\end{aligned}$$

That area equals the value of the determinant.

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

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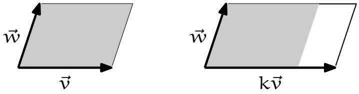
In line with the above use of column vectors, in this section we consider the determinant as a function of the columns of the matrix. Note that because the determinant of a matrix equals the determinant of its transpose, we can change the determinant properties from statements about rows to statements about columns. For instance, the first property as given earlier says that a determinant is unaffected by a row operation $k\rho_i + \rho_j$ (with $i \neq j$). By transposing we have the property that a determinant does not change under combinations of columns.

Definition of determinant reinterpreted

Recall property (3) from the definition of determinant, that rescaling a column rescales the entire determinant $\det(\ldots, k\vec{v}_i, \ldots) = k\det(\ldots, \vec{v}_i, \ldots)$. This fits with the idea that the determinant gives the size of the box formed by the columns of the matrix: if we scale a column by a factor k then the size of the box scales by that factor.



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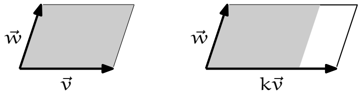
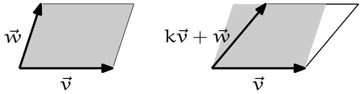


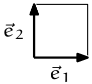
Property (1) says that the determinant is unaffected by combining columns. The picture



shows that the box formed by $\vec{v}$ and $k\vec{v} + \vec{w}$ is more slanted than the original box but the two have the same base and height, and hence the same area.

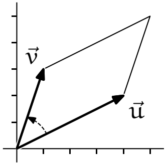
As we noted after the definition, property (2) is a consequence of the other properties so we leave it aside for the moment.

Property (4) says that the determinant of the identity matrix is 1.

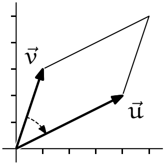


Aside: Orientation in two space

1.2 *Remark* Although property (2) is redundant, it says something notable. Consider these two.



$$\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$$



$$\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

Swapping changes the sign; on the left we take $\vec{u}$ first in the matrix and then follow the counterclockwise arc to $\vec{v}$, following the counterclockwise arc and and get a positive size, while on the right following the clockwise arc gives a negative size. The sign returned by the size function, the determinant, reflects the *orientation* or *sense* of the box.

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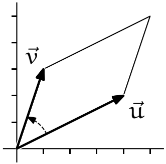

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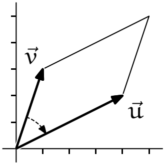

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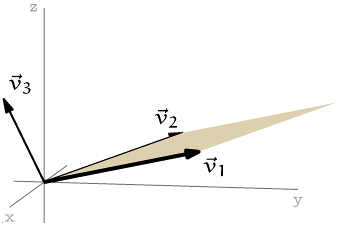
Aside: Orientation in three space

Starting with these two vectors we want to form a box with positive size.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$



Those two vectors span a plane, which divides three-space in two. The $\vec{v}_3$ shown is on the side of the plane containing vectors with this property:

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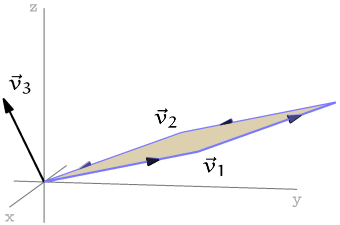

Those two vectors span a plane, which divides three-space in two. The $\vec{v}_3$ shown is on the side of the plane containing vectors with this property: if a person at the tip of $\vec{v}_3$ looks down at the box and traces out the parallelogram from $\vec{0}$, to $\vec{v}_1$, to $\vec{v}_1 + \vec{v}_2$, to $\vec{v}_2$, and back to $\vec{0}$ then their trace looks to them to be counterclockwise. Any such vector will make a positive-sized box.

The vector shown is this.

$$\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \qquad \begin{vmatrix} 1 & -2 & 0 \\ 4 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 25$$

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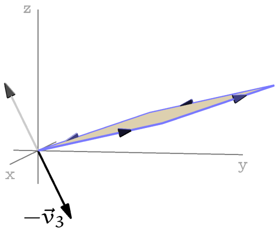

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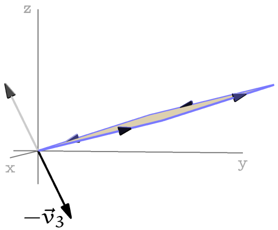

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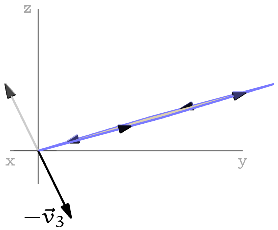

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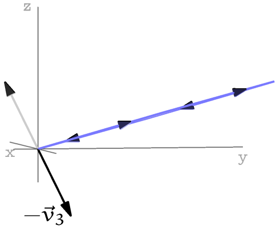

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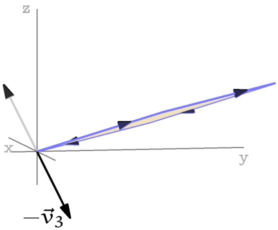

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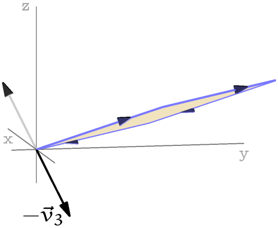

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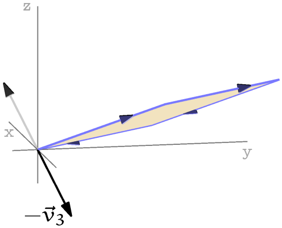

Determinants are multiplicative

1.5 *Theorem* A transformation $t\colon \mathbb{R}^n \to \mathbb{R}^n$ changes the size of all boxes by the same factor, namely the size of the image of a box $|t(S)|$ is $|T|$ times the size of the box $|S|$, where T is the matrix representing t with respect to the standard basis.

That is, for all $n \times n$ matrices, the determinant of a product is the product of the determinants $|TS| = |T| \cdot |S|$.

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Now consider the case that $|T| \neq 0$, that T is nonsingular. Recall that any nonsingular matrix factors into a product of elementary matrices $T = E_1E_2\cdots E_r$. To finish this argument we will verify that $|ES| = |E| \cdot |S|$ for all matrices S and elementary matrices E. The result will then follow because $|TS| = |E_1\cdots E_rS| = |E_1|\cdots|E_r| \cdot |S| = |E_1\cdots E_r| \cdot |S| = |T| \cdot |S|$.

There are three kinds of elementary matrix. We will cover the $M_i(k)$ case; the $P_{i,j}$ and $C_{i,j}(k)$ checks are similar. We have that $M_i(k)S$ equals S except that row i is multiplied by k. The third property of determinant functions then gives that $|M_i(k)S| = k \cdot |S|$. But $|M_i(k)| = k$, again by the third property because $M_i(k)$ is derived from the identity by multiplication of row i by k. Thus $|ES| = |E| \cdot |S|$ holds for $E = M_i(k)$. QED

Example The transformation $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through a counterclockwise angle θ is represented by this matrix.

$$T_\theta = \mathrm{Rep}_{\mathcal{E}_2}(t_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Observe that t_θ doesn't change the size of any boxes, it just rotates the entire box as a rigid whole. Note that $|T_\theta| = 1$.

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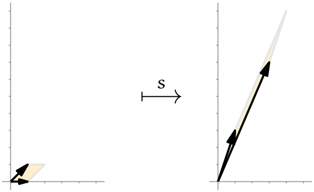
Example The linear transformation $s\colon \mathbb{R}^2 \to \mathbb{R}^2$ represented with respect to the standard basis by this matrix

$$S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

will, by the theorem, change the size of a box by a factor of $|S| = -2$. Here is s acting on a typical box.

The box defined by the two vectors $\vec{v}_1 = \binom{1}{0}$ and $\vec{v}_2 = \binom{1}{1}$ is transformed by s to the box defined by the two vectors $s(\vec{v}_1) = \binom{1}{3}$ and $s(\vec{v}_2) = \binom{3}{7}$.



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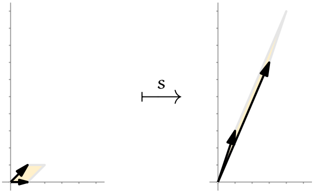

Note the change in orientation.

The two sizes are easy.

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} = -2$$

Determinant of the inverse is the inverse of the determinant

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Proof $1 = |I| = |TT^{-1}| = |T| \cdot |T^{-1}|$ QED

Volume

1.3 *Definition* The *volume* of a box is the absolute value of the determinant of a matrix with those vectors as columns.

Example The box formed by thevectors

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

gives this determinant

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

so its volume is 2.

Cramer's Rule

Geometric interpretation of linear systems

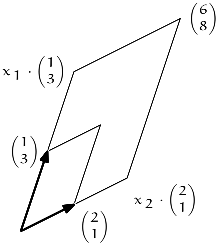
We have seen that a linear system

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 3x_1 + x_2 &= 8 \end{aligned}$$

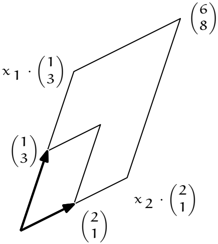
is equivalent to a linear relationship among vectors.

$$x_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

This pictures that vector equation. A parallelogram with sides formed from $\binom{1}{3}$ and $\binom{2}{1}$ is nested inside a parallelogram with sides formed from $x_1\binom{1}{3}$ and $x_2\binom{2}{1}$.



That is, we can restate the algebraic question of finding the solution of a linear system in geometric terms: by what factors x_1 and x_2 must we dilate the vectors to expand the small parallelogram so that it will fill the larger one?



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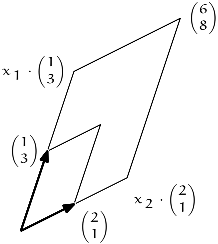
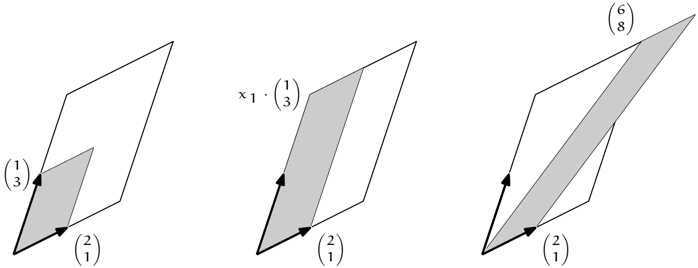


Next we consider expanding only one side of the parallelogram.

Compare the sizes of these shaded boxes.



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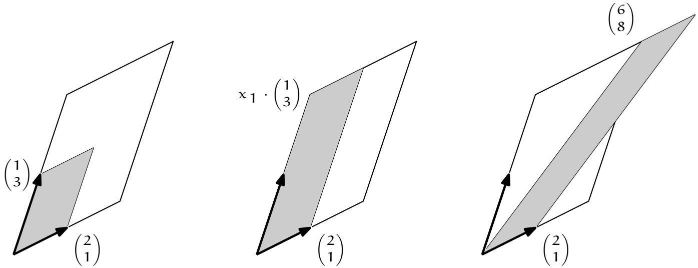

The second is defined by the vectors $x_1\binom{1}{3}$ and $\binom{2}{1}$, and one of the properties of the size function — the determinant — is that therefore the size of the second box is x_1 times the size of the first box. Since the third box is defined by the vector $x_1\binom{1}{3} + x_2\binom{2}{1} = \binom{6}{8}$ and the vector $\binom{2}{1}$, and since the determinant does not change when we add x_2 times the second column to the first column, the size of the third box equals that of the second.

$$\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix} = \begin{vmatrix} x_1 \cdot 1 & 2 \\ x_1 \cdot 3 & 1 \end{vmatrix} = x_1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

Solving gives the value of one of the variables.

$$x_1 = \frac{\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

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The symmetric argument for the other side gives this.

$$x_2 = \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = 2$$

Cramer's Rule

Let A be an $n \times n$ matrix, let $\vec{b}$ be an n-tall column vector, and consider the linear system $A\vec{x} = \vec{b}$. For any $i \in [1, \ldots, n]$ let B_i be the matrix obtained by substituting $\vec{b}$ for column i of A. Then the value of the i-th unknown is $x_i = |B_i|/|A|$.

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Exercise 3 gives the proof.

Example Given this system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 + 3x_2 \phantom{{}-5x_3} &= 2 \\ x_2 - 5x_3 &= 0 \end{aligned}$$

we can rewrite it as

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

and

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{vmatrix} = -26 \qquad |B_2| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -5 \end{vmatrix} = 0$$

so $x_2 = 0/-26 = 0$.

A caution

Cramer's Rule is an interesting application of the geometry that we have developed. And it allows us to mentally solve small systems, those with two or three variables.

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Cramer's Rule is an interesting application of the geometry that we have developed. And it allows us to mentally solve small systems, those with two or three variables.

But it is a poor choice for systems having many variables. Taking a determinant of a general large matrix is very slow.