

## Three.II Homomorphisms

*Linear Algebra*

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## Definition

# Homomorphism

1.1 *Definition* A function between vector spaces  $h: V \rightarrow W$  that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

*Example* The function  $h: \mathcal{P}_2 \rightarrow \mathbb{R}^2$  given by

$$h(a + bx + cx^2) = \begin{pmatrix} a + c \\ 0 \end{pmatrix}$$

is a homomorphism (it is neither one-to-one nor onto). We will verify that it respects the addition and scalar multiplication operations.

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Addition is routine.

$$\begin{aligned} & h((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) \\ &= h((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) \\ &= \begin{pmatrix} a_1 + a_2 + c_1 + c_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 + c_2 \\ 0 \end{pmatrix} \\ &= h(a_1 + b_1x + c_1x^2) + h(a_2 + b_2x + c_2x^2) \end{aligned}$$

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So is scalar multiplication.

$$r \cdot h(a + bx + cx^2) = r \cdot \begin{pmatrix} a + c \\ 0 \end{pmatrix} = \begin{pmatrix} ra + rc \\ 0 \end{pmatrix} = h(r(a + bx + cx^2))$$

*Example* Of these two maps  $h, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ , the first is linear while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{h} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{g} 2x - 3y + 1$$

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The map  $h$  respects addition

$$\begin{aligned} h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= h\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + h\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \end{aligned}$$

and scalar multiplication.

$$r \cdot h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h\left(r \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right)$$



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In contrast,  $g$  does not respect addition.

$$g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -17 \quad \text{while} \quad g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) + g\left(\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -16$$

We proved the following two in the context of studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map  $f: V \rightarrow W$  between vector spaces.

- (1)  $f$  is a homomorphism
- (2)  $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$  for any  $c_1, c_2 \in \mathbb{R}$  and  $\vec{v}_1, \vec{v}_2 \in V$
- (3)  $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$  for any  $c_1, \dots, c_n \in \mathbb{R}$  and  $\vec{v}_1, \dots, \vec{v}_n \in V$

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*Example* Between any two vector spaces the zero map  $Z: V \rightarrow W$ , defined by  $Z(\vec{v}) = \vec{0}_W$  is a homomorphism. The check is:  
 $Z(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2).$

*Example* The *inclusion map*  $\iota: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$\iota\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism. Here is the verification.

$$\begin{aligned} \iota\left(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= \iota\left(\begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_1 x_1 \\ c_1 y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 x_2 \\ c_2 y_2 \\ 0 \end{pmatrix} \\ &= c_1 \cdot \iota\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + c_2 \cdot \iota\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \end{aligned}$$

*Example* Consider this function  $h: \mathcal{P}_1 \rightarrow \mathcal{P}_1$ .

$$h(a + bx) = b + bx$$

Here are two examples of the action of this function:

$$h(1 + 2x) = 2 + 2x \text{ and } h(3 - x) = -1 - x.$$

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This function is linear.

$$\begin{aligned} h(c_1 \cdot (a_1 + b_1x) + c_2 \cdot (a_2 + b_2x)) \\ &= h((c_1 a_1 + c_2 a_2) + (c_1 b_1 + c_2 b_2)x) \\ &= (c_1 b_1 + c_2 b_2) + (c_1 b_1 + c_2 b_2)x \\ &= (c_1 b_1 + c_1 b_1x) + (c_2 b_2 + c_2 b_2x) \\ &= c_1 \cdot h(a_1 + b_1x) + c_2 \cdot h(a_2 + b_2x) \end{aligned}$$

*Example* The derivative map  $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$  is given by  $d/dx(ax^2 + bx + c) = 2ax + b$ . For instance,  $d/dx(3x^2 - 2x + 4) = 6x - 2$  and  $d/dx(x^2 + 1) = 2x$ .

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$$\begin{aligned} & d/dx \left( r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2) \right) \\ &= d/dx \left( (r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2) \right) \\ &= 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2) \\ &= (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2) \\ &= r_1 \cdot d/dx(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx(a_2x^2 + b_2x + c_2) \end{aligned}$$



*Example* The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus  $\text{Tr}: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}$  is this.

$$\text{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + b$$

It is linear.

$$\begin{aligned} \text{Tr}\left(r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \\ = \text{Tr}\left(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}\right) \\ = (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\ = r_1(a_1 + d_1) + r_2(a_2 + d_2) \\ = r_1 \cdot \text{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + r_2 \cdot \text{Tr}\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if  $V$  is a vector space with basis  $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$  and  $W$  is a vector space with elements  $\vec{w}_1, \dots, \vec{w}_n$  (perhaps not distinct elements) then there exists a homomorphism from  $V$  to  $W$  sending each  $\vec{\beta}_i$  to  $\vec{w}_i$ , and that homomorphism is unique.

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*Proof* For any input  $\vec{v} \in V$  let its expression with respect to the basis be  $\vec{v} = c_1 \vec{\beta}_1 + \dots + c_n \vec{\beta}_n$ . Define the associated output by using the same coordinates  $h(\vec{v}) = c_1 \vec{w}_1 + \dots + c_n \vec{w}_n$ . This is well defined because, with respect to the basis, the representation of each domain vector  $\vec{v}$  is unique.

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This map is a homomorphism because it preserves linear combinations: where  $\vec{v}_1 = c_1 \vec{\beta}_1 + \dots + c_n \vec{\beta}_n$  and  $\vec{v}_2 = d_1 \vec{\beta}_1 + \dots + d_n \vec{\beta}_n$ , here is the calculation.

$$\begin{aligned} h(r_1 \vec{v}_1 + r_2 \vec{v}_2) &= h((r_1 c_1 + r_2 d_1) \vec{\beta}_1 + \dots + (r_1 c_n + r_2 d_n) \vec{\beta}_n) \\ &= (r_1 c_1 + r_2 d_1) \vec{w}_1 + \dots + (r_1 c_n + r_2 d_n) \vec{w}_n \\ &= r_1 h(\vec{v}_1) + r_2 h(\vec{v}_2) \end{aligned}$$

This map is unique because if  $\hat{h}: V \rightarrow W$  is another homomorphism satisfying that  $h'(\vec{\beta}_i) = \vec{w}_i$  for each  $i$  then  $h$  and  $h'$  agree on all of the vectors in the domain.

$$\begin{aligned} h'(\vec{v}) &= h'(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h'(\vec{\beta}_1) + \cdots + c_nh'(\vec{\beta}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = h(\vec{v}) \end{aligned}$$

They have the same action so they are the same function. QED

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1.10 *Definition* Let  $V$  and  $W$  be vector spaces and let  $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$  be a basis for  $V$ . A function defined on that basis  $f: B \rightarrow W$  is *extended linearly* to a function  $\hat{f}: V \rightarrow W$  if for all  $\vec{v} \in V$  such that  $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ , the action of the map is  $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$ .

*Example* One basis of the space of quadratic polynomials  $\mathcal{P}_2$  is  $B = \langle x^2, x, 1 \rangle$ . We can define a map  $\text{eval}_3: \mathcal{P}_2 \rightarrow \mathbb{R}$  by specifying its action on that basis

$$x^2 \xrightarrow{\text{eval}_3} 9 \quad x \xrightarrow{\text{eval}_3} 3 \quad 1 \xrightarrow{\text{eval}_3} 1$$

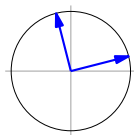
and then extending linearly.

$$\begin{aligned} \text{eval}_3(ax^2 + bx + c) &= a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) \\ &= 9a + 3b + c \end{aligned}$$

The action of this map on the basis elements is to plug the value 3 in for  $x$ . That remains true when we extend linearly, so  $\text{eval}_3(p(x)) = p(3)$ . For instance,  $\text{eval}_3(x^2 + 2x + 3) = 18$ .

*Example* Consider the standard basis  $\mathcal{E}_2$  for the vector space  $\mathbb{R}^2$ . We can specify a rotation of the two basis vectors as here.

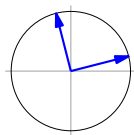
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$





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Extend linearly to get a homomorphism  $t_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .

$$\begin{aligned} t_\theta\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= t_\theta\left(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x \cdot t_\theta\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y \cdot t_\theta\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + y \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix} \end{aligned}$$

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*Example* In  $\mathbb{R}^3$  the function  $f_{yz}$  that reflects vectors over the  $yz$ -plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{f_{yz}} \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

is a linear transformation.

$$\begin{aligned} f_{yz}\left(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= f_{yz}\left(\begin{pmatrix} r_1 x_1 + r_2 x_2 \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix}\right) = \begin{pmatrix} -(r_1 x_1 + r_2 x_2) \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix} \\ &= r_1 \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} -x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 f_{yz}\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + r_2 f_{yz}\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) \end{aligned}$$

1.17 *Lemma* For vector spaces  $V$  and  $W$ , the set of linear functions from  $V$  to  $W$  is itself a vector space, a subspace of the space of all functions from  $V$  to  $W$ .

We denote the space of linear maps from  $V$  to  $W$  by  $\mathcal{L}(V, W)$ .

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*Proof* This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under the operations. Let  $f, g: V \rightarrow W$  be linear. Then the operation of function addition is preserved

$$\begin{aligned}(f + g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1f(\vec{v}_1) + c_2f(\vec{v}_2) + c_1g(\vec{v}_1) + c_2g(\vec{v}_2) \\ &= c_1(f + g)(\vec{v}_1) + c_2(f + g)(\vec{v}_2)\end{aligned}$$

as is the operation of scalar multiplication of a function.

$$\begin{aligned}(r \cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1f(\vec{v}_1) + c_2f(\vec{v}_2)) \\ &= c_1(r \cdot f)(\vec{v}_1) + c_2(r \cdot f)(\vec{v}_2)\end{aligned}$$

Hence  $\mathcal{L}(V, W)$  is a subspace.

QED

*Example* Consider  $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ . Fix these bases for the two spaces.

$$B_{\mathbb{R}} = \mathcal{E}_1 = \langle 1 \rangle \quad B_{\mathbb{R}^2} = \mathcal{E}_2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

A linear map is determined by its action on a basis of the domain space. Thus the elements of  $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$

$$1 \xrightarrow{t} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are determined by  $c_1, c_2$ ; we could write each such map as  $t = t_{c_1, c_2}$ . So  $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$  is a dimension 2 space.

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$$1 \xrightarrow{t} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are determined by  $c_1, c_2$ ; we could write each such map as  $t = t_{c_1, c_2}$ . So  $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$  is a dimension 2 space.

*Example* Similarly, for  $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ , fix  $\mathcal{E}_2 = \langle \vec{e}_1, \vec{e}_2 \rangle$  and  $\mathcal{E}_3 = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$  (note that one  $\vec{e}_1$  is two-tall while the other is three-tall). We can characterize the elements in this way.

$$\vec{e}_1 \xrightarrow{t} c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 \quad \vec{e}_2 \xrightarrow{t} d_1 \vec{e}_1 + d_2 \vec{e}_2 + d_3 \vec{e}_3$$

So,  $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$  is six-dimensional.



Range space and null space

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*Proof* Let  $h: V \rightarrow W$  be linear and let  $S$  be a subspace of the domain  $V$ . The image  $h(S)$  is a subset of the codomain  $W$ , which is nonempty because  $S$  is nonempty. Thus, to show that  $h(S)$  is a subspace of  $W$  we need only show that it is closed under linear combinations of two vectors. If  $h(\vec{s}_1)$  and  $h(\vec{s}_2)$  are members of  $h(S)$  then  $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$  is also a member of  $h(S)$  because it is the image of  $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$  from  $S$ . QED

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*Example* For any angle  $\theta$ , the function  $t_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that rotates vectors counterclockwise by  $\theta$  is a homomorphism. In the domain  $\mathbb{R}^2$  each line through the origin is a subspace. The image of that line under this map is another line through the origin, a subspace of the codomain  $\mathbb{R}^2$ .

## Range space

2.2 *Definition* The *range space* of a homomorphism  $h: V \rightarrow W$  is

$$\mathcal{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

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*Example* Projection  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is  $\mathcal{R}(\pi) = \mathbb{R}^2$  because given a vector  $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a  $\vec{v} \in \mathbb{R}^3$  that maps to it, specifically any vector with first component  $a$  and second component  $b$ . Thus the rank of  $\pi$  is 2.

*Example* The derivative map  $d/dx: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is linear. Its range is  $\mathcal{R}(d/dx) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$ . (Verifying that every member of that space is the derivative of a fourth degree polynomial is easy.) The rank of the derivative function is 3.

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*Example* This map from  $\mathcal{M}_{2 \times 2}$  to  $\mathbb{R}^2$  is linear; the check is routine.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The range space is this line through the origin

$$\left\{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

(every member of that set is the image

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right)$$

of a  $2 \times 2$  matrix). The rank of this map is 1.



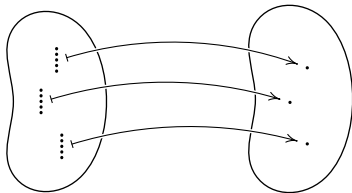
## Homomorphisms organize the domain

When we moved from studying isomorphisms to studying homomorphisms we dropped the requirements that the maps be onto and one-to-one. We've seen that dropping the onto condition has no effect in the sense that any homomorphism  $h: V \rightarrow W$  is onto some space, namely its range space  $\mathcal{R}(h)$ .

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Consider the effect of dropping the one-to-one condition. Now, for some vector  $\vec{w} \in W$  in the range there may be many vectors  $\vec{v} \in V$  mapped to  $\vec{w}$ .



Recall that for any function  $h: V \rightarrow W$ , the set of elements of  $V$  that map to  $\vec{w} \in W$  is the *inverse image*  $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$ . Above, the left side shows three inverse image sets.

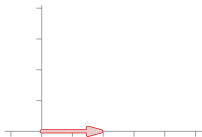
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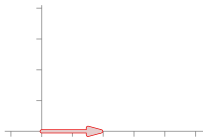
We can identify the codomain  $\mathbb{R}$  with the  $x$ -axis in  $\mathbb{R}^2$ . Here is a member of the  $x$ -axis, drawn in red.



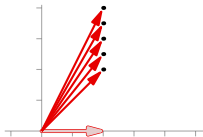
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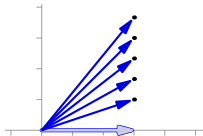


Next are some elements of  $\pi^{-1}(2)$ , shown both as dots (as in the bean diagram) and as vectors. These are also shown in red because they are associated by  $\pi$  with 2.

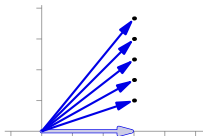


Think of these as “2 vectors.”

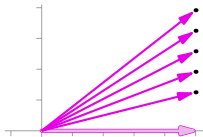
Here are some “3-vectors,” inverse images of 3.



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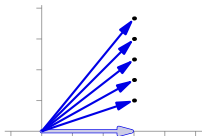


The definition of addition preservation is that  $\pi(\vec{u} + \vec{v}) = \pi(\vec{u}) + \pi(\vec{v})$ . Therefore where  $\pi(\vec{u}) = 2$  and  $\pi(\vec{v}) = 3$ , the vector sum  $\vec{u} + \vec{v}$  will be mapped by  $\pi$  to 5.

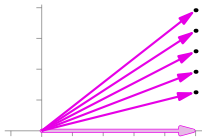


Thus, we can understand the definition of addition preservation as: red plus blue makes purple — a “2 vector” plus a “3 vector” sums to a “5 vector.”

Here are some “3-vectors,” inverse images of 3.



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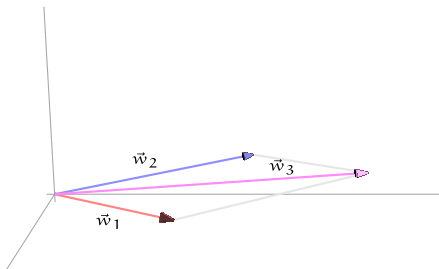


Thus, we can understand the definition of addition preservation as: red plus blue makes purple—a “2 vector” plus a “3 vector” sums to a “5 vector.” Preservation of scalar multiplication has a similar interpretation.



The same analysis holds for any homomorphism.

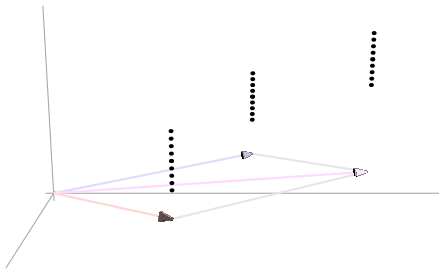
*Example* The projection  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is a linear map. As above we can identify the codomain with the  $xy$ -plane inside of  $\mathbb{R}^3$ .



In the  $xy$ -plane, red plus blue makes purple as shown by the parallelogram. We will illustrate that a  $\vec{w}_1$  vector plus a  $\vec{w}_2$  vector is a  $\vec{w}_3$  vector

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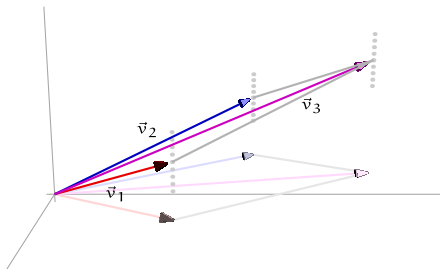


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Consider the inverse image sets; the diagram shows some of the infinitely many points in each  $\pi^{-1}(\vec{w}_i)$ . If we take a  $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$  and a  $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$  then they sum to a  $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$ .

This also holds when the spaces are not ones that we can conveniently draw.

*Example* Consider  $h: \mathcal{P}_2 \rightarrow \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ c \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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The inverse image of  $\vec{w}_1$  is  $h^{-1}(\vec{w}_1) = \{ax^2 + x + c_1 \mid a_1, c_1 \in \mathbb{R}\}$ .

Think of these as “ $\vec{w}_1$  vectors.” Some examples are

$3x^2 + x + 1$ ,  $3x^2 + x - 4$ , and  $-2x^2 + x$ .

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$h^{-1}(\vec{w}_3) = \{a_3x^2 + c_3 \mid a_3, c_3 \in \mathbb{R}\}$ .

As above, any  $\vec{v}_1 \in h^{-1}(\vec{w}_1)$  plus any  $\vec{v}_2 \in h^{-1}(\vec{w}_2)$  equals a  $\vec{v}_3 \in h^{-1}(\vec{w}_3)$ : a quadratic with an  $x$  coefficient of 1 plus a quadratic with an  $x$  coefficient of  $-1$  equals a quadratic with an  $x$  coefficient of 0. That is, a “ $\vec{w}_1$  vector” plus a “ $\vec{w}_2$  vector” is a “ $\vec{w}_3$  vector.”

In each of those examples, because there is a homomorphism  $h: V \rightarrow W$  we can view the domain  $V$  as organized into the inverse images  $h^{-1}(\vec{w})$  for each  $\vec{w} \in \mathcal{R}(h)$ .

It is “organized” because these inverse image sets reflect the structure of the range in that a “ $\vec{w}_1$  vector” plus a “ $\vec{w}_2$  vector” equals a “ $\vec{w}_1 + \vec{w}_2$  vector.”



## Null space

Vector spaces have a distinguished element,  $\vec{0}$ . So we next consider the inverse image of that element  $h^{-1}(\vec{0})$ .

2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

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**2.10 Lemma** For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

*Proof* Let  $h: V \rightarrow W$  be a homomorphism and let  $S$  be a subspace of the range space of  $h$ . Consider the inverse image of  $S$ . It is nonempty because it contains  $\vec{0}_V$ , since  $h(\vec{0}_V) = \vec{0}_W$  and  $\vec{0}_W$  is an element of  $S$  as  $S$  is a subspace. To finish we show that  $h^{-1}(S)$  is closed under linear combinations. Let  $\vec{v}_1$  and  $\vec{v}_2$  be two of its elements, so that  $h(\vec{v}_1)$  and  $h(\vec{v}_2)$  are elements of  $S$ . Then  $c_1\vec{v}_1 + c_2\vec{v}_2$  is an element of the inverse image  $h^{-1}(S)$  because  $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$  is a member of  $S$ . QED

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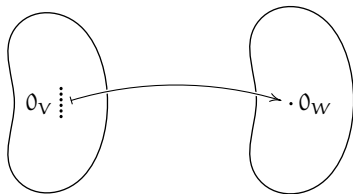
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*Note* This result complements Lemma 2.1 that for any subspace of the domain, its image is a subspace of the range.

2.11 *Definition* The *null space* or *kernel* of a linear map  $h: V \rightarrow W$  is the inverse image of  $\vec{0}_W$ .

$$\mathcal{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

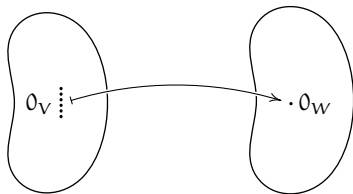
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The dimension of the null space is the map's *nullity*.



*Note* Strictly, the trivial subspace of the codomain is not  $\vec{0}_W$ , it is  $\{\vec{0}_W\}$ , and so we may think to write the nullspace as  $h^{-1}(\{\vec{0}_W\})$ . But we have defined the two sets  $h^{-1}(\vec{w})$  and  $h^{-1}(\{\vec{w}\})$  to be equal and the first is easier to write.

*Example* The derivative function  $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$  is linear.

$$\mathcal{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

The polynomial  $2ax + b$  equals the zero polynomial if and only if they have the same constant coefficient (which implies that  $b = 0$ ), the same coefficient of  $x$  (which implies that  $a = 0$ ), and the same coefficient of  $x^2$  (which gives no restriction). Thus the nullspace is this, and the nullity is 1.

$$\mathcal{N}(d/dx) = \{ax^2 + bx + c \mid a = 0, b = 0, c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

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*Example* The function  $h: \mathbb{R}^2 \rightarrow \mathbb{R}^1$  given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this null space.

$$\mathcal{N}(h) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0 \right\} = \left\{ \begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Its nullity is 1.

*Example* The homomorphism  $f: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a + b \\ c + d \end{pmatrix}$$

has this null space

$$\begin{aligned} \mathcal{N}(f) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = 0 \text{ and } c + d = 0 \right\} \\ &= \left\{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \right\} \end{aligned}$$

and a nullity of 2.



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has this null space

$$\begin{aligned} \mathcal{N}(f) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + b = 0 \text{ and } c + d = 0 \right\} \\ &= \left\{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \right\} \end{aligned}$$

and a nullity of 2.

*Example* The dilation function  $d_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

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## Rank plus nullity

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To see that  $B_R$  is linearly independent, consider  $\vec{0}_W = c_{k+1}h(\vec{\beta}_{k+1}) + \dots + c_n h(\vec{\beta}_n)$ . We have  $\vec{0}_W = h(c_{k+1}\vec{\beta}_{k+1} + \dots + c_n\vec{\beta}_n)$  and so  $c_{k+1}\vec{\beta}_{k+1} + \dots + c_n\vec{\beta}_n$  is in the null space of  $h$ . As  $B_N$  is a basis for the null space there are scalars  $c_1, \dots, c_k$  satisfying this relationship.

$$c_1\vec{\beta}_1 + \dots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \dots + c_n\vec{\beta}_n$$

But this is an equation among members of  $B_V$ , which is a basis for  $V$ , so each  $c_i$  equals 0. Therefore  $B_R$  is linearly independent.

To show that  $B_R$  spans the range space consider a member of the range space  $h(\vec{v})$ . Express  $\vec{v}$  as a linear combination  $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$  of members of  $B_V$ . This gives  $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h(\vec{\beta}_1) + \cdots + c_kh(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$  and since  $\vec{\beta}_1, \dots, \vec{\beta}_k$  are in the null space, we have that  $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$ . Thus,  $h(\vec{v})$  is a linear combination of members of  $B_R$ , and so  $B_R$  spans the range space. QED

*Example* Projection  $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a three-dimensional domain to a two-dimensional range. Its null space is the z-axis so its nullity is one.

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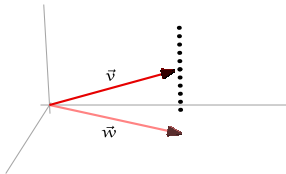


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This example illustrates the proof's steps particularly clearly. Take the basis  $B_N = \langle \vec{e}_3 \rangle$  for the null space. Expand that to the basis  $\mathcal{E}_3$  for the entire domain. The action of  $\pi$  is that the third dimension collapses: a linear combinations in the domain  $\vec{v} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3$  is sent to the combination  $\vec{w} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + \vec{0}$  in the range. Geometrically, the domain is organized into inverse images that are vertical lines, just like the null space.



*Example* The derivative function  $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$  has this range space

$$\mathcal{R}(d/dx) = \{2ax + b \mid a, b \in \mathbb{R}\} = \mathcal{P}_1$$

(any  $cx + d \in \mathcal{P}_1$  is the image of  $ax^2 + bx + c$  where  $a = c/2$ ,  $b = d$ , and  $c$  can be any real) and this null space (calculated earlier).

$$\mathcal{N}(d/dx) = \{c \mid c \in \mathbb{R}\}$$

The rank is 2 while the nullity is 1, and they add to the dimension of the domain  $\mathcal{P}_2$ .

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*Example* The function  $h: \mathbb{R}^2 \rightarrow \mathbb{R}^1$  given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this range space

$$\mathcal{R}(h) = \{2a + b \mid a, b \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

and this null space (calculated earlier).

$$\mathcal{N}(h) = \left\{ \begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Its rank is 1 and its nullity is 1. Its domain  $\mathbb{R}^2$  has dimension 2.

*Example* The dilation function  $d_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has range space  $\mathbb{R}^2$  and a trivial nullspace  $\mathcal{N}(d_3) = \{\vec{0}\}$ . So its rank is 2 and its nullity is 0.

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*Example* The homomorphism  $f: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{f} \begin{pmatrix} a + b \\ c + d \end{pmatrix}$$

has range space equal to  $\mathbb{R}^2$  (to get a vector with a first component of  $x$  and a second component of  $y$  we can take  $a = x$ ,  $b = 0$ ,  $c = y$ , and  $d = 0$ ). Thus  $f$ 's rank is 2. We found its null space earlier

$$\mathcal{N}(f) = \left\{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \right\}$$

and its nullity is 2.

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

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*Proof* Suppose that  $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$  with some  $c_i$  nonzero. Apply  $h$  to both sides:  $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n)$  and  $h(\vec{0}_V) = \vec{0}_W$ . Thus we have  $c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = \vec{0}_W$  with some  $c_i$  nonzero. QED

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*Example* The trace function  $\text{Tr}: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

is linear. This set of matrices is dependent.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

The three matrices map to 1, 0, and 2 respectively. The set  $\{1, 0, 2\}$  is linearly dependent in  $\mathbb{R}$ .



2.20 *Theorem* Where  $V$  is an  $n$ -dimensional vector space, these are equivalent statements about a linear map  $h: V \rightarrow W$ .

- (1)  $h$  is one-to-one
- (2)  $h$  has an inverse from its range to its domain that is a linear map
- (3)  $\mathcal{N}(h) = \{\vec{0}\}$ , that is,  $\text{nullity}(h) = 0$
- (4)  $\text{rank}(h) = n$
- (5) if  $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$  is a basis for  $V$  then  $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$  is a basis for  $\mathcal{R}(h)$

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*Proof* We will first show that  $(1) \iff (2)$ . We will then show that  $(1) \implies (3) \implies (4) \implies (5) \implies (2)$ .

For  $(1) \implies (2)$ , suppose that the linear map  $h$  is one-to-one, and therefore has an inverse  $h^{-1}: \mathcal{R}(h) \rightarrow V$ . The domain of that inverse is the range of  $h$  and thus a linear combination of two members of it has the form  $c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$ . On that combination, the inverse  $h^{-1}$  gives this.

$$\begin{aligned} h^{-1}(c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)) &= h^{-1}(h(c_1 \vec{v}_1 + c_2 \vec{v}_2)) \\ &= h^{-1} \circ h (c_1 \vec{v}_1 + c_2 \vec{v}_2) \\ &= c_1 \vec{v}_1 + c_2 \vec{v}_2 \\ &= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2)) \end{aligned}$$

Thus if a linear map has an inverse then the inverse must be linear. But this also gives the  $(2) \implies (1)$  implication, because the inverse itself must be one-to-one.

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Of the remaining implications,  $(1) \implies (3)$  holds because any homomorphism maps  $\vec{0}_V$  to  $\vec{0}_W$ , but a one-to-one map sends at most one member of  $V$  to  $\vec{0}_W$ .

Next,  $(3) \implies (4)$  is true since rank plus nullity equals the dimension of the domain.

For (4)  $\implies$  (5), to show that  $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$  is a basis for the range space we need only show that it is a spanning set, because by assumption the range has dimension  $n$ . Consider  $h(\vec{v}) \in \mathcal{R}(h)$ . Expressing  $\vec{v}$  as a linear combination of basis elements produces  $h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + c_n\vec{\beta}_n)$ , which gives that  $h(\vec{v}) = c_1h(\vec{\beta}_1) + \dots + c_nh(\vec{\beta}_n)$ , as desired. QED