Three.III Computing Linear Maps

Linear Algebra

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Representing Linear Maps with Matrices

Linear maps are determined by the action on a basis

Consider a domain space V and a codomain space W, and fix a basis $B_V = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ for the domain. Now for any linear map $h\colon V \to W$ consider the action of h on the basis elements $h(\vec{\beta}_1)$, $\dots$, $h(\vec{\beta}_n)$.

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$$h(\vec{v}) = h(c_1 \cdot \vec{\beta}_1 + \cdots + c_n \cdot \vec{\beta}_n) = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n) \quad (*)$$

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Example Let the domain be $V = \mathcal{P}_2$ with the basis $B_V = \langle 1, 1 + x, 1 + x + x^2 \rangle$. Let the codomain be $\mathbb{R}^2$ with this basis.

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We are developing a scheme to compute the action of this map h on any member $\vec{v} = c_1 \cdot \vec{\beta}_1 + c_2 \cdot \vec{\beta}_2 + c_3 \cdot \vec{\beta}_3$ of the domain. We want to detail the mechanics of calculating $\mathrm{Rep}_{B_W}(h(\vec{v}))$ from the $\mathrm{Rep}_{B_V}(h(\vec{\beta}_i))$.

Let the map h have this action on the domain basis.

$$h(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad h(1+x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad h(1+x+x^2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

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We first find the representation, with respect to the codomain's basis B_W, of the action of h on B_V.

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_1)) = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \quad \text{since } (1/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_2)) = \begin{pmatrix} 5/2 \\ 2 \end{pmatrix} \quad \text{since } (5/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_3)) = \begin{pmatrix} -3/2 \\ -1 \end{pmatrix} \quad \text{since } (-3/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

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We summarize by writing those three vectors side-by-side to make a matrix.

$$\begin{pmatrix} 1/2 & 5/2 & -3/2 \\ 1 & 2 & -1 \end{pmatrix}$$

Consider the domain vector $\vec{v} = c_1 \cdot \vec{\beta}_1 + c_2 \cdot \vec{\beta}_2 + c_3 \cdot \vec{\beta}_3$.

$$\mathrm{Rep}_{B_V}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Apply equation $(*)$.

$$\begin{aligned} c_1 h(\vec{\beta}_1) + c_2 h(\vec{\beta}_2) + c_3 h(\vec{\beta}_3) &= c_1(\,(1/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \\ &\quad + c_2(\,(5/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \\ &\quad + c_3(\,(-3/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \end{aligned}$$

Regroup.

$$((1/2)c_1 + (5/2)c_2 - (3/2)c_3) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (1c_1 + 2c_2 - 1c_3) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{v})) = \begin{pmatrix} (1/2)c_1 + (5/2)c_2 - (3/2)c_3 \\ 1c_1 + 2c_2 - 1c_3 \end{pmatrix}$$

So, the effect of the linear map summarized by this matrix

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on the domain vector summarized by this vector

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This is the computational scheme: we get the output vector by taking the dot product of each row of the matrix with the single column of the input vector.

Matrix representation of a linear map

1.2 *Definition* Suppose that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If

$$\mathrm{Rep}_D(h(\vec{\beta}_1)) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}_D \quad \ldots \quad \mathrm{Rep}_D(h(\vec{\beta}_n)) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}_D$$

then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

is the *matrix representation of* h *with respect to* B, D.

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is the *matrix representation of h with respect to B, D*.

We often omit the subscript on the matrix.

Example Consider projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis.

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

If we take the input and output bases to be

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle \text{ and } D = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

then we can compute

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{ so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

and the matrix representing π is this.

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

Example Again consider projection onto the x-axis

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

and this time we take the input and output bases to be the standard one.

$$B = D = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi)$ is this.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

1.4 *Theorem* Assume that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

and $\vec{v} \in V$ is represented by

$$\mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_{B}$$

then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_{D}(\,h(\vec{v})\,) = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}_{D}$$

Proof This formalizes the example that began this subsection. See Exercise 29 . QED

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1.5 *Definition* The *matrix-vector product* of a $m \times n$ matrix and a $n \times 1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + \cdots + a_{1,n}c_n \\ a_{2,1}c_1 + \cdots + a_{2,n}c_n \\ \vdots \\ a_{m,1}c_1 + \cdots + a_{m,n}c_n \end{pmatrix}$$

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Example We don't need to describe spaces and bases to perform the operation.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 1 \cdot (-1) + 2 \cdot (-3) \\ 0 \cdot 4 - 2 \cdot (-1) + 5 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

Example Recall the two matrices

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \quad \text{and} \quad \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

representing projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis with respect to

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and with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_2$. This domain vector has the given representations.

$$\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

and the matrix-vector products

$$\begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

give the representations $\mathrm{Rep}_D(\pi(\vec{v}))$ and $\mathrm{Rep}_{\mathcal{E}_2}(\pi(\vec{v}))$.

$$1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (1/2) \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2$$

Any Matrix Represents a Linear Map

Example The prior subsection shows how to start with a linear map and produce its matrix representation. Suppose instead that we start with a matrix.

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

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$$\mathcal{E}_2 \subset \mathbb{R}^2 \quad \langle 1-x, 1+x \rangle \subset \mathcal{P}_1$$

We will produce a function $m\colon \mathbb{R}^2 \to \mathcal{P}_1$ that is represented by the matrix M with respect to these bases. Then we will verify that m is linear.

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We define m in this way: first represent $\vec{u}, \vec{v} \in \mathbb{R}^2$ with respect to the domain basis, then multiply by M, and then take the resulting representations with respect to the codomain basis and covert over to members of the codomain.

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Now we will show that the map is linear $m(c \cdot \vec{u} + d \cdot \vec{v}) = c \cdot m(\vec{u}) + d \cdot m(\vec{v})$. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be elements of the domain. Because the domain basis is $\mathcal{E}_2$ the vectors are represented by themselves. We multiply by M.

$$\begin{aligned}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \left(c \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{pmatrix} \\
&= \begin{pmatrix} 1(cu_1 + dv_1) + 2(cu_2 + dv_2) \\ 3(cu_1 + dv_1) + 4(cu_2 + dv_2) \end{pmatrix} \\
&= \begin{pmatrix} 1cu_1 + 2cu_2 \\ 3cu_1 + 4cu_2 \end{pmatrix} + \begin{pmatrix} 1dv_1 + 2dv_2 \\ 3dv_1 + 4dv_2 \end{pmatrix} \\
&= c \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\end{aligned}$$

The result is $c \cdot \mathrm{Rep}_D(m(\vec{u})) + d\mathrm{Rep}_D(m(\vec{v}))$. We finish by using the basis D for the codomain to convert the representations to vectors in $\mathcal{P}_1$.

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Now assume that there are bases B, D such that $\mathrm{Rep}_{B,D}(h)$ is the zero matrix. Then for each $\vec{\beta}_i$ we have that $\mathrm{Rep}_D(h(\vec{\beta}_1))$ is a column vector of zeros, and so $h(\vec{\beta}_i)$ is $\vec{0}_W$. Extending linearly, we have that h maps each $\vec{v} \in V$ to $\vec{0}_W$, and h is the zero map.

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One thing this example does not illustrate is that typically a linear map will have many different matrices representing it, with respect to the many different pairs of bases B, D. A matrix property that derives from the map will be shared across all these representing matrices.

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Proof Suppose that the matrix H is $m \times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

$$\begin{aligned} \{h(\vec{v}) \bigm| \vec{v} \in V\} &= \{h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) \bigm| c_1, \ldots, c_n \in \mathbb{R}\} \\ &= \{c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) \bigm| c_1, \ldots, c_n \in \mathbb{R}\} \end{aligned}$$

is the span $[\{h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

2.4 *Theorem* The rank of a matrix equals the rank of any map that it represents.

Proof Suppose that the matrix H is $m \times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

$$\begin{aligned}\{h(\vec{v}) \mid \vec{v} \in V\} &= \{h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R}\} \\ &= \{c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n) \mid c_1, \ldots, c_n \in \mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

The rank of the matrix is the dimension of its column space, the span of the set of its columns $[\{\mathrm{Rep}_D(h(\vec{\beta}_1)), \ldots, \mathrm{Rep}_D(h(\vec{\beta}_n))\}]$.

To see that the two spans have the same dimension, recall from the proof of Lemma I.2.5 that if we fix a basis then representation with respect to that basis gives an isomorphism $\mathrm{Rep}_D\colon W \to \mathbb{R}^m$. Under this isomorphism there is a linear relationship among members of the range space if and only if the same relationship holds in the column space, e.g, $\vec{0} = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n)$ if and only if $\vec{0} = c_1 \cdot \mathrm{Rep}_D(h(\vec{\beta}_1)) + \cdots + c_n \cdot \mathrm{Rep}_D(h(\vec{\beta}_n))$. Hence, a subset of the range space is linearly independent if and only if the corresponding subset of the column space is linearly independent. Therefore the size of the largest linearly independent subset of the range space equals the size of the largest linearly independent subset of the column space, and so the two spaces have the same dimension. QED

Example The range of the linear transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a-b \\ 2a-b \end{pmatrix}$$

is the line $x=y$. Thus the map t's rank is 1.

Represent t first with respect the standard bases $\mathcal{E}_2, \mathcal{E}_2$ and then with respect to these.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle,\ D = \langle \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \rangle$$

The standard basis case is easy. This is the other calculation.

$$\mathrm{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} -3 \\ -3 \end{pmatrix}) = \begin{pmatrix} -6 \\ -9 \end{pmatrix}$$

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We end with these two representations, matrices of rank 1.

$$\mathrm{Rep}_{\mathcal{E}_2, \mathcal{E}_2}(t) = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \qquad \mathrm{Rep}(t) = \begin{pmatrix} 2 & -6 \\ 3 & -9 \end{pmatrix}$$

2.6 *Corollary* Let h be a linear map represented by a matrix H. Then h is onto if and only if the rank of H equals the number of its rows, and h is one-to-one if and only if the rank of H equals the number of its columns.

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Proof For the onto half, the dimension of the range space of h is the rank of h, which equals the rank of H by the theorem. Since the dimension of the codomain of h equals the number of rows in H, if the rank of H equals the number of rows then the dimension of the range space equals the dimension of the codomain. But a subspace with the same dimension as its superspace must equal that superspace (because any basis for the range space is a linearly independent subset of the codomain whose size is equal to the dimension of the codomain, and thus so this basis for the range space must also be a basis for the codomain).

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For the other half, a linear map is one-to-one if and only if it is an isomorphism between its domain and its range, that is, if and only if its domain has the same dimension as its range. But the number of columns in h is the dimension of h's domain, and by the theorem the rank of H equals the dimension of h's range. QED

2.7 *Definition* A linear map that is one-to-one and onto is *nonsingular*, otherwise it is *singular*. That is, a linear map is nonsingular if and only if it is an isomorphism.

2.9 *Lemma* A nonsingular linear map is represented by a square matrix. A square matrix represents nonsingular maps if and only if it is a nonsingular matrix. Thus, a matrix represents isomorphisms if and only if it is square and nonsingular.

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Proof Assume that the map $h \colon V \to W$ is nonsingular. Corollary 2.6 says that for any matrix H representing that map, because h is onto the number of rows of H equals the rank of H and because h is one-to-one the number of columns of H is also equal to the rank of H. Thus H is square.

2.9 *Lemma* A nonsingular linear map is represented by a square matrix. A square matrix represents nonsingular maps if and only if it is a nonsingular matrix. Thus, a matrix represents isomorphisms if and only if it is square and nonsingular.

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Next assume that H is square, $n \times n$. The matrix H is nonsingular if and only if its row rank is n, which is true if and only if H's rank is n by Theorem Two.III.3.11 , which is true if and only if h's rank is n by Theorem 2.4 , which is true if and only if h is an isomorphism by Theorem I.2.3 . (The last holds because the domain of h is n-dimensional as it is the number of columns in H.) QED

Example This matrix

$$\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

is nonsingular since by inspection its two rows form a linearly independent set. So any map, with any domain and codomain, and represented by this matrix with respect to any bases, is an isomorphism.

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Example Gauss's method will show that this matrix

$$\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 1 \\ -1 & 0 & 5 \end{pmatrix}$$

is singular so any map that it represents will be a homomorphism that is not an isomorphism.