Three.VI Projection

Linear Algebra

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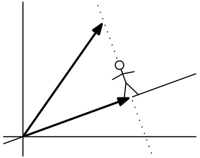
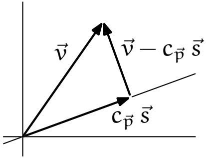
Orthogonal Projection Into a Line

Project a vector into a line

This shows a figure walking out on the line until they are at a point $\vec{p}$ such that the tip of $\vec{v}$ is directly above them, where "above" does not mean parallel to the y-axis but instead means orthogonal to the line.





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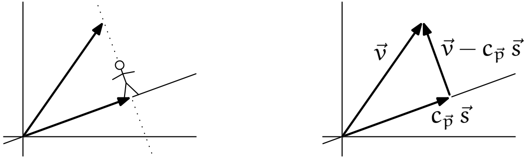

Since the line is the span of some vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, we have a coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to $c_{\vec{p}}\vec{s}$.

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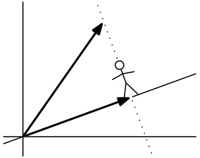

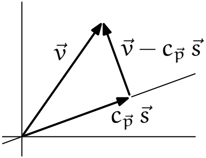

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To solve for this coefficient, observe that because $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$, it must be orthogonal to $\vec{s}$ itself. Then $(\vec{v} - c_{\vec{p}}\vec{s}) \bullet \vec{s} = 0$ gives that $c_{\vec{p}} = \vec{v} \bullet \vec{s} / \vec{s} \bullet \vec{s}$.

Note two things about orthogonal projection.

1) We have an idea of 'angle' in $\mathbb{R}^n$ but we have not given such a definition in some other spaces. This section's results will stick to spaces in which we have covered 'orthogonal,' namely the real spaces.

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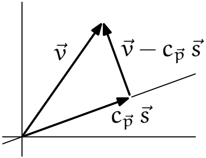
1) We have an idea of 'angle' in $\mathbb{R}^n$ but we have not given such a definition in some other spaces. This section's results will stick to spaces in which we have covered 'orthogonal,' namely the real spaces.
2) We have decomposed $\vec{v}$ into two parts.



Intuitively, some of $\vec{v}$ lies with the line and that gives the first part $c_{\vec{p}}\vec{s}$. The part of $\vec{v}$ that lies with a line orthogonal to the given line is $\vec{v} - c_{\vec{p}}\vec{s}$. What's compelling about pairing these two parts is that they don't interact in that the projection of one into the line spanned by the other is the zero vector.

1.1 *Definition* The *orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$* is this vector.

$$\mathrm{proj}_{[\vec{s}\,]}(\vec{v}) = \frac{\vec{v} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

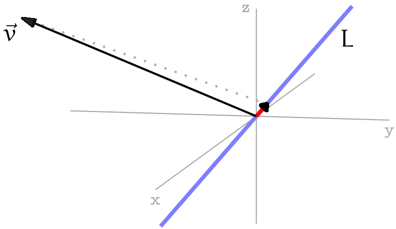
Example The projection of this $\mathbb{R}^3$ vector into the line

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$

This picture of that projection

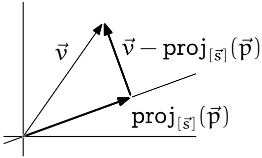


brings out that the projection vector is quite short: $\|\vec{v}\| = \sqrt(6) \approx 2.45$ while $\|\mathrm{proj}_{[\vec{s}\,]}(\vec{v})\| = \sqrt{1/6} \approx 0.41$. Only a small part of the vector $\vec{v}$ lies in the direction of the line L.

Gram-Schmidt Orthogonalization

Mutually orthogonal vectors

The prior subsection suggests that projecting a vector $\vec{v}$ into the line spanned by $\vec{s}$ decomposes $\vec{v}$ into two parts



$$\vec{v} = \mathrm{proj}_{[\vec{s}\,]}(\vec{v}) + \left(\vec{v} - \mathrm{proj}_{[\vec{s}\,]}(\vec{v})\right)$$

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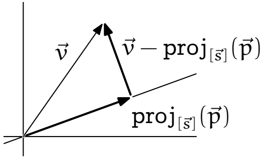

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that are orthogonal and so are not-interacting. We will now develop that suggestion.

2.1 *Definition* Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are *mutually orthogonal* when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

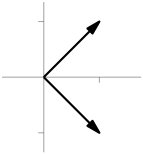
Example The vectors of the standard basis $\mathcal{E}_3 \subset \mathbb{R}^3$ are mutually orthogonal.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

This remains true if we rotate this basis.

Example These two vectors in $\mathbb{R}^2$ are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



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Proof Consider $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$. For $i \in \{1, .., k\}$, taking the dot product of $\vec{v}_i$ with both sides of the equation $\vec{v}_i \bullet (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = \vec{v}_i \bullet \vec{0}$, which gives $c_i \cdot (\vec{v}_i \bullet \vec{v}_i) = 0$, shows that $c_i = 0$ since $\vec{v}_i \neq \vec{0}$. QED

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2.5 *Definition* An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

2.7 *Theorem* If $\langle\vec{\beta}_1,\ldots\vec{\beta}_k\rangle$ is a basis for a subspace of $\mathbb{R}^n$ then the vectors

$$\begin{aligned}
\vec{\kappa}_1 &= \vec{\beta}_1 \\
\vec{\kappa}_2 &= \vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) \\
\vec{\kappa}_3 &= \vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\
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Proof We will use induction to check that each $\vec{\kappa}_i$ is nonzero, is in the span of $\langle\vec{\beta}_1,\ldots\,\vec{\beta}_i\rangle$, and is orthogonal to all preceding vectors $\vec{\kappa}_1\cdot\vec{\kappa}_i=\cdots=\vec{\kappa}_{i-1}\cdot\vec{\kappa}_i=0$. Then Corollary 2.3 gives that $\langle\vec{\kappa}_1,\ldots\,\vec{\kappa}_k\rangle$ is a basis for the same space as is the starting basis.

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We shall only cover the cases up to $i=3$, to give the sense of the argument. The full argument is Exercise 25 .

The $i = 1$ case is trivial; taking $\vec{\kappa}_1$ to be $\vec{\beta}_1$ makes it a nonzero vector since $\vec{\beta}_1$ is a member of a basis, it is obviously in the span of $\langle\vec{\beta}_1\rangle$, and the 'orthogonal to all preceding vectors' condition is satisfied vacuously.

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In the $i = 2$ case the expansion

$$\vec{\kappa}_2 = \vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \vec{\beta}_2 - \frac{\vec{\beta}_2 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\kappa}_1 = \vec{\beta}_2 - \frac{\vec{\beta}_2 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\beta}_1$$

shows that $\vec{\kappa}_2 \neq \vec{0}$ or else this would be a non-trivial linear dependence among the $\vec{\beta}$'s (it is nontrivial because the coefficient of $\vec{\beta}_2$ is 1). It also shows that $\vec{\kappa}_2$ is in the span of $\langle\vec{\beta}_1, \vec{\beta}_2\rangle$. And, $\vec{\kappa}_2$ is orthogonal to the only preceding vector

$$\vec{\kappa}_1 \bullet \vec{\kappa}_2 = \vec{\kappa}_1 \bullet (\vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)) = 0$$

because this projection is orthogonal.

The $i = 3$ case is the same as the $i = 2$ case except for one detail. As in the $i = 2$ case, expand the definition.

$$\begin{aligned}\vec{\kappa}_3 &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\kappa}_1 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_2}{\vec{\kappa}_2 \bullet \vec{\kappa}_2} \cdot \vec{\kappa}_2 \\ &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\beta}_1 - \frac{\vec{\beta}_3 \bullet \vec{\kappa}_2}{\vec{\kappa}_2 \bullet \vec{\kappa}_2} \cdot \bigl(\vec{\beta}_2 - \frac{\vec{\beta}_2 \bullet \vec{\kappa}_1}{\vec{\kappa}_1 \bullet \vec{\kappa}_1} \cdot \vec{\beta}_1\bigr)\end{aligned}$$

By the first line $\vec{\kappa}_3 \neq \vec{0}$, since $\vec{\beta}_3$ isn't in the span $[\vec{\beta}_1, \vec{\beta}_2]$ and therefore by the inductive hypothesis it isn't in the span $[\vec{\kappa}_1, \vec{\kappa}_2]$. By the second line $\vec{\kappa}_3$ is in the span of the first three $\vec{\beta}$'s. Finally, the calculation below shows that $\vec{\kappa}_3$ is orthogonal to $\vec{\kappa}_1$.

$$\begin{aligned}\vec{\kappa}_1 \cdot \vec{\kappa}_3 &= \vec{\kappa}_1 \cdot \left(\vec{\beta}_3 - \mbox{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mbox{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \right) \\ &= \vec{\kappa}_1 \cdot \bigl(\vec{\beta}_3 - \mbox{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3)\bigr) - \vec{\kappa}_1 \cdot \mbox{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\ &= 0\end{aligned}$$

(Here is the difference with the $i=2$ case: as happened for $i=2$ the first term is 0 because this projection is orthogonal, but here the second term in the second line is 0 because $\vec{\kappa}_1$ is orthogonal to $\vec{\kappa}_2$ and so is orthogonal to any vector in the line spanned by $\vec{\kappa}_2$.) A similar check shows that $\vec{\kappa}_3$ is also orthogonal to $\vec{\kappa}_2$. QED

Example This is a basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\3\\-1\end{pmatrix} \rangle$$

We produce the new basis by starting with $\vec{\beta}_1$.

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The next step is $\vec{\kappa}_2 = \vec{\beta}_2 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$.

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

The third step is $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$.

$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

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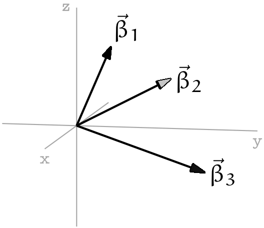
The members of this basis are mutually orthogonal.

$$K = \langle \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-3/2\\3/2\\0\end{pmatrix}, \begin{pmatrix}4/3\\4/3\\-4/3\end{pmatrix} \rangle$$



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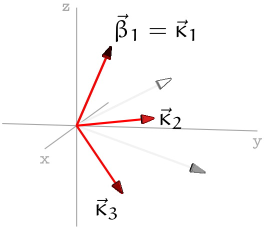

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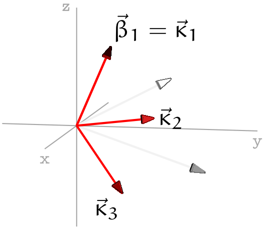

We could go on to make this basis even more like $\mathcal{E}_3$ by normalizing all of its members to have length 1, making an *orthonormal* basis.