One.II Linear Geometry

Linear Algebra
Jim Hefferon

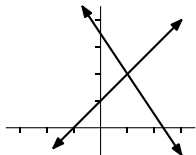http://joshua.smcvt.edu/linearalgebra

Geometry

We can draw two-unknown equations as lines. Then the three possibilities for solution sets become clear.
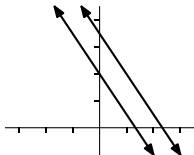
Geometry

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Unique solution
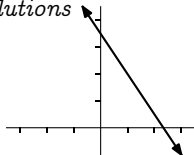

$$3x + 2y = 7$$
$$x - y = -1$$

No solutions

$$3x + 2y = 7$$
$$3x + 2y = 4$$

Infinitely many solutions

$$3x + 2y = 7$$
$$6x + 4y = 14$$

Besides being pretty, the geometry helps us understand what is happening.

Vectors in space

Vectors

A *vector* is an object consisting of a magnitude and a direction.

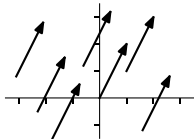For instance, a vector can model a displacement.

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Two vectors with the same magnitude and same direction, such as all of these, are equal.

For instance, each of the above could model a displacement of one over and two up.

Denote the vector that extends from (a_1, a_2) to (b_1, b_2) by

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so the "one over, two up" vector would be written in this way.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

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We often picture a vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as starting at the origin. From there $\vec{v}$ extends to (v_1, v_2) and we may refer to it as "the point $\vec{v}$" so that we may call each of these $\mathbb{R}^2$.

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

These definitions extend to higher dimensions. The vector that starts at $(a_1, \ldots, a_n)$ and ends at $(b_1, \ldots, b_n)$ is represented by this column

$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

and two vectors are equal if they have the same representation. Also, we aren't too careful about distinguishing between a point and the vector which, when it starts at the origin, ends at that point.

$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

Vector operations

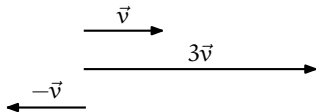Scalar multiplication makes a vector longer or shorter, including possibly flipping it around.
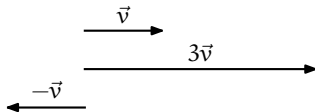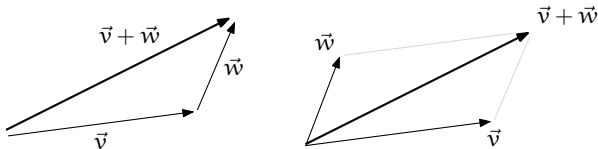
Vector operations

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Where $\vec{v}$ and $\vec{w}$ represent displacements, the vector sum $\vec{v} + \vec{w}$ represents those displacements combined.

The second drawing shows the *parallelogram rule* for vector addition.

Lines

The line in $\mathbb{R}^2$ through $(1,2)$ and $(3,1)$ is comprised of the vectors in this set

$$\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
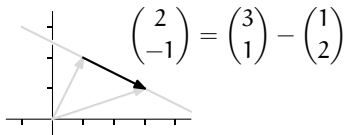
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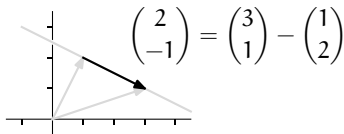
(that is, it is comprised of the endpoints of those vectors). The vector associated with the parameter t

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

is a *direction vector* for the line. Lines in higher dimensions work the same way.

Planes

The plane in $\mathbb{R}^3$ through the points $(1, 0, 5)$, $(2, 1, -3)$, and $(-2, 4, 0.5)$ consists of (endpoints of) the vectors in this set.

$$\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} + s \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} \mid t, s \in \mathbb{R} \}$$

The column vectors associated with the parameters

$$\begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 4 \\ -4.5 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0.5 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

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A set of the form $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$ where $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ and $k \leqslant n$ is a k-*dimensional linear surface* (or k-*flat*).

Length and angle measures

Length

2.1 *Definition*   The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components.

$$|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}$$

Example The length of

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For any nonzero vector $\vec{v}$, the length one vector with the same direction is $\vec{v}/|\vec{v}|$. We say that this *normalizes* $\vec{v}$ to unit length.

Dot product

Definition   The *dot product* (or *inner product* or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Example The dot product of two vectors

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The dot product of a vector with itself $\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2$ is the square of the vector's length.
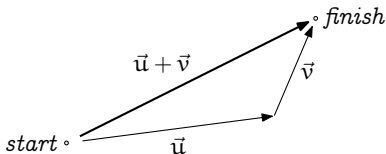
Triangle Inequality

For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u} + \vec{v}| \leqslant |\vec{u}| + |\vec{v}|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, "The shortest distance between two points is in a straight line."

2.5 *Proof*   Since all the numbers are positive, the inequality holds if and only if its square holds.

$$|\vec{u} + \vec{v}|^2 \leqslant (|\vec{u}| + |\vec{v}|)^2$$

$$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \leqslant |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2$$

$$\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \leqslant \vec{u} \cdot \vec{u} + 2|\vec{u}||\vec{v}| + \vec{v} \cdot \vec{v}$$

$$2\vec{u} \cdot \vec{v} \leqslant 2|\vec{u}||\vec{v}|$$

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$$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \leqslant |\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2$$
$$\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \leqslant \vec{u} \cdot \vec{u} + 2|\vec{u}||\vec{v}| + \vec{v} \cdot \vec{v}$$
$$2\vec{u} \cdot \vec{v} \leqslant 2|\vec{u}||\vec{v}|$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}|$ and $|\vec{v}|$

$$2(|\vec{v}|\vec{u}) \cdot (|\vec{u}|\vec{v}) \leqslant 2|\vec{u}|^2|\vec{v}|^2$$

and rewriting

$$0 \leqslant |\vec{u}|^2|\vec{v}|^2 - 2(|\vec{v}|\vec{u}) \cdot (|\vec{u}|\vec{v}) + |\vec{u}|^2|\vec{v}|^2$$

is true. But factoring shows that it is true

$$0 \leqslant (|\vec{u}|\vec{v} - |\vec{v}|\vec{u}) \cdot (|\vec{u}|\vec{v} - |\vec{v}|\vec{u})$$

since it only says that the square of the length of the vector $|\vec{u}|\vec{v} - |\vec{v}|\vec{u}$ is not negative.

As for equality, it holds when, and only when, $|\vec{u}|\,\vec{v} - |\vec{v}|\,\vec{u}$ is $\vec{0}$. The check that $|\vec{u}|\,\vec{v} = |\vec{v}|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy.   QED

Cauchy-Schwartz Inequality

2.6 *Corollary*   For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

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2.6 *Proof*   The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leqslant |\vec{u}||\vec{v}|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\vec{u} \cdot \vec{v}| = -(\vec{u} \cdot \vec{v}) = (-\vec{u}) \cdot \vec{v} \leqslant |-\vec{u}||\vec{v}| = |\vec{u}||\vec{v}|$$
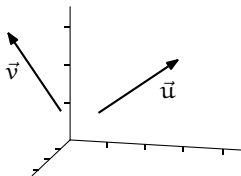
The equality condition is Exercise 19 .

Angle measure

Definition The **angle** between two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is this.

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$$

We motivate that definition with two vectors in $\mathbb{R}^3$.

If neither is a multiple of the other then they determine a plane, because if we put them in canonical position then the origin and the endpoints make three noncolinear points. Consider the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$.
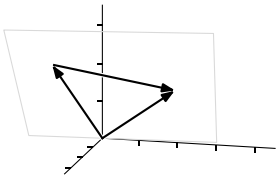
Apply the Law of Cosines: $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$ where θ is the angle that we want to find. The left side gives

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$
$$= (u_1^2 - 2u_1v_1 + v_1^2) + (u_2^2 - 2u_2v_2 + v_2^2) + (u_3^2 - 2u_3v_3 + v_3^2)$$

while the right side gives this.

$$(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2|\vec{u}||\vec{v}|\cos\theta$$

Canceling squares $u_1^2 \ldots, v_3^2$ and dividing by 2 gives the formula.

2.8 *Corollary* Vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero.