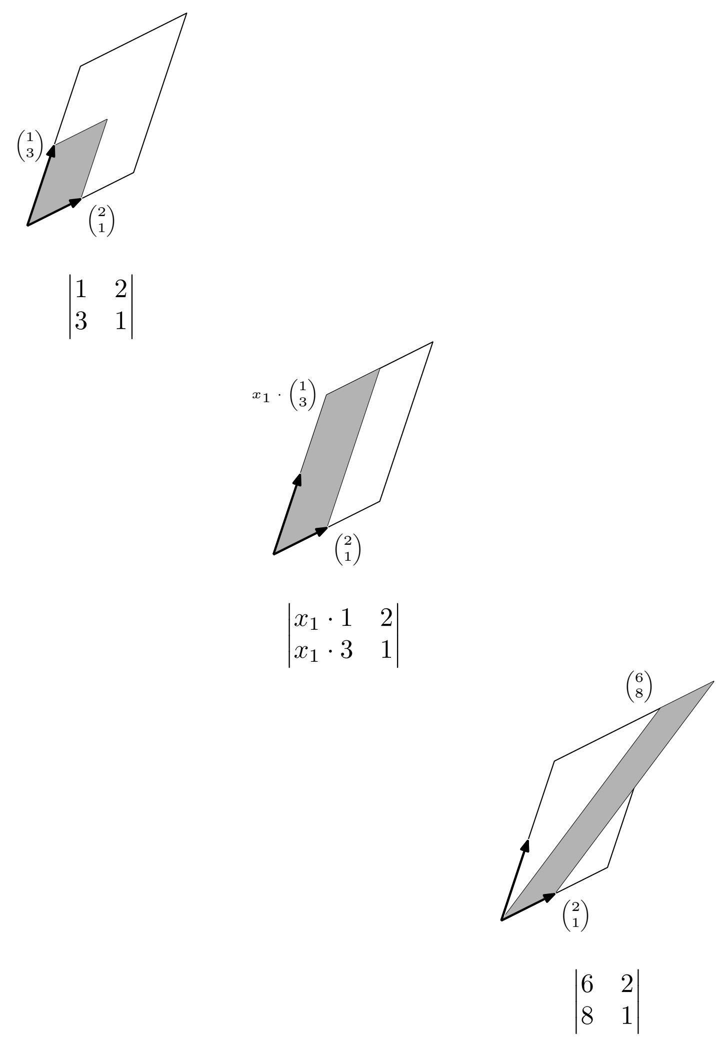
Linear Algebra

Jim Hefferon



Notation

$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$	real numbers, reals greater than 0, n-tuples of reals
$\mathbb{N}$	natural numbers: $\{0, 1, 2, \ldots\}$
$\mathbb{C}$	complex numbers
$\{\ldots \mid \ldots\}$	set of ... such that ...
$(a\,..\,b)$, $[a\,..\,b]$	interval (open or closed) of reals between a and b
$\langle \ldots \rangle$	sequence; like a set but order matters
V, W, U	vector spaces
$\vec{v}, \vec{w}$	vectors
$\vec{0}$, $\vec{0}_V$	zero vector, zero vector of V
B, D	bases
$\mathcal{E}_n = \langle \vec{e}_1, \ldots, \vec{e}_n \rangle$	standard basis for $\mathbb{R}^n$
$\vec{\beta}, \vec{\delta}$	basis vectors
$\mathrm{Rep}_B(\vec{v})$	matrix representing the vector
$\mathcal{P}_n$	set of n-th degree polynomials
$\mathcal{M}_{n\times m}$	set of $n\times m$ matrices
$[S]$	span of the set S
$M \oplus N$	direct sum of subspaces
$V \cong W$	isomorphic spaces
h, g	homomorphisms, linear maps
H, G	matrices
t, s	transformations; maps from a space to itself
T, S	square matrices
$\mathrm{Rep}_{B,D}(h)$	matrix representing the map h
$h_{i,j}$	matrix entry from row i, column j
$Z_{n\times m}, Z, I_{n\times n}, I$	zero matrix, identity matrix
$\lvert T \rvert$	determinant of the matrix T
$\mathscr{R}(h), \mathscr{N}(h)$	rangespace and nullspace of the map h
$\mathscr{R}_\infty(h), \mathscr{N}_\infty(h)$	generalized rangespace and nullspace

Lower case Greek alphabet

name	character	name	character	name	character
alpha	α	iota	ι	rho	ρ
beta	β	kappa	κ	sigma	σ
gamma	γ	lambda	λ	tau	τ
delta	δ	mu	μ	upsilon	υ
epsilon	ϵ	nu	ν	phi	ϕ
zeta	ζ	xi	ξ	chi	χ
eta	η	omicron	o	psi	ψ
theta	θ	pi	π	omega	ω

Cover. This is Cramer's Rule for the system $x_1 + 2x_2 = 6$, $3x_1 + x_2 = 8$. The size of the first box is the determinant shown (the absolute value of the size is the area). The size of the second box is x_1 times that, and equals the size of the final box. Hence, x_1 is the final determinant divided by the first determinant.

Preface

This book helps students to master the material of a standard US undergraduate linear algebra course.

The material is standard in that the topics covered are Gaussian reduction, vector spaces, linear maps, determinants, and eigenvalues and eigenvectors. Another standard is book's audience: sophomores or juniors, usually with a background of at least one semester of calculus. The help that it gives to students comes from taking a developmental approach — this book's presentation emphasizes motivation and naturalness, driven home by a wide variety of examples and by extensive and careful exercises.

The developmental approach is the feature that most recommends this book so I will say more. Courses in the beginning of most mathematics programs focus less on understanding theory and more on correctly applying formulas and algorithms. Later courses ask for mathematical maturity: the ability to follow different types of arguments, a familiarity with the themes that underlie many mathematical investigations such as elementary set and function facts, and a capacity for some independent reading and thinking. Linear algebra is an ideal spot to work on the transition. It comes early in a program so that progress made here pays off later, but also comes late enough that students are serious about mathematics, often majors and minors. The material is accessible, coherent, and elegant. There are a variety of argument styles, including proofs by contradiction, if and only if statements, and proofs by induction. And, examples are plentiful.

Helping readers start the transition to being serious students of the subject of mathematics itself means taking the mathematics seriously, so all of the results in this book are proved. On the other hand, we cannot assume that students have already arrived and so in contrast with more abstract texts, we give many examples and they are often quite detailed.

Some linear algebra books begin with extensive computations of linear systems, matrix multiplications, and determinants. Then, when the concepts — vector spaces and linear maps — finally appear, and definitions and proofs start, often the abrupt change brings students to a stop. In this book, while we start with a computational topic, linear reduction, from the first we do more than compute. We do linear systems quickly but completely, including the proofs needed to justify what we are computing. Then, with the linear systems work as motivation and at a point where the study of linear combinations seems natural, the second chapter starts with the definition of a real vector space. In the schedule below, this occurs by the end of the third week.

Another example of our emphasis on motivation and naturalness is that the third chapter on linear maps does not begin with the definition of homomorphism, but with isomorphism. The definition of isomorphism is easily motivated by the observation that some spaces are "just like" others. After that, the next section takes the reasonable step of defining homomorphism by isolating the operation-preservation idea. This approach loses mathematical slickness, but it is a good trade because it gives to students a large gain in sensibility.

One aim of our developmental approach is to present the material in such a way that students can see how the ideas arise, and perhaps can picture themselves doing the same type of work.

The clearest example of the developmental approach is the exercises. A student progresses most while doing the exercises, so the ones included here have been selected with great care. Each problem set ranges from simple checks to reasonably involved proofs. Since an instructor usually assigns about a dozen exercises after each lecture, each section ends with about twice that many, thereby providing a selection. There are even a few problems that are challenging puzzles taken from various journals, competitions, or problems collections. (These are marked with a '**?**' and as part of the fun, the original wording has been retained as much as possible.) In total, the exercises are aimed to both build an ability at, and help students experience the pleasure of, *doing* mathematics.

Applications and computers. The point of view taken here, that students should think of linear algebra as about vector spaces and linear maps, is not taken to the complete exclusion of others. Applications and computing are important and vital aspects of the subject. Consequently, each of this book's chapters closes with a few application or computer-related topics. Some are: network flows, the speed and accuracy of computer linear reductions, Leontief Input/Output analysis, dimensional analysis, Markov chains, voting paradoxes, analytic projective geometry, and difference equations.

These topics are brief enough to be done in a day's class or to be given as independent projects. Most simply give a reader a taste of the subject, discuss how linear algebra comes in, point to some further reading, and give a few exercises. In short, these topics invite readers to see for themselves that linear algebra is a tool that a professional must have.

The license. This book is freely available. You can download and read it without restriction. Class instructors can print copies for students and charge for those. See `http://joshua.smcvt.edu/linearalgebra` for more license information.

That page also contains the latest version of this book, and the latest version of the worked answers to every exercise. Also there, I provide the LaTeX source of the text and some instructors may wish to add their own material. If you like, you can send such additions to me and I may possibly incorporate them into future editions.

I am very glad for bug reports. I save them and periodically issue updates; people who contribute in this way are acknowledged in the text's source files.

For people reading this book on their own. This book's emphasis on motivation and development make it a good choice for self-study. But while a professional instructor can judge what pace and topics suit a class, if you are an independent student then you may find some advice helpful.

Here are two timetables for a semester. The first focuses on core material.

week	*Monday*	*Wednesday*	*Friday*
1	One.I.1	One.I.1, 2	One.I.2, 3
2	One.I.3	One.II.1	One.II.2
3	One.III.1, 2	One.III.2	Two.I.1
4	Two.I.2	Two.II	Two.III.1
5	Two.III.1, 2	Two.III.2	EXAM
6	Two.III.2, 3	Two.III.3	Three.I.1
7	Three.I.2	Three.II.1	Three.II.2
8	Three.II.2	Three.II.2	Three.III.1
9	Three.III.1	Three.III.2	Three.IV.1, 2
10	Three.IV.2, 3, 4	Three.IV.4	EXAM
11	Three.IV.4, Three.V.1	Three.V.1, 2	Four.I.1, 2
12	Four.I.3	Four.II	Four.II
13	Four.III.1	Five.I	Five.II.1
14	Five.II.2	Five.II.3	REVIEW

The second timetable is more ambitious. It supposes that you know One.II, the elements of vectors, usually covered in third semester calculus.

week	*Monday*	*Wednesday*	*Friday*
1	One.I.1	One.I.2	One.I.3
2	One.I.3	One.III.1, 2	One.III.2
3	Two.I.1	Two.I.2	Two.II
4	Two.III.1	Two.III.2	Two.III.3
5	Two.III.4	Three.I.1	EXAM
6	Three.I.2	Three.II.1	Three.II.2
7	Three.III.1	Three.III.2	Three.IV.1, 2
8	Three.IV.2	Three.IV.3	Three.IV.4
9	Three.V.1	Three.V.2	Three.VI.1
10	Three.VI.2	Four.I.1	EXAM
11	Four.I.2	Four.I.3	Four.I.4
12	Four.II	Four.II, Four.III.1	Four.III.2, 3
13	Five.II.1, 2	Five.II.3	Five.III.1
14	Five.III.2	Five.IV.1, 2	Five.IV.2

In the table of contents I have marked subsections as optional if some instructors will pass over them in favor of spending more time elsewhere.

You might pick one or two topics that appeal to you from the end of each chapter. You'll get more from these if you have access to computer software that can do any big calculations. I recommend *Sage*, freely available from http://sagemath.org.

My main advice is: do many exercises. I have marked a good sample with ✓'s in the margin. For all of them, you must justify your answer either with a computation or with a proof. Be aware that few inexperienced people can write correct proofs. Try to find someone with training to work with you on this.

Finally, if I may, a caution for all students, independent or not: I cannot overemphasize how much the statement that I sometimes hear, "I understand the material, but it's only that I have trouble with the problems" is mistaken. Being able to do things with the ideas is their entire point. The quotes below express this sentiment admirably. They state what I believe is the key to both the beauty and the power of mathematics and the sciences in general, and of linear algebra in particular; I took the liberty of formatting them as verse.

I know of no better tactic
than the illustration of exciting principles
by well-chosen particulars.
–Stephen Jay Gould

If you really wish to learn
then you must mount the machine
and become acquainted with its tricks
by actual trial.
–Wilbur Wright

Jim Hefferon
Mathematics, Saint Michael's College
Colchester, Vermont USA 05439
http://joshua.smcvt.edu
2011-Jan-01

Author's Note. Inventing a good exercise, one that enlightens as well as tests, is a creative act, and hard work. The inventor deserves recognition. But for some reason texts have traditionally not given attributions for questions. I have changed that here where I was sure of the source. I would be glad to hear from anyone who can help me to correctly attribute others of the questions.

Contents

Chapter One: Linear Systems **1**

I Solving Linear Systems . . . 1
1 Gauss' Method . . . 2
2 Describing the Solution Set . . . 11
3 General = Particular + Homogeneous . . . 20
II Linear Geometry of n-Space . . . 32
1 Vectors in Space . . . 32
2 Length and Angle Measures* . . . 39
III Reduced Echelon Form . . . 46
1 Gauss-Jordan Reduction . . . 46
2 Row Equivalence . . . 52
Topic: Computer Algebra Systems . . . 61
Topic: Input-Output Analysis . . . 63
Topic: Accuracy of Computations . . . 67
Topic: Analyzing Networks . . . 71

Chapter Two: Vector Spaces **77**

I Definition of Vector Space . . . 78
1 Definition and Examples . . . 78
2 Subspaces and Spanning Sets . . . 89
II Linear Independence . . . 99
1 Definition and Examples . . . 99
III Basis and Dimension . . . 110
1 Basis . . . 110
2 Dimension . . . 116
3 Vector Spaces and Linear Systems . . . 122
4 Combining Subspaces* . . . 129
Topic: Fields . . . 138
Topic: Crystals . . . 140
Topic: Voting Paradoxes . . . 144
Topic: Dimensional Analysis . . . 150

Chapter Three: Maps Between Spaces **157**
I Isomorphisms . . . 157
1 Definition and Examples . . . 157
2 Dimension Characterizes Isomorphism . . . 166
II Homomorphisms . . . 174
1 Definition . . . 174
2 Rangespace and Nullspace . . . 181
III Computing Linear Maps . . . 193
1 Representing Linear Maps with Matrices . . . 193
2 Any Matrix Represents a Linear Map* . . . 203
IV Matrix Operations . . . 210
1 Sums and Scalar Products . . . 210
2 Matrix Multiplication . . . 213
3 Mechanics of Matrix Multiplication . . . 220
4 Inverses . . . 229
V Change of Basis . . . 236
1 Changing Representations of Vectors . . . 236
2 Changing Map Representations . . . 240
VI Projection . . . 248
1 Orthogonal Projection Into a Line* . . . 248
2 Gram-Schmidt Orthogonalization* . . . 252
3 Projection Into a Subspace* . . . 258
Topic: Line of Best Fit . . . 267
Topic: Geometry of Linear Maps . . . 272
Topic: Markov Chains . . . 279
Topic: Orthonormal Matrices . . . 285

Chapter Four: Determinants **291**
I Definition . . . 292
1 Exploration* . . . 292
2 Properties of Determinants . . . 297
3 The Permutation Expansion . . . 301
4 Determinants Exist* . . . 309
II Geometry of Determinants . . . 317
1 Determinants as Size Functions . . . 317
III Other Formulas . . . 324
1 Laplace's Expansion* . . . 324
Topic: Cramer's Rule . . . 329
Topic: Speed of Calculating Determinants . . . 332
Topic: Projective Geometry . . . 335

Chapter Five: Similarity **347**
I Complex Vector Spaces . . . 347
1 Factoring and Complex Numbers; A Review* . . . 348
2 Complex Representations . . . 349
II Similarity . . . 351

1 Definition and Examples 351
2 Diagonalizability 353
3 Eigenvalues and Eigenvectors 357
III Nilpotence 365
1 Self-Composition* 365
2 Strings* 368
IV Jordan Form 379
1 Polynomials of Maps and Matrices* 379
2 Jordan Canonical Form* 386
Topic: Method of Powers 399
Topic: Stable Populations 403
Topic: Linear Recurrences 405

Appendix **A-1**
Propositions A-1
Quantifiers A-3
Techniques of Proof A-5
Sets, Functions, and Relations A-7

**Note:* starred subsections are optional.

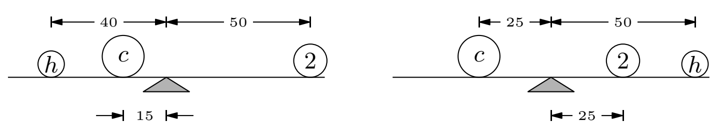
Chapter One

Linear Systems

I Solving Linear Systems

Systems of linear equations are common in science and mathematics. These two examples from high school science [Onan] give a sense of how they arise.

The first example is from Physics. Suppose that we are given three objects, one with a mass known to be 2 kg, and are asked to find the unknown masses. Suppose further that experimentation with a meter stick produces these two balances.



We know that the moment of each object is its mass times its distance from the balance point. We also know that for balance we must have that the sum of moments on the left equals the sum of moments on the right. That gives a system of two equations.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

The second example of a linear system is from Chemistry. We can mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water (conditions have to be controlled very well — trinitrotoluene is better known as TNT). In what proportion should we mix those components? The number of atoms of each element present before the reaction

$$x\,C_7H_8 \;+\; y\,HNO_3 \quad\longrightarrow\quad z\,C_7H_5O_6N_3 \;+\; w\,H_2O$$

must equal the number present afterward. Applying that to the elements C, H, N, and O in turn gives this system.

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

Finishing each of these examples requires solving a system of equations. In each system, the equations involve only the first power of the variables. This chapter shows how to solve any such system.

I.1 Gauss' Method

1.1 Definition A *linear combination* of $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the combination's *coefficients*. A *linear equation* has the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$ where $d \in \mathbb{R}$ is the *constant*.

An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a *solution* of, or *satisfies*, that equation if substituting the numbers s_1, ..., s_n for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.

A *system of linear equations*

$$\begin{array}{rcl} a_{1,1}x_1 + \ a_{1,2}x_2 + \cdots + \ a_{1,n}x_n &=& d_1 \\ a_{2,1}x_1 + \ a_{2,2}x_2 + \cdots + \ a_{2,n}x_n &=& d_2 \\ &\vdots& \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &=& d_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

1.2 Example The combination $3x_1 + 2x_2$ of x_1 and x_2 is linear. The combination $3x_1^2 + 2\sin(x_2)$ is not linear, nor is $3x_1^2 + 2x_2$.

1.3 Example The ordered pair $(-1, 5)$ is a solution of this system.

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -x_1 + \ \ x_2 &= 6 \end{aligned}$$

In contrast, $(5, -1)$ is not a solution.

Finding the set of all solutions is *solving* the system. No guesswork or good fortune is needed to solve a linear system. There is an algorithm that always works. The next example introduces that algorithm, called *Gauss' method* (or *Gaussian elimination* or *linear elimination*). It transforms the system, step by step, into one with a form that is easily solved. We will first illustrate how it goes and then we will see the formal statement.

1.4 Example To solve this system

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

we repeatedly transform it until it is in a form that is easy to solve. Below there are three transformations.

The first is to rewrite the system by interchanging the first and third row.

$$\xrightarrow{\text{swap row 1 with row 3}} \quad \begin{aligned} \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The second transformation is to rescale the first row by multiplying both sides of the equation by 3.

$$\xrightarrow{\text{multiply row 1 by 3}} \quad \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The third transformation is the only nontrivial one. We mentally multiply both sides of the first row by -1, mentally add that to the second row, and write the result in as the new second row.

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \quad \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ -x_2 - 2x_3 &= -7 \\ 3x_3 &= 9 \end{aligned}$$

The point of this sucession of steps is that system is now in a form where we can easily find the value of each variable. The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$ and so the system has a unique solution: the solution set is $\{\,(3, 1, 3)\,\}$.

Most of this subsection and the next one consists of examples of solving linear systems by Gauss' method. We will use it throughout this book. It is fast and easy.

But before we get to those examples, we will first show that this method is also safe in that it never loses solutions or picks up extraneous solutions.

1.5 Theorem (Gauss' method) If a linear system is changed to another by one of these operations

(1) an equation is swapped with another

(2) an equation has both sides multiplied by a nonzero constant

(3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

Each of those three operations has a restriction. Multiplying a row by 0 is not allowed because that can change the solution set of the system. Similarly, adding a multiple of a row to itself is not allowed because adding -1 times the row to itself has the effect of multiplying the row by 0. Finally, swapping a row with itself is disallowed to make some results in the fourth chapter easier to state and remember.

Proof. We will cover the equation swap operation here and save the other two cases for Exercise 30.

Consider this swap of row i with row j.

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \cdots \ a_{1,n}x_n = d_1 & & a_{1,1}x_1 + a_{1,2}x_2 + \cdots \ a_{1,n}x_n = d_1 \\ \vdots & & \vdots \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots \ a_{i,n}x_n = d_i & & a_{j,1}x_1 + a_{j,2}x_2 + \cdots \ a_{j,n}x_n = d_j \\ \vdots & \longrightarrow & \vdots \\ a_{j,1}x_1 + a_{j,2}x_2 + \cdots \ a_{j,n}x_n = d_j & & a_{i,1}x_1 + a_{i,2}x_2 + \cdots \ a_{i,n}x_n = d_i \\ \vdots & & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots \ a_{m,n}x_n = d_m & & a_{m,1}x_1 + a_{m,2}x_2 + \cdots \ a_{m,n}x_n = d_m \end{array}$$

The n-tuple $(s_1, \ldots, s_n)$ satisfies the system before the swap if and only if substituting the values, the s's, for the variables, the x's, gives true statements: $a_{1,1}s_1+a_{1,2}s_2+\cdots+a_{1,n}s_n = d_1$ and ... $a_{i,1}s_1+a_{i,2}s_2+\cdots+a_{i,n}s_n = d_i$ and ... $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and ... $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$.

In a requirement consisting of statements joined with 'and' we can rearrange the order of the statements, so that this requirement is met if and only if $a_{1,1}s_1+ a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and ... $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and ... $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and ... $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$. This is exactly the requirement that $(s_1, \ldots, s_n)$ solves the system after the row swap. QED

1.6 Definition The three operations from Theorem 1.5 are the *elementary reduction operations*, or *row operations*, or *Gaussian operations*. They are *swapping*, *multiplying by a scalar* (or *rescaling*), and *row combination*.

When writing out the calculations, we will abbreviate 'row i' by 'ρ_i'. For instance, we will denote a row combination operation by $k\rho_i + \rho_j$, with the row that is changed written second. We will also, to save writing, often list addition steps together when they use the same ρ_i.

1.7 Example Gauss' method is to systemmatically apply those row operations to solve a system. Here is a typical case.

$$\begin{aligned} x+\ \ y\quad\ \ &= 0 \\ 2x-\ \ y+3z &= 3 \\ x-2y-\ \ z &= 3 \end{aligned}$$

To start we use the first row to eliminate the $2x$ in the second row and the x in the third. To get rid of the $2x$, we mentally multiply the entire first row by -2, add that to the second row, and write the result in as the new second row. To get rid of the x, we multiply the first row by -1, add that to the third row, and write the result in as the new third row. (Using one entry to clear out the rest of a column is called *pivoting* on that entry.)

$$\xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad \begin{aligned} x+\ \ \ y\quad\ \ &= 0 \\ -3y+3z &= 3 \\ -3y-\ \ z &= 3 \end{aligned}$$

In this version of the system, the last two equations involve only two unknowns. To finish we transform the second system into a third system, where the last equation involves only one unknown. We use the second row to eliminate y from the third row.

$$\xrightarrow{-\rho_2+\rho_3}\quad \begin{aligned} x+\ \ \ y\quad\ \ \ &= 0 \\ -3y+\ 3z &= 3 \\ -4z &= 0 \end{aligned}$$

Now the third row shows that $z=0$. Substitute that back into the second row to get $y=-1$ and then substitute back into the first row to get $x=1$.

1.8 Example For the Physics problem from the start of this chapter, Gauss' method gives this.

$$\begin{aligned} 40h+15c &= 100 \\ -50h+25c &= 50 \end{aligned} \quad\xrightarrow{5/4\rho_1+\rho_2}\quad \begin{aligned} 40h+\qquad 15c &= 100 \\ (175/4)c &= 175 \end{aligned}$$

So $c=4$, and back-substitution gives that $h=1$. (The Chemistry problem is solved later.)

1.9 Example The reduction

$$\begin{aligned} x+\ \ y+\ \ z &= 9 \\ 2x+4y-3z &= 1 \\ 3x+6y-5z &= 0 \end{aligned} \quad\xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2}\quad \begin{aligned} x+\ \ y+\ \ z &= 9 \\ 2y-5z &= -17 \\ 3y-8z &= -27 \end{aligned}$$

$$\xrightarrow{-(3/2)\rho_2+\rho_3}\quad \begin{aligned} x+\ \ y+\qquad z &= 9 \\ 2y-\qquad 5z &= -17 \\ -(1/2)z &= -(3/2) \end{aligned}$$

shows that $z=3$, $y=-1$, and $x=7$.

As these examples illustrate, the point of Gauss' method is to use the elementary reduction operations to set up back-substitution.

1.10 Definition In each row of a system, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it (except for the leading variable in the first row).

1.11 Example The only operation needed in the example above is row combination. Here is a linear system that requires the operation of swapping equations to get it in echelon form. After the first combination

$$\begin{array}{rcl} x-\;y\qquad\quad &=0 \\ 2x-2y+\;z+2w&=4 \\ y\qquad+\;w&=0 \\ 2z+\;w&=5 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} x-y\qquad\quad &=0 \\ z+2w&=4 \\ y\qquad+\;w&=0 \\ 2z+\;w&=5 \end{array}$$

the second equation has no leading y. To get one, we look lower down in the system for a row that has a leading y and swap it in.

$$\xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{array}{rcl} x-y\qquad\quad &=0 \\ y\qquad+\;w&=0 \\ z+2w&=4 \\ 2z+\;w&=5 \end{array}$$

(Had there been more than one row below the second with a leading y then we could have swapped in any one.) The rest of Gauss' method goes as before.

$$\xrightarrow{-2\rho_3+\rho_4} \begin{array}{rcr} x-y\qquad\quad &=&0 \\ y\quad+\quad w&=&0 \\ z+\;2w&=&4 \\ -3w&=&-3 \end{array}$$

Back-substitution gives $w=1$, $z=2$, $y=-1$, and $x=-1$.

Strictly speaking, the operation of rescaling rows is not needed to solve linear systems. We have included it because we will use it later in this chapter as part of a variation on Gauss' method, the Gauss-Jordan method.

All of the systems seen so far have the same number of equations as unknowns. All of them have a solution, and for all of them there is only one solution. We finish this subsection by seeing for contrast some other things that can happen.

1.12 Example Linear systems need not have the same number of equations as unknowns. This system

$$\begin{array}{rcr} x+3y&=&1 \\ 2x+\;y&=&-3 \\ 2x+2y&=&-2 \end{array}$$

has more equations than variables. Gauss' method helps us understand this system also, since this

$$\xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2}\begin{aligned} x+\quad 3y&=\ \ 1\\ -5y&=-5\\ -4y&=-4 \end{aligned}$$

shows that one of the equations is redundant. Echelon form

$$\xrightarrow{-(4/5)\rho_2+\rho_3}\begin{aligned} x+\quad 3y&=\ \ 1\\ -5y&=-5\\ 0&=\ \ 0 \end{aligned}$$

gives that $y = 1$ and $x = -2$. The '$0 = 0$' reflects the redundancy.

That example's system has more equations than variables. Gauss' method is also useful on systems with more variables than equations. Many examples are in the next subsection.

Another way that linear systems can differ from the examples shown earlier is that some linear systems do not have a unique solution. This can happen in two ways.

The first is that a system can fail to have any solution at all.

1.13 Example Contrast the system in the last example with this one.

$$\begin{aligned} x+3y&=\ \ 1\\ 2x+\ \ y&=-3\\ 2x+2y&=\ \ 0 \end{aligned}\ \xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2}\ \begin{aligned} x+\quad 3y&=\ \ 1\\ -5y&=-5\\ -4y&=-2 \end{aligned}$$

Here the system is inconsistent: no pair of numbers satisfies all of the equations simultaneously. Echelon form makes this inconsistency obvious.

$$\xrightarrow{-(4/5)\rho_2+\rho_3}\begin{aligned} x+\quad 3y&=\ \ 1\\ -5y&=-5\\ 0&=\ \ 2 \end{aligned}$$

The solution set is empty.

1.14 Example The prior system has more equations than unknowns, but that is not what causes the inconsistency — Example 1.12 has more equations than unknowns and yet is consistent. Nor is having more equations than unknowns necessary for inconsistency, as is illustrated by this inconsistent system with the same number of equations as unknowns.

$$\begin{aligned} x+2y&=8\\ 2x+4y&=8 \end{aligned}\ \xrightarrow{-2\rho_1+\rho_2}\ \begin{aligned} x+2y&=\ \ 8\\ 0&=-8 \end{aligned}$$

The other way that a linear system can fail to have a unique solution is to have many solutions.

1.15 Example In this system

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 8 \end{aligned}$$

any pair of numbers satisfying the first equation automatically satisfies the second. The solution set $\{(x,y) \mid x + y = 4\}$ is infinite; some of its members are $(0,4)$, $(-1,5)$, and $(2.5,1.5)$. The result of applying Gauss' method here contrasts with the prior example because we do not get a contradictory equation.

$$\xrightarrow{-2\rho_1+\rho_2} \begin{aligned} x + y &= 4 \\ 0 &= 0 \end{aligned}$$

Don't be fooled by the '$0 = 0$' equation in that example. It is not the signal that a system has many solutions.

1.16 Example The absence of a '$0 = 0$' does not keep a system from having many different solutions. This system is in echelon form

$$\begin{aligned} x + y + z &= 0 \\ y + z &= 0 \end{aligned}$$

has no '$0 = 0$', and yet has infinitely many solutions. (For instance, each of these is a solution: $(0,1,-1)$, $(0,1/2,-1/2)$, $(0,0,0)$, and $(0,-\pi,\pi)$. There are infinitely many solutions because any triple whose first component is 0 and whose second component is the negative of the third is a solution.)

Nor does the presence of a '$0 = 0$' mean that the system must have many solutions. Example 1.12 shows that. So does this system, which does not have many solutions — in fact it has none — despite that when it is brought to echelon form it has a '$0 = 0$' row.

$$\begin{array}{rcl} \begin{aligned} 2x - 2z &= 6 \\ y + z &= 1 \\ 2x + y - z &= 7 \\ 3y + 3z &= 0 \end{aligned} & \xrightarrow{-\rho_1+\rho_3} & \begin{aligned} 2x - 2z &= 6 \\ y + z &= 1 \\ y + z &= 1 \\ 3y + 3z &= 0 \end{aligned} \\ & \xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} & \begin{aligned} 2x - 2z &= 6 \\ y + z &= 1 \\ 0 &= 0 \\ 0 &= -3 \end{aligned} \end{array}$$

We will finish this subsection with a summary of what we've seen so far about Gauss' method.

Gauss' method uses the three row operations to set a system up for back substitution. If any step shows a contradictory equation then we can stop with the conclusion that the system has no solutions. If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has a unique solution and we find it by back substitution.

Finally, if we reach echelon form without a contradictory equation, and there is not a unique solution (at least one variable is not a leading variable) then the system has many solutions.

The next subsection deals with the third case — we will see how to describe the solution set of a system with many solutions.

Note *For all exercises in this book, you must justify your answer. For instance, if a question asks whether a system has a solution then you must justify a yes response by producing the solution and must justify a no response by showing that no solution exists.*

Exercises

✓ **1.17** Use Gauss' method to find the unique solution for each system.

(a) $\begin{aligned} 2x + 3y &= 13 \\ x - y &= -1 \end{aligned}$ **(b)** $\begin{aligned} x \qquad - z &= 0 \\ 3x + y \qquad &= 1 \\ -x + y + z &= 4 \end{aligned}$

✓ **1.18** Use Gauss' method to solve each system or conclude 'many solutions' or 'no solutions'.

(a) $\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$ **(b)** $\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$ **(c)** $\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$

(d) $\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$ **(e)** $\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x \qquad + z &= 5 \\ x + y - z &= 10 \end{aligned}$ **(f)** $\begin{aligned} 2x \qquad + z + w &= 5 \\ y \qquad - w &= -1 \\ 3x \qquad - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$

✓ **1.19** There are methods for solving linear systems other than Gauss' method. One often taught in high school is to solve one of the equations for a variable, then substitute the resulting expression into other equations. That step is repeated until there is an equation with only one variable. From that, the first number in the solution is derived, and then back-substitution can be done. This method takes longer than Gauss' method, since it involves more arithmetic operations, and is also more likely to lead to errors. To illustrate how it can lead to wrong conclusions, we will use the system

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned}$$

from Example 1.13.

(a) Solve the first equation for x and substitute that expression into the second equation. Find the resulting y.

(b) Again solve the first equation for x, but this time substitute that expression into the third equation. Find this y.

What extra step must a user of this method take to avoid erroneously concluding a system has a solution?

✓ **1.20** For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= k \end{aligned}$$

✓ **1.21** This system is not linear, in some sense,

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 10 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

and yet we can nonetheless apply Gauss' method. Do so. Does the system have a solution?

✓ **1.22** What conditions must the constants, the b's, satisfy so that each of these systems has a solution? *Hint.* Apply Gauss' method and see what happens to the right side. [Anton]

(a)
$$\begin{aligned} x - 3y &= b_1 \\ 3x + y &= b_2 \\ x + 7y &= b_3 \\ 2x + 4y &= b_4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 + 8x_3 &= b_3 \end{aligned}$$

1.23 True or false: a system with more unknowns than equations has at least one solution. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample.)

1.24 Must any Chemistry problem like the one that starts this subsection — a balance the reaction problem — have infinitely many solutions?

✓ **1.25** Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

1.26 Gauss' method works by combining the equations in a system to make new equations.

(a) Can the equation $3x - 2y = 5$ be derived, by a sequence of Gaussian reduction steps, from the equations in this system?

$$\begin{aligned} x + y &= 1 \\ 4x - y &= 6 \end{aligned}$$

(b) Can the equation $5x - 3y = 2$ be derived, by a sequence of Gaussian reduction steps, from the equations in this system?

$$\begin{aligned} 2x + 2y &= 5 \\ 3x + y &= 4 \end{aligned}$$

(c) Can the equation $6x - 9y + 5z = -2$ be derived, by a sequence of Gaussian reduction steps, from the equations in the system?

$$\begin{aligned} 2x + y - z &= 4 \\ 6x - 3y + z &= 5 \end{aligned}$$

1.27 Prove that, where $a, b, \ldots, e$ are real numbers and $a \neq 0$, if

$$ax + by = c$$

has the same solution set as

$$ax + dy = e$$

then they are the same equation. What if $a = 0$?

✓ **1.28** Show that if $ad - bc \neq 0$ then

$$\begin{aligned} ax + by &= j \\ cx + dy &= k \end{aligned}$$

has a unique solution.

✓ **1.29** In the system

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

each of the equations describes a line in the xy-plane. By geometrical reasoning, show that there are three possibilities: there is a unique solution, there is no solution, and there are infinitely many solutions.

1.30 Finish the proof of Theorem 1.5.

1.31 Is there a two-unknowns linear system whose solution set is all of $\mathbb{R}^2$?

✓ **1.32** Are any of the operations used in Gauss' method redundant? That is, can any of the operations be made from a combination of the others?

1.33 Prove that each operation of Gauss' method is reversible. That is, show that if two systems are related by a row operation $S_1 \rightarrow S_2$ then there is a row operation to go back $S_2 \rightarrow S_1$.

? **1.34** A box holding pennies, nickels and dimes contains thirteen coins with a total value of 83 cents. How many coins of each type are in the box? [Anton]

? **1.35** Four positive integers are given. Select any three of the integers, find their arithmetic average, and add this result to the fourth integer. Thus the numbers 29, 23, 21, and 17 are obtained. One of the original integers is:

(a) 19 **(b)** 21 **(c)** 23 **(d)** 29 **(e)** 17

[Con. Prob. 1955]

? ✓ **1.36** Laugh at this: AHAHA + TEHE = TEHAW. It resulted from substituting a code letter for each digit of a simple example in addition, and it is required to identify the letters and prove the solution unique. [Am. Math. Mon., Jan. 1935]

? **1.37** The Wohascum County Board of Commissioners, which has 20 members, recently had to elect a President. There were three candidates (A, B, and C); on each ballot the three candidates were to be listed in order of preference, with no abstentions. It was found that 11 members, a majority, preferred A over B (thus the other 9 preferred B over A). Similarly, it was found that 12 members preferred C over A. Given these results, it was suggested that B should withdraw, to enable a runoff election between A and C. However, B protested, and it was then found that 14 members preferred B over C! The Board has not yet recovered from the resulting confusion. Given that every possible order of A, B, C appeared on at least one ballot, how many members voted for B as their first choice? [Wohascum no. 2]

? **1.38** "This system of n linear equations with n unknowns," said the Great Mathematician, "has a curious property."

"Good heavens!" said the Poor Nut, "What is it?"

"Note," said the Great Mathematician, "that the constants are in arithmetic progression."

"It's all so clear when you explain it!" said the Poor Nut. "Do you mean like $6x + 9y = 12$ and $15x + 18y = 21$?"

"Quite so," said the Great Mathematician, pulling out his bassoon. "Indeed, the system has a unique solution. Can you find it?"

"Good heavens!" cried the Poor Nut, "I am baffled."

Are you? [Am. Math. Mon., Jan. 1963]

I.2 Describing the Solution Set

A linear system with a unique solution has a solution set with one element. A linear system with no solution has a solution set that is empty. In these cases the solution set is easy to describe. Solution sets are a challenge to describe only when they contain many elements.

2.1 Example This system has many solutions because in echelon form

$$\begin{array}{rcl} 2x + z = 3 \\ x - y - z = 1 \\ 3x - y = 4 \end{array} \xrightarrow[-(3/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \begin{array}{rcl} 2x + z = 3 \\ -y - (3/2)z = -1/2 \\ -y - (3/2)z = -1/2 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{rcl} 2x + z = 3 \\ -y - (3/2)z = -1/2 \\ 0 = 0 \end{array}$$

not all of the variables are leading variables. The Gauss' method theorem showed that a triple (x, y, z) satisfies the first system if and only if it satisfies the third. Thus, the solution set $\{(x, y, z) \mid 2x + z = 3 \text{ and } x - y - z = 1 \text{ and } 3x - y = 4\}$ can also be described as $\{(x, y, z) \mid 2x + z = 3 \text{ and } -y - 3z/2 = -1/2\}$. However, this second description is not much of an improvement. It has two equations instead of three, but it still involves some hard-to-understand interaction among the variables.

To get a description that is free of any such interaction, we take the variable that does not lead any equation, z, and use it to describe the variables that do lead, x and y. The second equation gives $y = (1/2) - (3/2)z$ and the first equation gives $x = (3/2) - (1/2)z$. Thus, the solution set can be described as $\{(x, y, z) = ((3/2) - (1/2)z, (1/2) - (3/2)z, z) \mid z \in \mathbb{R}\}$. For instance, $(1/2, -5/2, 2)$ is a solution because taking $z = 2$ gives a first component of $1/2$ and a second component of $-5/2$.

The advantage of this description over the ones above is that the only variable appearing, z, is unrestricted — it can be any real number.

2.2 Definition The non-leading variables in an echelon-form linear system are *free variables.*

In the echelon form system derived in the above example, x and y are leading variables and z is free.

2.3 Example A linear system can end with more than one variable free. This row reduction

$$\begin{array}{rcl} x + y + z - w = 1 \\ y - z + w = -1 \\ 3x + 6z - 6w = 6 \\ -y + z - w = 1 \end{array} \xrightarrow{-3\rho_1+\rho_3} \begin{array}{rcl} x + y + z - w = 1 \\ y - z + w = -1 \\ -3y + 3z - 3w = 3 \\ -y + z - w = 1 \end{array}$$

$$\xrightarrow[\rho_2+\rho_4]{3\rho_2+\rho_3} \begin{array}{rcl} x + y + z - w = 1 \\ y - z + w = -1 \\ 0 = 0 \\ 0 = 0 \end{array}$$

ends with x and y leading, and with both z and w free. To get the description that we prefer we will start at the bottom. We first express y in terms of the free variables z and w with $y = -1 + z - w$. Next, moving up to the top equation, substituting for y in the first equation $x + (-1 + z - w) + z - w = 1$ and solving for x yields $x = 2 - 2z + 2w$. Thus, the solution set is $\{(2 - 2z + 2w, -1 + z - w, z, w) \mid z, w \in \mathbb{R}\}$.

We prefer this description because the only variables that appear, z and w, are unrestricted. This makes the job of deciding which four-tuples are system solutions into an easy one. For instance, taking $z = 1$ and $w = 2$ gives the solution $(4, -2, 1, 2)$. In contrast, $(3, -2, 1, 2)$ is not a solution, since the first component of any solution must be 2 minus twice the third component plus twice the fourth.

2.4 Example After this reduction

$$\begin{array}{rcl} \begin{aligned} 2x - 2y \qquad\qquad &= 0 \\ z + 3w &= 2 \\ 3x - 3y \qquad\qquad &= 0 \\ x - y + 2z + 6w &= 4 \end{aligned} & \xrightarrow[-(1/2)\rho_1+\rho_4]{-(3/2)\rho_1+\rho_3} & \begin{aligned} 2x - 2y \qquad\qquad &= 0 \\ z + 3w &= 2 \\ 0 &= 0 \\ 2z + 6w &= 4 \end{aligned} \\ & \xrightarrow{-2\rho_2+\rho_4} & \begin{aligned} 2x - 2y \qquad\qquad &= 0 \\ z + 3w &= 2 \\ 0 &= 0 \\ 0 &= 0 \end{aligned} \end{array}$$

x and z lead, y and w are free. The solution set is $\{(y, y, 2 - 3w, w) \mid y, w \in \mathbb{R}\}$. For instance, $(1, 1, 2, 0)$ satisfies the system — take $y = 1$ and $w = 0$. The four-tuple $(1, 0, 5, 4)$ is not a solution since its first coordinate does not equal its second.

We refer to a variable used to describe a family of solutions as a *parameter* and we say that the set above is *parametrized* with y and w. (The terms 'parameter' and 'free variable' do not mean the same thing. Above, y and w are free because in the echelon form system they do not lead any row. They are parameters because they are used in the solution set description. We could have instead parametrized with y and z by rewriting the second equation as $w = 2/3 - (1/3)z$. In that case, the free variables are still y and w, but the parameters are y and z. Notice that we could not have parametrized with x and y, so there is sometimes a restriction on the choice of parameters. The terms 'parameter' and 'free' are related because, as we shall show later in this chapter, the solution set of a system can always be parametrized with the free variables. Consequently, we shall parametrize all of our descriptions in this way.)

2.5 Example This is another system with infinitely many solutions.

$$\begin{array}{rcl} \begin{aligned} x + 2y \qquad\qquad &= 1 \\ 2x \qquad + z \qquad &= 2 \\ 3x + 2y + z - w &= 4 \end{aligned} & \xrightarrow[-3\rho_1+\rho_3]{-2\rho_1+\rho_2} & \begin{aligned} x + 2y \qquad\qquad &= 1 \\ -4y + z \qquad &= 0 \\ -4y + z - w &= 1 \end{aligned} \\ & \xrightarrow{-\rho_2+\rho_3} & \begin{aligned} x + 2y \qquad\qquad &= 1 \\ -4y + z \qquad &= 0 \\ -w &= 1 \end{aligned} \end{array}$$

The leading variables are x, y, and w. The variable z is free. (Notice here that, although there are infinitely many solutions, the value of one of the variables is fixed — $w = -1$.) Write w in terms of z with $w = -1 + 0z$. Then $y = (1/4)z$. To express x in terms of z, substitute for y into the first equation to get $x = 1 - (1/2)z$. The solution set is $\{(1 - (1/2)z, (1/4)z, z, -1) \mid z \in \mathbb{R}\}$.

We finish this subsection by developing the notation for linear systems and their solution sets that we shall use in the rest of this book.

2.6 Definition An $m{\times}n$ *matrix* is a rectangular array of numbers with m *rows* and n *columns*. Each number in the matrix is an *entry*,

Matrices are usually named by upper case roman letters, e.g. A. Each entry is denoted by the corresponding lower-case letter, e.g. $a_{i,j}$ is the number in row i and column j of the array. For instance,

$$A = \begin{pmatrix} 1 & 2.2 & 5 \\ 3 & 4 & -7 \end{pmatrix}$$

has two rows and three columns, and so is a 2×3 matrix. (Read that as "two-by-three"; the number of rows is always stated first.) The entry in the second row and first column is $a_{2,1} = 3$. Note that the order of the subscripts matters: $a_{1,2} \neq a_{2,1}$ since $a_{1,2} = 2.2$. (The parentheses around the array are a typographic device so that when two matrices are side by side we can tell where one ends and the other starts.)

Matrices occur throughout this book. We shall use $\mathcal{M}_{n\times m}$ to denote the collection of $n\times m$ matrices.

2.7 Example We can abbreviate this linear system

$$\begin{aligned} x + 2y \phantom{{}+2z} &= 4 \\ y - z &= 0 \\ x \phantom{{}+2y} + 2z &= 4 \end{aligned}$$

with this matrix.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right)$$

The vertical bar just reminds a reader of the difference between the coefficients on the systems's left hand side and the constants on the right. When a bar is used to divide a matrix into parts, we call it an *augmented* matrix. In this notation, Gauss' method goes this way.

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The second row stands for $y - z = 0$ and the first row stands for $x + 2y = 4$ so the solution set is $\{(4 - 2z, z, z) \mid z \in \mathbb{R}\}$. One advantage of the new notation is that the clerical load of Gauss' method — the copying of variables, the writing of $+$'s and $=$'s, etc. — is lighter.

We will also use the array notation to clarify the descriptions of solution sets. A description like $\{(2-2z+2w, -1+z-w, z, w) \mid z, w \in \mathbb{R}\}$ from Example 2.3 is hard to read. We will rewrite it to group all the constants together, all the coefficients of z together, and all the coefficients of w together. We will write them vertically, in one-column wide matrices.

$$\{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\}$$

For instance, the top line says that $x = 2 - 2z + 2w$ and the second line says that $y = -1 + z - w$. The next section gives a geometric interpretation that will help us picture the solution sets when they are written in this way.

2.8 Definition A *vector* (or *column vector*) is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*.

Vectors are an exception to the convention of representing matrices with capital roman letters. We use lower-case roman or greek letters overlined with an arrow: $\vec{a}$, $\vec{b}$, ... or $\vec{\alpha}$, $\vec{\beta}$, ... (boldface is also common: $\boldsymbol{a}$ or $\boldsymbol{\alpha}$). For instance, this is a column vector with a third component of 7.

$$\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$$

2.9 Definition The linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ with unknowns $x_1, \ldots, x_n$ is *satisfied* by

$$\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

if $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A vector satisfies a linear system if it satisfies each equation in the system.

The style of description of solution sets that we use involves adding the vectors, and also multiplying them by real numbers, such as the z and w. We need to define these operations.

2.10 Definition The *vector sum* of $\vec{u}$ and $\vec{v}$ is this.

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

Note that the vectors have to have the same number of entries for the addition to be defined. This entry-by-entry addition works for any pair of matrices, not just vectors, provided that they have the same number of rows and columns.

2.11 Definition The *scalar multiplication* of the real number r and the vector $\vec{v}$ is this.

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

As with the addition operation, this entry-by-entry scalar multiplication operation extends beyond vectors to any matrix.

Scalar multiplication can be written in either order: $r \cdot \vec{v}$ or $\vec{v} \cdot r$, or without the '$\cdot$' symbol: $r\vec{v}$. (Do not refer to scalar multiplication as 'scalar product' because that name is used for a different operation.)

2.12 Example

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 3-1 \\ 1+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} \qquad 7 \cdot \begin{pmatrix} 1 \\ 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 28 \\ -7 \\ -21 \end{pmatrix}$$

Notice that the definitions of vector addition and scalar multiplication agree where they overlap, for instance, $\vec{v} + \vec{v} = 2\vec{v}$.

With the notation defined, we can now solve systems in the way that we will use throughout this book.

2.13 Example This system

$$\begin{array}{rcrcrcrcrcl} 2x & + & y & & & - & w & & & = & 4 \\ & & y & & & + & w & + & u & = & 4 \\ x & & & - & z & + & 2w & & & = & 0 \end{array}$$

reduces in this way.

$$\left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 1 & 0 & -1 & 2 & 0 & 0 \end{array}\right) \xrightarrow{-(1/2)\rho_1+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & -1/2 & -1 & 5/2 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & 3 & 1/2 & 0 \end{array}\right)$$

The solution set is $\{(w + (1/2)u, 4 - w - u, 3w + (1/2)u, w, u) \mid w, u \in \mathbb{R}\}$. We write that in vector form.

$$\{\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} w + \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} u \mid w, u \in \mathbb{R}\}$$

Note again how well vector notation sets off the coefficients of each parameter. For instance, the third row of the vector form shows plainly that if u is held fixed then z increases three times as fast as w.

That format also shows plainly that there are infinitely many solutions. For example, we can fix u as 0, let w range over the real numbers, and consider the first component x. We get infinitely many first components and hence infinitely many solutions.

Another thing shown plainly is that setting both w and u to zero gives that this vector

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is a particular solution of the linear system.

2.14 Example In the same way, this system

$$\begin{aligned} x - y + z &= 1 \\ 3x + z &= 3 \\ 5x - 2y + 3z &= 5 \end{aligned}$$

reduces

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & -2 & 3 & 5 \end{array}\right) \xrightarrow[-5\rho_1+\rho_3]{-3\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 3 & -2 & 0 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

to a one-parameter solution set.

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix} z \;\big|\; z \in \mathbb{R}\}$$

Before the exercises, we pause to point out some things that we have yet to do.

The first two subsections have been on the mechanics of Gauss' method. Except for one result, Theorem 1.5 — without which developing the method doesn't make sense since it says that the method gives the right answers — we have not stopped to consider any of the interesting questions that arise.

For example, can we always describe solution sets as above, with a particular solution vector added to an unrestricted linear combination of some other vectors? The solution sets we described with unrestricted parameters were easily seen to have infinitely many solutions so an answer to this question could tell us something about the size of solution sets. An answer to that question could also help us picture the solution sets, in $\mathbb{R}^2$, or in $\mathbb{R}^3$, etc.

Many questions arise from the observation that Gauss' method can be done in more than one way (for instance, when swapping rows, we may have a choice of which row to swap with). Theorem 1.5 says that we must get the same solution set no matter how we proceed, but if we do Gauss' method in two different ways must we get the same number of free variables both times, so that any two solution set descriptions have the same number of parameters? Must those be the same variables (e.g., is it impossible to solve a problem one way and get y and w free or solve it another way and get y and z free)?

In the rest of this chapter we answer these questions. The answer to each is 'yes'. The first question is answered in the last subsection of this section. In the second section we give a geometric description of solution sets. In the final section of this chapter we tackle the last set of questions. Consequently, by the end of the first chapter we will not only have a solid grounding in the practice of Gauss' method, we will also have a solid grounding in the theory. We will be sure of what can and cannot happen in a reduction.

Exercises

✓ **2.15** Find the indicated entry of the matrix, if it is defined.

$$A=\begin{pmatrix}1&3&1\\2&-1&4\end{pmatrix}$$

(a) $a_{2,1}$ **(b)** $a_{1,2}$ **(c)** $a_{2,2}$ **(d)** $a_{3,1}$

✓ **2.16** Give the size of each matrix.

(a) $\begin{pmatrix}1&0&4\\2&1&5\end{pmatrix}$ **(b)** $\begin{pmatrix}1&1\\-1&1\\3&-1\end{pmatrix}$ **(c)** $\begin{pmatrix}5&10\\10&5\end{pmatrix}$

✓ **2.17** Do the indicated vector operation, if it is defined.

(a) $\begin{pmatrix}2\\1\\1\end{pmatrix}+\begin{pmatrix}3\\0\\4\end{pmatrix}$ **(b)** $5\begin{pmatrix}4\\-1\end{pmatrix}$ **(c)** $\begin{pmatrix}1\\5\\1\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}$ **(d)** $7\begin{pmatrix}2\\1\end{pmatrix}+9\begin{pmatrix}3\\5\end{pmatrix}$

(e) $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\2\\3\end{pmatrix}$ **(f)** $6\begin{pmatrix}3\\1\\1\end{pmatrix}-4\begin{pmatrix}2\\0\\3\end{pmatrix}+2\begin{pmatrix}1\\1\\5\end{pmatrix}$

✓ **2.18** Solve each system using matrix notation. Express the solution using vectors.

(a) $\begin{aligned}3x+6y&=18\\x+2y&=6\end{aligned}$ **(b)** $\begin{aligned}x+y&=1\\x-y&=-1\end{aligned}$ **(c)** $\begin{aligned}x_1\qquad\;\;+x_3&=4\\x_1-x_2+2x_3&=5\\4x_1-x_2+5x_3&=17\end{aligned}$

(d) $\begin{aligned}2a+b-c&=2\\2a\qquad+c&=3\\a-b\qquad&=0\end{aligned}$ **(e)** $\begin{aligned}x+2y-z\qquad&=3\\2x+y\qquad+w&=4\\x-y+z+w&=1\end{aligned}$ **(f)** $\begin{aligned}x\qquad+z+w&=4\\2x+y\qquad-w&=2\\3x+y+z\qquad&=7\end{aligned}$

✓ **2.19** Solve each system using matrix notation. Give each solution set in vector notation.

(a) $\begin{aligned}2x+y-z&=1\\4x-y\qquad&=3\end{aligned}$ **(b)** $\begin{aligned}x\qquad-z\qquad&=1\\y+2z-w&=3\\x+2y+3z-w&=7\end{aligned}$ **(c)** $\begin{aligned}x-y+z\qquad&=0\\y\qquad+w&=0\\3x-2y+3z+w&=0\\-y\qquad-w&=0\end{aligned}$

(d) $\begin{aligned}a+2b+3c+d-e&=1\\3a-b+c+d+e&=3\end{aligned}$

✓ **2.20** The vector is in the set. What value of the parameters produces that vector?

(a) $\begin{pmatrix}5\\-5\end{pmatrix}$, $\{\begin{pmatrix}1\\-1\end{pmatrix}k \bigm| k\in\mathbb{R}\}$

(b) $\begin{pmatrix}-1\\2\\1\end{pmatrix}$, $\{\begin{pmatrix}-2\\1\\0\end{pmatrix}i+\begin{pmatrix}3\\0\\1\end{pmatrix}j \bigm| i,j\in\mathbb{R}\}$

(c) $\begin{pmatrix}0\\-4\\2\end{pmatrix}$, $\{\begin{pmatrix}1\\1\\0\end{pmatrix}m+\begin{pmatrix}2\\0\\1\end{pmatrix}n \bigm| m,n\in\mathbb{R}\}$

2.21 Decide if the vector is in the set.

(a) $\begin{pmatrix}3\\-1\end{pmatrix}$, $\{\begin{pmatrix}-6\\2\end{pmatrix}k \bigm| k\in\mathbb{R}\}$

(b) $\begin{pmatrix}5\\4\end{pmatrix}$, $\{\begin{pmatrix}5\\-4\end{pmatrix}j \bigm| j\in\mathbb{R}\}$

(c) $\begin{pmatrix}2\\1\\-1\end{pmatrix}$, $\{\begin{pmatrix}0\\3\\-7\end{pmatrix}+\begin{pmatrix}1\\-1\\3\end{pmatrix}r \bigm| r\in\mathbb{R}\}$

(d) $\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\{\begin{pmatrix}2\\0\\1\end{pmatrix}j+\begin{pmatrix}-3\\-1\\1\end{pmatrix}k \bigm| j,k\in\mathbb{R}\}$

2.22 Parametrize the solution set of this one-equation system.

$$x_1+x_2+\cdots+x_n=0$$

✓ **2.23** **(a)** Apply Gauss' method to the left-hand side to solve

$$\begin{array}{rcrcrcrcl} x&+&2y& & &-&w&=&a\\ 2x& & &+&z& & &=&b\\ x&+&y& & &+&2w&=&c \end{array}$$

for x, y, z, and w, in terms of the constants a, b, and c.

(b) Use your answer from the prior part to solve this.

$$\begin{array}{rcrcrcrcr} x&+&2y& & &-&w&=&3\\ 2x& & &+&z& & &=&1\\ x&+&y& & &+&2w&=&-2 \end{array}$$

✓ **2.24** Why is the comma needed in the notation '$a_{i,j}$' for matrix entries?

✓ **2.25** Give the 4×4 matrix whose i,j-th entry is

(a) $i+j$; **(b)** -1 to the $i+j$ power.

2.26 For any matrix A, the *transpose* of A, written A^{trans}, is the matrix whose columns are the rows of A. Find the transpose of each of these.

(a) $\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$ **(b)** $\begin{pmatrix}2&-3\\1&1\end{pmatrix}$ **(c)** $\begin{pmatrix}5&10\\10&5\end{pmatrix}$ **(d)** $\begin{pmatrix}1\\1\\0\end{pmatrix}$

✓ **2.27** **(a)** Describe all functions $f(x)=ax^2+bx+c$ such that $f(1)=2$ and $f(-1)=6$.

(b) Describe all functions $f(x)=ax^2+bx+c$ such that $f(1)=2$.

2.28 Show that any set of five points from the plane $\mathbb{R}^2$ lie on a common conic section, that is, they all satisfy some equation of the form $ax^2+by^2+cxy+dx+ey+f=0$ where some of $a,\ldots,f$ are nonzero.

2.29 Make up a four equations/four unknowns system having

(a) a one-parameter solution set;

(b) a two-parameter solution set;
(c) a three-parameter solution set.

? **2.30** This puzzle is from a Russian web-site `http://www.arbuz.uz/`, and there are many solutions to it, but mine uses linear algebra and is very naive. There's a planet inhabited by arbuzoids (watermeloners, if to translate from Russian). Those creatures are found in three colors: red, green and blue. There are 13 red arbuzoids, 15 blue ones, and 17 green. When two differently coloured arbuzoids meet, they both change to the third color.

The question is, can it ever happen that all of them assume the same color? [Shepelev]

? **2.31** **(a)** Solve the system of equations.

$$\begin{aligned} ax + y &= a^2 \\ x + ay &= 1 \end{aligned}$$

For what values of a does the system fail to have solutions, and for what values of a are there infinitely many solutions?

(b) Answer the above question for the system.

$$\begin{aligned} ax + y &= a^3 \\ x + ay &= 1 \end{aligned}$$

[USSR Olympiad no. 174]

? **2.32** In air a gold-surfaced sphere weighs 7588 grams. It is known that it may contain one or more of the metals aluminum, copper, silver, or lead. When weighed successively under standard conditions in water, benzene, alcohol, and glycerine its respective weights are 6588, 6688, 6778, and 6328 grams. How much, if any, of the forenamed metals does it contain if the specific gravities of the designated substances are taken to be as follows?

Aluminum	2.7	Alcohol	0.81
Copper	8.9	Benzene	0.90
Gold	19.3	Glycerine	1.26
Lead	11.3	Water	1.00
Silver	10.8		

[Math. Mag., Sept. 1952]

I.3 General = Particular + Homogeneous

The prior subsection has many descriptions of solution sets. They all fit a pattern. They have a vector that is a particular solution of the system added to an unrestricted combination of some other vectors. The solution set from Example 2.13 illustrates.

$$\{\underbrace{\begin{pmatrix}0\\4\\0\\0\\0\end{pmatrix}}_{\substack{\text{particular}\\\text{solution}}} + \underbrace{w\begin{pmatrix}1\\-1\\3\\1\\0\end{pmatrix} + u\begin{pmatrix}1/2\\-1\\1/2\\0\\1\end{pmatrix}}_{\substack{\text{unrestricted}\\\text{combination}}} \;\Big|\; w,u\in\mathbb{R}\}$$

The combination is unrestricted in that w and u can be any real numbers — there is no condition like "such that $2w-u=0$" that would restrict which pairs w, u can be used to form combinations.

That example shows an infinite solution set conforming to the pattern. We can think of the other two kinds of solution sets as fitting the same pattern. A one-element solution set fits the pattern in that it has a particular solution, and the unrestricted combination part is a trivial sum. (That is, instead of being a combination of two vectors, as above, or a combination of one vector, it is a combination of no vectors. We will use the convention that the sum of an empty set of vectors is the vector of all zeros.) A zero-element solution set fits the pattern since there is no particular solution, and so there are no sums of that form.

This subsection formally proves what the prior paragraph outlines: every solution set can be written as a vector that is a particular solution of the system added to an unrestricted combination of some other vectors.

3.1 Theorem Any linear system's solution set can be described as

$$\{\vec{p}+c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k \bigm| c_1,\ldots,c_k\in\mathbb{R}\}$$

where $\vec{p}$ is any particular solution, and where the number of vectors $\vec{\beta}_1$, ..., $\vec{\beta}_k$ equals the number of free variables that the system has after a Gaussian reduction.

The solution description has two parts, the particular solution $\vec{p}$ and also the unrestricted linear combination of the $\vec{\beta}$'s. We shall prove the theorem in two corresponding parts, with two lemmas.

We will focus first on the unrestricted combination part. To do that, we consider systems that have the vector of zeroes as one of the particular solutions, so that $\vec{p}+c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k$ can be shortened to $c_1\vec{\beta}_1+\cdots+c_k\vec{\beta}_k$.

3.2 Definition A linear equation is *homogeneous* if it has a constant of zero, that is, if it can be put in the form $a_1x_1+a_2x_2+\cdots+a_nx_n=0$.

3.3 Example With any linear system like

$$\begin{aligned} 3x+4y&=3\\ 2x-y&=1 \end{aligned}$$

we associate a system of homogeneous equations by setting the right side to zeros.

$$\begin{aligned} 3x+4y&=0\\ 2x-y&=0 \end{aligned}$$

Our interest in the homogeneous system associated with a linear system can be understood by comparing the reduction of the system

$$\begin{array}{rl} 3x+4y&=3\\ 2x-y&=1 \end{array} \xrightarrow{-(2/3)\rho_1+\rho_2} \begin{array}{rrl} 3x+&4y&=3\\ &-(11/3)y&=-1 \end{array}$$

with the reduction of the associated homogeneous system.

$$\begin{array}{rcl} 3x+4y=0 & \xrightarrow{-(2/3)\rho_1+\rho_2} & 3x+4y=0 \\ 2x-y=0 & & -(11/3)y=0 \end{array}$$

Obviously the two reductions go in the same way. We can study how linear systems are reduced by instead studying how the associated homogeneous systems are reduced.

Studying the associated homogeneous system has a great advantage over studying the original system. Nonhomogeneous systems can be inconsistent. But a homogeneous system must be consistent since there is always at least one solution, the vector of zeros.

3.4 Definition A column or row vector of all zeros is a *zero vector*, denoted $\vec{0}$.

There are many different zero vectors, e.g., the one-tall zero vector, the two-tall zero vector, etc. Nonetheless, people often refer to "the" zero vector, expecting that the size of the one being discussed will be clear from the context.

3.5 Example Some homogeneous systems have the zero vector as their only solution.

$$\begin{array}{rcl} 3x+2y+z=0 \\ 6x+4y=0 \\ y+z=0 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{rcl} 3x+2y+z=0 \\ -2z=0 \\ y+z=0 \end{array} \xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{array}{rcl} 3x+2y+z=0 \\ y+z=0 \\ -2z=0 \end{array}$$

3.6 Example Some homogeneous systems have many solutions. One example is the Chemistry problem from the first page of this book.

$$\begin{array}{rl} 7x-7z&=0 \\ 8x+y-5z-2w&=0 \\ y-3z&=0 \\ 3y-6z-w&=0 \end{array} \xrightarrow{-(8/7)\rho_1+\rho_2} \begin{array}{rl} 7x-7z&=0 \\ y+3z-2w&=0 \\ y-3z&=0 \\ 3y-6z-w&=0 \end{array}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{array}{rl} 7x-7z&=0 \\ y+3z-2w&=0 \\ -6z+2w&=0 \\ -15z+5w&=0 \end{array}$$

$$\xrightarrow{-(5/2)\rho_3+\rho_4} \begin{array}{rl} 7x-7z&=0 \\ y+3z-2w&=0 \\ -6z+2w&=0 \\ 0&=0 \end{array}$$

The solution set:

$$\{\begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{pmatrix} w \bigm| w\in\mathbb{R}\}$$

has many vectors besides the zero vector (if we interpret w as a number of molecules then solutions make sense only when w is a nonnegative multiple of 3).

We now have the terminology to prove the two parts of Theorem 3.1. The first lemma deals with unrestricted combinations.

3.7 Lemma For any homogeneous linear system there exist vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where k is the number of free variables in an echelon form version of the system.

Before the proof, we will recall the back substitution calculations that were done in the prior subsection. Imagine that we have brought a system to this echelon form.

$$\begin{array}{rcl} x + \quad 2y - z + \ 2w &=& 0 \\ -3y + z \qquad\quad &=& 0 \\ -w &=& 0 \end{array}$$

We next perform back-substitution to express each variable in terms of the free variable z. Working from the bottom up, we get first that w is $0 \cdot z$, next that y is $(1/3) \cdot z$, and then substituting those two into the top equation $x + 2((1/3)z) - z + 2(0) = 0$ gives $x = (1/3) \cdot z$. So, back substitution gives a parametrization of the solution set by starting at the bottom equation and using the free variables as the parameters to work row-by-row to the top. The proof below follows this pattern.

Comment: That is, this proof just does a verification of the bookkeeping in back substitution to show that we haven't overlooked any obscure cases where this procedure fails, say, by leading to a division by zero. So this argument, while quite detailed, doesn't give us any new insights. Nevertheless, we have written it out for two reasons. The first reason is that we need the result — the computational procedure that we employ must be verified to work as promised.

The second reason is that the row-by-row nature of back substitution leads to a proof that uses the technique of mathematical induction.* This is an important, and non-obvious, proof technique that we shall use a number of times in this book. Doing an induction argument here gives us a chance to see one in a setting where the proof material is easy to follow, and so the technique can be studied. Readers who are unfamiliar with induction arguments should be sure to master this one and the ones later in this chapter before going on to the second chapter.

Proof. First use Gauss' method to reduce the homogeneous system to echelon form. We will show that each leading variable can be expressed in terms of free variables. That will finish the argument because then we can use those free variables as the parameters. That is, the $\vec{\beta}$'s are the vectors of coefficients of

* More information on mathematical induction is in the appendix.

the free variables (as in Example 3.6, where the solution is $x = (1/3)w$, $y = w$, $z = (1/3)w$, and $w = w$).

We will proceed by mathematical induction, which has two steps. The base step of the argument will be to focus on the bottom-most non-'$0 = 0$' equation and write its leading variable in terms of the free variables. The inductive step of the argument will be to argue that if we can express the leading variables from the bottom t rows in terms of free variables, then we can express the leading variable of the next row up — the $t+1$-th row up from the bottom — in terms of free variables. With those two steps, the theorem will be proved because by the base step it is true for the bottom equation, and by the inductive step the fact that it is true for the bottom equation shows that it is true for the next one up, and then another application of the inductive step implies it is true for the third equation up, etc.

For the base step, consider the bottom-most non-'$0 = 0$' equation (the case where all the equations are '$0 = 0$' is trivial). We call that the m-th row:

$$a_{m,\ell_m}x_{\ell_m} + a_{m,\ell_m+1}x_{\ell_m+1} + \cdots + a_{m,n}x_n = 0$$

where $a_{m,\ell_m} \neq 0$. (The notation here has 'ℓ' stand for 'leading', so a_{m,ℓ_m} means "the coefficient from the row m of the variable leading row m".) Either there are variables in this equation other than the leading one x_{ℓ_m} or else there are not. If there are other variables x_{ℓ_m+1}, etc., then they must be free variables because this is the bottom non-'$0 = 0$' row. Move them to the right and divide by a_{m,ℓ_m}

$$x_{\ell_m} = (-a_{m,\ell_m+1}/a_{m,\ell_m})x_{\ell_m+1} + \cdots + (-a_{m,n}/a_{m,\ell_m})x_n$$

to express this leading variable in terms of free variables. If there are no free variables in this equation then $x_{\ell_m} = 0$ (see the "tricky point" noted following this proof).

For the inductive step, we assume that for the m-th equation, and for the $(m-1)$-th equation, ..., and for the $(m-t)$-th equation, we can express the leading variable in terms of free variables (where $0 \leq t < m$). To prove that the same is true for the next equation up, the $(m-(t+1))$-th equation, we take each variable that leads in a lower-down equation $x_{\ell_m}, \ldots, x_{\ell_{m-t}}$ and substitute its expression in terms of free variables. The result has the form

$$a_{m-(t+1),\ell_{m-(t+1)}}x_{\ell_{m-(t+1)}} + \text{sums of multiples of free variables} = 0$$

where $a_{m-(t+1),\ell_{m-(t+1)}} \neq 0$. We move the free variables to the right-hand side and divide by $a_{m-(t+1),\ell_{m-(t+1)}}$, to end with $x_{\ell_{m-(t+1)}}$ expressed in terms of free variables.

Because we have shown both the base step and the inductive step, by the principle of mathematical induction the proposition is true. QED

We say that the set $\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$ is *generated by* or *spanned by* the set of vectors $\{\vec{\beta}_1, \ldots, \vec{\beta}_k\}$.

There is a tricky point to this. We rely on the convention that the sum of an empty set of vectors is the zero vector. In particular, we need this in the case where a homogeneous system has a unique solution. Then the homogeneous case fits the pattern of the other solution sets: in the proof above, the solution set is derived by taking the c's to be the free variables and if there is a unique solution then there are no free variables.

The proof incidentally shows, as discussed after Example 2.4, that solution sets can always be parametrized using the free variables.

The next lemma finishes the proof of Theorem 3.1 by considering the particular solution part of the solution set's description.

3.8 Lemma For a linear system, where $\vec{p}$ is any particular solution, the solution set equals this set.

$$\{\vec{p}+\vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$$

Proof. We will show mutual set inclusion, that any solution to the system is in the above set and that anything in the set is a solution to the system.*

For set inclusion the first way, that if a vector solves the system then it is in the set described above, assume that $\vec{s}$ solves the system. Then $\vec{s}-\vec{p}$ solves the associated homogeneous system since for each equation index i,

$$\begin{aligned} a_{i,1}(s_1-p_1)+\cdots+a_{i,n}(s_n-p_n) &= (a_{i,1}s_1+\cdots+a_{i,n}s_n) \\ &\quad -(a_{i,1}p_1+\cdots+a_{i,n}p_n) \\ &= d_i-d_i \\ &= 0 \end{aligned}$$

where p_j and s_j are the j-th components of $\vec{p}$ and $\vec{s}$. We can write $\vec{s}-\vec{p}$ as $\vec{h}$, where $\vec{h}$ solves the associated homogeneous system, to express $\vec{s}$ in the required $\vec{p}+\vec{h}$ form.

For set inclusion the other way, take a vector of the form $\vec{p}+\vec{h}$, where $\vec{p}$ solves the system and $\vec{h}$ solves the associated homogeneous system, and note that it solves the given system: for any equation index i,

$$\begin{aligned} a_{i,1}(p_1+h_1)+\cdots+a_{i,n}(p_n+h_n) &= (a_{i,1}p_1+\cdots+a_{i,n}p_n) \\ &\quad +(a_{i,1}h_1+\cdots+a_{i,n}h_n) \\ &= d_i+0 \\ &= d_i \end{aligned}$$

where h_j is the j-th component of $\vec{h}$. QED

The two lemmas above together establish Theorem 3.1. We remember that theorem with the slogan "General = Particular + Homogeneous".

* More information on equality of sets is in the appendix.

3.9 Example This system illustrates Theorem 3.1.

$$\begin{aligned} x+2y-z&=1\\ 2x+4y&=2\\ y-3z&=0 \end{aligned}$$

Gauss' method

$$\xrightarrow{-2\rho_1+\rho_2}\ \begin{aligned} x+2y-z&=1\\ 2z&=0\\ y-3z&=0 \end{aligned}\ \xrightarrow{\rho_2\leftrightarrow\rho_3}\ \begin{aligned} x+2y-z&=1\\ y-3z&=0\\ 2z&=0 \end{aligned}$$

shows that the general solution is a singleton set.

$$\{\begin{pmatrix}1\\0\\0\end{pmatrix}\}$$

That single vector is, of course, a particular solution. The associated homogeneous system reduces via the same row operations

$$\begin{aligned} x+2y-z&=0\\ 2x+4y&=0\\ y-3z&=0 \end{aligned}\ \xrightarrow{-2\rho_1+\rho_2}\ \xrightarrow{\rho_2\leftrightarrow\rho_3}\ \begin{aligned} x+2y-z&=0\\ y-3z&=0\\ 2z&=0 \end{aligned}$$

to also give a singleton set.

$$\{\begin{pmatrix}0\\0\\0\end{pmatrix}\}$$

As the theorem states, and as discussed at the start of this subsection, in this single-solution case the general solution results from taking the particular solution and adding to it the unique solution of the associated homogeneous system.

3.10 Example Also discussed at the start of this subsection is that the case where the general solution set is empty fits the 'General = Particular + Homogeneous' pattern. This system illustrates. Gauss' method

$$\begin{aligned} x\phantom{{}-y}+z+w&=-1\\ 2x-y\phantom{{}+3z}+w&=3\\ x+y+3z+2w&=1 \end{aligned}\ \underset{-\rho_1+\rho_3}{\xrightarrow{-2\rho_1+\rho_2}}\ \begin{aligned} x\phantom{{}-y}+z+w&=-1\\ -y-2z-w&=5\\ y+2z+w&=2 \end{aligned}$$

shows that it has no solutions because the final two equations are in conflict. The associated homogeneous system, of course, has a solution.

$$\begin{aligned} x\phantom{{}-y}+z+w&=0\\ 2x-y\phantom{{}+3z}+w&=0\\ x+y+3z+2w&=0 \end{aligned}\ \underset{-\rho_1+\rho_3}{\xrightarrow{-2\rho_1+\rho_2}}\ \xrightarrow{\rho_2+\rho_3}\ \begin{aligned} x\phantom{{}-y}+z+w&=0\\ -y-2z-w&=0\\ 0&=0 \end{aligned}$$

In fact, the solution set of the homogeneous system is infinite.

$$\{\begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}z+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}w \mid z,w\in\mathbb{R}\}$$

However, because no particular solution of the original system exists, the general solution set is empty — there are no vectors of the form $\vec{p}+\vec{h}$ because there are no $\vec{p}$'s.

3.11 Corollary Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

Proof. We've seen examples of all three happening so we need only prove that those are the only possibilities.

First, notice a homogeneous system with at least one non-$\vec{0}$ solution $\vec{v}$ has infinitely many solutions because the set of multiples $s\vec{v}$ is infinite — if $s\neq 1$ then $s\vec{v}-\vec{v}=(s-1)\vec{v}$ is easily seen to be non-$\vec{0}$, and so $s\vec{v}\neq\vec{v}$.

Now, apply Lemma 3.8 to conclude that a solution set

$$\{\vec{p}+\vec{h} \mid \vec{h} \text{ solves the associated homogeneous system}\}$$

is either empty (if there is no particular solution $\vec{p}$), or has one element (if there is a $\vec{p}$ and the homogeneous system has the unique solution $\vec{0}$), or is infinite (if there is a $\vec{p}$ and the homogeneous system has a non-$\vec{0}$ solution, and thus by the prior paragraph has infinitely many solutions). QED

This table summarizes the factors affecting the size of a general solution.

		number of solutions of the associated homogeneous system	
		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
	no	no solutions	no solutions

The factor on the top of the table is the simpler one. When we perform Gauss' method on a linear system, ignoring the constants on the right side and so paying attention only to the coefficients on the left-hand side, we either end with every variable leading some row or else we find that some variable does not lead a row, that is, that some variable is free. (Of course, "ignoring the constants on the right" is formalized by considering the associated homogeneous system. We are simply putting aside for the moment the possibility of a contradictory equation.)

A nice insight into the factor on the top of this table at work comes from considering the case of a system having the same number of equations as variables.

This system will have a solution, and the solution will be unique, if and only if it reduces to an echelon form system where every variable leads its row, which will happen if and only if the associated homogeneous system has a unique solution. Thus, the question of uniqueness of solution is especially interesting when the system has the same number of equations as variables.

3.12 Definition A square matrix is *nonsingular* if it is the matrix of coefficients of a homogeneous system with a unique solution. It is *singular* otherwise, that is, if it is the matrix of coefficients of a homogeneous system with infinitely many solutions.

The word singular means "departing from general expectation" and here expresses that we could expect that systems with the same number of equations as unknowns will typically have a unique solution. (That 'singular' applies to systems having more than one solution is ironic, but it is the standard term.)

3.13 Example The systems from Example 3.3, Example 3.5, and Example 3.9 each have an associated homogeneous system with a unique solution. Thus these matrices are nonsingular.

$$\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 & 1 \\ 6 & -4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -3 \end{pmatrix}$$

The Chemistry problem from Example 3.6 is a homogeneous system with more than one solution so its matrix is singular.

$$\begin{pmatrix} 7 & 0 & -7 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{pmatrix}$$

3.14 Example The first of these matrices is nonsingular while the second is singular

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

because the first of these homogeneous systems has a unique solution while the second has infinitely many solutions.

$$\begin{aligned} x + 2y &= 0 \\ 3x + 4y &= 0 \end{aligned} \qquad \begin{aligned} x + 2y &= 0 \\ 3x + 6y &= 0 \end{aligned}$$

We have made the distinction in the definition because a system (with the same number of equations as variables) behaves in one of two ways, depending on whether its matrix of coefficients is nonsingular or singular. A system where the matrix of coefficients is nonsingular has a unique solution for any constants on the right side: for instance, Gauss' method shows that this system

$$\begin{aligned} x + 2y &= a \\ 3x + 4y &= b \end{aligned}$$

has the unique solution $x = b - 2a$ and $y = (3a - b)/2$. On the other hand, a system where the matrix of coefficients is singular never has a unique solution — it has either no solutions or else has infinitely many, as with these.

$$\begin{aligned} x + 2y &= 1 \\ 3x + 6y &= 2 \end{aligned} \qquad \begin{aligned} x + 2y &= 1 \\ 3x + 6y &= 3 \end{aligned}$$

Thus, 'singular' can be thought of as connoting "troublesome", or at least "not ideal".

The above table has two factors. We have already considered the factor along the top: we can tell which column a given linear system goes in solely by considering the system's left-hand side — the constants on the right-hand side play no role in this factor. The table's other factor, determining whether a particular solution exists, is tougher. Consider these two

$$\begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 5 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 4 \end{aligned}$$

with the same left sides but different right sides. Obviously, the first has a solution while the second does not, so here the constants on the right side decide if the system has a solution. We could conjecture that the left side of a linear system determines the number of solutions while the right side determines if solutions exist, but that guess is not correct. Compare these two systems

$$\begin{aligned} 3x + 2y &= 5 \\ 4x + 2y &= 4 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 5 \\ 3x + 2y &= 4 \end{aligned}$$

with the same right sides but different left sides. The first has a solution but the second does not. Thus the constants on the right side of a system don't decide alone whether a solution exists; rather, it depends on some interaction between the left and right sides.

For some intuition about that interaction, consider this system with one of the coefficients left as the parameter c.

$$\begin{aligned} x + 2y + 3z &= 1 \\ x + y + z &= 1 \\ cx + 3y + 4z &= 0 \end{aligned}$$

If $c = 2$ then this system has no solution because the left-hand side has the third row as a sum of the first two, while the right-hand does not. If $c \neq 2$ then this system has a unique solution (try it with $c = 1$). For a system to have a solution, if one row of the matrix of coefficients on the left is a linear combination of other rows, then on the right the constant from that row must be the same combination of constants from the same rows.

More intuition about the interaction comes from studying linear combinations. That will be our focus in the second chapter, after we finish the study of Gauss' method itself in the rest of this chapter.

Exercises

✓ **3.15** Solve each system. Express the solution set using vectors. Identify the particular solution and the solution set of the homogeneous system.

(a) $\begin{aligned} 3x+6y&=18 \\ x+2y&=6 \end{aligned}$ **(b)** $\begin{aligned} x+y&=1 \\ x-y&=-1 \end{aligned}$ **(c)** $\begin{aligned} x_1 + x_3&=4 \\ x_1-x_2+2x_3&=5 \\ 4x_1-x_2+5x_3&=17 \end{aligned}$

(d) $\begin{aligned} 2a+b-c&=2 \\ 2a +c&=3 \\ a-b &=0 \end{aligned}$ **(e)** $\begin{aligned} x+2y-z &=3 \\ 2x+y +w&=4 \\ x-y+z+w&=1 \end{aligned}$ **(f)** $\begin{aligned} x +z+w&=4 \\ 2x+y -w&=2 \\ 3x+y+z &=7 \end{aligned}$

3.16 Solve each system, giving the solution set in vector notation. Identify the particular solution and the solution of the homogeneous system.

(a) $\begin{aligned} 2x+y-z&=1 \\ 4x-y &=3 \end{aligned}$ **(b)** $\begin{aligned} x -z &=1 \\ y+2z-w&=3 \\ x+2y+3z-w&=7 \end{aligned}$ **(c)** $\begin{aligned} x-y+z &=0 \\ y +w&=0 \\ 3x-2y+3z+w&=0 \\ -y -w&=0 \end{aligned}$

(d) $\begin{aligned} a+2b+3c+d-e&=1 \\ 3a-b+c+d+e&=3 \end{aligned}$

✓ **3.17** For the system

$$\begin{aligned} 2x-y -w&=3 \\ y+z+2w&=2 \\ x-2y-z &=-1 \end{aligned}$$

which of these can be used as the particular solution part of some general solution?

(a) $\begin{pmatrix} 0 \\ -3 \\ 5 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 \\ -4 \\ 8 \\ -1 \end{pmatrix}$

✓ **3.18** Lemma 3.8 says that any particular solution may be used for $\vec{p}$. Find, if possible, a general solution to this system

$$\begin{aligned} x-y +w&=4 \\ 2x+3y-z &=0 \\ y+z+w&=4 \end{aligned}$$

that uses the given vector as its particular solution.

(a) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

3.19 One of these is nonsingular while the other is singular. Which is which?

(a) $\begin{pmatrix} 1 & 3 \\ 4 & -12 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

✓ **3.20** Singular or nonsingular?

(a) $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ (Careful!)

(d) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 3 & 4 & 7 \end{pmatrix}$ **(e)** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 5 \\ -1 & 1 & 4 \end{pmatrix}$

✓ **3.21** Is the given vector in the set generated by the given set?

(a) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}\}$

(b) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$

(c) $\begin{pmatrix}1\\3\\0\end{pmatrix}$, $\{\begin{pmatrix}1\\0\\4\end{pmatrix}, \begin{pmatrix}2\\1\\5\end{pmatrix}, \begin{pmatrix}3\\3\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}\}$

(d) $\begin{pmatrix}1\\0\\1\\1\end{pmatrix}$, $\{\begin{pmatrix}2\\1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\0\\0\\2\end{pmatrix}\}$

3.22 Prove that any linear system with a nonsingular matrix of coefficients has a solution, and that the solution is unique.

3.23 To tell the whole truth, there is another tricky point to the proof of Lemma 3.7. What happens if there are no non-'$0 = 0$' equations? (There aren't any more tricky points after this one.)

✓ **3.24** Prove that if $\vec{s}$ and $\vec{t}$ satisfy a homogeneous system then so do these vectors.

(a) $\vec{s}+\vec{t}$ **(b)** $3\vec{s}$ **(c)** $k\vec{s}+m\vec{t}$ for $k, m \in \mathbb{R}$

What's wrong with: "These three show that if a homogeneous system has one solution then it has many solutions — any multiple of a solution is another solution, and any sum of solutions is a solution also — so there are no homogeneous systems with exactly one solution."?

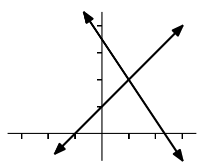
3.25 Prove that if a system with only rational coefficients and constants has a solution then it has at least one all-rational solution. Must it have infinitely many?

II Linear Geometry of n-Space

For readers who have seen the elements of vectors before, in calculus or physics, this section is an optional review. However, later work will refer to this material so if it is not a review then it is not optional.

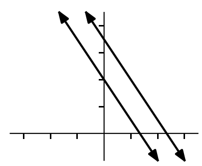
In the first section, we had to do a bit of work to show that there are only three types of solution sets — singleton, empty, and infinite. But in the special case of systems with two equations and two unknowns this is easy to see with a picture. Draw each two-unknowns equation as a line in the plane and then the two lines could have a unique intersection, be parallel, or be the same line.

Unique solution



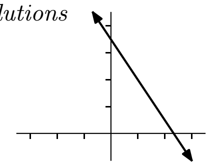
$$\begin{aligned} 3x + 2y &= 7 \\ x - y &= -1 \end{aligned}$$

No solutions



$$\begin{aligned} 3x + 2y &= 7 \\ 3x + 2y &= 4 \end{aligned}$$

Infinitely many solutions



$$\begin{aligned} 3x + 2y &= 7 \\ 6x + 4y &= 14 \end{aligned}$$

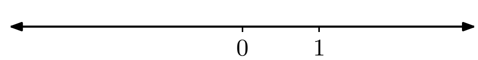
These pictures don't prove the results from the prior section, which apply to any number of linear equations and any number of unknowns, but nonetheless they do help us to understand those results. This section develops the ideas that we need to express our results from the prior section, and from some future sections, geometrically. In particular, while the two-dimensional case is familiar, to extend to systems with more than two unknowns we shall need some higher-dimensional geometry.

II.1 Vectors in Space

"Higher-dimensional geometry" sounds exotic. It is exotic — interesting and eye-opening. But it isn't distant or unreachable.

We begin by defining one-dimensional space to be the set $\mathbb{R}^1$. To see that definition is reasonable, draw a one-dimensional space

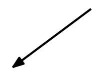
and make the usual correspondence with $\mathbb{R}$: pick a point to label 0 and another to label 1.



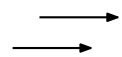
Now, with a scale and a direction, finding the point corresponding to, say $+2.17$, is easy — start at 0 and head in the direction of 1 (i.e., the positive direction), but don't stop there, go 2.17 times as far.

The basic idea here, combining magnitude with direction, is the key to extending to higher dimensions.

An object comprised of a magnitude and a direction is a *vector* (we will use the same word as in the previous section because we shall show below how to describe such an object with a column vector). We can draw a vector as having some length, and pointing somewhere.

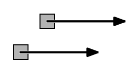


There is a subtlety here — these vectors

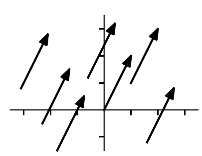


are equal, even though they start in different places, because they have equal lengths and equal directions. Again: those vectors are not just alike, they are equal.

How can things that are in different places be equal? Think of a vector as representing a displacement ('vector' is Latin for "carrier" or "traveler"). These squares undergo the same displacement, despite that those displacements start in different places.



Sometimes, to emphasize this property vectors have of not being anchored, they are referred to as *free* vectors. Thus, these free vectors are equal as each is a displacement of one over and two up.



More generally, vectors in the plane are the same if and only if they have the same change in first components and the same change in second components: the vector extending from (a_1, a_2) to (b_1, b_2) equals the vector from (c_1, c_2) to (d_1, d_2) if and only if $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$.

An expression like 'the vector that, were it to start at (a_1, a_2), would extend to (b_1, b_2)' is awkward. We instead describe such a vector as

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

so that, for instance, the 'one over and two up' arrows shown above picture this vector.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We often draw the arrow as starting at the origin, and we then say it is in the *canonical position* (or *natural position* or *standard position*). When the vector

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

is in its canonical position then it extends to the endpoint $(b_1 - a_1, b_2 - a_2)$.

We typically just refer to "the point

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\text{''}$$

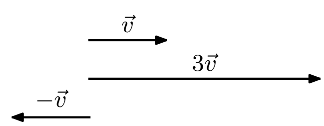
rather than "the endpoint of the canonical position of" that vector. Thus, we will call both of these sets $\mathbb{R}^2$.

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}$$

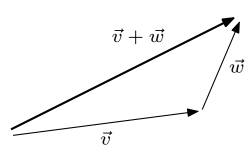
In the prior section we defined vectors and vector operations with an algebraic motivation;

$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

we can now interpret those operations geometrically. For instance, if $\vec{v}$ represents a displacement then $3\vec{v}$ represents a displacement in the same direction but three times as far, and $-1\vec{v}$ represents a displacement of the same distance as $\vec{v}$ but in the opposite direction.

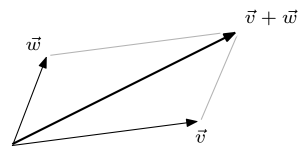


And, where $\vec{v}$ and $\vec{w}$ represent displacements, $\vec{v} + \vec{w}$ represents those displacements combined.



The long arrow is the combined displacement in this sense: if, in one minute, a ship's motion gives it the displacement relative to the earth of $\vec{v}$ and a passenger's motion gives a displacement relative to the ship's deck of $\vec{w}$, then $\vec{v} + \vec{w}$ is the displacement of the passenger relative to the earth.

Another way to understand the vector sum is with the *parallelogram rule*. Draw the parallelogram formed by the vectors $\vec{v}_1, \vec{v}_2$ and then the sum $\vec{v}_1 + \vec{v}_2$ extends along the diagonal to the far corner.



The above drawings show how vectors and vector operations behave in $\mathbb{R}^2$. We can extend to $\mathbb{R}^3$, or to even higher-dimensional spaces where we have no pictures, with the obvious generalization: the free vector that, if it starts at $(a_1, \ldots, a_n)$, ends at $(b_1, \ldots, b_n)$, is represented by this column

$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

(vectors are equal if they have the same representation), we aren't too careful to distinguish between a point and the vector whose canonical representation ends at that point,

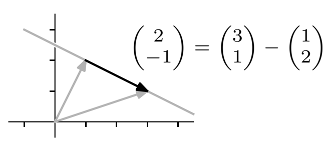
$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

and addition and scalar multiplication are done component-wise.

Having considered points, we now turn to the lines. In $\mathbb{R}^2$, the line through $(1, 2)$ and $(3, 1)$ is comprised of (the endpoints of) the vectors in this set.

$$\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \}$$

That description expresses this picture.



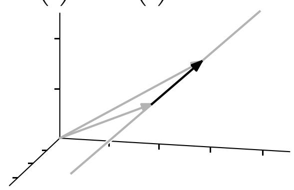
The vector associated with the parameter t

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

has its whole body in the line — it is a *direction vector* for the line. Note that points on the line to the left of $x = 1$ are described using negative values of t. Note also that this description of lines generalizes the familiar $y = b + mx$ form for lines in the plane.

In $\mathbb{R}^3$, the line through $(1, 2, 1)$ and $(2, 3, 2)$ is the set of (endpoints of) vectors of this form

$$\{\begin{pmatrix}1\\2\\1\end{pmatrix}+t\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\mid t\in\mathbb{R}\}$$



and lines in even higher-dimensional spaces work in the same way.

In $\mathbb{R}^3$, a line uses one parameter so that there is freedom to move back and forth in one dimension, and a plane involves two parameters. For example, the plane through the points $(1,0,5)$, $(2,1,-3)$, and $(-2,4,0.5)$ consists of (endpoints of) the vectors in

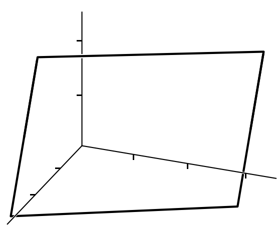
$$\{\begin{pmatrix}1\\0\\5\end{pmatrix}+t\cdot\begin{pmatrix}1\\1\\-8\end{pmatrix}+s\cdot\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}\mid t,s\in\mathbb{R}\}$$

(the column vectors associated with the parameters

$$\begin{pmatrix}1\\1\\-8\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}\qquad\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}=\begin{pmatrix}-2\\4\\0.5\end{pmatrix}-\begin{pmatrix}1\\0\\5\end{pmatrix}$$

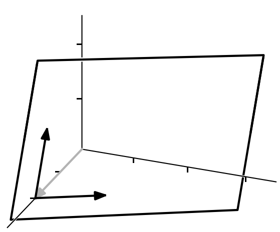
are two vectors whose whole bodies lie in the plane). As with the line, note that some points in this plane are described with negative t's or negative s's or both.

In algebra and calculus we often use a description of planes involving a single equation as the condition that describes the relationship among the first, second, and third coordinates of points in a plane.



$$P=\{\begin{pmatrix}x\\y\\z\end{pmatrix}\mid 2x+y+z=4\}$$

The translation from such a description to the vector description that we favor in this book is to think of the condition as a one-equation linear system and parametrize $x=(1/2)(4-y-z)$.



$$P=\{\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}y+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}z\mid y,z\in\mathbb{R}\}$$

Generalizing from lines and planes, we define a *k-dimensional linear surface* (or *k-flat*) in $\mathbb{R}^n$ to be $\{\vec{p}+t_1\vec{v}_1+t_2\vec{v}_2+\cdots+t_k\vec{v}_k \bigm| t_1,\ldots,t_k\in\mathbb{R}\}$ where $\vec{v}_1,\ldots,\vec{v}_k\in\mathbb{R}^n$. For example, in $\mathbb{R}^4$,

$$\{\begin{pmatrix}2\\ \pi\\ 3\\ -0.5\end{pmatrix}+t\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\bigm| t\in\mathbb{R}\}$$

is a line,

$$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}+s\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\bigm| t,s\in\mathbb{R}\}$$

is a plane, and

$$\{\begin{pmatrix}3\\1\\-2\\0.5\end{pmatrix}+r\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}+s\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\bigm| r,s,t\in\mathbb{R}\}$$

is a three-dimensional linear surface. Again, the intuition is that a line permits motion in one direction, a plane permits motion in combinations of two directions, etc. (When k is one less than the dimension of the space, that is in $\mathbb{R}^n$ when $k=n-1$, then a k-dimensional linear surface is called a *hyperplane*.)

The description of a linear surface can be misleading about the dimension — this

$$L=\{\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}+t\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}+s\begin{pmatrix}2\\2\\0\\-2\end{pmatrix}\bigm| t,s\in\mathbb{R}\}$$

is a *degenerate* plane because it is actually a line — the vectors are multiples of each other so we can merge the two into one.

$$L=\{\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}+r\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}\bigm| r\in\mathbb{R}\}$$

We shall see in the Linear Independence section of Chapter Two what relationships among vectors causes the linear surface they generate to be degenerate.

We finish this subsection by restating our conclusions from the first section in geometric terms. First, the solution set of a linear system with n unknowns is a linear surface in $\mathbb{R}^n$. Specifically, it is a k-dimensional linear surface, where k is the number of free variables in an echelon form version of the system. Second, the solution set of a homogeneous linear system is a linear surface passing through the origin. Finally, we can view the general solution set of any linear system as being the solution set of its associated homogeneous system offset from the origin by a vector, namely by any particular solution.

Exercises

✓ **1.1** Find the canonical name for each vector.

(a) the vector from $(2,1)$ to $(4,2)$ in $\mathbb{R}^2$
(b) the vector from $(3,3)$ to $(2,5)$ in $\mathbb{R}^2$
(c) the vector from $(1,0,6)$ to $(5,0,3)$ in $\mathbb{R}^3$
(d) the vector from $(6,8,8)$ to $(6,8,8)$ in $\mathbb{R}^3$

✓ **1.2** Decide if the two vectors are equal.

(a) the vector from $(5,3)$ to $(6,2)$ and the vector from $(1,-2)$ to $(1,1)$
(b) the vector from $(2,1,1)$ to $(3,0,4)$ and the vector from $(5,1,4)$ to $(6,0,7)$

✓ **1.3** Does $(1,0,2,1)$ lie on the line through $(-2,1,1,0)$ and $(5,10,-1,4)$?

✓ **1.4** **(a)** Describe the plane through $(1,1,5,-1)$, $(2,2,2,0)$, and $(3,1,0,4)$.
(b) Is the origin in that plane?

1.5 Describe the plane that contains this point and line.

$$\begin{pmatrix}2\\0\\3\end{pmatrix} \qquad \{\begin{pmatrix}-1\\0\\-4\end{pmatrix}+\begin{pmatrix}1\\1\\2\end{pmatrix}t \mid t\in\mathbb{R}\}$$

✓ **1.6** Intersect these planes.

$$\{\begin{pmatrix}1\\1\\1\end{pmatrix}t+\begin{pmatrix}0\\1\\3\end{pmatrix}s \mid t,s\in\mathbb{R}\} \qquad \{\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}0\\3\\0\end{pmatrix}k+\begin{pmatrix}2\\0\\4\end{pmatrix}m \mid k,m\in\mathbb{R}\}$$

✓ **1.7** Intersect each pair, if possible.

(a) $\{\begin{pmatrix}1\\1\\2\end{pmatrix}+t\begin{pmatrix}0\\1\\1\end{pmatrix} \mid t\in\mathbb{R}\}$, $\{\begin{pmatrix}1\\3\\-2\end{pmatrix}+s\begin{pmatrix}0\\1\\2\end{pmatrix} \mid s\in\mathbb{R}\}$

(b) $\{\begin{pmatrix}2\\0\\1\end{pmatrix}+t\begin{pmatrix}1\\1\\-1\end{pmatrix} \mid t\in\mathbb{R}\}$, $\{s\begin{pmatrix}0\\1\\2\end{pmatrix}+w\begin{pmatrix}0\\4\\1\end{pmatrix} \mid s,w\in\mathbb{R}\}$

1.8 When a plane does not pass through the origin, performing operations on vectors whose bodies lie in it is more complicated than when the plane passes through the origin. Consider the picture in this subsection of the plane

$$\{\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}y+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}z \mid y,z\in\mathbb{R}\}$$

and the three vectors it shows, with endpoints $(2,0,0)$, $(1.5,1,0)$, and $(1.5,0,1)$.

(a) Redraw the picture, including the vector in the plane that is twice as long as the one with endpoint $(1.5,1,0)$. The endpoint of your vector is not $(3,2,0)$; what is it?
(b) Redraw the picture, including the parallelogram in the plane that shows the sum of the vectors ending at $(1.5,0,1)$ and $(1.5,1,0)$. The endpoint of the sum, on the diagonal, is not $(3,1,1)$; what is it?

1.9 Show that the line segments $\overline{(a_1,a_2)(b_1,b_2)}$ and $\overline{(c_1,c_2)(d_1,d_2)}$ have the same lengths and slopes if $b_1-a_1=d_1-c_1$ and $b_2-a_2=d_2-c_2$. Is that only if?

1.10 How should $\mathbb{R}^0$ be defined?

? ✓ **1.11** A person traveling eastward at a rate of 3 miles per hour finds that the wind appears to blow directly from the north. On doubling his speed it appears to come from the north east. What was the wind's velocity? [Math. Mag., Jan. 1957]

1.12 Euclid describes a plane as "a surface which lies evenly with the straight lines on itself". Commentators (e.g., Heron) have interpreted this to mean "(A plane surface is) such that, if a straight line pass through two points on it, the line coincides wholly with it at every spot, all ways". (Translations from [Heath], pp. 171-172.) Do planes, as described in this section, have that property? Does this description adequately define planes?

II.2 Length and Angle Measures

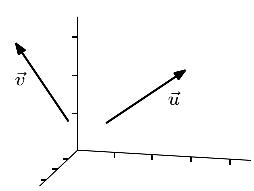
We've translated the first section's results about solution sets into geometric terms for insight into how those sets look. But we must watch out not to be misled by our own terms; labeling subsets of $\mathbb{R}^k$ of the forms $\{\vec{p}+t\vec{v} \mid t\in\mathbb{R}\}$ and $\{\vec{p}+t\vec{v}+s\vec{w} \mid t,s\in\mathbb{R}\}$ as "lines" and "planes" doesn't make them act like the lines and planes of our prior experience. Rather, we must ensure that the names suit the sets. While we can't prove that the sets satisfy our intuition — we can't prove anything about intuition — in this subsection we'll observe that a result familiar from $\mathbb{R}^2$ and $\mathbb{R}^3$, when generalized to arbitrary $\mathbb{R}^k$, supports the idea that a line is straight and a plane is flat. Specifically, we'll see how to do Euclidean geometry in a "plane" by giving a definition of the angle between two $\mathbb{R}^n$ vectors in the plane that they generate.

2.1 Definition The *length* of a vector $\vec{v}\in\mathbb{R}^n$ is this.

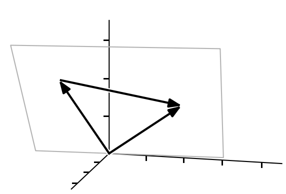
$$\|\vec{v}\,\| = \sqrt{v_1^2+\cdots+v_n^2}$$

2.2 Remark This is a natural generalization of the Pythagorean Theorem. A classic discussion is in [Polya].

We can use that definition to derive a formula for the angle between two vectors. For a model of what to do, consider two vectors in $\mathbb{R}^3$.



Put them in canonical position and, in the plane that they determine, consider the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$.



Apply the Law of Cosines, $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta$, where θ is the angle between the vectors. Expand both sides

$$\begin{aligned}&(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2\\&\qquad= (u_1^2+u_2^2+u_3^2)+(v_1^2+v_2^2+v_3^2) - 2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta\end{aligned}$$

and simplify.

$$\theta = \arccos\Big(\,\frac{u_1v_1+u_2v_2+u_3v_3}{\|\vec{u}\|\,\|\vec{v}\|}\,\Big)$$

In higher dimensions no picture suffices but we can make the same argument analytically. First, the form of the numerator is clear — it comes from the middle terms of the squares $(u_1-v_1)^2$, $(u_2-v_2)^2$, etc.

2.3 Definition The *dot product* (or *inner product*, or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Note that the dot product of two vectors is a real number, not a vector, and that the dot product of a vector from $\mathbb{R}^n$ with a vector from $\mathbb{R}^m$ is defined only when n equals m. Note also this relationship between dot product and length: dotting a vector with itself gives its length squared $\vec{u}\cdot\vec{u} = u_1u_1 + \cdots + u_nu_n = \|\vec{u}\|^2$.

2.4 Remark The wording in that definition allows one or both of the two to be a row vector instead of a column vector. Some books require that the first vector be a row vector and that the second vector be a column vector. We shall not be that strict.

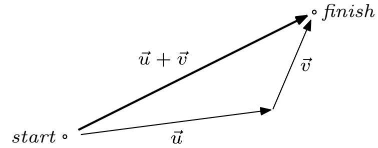
Still reasoning with letters, but guided by the pictures, we use the next theorem to argue that the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$ in $\mathbb{R}^n$ lies in the planar subset of $\mathbb{R}^n$ generated by $\vec{u}$ and $\vec{v}$.

2.5 Theorem (Triangle Inequality) For any $\vec{u},\vec{v}\in\mathbb{R}^n$,

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This inequality is the source of the familiar saying, "The shortest distance between two points is in a straight line."



Proof. (We'll use some algebraic properties of dot product that we have not yet checked, for instance that $\vec{u}\cdot(\vec{a}+\vec{b})=\vec{u}\cdot\vec{a}+\vec{u}\cdot\vec{b}$ and that $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$. See Exercise 17.) The desired inequality holds if and only if its square holds.

$$\begin{aligned}
|\vec{u}+\vec{v}\,|^2 &\leq (\,|\vec{u}\,|+|\vec{v}\,|\,)^2\\
(\,\vec{u}+\vec{v}\,)\cdot(\,\vec{u}+\vec{v}\,) &\leq |\vec{u}\,|^2+2\,|\vec{u}\,|\,|\vec{v}\,|+|\vec{v}\,|^2\\
\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} &\leq \vec{u}\cdot\vec{u}+2\,|\vec{u}\,|\,|\vec{v}\,|+\vec{v}\cdot\vec{v}\\
2\,\vec{u}\cdot\vec{v} &\leq 2\,|\vec{u}\,|\,|\vec{v}\,|
\end{aligned}$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}\,|$ and $|\vec{v}\,|$

$$2\,(\,|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}\,)\leq 2\,|\vec{u}\,|^2\,|\vec{v}\,|^2$$

and rewriting

$$0\leq |\vec{u}\,|^2\,|\vec{v}\,|^2-2\,(\,|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}\,)+|\vec{u}\,|^2\,|\vec{v}\,|^2$$

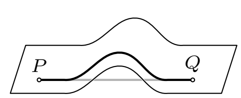
is true. But factoring

$$0\leq (\,|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}\,)\cdot(\,|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}\,)$$

shows that this certainly is true since it only says that the square of the length of the vector $|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}$ is not negative.

As for equality, it holds when, and only when, $|\vec{u}\,|\,\vec{v}-|\vec{v}\,|\,\vec{u}$ is $\vec{0}$. The check that $|\vec{u}\,|\,\vec{v}=|\vec{v}\,|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

This result supports the intuition that even in higher-dimensional spaces, lines are straight and planes are flat. For any two points in a linear surface, the line segment connecting them is contained in that surface (this is easily checked from the definition). But if the surface has a bend then that would allow for a shortcut (shown here grayed, while the segment from P to Q that is contained in the surface is solid).



Because the Triangle Inequality says that in any $\mathbb{R}^n$, the shortest cut between two endpoints is simply the line segment connecting them, linear surfaces have no such bends.

Back to the definition of angle measure. The heart of the Triangle Inequality's proof is the '$\vec{u}\cdot\vec{v}\leq|\vec{u}\,|\,|\vec{v}\,|$' line. At first glance, a reader might wonder if some pairs of vectors satisfy the inequality in this way: while $\vec{u}\cdot\vec{v}$ is a large number, with absolute value bigger than the right-hand side, it is a negative large number. The next result says that no such pair of vectors exists.

2.6 Corollary (Cauchy-Schwartz Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\,\vec{u} \cdot \vec{v}\,| \leq \|\,\vec{u}\,\|\,\|\vec{v}\,\|$$

with equality if and only if one vector is a scalar multiple of the other.

Proof. The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leq \|\vec{u}\,\|\,\|\vec{v}\,\|$ so if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds.

$$|\,\vec{u} \cdot \vec{v}\,| = -(\,\vec{u} \cdot \vec{v}\,) = (-\vec{u}\,) \cdot \vec{v} \leq \|-\vec{u}\,\|\,\|\vec{v}\,\| = \|\vec{u}\,\|\,\|\vec{v}\,\|$$

The equality condition is Exercise 18. QED

The Cauchy-Schwartz inequality assures us that the next definition makes sense because the fraction has absolute value less than or equal to one.

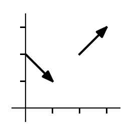
2.7 Definition The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\,\|\,\|\vec{v}\,\|}\right)$$

(the angle between the zero vector and any other vector is defined to be a right angle).

Thus vectors from $\mathbb{R}^n$ are orthogonal, that is, perpendicular, if and only if their dot product is zero.

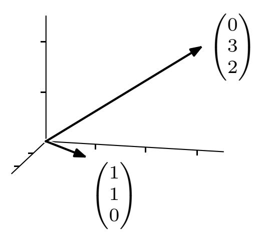
2.8 Example These vectors are orthogonal.



$$\begin{pmatrix}1\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\1\end{pmatrix} = 0$$

The arrows are shown away from canonical position but nevertheless the vectors are orthogonal.

2.9 Example The $\mathbb{R}^3$ angle formula given at the start of this subsection is a special case of the definition. Between these two



the angle is

$$\arccos\left(\frac{(1)(0)+(1)(3)+(0)(2)}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+3^2+2^2}}\right)=\arccos\left(\frac{3}{\sqrt{2}\sqrt{13}}\right)$$

approximately 0.94 radians. Notice that these vectors are not orthogonal. Although the yz-plane may appear to be perpendicular to the xy-plane, in fact the two planes are that way only in the weak sense that there are vectors in each orthogonal to all vectors in the other. Not every vector in each is orthogonal to all vectors in the other.

Exercises

✓ **2.10** Find the length of each vector.

(a) $\begin{pmatrix}3\\1\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\2\end{pmatrix}$ **(c)** $\begin{pmatrix}4\\1\\1\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$ **(e)** $\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}$

✓ **2.11** Find the angle between each two, if it is defined.

(a) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\4\\1\end{pmatrix}$ **(c)** $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\4\\-1\end{pmatrix}$

✓ **2.12** During maneuvers preceding the Battle of Jutland, the British battle cruiser *Lion* moved as follows (in nautical miles): 1.2 miles north, 6.1 miles 38 degrees east of south, 4.0 miles at 89 degrees east of north, and 6.5 miles at 31 degrees east of north. Find the distance between starting and ending positions. [Ohanian]

2.13 Find k so that these two vectors are perpendicular.

$$\begin{pmatrix}k\\1\end{pmatrix}\qquad\begin{pmatrix}4\\3\end{pmatrix}$$

2.14 Describe the set of vectors in $\mathbb{R}^3$ orthogonal to this one.

$$\begin{pmatrix}1\\3\\-1\end{pmatrix}$$

✓ **2.15** **(a)** Find the angle between the diagonal of the unit square in $\mathbb{R}^2$ and one of the axes.

(b) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the axes.

(c) Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

(d) What is the limit, as n goes to ∞, of the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes?

2.16 Is any vector perpendicular to itself?

✓ **2.17** Describe the algebraic properties of dot product.

(a) Is it right-distributive over addition: $(\vec{u}+\vec{v})\cdot\vec{w}=\vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$?

(b) Is it left-distributive (over addition)?

(c) Does it commute?

(d) Associate?

(e) How does it interact with scalar multiplication?

As always, any assertion must be backed by either a proof or an example.

2.18 Verify the equality condition in Corollary 2.6, the Cauchy-Schwartz Inequality.

(a) Show that if $\vec{u}$ is a negative scalar multiple of $\vec{v}$ then $\vec{u}\cdot\vec{v}$ and $\vec{v}\cdot\vec{u}$ are less than or equal to zero.

(b) Show that $|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\,\|\vec{v}\|$ if and only if one vector is a scalar multiple of the other.

2.19 Suppose that $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$ and $\vec{u} \neq \vec{0}$. Must $\vec{v} = \vec{w}$?

✓ **2.20** Does any vector have length zero except a zero vector? (If "yes", produce an example. If "no", prove it.)

✓ **2.21** Find the midpoint of the line segment connecting (x_1, y_1) with (x_2, y_2) in $\mathbb{R}^2$. Generalize to $\mathbb{R}^n$.

2.22 Show that if $\vec{v} \neq \vec{0}$ then $\vec{v}/\|\vec{v}\|$ has length one. What if $\vec{v} = \vec{0}$?

2.23 Show that if $r \geq 0$ then $r\vec{v}$ is r times as long as $\vec{v}$. What if $r < 0$?

✓ **2.24** A vector $\vec{v} \in \mathbb{R}^n$ of length one is a *unit* vector. Show that the dot product of two unit vectors has absolute value less than or equal to one. Can 'less than' happen? Can 'equal to'?

2.25 Prove that $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$.

2.26 Show that if $\vec{x}\cdot\vec{y} = 0$ for every $\vec{y}$ then $\vec{x} = \vec{0}$.

2.27 Is $\|\vec{u}_1 + \cdots + \vec{u}_n\| \leq \|\vec{u}_1\| + \cdots + \|\vec{u}_n\|$? If it is true then it would generalize the Triangle Inequality.

2.28 What is the ratio between the sides in the Cauchy-Schwartz inequality?

2.29 Why is the zero vector defined to be perpendicular to every vector?

2.30 Describe the angle between two vectors in $\mathbb{R}^1$.

2.31 Give a simple necessary and sufficient condition to determine whether the angle between two vectors is acute, right, or obtuse.

✓ **2.32** Generalize to $\mathbb{R}^n$ the converse of the Pythagorean Theorem, that if $\vec{u}$ and $\vec{v}$ are perpendicular then $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

2.33 Show that $\|\vec{u}\| = \|\vec{v}\|$ if and only if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular. Give an example in $\mathbb{R}^2$.

2.34 Show that if a vector is perpendicular to each of two others then it is perpendicular to each vector in the plane they generate. (*Remark.* They could generate a degenerate plane — a line or a point — but the statement remains true.)

2.35 Prove that, where $\vec{u}, \vec{v} \in \mathbb{R}^n$ are nonzero vectors, the vector

$$\frac{\vec{u}}{\|\vec{u}\|} + \frac{\vec{v}}{\|\vec{v}\|}$$

bisects the angle between them. Illustrate in $\mathbb{R}^2$.

2.36 Verify that the definition of angle is dimensionally correct: (1) if $k > 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ equals the cosine of the angle between $\vec{u}$ and $\vec{v}$, and (2) if $k < 0$ then the cosine of the angle between $k\vec{u}$ and $\vec{v}$ is the negative of the cosine of the angle between $\vec{u}$ and $\vec{v}$.

✓ **2.37** Show that the inner product operation is *linear*: for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $k, m \in \mathbb{R}$, $\vec{u}\cdot(k\vec{v} + m\vec{w}) = k(\vec{u}\cdot\vec{v}) + m(\vec{u}\cdot\vec{w})$.

✓ **2.38** The *geometric mean* of two positive reals x, y is $\sqrt{xy}$. It is analogous to the *arithmetic mean* $(x+y)/2$. Use the Cauchy-Schwartz inequality to show that the geometric mean of any $x, y \in \mathbb{R}$ is less than or equal to the arithmetic mean.

? **2.39** A ship is sailing with speed and direction $\vec{v}_1$; the wind blows apparently (judging by the vane on the mast) in the direction of a vector $\vec{a}$; on changing the direction and speed of the ship from $\vec{v}_1$ to $\vec{v}_2$ the apparent wind is in the direction of a vector $\vec{b}$.

Find the vector velocity of the wind. [Am. Math. Mon., Feb. 1933]

2.40 Verify the Cauchy-Schwartz inequality by first proving Lagrange's identity:

$$\left(\sum_{1\leq j\leq n} a_j b_j\right)^2 = \left(\sum_{1\leq j\leq n} a_j^2\right)\left(\sum_{1\leq j\leq n} b_j^2\right) - \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2$$

and then noting that the final term is positive. (Recall the meaning

$$\sum_{1\leq j\leq n} a_j b_j = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

and

$$\sum_{1\leq j\leq n} {a_j}^2 = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2$$

of the Σ notation.) This result is an improvement over Cauchy-Schwartz because it gives a formula for the difference between the two sides. Interpret that difference in $\mathbb{R}^2$.

III Reduced Echelon Form

After developing the mechanics of Gauss' method, we observed that it can be done in more than one way. One example is that we sometimes have to swap rows and there can be more than one row to choose from. Another example is that from this matrix

$$\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$$

Gauss' method could derive any of these echelon form matrices.

$$\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

The first results from $-2\rho_1+\rho_2$. The second comes from following $(1/2)\rho_1$ with $-4\rho_1+\rho_2$. The third comes from $-2\rho_1+\rho_2$ followed by $2\rho_2+\rho_1$ (after the first row combination the matrix is already in echelon form so the second one is extra work but it is nonetheless a legal row operation).

The fact that the echelon form outcome of Gauss' method is not unique leaves us with some questions. Will any two echelon form versions of a system have the same number of free variables? Will they in fact have exactly the same variables free? In this section we will answer both questions "yes". We will do more than answer the questions. We will give a way to decide if one linear system can be derived from another by row operations. The answers to the two questions will follow from this larger result.

III.1 Gauss-Jordan Reduction

Gaussian elimination coupled with back-substitution solves linear systems, but it's not the only method possible. Here is an extension of Gauss' method that has some advantages.

1.1 Example To solve

$$\begin{aligned} x + y - 2z &= -2 \\ y + 3z &= 7 \\ x \phantom{{}+y} - z &= -1 \end{aligned}$$

we can start by going to echelon form as usual.

$$\xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right)$$

We can keep going to a second stage by making the leading entries into ones

$$\xrightarrow{(1/4)\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

and then to a third stage that uses the leading entries to eliminate all of the other entries in each column by combining upwards.

$$\xrightarrow[2\rho_3+\rho_1]{-3\rho_3+\rho_2}\left(\begin{array}{ccc|c}1&1&0&2\\0&1&0&1\\0&0&1&2\end{array}\right)\xrightarrow{-\rho_2+\rho_1}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&2\end{array}\right)$$

The answer is $x=1$, $y=1$, and $z=2$.

Note that the row combination operations in the first stage proceed from column one to column three while the combination operations in the third stage proceed from column three to column one.

1.2 Example We often combine the operations of the middle stage into a single step, even though they are operations on different rows.

$$\begin{aligned}\left(\begin{array}{cc|c}2&1&7\\4&-2&6\end{array}\right)&\xrightarrow{-2\rho_1+\rho_2}\left(\begin{array}{cc|c}2&1&7\\0&-4&-8\end{array}\right)\\&\xrightarrow[(-1/4)\rho_2]{(1/2)\rho_1}\left(\begin{array}{cc|c}1&1/2&7/2\\0&1&2\end{array}\right)\\&\xrightarrow{-(1/2)\rho_2+\rho_1}\left(\begin{array}{cc|c}1&0&5/2\\0&1&2\end{array}\right)\end{aligned}$$

The answer is $x=5/2$ and $y=2$.

This extension of Gauss' method is *Gauss-Jordan reduction*. It goes past echelon form to a more refined, more specialized, matrix form.

1.3 Definition A matrix is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a one and is the only nonzero entry in its column.

The disadvantage of using Gauss-Jordan reduction to solve a system is that the additional row operations mean additional arithmetic. The advantage is that the solution set can just be read off.

In any echelon form, plain or reduced, we can read off when a system has an empty solution set because there is a contradictory equation, we can read off when a system has a one-element solution set because there is no contradiction and every variable is the leading variable in some row, and we can read off when a system has an infinite solution set because there is no contradiction and at least one variable is free.

In reduced echelon form we can read off not just what kind of solution set the system has, but also its description. Whether or not the echelon form is reduced, we have no trouble describing the solution set when it is empty, of course. The two examples above show that when the system has a single solution then the solution can be read off from the right-hand column. In the case when the solution set is infinite, its parametrization can also be read off of the reduced echelon form. Consider, for example, this system that is shown brought to echelon form and then to reduced echelon form.

$$\left(\begin{array}{cccc|c}2&6&1&2&5\\0&3&1&4&1\\0&3&1&2&5\end{array}\right)\xrightarrow{-\rho_2+\rho_3}\left(\begin{array}{cccc|c}2&6&1&2&5\\0&3&1&4&1\\0&0&0&-2&4\end{array}\right)$$

$$\mathop{\longrightarrow}\limits^{(1/2)\rho_1}_{\substack{(1/3)\rho_2\\-(1/2)\rho_3}}\ \mathop{\longrightarrow}\limits^{(4/3)\rho_3+\rho_2}_{-\rho_3+\rho_1}\ \xrightarrow{-3\rho_2+\rho_1}\left(\begin{array}{cccc|c}1&0&-1/2&0&-9/2\\0&1&1/3&0&3\\0&0&0&1&-2\end{array}\right)$$

Starting with the middle matrix, the echelon form version, back substitution produces $-2x_4 = 4$ so that $x_4 = -2$, then another back substitution gives $3x_2 + x_3 + 4(-2) = 1$ implying that $x_2 = 3 - (1/3)x_3$, and then the final back substitution gives $2x_1 + 6(3 - (1/3)x_3) + x_3 + 2(-2) = 5$ implying that $x_1 = -(9/2) + (1/2)x_3$. Thus the solution set is this.

$$S = \{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-9/2\\3\\0\\-2\end{pmatrix} + \begin{pmatrix}1/2\\-1/3\\1\\0\end{pmatrix}x_3 \bigm| x_3 \in \mathbb{R}\}$$

Now, considering the final matrix, the reduced echelon form version, note that adjusting the parametrization by moving the x_3 terms to the other side does indeed give the description of this infinite solution set.

Part of the reason that this works is straightforward. While a set can have many parametrizations that describe it, e.g., both of these also describe the above set S (take t to be $x_3/6$ and s to be $x_3 - 1$)

$$\{\begin{pmatrix}-9/2\\3\\0\\-2\end{pmatrix} + \begin{pmatrix}3\\-2\\6\\0\end{pmatrix}t \bigm| t \in \mathbb{R}\} \qquad \{\begin{pmatrix}-4\\8/3\\1\\-2\end{pmatrix} + \begin{pmatrix}1/2\\-1/3\\1\\0\end{pmatrix}s \bigm| s \in \mathbb{R}\}$$

nonetheless we have in this book stuck to a convention of parametrizing using the unmodified free variables (that is, $x_3 = x_3$ instead of $x_3 = 6t$). We can easily see that a reduced echelon form version of a system is equivalent to a parametrization in terms of unmodified free variables. For instance,

$$\begin{aligned}x_1 &= 4 - 2x_3\\ x_2 &= 3 - x_3\end{aligned} \quad\Longleftrightarrow\quad \left(\begin{array}{ccc|c}1&0&2&4\\0&1&1&3\\0&0&0&0\end{array}\right)$$

(to move from left to right we also need to know how many equations are in the system). So, the convention of parametrizing with the free variables by solving each equation for its leading variable and then eliminating that leading variable from every other equation is exactly equivalent to the reduced echelon form conditions that each leading entry must be a one and must be the only nonzero entry in its column.

Not as straightforward is the other part of the reason that the reduced echelon form version allows us to read off the parametrization that we would have gotten had we stopped at echelon form and then done back substitution. The prior paragraph shows that reduced echelon form corresponds to some parametrization, but why the same parametrization? A solution set can be parametrized in many ways, and Gauss' method or the Gauss-Jordan method can be done in many ways, so a first guess might be that we could derive many different reduced echelon form versions of the same starting system and many different parametrizations. But we never do. Experience shows that starting with the same system and proceeding with row operations in many different ways always yields the same reduced echelon form and the same parametrization (using the unmodified free variables).

In the rest of this section we will show that the reduced echelon form version of a matrix is unique. It follows that the parametrization of a linear system in terms of its unmodified free variables is unique because two different ones would give two different reduced echelon forms.

We shall use this result, and the ones that lead up to it, in the rest of the book but perhaps a restatement in a way that makes it seem more immediately useful may be encouraging. Imagine that we solve a linear system, parametrize, and check in the back of the book for the answer. But the parametrization there appears different. Have we made a mistake, or could these be different-looking descriptions of the same set, as with the three descriptions above of S? The prior paragraph notes that we will show here that different-looking parametrizations (using the unmodified free variables) describe genuinely different sets.

Here is an informal argument that the reduced echelon form version of a matrix is unique. Consider again the example that started this section of a matrix that reduces to three different echelon form matrices. The first matrix of the three is the natural echelon form version. The second matrix is the same as the first except that a row has been halved. The third matrix, too, is just a cosmetic variant of the first. The definition of reduced echelon form outlaws this kind of fooling around. In reduced echelon form, halving a row is not possible because that would change the row's leading entry away from one, and neither is combining rows possible, because then a leading entry would no longer be alone in its column.

This informal justification is not a proof; the argument shows that no two different reduced echelon form matrices are related by a single row operation step, but the argument does not ruled out the possibility that two different reduced echelon form matrices could be related by multiple steps. Before we go to the proof, we finish this subsection by rephrasing our work in a terminology that will be enlightening.

Many different matrices yield the same reduced echelon form matrix. The three echelon form matrices from the start of this section, and the matrix they were derived from, all give this reduced echelon form matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We think of these matrices as related to each other. The next result speaks to this relationship.

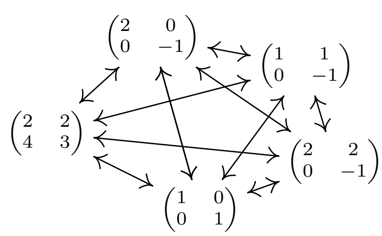
1.4 Lemma Elementary row operations are reversible.

PROOF. For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i+\rho_j} \xrightarrow{-k\rho_i+\rho_j} A$$

(The $i \neq j$ conditions is needed. See Exercise 13.) QED

This lemma suggests that 'reduces to' is misleading — where $A \longrightarrow B$, we shouldn't think of B as "after" A or "simpler than" A. Instead we should think of them as interreducible or interrelated. Below is a picture of the idea. The matrices from the start of this section and their reduced echelon form version are shown in a cluster. They are all interreducible; these relationships are shown also.



We say that matrices that reduce to each other are 'equivalent with respect to the relationship of row reducibility'. The next result verifies this statement using the definition of an equivalence.*

1.5 Lemma Between matrices, 'reduces to' is an equivalence relation.

PROOF. We must check the conditions (i) reflexivity, that any matrix reduces to itself, (ii) symmetry, that if A reduces to B then B reduces to A, and (iii) transitivity, that if A reduces to B and B reduces to C then A reduces to C.

Reflexivity is easy; any matrix reduces to itself in zero row operations.

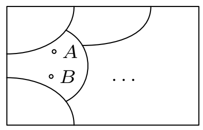
That the relationship is symmetric is Lemma 1.4 — if A reduces to B by some row operations then also B reduces to A by reversing those operations.

For transitivity, suppose that A reduces to B and that B reduces to C. Linking the reduction steps from $A \to \cdots \to B$ with those from $B \to \cdots \to C$ gives a reduction from A to C. QED

1.6 Definition Two matrices that are interreducible by the elementary row operations are *row equivalent*.

* More information on equivalence relations is in the appendix.

The diagram below shows the collection of all matrices as a box. Inside that box, each matrix lies in some class. Matrices are in the same class if and only if they are interreducible. The classes are disjoint — no matrix is in two distinct classes. The collection of matrices has been partitioned into *row equivalence classes.*[*]



One of the classes in this partition is the cluster of matrices shown above, expanded to include all of the nonsingular 2×2 matrices.

The next subsection proves that the reduced echelon form of a matrix is unique; that every matrix reduces to one and only one reduced echelon form matrix. Rephrased in terms of the row-equivalence relationship, we shall prove that every matrix is row equivalent to one and only one reduced echelon form matrix. In terms of the partition what we shall prove is: every equivalence class contains one and only one reduced echelon form matrix. So each reduced echelon form matrix serves as a representative of its class.

After that proof we shall, as mentioned in the introduction to this section, have a way to decide if one matrix can be derived from another by row reduction. We just apply the Gauss-Jordan procedure to both and see whether or not they come to the same reduced echelon form.

Exercises

✓ **1.7** Use Gauss-Jordan reduction to solve each system.

(a) $\begin{aligned} x+y&=2\\ x-y&=0 \end{aligned}$ **(b)** $\begin{aligned} x\phantom{{}+2y}-z&=4\\ 2x+2y\phantom{{}-z}&=1 \end{aligned}$ **(c)** $\begin{aligned} 3x-2y&=1\\ 6x+y&=1/2 \end{aligned}$

(d) $\begin{aligned} 2x-y\phantom{{}-z}&=-1\\ x+3y-z&=5\\ y+2z&=5 \end{aligned}$

✓ **1.8** Find the reduced echelon form of each matrix.

(a) $\begin{pmatrix} 2&1\\ 1&3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1&3&1\\ 2&0&4\\ -1&-3&-3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1&0&3&1&2\\ 1&4&2&1&5\\ 3&4&8&1&2 \end{pmatrix}$

(d) $\begin{pmatrix} 0&1&3&2\\ 0&0&5&6\\ 1&5&1&5 \end{pmatrix}$

✓ **1.9** Find each solution set by using Gauss-Jordan reduction, then reading off the parametrization.

(a) $\begin{aligned} 2x+y-z&=1\\ 4x-y\phantom{{}-z}&=3 \end{aligned}$ **(b)** $\begin{aligned} x\phantom{{}+2y}-z\phantom{{}-w}&=1\\ y+2z-w&=3\\ x+2y+3z-w&=7 \end{aligned}$ **(c)** $\begin{aligned} x-y+z\phantom{{}+w}&=0\\ y\phantom{{}+3z}+w&=0\\ 3x-2y+3z+w&=0\\ -y\phantom{{}+3z}-w&=0 \end{aligned}$

(d) $\begin{aligned} a+2b+3c+d-e&=1\\ 3a-b+c+d+e&=3 \end{aligned}$

[*] More information on partitions and class representatives is in the appendix.

1.10 Give two distinct echelon form versions of this matrix.

$$\begin{pmatrix} 2 & 1 & 1 & 3 \\ 6 & 4 & 1 & 2 \\ 1 & 5 & 1 & 5 \end{pmatrix}$$

✓ **1.11** List the reduced echelon forms possible for each size.
(a) 2×2 **(b)** 2×3 **(c)** 3×2 **(d)** 3×3

✓ **1.12** What results from applying Gauss-Jordan reduction to a nonsingular matrix?

1.13 The proof of Lemma 1.4 contains a reference to the $i \neq j$ condition on the row combination operation.

(a) The definition of row operations has an $i \neq j$ condition on the swap operation $\rho_i \leftrightarrow \rho_j$. Show that in $A \stackrel{\rho_i \leftrightarrow \rho_j}{\longrightarrow} \stackrel{\rho_i \leftrightarrow \rho_j}{\longrightarrow} A$ this condition is not needed.

(b) Write down a 2×2 matrix with nonzero entries, and show that the $-1\cdot\rho_1+\rho_1$ operation is not reversed by $1\cdot\rho_1+\rho_1$.

(c) Expand the proof of that lemma to make explicit exactly where the $i \neq j$ condition on combining is used.

III.2 Row Equivalence

We will close this section and this chapter by proving that every matrix is row equivalent to one and only one reduced echelon form matrix. The ideas that appear here will reappear, and be further developed, in the next chapter.

The underlying theme here is that one way to understand a mathematical situation is by being able to classify the cases that can happen. We have met this theme several times already. We have classified solution sets of linear systems into the no-elements, one-element, and infinitely-many elements cases. We have also classified linear systems with the same number of equations as unknowns into the nonsingular and singular cases. We adopted these classifications because they give us a way to understand the situations that we were investigating. Here, where we are investigating row equivalence, we know that the set of all matrices breaks into the row equivalence classes. When we finish the proof here, we will have a way to understand each of those classes — its matrices can be thought of as derived by row operations from the unique reduced echelon form matrix in that class.

To understand how row operations act to transform one matrix into another, we consider the effect that they have on the parts of a matrix. The crucial observation is that row operations combine the rows linearly.

2.1 Lemma (Linear Combination Lemma) A linear combination of linear combinations is a linear combination.

Proof. Given the linear combinations $c_{1,1}x_1 + \cdots + c_{1,n}x_n$ through $c_{m,1}x_1 + \cdots + c_{m,n}x_n$, consider a combination of those

$$d_1(c_{1,1}x_1 + \cdots + c_{1,n}x_n) + \cdots + d_m(c_{m,1}x_1 + \cdots + c_{m,n}x_n)$$

where the d's are scalars along with the c's. Distributing those d's and regrouping gives

$$= (d_1c_{1,1} + \cdots + d_mc_{m,1})x_1 \; + \cdots + \; (d_1c_{1,n} + \cdots + d_mc_{m,n})x_n$$

which is a linear combination of the x's. QED

In this subsection we will use the convention that, where a matrix is named with an upper case roman letter, the matching lower-case greek letter names the rows.

$$A = \begin{pmatrix} \cdots \; \alpha_1 \; \cdots \\ \cdots \; \alpha_2 \; \cdots \\ \vdots \\ \cdots \; \alpha_m \; \cdots \end{pmatrix} \qquad B = \begin{pmatrix} \cdots \; \beta_1 \; \cdots \\ \cdots \; \beta_2 \; \cdots \\ \vdots \\ \cdots \; \beta_m \; \cdots \end{pmatrix}$$

2.2 Corollary Where one matrix reduces to another, each row of the second is a linear combination of the rows of the first.

The proof below uses induction on the number of row operations used to reduce one matrix to the other. Before we proceed, here is an outline of the argument (readers unfamiliar with induction may want to compare this argument with the one used in the 'General = Particular + Homogeneous' proof).*

First, for the base step of the argument, we will verify that the proposition is true when reduction can be done in zero row operations. Second, for the inductive step, we will argue that if being able to reduce the first matrix to the second in some number $t \geq 0$ of operations implies that each row of the second is a linear combination of the rows of the first, then being able to reduce the first to the second in $t+1$ operations implies the same thing.

Together, this base step and induction step prove the result because by the inductive step the fact that it is true in the zero operations case (that's shown in the base step) implies that it is true in the one operation case, and then the inductive step applied again gives that it is therefore true in the two operations case, etc.

Proof. We proceed by induction on the minimum number of row operations that take a first matrix A to a second one B.

In the base step, that zero reduction operations suffice, the two matrices are equal and each row of B is obviously a combination of A's rows: $\vec{\beta}_i = 0 \cdot \vec{\alpha}_1 + \cdots + 1 \cdot \vec{\alpha}_i + \cdots + 0 \cdot \vec{\alpha}_m$.

For the inductive step, assume the inductive hypothesis: with $t \geq 0$, if a matrix can be derived from A in t or fewer operations then its rows are linear combinations of the A's rows. Consider a B that takes $t+1$ operations. Because there are more than zero operations, there must be a next-to-last matrix G so that $A \longrightarrow \cdots \longrightarrow G \longrightarrow B$. This G is only t operations away from A and so the

* More information on mathematical induction is in the appendix.

inductive hypothesis applies to it, that is, each row of G is a linear combination of the rows of A.

If the last operation, the one from G to B, is a row swap then the rows of B are just the rows of G reordered and thus each row of B is also a linear combination of the rows of A. The other two possibilities for this last operation, that it multiplies a row by a scalar and that it adds a multiple of one row to another, both result in the rows of B being linear combinations of the rows of G. But therefore, by the Linear Combination Lemma, each row of B is a linear combination of the rows of A.

With that, we have both the base step and the inductive step, and so the proposition follows. QED

2.3 Example In the reduction

$$\begin{pmatrix}0&2\\1&1\end{pmatrix}\xrightarrow{\rho_1\leftrightarrow\rho_2}\begin{pmatrix}1&1\\0&2\end{pmatrix}\xrightarrow{(1/2)\rho_2}\begin{pmatrix}1&1\\0&1\end{pmatrix}\xrightarrow{-\rho_2+\rho_1}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

call the matrices A, D, G, and B. The methods of the proof show that there are three sets of linear relationships.

$$\begin{array}{lll}\delta_1=0\cdot\alpha_1+1\cdot\alpha_2 & \gamma_1=0\cdot\alpha_1+1\cdot\alpha_2 & \beta_1=(-1/2)\alpha_1+1\cdot\alpha_2\\ \delta_2=1\cdot\alpha_1+0\cdot\alpha_2 & \gamma_2=(1/2)\alpha_1+0\cdot\alpha_2 & \beta_2=(1/2)\alpha_1+0\cdot\alpha_2\end{array}$$

The prior result gives us the insight that Gauss' method works by taking linear combinations of the rows. But to what end; why do we go to echelon form as a particularly simple, or basic, version of a linear system? The answer, of course, is that echelon form is suitable for back substitution, because we have isolated the variables. For instance, in this matrix

$$R=\begin{pmatrix}2&3&7&8&0&0\\0&0&1&5&1&1\\0&0&0&3&3&0\\0&0&0&0&2&1\end{pmatrix}$$

x_1 has been removed from x_5's equation. That is, Gauss' method has made x_5's row independent of x_1's row.

Independence of a collection of row vectors, or of any kind of vectors, will be precisely defined and explored in the next chapter. But a first take on it is that we can show that, say, the third row above is not comprised of the other rows, that $\rho_3\neq c_1\rho_1+c_2\rho_2+c_4\rho_4$. For, suppose that there are scalars c_1, c_2, and c_4 such that this relationship holds.

$$\begin{aligned}\begin{pmatrix}0&0&0&3&3&0\end{pmatrix}&=c_1\begin{pmatrix}2&3&7&8&0&0\end{pmatrix}\\&\quad+c_2\begin{pmatrix}0&0&1&5&1&1\end{pmatrix}\\&\quad+c_4\begin{pmatrix}0&0&0&0&2&1\end{pmatrix}\end{aligned}$$

The first row's leading entry is in the first column and narrowing our consideration of the above relationship to consideration only of the entries from the first column $0 = 2c_1 + 0c_2 + 0c_4$ gives that $c_1 = 0$. The second row's leading entry is in the third column and the equation of entries in that column $0 = 7c_1 + 1c_2 + 0c_4$, along with the knowledge that $c_1 = 0$, gives that $c_2 = 0$. Now, to finish, the third row's leading entry is in the fourth column and the equation of entries in that column $3 = 8c_1 + 5c_2 + 0c_4$, along with $c_1 = 0$ and $c_2 = 0$, gives an impossibility.

The following result shows that this effect always holds. It shows that what Gauss' linear elimination method eliminates is linear relationships among the rows.

2.4 Lemma In an echelon form matrix, no nonzero row is a linear combination of the other rows.

Proof. Let R be in echelon form. Suppose, to obtain a contradiction, that some nonzero row is a linear combination of the others.

$$\rho_i = c_1\rho_1 + \ldots + c_{i-1}\rho_{i-1} + c_{i+1}\rho_{i+1} + \ldots + c_m\rho_m$$

We will first use induction to show that the coefficients $c_1, \ldots, c_{i-1}$ associated with rows above ρ_i are all zero. The contradiction will come from consideration of ρ_i and the rows below it.

The base step of the induction argument is to show that the first coefficient c_1 is zero. Let the first row's leading entry be in column number ℓ_1 and consider the equation of entries in that column.

$$\rho_{i,\ell_1} = c_1\rho_{1,\ell_1} + \ldots + c_{i-1}\rho_{i-1,\ell_1} + c_{i+1}\rho_{i+1,\ell_1} + \ldots + c_m\rho_{m,\ell_1}$$

The matrix is in echelon form so the entries $\rho_{2,\ell_1}, \ldots, \rho_{m,\ell_1}$, including ρ_{i,ℓ_1}, are all zero.

$$0 = c_1\rho_{1,\ell_1} + \cdots + c_{i-1}\cdot 0 + c_{i+1}\cdot 0 + \cdots + c_m\cdot 0$$

Because the entry ρ_{1,ℓ_1} is nonzero as it leads its row, the coefficient c_1 must be zero.

The inductive step is to show that for each row index k between 1 and $i-2$, if the coefficient c_1 and the coefficients $c_2, \ldots, c_k$ are all zero then c_{k+1} is also zero. That argument, and the contradiction that finishes this proof, is saved for Exercise 20. QED

We can now prove that each matrix is row equivalent to one and only one reduced echelon form matrix. We will find it convenient to break the first half of the argument off as a preliminary lemma. For one thing, it holds for any echelon form whatever, not just reduced echelon form.

2.5 Lemma If two echelon form matrices are row equivalent then the leading entries in their first rows lie in the same column. The same is true of all the nonzero rows — the leading entries in their second rows lie in the same column, etc.

For the proof we rephrase the result in more technical terms. Define the *form* of an $m\times n$ matrix to be the sequence $\langle \ell_1, \ell_2, \ldots, \ell_m \rangle$ where ℓ_i is the column number of the leading entry in row i and $\ell_i = \infty$ if there is no leading entry in that row. The lemma says that if two echelon form matrices are row equivalent then their forms are equal sequences.

Proof. Let B and D be echelon form matrices that are row equivalent. Because they are row equivalent they must be the same size, say $m\times n$. Let the column number of the leading entry in row i of B be ℓ_i and let the column number of the leading entry in row j of D be k_j. We will show that $\ell_1 = k_1$, that $\ell_2 = k_2$, etc., by induction.

This induction argument relies on the fact that the matrices are row equivalent, because the Linear Combination Lemma and its corollary therefore give that each row of B is a linear combination of the rows of D and vice versa:

$$\beta_i = s_{i,1}\delta_1 + s_{i,2}\delta_2 + \cdots + s_{i,m}\delta_m \quad \text{and} \quad \delta_j = t_{j,1}\beta_1 + t_{j,2}\beta_2 + \cdots + t_{j,m}\beta_m$$

where the s's and t's are scalars.

The base step of the induction is to verify the lemma for the first rows of the matrices, that is, to verify that $\ell_1 = k_1$. If either row is a zero row then every entry in the matrix is a zero since it is in echelon form, and therefore both matrices consist solely of zero entries (by Corollary 2.2), and so both ℓ_1 and k_1 are ∞. For the case where neither β_1 nor δ_1 is a zero row, consider the $i = 1$ instance of the linear relationship above.

$$\begin{aligned}
\beta_1 &= s_{1,1}\delta_1 + s_{1,2}\delta_2 + \cdots + s_{1,m}\delta_m \\
\begin{pmatrix} 0 & \cdots & b_{1,\ell_1} & \cdots \end{pmatrix} &= s_{1,1}\begin{pmatrix} 0 & \cdots & d_{1,k_1} & \cdots \end{pmatrix} \\
&\quad + s_{1,2}\begin{pmatrix} 0 & \cdots & 0 & \cdots \end{pmatrix} \\
&\quad\;\vdots \\
&\quad + s_{1,m}\begin{pmatrix} 0 & \cdots & 0 & \cdots \end{pmatrix}
\end{aligned}$$

First, note that $\ell_1 < k_1$ is impossible: in the columns of D to the left of column k_1 the entries are all zeroes (as d_{1,k_1} leads the first row) and so if $\ell_1 < k_1$ then the equation of entries from column ℓ_1 would be $b_{1,\ell_1} = s_{1,1}\cdot 0 + \cdots + s_{1,m}\cdot 0$, but b_{1,ℓ_1} isn't zero since it leads its row and so this is an impossibility. Next, a symmetric argument shows that $k_1 < \ell_1$ also is impossible. Thus the $\ell_1 = k_1$ base case holds.

The inductive step is to show that if $\ell_1 = k_1$, and $\ell_2 = k_2$, ..., and $\ell_r = k_r$, then also $\ell_{r+1} = k_{r+1}$ (for r in the interval $1\,..\,m-1$). This argument is saved for Exercise 21. QED

That lemma answers two of the questions that we have posed: (i) any two echelon form versions of a matrix have the same free variables, and consequently, and (ii) any two echelon form versions have the same number of free variables. There is no linear system and no combination of row operations such that, say, we could solve the system one way and get y and z free but solve it another way and get y and w free, or solve it one way and get two free variables while solving it another way yields three.

We finish now by specializing to the case of reduced echelon form matrices.

2.6 Theorem Each matrix is row equivalent to a unique reduced echelon form matrix.

Proof. Clearly any matrix is row equivalent to at least one reduced echelon form matrix, via Gauss-Jordan reduction. For the other half, that any matrix is equivalent to at most one reduced echelon form matrix, we will show that if a matrix Gauss-Jordan reduces to each of two others then those two are equal.

Suppose that a matrix is row equivalent to two reduced echelon form matrices B and D, which are therefore row equivalent to each other. The Linear Combination Lemma and its corollary allow us to write the rows of one, say B, as a linear combination of the rows of the other $\beta_i = c_{i,1}\delta_1 + \cdots + c_{i,m}\delta_m$. The preliminary result, Lemma 2.5, says that in the two matrices, the same collection of rows are nonzero. Thus, if β_1 through β_r are the nonzero rows of B then the nonzero rows of D are δ_1 through δ_r. Zero rows don't contribute to the sum so we can rewrite the relationship to include just the nonzero rows.

$$\beta_i = c_{i,1}\delta_1 + \cdots + c_{i,r}\delta_r \tag{$*$}$$

The preliminary result also says that for each row j between 1 and r, the leading entries of the j-th row of B and D appear in the same column, denoted ℓ_j. Rewriting the above relationship to focus on the entries in the ℓ_j-th column

$$\begin{aligned}
\begin{pmatrix} \cdots & b_{i,\ell_j} & \cdots \end{pmatrix} = c_{i,1} & \begin{pmatrix} \cdots & d_{1,\ell_j} & \cdots \end{pmatrix} \\
+ c_{i,2} & \begin{pmatrix} \cdots & d_{2,\ell_j} & \cdots \end{pmatrix} \\
& \vdots \\
+ c_{i,r} & \begin{pmatrix} \cdots & d_{r,\ell_j} & \cdots \end{pmatrix}
\end{aligned}$$

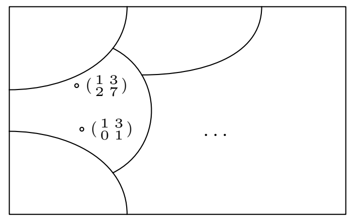
gives this set of equations for $i = 1$ up to $i = r$.

$$\begin{aligned}
b_{1,\ell_j} &= c_{1,1}d_{1,\ell_j} + \cdots + c_{1,j}d_{j,\ell_j} + \cdots + c_{1,r}d_{r,\ell_j} \\
&\vdots \\
b_{j,\ell_j} &= c_{j,1}d_{1,\ell_j} + \cdots + c_{j,j}d_{j,\ell_j} + \cdots + c_{j,r}d_{r,\ell_j} \\
&\vdots \\
b_{r,\ell_j} &= c_{r,1}d_{1,\ell_j} + \cdots + c_{r,j}d_{j,\ell_j} + \cdots + c_{r,r}d_{r,\ell_j}
\end{aligned}$$

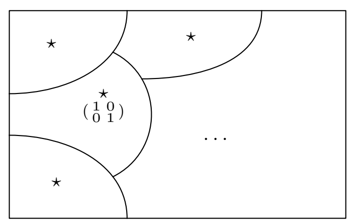
Since D is in reduced echelon form, all of the d's in column ℓ_j are zero except for d_{j,ℓ_j}, which is 1. Thus each equation above simplifies to $b_{i,\ell_j} = c_{i,j}d_{j,\ell_j} = c_{i,j} \cdot 1$. But B is also in reduced echelon form and so all of the b's in column ℓ_j are zero except for b_{j,ℓ_j}, which is 1. Therefore, each $c_{i,j}$ is zero, except that $c_{1,1} = 1$, and $c_{2,2} = 1$, ..., and $c_{r,r} = 1$.

We have shown that the only nonzero coefficient in the linear combination labelled $(*)$ is $c_{j,j}$, which is 1. Therefore $\beta_j = \delta_j$. Because this holds for all nonzero rows, $B = D$. QED

We end with a recap. In Gauss' method we start with a matrix and then derive a sequence of other matrices. We defined two matrices to be related if one can be derived from the other. That relation is an equivalence relation, called row equivalence, and so partitions the set of all matrices into row equivalence classes.



(There are infinitely many matrices in the pictured class, but we've only got room to show two.) We have proved there is one and only one reduced echelon form matrix in each row equivalence class. So the reduced echelon form is a *canonical form** for row equivalence: the reduced echelon form matrices are representatives of the classes.



We can answer questions about the classes by translating them into questions about the representatives.

2.7 Example We can decide if matrices are interreducible by seeing if Gauss-Jordan reduction produces the same reduced echelon form result. Thus, these are not row equivalent

$$\begin{pmatrix}1&-3\\-2&6\end{pmatrix}\qquad\begin{pmatrix}1&-3\\-2&5\end{pmatrix}$$

because their reduced echelon forms are not equal.

$$\begin{pmatrix}1&-3\\0&0\end{pmatrix}\qquad\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

2.8 Example Any nonsingular 3×3 matrix Gauss-Jordan reduces to this.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

* More information on canonical representatives is in the appendix.

2.9 Example We can describe the classes by listing all possible reduced echelon form matrices. Any 2×2 matrix lies in one of these: the class of matrices row equivalent to this,

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

the infinitely many classes of matrices row equivalent to one of this type

$$\begin{pmatrix}1&a\\0&0\end{pmatrix}$$

where $a\in\mathbb{R}$ (including $a=0$), the class of matrices row equivalent to this,

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

and the class of matrices row equivalent to this

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

(this is the class of nonsingular 2×2 matrices).

Exercises

✓ **2.10** Decide if the matrices are row equivalent.

(a) $\begin{pmatrix}1&2\\4&8\end{pmatrix},\begin{pmatrix}0&1\\1&2\end{pmatrix}$ **(b)** $\begin{pmatrix}1&0&2\\3&-1&1\\5&-1&5\end{pmatrix},\begin{pmatrix}1&0&2\\0&2&10\\2&0&4\end{pmatrix}$

(c) $\begin{pmatrix}2&1&-1\\1&1&0\\4&3&-1\end{pmatrix},\begin{pmatrix}1&0&2\\0&2&10\end{pmatrix}$ **(d)** $\begin{pmatrix}1&1&1\\-1&2&2\end{pmatrix},\begin{pmatrix}0&3&-1\\2&2&5\end{pmatrix}$

(e) $\begin{pmatrix}1&1&1\\0&0&3\end{pmatrix},\begin{pmatrix}0&1&2\\1&-1&1\end{pmatrix}$

2.11 Describe the matrices in each of the classes represented in Example 2.9.

2.12 Describe all matrices in the row equivalence class of these.

(a) $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ **(b)** $\begin{pmatrix}1&2\\2&4\end{pmatrix}$ **(c)** $\begin{pmatrix}1&1\\1&3\end{pmatrix}$

2.13 How many row equivalence classes are there?

2.14 Can row equivalence classes contain different-sized matrices?

2.15 How big are the row equivalence classes?

(a) Show that for any matrix of all zeros, the class is finite.

(b) Do any other classes contain only finitely many members?

✓ **2.16** Give two reduced echelon form matrices that have their leading entries in the same columns, but that are not row equivalent.

✓ **2.17** Show that any two $n\times n$ nonsingular matrices are row equivalent. Are any two singular matrices row equivalent?

✓ **2.18** Describe all of the row equivalence classes containing these.

(a) 2×2 matrices **(b)** 2×3 matrices **(c)** 3×2 matrices
(d) 3×3 matrices

2.19 **(a)** Show that a vector $\vec{\beta}_0$ is a linear combination of members of the set $\{\vec{\beta}_1, \ldots, \vec{\beta}_n\}$ if and only if there is a linear relationship $\vec{0} = c_0\vec{\beta}_0 + \cdots + c_n\vec{\beta}_n$ where c_0 is not zero. (*Hint.* Watch out for the $\vec{\beta}_0 = \vec{0}$ case.)
(b) Use that to simplify the proof of Lemma 2.4.

✓ **2.20** Finish the proof of Lemma 2.4.
(a) First illustrate the inductive step by showing that $c_2 = 0$.
(b) Do the full inductive step: where $1 \leq n < i - 1$, assume that $c_k = 0$ for $1 < k < n$ and deduce that $c_{n+1} = 0$ also.
(c) Find the contradiction.

2.21 Finish the induction argument in Lemma 2.5.
(a) State the inductive hypothesis, Also state what must be shown to follow from that hypothesis.
(b) Check that the inductive hypothesis implies that in the relationship $\beta_{r+1} = s_{r+1,1}\delta_1 + s_{r+2,2}\delta_2 + \cdots + s_{r+1,m}\delta_m$ the coefficients $s_{r+1,1}, \ldots, s_{r+1,r}$ are each zero.
(c) Finish the inductive step by arguing, as in the base case, that $\ell_{r+1} < k_{r+1}$ and $k_{r+1} < \ell_{r+1}$ are impossible.

2.22 Why, in the proof of Theorem 2.6, do we bother to restrict to the nonzero rows? Why not just stick to the relationship that we began with, $\beta_i = c_{i,1}\delta_1 + \cdots + c_{i,m}\delta_m$, with m instead of r, and argue using it that the only nonzero coefficient is $c_{i,i}$, which is 1?

✓ **2.23** Three truck drivers went into a roadside cafe. One truck driver purchased four sandwiches, a cup of coffee, and ten doughnuts for \$8.45. Another driver purchased three sandwiches, a cup of coffee, and seven doughnuts for \$6.30. What did the third truck driver pay for a sandwich, a cup of coffee, and a doughnut? [Trono]

2.24 The fact that Gaussian reduction disallows multiplication of a row by zero is needed for the proof of uniqueness of reduced echelon form, or else every matrix would be row equivalent to a matrix of all zeros. Where is it used?

✓ **2.25** The Linear Combination Lemma says which equations can be gotten from Gaussian reduction from a given linear system.

(1) Produce an equation not implied by this system.

$$\begin{aligned} 3x + 4y &= 8 \\ 2x + y &= 3 \end{aligned}$$

(2) Can any equation be derived from an inconsistent system?

2.26 Extend the definition of row equivalence to linear systems. Under your definition, do equivalent systems have the same solution set? [Hoffman & Kunze]

✓ **2.27** In this matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ 1 & 4 & 5 \end{pmatrix}$$

the first and second columns add to the third.
(a) Show that remains true under any row operation.
(b) Make a conjecture.
(c) Prove that it holds.

Topic: Computer Algebra Systems

The linear systems in this chapter are small enough that their solution by hand is easy. But large systems are easiest, and safest, to do on a computer. There are special purpose programs such as LINPACK for this job. Another popular tool is a general purpose computer algebra system, including both commercial packages such as Maple, Mathematica, or MATLAB, or free packages such as Sage.

For example, in the Topic on Networks, we need to solve this.

$$\begin{array}{rrrrrrrl}
i_0 - & i_1 - & i_2 & & & & & = 0 \\
& i_1 & & - \; i_3 & & - \; i_5 & & = 0 \\
& & i_2 & & - \; i_4 + & i_5 & & = 0 \\
& & & i_3 + & i_4 & & - \; i_6 & = 0 \\
& 5i_1 & & + 10i_3 & & & & = 10 \\
& & 2i_2 & & + 4i_4 & & & = 10 \\
& 5i_1 - & 2i_2 & & & + 50i_5 & & = 0
\end{array}$$

It can be done by hand, but it would take a while and be error-prone. Using a computer is better.

We illustrate by solving that system under Maple (for another system, a user's manual would obviously detail the exact syntax needed). The array of coefficients can be entered in this way

```
> A:=array( [[1,-1,-1,0,0,0,0],
             [0,1,0,-1,0,-1,0],
             [0,0,1,0,-1,1,0],
             [0,0,0,1,1,0,-1],
             [0,5,0,10,0,0,0],
             [0,0,2,0,4,0,0],
             [0,5,-1,0,0,10,0]] );
```

(putting the rows on separate lines is not necessary, but is done for clarity). The vector of constants is entered similarly.

```
> u:=array( [0,0,0,0,10,10,0] );
```

Then the system is solved, like magic.

```
> linsolve(A,u);
      7  2  5  2  5     7
    [ -, -, -, -, -, 0, - ]
      3  3  3  3  3     3
```

Systems with infinitely many solutions are solved in the same way — the computer simply returns a parametrization.

Exercises

Answers for this Topic use Maple as the computer algebra system. In particular, all of these were tested on Maple V running under MS-DOS NT version 4.0. *(On all of them, the preliminary command to load the linear algebra package along with Maple's responses to the Enter key, have been omitted.) Other systems have similar commands.*

1 Use the computer to solve the two problems that opened this chapter.

(a) This is the Statics problem.

$$\begin{aligned} 40h + 15c &= 100 \\ 25c &= 50 + 50h \end{aligned}$$

(b) This is the Chemistry problem.

$$\begin{aligned} 7h &= 7j \\ 8h + 1i &= 5j + 2k \\ 1i &= 3j \\ 3i &= 6j + 1k \end{aligned}$$

2 Use the computer to solve these systems from the first subsection, or conclude 'many solutions' or 'no solutions'.

(a) $\begin{aligned} 2x + 2y &= 5 \\ x - 4y &= 0 \end{aligned}$ **(b)** $\begin{aligned} -x + y &= 1 \\ x + y &= 2 \end{aligned}$ **(c)** $\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$

(d) $\begin{aligned} -x - y &= 1 \\ -3x - 3y &= 2 \end{aligned}$ **(e)** $\begin{aligned} 4y + z &= 20 \\ 2x - 2y + z &= 0 \\ x + z &= 5 \\ x + y - z &= 10 \end{aligned}$ **(f)** $\begin{aligned} 2x + z + w &= 5 \\ y - w &= -1 \\ 3x - z - w &= 0 \\ 4x + y + 2z + w &= 9 \end{aligned}$

3 Use the computer to solve these systems from the second subsection.

(a) $\begin{aligned} 3x + 6y &= 18 \\ x + 2y &= 6 \end{aligned}$ **(b)** $\begin{aligned} x + y &= 1 \\ x - y &= -1 \end{aligned}$ **(c)** $\begin{aligned} x_1 + x_3 &= 4 \\ x_1 - x_2 + 2x_3 &= 5 \\ 4x_1 - x_2 + 5x_3 &= 17 \end{aligned}$

(d) $\begin{aligned} 2a + b - c &= 2 \\ 2a + c &= 3 \\ a - b &= 0 \end{aligned}$ **(e)** $\begin{aligned} x + 2y - z &= 3 \\ 2x + y + w &= 4 \\ x - y + z + w &= 1 \end{aligned}$ **(f)** $\begin{aligned} x + z + w &= 4 \\ 2x + y - w &= 2 \\ 3x + y + z &= 7 \end{aligned}$

4 What does the computer give for the solution of the general 2×2 system?

$$\begin{aligned} ax + cy &= p \\ bx + dy &= q \end{aligned}$$

Topic: Input-Output Analysis

An economy is an immensely complicated network of interdependences. Changes in one part can ripple out to affect other parts. Economists have struggled to be able to describe, and to make predictions about, such a complicated object. Mathematical models using systems of linear equations have emerged as a key tool. One is Input-Output Analysis, pioneered by W. Leontief, who won the 1973 Nobel Prize in Economics.

Consider an economy with many parts, two of which are the steel industry and the auto industry. As they work to meet the demand for their product from other parts of the economy, that is, from users external to the steel and auto sectors, these two interact tightly. For instance, should the external demand for autos go up, that would lead to an increase in the auto industry's usage of steel. Or, should the external demand for steel fall, then it would lead to a fall in steel's purchase of autos. The type of Input-Output model we will consider takes in the external demands and then predicts how the two interact to meet those demands.

We start with a listing of production and consumption statistics. (These numbers, giving dollar values in millions, are excerpted from [Leontief 1965], describing the 1958 U.S. economy. Today's statistics would be quite different, both because of inflation and because of technical changes in the industries.)

	used by steel	*used by auto*	*used by others*	*total*
value of steel	5 395	2 664		25 448
value of auto	48	9 030		30 346

For instance, the dollar value of steel used by the auto industry in this year is 2,664 million. Note that industries may consume some of their own output.

We can fill in the blanks for the external demand. This year's value of the steel used by others this year is 17,389 and this year's value of the auto used by others is 21,268. With that, we have a complete description of the external demands and of how auto and steel interact, this year, to meet them.

Now, imagine that the external demand for steel has recently been going up by 200 per year and so we estimate that next year it will be 17,589. Imagine also that for similar reasons we estimate that next year's external demand for autos will be down 25 to 21,243. We wish to predict next year's total outputs.

That prediction isn't as simple as adding 200 to this year's steel total and subtracting 25 from this year's auto total. For one thing, a rise in steel will cause that industry to have an increased demand for autos, which will mitigate, to some extent, the loss in external demand for autos. On the other hand, the drop in external demand for autos will cause the auto industry to use less steel, and so lessen somewhat the upswing in steel's business. In short, these two industries form a system, and we need to predict the totals at which the system as a whole will settle.

For that prediction, let s be next years total production of steel and let a be next year's total output of autos. We form these equations.

$$\begin{aligned}
\text{next year's production of steel} &= \text{next year's use of steel by steel} \\
&\quad + \text{next year's use of steel by auto} \\
&\quad + \text{next year's use of steel by others} \\
\text{next year's production of autos} &= \text{next year's use of autos by steel} \\
&\quad + \text{next year's use of autos by auto} \\
&\quad + \text{next year's use of autos by others}
\end{aligned}$$

On the left side of those equations go the unknowns s and a. At the ends of the right sides go our external demand estimates for next year $17,589$ and $21,243$. For the remaining four terms, we look to the table of this year's information about how the industries interact.

For instance, for next year's use of steel by steel, we note that this year the steel industry used 5395 units of steel input to produce $25,448$ units of steel output. So next year, when the steel industry will produce s units out, we expect that doing so will take $s \cdot (5395)/(25\,448)$ units of steel input — this is simply the assumption that input is proportional to output. (We are assuming that the ratio of input to output remains constant over time; in practice, models may try to take account of trends of change in the ratios.)

Next year's use of steel by the auto industry is similar. This year, the auto industry uses 2664 units of steel input to produce 30346 units of auto output. So next year, when the auto industry's total output is a, we expect it to consume $a \cdot (2664)/(30346)$ units of steel.

Filling in the other equation in the same way, we get this system of linear equation.

$$\begin{aligned}
\frac{5\,395}{25\,448} \cdot s + \frac{2\,664}{30\,346} \cdot a + 17\,589 &= s \\
\frac{48}{25\,448} \cdot s + \frac{9\,030}{30\,346} \cdot a + 21\,243 &= a
\end{aligned}$$

Gauss' method on this system.

$$\begin{aligned}
(20\,053/25\,448)s - (2\,664/30\,346)a &= 17\,589 \\
-(48/25\,448)s + (21\,316/30\,346)a &= 21\,243
\end{aligned}$$

gives $s = 25\,698$ and $a = 30\,311$.

Looking back, recall that above we described why the prediction of next year's totals isn't as simple as adding 200 to last year's steel total and subtracting 25 from last year's auto total. In fact, comparing these totals for next year to the ones given at the start for the current year shows that, despite the drop in external demand, the total production of the auto industry is predicted to rise. The increase in internal demand for autos caused by steel's sharp rise in business more than makes up for the loss in external demand for autos.

One of the advantages of having a mathematical model is that we can ask "What if …?" questions. For instance, we can ask "What if the estimates for next year's external demands are somewhat off?" To try to understand how much the model's predictions change in reaction to changes in our estimates, we can try revising our estimate of next year's external steel demand from 17, 589 down to 17, 489, while keeping the assumption of next year's external demand for autos fixed at 21, 243. The resulting system

$$\begin{aligned} (20\,053/25\,448)s - \;\; (2\,664/30\,346)a &= 17\,489 \\ -(48/25\,448)s + (21\,316/30\,346)a &= 21\,243 \end{aligned}$$

when solved gives $s = 25\,571$ and $a = 30\,311$. This kind of exploration of the model is *sensitivity analysis*. We are seeing how sensitive the predictions of our model are to the accuracy of the assumptions.

Obviously, we can consider larger models that detail the interactions among more sectors of an economy. These models are typically solved on a computer, using the techniques of matrix algebra that we will develop in Chapter Three. Some examples are given in the exercises. Obviously also, a single model does not suit every case; expert judgment is needed to see if the assumptions underlying the model are reasonable for a particular case. With those caveats, however, this model has proven in practice to be a useful and accurate tool for economic analysis. For further reading, try [Leontief 1951] and [Leontief 1965].

Exercises

Hint: these systems are easiest to solve on a computer.

1 With the steel-auto system given above, estimate next year's total productions in these cases.

(a) Next year's external demands are: up 200 from this year for steel, and unchanged for autos.

(b) Next year's external demands are: up 100 for steel, and up 200 for autos.

(c) Next year's external demands are: up 200 for steel, and up 200 for autos.

2 In the steel-auto system, the ratio for the use of steel by the auto industry is 2 664/30 346, about 0.0878. Imagine that a new process for making autos reduces this ratio to .0500.

(a) How will the predictions for next year's total productions change compared to the first example discussed above (i.e., taking next year's external demands to be 17, 589 for steel and 21, 243 for autos)?

(b) Predict next year's totals if, in addition, the external demand for autos rises to be 21, 500 because the new cars are cheaper.

3 This table gives the numbers for the auto-steel system from a different year, 1947 (see [Leontief 1951]). The units here are billions of 1947 dollars.

	used by steel	*used by auto*	*used by others*	*total*
value of steel	6.90	1.28		18.69
value of autos	0	4.40		14.27

(a) Solve for total output if next year's external demands are: steel's demand up 10% and auto's demand up 15%.

(b) How do the ratios compare to those given above in the discussion for the 1958 economy?

(c) Solve the 1947 equations with the 1958 external demands (note the difference in units; a 1947 dollar buys about what \$1.30 in 1958 dollars buys). How far off are the predictions for total output?

4 Predict next year's total productions of each of the three sectors of the hypothetical economy shown below

	used by farm	*used by rail*	*used by shipping*	*used by others*	*total*
value of farm	25	50	100		800
value of rail	25	50	50		300
value of shipping	15	10	0		500

if next year's external demands are as stated.

(a) 625 for farm, 200 for rail, 475 for shipping

(b) 650 for farm, 150 for rail, 450 for shipping

5 This table gives the interrelationships among three segments of an economy (see [Clark & Coupe]).

	used by food	*used by wholesale*	*used by retail*	*used by others*	*total*
value of food	0	2 318	4 679		11 869
value of wholesale	393	1 089	22 459		122 242
value of retail	3	53	75		116 041

We will do an Input-Output analysis on this system.

(a) Fill in the numbers for this year's external demands.

(b) Set up the linear system, leaving next year's external demands blank.

(c) Solve the system where next year's external demands are calculated by taking this year's external demands and inflating them 10%. Do all three sectors increase their total business by 10%? Do they all even increase at the same rate?

(d) Solve the system where next year's external demands are calculated by taking this year's external demands and reducing them 7%. (The study from which these numbers are taken concluded that because of the closing of a local military facility, overall personal income in the area would fall 7%, so this might be a first guess at what would actually happen.)

Topic: Accuracy of Computations

Gauss' method lends itself nicely to computerization. The code below illustrates. It operates on an $n \times n$ matrix `a`, doing row combinations using the first row, then the second row, etc.

```
for(row=1;row<=n-1;row++){
  for(row_below=row+1;row_below<=n;row_below++){
    multiplier=a[row_below,row]/a[row,row];
    for(col=row; col<=n; col++){
      a[row_below,col]-=multiplier*a[row,col];
    }
  }
}
```

(This code is in the C language. Here is a brief translation. The loop construct `for(row=1;row<=n-1;row++){···}` sets `row` to 1 and then iterates while `row` is less than or equal to $n-1$, each time through incrementing the variable `row` by one with the '`++`' operation. The other non-obvious construct is that the '`-=`' in the innermost loop amounts to the `a[row_below,col]` $= -$`multiplier` $*$ `a[row,col]` $+$ `a[row_below,col]` operation.)

While this code provides a quick take on how Gauss' method can be mechanized, it is not ready to use. It is naive in many ways. The most glaring way is that it assumes that a nonzero number is always found in the `row,row` position. To make it practical, one way in which this code needs to be reworked is to cover the case where finding a zero in that location leads to a row swap, or to the conclusion that the matrix is singular.

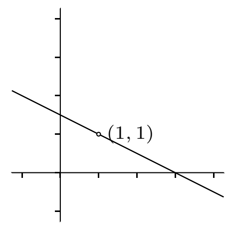
Adding some `if` $\cdots$ statements to cover those cases is not hard, but we will instead consider some more subtle ways in which the code is naive. There are pitfalls arising from the computer's reliance on finite-precision floating point arithmetic.

For example, we have seen above that we must handle as a separate case a system that is singular. But systems that are nearly singular also require care. Consider this one.

$$\begin{aligned} x + 2y &= 3 \\ 1.000\,000\,01x + 2y &= 3.000\,000\,01 \end{aligned}$$

By eye we get the solution $x = 1$ and $y = 1$. But a computer has more trouble. A computer that represents real numbers to eight significant places (as is common, usually called *single precision*) will represent the second equation internally as $1.000\,000\,0x + 2y = 3.000\,000\,0$, losing the digits in the ninth place. Instead of reporting the correct solution, this computer will report something that is not even close — this computer thinks that the system is singular because the two equations are represented internally as equal.

For some intuition about how the computer could come up with something that far off, we can graph the system.



At the scale of this graph, the two lines cannot be resolved apart. This system is nearly singular in the sense that the two lines are nearly the same line. Near-singularity gives this system the property that a small change in the system can cause a large change in its solution; for instance, changing the $3.000\,000\,01$ to $3.000\,000\,03$ changes the intersection point from $(1,1)$ to $(3,0)$. This system changes radically depending on a ninth digit, which explains why the eight-place computer has trouble. A problem that is very sensitive to inaccuracy or uncertainties in the input values is *ill-conditioned*.

The above example gives one way in which a system can be difficult to solve on a computer. It has the advantage that the picture of nearly-equal lines gives a memorable insight into one way that numerical difficulties can arise. Unfortunately this insight isn't very useful when we wish to solve some large system. We cannot, typically, hope to understand the geometry of an arbitrary large system. In addition, there are ways that a computer's results may be unreliable other than that the angle between some of the linear surfaces is quite small.

For an example, consider the system below, from [Hamming].

$$\begin{aligned} 0.001x + y &= 1 \\ x - y &= 0 \end{aligned} \qquad (*)$$

The second equation gives $x = y$, so $x = y = 1/1.001$ and thus both variables have values that are just less than 1. A computer using two digits represents the system internally in this way (we will do this example in two-digit floating point arithmetic, but a similar one with eight digits is easy to invent).

$$\begin{aligned} (1.0 \times 10^{-2})x + (1.0 \times 10^{0})y &= 1.0 \times 10^{0} \\ (1.0 \times 10^{0})x - (1.0 \times 10^{0})y &= 0.0 \times 10^{0} \end{aligned}$$

The computer's row reduction step $-1000\rho_1 + \rho_2$ produces a second equation $-1001y = -999$, which the computer rounds to two places as $(-1.0 \times 10^{3})y = -1.0 \times 10^{3}$. Then the computer decides from the second equation that $y = 1$ and from the first equation that $x = 0$. This y value is fairly good, but the x is quite bad. Thus, another cause of unreliable output is a mixture of floating point arithmetic and a reliance on using leading entries that are small.

An experienced programmer may respond that we should go to *double precision* where sixteen significant digits are retained. This will indeed solve many problems. However, there are some difficulties with it as a general approach. For one thing, double precision takes longer than single precision (on a '486 chip, multiplication takes eleven ticks in single precision but fourteen in double precision [Programmer's Ref.]) and has twice the memory requirements. So attempting to do all calculations in double precision is just not practical. And besides, the above systems can obviously be tweaked to give the same trouble in the seventeenth digit, so double precision won't fix all problems. What we need is a strategy to minimize the numerical trouble arising from solving systems on a computer, and some guidance as to how far the reported solutions can be trusted.

Mathematicians have made a careful study of how to get the most reliable results. A basic improvement on the naive code above is to not simply take the entry in the *row*, *row* position to determine the factor to use for the row combination, but rather to look at all of the entries in the *row* column below the *row* row, and take the one that is most likely to give reliable results (e.g., take one that is not too small). This strategy is called *partial pivoting*.

For example, to solve the troublesome system $(*)$ above, we start by looking at both equations for a best entry to use, and taking the 1 in the second equation as more likely to give good results. Then, the combination step of $-.001\rho_2 + \rho_1$ gives a first equation of $1.001y = 1$, which the computer will represent as $(1.0 \times 10^0)y = 1.0 \times 10^0$, leading to the conclusion that $y = 1$ and, after back-substitution, $x = 1$, both of which are close to right. The code from above can be adapted to this purpose.

```
for(row=1;row<=n-1;row++){
/* find the largest entry in this column (in row max) */
  max=row;
  for(row_below=row+1;row_below<=n;row_below++){
    if (abs(a[row_below,row]) > abs(a[max,row]));
      max = row_below;
  }
/* swap rows to move that best entry up */
  for(col=row;col<=n;col++){
    temp=a[row,col];
    a[row,col]=a[max,col];
    a[max,col]=temp;
  }
/* proceed as before */
  for(row_below=row+1;row_below<=n;row_below++){
    multiplier=a[row_below,row]/a[row,row];
     for(col=row;col<=n;col++){
       a[row_below,col]-=multiplier*a[row,col];
    }
  }
}
```

A full analysis of the best way to implement Gauss' method is outside the scope of the book (see [Wilkinson 1965]), but the method recommended by most experts is a variation on the code above that first finds the best entry among the candidates, and then scales it to a number that is less likely to give trouble. This is *scaled partial pivoting*.

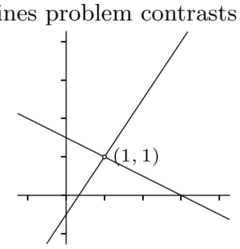
In addition to returning a result that is likely to be reliable, most well-done code will return a number, called the *conditioning number* that describes the factor by which uncertainties in the input numbers could be magnified to become inaccuracies in the results returned (see [Rice]).

The lesson of this discussion is that just because Gauss' method always works in theory, and just because computer code correctly implements that method, doesn't mean that the answer is reliable. In practice, always use a package where experts have worked hard to counter what can go wrong.

Exercises

1 Using two decimal places, add 253 and 2/3.

2 This intersect-the-lines problem contrasts with the example discussed above.



$$\begin{aligned} x + 2y &= 3 \\ 3x - 2y &= 1 \end{aligned}$$

Illustrate that in this system some small change in the numbers will produce only a small change in the solution by changing the constant in the bottom equation to 1.008 and solving. Compare it to the solution of the unchanged system.

3 Solve this system by hand ([Rice]).

$$\begin{aligned} 0.000\,3x + 1.556y &= 1.569 \\ 0.345\,4x - 2.346y &= 1.018 \end{aligned}$$

(a) Solve it accurately, by hand. **(b)** Solve it by rounding at each step to four significant digits.

4 Rounding inside the computer often has an effect on the result. Assume that your machine has eight significant digits.

(a) Show that the machine will compute $(2/3) + ((2/3) - (1/3))$ as unequal to $((2/3) + (2/3)) - (1/3)$. Thus, computer arithmetic is not associative.

(b) Compare the computer's version of $(1/3)x + y = 0$ and $(2/3)x + 2y = 0$. Is twice the first equation the same as the second?

5 Ill-conditioning is not only dependent on the matrix of coefficients. This example [Hamming] shows that it can arise from an interaction between the left and right sides of the system. Let ε be a small real.

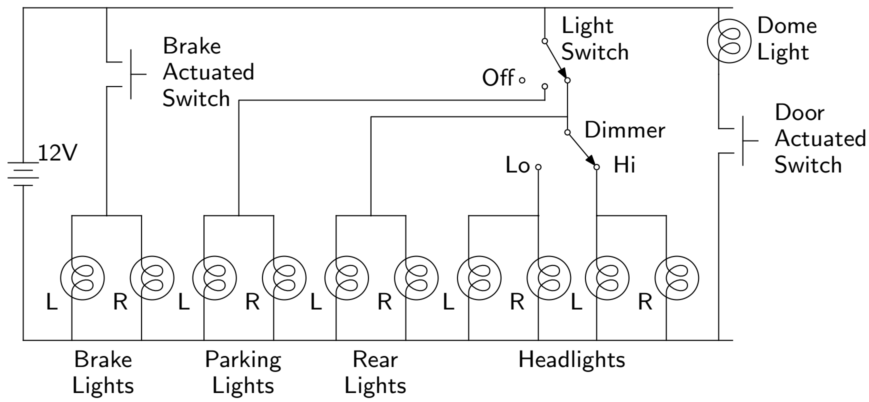
$$\begin{aligned} 3x + 2y + z &= 6 \\ 2x + 2\varepsilon y + 2\varepsilon z &= 2 + 4\varepsilon \\ x + 2\varepsilon y - \varepsilon z &= 1 + \varepsilon \end{aligned}$$

(a) Solve the system by hand. Notice that the ε's divide out only because there is an exact cancelation of the integer parts on the right side as well as on the left.

(b) Solve the system by hand, rounding to two decimal places, and with $\varepsilon = 0.001$.

Topic: Analyzing Networks

The diagram below shows some of a car's electrical network. The battery is on the left, drawn as stacked line segments. The wires are drawn as lines, shown straight and with sharp right angles for neatness. Each light is a circle enclosing a loop.



The designer of such a network needs to answer questions like: How much electricity flows when both the hi-beam headlights and the brake lights are on? Below, we will use linear systems to analyze simpler versions of electrical networks.

For the analysis we need two facts about electricity and two facts about electrical networks.

The first fact about electricity is that a battery is like a pump: it provides a force impelling the electricity to flow through the circuits connecting the battery's ends, if there are any such circuits. We say that the battery provides a *potential* to flow. Of course, this network accomplishes its function when, as the electricity flows through a circuit, it goes through a light. For instance, when the driver steps on the brake then the switch makes contact and a circuit is formed on the left side of the diagram, and the electrical current flowing through that circuit will make the brake lights go on, warning drivers behind.

The second electrical fact is that in some kinds of network components the amount of flow is proportional to the force provided by the battery. That is, for each such component there is a number, it's *resistance*, such that the potential is equal to the flow times the resistance. The units of measurement are: potential is described in *volts*, the rate of flow is in *amperes*, and resistance to the flow is in *ohms*. These units are defined so that volts = amperes · ohms.

Components with this property, that the voltage-amperage response curve is a line through the origin, are called *resistors*. (Light bulbs such as the ones shown above are not this kind of component, because their ohmage changes as they heat up.) For example, if a resistor measures 2 ohms then wiring it to a 12 volt battery results in a flow of 6 amperes. Conversely, if we have flow of electrical current of 2 amperes through it then there must be a 4 volt potential difference between it's ends. This is the *voltage drop* across the resistor. One way to think of a electrical circuits like the one above is that the battery provides a voltage rise while the other components are voltage drops.

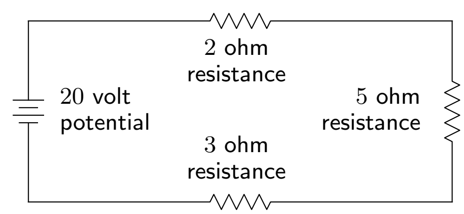
The two facts that we need about networks are Kirchhoff's Laws.

> *Current Law.* For any point in a network, the flow in equals the flow out.

> *Voltage Law.* Around any circuit the total drop equals the total rise.

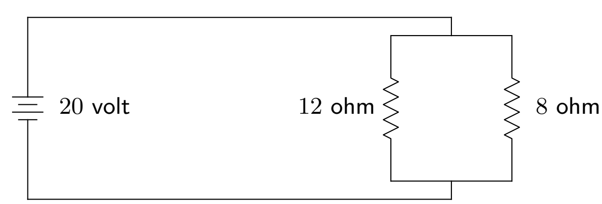
In the above network there is only one voltage rise, at the battery, but some networks have more than one.

For a start we can consider the network below. It has a battery that provides the potential to flow and three resistors (resistors are drawn as zig-zags). When components are wired one after another, as here, they are said to be in *series*.

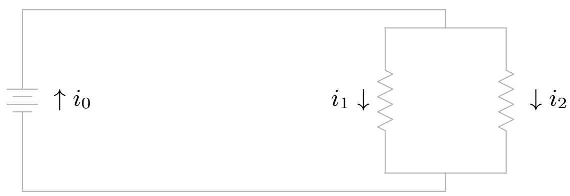


By Kirchhoff's Voltage Law, because the voltage rise is 20 volts, the total voltage drop must also be 20 volts. Since the resistance from start to finish is 10 ohms (the resistance of the wires is negligible), we get that the current is $(20/10) = 2$ amperes. Now, by Kirchhoff's Current Law, there are 2 amperes through each resistor. (And therefore the voltage drops are: 4 volts across the 2 oh m resistor, 10 volts across the 5 ohm resistor, and 6 volts across the 3 ohm resistor.)

The prior network is so simple that we didn't use a linear system, but the next network is more complicated. In this one, the resistors are in *parallel*. This network is more like the car lighting diagram shown earlier.



We begin by labeling the branches, shown below. Let the current through the left branch of the parallel portion be i_1 and that through the right branch be i_2, and also let the current through the battery be i_0. (We are following Kirchoff's Current Law; for instance, all points in the right branch have the same current, which we call i_2. Note that we don't need to know the actual direction of flow — if current flows in the direction opposite to our arrow then we will simply get a negative number in the solution.)

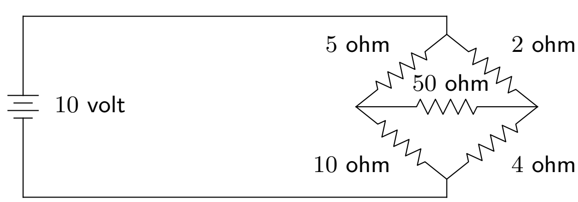


The Current Law, applied to the point in the upper right where the flow i_0 meets i_1 and i_2, gives that $i_0 = i_1 + i_2$. Applied to the lower right it gives $i_1 + i_2 = i_0$. In the circuit that loops out of the top of the battery, down the left branch of the parallel portion, and back into the bottom of the battery, the voltage rise is 20 while the voltage drop is $i_1 \cdot 12$, so the Voltage Law gives that $12i_1 = 20$. Similarly, the circuit from the battery to the right branch and back to the battery gives that $8i_2 = 20$. And, in the circuit that simply loops around in the left and right branches of the parallel portion (arbitrarily taken clockwise), there is a voltage rise of 0 and a voltage drop of $8i_2 - 12i_1$ so the Voltage Law gives that $8i_2 - 12i_1 = 0$.

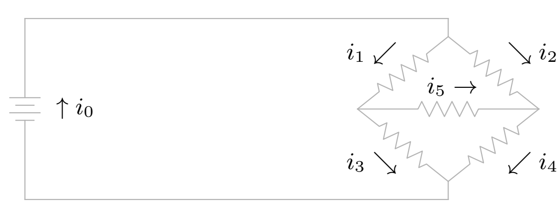
$$\begin{array}{rrrl}
i_0 - & i_1 - & i_2 = & 0 \\
-i_0 + & i_1 + & i_2 = & 0 \\
 & 12i_1 & = & 20 \\
 & & 8i_2 = & 20 \\
 & -12i_1 + & 8i_2 = & 0
\end{array}$$

The solution is $i_0 = 25/6$, $i_1 = 5/3$, and $i_2 = 5/2$, all in amperes. (Incidentally, this illustrates that redundant equations do indeed arise in practice.)

Kirchhoff's laws can be used to establish the electrical properties of networks of great complexity. The next diagram shows five resistors, wired in a *series-parallel* way.



This network is a *Wheatstone bridge* (see Exercise 4). To analyze it, we can place the arrows in this way.



Kirchoff's Current Law, applied to the top node, the left node, the right node, and the bottom node gives these.

$$\begin{aligned} i_0 &= i_1 + i_2 \\ i_1 &= i_3 + i_5 \\ i_2 + i_5 &= i_4 \\ i_3 + i_4 &= i_0 \end{aligned}$$

Kirchhoff's Voltage Law, applied to the inside loop (the i_0 to i_1 to i_3 to i_0 loop), the outside loop, and the upper loop not involving the battery, gives these.

$$\begin{aligned} 5i_1 + 10i_3 &= 10 \\ 2i_2 + 4i_4 &= 10 \\ 5i_1 + 50i_5 - 2i_2 &= 0 \end{aligned}$$

Those suffice to determine the solution $i_0 = 7/3$, $i_1 = 2/3$, $i_2 = 5/3$, $i_3 = 2/3$, $i_4 = 5/3$, and $i_5 = 0$.

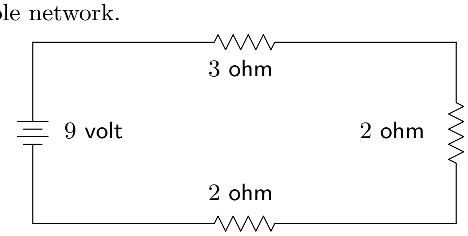
Networks of other kinds, not just electrical ones, can also be analyzed in this way. For instance, networks of streets are given in the exercises.

Exercises

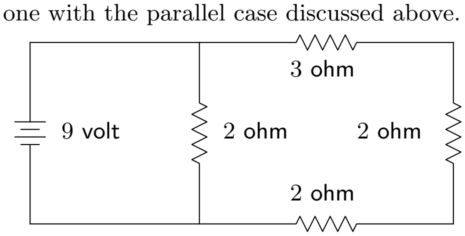
Many of the systems for these problems are mostly easily solved on a computer.

1 Calculate the amperages in each part of each network.

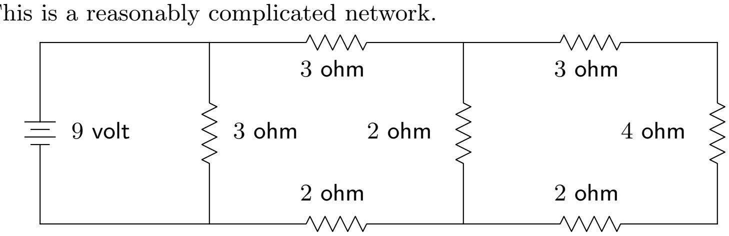
(a) This is a simple network.



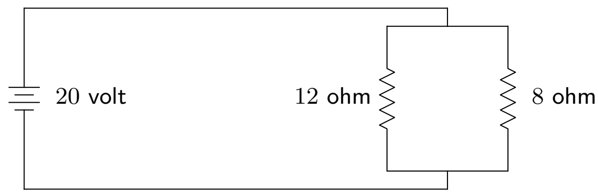
(b) Compare this one with the parallel case discussed above.



(c) This is a reasonably complicated network.



2 In the first network that we analyzed, with the three resistors in series, we just added to get that they acted together like a single resistor of 10 ohms. We can do a similar thing for parallel circuits. In the second circuit analyzed,

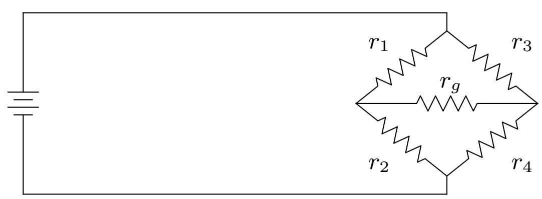


the electric current through the battery is $25/6$ amperes. Thus, the parallel portion is *equivalent* to a single resistor of $20/(25/6) = 4.8$ ohms.

(a) What is the equivalent resistance if we change the 12 ohm resistor to 5 ohms?
(b) What is the equivalent resistance if the two are each 8 ohms?
(c) Find the formula for the equivalent resistance if the two resistors in parallel are r_1 ohms and r_2 ohms.

3 For the car dashboard example that opens this Topic, solve for these amperages (assume that all resistances are 2 ohms).
(a) If the driver is stepping on the brakes, so the brake lights are on, and no other circuit is closed.
(b) If the hi-beam headlights and the brake lights are on.

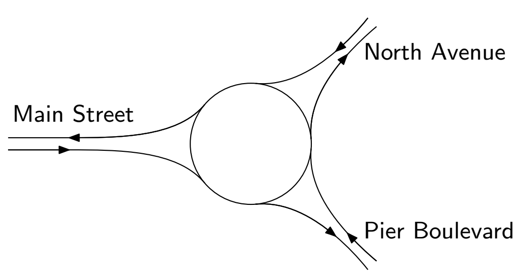
4 Show that, in this Wheatstone Bridge,



r_2/r_1 equals r_4/r_3 if and only if the current flowing through r_g is zero. (The way that this device is used in practice is that an unknown resistance at r_4 is compared to the other three r_1, r_2, and r_3. At r_g is placed a meter that shows the current. The three resistances r_1, r_2, and r_3 are varied — typically they each have a calibrated knob — until the current in the middle reads 0, and then the above equation gives the value of r_4.)

There are networks other than electrical ones, and we can ask how well Kirchoff's laws apply to them. The remaining questions consider an extension to networks of streets.

5 Consider this traffic circle.



This is the traffic volume, in units of cars per five minutes.

	North	*Pier*	*Main*
into	100	150	25
out of	75	150	50

We can set up equations to model how the traffic flows.

(a) Adapt Kirchoff's Current Law to this circumstance. Is it a reasonable modelling assumption?

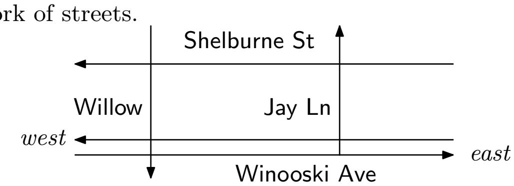
(b) Label the three between-road arcs in the circle with a variable. Using the (adapted) Current Law, for each of the three in-out intersections state an equation describing the traffic flow at that node.

(c) Solve that system.

(d) Interpret your solution.

(e) Restate the Voltage Law for this circumstance. How reasonable is it?

6 This is a network of streets.



The hourly flow of cars into this network's entrances, and out of its exits can be observed.

	east Winooski	*west Winooski*	*Willow*	*Jay*	*Shelburne*
into	80	50	65	–	40
out of	30	5	70	55	75

(Note that to reach Jay a car must enter the network via some other road first, which is why there is no 'into Jay' entry in the table. Note also that over a long period of time, the total in must approximately equal the total out, which is why both rows add to 235 cars.) Once inside the network, the traffic may flow in different ways, perhaps filling Willow and leaving Jay mostly empty, or perhaps flowing in some other way. Kirchhoff's Laws give the limits on that freedom.

(a) Determine the restrictions on the flow inside this network of streets by setting up a variable for each block, establishing the equations, and solving them. Notice that some streets are one-way only. (*Hint:* this will not yield a unique solution, since traffic can flow through this network in various ways; you should get at least one free variable.)

(b) Suppose that some construction is proposed for Winooski Avenue East between Willow and Jay, so traffic on that block will be reduced. What is the least amount of traffic flow that can be allowed on that block without disrupting the hourly flow into and out of the network?

Chapter Two

Vector Spaces

The first chapter began by introducing Gauss' method and finished with a fair understanding, keyed on the Linear Combination Lemma, of how it finds the solution set of a linear system. Gauss' method systematically takes linear combinations of the rows. With that insight, we now move to a general study of linear combinations.

We need a setting for this study. At times in the first chapter, we've combined vectors from $\mathbb{R}^2$, at other times vectors from $\mathbb{R}^3$, and at other times vectors from even higher-dimensional spaces. Thus, our first impulse might be to work in $\mathbb{R}^n$, leaving n unspecified. This would have the advantage that any of the results would hold for $\mathbb{R}^2$ and for $\mathbb{R}^3$ and for many other spaces, simultaneously.

But, if having the results apply to many spaces at once is advantageous then sticking only to $\mathbb{R}^n$'s is overly restrictive. We'd like the results to also apply to combinations of row vectors, as in the final section of the first chapter. We've even seen some spaces that are not just a collection of all of the same-sized column vectors or row vectors. For instance, we've seen a solution set of a homogeneous system that is a plane, inside of $\mathbb{R}^3$. This solution set is a closed system in the sense that a linear combination of these solutions is also a solution. But it is not just a collection of all of the three-tall column vectors; only some of them are in this solution set.

We want the results about linear combinations to apply anywhere that linear combinations are sensible. We shall call any such set a *vector space*. Our results, instead of being phrased as "Whenever we have a collection in which we can sensibly take linear combinations ...", will be stated as "In any vector space ...".

Such a statement describes at once what happens in many spaces. The step up in abstraction from studying a single space at a time to studying a class of spaces can be hard to make. To understand its advantages, consider this analogy. Imagine that the government made laws one person at a time: "Leslie Jones can't jay walk." That would be a bad idea; statements have the virtue of economy when they apply to many cases at once. Or, suppose that they ruled, "Kim Ke must stop when passing the scene of an accident." Contrast that with, "Any doctor must stop when passing the scene of an accident." More general statements, in some ways, are clearer.

I Definition of Vector Space

We shall study structures with two operations, an addition and a scalar multiplication, that are subject to some simple conditions. We will reflect more on the conditions later, but on first reading notice how reasonable they are. For instance, surely any operation that can be called an addition (e.g., column vector addition, row vector addition, or real number addition) will satisfy conditions (1) through (5) below.

I.1 Definition and Examples

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '$\cdot$' subject to these conditions.

Where $\vec{v},\vec{w}\in V$, (1) their *vector sum* $\vec{v}+\vec{w}$ is an element of V. If $\vec{u},\vec{v},\vec{w}\in V$ then (2) $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ and (3) $(\vec{v}+\vec{w})+\vec{u}=\vec{v}+(\vec{w}+\vec{u})$. (4) There is a *zero vector* $\vec{0}\in V$ such that $\vec{v}+\vec{0}=\vec{v}$ for all $\vec{v}\in V$. (5) Each $\vec{v}\in V$ has an *additive inverse* $\vec{w}\in V$ such that $\vec{w}+\vec{v}=\vec{0}$.

If r,s are *scalars*, members of $\mathbb{R}$, and $\vec{v},\vec{w}\in V$ then (6) each *scalar multiple* $r\cdot\vec{v}$ is in V. If $r,s\in\mathbb{R}$ and $\vec{v},\vec{w}\in V$ then (7) $(r+s)\cdot\vec{v}=r\cdot\vec{v}+s\cdot\vec{v}$, and (8) $r\cdot(\vec{v}+\vec{w})=r\cdot\vec{v}+r\cdot\vec{w}$, and (9) $(rs)\cdot\vec{v}=r\cdot(s\cdot\vec{v})$, and (10) $1\cdot\vec{v}=\vec{v}$.

1.2 Remark Because it involves two kinds of addition and two kinds of multiplication, that definition may seem confused. For instance, in condition (7) '$(r+s)\cdot\vec{v}=r\cdot\vec{v}+s\cdot\vec{v}$', the first '+' is the real number addition operator while the '+' to the right of the equals sign represents vector addition in the structure V. These expressions aren't ambiguous because, e.g., r and s are real numbers so '$r+s$' can only mean real number addition.

The best way to go through the examples below is to check all ten conditions in the definition. That check is written out at length in the first example. Use it as a model for the others. Especially important are the first condition '$\vec{v}+\vec{w}$ is in V' and the sixth condition '$r\cdot\vec{v}$ is in V'. These are the *closure* conditions. They specify that the addition and scalar multiplication operations are always sensible — they are defined for every pair of vectors, and every scalar and vector, and the result of the operation is a member of the set (see Example 1.4).

1.3 Example The set $\mathbb{R}^2$ is a vector space if the operations '+' and '$\cdot$' have their usual meaning.

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix}\qquad r\cdot\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}rx_1\\rx_2\end{pmatrix}$$

We shall check all of the conditions.

There are five conditions in item (1). For (1), closure of addition, note that for any $v_1,v_2,w_1,w_2\in\mathbb{R}$ the result of the sum

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}v_1+w_1\\v_2+w_2\end{pmatrix}$$

is a column array with two real entries, and so is in $\mathbb{R}^2$. For (2), that addition of vectors commutes, take all entries to be real numbers and compute

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix}=\begin{pmatrix}v_1+w_1\\v_2+w_2\end{pmatrix}=\begin{pmatrix}w_1+v_1\\w_2+v_2\end{pmatrix}=\begin{pmatrix}w_1\\w_2\end{pmatrix}+\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

(the second equality follows from the fact that the components of the vectors are real numbers, and the addition of real numbers is commutative). Condition (3), associativity of vector addition, is similar.

$$\begin{aligned}(\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix})+\begin{pmatrix}u_1\\u_2\end{pmatrix}&=\begin{pmatrix}(v_1+w_1)+u_1\\(v_2+w_2)+u_2\end{pmatrix}\\&=\begin{pmatrix}v_1+(w_1+u_1)\\v_2+(w_2+u_2)\end{pmatrix}\\&=\begin{pmatrix}v_1\\v_2\end{pmatrix}+(\begin{pmatrix}w_1\\w_2\end{pmatrix}+\begin{pmatrix}u_1\\u_2\end{pmatrix})\end{aligned}$$

For the fourth condition we must produce a zero element — the vector of zeroes is it.

$$\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

For (5), to produce an additive inverse, note that for any $v_1,v_2\in\mathbb{R}$ we have

$$\begin{pmatrix}-v_1\\-v_2\end{pmatrix}+\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

so the first vector is the desired additive inverse of the second.

The checks for the five conditions having to do with scalar multiplication are just as routine. For (6), closure under scalar multiplication, where $r,v_1,v_2\in\mathbb{R}$,

$$r\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}rv_1\\rv_2\end{pmatrix}$$

is a column array with two real entries, and so is in $\mathbb{R}^2$. Next, this checks (7).

$$(r+s)\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}(r+s)v_1\\(r+s)v_2\end{pmatrix}=\begin{pmatrix}rv_1+sv_1\\rv_2+sv_2\end{pmatrix}=r\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}+s\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

For (8), that scalar multiplication distributes from the left over vector addition, we have this.

$$r\cdot(\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}w_1\\w_2\end{pmatrix})=\begin{pmatrix}r(v_1+w_1)\\r(v_2+w_2)\end{pmatrix}=\begin{pmatrix}rv_1+rw_1\\rv_2+rw_2\end{pmatrix}=r\cdot\begin{pmatrix}v_1\\v_2\end{pmatrix}+r\cdot\begin{pmatrix}w_1\\w_2\end{pmatrix}$$

The ninth

$$(rs)\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}(rs)v_1\\ (rs)v_2\end{pmatrix}=\begin{pmatrix}r(sv_1)\\ r(sv_2)\end{pmatrix}=r\cdot(s\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix})$$

and tenth conditions are also straightforward.

$$1\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}1v_1\\ 1v_2\end{pmatrix}=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$$

In a similar way, each $\mathbb{R}^n$ is a vector space with the usual operations of vector addition and scalar multiplication. (In $\mathbb{R}^1$, we usually do not write the members as column vectors, i.e., we usually do not write '(π)'. Instead we just write 'π'.)

1.4 Example This subset of $\mathbb{R}^3$ that is a plane through the origin

$$P=\{\begin{pmatrix}x\\ y\\ z\end{pmatrix}\mid x+y+z=0\}$$

is a vector space if '$+$' and '$\cdot$' are interpreted in this way.

$$\begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}+\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2\\ y_1+y_2\\ z_1+z_2\end{pmatrix}\qquad r\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}rx\\ ry\\ rz\end{pmatrix}$$

The addition and scalar multiplication operations here are just the ones of $\mathbb{R}^3$, reused on its subset P. We say that P *inherits* these operations from $\mathbb{R}^3$. This example of an addition in P

$$\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}+\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$$

illustrates that P is closed under addition. We've added two vectors from P—that is, with the property that the sum of their three entries is zero—and the result is a vector also in P. Of course, this example of closure is not a proof of closure. To prove that P is closed under addition, take two elements of P

$$\begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}\qquad\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}$$

(membership in P means that $x_1+y_1+z_1=0$ and $x_2+y_2+z_2=0$), and observe that their sum

$$\begin{pmatrix}x_1+x_2\\ y_1+y_2\\ z_1+z_2\end{pmatrix}$$

is also in P since its entries add $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2)$ to 0. To show that P is closed under scalar multiplication, start with a vector from P

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(so that $x + y + z = 0$) and then for $r \in \mathbb{R}$ observe that the scalar multiple

$$r \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$$

satisfies that $rx + ry + rz = r(x + y + z) = 0$. Thus the two closure conditions are satisfied. Verification of the other conditions in the definition of a vector space are just as straightforward.

1.5 Example Example 1.3 shows that the set of all two-tall vectors with real entries is a vector space. Example 1.4 gives a subset of an $\mathbb{R}^n$ that is also a vector space. In contrast with those two, consider the set of two-tall columns with entries that are integers (under the obvious operations). This is a subset of a vector space, but it is not itself a vector space. The reason is that this set is not closed under scalar multiplication, that is, it does not satisfy condition (6). Here is a column with integer entries, and a scalar, such that the outcome of the operation

$$0.5 \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix}$$

is not a member of the set, since its entries are not all integers.

1.6 Example The singleton set

$$\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$$

is a vector space under the operations

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

that it inherits from $\mathbb{R}^4$.

A vector space must have at least one element, its zero vector. Thus a one-element vector space is the smallest one possible.

1.7 Definition A one-element vector space is a *trivial* space.

Warning! The examples so far involve sets of column vectors with the usual operations. But vector spaces need not be collections of column vectors, or even of row vectors. Below are some other types of vector spaces. The term 'vector space' does not mean 'collection of columns of reals'. It means something more like 'collection in which any linear combination is sensible'.

1.8 Example Consider $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0, \ldots, a_3 \in \mathbb{R}\}$, the set of polynomials of degree three or less (in this book, we'll take constant polynomials, including the zero polynomial, to be of degree zero). It is a vector space under the operations

$$\begin{aligned}(a_0 + a_1x + a_2x^2 + a_3x^3) &+ (b_0 + b_1x + b_2x^2 + b_3x^3) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\end{aligned}$$

and

$$r \cdot (a_0 + a_1x + a_2x^2 + a_3x^3) = (ra_0) + (ra_1)x + (ra_2)x^2 + (ra_3)x^3$$

(the verification is easy). This vector space is worthy of attention because these are the polynomial operations familiar from high school algebra. For instance, $3 \cdot (1 - 2x + 3x^2 - 4x^3) - 2 \cdot (2 - 3x + x^2 - (1/2)x^3) = -1 + 7x^2 - 11x^3$.

Although this space is not a subset of any $\mathbb{R}^n$, there is a sense in which we can think of $\mathcal{P}_3$ as "the same" as $\mathbb{R}^4$. If we identify these two spaces's elements in this way

$$a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{corresponds to} \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

then the operations also correspond. Here is an example of corresponding additions.

$$\begin{array}{rc} & 1 - 2x + 0x^2 + 1x^3 \\ + & 2 + 3x + 7x^2 - 4x^3 \\ \hline & 3 + 1x + 7x^2 - 3x^3 \end{array} \quad \text{corresponds to} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 7 \\ -3 \end{pmatrix}$$

Things we are thinking of as "the same" add to "the same" sum. Chapter Three makes precise this idea of vector space correspondence. For now we shall just leave it as an intuition.

1.9 Example The set $\mathcal{M}_{2\times2}$ of 2×2 matrices with real number entries is a vector space under the natural entry-by-entry operations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a + w & b + x \\ c + y & d + z \end{pmatrix} \qquad r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$$

As in the prior example, we can think of this space as "the same" as $\mathbb{R}^4$.

1.10 Example The set $\{f \mid f\colon \mathbb{N} \to \mathbb{R}\}$ of all real-valued functions of one natural number variable is a vector space under the operations

$$(f_1 + f_2)\,(n) = f_1(n) + f_2(n) \qquad (r \cdot f)\,(n) = r\,f(n)$$

so that if, for example, $f_1(n) = n^2 + 2\sin(n)$ and $f_2(n) = -\sin(n) + 0.5$ then $(f_1 + 2f_2)\,(n) = n^2 + 1$.

We can view this space as a generalization of Example 1.3 — instead of 2-tall vectors, these functions are like infinitely-tall vectors.

$$\begin{array}{c|c} n & f(n) = n^2 + 1 \\ \hline 0 & 1 \\ 1 & 2 \\ 2 & 5 \\ 3 & 10 \\ \vdots & \vdots \end{array} \qquad \text{corresponds to} \qquad \begin{pmatrix} 1 \\ 2 \\ 5 \\ 10 \\ \vdots \end{pmatrix}$$

Addition and scalar multiplication are component-wise, as in Example 1.3. (We can formalize "infinitely-tall" by saying that it means an infinite sequence, or that it means a function from $\mathbb{N}$ to $\mathbb{R}$.)

1.11 Example The set of polynomials with real coefficients

$$\{a_0 + a_1x + \cdots + a_nx^n \mid n \in \mathbb{N} \text{ and } a_0, \ldots, a_n \in \mathbb{R}\}$$

makes a vector space when given the natural '+'

$$\begin{aligned} (a_0 + a_1x + \cdots + a_nx^n) &+ (b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \end{aligned}$$

and '$\cdot$'.

$$r \cdot (a_0 + a_1x + \ldots a_nx^n) = (ra_0) + (ra_1)x + \ldots (ra_n)x^n$$

This space differs from the space $\mathcal{P}_3$ of Example 1.8. This space contains not just degree three polynomials, but degree thirty polynomials and degree three hundred polynomials, too. Each individual polynomial of course is of a finite degree, but the set has no single bound on the degree of all of its members.

This example, like the prior one, can be thought of in terms of infinite-tuples. For instance, we can think of $1 + 3x + 5x^2$ as corresponding to $(1, 3, 5, 0, 0, \ldots)$. However, this space differs from the one in Example 1.10. Here, each member of the set has a finite degree, that is, under the correspondence there is no element from this space matching $(1, 2, 5, 10, \ldots)$. Vectors in this space correspond to infinite-tuples that end in zeroes.

1.12 Example The set $\{f \mid f\colon \mathbb{R} \to \mathbb{R}\}$ of all real-valued functions of one real variable is a vector space under these.

$$(f_1 + f_2)\,(x) = f_1(x) + f_2(x) \qquad (r \cdot f)\,(x) = r\,f(x)$$

The difference between this and Example 1.10 is the domain of the functions.

1.13 Example The set $F = \{a\cos\theta + b\sin\theta \bigm| a, b \in \mathbb{R}\}$ of real-valued functions of the real variable θ is a vector space under the operations

$$(a_1\cos\theta + b_1\sin\theta) + (a_2\cos\theta + b_2\sin\theta) = (a_1 + a_2)\cos\theta + (b_1 + b_2)\sin\theta$$

and

$$r \cdot (a\cos\theta + b\sin\theta) = (ra)\cos\theta + (rb)\sin\theta$$

inherited from the space in the prior example. (We can think of F as "the same" as $\mathbb{R}^2$ in that $a\cos\theta + b\sin\theta$ corresponds to the vector with components a and b.)

1.14 Example The set

$$\{f\colon \mathbb{R} \to \mathbb{R} \bigm| \frac{d^2 f}{dx^2} + f = 0\}$$

is a vector space under the, by now natural, interpretation.

$$(f + g)\,(x) = f(x) + g(x) \qquad (r \cdot f)\,(x) = r\,f(x)$$

In particular, notice that closure is a consequence

$$\frac{d^2(f+g)}{dx^2} + (f + g) = (\frac{d^2 f}{dx^2} + f) + (\frac{d^2 g}{dx^2} + g)$$

and

$$\frac{d^2(rf)}{dx^2} + (rf) = r(\frac{d^2 f}{dx^2} + f)$$

of basic Calculus. This turns out to equal the space from the prior example— functions satisfying this differential equation have the form $a\cos\theta + b\sin\theta$— but this description suggests an extension to solutions sets of other differential equations.

1.15 Example The set of solutions of a homogeneous linear system in n variables is a vector space under the operations inherited from $\mathbb{R}^n$. For example, for closure under addition consider a typical equation in that system $c_1x_1 + \cdots + c_nx_n = 0$ and suppose that both these vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

satisfy the equation. Then their sum $\vec{v} + \vec{w}$ also satisfies that equation: $c_1(v_1 + w_1) + \cdots + c_n(v_n + w_n) = (c_1v_1 + \cdots + c_nv_n) + (c_1w_1 + \cdots + c_nw_n) = 0$. The checks of the other vector space conditions are just as routine.

As we've done in those equations, we often omit the multiplication symbol '$\cdot$'. We can distinguish the multiplication in 'c_1v_1' from that in '$r\vec{v}$' since if both multiplicands are real numbers then real-real multiplication must be meant, while if one is a vector then scalar-vector multiplication must be meant.

The prior example has brought us full circle since it is one of our motivating examples.

1.16 Remark Now, with some feel for the kinds of structures that satisfy the definition of a vector space, we can reflect on that definition. For example, why specify in the definition the condition that $1 \cdot \vec{v} = \vec{v}$ but not a condition that $0 \cdot \vec{v} = \vec{0}$?

One answer is that this is just a definition — it gives the rules of the game from here on, and if you don't like it, put the book down and walk away.

Another answer is perhaps more satisfying. People in this area have worked hard to develop the right balance of power and generality. This definition has been shaped so that it contains the conditions needed to prove all of the interesting and important properties of spaces of linear combinations. As we proceed, we shall derive all of the properties natural to collections of linear combinations from the conditions given in the definition.

The next result is an example. We do not need to include these properties in the definition of vector space because they follow from the properties already listed there.

1.17 Lemma In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof. For (1), note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

The second item is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$ shows that we can write '$-\vec{v}$' for the additive inverse of $\vec{v}$ without worrying about possible confusion with $(-1) \cdot \vec{v}$.

For (3), this $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

We finish with a recap.

Our study in Chapter One of Gaussian reduction led us to consider collections of linear combinations. So in this chapter we have defined a vector space to be a structure in which we can form such combinations, expressions of the form $c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n$ (subject to simple conditions on the addition and scalar multiplication operations). In a phrase: vector spaces are the right context in which to study linearity.

Finally, a comment. From the fact that it forms a whole chapter, and especially because that chapter is the first one, a reader could come to think that the study of linear systems is our purpose. The truth is, we will not so much use vector spaces in the study of linear systems as we will instead have linear systems start us on the study of vector spaces. The wide variety of examples from this subsection shows that the study of vector spaces is interesting and important in its own right, aside from how it helps us understand linear systems. Linear systems won't go away. But from now on our primary objects of study will be vector spaces.

Exercises

1.18 Name the zero vector for each of these vector spaces.

(a) The space of degree three polynomials under the natural operations
(b) The space of 2×4 matrices
(c) The space $\{f\colon [0..1]\to\mathbb{R} \mid f \text{ is continuous}\}$
(d) The space of real-valued functions of one natural number variable

✓ **1.19** Find the additive inverse, in the vector space, of the vector.

(a) In $\mathcal{P}_3$, the vector $-3-2x+x^2$.
(b) In the space 2×2,

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

(c) In $\{ae^x+be^{-x} \mid a,b\in\mathbb{R}\}$, the space of functions of the real variable x under the natural operations, the vector $3e^x-2e^{-x}$.

✓ **1.20** Show that each of these is a vector space.

(a) The set of linear polynomials $\mathcal{P}_1=\{a_0+a_1x \mid a_0,a_1\in\mathbb{R}\}$ under the usual polynomial addition and scalar multiplication operations.
(b) The set of 2×2 matrices with real entries under the usual matrix operations.
(c) The set of three-component row vectors with their usual operations.
(d) The set

$$L=\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\in\mathbb{R}^4 \mid x+y-z+w=0\}$$

under the operations inherited from $\mathbb{R}^4$.

✓ **1.21** Show that each of these is not a vector space. (*Hint.* Start by listing two members of each set.)

(a) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\in\mathbb{R}^3 \mid x+y+z=1\}$$

(b) Under the operations inherited from $\mathbb{R}^3$, this set

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$$

(c) Under the usual matrix operations,

$$\{\begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \mid a,b,c\in\mathbb{R}\}$$

(d) Under the usual polynomial operations,

$$\{a_0+a_1x+a_2x^2 \mid a_0,a_1,a_2\in\mathbb{R}^+\}$$

where $\mathbb{R}^+$ is the set of reals greater than zero

(e) Under the inherited operations,

$$\{\begin{pmatrix} x \\ y \end{pmatrix}\in\mathbb{R}^2 \mid x+3y=4 \text{ and } 2x-y=3 \text{ and } 6x+4y=10\}$$

1.22 Define addition and scalar multiplication operations to make the complex numbers a vector space over $\mathbb{R}$.

✓ **1.23** Is the set of rational numbers a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication operations?

1.24 Show that the set of linear combinations of the variables x, y, z is a vector space under the natural addition and scalar multiplication operations.

1.25 Prove that this is not a vector space: the set of two-tall column vectors with real entries subject to these operations.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx \\ ry \end{pmatrix}$$

1.26 Prove or disprove that $\mathbb{R}^3$ is a vector space under these operations.

(a) $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rx \\ ry \\ rz \end{pmatrix}$

(b) $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $r \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

✓ **1.27** For each, decide if it is a vector space; the intended operations are the natural ones.

(a) The *diagonal* 2×2 matrices

$$\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\}$$

(b) This set of 2×2 matrices

$$\{\begin{pmatrix} x & x+y \\ x+y & y \end{pmatrix} \mid x, y \in \mathbb{R}\}$$

(c) This set

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + y + w = 1\}$$

(d) The set of functions $\{f\colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 0\}$

(e) The set of functions $\{f\colon \mathbb{R} \to \mathbb{R} \mid df/dx + 2f = 1\}$

✓ **1.28** Prove or disprove that this is a vector space: the real-valued functions f of one real variable such that $f(7) = 0$.

✓ **1.29** Show that the set $\mathbb{R}^+$ of positive reals is a vector space when '$x + y$' is interpreted to mean the product of x and y (so that $2 + 3$ is 6), and '$r \cdot x$' is interpreted as the r-th power of x.

1.30 Is $\{(x, y) \mid x, y \in \mathbb{R}\}$ a vector space under these operations?

(a) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, y)$

(b) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $r \cdot (x, y) = (rx, 0)$

1.31 Prove or disprove that this is a vector space: the set of polynomials of degree greater than or equal to two, along with the zero polynomial.

1.32 At this point "the same" is only an intuition, but nonetheless for each vector space identify the k for which the space is "the same" as $\mathbb{R}^k$.

(a) The 2×3 matrices under the usual operations

(b) The $n\times m$ matrices (under their usual operations)

(c) This set of 2×2 matrices

$$\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathbb{R}\}$$

(d) This set of 2×2 matrices

$$\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a+b+c=0\}$$

✓ **1.33** Using $\vec{+}$ to represent vector addition and $\vec{\cdot}$ for scalar multiplication, restate the definition of vector space.

✓ **1.34** Prove these.

(a) Any vector is the additive inverse of the additive inverse of itself.

(b) Vector addition left-cancels: if $\vec{v}, \vec{s}, \vec{t} \in V$ then $\vec{v}+\vec{s}=\vec{v}+\vec{t}$ implies that $\vec{s}=\vec{t}$.

1.35 The definition of vector spaces does not explicitly say that $\vec{0}+\vec{v}=\vec{v}$ (it instead says that $\vec{v}+\vec{0}=\vec{v}$). Show that it must nonetheless hold in any vector space.

✓ **1.36** Prove or disprove that this is a vector space: the set of all matrices, under the usual operations.

1.37 In a vector space every element has an additive inverse. Can some elements have two or more?

1.38 **(a)** Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a vector space under the inherited operations.

(b) What if it doesn't contain the origin?

✓ **1.39** Using the idea of a vector space we can easily reprove that the solution set of a homogeneous linear system has either one element or infinitely many elements. Assume that $\vec{v} \in V$ is not $\vec{0}$.

(a) Prove that $r \cdot \vec{v} = \vec{0}$ if and only if $r=0$.

(b) Prove that $r_1 \cdot \vec{v} = r_2 \cdot \vec{v}$ if and only if $r_1 = r_2$.

(c) Prove that any nontrivial vector space is infinite.

(d) Use the fact that a nonempty solution set of a homogeneous linear system is a vector space to draw the conclusion.

1.40 Is this a vector space under the natural operations: the real-valued functions of one real variable that are differentiable?

1.41 A *vector space over the complex numbers* $\mathbb{C}$ has the same definition as a vector space over the reals except that scalars are drawn from $\mathbb{C}$ instead of from $\mathbb{R}$. Show that each of these is a vector space over the complex numbers. (Recall how complex numbers add and multiply: $(a_0+a_1i)+(b_0+b_1i)=(a_0+b_0)+(a_1+b_1)i$ and $(a_0+a_1i)(b_0+b_1i)=(a_0b_0-a_1b_1)+(a_0b_1+a_1b_0)i$.)

(a) The set of degree two polynomials with complex coefficients

(b) This set

$$\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a,b \in \mathbb{C} \text{ and } a+b=0+0i\}$$

1.42 Name a property shared by all of the $\mathbb{R}^n$'s but not listed as a requirement for a vector space.

✓ **1.43** **(a)** Prove that a sum of four vectors $\vec{v}_1, \ldots, \vec{v}_4 \in V$ can be associated in any way without changing the result.

$$\begin{aligned}((\vec{v}_1+\vec{v}_2)+\vec{v}_3)+\vec{v}_4 &= (\vec{v}_1+(\vec{v}_2+\vec{v}_3))+\vec{v}_4 \\ &= (\vec{v}_1+\vec{v}_2)+(\vec{v}_3+\vec{v}_4) \\ &= \vec{v}_1+((\vec{v}_2+\vec{v}_3)+\vec{v}_4) \\ &= \vec{v}_1+(\vec{v}_2+(\vec{v}_3+\vec{v}_4))\end{aligned}$$

This allows us to simply write '$\vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4$' without ambiguity.

(b) Prove that any two ways of associating a sum of any number of vectors give the same sum. (*Hint.* Use induction on the number of vectors.)

1.44 Example 1.5 gives a subset of $\mathbb{R}^2$ that is not a vector space, under the obvious operations, because while it is closed under addition, it is not closed under scalar multiplication. Consider the set of vectors in the plane whose components have the same sign or are 0. Show that this set is closed under scalar multiplication but not addition.

1.45 For any vector space, a subset that is itself a vector space under the inherited operations (e.g., a plane through the origin inside of $\mathbb{R}^3$) is a *subspace*.

(a) Show that $\{a_0 + a_1x + a_2x^2 \bigm| a_0 + a_1 + a_2 = 0\}$ is a subspace of the vector space of degree two polynomials.

(b) Show that this is a subspace of the 2×2 matrices.

$$\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \bigm| a + b = 0\}$$

(c) Show that a nonempty subset S of a real vector space is a subspace if and only if it is closed under linear combinations of pairs of vectors: whenever $c_1, c_2 \in \mathbb{R}$ and $\vec{s}_1, \vec{s}_2 \in S$ then the combination $c_1\vec{v}_1 + c_2\vec{v}_2$ is in S.

I.2 Subspaces and Spanning Sets

One of the examples that led us to introduce the idea of a vector space was the solution set of a homogeneous system. For instance, we've seen in Example 1.4 such a space that is a planar subset of $\mathbb{R}^3$. There, the vector space $\mathbb{R}^3$ contains inside it another vector space, the plane.

2.1 Definition For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

2.2 Example The plane from the prior subsection,

$$P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| x + y + z = 0\}$$

is a subspace of $\mathbb{R}^3$. As specified in the definition, the operations are the ones that are inherited from the larger space, that is, vectors add in P as they add in $\mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

and scalar multiplication is also the same as it is in $\mathbb{R}^3$. To show that P is a subspace, we need only note that it is a subset and then verify that it is a space. Checking that P satisfies the conditions in the definition of a vector space is routine. For instance, for closure under addition, just note that if the summands satisfy that $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then the sum satisfies that $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0$.

2.3 Example The x-axis in $\mathbb{R}^2$ is a subspace where the addition and scalar multiplication operations are the inherited ones.

$$\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 0 \end{pmatrix} \qquad r \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} rx \\ 0 \end{pmatrix}$$

As above, to verify that this is a subspace, we simply note that it is a subset and then check that it satisfies the conditions in definition of a vector space. For instance, the two closure conditions are satisfied: (1) adding two vectors with a second component of zero results in a vector with a second component of zero, and (2) multiplying a scalar times a vector with a second component of zero results in a vector with a second component of zero.

2.4 Example Another subspace of $\mathbb{R}^2$ is

$$\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

its trivial subspace.

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

2.5 Example The condition in the definition requiring that the addition and scalar multiplication operations must be the ones inherited from the larger space is important. Consider the subset $\{1\}$ of the vector space $\mathbb{R}^1$. Under the operations $1+1=1$ and $r \cdot 1 = 1$ that set is a vector space, specifically, a trivial space. But it is not a subspace of $\mathbb{R}^1$ because those aren't the inherited operations, since of course $\mathbb{R}^1$ has $1+1=2$.

2.6 Example All kinds of vector spaces, not just $\mathbb{R}^n$'s, have subspaces. The vector space of cubic polynomials $\{a + bx + cx^2 + dx^3 \bigm| a, b, c, d \in \mathbb{R}\}$ has a subspace comprised of all linear polynomials $\{m + nx \bigm| m, n \in \mathbb{R}\}$.

2.7 Example Another example of a subspace not taken from an $\mathbb{R}^n$ is one from the examples following the definition of a vector space. The space of all real-valued functions of one real variable $f\colon \mathbb{R} \to \mathbb{R}$ has a subspace of functions satisfying the restriction $(d^2\, f/dx^2) + f = 0$.

2.8 Example Being vector spaces themselves, subspaces must satisfy the closure conditions. The set $\mathbb{R}^+$ is not a subspace of the vector space $\mathbb{R}^1$ because with the inherited operations it is not closed under scalar multiplication: if $\vec{v} = 1$ then $-1 \cdot \vec{v} \not\in \mathbb{R}^+$.

The next result says that Example 2.8 is prototypical. The only way that a subset can fail to be a subspace (if it is nonempty and the inherited operations are used) is if it isn't closed.

2.9 Lemma For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements.*

(1) S is a subspace of that vector space
(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S
(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \dots, \vec{s}_n \in S$ and scalars $r_1, \dots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is in S.

Briefly, the way that a subset gets to be a subspace is by being closed under linear combinations.

Proof. 'The following are equivalent' means that each pair of statements are equivalent.

$$(1) \iff (2) \qquad (2) \iff (3) \qquad (3) \iff (1)$$

We will show this equivalence by establishing that $(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by noticing that $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue the single implication $(2) \implies (1)$.

For that argument, assume that S is a nonempty subset of a vector space V and that S is closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The first item in the vector space definition has five conditions. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V, and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that (where $\vec{s}$ is any member of the nonempty set S) $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is in S; showing that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S; showing that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

The checks for item (2) are similar and are saved for Exercise 32. QED

We usually show that a subset is a subspace with $(2) \implies (1)$.

2.10 Remark At the start of this chapter we introduced vector spaces as collections in which linear combinations are "sensible". The above result speaks to this.

The vector space definition has ten conditions but eight of them — the conditions not about closure — simply ensure that referring to the operations as an 'addition' and a 'scalar multiplication' is sensible. The proof above checks that these eight are inherited from the surrounding vector space provided that the

* More information on equivalence of statements is in the appendix.

nonempty set S satisfies Theorem 2.9's statement (2) (e.g., commutativity of addition in S follows right from commutativity of addition in V). So, in this context, this meaning of "sensible" is automatically satisfied.

In assuring us that this first meaning of the word is met, the result draws our attention to the second meaning of "sensible". It has to do with the two remaining conditions, the closure conditions. Above, the two separate closure conditions inherent in statement (1) are combined in statement (2) into the single condition of closure under all linear combinations of two vectors, which is then extended in statement (3) to closure under combinations of any number of vectors. The latter two statements say that we can always make sense of an expression like $r_1\vec{s}_1 + r_2\vec{s}_2$, without restrictions on the r's — such expressions are "sensible" in that the vector described is defined and is in the set S.

This second meaning suggests that a good way to think of a vector space is as a collection of unrestricted linear combinations. The next two examples take some spaces and describe them in this way. That is, in these examples we parametrize, just as we did in Chapter One to describe the solution set of a homogeneous linear system.

2.11 Example This subset of $\mathbb{R}^3$

$$S = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x - 2y + z = 0\}$$

is a subspace under the usual addition and scalar multiplication operations of column vectors (the check that it is nonempty and closed under linear combinations of two vectors is just like the one in Example 2.2). To parametrize, we can take $x - 2y + z = 0$ to be a one-equation linear system and expressing the leading variable in terms of the free variables $x = 2y - z$.

$$S = \{\begin{pmatrix} 2y - z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R}\} = \{y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R}\}$$

Now the subspace is described as the collection of unrestricted linear combinations of those two vectors. Of course, in either description, this is a plane through the origin.

2.12 Example This is a subspace of the 2×2 matrices

$$L = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a + b + c = 0\}$$

(checking that it is nonempty and closed under linear combinations is easy). To parametrize, express the condition as $a = -b - c$.

$$L = \{\begin{pmatrix} -b - c & 0 \\ b & c \end{pmatrix} \mid b, c \in \mathbb{R}\} = \{b\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{R}\}$$

As above, we've described the subspace as a collection of unrestricted linear combinations (by coincidence, also of two elements).

Parametrization is an easy technique, but it is important. We shall use it often.

2.13 Definition The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \bigm| c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common, but so are 'span(S)' and 'sp(S)'.

2.14 Remark In Chapter One, after we showed that the solution set of a homogeneous linear system can be written as $\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \bigm| c_1, \ldots, c_k \in \mathbb{R}\}$, we described that as the set 'generated' by the $\vec{\beta}$'s. We now have the technical term; we call that the 'span' of the set $\{\vec{\beta}_1, \ldots, \vec{\beta}_k\}$.

Recall also the discussion of the "tricky point" in that proof. The span of the empty set is defined to be the set $\{\vec{0}\}$ because we follow the convention that a linear combination of no vectors sums to $\vec{0}$. Besides, defining the empty set's span to be the trivial subspace is a convienence in that it keeps results like the next one from having annoying exceptional cases.

2.15 Lemma In a vector space, the span of any subset is a subspace.

PROOF. Call the subset S. If S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

(p, r scalars) is a linear combination of elements of S and so is in $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$, but it does not matter). QED

The converse of the lemma holds: any subspace is the span of some set, because a subspace is obviously the span of the set of its members. Thus a subset of a vector space is a subspace if and only if it is a span. This fits the intuition that a good way to think of a vector space is as a collection in which linear combinations are sensible.

Taken together, Lemma 2.9 and Lemma 2.15 show that the span of a subset S of a vector space is the smallest subspace containing all the members of S.

2.16 Example In any vector space V, for any vector $\vec{v}$, the set $\{r \cdot \vec{v} \bigm| r \in \mathbb{R}\}$ is a subspace of V. For instance, for any vector $\vec{v} \in \mathbb{R}^3$, the line through the origin containing that vector, $\{k\vec{v} \bigm| k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. This is true even when $\vec{v}$ is the zero vector, in which case the subspace is the degenerate line, the trivial subspace.

2.17 Example The span of this set is all of $\mathbb{R}^2$.

$$\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\}$$

To check this we must show that any member of $\mathbb{R}^2$ is a linear combination of these two vectors. So we ask: for which vectors (with real components x and y) are there scalars c_1 and c_2 such that this holds?

$$c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$$

Gauss' method

$$\begin{array}{rl} c_1+c_2=x \\ c_1-c_2=y \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1+c_2= & x \\ -2c_2= & -x+y \end{array}$$

with back substitution gives $c_2=(x-y)/2$ and $c_1=(x+y)/2$. These two equations show that for any x and y that we start with, there are appropriate coefficients c_1 and c_2 making the above vector equation true. For instance, for $x=1$ and $y=2$ the coefficients $c_2=-1/2$ and $c_1=3/2$ will do. That is, any vector in $\mathbb{R}^2$ can be written as a linear combination of the two given vectors.

Since spans are subspaces, and we know that a good way to understand a subspace is to parametrize its description, we can try to understand a set's span in that way.

2.18 Example Consider, in $\mathcal{P}_2$, the span of the set $\{3x-x^2,2x\}$. By the definition of span, it is the set of unrestricted linear combinations of the two $\{c_1(3x-x^2)+c_2(2x)\bigm| c_1,c_2\in\mathbb{R}\}$. Clearly polynomials in this span must have a constant term of zero. Is that necessary condition also sufficient?

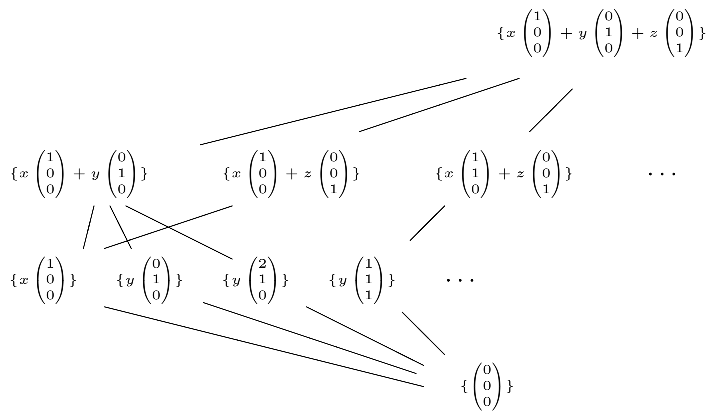
We are asking: for which members $a_2x^2+a_1x+a_0$ of $\mathcal{P}_2$ are there c_1 and c_2 such that $a_2x^2+a_1x+a_0=c_1(3x-x^2)+c_2(2x)$? Since polynomials are equal if and only if their coefficients are equal, we are looking for conditions on a_2, a_1, and a_0 satisfying these.

$$\begin{aligned} -c_1 \qquad &= a_2 \\ 3c_1+2c_2 &= a_1 \\ 0 &= a_0 \end{aligned}$$

Gauss' method gives that $c_1=-a_2$, $c_2=(3/2)a_2+(1/2)a_1$, and $0=a_0$. Thus the only condition on polynomials in the span is the condition that we knew of — as long as $a_0=0$, we can give appropriate coefficients c_1 and c_2 to describe the polynomial $a_0+a_1x+a_2x^2$ as in the span. For instance, for the polynomial $0-4x+3x^2$, the coefficients $c_1=-3$ and $c_2=5/2$ will do. So the span of the given set is $\{a_1x+a_2x^2\bigm| a_1,a_2\in\mathbb{R}\}$.

This shows, incidentally, that the set $\{x,x^2\}$ also spans this subspace. A space can have more than one spanning set. Two other sets spanning this subspace are $\{x,x^2,-x+2x^2\}$ and $\{x,x+x^2,x+2x^2,\dots\}$. (Naturally, we usually prefer to work with spanning sets that have only a few members.)

2.19 Example These are the subspaces of $\mathbb{R}^3$ that we now know of, the trivial subspace, the lines through the origin, the planes through the origin, and the whole space (of course, the picture shows only a few of the infinitely many subspaces). In the next section we will prove that $\mathbb{R}^3$ has no other type of subspaces, so in fact this picture shows them all.



The subsets are described as spans of sets, using a minimal number of members, and are shown connected to their supersets. Note that these subspaces fall naturally into levels — planes on one level, lines on another, etc. — according to how many vectors are in a minimal-sized spanning set.

So far in this chapter we have seen that to study the properties of linear combinations, the right setting is a collection that is closed under these combinations. In the first subsection we introduced such collections, vector spaces, and we saw a great variety of examples. In this subsection we saw still more spaces, ones that happen to be subspaces of others. In all of the variety we've seen a commonality. Example 2.19 above brings it out: vector spaces and subspaces are best understood as a span, and especially as a span of a small number of vectors. The next section studies spanning sets that are minimal.

Exercises

✓ **2.20** Which of these subsets of the vector space of 2×2 matrices are subspaces under the inherited operations? For each one that is a subspace, parametrize its description. For each that is not, give a condition that fails.

(a) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \bigm| a,b\in\mathbb{R}\}$

(b) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \bigm| a+b=0\}$

(c) $\{\begin{pmatrix}a&0\\0&b\end{pmatrix} \bigm| a+b=5\}$

(d) $\{\begin{pmatrix}a&c\\0&b\end{pmatrix} \bigm| a+b=0, c\in\mathbb{R}\}$

✓ **2.21** Is this a subspace of $\mathcal{P}_2$: $\{a_0+a_1x+a_2x^2 \bigm| a_0+2a_1+a_2=4\}$? If it is then parametrize its description.

✓ **2.22** Decide if the vector lies in the span of the set, inside of the space.

(a) $\begin{pmatrix}2\\0\\1\end{pmatrix}$, $\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$, in $\mathbb{R}^3$

(b) $x-x^3$, $\{x^2, 2x+x^2, x+x^3\}$, in $\mathcal{P}_3$

(c) $\begin{pmatrix}0&1\\4&2\end{pmatrix}$, $\{\begin{pmatrix}1&0\\1&1\end{pmatrix},\begin{pmatrix}2&0\\2&3\end{pmatrix}\}$, in $\mathcal{M}_{2\times2}$

2.23 Which of these are members of the span $[\{\cos^2 x, \sin^2 x\}]$ in the vector space of real-valued functions of one real variable?

(a) $f(x)=1$ **(b)** $f(x)=3+x^2$ **(c)** $f(x)=\sin x$ **(d)** $f(x)=\cos(2x)$

✓ **2.24** Which of these sets spans $\mathbb{R}^3$? That is, which of these sets has the property that any three-tall vector can be expressed as a suitable linear combination of the set's elements?

(a) $\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\end{pmatrix}\}$ **(b)** $\{\begin{pmatrix}2\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\}$ **(c)** $\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}3\\0\\0\end{pmatrix}\}$

(d) $\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\0\end{pmatrix},\begin{pmatrix}2\\1\\5\end{pmatrix}\}$ **(e)** $\{\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}3\\0\\1\end{pmatrix},\begin{pmatrix}5\\1\\2\end{pmatrix},\begin{pmatrix}6\\0\\2\end{pmatrix}\}$

✓ **2.25** Parametrize each subspace's description. Then express each subspace as a span.

(a) The subset $\{\begin{pmatrix}a&b&c\end{pmatrix} \mid a-c=0\}$ of the three-wide row vectors

(b) This subset of $\mathcal{M}_{2\times2}$

$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid a+d=0\}$$

(c) This subset of $\mathcal{M}_{2\times2}$

$$\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \mid 2a-c-d=0 \text{ and } a+3b=0\}$$

(d) The subset $\{a+bx+cx^3 \mid a-2b+c=0\}$ of $\mathcal{P}_3$

(e) The subset of $\mathcal{P}_2$ of quadratic polynomials p such that $p(7)=0$

✓ **2.26** Find a set to span the given subspace of the given space. (*Hint.* Parametrize each.)

(a) the xz-plane in $\mathbb{R}^3$

(b) $\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid 3x+2y+z=0\}$ in $\mathbb{R}^3$

(c) $\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \mid 2x+y+w=0 \text{ and } y+2z=0\}$ in $\mathbb{R}^4$

(d) $\{a_0+a_1x+a_2x^2+a_3x^3 \mid a_0+a_1=0 \text{ and } a_2-a_3=0\}$ in $\mathcal{P}_3$

(e) The set $\mathcal{P}_4$ in the space $\mathcal{P}_4$

(f) $\mathcal{M}_{2\times2}$ in $\mathcal{M}_{2\times2}$

2.27 Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

✓ **2.28** Decide if each is a subspace of the vector space of real-valued functions of one real variable.

(a) The *even* functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid f(-x)=f(x) \text{ for all } x\}$. For example, two members of this set are $f_1(x)=x^2$ and $f_2(x)=\cos(x)$.

(b) The *odd* functions $\{f\colon \mathbb{R}\to\mathbb{R} \mid f(-x)=-f(x) \text{ for all } x\}$. Two members are $f_3(x)=x^3$ and $f_4(x)=\sin(x)$.

2.29 Example 2.16 says that for any vector $\vec{v}$ that is an element of a vector space V, the set $\{r\cdot\vec{v} \mid r\in\mathbb{R}\}$ is a subspace of V. (This is of course, simply the span of the singleton set $\{\vec{v}\}$.) Must any such subspace be a proper subspace, or can it be improper?

2.30 An example following the definition of a vector space shows that the solution set of a homogeneous linear system is a vector space. In the terminology of this subsection, it is a subspace of $\mathbb{R}^n$ where the system has n variables. What about a non-homogeneous linear system; do its solutions form a subspace (under the inherited operations)?

2.31 Example 2.19 shows that $\mathbb{R}^3$ has infinitely many subspaces. Does every nontrivial space have infinitely many subspaces?

2.32 Finish the proof of Lemma 2.9.

2.33 Show that each vector space has only one trivial subspace.

✓ **2.34** Show that for any subset S of a vector space, the span of the span equals the span $[[S]]=[S]$. (*Hint.* Members of $[S]$ are linear combinations of members of S. Members of $[[S]]$ are linear combinations of linear combinations of members of S.)

2.35 All of the subspaces that we've seen use zero in their description in some way. For example, the subspace in Example 2.3 consists of all the vectors from $\mathbb{R}^2$ with a second component of zero. In contrast, the collection of vectors from $\mathbb{R}^2$ with a second component of one does not form a subspace (it is not closed under scalar multiplication). Another example is Example 2.2, where the condition on the vectors is that the three components add to zero. If the condition were that the three components add to one then it would not be a subspace (again, it would fail to be closed). This exercise shows that a reliance on zero is not strictly necessary. Consider the set

$$\{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid x+y+z=1\}$$

under these operations.

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2-1\\y_1+y_2\\z_1+z_2\end{pmatrix} \qquad r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx-r+1\\ry\\rz\end{pmatrix}$$

(a) Show that it is not a subspace of $\mathbb{R}^3$. (*Hint.* See Example 2.5).

(b) Show that it is a vector space. Note that by the prior item, Lemma 2.9 can not apply.

(c) Show that any subspace of $\mathbb{R}^3$ must pass through the origin, and so any subspace of $\mathbb{R}^3$ must involve zero in its description. Does the converse hold? Does any subset of $\mathbb{R}^3$ that contains the origin become a subspace when given the inherited operations?

2.36 We can give a justification for the convention that the sum of zero-many vectors equals the zero vector. Consider this sum of three vectors $\vec{v}_1+\vec{v}_2+\vec{v}_3$.

(a) What is the difference between this sum of three vectors and the sum of the first two of these three?

(b) What is the difference between the prior sum and the sum of just the first one vector?

(c) What should be the difference between the prior sum of one vector and the sum of no vectors?

(d) So what should be the definition of the sum of no vectors?

2.37 Is a space determined by its subspaces? That is, if two vector spaces have the same subspaces, must the two be equal?

2.38 **(a)** Give a set that is closed under scalar multiplication but not addition.

(b) Give a set closed under addition but not scalar multiplication.

(c) Give a set closed under neither.

2.39 Show that the span of a set of vectors does not depend on the order in which the vectors are listed in that set.

2.40 Which trivial subspace is the span of the empty set? Is it

$$\{\begin{pmatrix}0\\0\\0\end{pmatrix}\} \subseteq \mathbb{R}^3, \quad \text{or} \quad \{0+0x\} \subseteq \mathcal{P}_1,$$

or some other subspace?

2.41 Show that if a vector is in the span of a set then adding that vector to the set won't make the span any bigger. Is that also 'only if'?

✓ **2.42** Subspaces are subsets and so we naturally consider how 'is a subspace of' interacts with the usual set operations.

(a) If A, B are subspaces of a vector space, must their interesction $A \cap B$ be a subspace? Always? Sometimes? Never?

(b) Must the union $A \cup B$ be a subspace?

(c) If A is a subspace, must its complement be a subspace?

(*Hint.* Try some test subspaces from Example 2.19.)

✓ **2.43** Does the span of a set depend on the enclosing space? That is, if W is a subspace of V and S is a subset of W (and so also a subset of V), might the span of S in W differ from the span of S in V?

2.44 Is the relation 'is a subspace of' transitive? That is, if V is a subspace of W and W is a subspace of X, must V be a subspace of X?

✓ **2.45** Because 'span of' is an operation on sets we naturally consider how it interacts with the usual set operations.

(a) If $S \subseteq T$ are subsets of a vector space, is $[S] \subseteq [T]$? Always? Sometimes? Never?

(b) If S, T are subsets of a vector space, is $[S \cup T] = [S] \cup [T]$?

(c) If S, T are subsets of a vector space, is $[S \cap T] = [S] \cap [T]$?

(d) Is the span of the complement equal to the complement of the span?

2.46 Reprove Lemma 2.15 without doing the empty set separately.

2.47 Find a structure that is closed under linear combinations, and yet is not a vector space. (*Remark.* This is a bit of a trick question.)

II Linear Independence

The prior section shows that a vector space can be understood as an unrestricted linear combination of some of its elements — that is, as a span. For example, the space of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$ is spanned by the set $\{1, x\}$. The prior section also showed that a space can have many sets that span it. The space of linear polynomials is also spanned by $\{1, 2x\}$ and $\{1, x, 2x\}$.

At the end of that section we described some spanning sets as 'minimal', but we never precisely defined that word. We could take 'minimal' to mean one of two things. We could mean that a spanning set is minimal if it contains the smallest number of members of any set with the same span. With this meaning $\{1, x, 2x\}$ is not minimal because it has one member more than the other two. Or we could mean that a spanning set is minimal when it has no elements that can be removed without changing the span. Under this meaning $\{1, x, 2x\}$ is not minimal because removing the $2x$ and getting $\{1, x\}$ leaves the span unchanged.

The first sense of minimality appears to be a global requirement, in that to check if a spanning set is minimal we seemingly must look at all the spanning sets of a subspace and find one with the least number of elements. The second sense of minimality is local in that we need to look only at the set under discussion and consider the span with and without various elements. For instance, using the second sense, we could compare the span of $\{1, x, 2x\}$ with the span of $\{1, x\}$ and note that the $2x$ is a "repeat" in that its removal doesn't shrink the span.

In this section we will use the second sense of 'minimal spanning set' because of this technical convenience. However, the most important result of this book is that the two senses coincide; we will prove that in the section after this one.

II.1 Definition and Examples

We first characterize when a vector can be removed from a set without changing the span of that set. For that, note that if a vector $\vec{v}$ is not a member of a set S then the union $S \cup \{\vec{v}\}$ and the set S differ only in that the former contains $\vec{v}$.

1.1 Lemma Where S is a subset of a vector space V,

$$[S] = [S \cup \{\vec{v}\}] \quad \text{if and only if} \quad \vec{v} \in [S]$$

for any $\vec{v} \in V$.

Proof. The left to right implication is easy. If $[S] = [S \cup \{\vec{v}\}]$ then, since $\vec{v} \in [S \cup \{\vec{v}\}]$, the equality of the two sets gives that $\vec{v} \in [S]$.

For the right to left implication assume that $\vec{v} \in [S]$ to show that $[S] = [S \cup \{\vec{v}\}]$ by mutual inclusion. The inclusion $[S] \subseteq [S \cup \{\vec{v}\}]$ is obvious. For the other inclusion $[S] \supseteq [S \cup \{\vec{v}\}]$, write an element of $[S \cup \{\vec{v}\}]$ as $d_0\vec{v} + d_1\vec{s}_1 + \cdots + d_m\vec{s}_m$ and substitute $\vec{v}$'s expansion as a linear combination of members of the same set $d_0(c_0\vec{t}_0 + \cdots + c_k\vec{t}_k) + d_1\vec{s}_1 + \cdots + d_m\vec{s}_m$. This is a linear combination of linear combinations and so distributing d_0 results in a linear combination of vectors from S. Hence each member of $[S \cup \{\vec{v}\}]$ is also a member of $[S]$. QED

1.2 Example In $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

the spans $[\{\vec{v}_1, \vec{v}_2\}]$ and $[\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}]$ are equal since $\vec{v}_3$ is in the span $[\{\vec{v}_1, \vec{v}_2\}]$.

The lemma says that if we have a spanning set then we can remove a $\vec{v}$ to get a new set S with the same span if and only if $\vec{v}$ is a linear combination of vectors from S. Thus, under the second sense described above, a spanning set is minimal if and only if it contains no vectors that are linear combinations of the others in that set. We have a term for this important property.

1.3 Definition A subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.

Here is an important observation: although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets $\vec{s}_0$ off from the other vectors, algebraically there is nothing special in that equation about $\vec{s}_0$. For any $\vec{s}_i$ with a coefficient c_i that is nonzero, we can rewrite the relationship to set off $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + (-c_1/c_i)\vec{s}_1 + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector by writing it alone on one side of the equation we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and write the relationship with all of the vectors on the same side. The next result rephrases the linear independence definition in this style. It gives what is usually the easiest way to compute whether a finite set is dependent or independent.

1.4 Lemma A subset S of a vector space is linearly independent if and only if for any distinct $\vec{s}_1, \ldots, \vec{s}_n \in S$ the only linear relationship among those vectors

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0} \qquad c_1, \ldots, c_n \in \mathbb{R}$$

is the trivial one: $c_1 = 0, \ldots, c_n = 0$.

Proof. This is a direct consequence of the observation above.

If the set S is linearly independent then no vector $\vec{s}_i$ can be written as a linear combination of the other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients. If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S, and subtracting $\vec{s}_i$ from both sides of that equation gives a linear relationship involving a nonzero coefficient, namely the -1 in front of $\vec{s}_i$. QED

1.5 Example In the vector space of two-wide row vectors, the two-element set $\{\begin{pmatrix}40 & 15\end{pmatrix}, \begin{pmatrix}-50 & 25\end{pmatrix}\}$ is linearly independent. To check this, set

$$c_1 \cdot \begin{pmatrix}40 & 15\end{pmatrix} + c_2 \cdot \begin{pmatrix}-50 & 25\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}$$

and solving the resulting system

$$\begin{array}{rl} 40c_1 - 50c_2 = 0 \\ 15c_1 + 25c_2 = 0 \end{array} \xrightarrow{-(15/40)\rho_1+\rho_2} \begin{array}{rl} 40c_1 - \quad 50c_2 = 0 \\ (175/4)c_2 = 0 \end{array}$$

shows that both c_1 and c_2 are zero. So the only linear relationship between the two given row vectors is the trivial relationship.

In the same vector space, $\{\begin{pmatrix}40 & 15\end{pmatrix}, \begin{pmatrix}20 & 7.5\end{pmatrix}\}$ is linearly dependent since we can satisfy

$$c_1 \begin{pmatrix}40 & 15\end{pmatrix} + c_2 \cdot \begin{pmatrix}20 & 7.5\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}$$

with $c_1 = 1$ and $c_2 = -2$.

1.6 Remark Recall the Statics example that began this book. We first set the unknown-mass objects at 40 cm and 15 cm and got a balance, and then we set the objects at -50 cm and 25 cm and got a balance. With those two pieces of information we could compute values of the unknown masses. Had we instead first set the unknown-mass objects at 40 cm and 15 cm, and then at 20 cm and 7.5 cm, we would not have been able to compute the values of the unknown masses (try it). Intuitively, the problem is that the $\begin{pmatrix}20 & 7.5\end{pmatrix}$ information is a "repeat" of the $\begin{pmatrix}40 & 15\end{pmatrix}$ information — that is, $\begin{pmatrix}20 & 7.5\end{pmatrix}$ is in the span of the set $\{\begin{pmatrix}40 & 15\end{pmatrix}\}$ — and so we would be trying to solve a two-unknowns problem with what is essentially one piece of information.

1.7 Example The set $\{1+x, 1-x\}$ is linearly independent in $\mathcal{P}_2$, the space of quadratic polynomials with real coefficients, because

$$0 + 0x + 0x^2 = c_1(1+x) + c_2(1-x) = (c_1+c_2) + (c_1-c_2)x + 0x^2$$

gives

$$\begin{array}{rl} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + \quad c_2 = 0 \\ 2c_2 = 0 \end{array}$$

since polynomials are equal only if their coefficients are equal. Thus, the only linear relationship between these two members of $\mathcal{P}_2$ is the trivial one.

1.8 Example In $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{pmatrix}3\\4\\5\end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix}2\\9\\2\end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix}4\\18\\4\end{pmatrix}$$

the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent because this is a relationship

$$0 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 - 1 \cdot \vec{v}_3 = \vec{0}$$

where not all of the scalars are zero (the fact that some of the scalars are zero doesn't matter).

1.9 Remark That example illustrates why, although Definition 1.3 is a clearer statement of what independence is, Lemma 1.4 is more useful for computations. Working straight from the definition, someone trying to compute whether S is linearly independent would start by setting $\vec{v}_1 = c_2\vec{v}_2 + c_3\vec{v}_3$ and concluding that there are no such c_2 and c_3. But knowing that the first vector is not dependent on the other two is not enough. This person would have to go on to try $\vec{v}_2 = c_1\vec{v}_1 + c_3\vec{v}_3$ to find the dependence $c_1 = 0$, $c_3 = 1/2$. Lemma 1.4 gets the same conclusion with only one computation.

1.10 Example The empty subset of a vector space is linearly independent. There is no nontrivial linear relationship among its members as it has no members.

1.11 Example In any vector space, any subset containing the zero vector is linearly dependent. For example, in the space $\mathcal{P}_2$ of quadratic polynomials, consider the subset $\{1+x, x+x^2, 0\}$.

One way to see that this subset is linearly dependent is to use Lemma 1.4: we have $0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + 1\cdot\vec{0} = \vec{0}$, and this is a nontrivial relationship as not all of the coefficients are zero. Another way to see that this subset is linearly dependent is to go straight to Definition 1.3: we can express the third member of the subset as a linear combination of the first two, namely, $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ is satisfied by taking $c_1 = 0$ and $c_2 = 0$ (in contrast to the lemma, the definition allows all of the coefficients to be zero).

(There is subtler way to see that this subset is dependent. The zero vector is equal to the trivial sum, the sum of the empty set. So a set containing the zero vector has an element that can be written as a combination of a set of other vectors from the set, specifically, the zero vector can be written as a combination of the empty set.)

The above examples, especially Example 1.5, underline the discussion that begins this section. The next result says that given a finite set, we can produce a linearly independent subset by discarding what Remark 1.6 calls "repeats".

1.12 Theorem In a vector space, any finite subset has a linearly independent subset with the same span.

Proof. If the set $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependence, there is a vector $\vec{s}_i$ that is a linear combination of the others. Call that vector $\vec{v}_1$. Discard it — define the set $S_1 = S - \{\vec{v}_1\}$. By Lemma 1.1, the span does not shrink $[S_1] = [S]$.

Now, if S_1 is linearly independent then we are finished. Otherwise iterate the prior paragraph: take a vector $\vec{v}_2$ that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. (Formally, this argument uses induction on n, the number of elements in the starting set. Exercise 37 asks for the details.) QED

1.13 Example This set spans $\mathbb{R}^3$ (the check of this is easy).

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \}$$

Looking for a linear relationship

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_5 \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

gives a system

$$\begin{array}{rcrcrcrcl} c_1 & & & + & c_3 & + & & + & 3c_5 = 0 \\ & & 2c_2 & + & 2c_3 & - & c_4 & + & 3c_5 = 0 \\ & & & & & & c_4 & + & \quad\;\, = 0 \end{array}$$

with leading variables c_1, c_2, and c_4 and free variables c_3 and c_5. We can paramatrize the solution set in this way.

$$\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = c_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} -3 \\ -3/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \bigm| c_3, c_5 \in \mathbb{R} \}$$

So S is linearly dependent.

To find something to discard, consider the vectors associated with the free variables c_3 and c_5. Setting $c_3 = 0$ and $c_5 = 1$ shows that that $c_1 = -3$, $c_2 = -3/2$, $c_3 = 0$, $c_4 = 0$, and $c_5 = 1$ is a linear dependence in equation $(*)$ above, that is, c_5's vector is a linear combination of the first two. Lemma 1.1 says that discarding this fifth vector

$$S_1 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \}$$

leaves the span unchanged $[S_1] = [S]$.

Similarly, setting $c_3 = 1$ and $c_5 = 0$ gives a linear dependence in equation $(*)$ above. Since $c_5 = 0$ this is a relationship among the first four vectors, the members of S_1. Thus we can discard c_3's vector from S_1 to get

$$S_2 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \}$$

with the same span as S_1, and therefore the same span as S, but with one difference. We can easily check that S_2 is linearly independent and so discarding any of its elements will shrink the span.

That example makes clear the general method: given a finite set of vectors, we first write the system to find a linear dependence. Then discarding any vectors associated with the free variables of that system will leave the span unchanged.

Theorem 1.12 describes producing a linearly independent set by shrinking, that is, by taking subsets. We finish this subsection by considering how linear independence and dependence, which are properties of sets, interact with the subset relation between sets.

1.14 Lemma Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof. This is clear. QED

Restated, independence is preserved by subset and dependence is preserved by superset.

Those are two of the four possible cases of interaction that we can consider. The third case, whether linear dependence is preserved by the subset operation, is covered by Example 1.13, which gives a linearly dependent set S with a subset S_1 that is linearly dependent and another subset S_2 that is linearly independent.

That leaves one case, whether linear independence is preserved by superset. The next example shows what can happen.

1.15 Example In each of these three paragraphs the subset S is linearly independent.

For the set

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$$

the span $[S]$ is the x axis. Here are two supersets of S, one linearly dependent and the other linearly independent.

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \} \qquad \text{independent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

Checking the dependence or independence of these sets is easy.

For

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

the span $[S]$ is the xy plane. These are two supersets.

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \} \qquad \text{independent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

If

$$S = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

then $[S] = \mathbb{R}^3$. A linearly dependent superset is

$$\text{dependent: } \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \}$$

but there are no linearly independent supersets of S. The reason is that for any vector that we would add to make a superset, the linear dependence equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

has a solution $c_1 = x$, $c_2 = y$, and $c_3 = z$.

So, in general, a linearly independent set may have a superset that is dependent. And, in general, a linearly independent set may have a superset that is independent. We can characterize when the superset is one and when it is the other.

1.16 Lemma Where S is a linearly independent subset of a vector space V,

$$S \cup \{\vec{v}\} \text{ is linearly dependent} \quad \text{if and only if} \quad \vec{v} \in [S]$$

for any $\vec{v} \in V$ with $\vec{v} \notin S$.

Proof. One implication is clear: if $\vec{v} \in [S]$ then $\vec{v} = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$ where each $\vec{s}_i \in S$ and $c_i \in \mathbb{R}$, and so $\vec{0} = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n + (-1)\vec{v}$ is a nontrivial linear relationship among elements of $S \cup \{\vec{v}\}$.

The other implication requires the assumption that S is linearly independent. With $S \cup \{\vec{v}\}$ linearly dependent, there is a nontrivial linear relationship $c_0\vec{v} + c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n = \vec{0}$ and independence of S then implies that $c_0 \neq 0$, or else that would be a nontrivial relationship among members of S. Now rewriting this equation as $\vec{v} = -(c_1/c_0)\vec{s}_1 - \cdots - (c_n/c_0)\vec{s}_n$ shows that $\vec{v} \in [S]$. QED

(Compare this result with Lemma 1.1. Both say, roughly, that $\vec{v}$ is a "repeat" if it is in the span of S. However, note the additional hypothesis here of linear independence.)

1.17 Corollary A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s}_i$ is a linear combination of the vectors $\vec{s}_1, \ldots, \vec{s}_{i-1}$ listed before it.

Proof. Consider $S_0 = \{\}$, $S_1 = \{\vec{s}_1\}$, $S_2 = \{\vec{s}_1, \vec{s}_2\}$, etc. Some index $i \geq 1$ is the first one with $S_{i-1} \cup \{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i \in [S_{i-1}]$. QED

Lemma 1.16 can be restated in terms of independence instead of dependence: if S is linearly independent and $\vec{v} \notin S$ then the set $S \cup \{\vec{v}\}$ is also linearly independent if and only if $\vec{v} \notin [S]$. Applying Lemma 1.1, we conclude that if S is linearly independent and $\vec{v} \notin S$ then $S \cup \{\vec{v}\}$ is also linearly independent if and only if $[S \cup \{\vec{v}\}] \neq [S]$. Briefly, when passing from S to a superset S_1, to preserve linear independence we must expand the span $[S_1] \supset [S]$.

Example 1.15 shows that some linearly independent sets are maximal — have as many elements as possible — in that they have no supersets that are linearly independent. By the prior paragraph, a linearly independent sets is maximal if and only if it spans the entire space, because then no vector exists that is not already in the span.

This table summarizes the interaction between the properties of independence and dependence and the relations of subset and superset.

	$S_1 \subset S$	$S_1 \supset S$
S independent	S_1 must be independent	S_1 may be either
S dependent	S_1 may be either	S_1 must be dependent

In developing this table we've uncovered an intimate relationship between linear independence and span. Complementing the fact that a spanning set is minimal if and only if it is linearly independent, a linearly independent set is maximal if and only if it spans the space.

In summary, we have introduced the definition of linear independence to formalize the idea of the minimality of a spanning set. We have developed some properties of this idea. The most important is Lemma 1.16, which tells us that a linearly independent set is maximal when it spans the space.

Exercises

✓ **1.18** Decide whether each subset of $\mathbb{R}^3$ is linearly dependent or linearly independent.

(a) $\{\begin{pmatrix}1\\-3\\5\end{pmatrix}, \begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}4\\-4\\14\end{pmatrix}\}$

(b) $\{\begin{pmatrix}1\\7\\7\end{pmatrix}, \begin{pmatrix}2\\7\\7\end{pmatrix}, \begin{pmatrix}3\\7\\7\end{pmatrix}\}$

(c) $\{\begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\4\end{pmatrix}\}$

(d) $\{\begin{pmatrix}9\\9\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}3\\5\\-4\end{pmatrix}, \begin{pmatrix}12\\12\\-1\end{pmatrix}\}$

✓ **1.19** Which of these subsets of $\mathcal{P}_3$ are linearly dependent and which are independent?

(a) $\{3 - x + 9x^2, 5 - 6x + 3x^2, 1 + 1x - 5x^2\}$

(b) $\{-x^2, 1 + 4x^2\}$

(c) $\{2 + x + 7x^2, 3 - x + 2x^2, 4 - 3x^2\}$

(d) $\{8+3x+3x^2, x+2x^2, 2+2x+2x^2, 8-2x+5x^2\}$

✓ **1.20** Prove that each set $\{f, g\}$ is linearly independent in the vector space of all functions from $\mathbb{R}^+$ to $\mathbb{R}$.

(a) $f(x) = x$ and $g(x) = 1/x$
(b) $f(x) = \cos(x)$ and $g(x) = \sin(x)$
(c) $f(x) = e^x$ and $g(x) = \ln(x)$

✓ **1.21** Which of these subsets of the space of real-valued functions of one real variable is linearly dependent and which is linearly independent? (Note that we have abbreviated some constant functions; e.g., in the first item, the '2' stands for the constant function $f(x) = 2$.)

(a) $\{2, 4\sin^2(x), \cos^2(x)\}$ **(b)** $\{1, \sin(x), \sin(2x)\}$ **(c)** $\{x, \cos(x)\}$
(d) $\{(1+x)^2, x^2+2x, 3\}$ **(e)** $\{\cos(2x), \sin^2(x), \cos^2(x)\}$ **(f)** $\{0, x, x^2\}$

1.22 Does the equation $\sin^2(x)/\cos^2(x) = \tan^2(x)$ show that this set of functions $\{\sin^2(x), \cos^2(x), \tan^2(x)\}$ is a linearly dependent subset of the set of all real-valued functions with domain the interval $(-\pi/2..\pi/2)$ of real numbers between $-\pi/2$ and $\pi/2)$?

1.23 Why does Lemma 1.4 say "distinct"?

✓ **1.24** Show that the nonzero rows of an echelon form matrix form a linearly independent set.

✓ **1.25** **(a)** Show that if the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent set then so is the set $\{\vec{u}, \vec{u}+\vec{v}, \vec{u}+\vec{v}+\vec{w}\}$.

(b) What is the relationship between the linear independence or dependence of the set $\{\vec{u}, \vec{v}, \vec{w}\}$ and the independence or dependence of $\{\vec{u}-\vec{v}, \vec{v}-\vec{w}, \vec{w}-\vec{u}\}$?

1.26 Example 1.10 shows that the empty set is linearly independent.

(a) When is a one-element set linearly independent?
(b) How about a set with two elements?

1.27 In any vector space V, the empty set is linearly independent. What about all of V?

1.28 Show that if $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent then so are all of its proper subsets: $\{\vec{x}, \vec{y}\}$, $\{\vec{x}, \vec{z}\}$, $\{\vec{y}, \vec{z}\}$, $\{\vec{x}\}$,$\{\vec{y}\}$, $\{\vec{z}\}$, and $\{\}$. Is that 'only if' also?

1.29 **(a)** Show that this

$$S = \{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}\}$$

is a linearly independent subset of $\mathbb{R}^3$.

(b) Show that

$$\begin{pmatrix}3\\2\\0\end{pmatrix}$$

is in the span of S by finding c_1 and c_2 giving a linear relationship.

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$$

Show that the pair c_1, c_2 is unique.

(c) Assume that S is a subset of a vector space and that $\vec{v}$ is in $[S]$, so that $\vec{v}$ is a linear combination of vectors from S. Prove that if S is linearly independent then a linear combination of vectors from S adding to $\vec{v}$ is unique (that is, unique up to reordering and adding or taking away terms of the form $0\cdot\vec{s}$). Thus S

as a spanning set is minimal in this strong sense: each vector in $[S]$ is "hit" a minimum number of times — only once.

(d) Prove that it can happen when S is not linearly independent that distinct linear combinations sum to the same vector.

1.30 Prove that a polynomial gives rise to the zero function if and only if it is the zero polynomial. (*Comment.* This question is not a Linear Algebra matter, but we often use the result. A polynomial gives rise to a function in the obvious way: $x \mapsto c_n x^n + \cdots + c_1 x + c_0$.)

1.31 Return to Section 1.2 and redefine point, line, plane, and other linear surfaces to avoid degenerate cases.

1.32 **(a)** Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

(b) Is this true for any set of five? Any set of three?

(c) What is the most number of elements that a linearly independent subset of $\mathbb{R}^2$ can have?

✓ **1.33** Is there a set of four vectors in $\mathbb{R}^3$, any three of which form a linearly independent set?

1.34 Must every linearly dependent set have a subset that is dependent and a subset that is independent?

1.35 In $\mathbb{R}^4$, what is the biggest linearly independent set you can find? The smallest? The biggest linearly dependent set? The smallest? ('Biggest' and 'smallest' mean that there are no supersets or subsets with the same property.)

✓ **1.36** Linear independence and linear dependence are properties of sets. We can thus naturally ask how those properties act with respect to the familiar elementary set relations and operations. In this body of this subsection we have covered the subset and superset relations. We can also consider the operations of intersection, complementation, and union.

(a) How does linear independence relate to intersection: can an intersection of linearly independent sets be independent? Must it be?

(b) How does linear independence relate to complementation?

(c) Show that the union of two linearly independent sets need not be linearly independent.

(d) Characterize when the union of two linearly independent sets is linearly independent, in terms of the intersection of the span of each.

✓ **1.37** For Theorem 1.12,

(a) fill in the induction for the proof;

(b) give an alternate proof that starts with the empty set and builds a sequence of linearly independent subsets of the given finite set until one appears with the same span as the given set.

1.38 With a little calculation we can get formulas to determine whether or not a set of vectors is linearly independent.

(a) Show that this subset of $\mathbb{R}^2$

$$\{\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\}$$

is linearly independent if and only if $ad - bc \neq 0$.

(b) Show that this subset of $\mathbb{R}^3$

$$\{\begin{pmatrix} a \\ d \\ g \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}, \begin{pmatrix} c \\ f \\ i \end{pmatrix}\}$$

is linearly independent iff $aei + bfg + cdh - hfa - idb - gec \neq 0$.

(c) When is this subset of $\mathbb{R}^3$

$$\{\begin{pmatrix}a\\d\\g\end{pmatrix},\begin{pmatrix}b\\e\\h\end{pmatrix}\}$$

linearly independent?

(d) This is an opinion question: for a set of four vectors from $\mathbb{R}^4$, must there be a formula involving the sixteen entries that determines independence of the set? (You needn't produce such a formula, just decide if one exists.)

✓ **1.39** **(a)** Prove that a set of two perpendicular nonzero vectors from $\mathbb{R}^n$ is linearly independent when $n > 1$.

(b) What if $n = 1$? $n = 0$?

(c) Generalize to more than two vectors.

1.40 Consider the set of functions from the open interval $(-1..1)$ to $\mathbb{R}$.

(a) Show that this set is a vector space under the usual operations.

(b) Recall the formula for the sum of an infinite geometric series: $1+x+x^2+\cdots = 1/(1-x)$ for all $x \in (-1..1)$. Why does this not express a dependence inside of the set $\{g(x) = 1/(1-x), f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, \ldots\}$ (in the vector space that we are considering)? (*Hint.* Review the definition of linear combination.)

(c) Show that the set in the prior item is linearly independent.

This shows that some vector spaces exist with linearly independent subsets that are infinite.

1.41 Show that, where S is a subspace of V, if a subset T of S is linearly independent in S then T is also linearly independent in V. Is that 'only if'?

III Basis and Dimension

The prior section ends with the statement that a spanning set is minimal when it is linearly independent and a linearly independent set is maximal when it spans the space. So the notions of minimal spanning set and maximal independent set coincide. In this section we will name this idea and study its properties.

III.1 Basis

1.1 Definition A *basis* for a vector space is a sequence of vectors that form a set that is linearly independent and that spans the space.

We denote a basis with angle brackets $\langle \vec{\beta}_1, \vec{\beta}_2, \ldots \rangle$ to signify that this collection is a sequence* — the order of the elements is significant. (The requirement that a basis be ordered will be needed, for instance, in Definition 1.13.)

1.2 Example This is a basis for $\mathbb{R}^2$.

$$\langle \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

It is linearly independent

$$c_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{aligned} 2c_1 + 1c_2 &= 0 \\ 4c_1 + 1c_2 &= 0 \end{aligned} \quad \Longrightarrow \quad c_1 = c_2 = 0$$

and it spans $\mathbb{R}^2$.

$$\begin{aligned} 2c_1 + 1c_2 &= x \\ 4c_1 + 1c_2 &= y \end{aligned} \quad \Longrightarrow \quad c_2 = 2x - y \text{ and } c_1 = (y - x)/2$$

1.3 Example This basis for $\mathbb{R}^2$

$$\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \rangle$$

differs from the prior one because the vectors are in a different order. The verification that it is a basis is just as in the prior example.

1.4 Example The space $\mathbb{R}^2$ has many bases. Another one is this.

$$\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

The verification is easy.

* More information on sequences is in the appendix.

1.5 Definition For any $\mathbb{R}^n$,

$$\mathcal{E}_n = \langle \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors by $\vec{e}_1, \dots, \vec{e}_n$.

(Calculus books refer to $\mathbb{R}^2$'s standard basis vectors $\vec{\imath}$ and $\vec{\jmath}$ instead of $\vec{e}_1$ and $\vec{e}_2$, and they refer to $\mathbb{R}^3$'s standard basis vectors $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ instead of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$.) Note that the symbol '$\vec{e}_1$' means something different in a discussion of $\mathbb{R}^3$ than it means in a discussion of $\mathbb{R}^2$.

1.6 Example Consider the space $\{a \cdot \cos\theta + b \cdot \sin\theta \bigm| a, b \in \mathbb{R}\}$ of functions of the real variable θ. This is a natural basis.

$$\langle 1 \cdot \cos\theta + 0 \cdot \sin\theta, 0 \cdot \cos\theta + 1 \cdot \sin\theta \rangle = \langle \cos\theta, \sin\theta \rangle$$

Another, more generic, basis is $\langle \cos\theta - \sin\theta, 2\cos\theta + 3\sin\theta \rangle$. Verfication that these two are bases is Exercise 22.

1.7 Example A natural basis for the vector space of cubic polynomials $\mathcal{P}_3$ is $\langle 1, x, x^2, x^3 \rangle$. Two other bases for this space are $\langle x^3, 3x^2, 6x, 6 \rangle$ and $\langle 1, 1+x, 1+x+x^2, 1+x+x^2+x^3 \rangle$. Checking that these are linearly independent and span the space is easy.

1.8 Example The trivial space $\{\vec{0}\}$ has only one basis, the empty one $\langle\rangle$.

1.9 Example The space of finite-degree polynomials has a basis with infinitely many elements $\langle 1, x, x^2, \dots \rangle$.

1.10 Example We have seen bases before. In the first chapter we described the solution set of homogeneous systems such as this one

$$\begin{array}{rl} x + y - w &= 0 \\ z + w &= 0 \end{array}$$

by parametrizing.

$$\{ \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} y + \begin{pmatrix}1\\0\\-1\\1\end{pmatrix} w \bigm| y, w \in \mathbb{R} \}$$

That is, we described the vector space of solutions as the span of a two-element set. We can easily check that this two-vector set is also linearly independent. Thus the solution set is a subspace of $\mathbb{R}^4$ with a two-element basis.

1.11 Example Parameterization helps find bases for other vector spaces, not just for solution sets of homogeneous systems. To find a basis for this subspace of $\mathcal{M}_{2\times 2}$

$$\{\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a+b-2c=0\}$$

we rewrite the condition as $a=-b+2c$.

$$\{\begin{pmatrix} -b+2c & b \\ c & 0 \end{pmatrix} \mid b,c\in\mathbb{R}\} = \{b\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mid b,c\in\mathbb{R}\}$$

Thus, this is a natural candidate for a basis.

$$\langle \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \rangle$$

The above work shows that it spans the space. To show that it is linearly independent is routine.

Consider again Example 1.2. It involves two verifications.

In the first, to check that the set is linearly independent we looked at linear combinations of the set's members that total to the zero vector $c_1\vec{\beta}_1+c_2\vec{\beta}_2=\binom{0}{0}$. The resulting calculation shows that such a combination is unique, that c_1 must be 0 and c_2 must be 0.

The second verification, that the set spans the space, looks at linear combinations that total to any member of the space $c_1\vec{\beta}_1+c_2\vec{\beta}_2=\binom{x}{y}$. In Example 1.2 we noted only that the resulting calculation shows that such a combination exists, that for each x,y there is a c_1,c_2. However, in fact the calculation also shows that the combination is unique: c_1 must be $(y-x)/2$ and c_2 must be $2x-y$.

That is, the first calculation is a special case of the second. The next result says that this holds in general for a spanning set: the combination totaling to the zero vector is unique if and only if the combination totaling to any vector is unique.

1.12 Theorem In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in a unique way.

We consider combinations to be the same if they differ only in the order of summands or in the addition or deletion of terms of the form '$0\cdot\vec{\beta}$'.

Proof. By definition, a sequence is a basis if and only if its vectors form both a spanning set and a linearly independent set. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way.

Thus, to finish we need only show that a subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way. Consider two expressions of a vector as a linear combination of the members of the basis. We can rearrange the two sums, and if necessary add some $0\vec{\beta}_i$ terms, so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ and $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$$

holds if and only if

$$(c_1 - d_1)\vec{\beta}_1 + \cdots + (c_n - d_n)\vec{\beta}_n = \vec{0}$$

holds, and so asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i. QED

1.13 Definition In a vector space with basis B the *representation of $\vec{v}$ with respect to B* is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $B = \langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to B*.

We will later do representations in contexts that involve more than one basis. To help with the bookkeeping, we shall often attach a subscript B to the column vector.

1.14 Example In $\mathcal{P}_3$, with respect to the basis $B = \langle 1, 2x, 2x^2, 2x^3\rangle$, the representation of $x + x^2$ is

$$\mathrm{Rep}_B(x + x^2) = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix}_B$$

(note that the coordinates are scalars, not vectors). With respect to a different basis $D = \langle 1 + x, 1 - x, x + x^2, x + x^3\rangle$, the representation

$$\mathrm{Rep}_D(x + x^2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}_D$$

is different.

1.15 Remark This use of column notation and the term 'coordinates' has both a down side and an up side.

The down side is that representations look like vectors from $\mathbb{R}^n$, which can be confusing when the vector space we are working with is $\mathbb{R}^n$, especially since we sometimes omit the subscript base. We must then infer the intent from the context. For example, the phrase 'in $\mathbb{R}^2$, where $\vec{v} = \binom{3}{2}$' refers to the plane vector that, when in canonical position, ends at $(3,2)$. To find the coordinates of that vector with respect to the basis

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

we solve

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

to get that $c_1 = 3$ and $c_2 = 1/2$. Then we have this.

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 3 \\ -1/2 \end{pmatrix}$$

Here, although we've ommited the subscript B from the column, the fact that the right side is a representation is clear from the context.

The up side of the notation and the term 'coordinates' is that they generalize the use that we are familiar with: in $\mathbb{R}^n$ and with respect to the standard basis $\mathcal{E}_n$, the vector starting at the origin and ending at $(v_1, \ldots, v_n)$ has this representation.

$$\mathrm{Rep}_{\mathcal{E}_n}(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_{\mathcal{E}_n}$$

Our main use of representations will come in the third chapter. The definition appears here because the fact that every vector is a linear combination of basis vectors in a unique way is a crucial property of bases, and also to help make two points. First, we fix an order for the elements of a basis so that coordinates can be stated in that order. Second, for calculation of coordinates, among other things, we shall restrict our attention to spaces with bases having only finitely many elements. We will see that in the next subsection.

Exercises

✓ **1.16** Decide if each is a basis for $\mathbb{R}^3$.

(a) $\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$ **(b)** $\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \rangle$ **(c)** $\langle \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} \rangle$

(d) $\langle \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \rangle$

✓ **1.17** Represent the vector with respect to the basis.

(a) $\begin{pmatrix}1\\2\end{pmatrix}$, $B = \langle\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\rangle \subseteq \mathbb{R}^2$

(b) $x^2 + x^3$, $D = \langle 1, 1+x, 1+x+x^2, 1+x+x^2+x^3\rangle \subseteq \mathcal{P}_3$

(c) $\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}$, $\mathcal{E}_4 \subseteq \mathbb{R}^4$

1.18 Find a basis for $\mathcal{P}_2$, the space of all quadratic polynomials. Must any such basis contain a polynomial of each degree: degree zero, degree one, and degree two?

1.19 Find a basis for the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

✓ **1.20** Find a basis for $\mathcal{M}_{2\times2}$, the space of 2×2 matrices.

✓ **1.21** Find a basis for each.

(a) The subspace $\{a_2x^2 + a_1x + a_0 \mid a_2 - 2a_1 = a_0\}$ of $\mathcal{P}_2$

(b) The space of three-wide row vectors whose first and second components add to zero

(c) This subspace of the 2×2 matrices

$$\{\begin{pmatrix}a & b\\0 & c\end{pmatrix} \mid c - 2b = 0\}$$

1.22 Check Example 1.6.

✓ **1.23** Find the span of each set and then find a basis for that span.

(a) $\{1+x, 1+2x\}$ in $\mathcal{P}_2$ **(b)** $\{2-2x, 3+4x^2\}$ in $\mathcal{P}_2$

✓ **1.24** Find a basis for each of these subspaces of the space $\mathcal{P}_3$ of cubic polynomials.

(a) The subspace of cubic polynomials $p(x)$ such that $p(7) = 0$

(b) The subspace of polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$

(c) The subspace of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$

(d) The space of polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

1.25 We've seen that it is possible for a basis to remain a basis when it is reordered. Must it remain a basis?

1.26 Can a basis contain a zero vector?

✓ **1.27** Let $\langle\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ be a basis for a vector space.

(a) Show that $\langle c_1\vec{\beta}_1, c_2\vec{\beta}_2, c_3\vec{\beta}_3\rangle$ is a basis when $c_1, c_2, c_3 \neq 0$. What happens when at least one c_i is 0?

(b) Prove that $\langle\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\rangle$ is a basis where $\vec{\alpha}_i = \vec{\beta}_1 + \vec{\beta}_i$.

1.28 Find one vector $\vec{v}$ that will make each into a basis for the space.

(a) $\langle\begin{pmatrix}1\\1\end{pmatrix}, \vec{v}\rangle$ in $\mathbb{R}^2$ **(b)** $\langle\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{v}\rangle$ in $\mathbb{R}^3$ **(c)** $\langle x, 1+x^2, \vec{v}\rangle$ in $\mathcal{P}_2$

✓ **1.29** Where $\langle\vec{\beta}_1, \dots, \vec{\beta}_n\rangle$ is a basis, show that in this equation

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

each of the c_i's is zero. Generalize.

1.30 A basis contains some of the vectors from a vector space; can it contain them all?

1.31 Theorem 1.12 shows that, with respect to a basis, every linear combination is unique. If a subset is not a basis, can linear combinations be not unique? If so, must they be?

✓ **1.32** A square matrix is *symmetric* if for all indices i and j, entry i,j equals entry j,i.

(a) Find a basis for the vector space of symmetric 2×2 matrices.

(b) Find a basis for the space of symmetric 3×3 matrices.

(c) Find a basis for the space of symmetric $n\times n$ matrices.

✓ **1.33** We can show that every basis for $\mathbb{R}^3$ contains the same number of vectors.

(a) Show that no linearly independent subset of $\mathbb{R}^3$ contains more than three vectors.

(b) Show that no spanning subset of $\mathbb{R}^3$ contains fewer than three vectors. *Hint:* recall how to calculate the span of a set and show that this method cannot yield all of $\mathbb{R}^3$ when it is applied to fewer than three vectors.

1.34 One of the exercises in the Subspaces subsection shows that the set

$$\{\begin{pmatrix}x\\y\\z\end{pmatrix} \bigm| x+y+z=1\}$$

is a vector space under these operations.

$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2-1\\y_1+y_2\\z_1+z_2\end{pmatrix}\qquad r\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}rx-r+1\\ry\\rz\end{pmatrix}$$

Find a basis.

III.2 Dimension

In the prior subsection we defined the basis of a vector space, and we saw that a space can have many different bases. For example, following the definition of a basis, we saw three different bases for $\mathbb{R}^2$. So we cannot talk about "the" basis for a vector space. True, some vector spaces have bases that strike us as more natural than others, for instance, $\mathbb{R}^2$'s basis $\mathcal{E}_2$ or $\mathbb{R}^3$'s basis $\mathcal{E}_3$ or $\mathcal{P}_2$'s basis $\langle 1,x,x^2\rangle$. But, for example in the space $\{a_2x^2+a_1x+a_0 \bigm| 2a_2-a_0=a_1\}$, no particular basis leaps out at us as the most natural one. We cannot, in general, associate with a space any single basis that best describes that space.

We can, however, find something about the bases that is uniquely associated with the space. This subsection shows that any two bases for a space have the same number of elements. So, with each space we can associate a number, the number of vectors in any of its bases.

This brings us back to when we considered the two things that could be meant by the term 'minimal spanning set'. At that point we defined 'minimal' as linearly independent, but we noted that another reasonable interpretation of the term is that a spanning set is 'minimal' when it has the fewest number of elements of any set with the same span. At the end of this subsection, after we have shown that all bases have the same number of elements, then we will have shown that the two senses of 'minimal' are equivalent.

Before we start, we first limit our attention to spaces where at least one basis has only finitely many members.

2.1 Definition A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

(One reason for sticking to finite-dimensional spaces is so that the representation of a vector with respect to a basis is a finitely-tall vector, and so can be easily written.) From now on we study only finite-dimensional vector spaces. We shall take the term 'vector space' to mean 'finite-dimensional vector space'. Other spaces are interesting and important, but they lie outside of our scope.

To prove the main theorem we shall use a technical result, the Exchange Lemma. We first illustrate its conclusion with an example.

2.2 Example Here is a basis for $\mathbb{R}^3$ and a vector given as a linear combination of members of that basis.

$$B = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \rangle \qquad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

In that combination two of the basis vectors have non-zero coefficients. We can pick either one, here we pick the first. Replacing it with the vector we've expressed as the combination

$$\hat{B} = \langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \rangle$$

gives a new basis for the space.

2.3 Lemma (Exchange Lemma) Assume that $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Proof. Call the outcome of the exchange $\hat{B} = \langle \vec{\beta}_1, \dots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \dots, \vec{\beta}_n \rangle$.

We first show that $\hat{B}$ is linearly independent. Any relationship $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n = \vec{0}$ among the members of $\hat{B}$, after substitution for $\vec{v}$,

$$d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_i\vec{\beta}_i + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n = \vec{0} \qquad (*)$$

gives a linear relationship among the members of B. The basis B is linearly independent, so the coefficient d_ic_i of $\vec{\beta}_i$ is zero. Because c_i is assumed to be nonzero, $d_i = 0$. Using this in equation $(*)$ above gives that all of the other d's are also zero. Therefore $\hat{B}$ is linearly independent.

We finish by showing that $\hat{B}$ has the same span as B. Half of this argument, that $[\hat{B}] \subseteq [B]$, is easy; any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n$ of $[\hat{B}]$ can be written $d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n$, which is a linear combination of linear combinations of members of B, and hence is in $[B]$. For the $[B] \subseteq [\hat{B}]$ half of the argument, recall that when $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ with $c_i \neq 0$, then the equation can be rearranged to $\vec{\beta}_i = (-c_1/c_i)\vec{\beta}_1 + \cdots + (1/c_i)\vec{v} + \cdots + (-c_n/c_i)\vec{\beta}_n$. Now, consider any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{\beta}_i + \cdots + d_n\vec{\beta}_n$ of $[B]$, substitute for $\vec{\beta}_i$ its expression as a linear combination of the members of $\hat{B}$, and recognize (as in the first half of this argument) that the result is a linear combination of linear combinations, of members of $\hat{B}$, and hence is in $[\hat{B}]$. QED

2.4 Theorem In any finite-dimensional vector space, all of the bases have the same number of elements.

Proof. Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

The basis B spans the space and $\vec{\delta}_1$ is in the space, so $\vec{\delta}_1$ is a nontrivial linear combination of elements of B. By the Exchange Lemma, $\vec{\delta}_1$ can be swapped for a vector from B, resulting in a basis B_1, where one element is $\vec{\delta}$ and all of the $n-1$ other elements are $\vec{\beta}$'s.

The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leq k < n$) containing k members of D and $n-k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

Repeat the inductive step until no $\vec{\beta}$'s remain, so that B_n contains $\vec{\delta}_1, \ldots, \vec{\delta}_n$. Now, D cannot have more than these n vectors because any $\vec{\delta}_{n+1}$ that remains would be in the span of B_n (since it is a basis) and hence would be a linear combination of the other $\vec{\delta}$'s, contradicting that D is linearly independent. QED

2.5 Definition The *dimension* of a vector space is the number of vectors in any of its bases.

2.6 Example Any basis for $\mathbb{R}^n$ has n vectors since the standard basis $\mathcal{E}_n$ has n vectors. Thus, this definition generalizes the most familiar use of term, that $\mathbb{R}^n$ is n-dimensional.

2.7 Example The space $\mathcal{P}_n$ of polynomials of degree at most n has dimension $n+1$. We can show this by exhibiting any basis — $\langle 1, x, \ldots, x^n \rangle$ comes to mind — and counting its members.

2.8 Example A trivial space is zero-dimensional since its basis is empty.

Again, although we sometimes say 'finite-dimensional' as a reminder, in the rest of this book all vector spaces are assumed to be finite-dimensional. An instance of this is that in the next result the word 'space' should be taken to mean 'finite-dimensional vector space'.

2.9 Corollary No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof. Inspection of the above proof shows that it never uses that D spans the space, only that D is linearly independent. QED

2.10 Example Recall the subspace diagram from the prior section showing the subspaces of $\mathbb{R}^3$. Each subspace shown is described with a minimal spanning set, for which we now have the term 'basis'. The whole space has a basis with three members, the plane subspaces have bases with two members, the line subspaces have bases with one member, and the trivial subspace has a basis with zero members. When we saw that diagram we could not show that these are the only subspaces that this space has. We can show it now. The prior corollary proves that the only subspaces of $\mathbb{R}^3$ are either three-, two-, one-, or zero-dimensional. Therefore, the diagram indicates all of the subspaces. There are no subspaces somehow, say, between lines and planes.

2.11 Corollary Any linearly independent set can be expanded to make a basis.

Proof. If a linearly independent set is not already a basis then it must not span the space. Adding to it a vector that is not in the span preserves linear independence. Keep adding, until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps. QED

2.12 Corollary Any spanning set can be shrunk to a basis.

Proof. Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done.

If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s}_2 \rangle$; if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.13 Corollary In an n-dimensional space, a set of n vectors is linearly independent if and only if it spans the space.

Proof. First we will show that a subset with n vectors is linearly independent if and only if it is a basis. 'If' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with. QED

The main result of this subsection, that all of the bases in a finite-dimensional vector space have the same number of elements, is the single most important result in this book because, as Example 2.10 shows, it describes what vector spaces and subspaces there can be. We will see more in the next chapter.

2.14 Remark The case of infinite-dimensional vector spaces is somewhat controversial. The statement 'any infinite-dimensional vector space has a basis' is known to be equivalent to a statement called the Axiom of Choice (see [Blass 1984]). Mathematicians differ philosophically on whether to accept or reject this statement as an axiom on which to base mathematics (although, the great majority seem to accept it). Consequently the question about infinite-dimensional vector spaces is still somewhat up in the air. (A discussion of the Axiom of Choice can be found in the Frequently Asked Questions list for the Usenet group `sci.math`. Another accessible reference is [Rucker].)

Exercises

Assume that all spaces are finite-dimensional unless otherwise stated.

✓ **2.15** Find a basis for, and the dimension of, $\mathcal{P}_2$.

2.16 Find a basis for, and the dimension of, the solution set of this system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

✓ **2.17** Find a basis for, and the dimension of, $\mathcal{M}_{2\times2}$, the vector space of 2×2 matrices.

2.18 Find the dimension of the vector space of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

subject to each condition.

(a) $a, b, c, d \in \mathbb{R}$
(b) $a - b + 2c = 0$ and $d \in \mathbb{R}$
(c) $a + b + c = 0$, $a + b - c = 0$, and $d \in \mathbb{R}$

✓ **2.19** Find the dimension of each.

(a) The space of cubic polynomials $p(x)$ such that $p(7) = 0$
(b) The space of cubic polynomials $p(x)$ such that $p(7) = 0$ and $p(5) = 0$
(c) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, and $p(3) = 0$
(d) The space of cubic polynomials $p(x)$ such that $p(7) = 0$, $p(5) = 0$, $p(3) = 0$, and $p(1) = 0$

2.20 What is the dimension of the span of the set $\{\cos^2\theta, \sin^2\theta, \cos 2\theta, \sin 2\theta\}$? This span is a subspace of the space of all real-valued functions of one real variable.

2.21 Find the dimension of $\mathbb{C}^{47}$, the vector space of 47-tuples of complex numbers.

2.22 What is the dimension of the vector space $\mathcal{M}_{3\times5}$ of 3×5 matrices?

✓ **2.23** Show that this is a basis for $\mathbb{R}^4$.

$$\langle\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\rangle$$

(The results of this subsection can be used to simplify this job.)

2.24 Refer to Example 2.10.

(a) Sketch a similar subspace diagram for $\mathcal{P}_2$.

(b) Sketch one for $\mathcal{M}_{2\times2}$.

✓ **2.25** Where S is a set, the functions $f\colon S\to\mathbb{R}$ form a vector space under the natural operations: the sum $f+g$ is the function given by $f+g\,(s)=f(s)+g(s)$ and the scalar product is given by $r\cdot f\,(s)=r\cdot f(s)$. What is the dimension of the space resulting for each domain?

(a) $S=\{1\}$ **(b)** $S=\{1,2\}$ **(c)** $S=\{1,\ldots,n\}$

2.26 (See Exercise 25.) Prove that this is an infinite-dimensional space: the set of all functions $f\colon\mathbb{R}\to\mathbb{R}$ under the natural operations.

2.27 (See Exercise 25.) What is the dimension of the vector space of functions $f\colon S\to\mathbb{R}$, under the natural operations, where the domain S is the empty set?

2.28 Show that any set of four vectors in $\mathbb{R}^2$ is linearly dependent.

2.29 Show that $\langle\vec{\alpha}_1,\vec{\alpha}_2,\vec{\alpha}_3\rangle\subset\mathbb{R}^3$ is a basis if and only if there is no plane through the origin containing all three vectors.

2.30 **(a)** Prove that any subspace of a finite dimensional space has a basis.

(b) Prove that any subspace of a finite dimensional space is finite dimensional.

2.31 Where is the finiteness of B used in Theorem 2.4?

✓ **2.32** Prove that if U and W are both three-dimensional subspaces of $\mathbb{R}^5$ then $U\cap W$ is non-trivial. Generalize.

2.33 A basis for a space consists of elements of that space. So we are naturally led to how the property 'is a basis' interacts with operations $\subseteq$ and $\cap$ and $\cup$. (Of course, a basis is actually a sequence in that it is ordered, but there is a natural extension of these operations.)

(a) Consider first how bases might be related by $\subseteq$. Assume that U,W are subspaces of some vector space and that $U\subseteq W$. Can there exist bases B_U for U and B_W for W such that $B_U\subseteq B_W$? Must such bases exist?

For any basis B_U for U, must there be a basis B_W for W such that $B_U\subseteq B_W$?

For any basis B_W for W, must there be a basis B_U for U such that $B_U\subseteq B_W$?

For any bases B_U,B_W for U and W, must B_U be a subset of B_W?

(b) Is the $\cap$ of bases a basis? For what space?

(c) Is the $\cup$ of bases a basis? For what space?

(d) What about the complement operation?

(*Hint.* Test any conjectures against some subspaces of $\mathbb{R}^3$.)

✓ **2.34** Consider how 'dimension' interacts with 'subset'. Assume U and W are both subspaces of some vector space, and that $U\subseteq W$.

(a) Prove that $\dim(U)\leq\dim(W)$.

(b) Prove that equality of dimension holds if and only if $U=W$.

(c) Show that the prior item does not hold if they are infinite-dimensional.

? **2.35** For any vector $\vec{v}$ in $\mathbb{R}^n$ and any permutation σ of the numbers 1, 2, …, n (that is, σ is a rearrangement of those numbers into a new order), define $\sigma(\vec{v})$

to be the vector whose components are $v_{\sigma(1)}$, $v_{\sigma(2)}$, ..., and $v_{\sigma(n)}$ (where $\sigma(1)$ is the first number in the rearrangement, etc.). Now fix $\vec{v}$ and let V be the span of $\{\sigma(\vec{v}) \mid \sigma \text{ permutes } 1, \ldots, n\}$. What are the possibilities for the dimension of V? [Wohascum no. 47]

III.3 Vector Spaces and Linear Systems

We will now reconsider linear systems and Gauss' method, aided by the tools and terms of this chapter. We will make three points.

For the first point, recall the first chapter's Linear Combination Lemma and its corollary: if two matrices are related by row operations $A \longrightarrow \cdots \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, Gauss' method works by taking linear combinations of rows. Therefore, the right setting in which to study row operations in general, and Gauss' method in particular, is the following vector space.

3.1 Definition The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of the row space, the number of linearly independent rows.

3.2 Example If

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

then $\text{Rowspace}(A)$ is this subspace of the space of two-component row vectors.

$$\{c_1 \cdot \begin{pmatrix} 2 & 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 4 & 6 \end{pmatrix} \mid c_1, c_2 \in \mathbb{R}\}$$

The linear dependence of the second on the first is obvious and so we can simplify this description to $\{c \cdot \begin{pmatrix} 2 & 3 \end{pmatrix} \mid c \in \mathbb{R}\}$.

3.3 Lemma If the matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space, and hence also, the same row rank.

Proof. By the Linear Combination Lemma's corollary, each row of B is in the row space of A. Further, $\text{Rowspace}(B) \subseteq \text{Rowspace}(A)$ because a member of the set $\text{Rowspace}(B)$ is a linear combination of the rows of B, which means it is a combination of a combination of the rows of A, and hence, by the Linear Combination Lemma, is also a member of $\text{Rowspace}(A)$.

For the other containment, recall that row operations are reversible: $A \longrightarrow B$ if and only if $B \longrightarrow A$. With that, $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$ also follows from the prior paragraph, and so the two sets are equal. QED

Thus, row operations leave the row space unchanged. But of course, Gauss' method performs the row operations systematically, with a specific goal in mind, echelon form.

3.4 Lemma The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof. A result in the first chapter, Lemma III.2.4, states that in an echelon form matrix, no nonzero row is a linear combination of the other rows. This is a restatement of that result into new terminology. QED

Thus, in the language of this chapter, Gaussian reduction works by eliminating linear dependences among rows, leaving the span unchanged, until no nontrivial linear relationships remain (among the nonzero rows). That is, Gauss' method produces a basis for the row space.

3.5 Example From any matrix, we can produce a basis for the row space by performing Gauss' method and taking the nonzero rows of the resulting echelon form matrix. For instance,

$$\begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 1 \\ 2 & 0 & 5 \end{pmatrix} \xrightarrow[-2\rho_1+\rho_3]{-\rho_1+\rho_2} \xrightarrow{6\rho_2+\rho_3} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

produces the basis $\langle \begin{pmatrix} 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 \end{pmatrix} \rangle$ for the row space. This is a basis for the row space of both the starting and ending matrices, since the two row spaces are equal.

Using this technique, we can also find bases for spans not directly involving row vectors.

3.6 Definition The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Our interest in column spaces stems from our study of linear systems. An example is that this system

$$\begin{aligned} c_1 + 3c_2 + 7c_3 &= d_1 \\ 2c_1 + 3c_2 + 8c_3 &= d_2 \\ c_2 + 2c_3 &= d_3 \\ 4c_1 \qquad + 4c_3 &= d_4 \end{aligned}$$

has a solution if and only if the vector of d's is a linear combination of the other column vectors,

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 7 \\ 8 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

meaning that the vector of d's is in the column space of the matrix of coefficients.

3.7 Example Given this matrix,

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 3 & 8 \\ 0 & 1 & 2 \\ 4 & 0 & 4 \end{pmatrix}$$

to get a basis for the column space, temporarily turn the columns into rows and reduce.

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 1 & 0 \\ 7 & 8 & 2 & 4 \end{pmatrix} \xrightarrow[-7\rho_1+\rho_3]{-3\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 1 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now turn the rows back to columns.

$$\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ -12 \end{pmatrix} \rangle$$

The result is a basis for the column space of the given matrix.

3.8 Definition The *transpose* of a matrix is the result of interchanging the rows and columns of that matrix. That is, column j of the matrix A is row j of A^{trans}, and vice versa.

So the instructions for the prior example are "transpose, reduce, and transpose back".

We can even, at the price of tolerating the as-yet-vague idea of vector spaces being "the same", use Gauss' method to find bases for spans in other types of vector spaces.

3.9 Example To get a basis for the span of $\{x^2 + x^4, 2x^2 + 3x^4, -x^2 - 3x^4\}$ in the space $\mathcal{P}_4$, think of these three polynomials as "the same" as the row vectors $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 2 & 0 & 3 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & -1 & 0 & -3 \end{pmatrix}$, apply Gauss' method

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & -1 & 0 & -3 \end{pmatrix} \xrightarrow[\rho_1+\rho_3]{-2\rho_1+\rho_2} \xrightarrow{2\rho_2+\rho_3} \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and translate back to get the basis $\langle x^2 + x^4, x^4 \rangle$. (As mentioned earlier, we will make the phrase "the same" precise at the start of the next chapter.)

Thus, our first point in this subsection is that the tools of this chapter give us a more conceptual understanding of Gaussian reduction.

For the second point of this subsection, consider the effect on the column space of this row reduction.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

The column space of the left-hand matrix contains vectors with a second component that is nonzero. But the column space of the right-hand matrix is different because it contains only vectors whose second component is zero. It is this knowledge that row operations can change the column space that makes next result surprising.

3.10 Lemma Row operations do not change the column rank.

PROOF. Restated, if A reduces to B then the column rank of B equals the column rank of A.

We will be done if we can show that row operations do not affect linear relationships among columns because the column rank is just the size of the largest set of unrelated columns. That is, we will show that a relationship exists among columns (such as that the fifth column is twice the second plus the fourth) if and only if that relationship exists after the row operation. But this is exactly the first theorem of this book: in a relationship among columns,

$$c_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \cdot \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

row operations leave unchanged the set of solutions $(c_1, \ldots, c_n)$. QED

Another way, besides the prior result, to state that Gauss' method has something to say about the column space as well as about the row space is to consider again Gauss-Jordan reduction. Recall that it ends with the reduced echelon form of a matrix, as here.

$$\begin{pmatrix} 1 & 3 & 1 & 6 \\ 2 & 6 & 3 & 16 \\ 1 & 3 & 1 & 6 \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider the row space and the column space of this result. Our first point made above says that a basis for the row space is easy to get: simply collect together all of the rows with leading entries. However, because this is a reduced echelon form matrix, a basis for the column space is just as easy: take the columns containing the leading entries, that is, $\langle \vec{e}_1, \vec{e}_2 \rangle$. (Linear independence is obvious. The other columns are in the span of this set, since they all have a third component of zero.) Thus, for a reduced echelon form matrix, bases for the row and column spaces can be found in essentially the same way — by taking the parts of the matrix, the rows or columns, containing the leading entries.

3.11 Theorem The row rank and column rank of a matrix are equal.

PROOF. First bring the matrix to reduced echelon form. At that point, the row rank equals the number of leading entries since each equals the number of nonzero rows. Also at that point, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.12 Definition The *rank* of a matrix is its row rank or column rank.

So our second point in this subsection is that the column space and row space of a matrix have the same dimension. Our third and final point is that the concepts that we've seen arising naturally in the study of vector spaces are exactly the ones that we have studied with linear systems.

3.13 Theorem For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

So if the system has at least one particular solution then for the set of solutions, the number of parameters equals $n - r$, the number of variables minus the rank of the matrix of coefficients.

Proof. The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables. QED

3.14 Remark [Munkres] Sometimes that result is mistakenly remembered to say that the general solution of an n unknown system of m equations uses $n-m$ parameters. The number of equations is not the relevant figure, rather, what matters is the number of independent equations (the number of equations in a maximal independent set). Where there are r independent equations, the general solution involves $n - r$ parameters.

3.15 Corollary Where the matrix A is $n \times n$, the statements

(1) the rank of A is n

(2) A is nonsingular

(3) the rows of A form a linearly independent set

(4) the columns of A form a linearly independent set

(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

Proof. Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1\begin{pmatrix}a_{1,1}\\a_{2,1}\\\vdots\\a_{m,1}\end{pmatrix}+\cdots+c_n\begin{pmatrix}a_{1,n}\\a_{2,n}\\\vdots\\a_{m,n}\end{pmatrix}=\begin{pmatrix}d_1\\d_2\\\vdots\\d_m\end{pmatrix}$$

has a unique solution for all choices of $d_1,\ldots,d_n\in\mathbb{R}$ if and only if the vectors of a's form a basis. QED

Exercises

3.16 Transpose each.

(a) $\begin{pmatrix}2&1\\3&1\end{pmatrix}$ **(b)** $\begin{pmatrix}2&1\\1&3\end{pmatrix}$ **(c)** $\begin{pmatrix}1&4&3\\6&7&8\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\0\\0\end{pmatrix}$

(e) $\begin{pmatrix}-1&-2\end{pmatrix}$

✓ **3.17** Decide if the vector is in the row space of the matrix.

(a) $\begin{pmatrix}2&1\\3&1\end{pmatrix}$, $\begin{pmatrix}1&0\end{pmatrix}$ **(b)** $\begin{pmatrix}0&1&3\\-1&0&1\\-1&2&7\end{pmatrix}$, $\begin{pmatrix}1&1&1\end{pmatrix}$

✓ **3.18** Decide if the vector is in the column space.

(a) $\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $\begin{pmatrix}1\\3\end{pmatrix}$ **(b)** $\begin{pmatrix}1&3&1\\2&0&4\\1&-3&-3\end{pmatrix}$, $\begin{pmatrix}1\\0\\0\end{pmatrix}$

✓ **3.19** Find a basis for the row space of this matrix.

$$\begin{pmatrix}2&0&3&4\\0&1&1&-1\\3&1&0&2\\1&0&-4&-1\end{pmatrix}$$

✓ **3.20** Find the rank of each matrix.

(a) $\begin{pmatrix}2&1&3\\1&-1&2\\1&0&3\end{pmatrix}$ **(b)** $\begin{pmatrix}1&-1&2\\3&-3&6\\-2&2&-4\end{pmatrix}$ **(c)** $\begin{pmatrix}1&3&2\\5&1&1\\6&4&3\end{pmatrix}$

(d) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

✓ **3.21** Find a basis for the span of each set.

(a) $\{\begin{pmatrix}1&3\end{pmatrix},\begin{pmatrix}-1&3\end{pmatrix},\begin{pmatrix}1&4\end{pmatrix},\begin{pmatrix}2&1\end{pmatrix}\}\subseteq\mathcal{M}_{1\times2}$

(b) $\{\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}3\\1\\-1\end{pmatrix},\begin{pmatrix}1\\-3\\-3\end{pmatrix}\}\subseteq\mathbb{R}^3$

(c) $\{1+x,1-x^2,3+2x-x^2\}\subseteq\mathcal{P}_3$

(d) $\{\begin{pmatrix}1&0&1\\3&1&-1\end{pmatrix},\begin{pmatrix}1&0&3\\2&1&4\end{pmatrix},\begin{pmatrix}-1&0&-5\\-1&-1&-9\end{pmatrix}\}\subseteq\mathcal{M}_{2\times3}$

3.22 Which matrices have rank zero? Rank one?

✓ **3.23** Given $a, b, c \in \mathbb{R}$, what choice of d will cause this matrix to have the rank of one?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

3.24 Find the column rank of this matrix.

$$\begin{pmatrix} 1 & 3 & -1 & 5 & 0 & 4 \\ 2 & 0 & 1 & 0 & 4 & 1 \end{pmatrix}$$

3.25 Show that a linear system with at least one solution has at most one solution if and only if the matrix of coefficients has rank equal to the number of its columns.

✓ **3.26** If a matrix is 5×9, which set must be dependent, its set of rows or its set of columns?

3.27 Give an example to show that, despite that they have the same dimension, the row space and column space of a matrix need not be equal. Are they ever equal?

3.28 Show that the set $\{(1,-1,2,-3),(1,1,2,0),(3,-1,6,-6)\}$ does not have the same span as $\{(1,0,1,0),(0,2,0,3)\}$. What, by the way, is the vector space?

✓ **3.29** Show that this set of column vectors

$$\left\{ \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \;\Big|\; \text{there are } x,\ y, \text{ and } z \text{ such that } \begin{array}{rl} 3x+2y+4z &= d_1 \\ x - z &= d_2 \\ 2x+2y+5z &= d_3 \end{array} \right\}$$

is a subspace of $\mathbb{R}^3$. Find a basis.

3.30 Show that the transpose operation is *linear*:

$$(rA+sB)^{\text{trans}} = rA^{\text{trans}} + sB^{\text{trans}}$$

for $r, s \in \mathbb{R}$ and $A, B \in \mathcal{M}_{m\times n}$,

✓ **3.31** In this subsection we have shown that Gaussian reduction finds a basis for the row space.

(a) Show that this basis is not unique — different reductions may yield different bases.

(b) Produce matrices with equal row spaces but unequal numbers of rows.

(c) Prove that two matrices have equal row spaces if and only if after Gauss-Jordan reduction they have the same nonzero rows.

3.32 Why is there not a problem with Remark 3.14 in the case that r is bigger than n?

3.33 Show that the row rank of an $m\times n$ matrix is at most m. Is there a better bound?

✓ **3.34** Show that the rank of a matrix equals the rank of its transpose.

3.35 True or false: the column space of a matrix equals the row space of its transpose.

✓ **3.36** We have seen that a row operation may change the column space. Must it?

3.37 Prove that a linear system has a solution if and only if that system's matrix of coefficients has the same rank as its augmented matrix.

3.38 An $m\times n$ matrix has *full row rank* if its row rank is m, and it has *full column rank* if its column rank is n.

(a) Show that a matrix can have both full row rank and full column rank only if it is square.

(b) Prove that the linear system with matrix of coefficients A has a solution for any $d_1, \ldots, d_n$'s on the right side if and only if A has full row rank.

(c) Prove that a homogeneous system has a unique solution if and only if its matrix of coefficients A has full column rank.
(d) Prove that the statement "if a system with matrix of coefficients A has any solution then it has a unique solution" holds if and only if A has full column rank.

3.39 How would the conclusion of Lemma 3.3 change if Gauss' method is changed to allow multiplying a row by zero?

✓ **3.40** What is the relationship between $\text{rank}(A)$ and $\text{rank}(-A)$? Between $\text{rank}(A)$ and $\text{rank}(kA)$? What, if any, is the relationship between $\text{rank}(A)$, $\text{rank}(B)$, and $\text{rank}(A+B)$?

III.4 Combining Subspaces

This subsection is optional. It is required only for the last sections of Chapter Three and Chapter Five and for occasional exercises, and can be passed over without loss of continuity.

This chapter opened with the definition of a vector space, and the middle consisted of a first analysis of the idea. This subsection closes the chapter by finishing the analysis, in the sense that 'analysis' means "method of determining the ... essential features of something by separating it into parts" [Macmillan Dictionary].

A common way to understand things is to see how they can be built from component parts. For instance, we think of $\mathbb{R}^3$ as put together, in some way, from the x-axis, the y-axis, and z-axis. In this subsection we will make this precise; we will describe how to decompose a vector space into a combination of some of its subspaces. In developing this idea of subspace combination, we will keep the $\mathbb{R}^3$ example in mind as a benchmark model.

Subspaces are subsets and sets combine via union. But taking the combination operation for subspaces to be the simple union operation isn't what we want. For one thing, the union of the x-axis, the y-axis, and z-axis is not all of $\mathbb{R}^3$, so the benchmark model would be left out. Besides, union is all wrong for this reason: a union of subspaces need not be a subspace (it need not be closed; for instance, this $\mathbb{R}^3$ vector

$$\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

is in none of the three axes and hence is not in the union). In addition to the members of the subspaces, we must at least also include all of the linear combinations.

4.1 Definition Where $W_1,\ldots,W_k$ are subspaces of a vector space, their *sum* is the span of their union $W_1+W_2+\cdots+W_k=[W_1\cup W_2\cup\ldots W_k]$.

(The notation, writing the '+' between sets in addition to using it between vectors, fits with the practice of using this symbol for any natural accumulation operation.)

4.2 Example The $\mathbb{R}^3$ model fits with this operation. Any vector $\vec{w} \in \mathbb{R}^3$ can be written as a linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where $\vec{v}_1$ is a member of the x-axis, etc., in this way

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} w_1 \\ 0 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ w_2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix}$$

and so $\mathbb{R}^3 = x\text{-axis} + y\text{-axis} + z\text{-axis}$.

4.3 Example A sum of subspaces can be less than the entire space. Inside of $\mathcal{P}_4$, let L be the subspace of linear polynomials $\{a + bx \bigm| a, b \in \mathbb{R}\}$ and let C be the subspace of purely-cubic polynomials $\{cx^3 \bigm| c \in \mathbb{R}\}$. Then $L + C$ is not all of $\mathcal{P}_4$. Instead, it is the subspace $L + C = \{a + bx + cx^3 \bigm| a, b, c \in \mathbb{R}\}$.

4.4 Example A space can be described as a combination of subspaces in more than one way. Besides the decomposition $\mathbb{R}^3 = x\text{-axis} + y\text{-axis} + z\text{-axis}$, we can also write $\mathbb{R}^3 = xy\text{-plane} + yz\text{-plane}$. To check this, note that any $\vec{w} \in \mathbb{R}^3$ can be written as a linear combination of a member of the xy-plane and a member of the yz-plane; here are two such combinations.

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 0 \\ w_3 \end{pmatrix} \qquad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 1 \cdot \begin{pmatrix} w_1 \\ w_2/2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ w_2/2 \\ w_3 \end{pmatrix}$$

The above definition gives one way in which a space can be thought of as a combination of some of its parts. However, the prior example shows that there is at least one interesting property of our benchmark model that is not captured by the definition of the sum of subspaces. In the familiar decomposition of $\mathbb{R}^3$, we often speak of a vector's 'x part' or 'y part' or 'z part'. That is, in this model, each vector has a unique decomposition into parts that come from the parts making up the whole space. But in the decomposition used in Example 4.4, we cannot refer to the "xy part" of a vector — these three sums

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

all describe the vector as comprised of something from the first plane plus something from the second plane, but the "xy part" is different in each.

That is, when we consider how $\mathbb{R}^3$ is put together from the three axes "in some way", we might mean "in such a way that every vector has at least one decomposition", and that leads to the definition above. But if we take it to mean "in such a way that every vector has one and only one decomposition" then we need another condition on combinations. To see what this condition is, recall that vectors are uniquely represented in terms of a basis. We can use this to break a space into a sum of subspaces such that any vector in the space breaks uniquely into a sum of members of those subspaces.

4.5 Example The benchmark is $\mathbb{R}^3$ with its standard basis $\mathcal{E}_3 = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$. The subspace with the basis $B_1 = \langle \vec{e}_1 \rangle$ is the x-axis. The subspace with the basis $B_2 = \langle \vec{e}_2 \rangle$ is the y-axis. The subspace with the basis $B_3 = \langle \vec{e}_3 \rangle$ is the z-axis. The fact that any member of $\mathbb{R}^3$ is expressible as a sum of vectors from these subspaces

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$$

is a reflection of the fact that $\mathcal{E}_3$ spans the space — this equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

has a solution for any $x, y, z \in \mathbb{R}$. And, the fact that each such expression is unique reflects that fact that $\mathcal{E}_3$ is linearly independent — any equation like the one above has a unique solution.

4.6 Example We don't have to take the basis vectors one at a time, the same idea works if we conglomerate them into larger sequences. Consider again the space $\mathbb{R}^3$ and the vectors from the standard basis $\mathcal{E}_3$. The subspace with the basis $B_1 = \langle \vec{e}_1, \vec{e}_3 \rangle$ is the xz-plane. The subspace with the basis $B_2 = \langle \vec{e}_2 \rangle$ is the y-axis. As in the prior example, the fact that any member of the space is a sum of members of the two subspaces in one and only one way

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

is a reflection of the fact that these vectors form a basis — this system

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

has one and only one solution for any $x, y, z \in \mathbb{R}$.

These examples illustrate a natural way to decompose a space into a sum of subspaces in such a way that each vector decomposes uniquely into a sum of vectors from the parts. The next result says that this way is the only way.

4.7 Definition The *concatenation* of the sequences $B_1 = \langle \vec{\beta}_{1,1}, \dots, \vec{\beta}_{1,n_1} \rangle$, $\dots$, $B_k = \langle \vec{\beta}_{k,1}, \dots, \vec{\beta}_{k,n_k} \rangle$ is their adjoinment.

$$B_1 {}^\frown B_2 {}^\frown \cdots {}^\frown B_k = \langle \vec{\beta}_{1,1}, \dots, \vec{\beta}_{1,n_1}, \vec{\beta}_{2,1}, \dots, \vec{\beta}_{k,n_k} \rangle$$

4.8 Lemma Let V be a vector space that is the sum of some of its subspaces $V = W_1 + \cdots + W_k$. Let $B_1, \ldots, B_k$ be any bases for these subspaces. Then the following are equivalent.

(1) For every $\vec{v} \in V$, the expression $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$ (with $\vec{w}_i \in W_i$) is unique.

(2) The concatenation $B_1 \frown \cdots \frown B_k$ is a basis for V.

(3) The nonzero members of $\{\vec{w}_1, \ldots, \vec{w}_k\}$ (with $\vec{w}_i \in W_i$) form a linearly independent set — among nonzero vectors from different W_i's, every linear relationship is trivial.

Proof. We will show that (1) $\implies$ (2), that (2) $\implies$ (3), and finally that (3) $\implies$ (1). For these arguments, observe that we can pass from a combination of $\vec{w}$'s to a combination of $\vec{\beta}$'s

$$\begin{aligned}
&d_1\vec{w}_1 + \cdots + d_k\vec{w}_k \\
&\quad = d_1(c_{1,1}\vec{\beta}_{1,1} + \cdots + c_{1,n_1}\vec{\beta}_{1,n_1}) + \cdots + d_k(c_{k,1}\vec{\beta}_{k,1} + \cdots + c_{k,n_k}\vec{\beta}_{k,n_k}) \\
&\quad = d_1c_{1,1} \cdot \vec{\beta}_{1,1} + \cdots + d_kc_{k,n_k} \cdot \vec{\beta}_{k,n_k} \qquad (*)
\end{aligned}$$

and vice versa.

For (1) $\implies$ (2), assume that all decompositions are unique. We will show that $B_1 \frown \cdots \frown B_k$ spans the space and is linearly independent. It spans the space because the assumption that $V = W_1 + \cdots + W_k$ means that every $\vec{v}$ can be expressed as $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$, which translates by equation $(*)$ to an expression of $\vec{v}$ as a linear combination of the $\vec{\beta}$'s from the concatenation. For linear independence, consider this linear relationship.

$$\vec{0} = c_{1,1}\vec{\beta}_{1,1} + \cdots + c_{k,n_k}\vec{\beta}_{k,n_k}$$

Regroup as in $(*)$ (that is, take $d_1, \ldots, d_k$ to be 1 and move from bottom to top) to get the decomposition $\vec{0} = \vec{w}_1 + \cdots + \vec{w}_k$. Because of the assumption that decompositions are unique, and because the zero vector obviously has the decomposition $\vec{0} = \vec{0} + \cdots + \vec{0}$, we now have that each $\vec{w}_i$ is the zero vector. This means that $c_{i,1}\vec{\beta}_{i,1} + \cdots + c_{i,n_i}\vec{\beta}_{i,n_i} = \vec{0}$. Thus, since each B_i is a basis, we have the desired conclusion that all of the c's are zero.

For (2) $\implies$ (3), assume that $B_1 \frown \cdots \frown B_k$ is a basis for the space. Consider a linear relationship among nonzero vectors from different W_i's,

$$\vec{0} = \cdots + d_i\vec{w}_i + \cdots$$

in order to show that it is trivial. (The relationship is written in this way because we are considering a combination of nonzero vectors from only some of the W_i's; for instance, there might not be a $\vec{w}_1$ in this combination.) As in $(*)$, $\vec{0} = \cdots + d_i(c_{i,1}\vec{\beta}_{i,1} + \cdots + c_{i,n_i}\vec{\beta}_{i,n_i}) + \cdots = \cdots + d_ic_{i,1} \cdot \vec{\beta}_{i,1} + \cdots + d_ic_{i,n_i} \cdot \vec{\beta}_{i,n_i} + \cdots$ and the linear independence of $B_1 \frown \cdots \frown B_k$ gives that each coefficient $d_ic_{i,j}$ is zero. Now, $\vec{w}_i$ is a nonzero vector, so at least one of the $c_{i,j}$'s is not zero, and thus d_i is zero. This holds for each d_i, and therefore the linear relationship is trivial.

Finally, for (3) $\implies$ (1), assume that, among nonzero vectors from different W_i's, any linear relationship is trivial. Consider two decompositions of a vector $\vec{v} = \vec{w}_1 + \cdots + \vec{w}_k$ and $\vec{v} = \vec{u}_1 + \cdots + \vec{u}_k$ in order to show that the two are the same. We have

$$\vec{0} = (\vec{w}_1 + \cdots + \vec{w}_k) - (\vec{u}_1 + \cdots + \vec{u}_k) = (\vec{w}_1 - \vec{u}_1) + \cdots + (\vec{w}_k - \vec{u}_k)$$

which violates the assumption unless each $\vec{w}_i - \vec{u}_i$ is the zero vector. Hence, decompositions are unique. QED

4.9 Definition A collection of subspaces $\{W_1, \ldots, W_k\}$ is *independent* if no nonzero vector from any W_i is a linear combination of vectors from the other subspaces $W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_k$.

4.10 Definition A vector space V is the *direct sum* (or *internal direct sum*) of its subspaces $W_1, \ldots, W_k$ if $V = W_1 + W_2 + \cdots + W_k$ and the collection $\{W_1, \ldots, W_k\}$ is independent. We write $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$.

4.11 Example The benchmark model fits: $\mathbb{R}^3 = x\text{-axis} \oplus y\text{-axis} \oplus z\text{-axis}$.

4.12 Example The space of 2×2 matrices is this direct sum.

$$\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}\} \oplus \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{R}\} \oplus \{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \in \mathbb{R}\}$$

It is the direct sum of subspaces in many other ways as well; direct sum decompositions are not unique.

4.13 Corollary The dimension of a direct sum is the sum of the dimensions of its summands.

Proof. In Lemma 4.8, the number of basis vectors in the concatenation equals the sum of the number of vectors in the subbases that make up the concatenation. QED

The special case of two subspaces is worth mentioning separately.

4.14 Definition When a vector space is the direct sum of two of its subspaces, then they are said to be *complements*.

4.15 Lemma A vector space V is the direct sum of two of its subspaces W_1 and W_2 if and only if it is the sum of the two $V = W_1 + W_2$ and their intersection is trivial $W_1 \cap W_2 = \{\vec{0}\,\}$.

Proof. Suppose first that $V = W_1 \oplus W_2$. By definition, V is the sum of the two. To show that the two have a trivial intersection, let $\vec{v}$ be a vector from $W_1 \cap W_2$ and consider the equation $\vec{v} = \vec{v}$. On the left side of that equation is a member of W_1, and on the right side is a linear combination of members (actually, of only one member) of W_2. But the independence of the spaces then implies that $\vec{v} = \vec{0}$, as desired.

For the other direction, suppose that V is the sum of two spaces with a trivial intersection. To show that V is a direct sum of the two, we need only show that the spaces are independent — no nonzero member of the first is expressible as a linear combination of members of the second, and vice versa. This is true because any relationship $\vec{w}_1 = c_1\vec{w}_{2,1} + \cdots + d_k\vec{w}_{2,k}$ (with $\vec{w}_1 \in W_1$ and $\vec{w}_{2,j} \in W_2$ for all j) shows that the vector on the left is also in W_2, since the right side is a combination of members of W_2. The intersection of these two spaces is trivial, so $\vec{w}_1 = \vec{0}$. The same argument works for any $\vec{w}_2$. QED

4.16 Example In the space $\mathbb{R}^2$, the x-axis and the y-axis are complements, that is, $\mathbb{R}^2 = x\text{-axis} \oplus y\text{-axis}$. A space can have more than one pair of complementary subspaces; another pair here are the subspaces consisting of the lines $y = x$ and $y = 2x$.

4.17 Example In the space $F = \{a\cos\theta + b\sin\theta \mid a, b \in \mathbb{R}\}$, the subspaces $W_1 = \{a\cos\theta \mid a \in \mathbb{R}\}$ and $W_2 = \{b\sin\theta \mid b \in \mathbb{R}\}$ are complements. In addition to the fact that a space like F can have more than one pair of complementary subspaces, inside of the space a single subspace like W_1 can have more than one complement — another complement of W_1 is $W_3 = \{b\sin\theta + b\cos\theta \mid b \in \mathbb{R}\}$.

4.18 Example In $\mathbb{R}^3$, the xy-plane and the yz-planes are not complements, which is the point of the discussion following Example 4.4. One complement of the xy-plane is the z-axis. A complement of the yz-plane is the line through $(1, 1, 1)$.

4.19 Example Following Lemma 4.15, here is a natural question: is the simple sum $V = W_1 + \cdots + W_k$ also a direct sum if and only if the intersection of the subspaces is trivial? The answer is that if there are more than two subspaces then having a trivial intersection is not enough to guarantee unique decomposition (i.e., is not enough to ensure that the spaces are independent). In $\mathbb{R}^3$, let W_1 be the x-axis, let W_2 be the y-axis, and let W_3 be this.

$$W_3 = \{\begin{pmatrix} q \\ q \\ r \end{pmatrix} \mid q, r \in \mathbb{R}\}$$

The check that $\mathbb{R}^3 = W_1 + W_2 + W_3$ is easy. The intersection $W_1 \cap W_2 \cap W_3$ is trivial, but decompositions aren't unique.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y - x \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ x \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} y \\ y \\ z \end{pmatrix}$$

(This example also shows that this requirement is also not enough: that all pairwise intersections of the subspaces be trivial. See Exercise 30.)

In this subsection we have seen two ways to regard a space as built up from component parts. Both are useful; in particular, in this book the direct sum definition is needed to do the Jordan Form construction in the fifth chapter.

Exercises

✓ **4.20** Decide if $\mathbb{R}^2$ is the direct sum of each W_1 and W_2.

(a) $W_1 = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \bigm| x \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} x \\ x \end{pmatrix} \bigm| x \in \mathbb{R}\}$

(b) $W_1 = \{\begin{pmatrix} s \\ s \end{pmatrix} \bigm| s \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} s \\ 1.1s \end{pmatrix} \bigm| s \in \mathbb{R}\}$

(c) $W_1 = \mathbb{R}^2$, $W_2 = \{\vec{0}\}$

(d) $W_1 = W_2 = \{\begin{pmatrix} t \\ t \end{pmatrix} \bigm| t \in \mathbb{R}\}$

(e) $W_1 = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ 0 \end{pmatrix} \bigm| x \in \mathbb{R}\}$, $W_2 = \{\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} \bigm| y \in \mathbb{R}\}$

✓ **4.21** Show that $\mathbb{R}^3$ is the direct sum of the xy-plane with each of these.

(a) the z-axis

(b) the line

$$\{\begin{pmatrix} z \\ z \\ z \end{pmatrix} \bigm| z \in \mathbb{R}\}$$

4.22 Is $\mathcal{P}_2$ the direct sum of $\{a + bx^2 \bigm| a, b \in \mathbb{R}\}$ and $\{cx \bigm| c \in \mathbb{R}\}$?

✓ **4.23** In $\mathcal{P}_n$, the *even* polynomials are the members of this set

$$\mathcal{E} = \{p \in \mathcal{P}_n \bigm| p(-x) = p(x) \text{ for all } x\}$$

and the *odd* polynomials are the members of this set.

$$\mathcal{O} = \{p \in \mathcal{P}_n \bigm| p(-x) = -p(x) \text{ for all } x\}$$

Show that these are complementary subspaces.

4.24 Which of these subspaces of $\mathbb{R}^3$

W_1: the x-axis, W_2: the y-axis, W_3: the z-axis,
W_4: the plane $x + y + z = 0$, W_5: the yz-plane

can be combined to

(a) sum to $\mathbb{R}^3$? **(b)** direct sum to $\mathbb{R}^3$?

✓ **4.25** Show that $\mathcal{P}_n = \{a_0 \bigm| a_0 \in \mathbb{R}\} \oplus \ldots \oplus \{a_n x^n \bigm| a_n \in \mathbb{R}\}$.

4.26 What is $W_1 + W_2$ if $W_1 \subseteq W_2$?

4.27 Does Example 4.5 generalize? That is, is this true or false: if a vector space V has a basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ then it is the direct sum of the spans of the one-dimensional subspaces $V = [\{\vec{\beta}_1\}] \oplus \ldots \oplus [\{\vec{\beta}_n\}]$?

4.28 Can $\mathbb{R}^4$ be decomposed as a direct sum in two different ways? Can $\mathbb{R}^1$?

4.29 This exercise makes the notation of writing '+' between sets more natural. Prove that, where $W_1, \ldots, W_k$ are subspaces of a vector space,

$$W_1 + \cdots + W_k = \{\vec{w}_1 + \vec{w}_2 + \cdots + \vec{w}_k \bigm| \vec{w}_1 \in W_1, \ldots, \vec{w}_k \in W_k\},$$

and so the sum of subspaces is the subspace of all sums.

4.30 (Refer to Example 4.19. This exercise shows that the requirement that pariwise intersections be trivial is genuinely stronger than the requirement only that the intersection of all of the subspaces be trivial.) Give a vector space and three subspaces W_1, W_2, and W_3 such that the space is the sum of the subspaces, the intersection of all three subspaces $W_1 \cap W_2 \cap W_3$ is trivial, but the pairwise intersections $W_1 \cap W_2$, $W_1 \cap W_3$, and $W_2 \cap W_3$ are nontrivial.

✓ **4.31** Prove that if $V = W_1 \oplus \ldots \oplus W_k$ then $W_i \cap W_j$ is trivial whenever $i \neq j$. This shows that the first half of the proof of Lemma 4.15 extends to the case of more than two subspaces. (Example 4.19 shows that this implication does not reverse; the other half does not extend.)

4.32 Recall that no linearly independent set contains the zero vector. Can an independent set of subspaces contain the trivial subspace?

✓ **4.33** Does every subspace have a complement?

✓ **4.34** Let W_1, W_2 be subspaces of a vector space.

(a) Assume that the set S_1 spans W_1, and that the set S_2 spans W_2. Can $S_1 \cup S_2$ span $W_1 + W_2$? Must it?

(b) Assume that S_1 is a linearly independent subset of W_1 and that S_2 is a linearly independent subset of W_2. Can $S_1 \cup S_2$ be a linearly independent subset of $W_1 + W_2$? Must it?

4.35 When a vector space is decomposed as a direct sum, the dimensions of the subspaces add to the dimension of the space. The situation with a space that is given as the sum of its subspaces is not as simple. This exercise considers the two-subspace special case.

(a) For these subspaces of $\mathcal{M}_{2\times2}$ find $W_1 \cap W_2$, $\dim(W_1 \cap W_2)$, $W_1 + W_2$, and $\dim(W_1 + W_2)$.

$$W_1 = \{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R}\} \qquad W_2 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in \mathbb{R}\}$$

(b) Suppose that U and W are subspaces of a vector space. Suppose that the sequence $\langle \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ is a basis for $U \cap W$. Finally, suppose that the prior sequence has been expanded to give a sequence $\langle \vec{\mu}_1, \ldots, \vec{\mu}_j, \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ that is a basis for U, and a sequence $\langle \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\omega}_1, \ldots, \vec{\omega}_p \rangle$ that is a basis for W. Prove that this sequence

$$\langle \vec{\mu}_1, \ldots, \vec{\mu}_j, \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\omega}_1, \ldots, \vec{\omega}_p \rangle$$

is a basis for for the sum $U + W$.

(c) Conclude that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

(d) Let W_1 and W_2 be eight-dimensional subspaces of a ten-dimensional space. List all values possible for $\dim(W_1 \cap W_2)$.

4.36 Let $V = W_1 \oplus \ldots \oplus W_k$ and for each index i suppose that S_i is a linearly independent subset of W_i. Prove that the union of the S_i's is linearly independent.

4.37 A matrix is *symmetric* if for each pair of indices i and j, the i, j entry equals the j, i entry. A matrix is *antisymmetric* if each i, j entry is the negative of the j, i entry.

(a) Give a symmetric 2×2 matrix and an antisymmetric 2×2 matrix. (*Remark.* For the second one, be careful about the entries on the diagional.)

(b) What is the relationship between a square symmetric matrix and its transpose? Between a square antisymmetric matrix and its transpose?

(c) Show that $\mathcal{M}_{n\times n}$ is the direct sum of the space of symmetric matrices and the space of antisymmetric matrices.

4.38 Let W_1, W_2, W_3 be subspaces of a vector space. Prove that $(W_1 \cap W_2) + (W_1 \cap W_3) \subseteq W_1 \cap (W_2 + W_3)$. Does the inclusion reverse?

4.39 The example of the x-axis and the y-axis in $\mathbb{R}^2$ shows that $W_1 \oplus W_2 = V$ does not imply that $W_1 \cup W_2 = V$. Can $W_1 \oplus W_2 = V$ and $W_1 \cup W_2 = V$ happen?

✓ **4.40** Consider Corollary 4.13. Does it work both ways — that is, supposing that $V = W_1 + \cdots + W_k$, is $V = W_1 \oplus \ldots \oplus W_k$ if and only if $\dim(V) = \dim(W_1) + \cdots + \dim(W_k)$?

4.41 We know that if $V = W_1 \oplus W_2$ then there is a basis for V that splits into a basis for W_1 and a basis for W_2. Can we make the stronger statement that every basis for V splits into a basis for W_1 and a basis for W_2?

4.42 We can ask about the algebra of the '+' operation.
- **(a)** Is it commutative; is $W_1 + W_2 = W_2 + W_1$?
- **(b)** Is it associative; is $(W_1 + W_2) + W_3 = W_1 + (W_2 + W_3)$?
- **(c)** Let W be a subspace of some vector space. Show that $W + W = W$.
- **(d)** Must there be an identity element, a subspace I such that $I + W = W + I = W$ for all subspaces W?
- **(e)** Does left-cancelation hold: if $W_1 + W_2 = W_1 + W_3$ then $W_2 = W_3$? Right cancelation?

4.43 Consider the algebraic properties of the direct sum operation.
- **(a)** Does direct sum commute: does $V = W_1 \oplus W_2$ imply that $V = W_2 \oplus W_1$?
- **(b)** Prove that direct sum is associative: $(W_1 \oplus W_2) \oplus W_3 = W_1 \oplus (W_2 \oplus W_3)$.
- **(c)** Show that $\mathbb{R}^3$ is the direct sum of the three axes (the relevance here is that by the previous item, we needn't specify which two of the threee axes are combined first).
- **(d)** Does the direct sum operation left-cancel: does $W_1 \oplus W_2 = W_1 \oplus W_3$ imply $W_2 = W_3$? Does it right-cancel?
- **(e)** There is an identity element with respect to this operation. Find it.
- **(f)** Do some, or all, subspaces have inverses with respect to this operation: is there a subspace W of some vector space such that there is a subspace U with the property that $U \oplus W$ equals the identity element from the prior item?

Topic: Fields

Linear combinations involving only fractions or only integers are much easier for computations than combinations involving real numbers, because computing with irrational numbers is awkward. Could other number systems, like the rationals or the integers, work in the place of $\mathbb{R}$ in the definition of a vector space?

Yes and no. If we take "work" to mean that the results of this chapter remain true then an analysis of which properties of the reals we have used in this chapter gives the following list of conditions an algebraic system needs in order to "work" in the place of $\mathbb{R}$.

Definition. A *field* is a set $\mathcal{F}$ with two operations '$+$' and '$\cdot$' such that

(1) for any $a, b \in \mathcal{F}$ the result of $a + b$ is in $\mathcal{F}$ and

- $a + b = b + a$
- if $c \in \mathcal{F}$ then $a + (b + c) = (a + b) + c$

(2) for any $a, b \in \mathcal{F}$ the result of $a \cdot b$ is in $\mathcal{F}$ and

- $a \cdot b = b \cdot a$
- if $c \in \mathcal{F}$ then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(3) if $a, b, c \in \mathcal{F}$ then $a \cdot (b + c) = a \cdot b + a \cdot c$

(4) there is an element $0 \in \mathcal{F}$ such that

- if $a \in \mathcal{F}$ then $a + 0 = a$
- for each $a \in \mathcal{F}$ there is an element $-a \in \mathcal{F}$ such that $(-a) + a = 0$

(5) there is an element $1 \in \mathcal{F}$ such that

- if $a \in \mathcal{F}$ then $a \cdot 1 = a$
- for each element $a \neq 0$ of $\mathcal{F}$ there is an element $a^{-1} \in \mathcal{F}$ such that $a^{-1} \cdot a = 1$.

The number system consisting of the set of real numbers along with the usual addition and multiplication operation is a field, naturally. Another field is the set of rational numbers with its usual addition and multiplication operations. An example of an algebraic structure that is not a field is the integer number system—it fails the final condition.

Some examples are surprising. The set $\{0, 1\}$ under these operations:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

is a field (see Exercise 4).

We could develop Linear Algebra as the theory of vector spaces with scalars from an arbitrary field, instead of sticking to taking the scalars only from $\mathbb{R}$. In that case, almost all of the statements in this book would carry over by replacing '$\mathbb{R}$' with '$\mathcal{F}$', and thus by taking coefficients, vector entries, and matrix entries to be elements of $\mathcal{F}$ ("almost" because statements involving distances or angles are exceptions). Here are some examples; each applies to a vector space V over a field $\mathcal{F}$.

* For any $\vec{v} \in V$ and $a \in \mathcal{F}$, (i) $0 \cdot \vec{v} = \vec{0}$, and (ii) $-1 \cdot \vec{v} + \vec{v} = \vec{0}$, and (iii) $a \cdot \vec{0} = \vec{0}$.
* The span (the set of linear combinations) of a subset of V is a subspace of V.
* Any subset of a linearly independent set is also linearly independent.
* In a finite-dimensional vector space, any two bases have the same number of elements.

(Even statements that don't explicitly mention $\mathcal{F}$ use field properties in their proof.)

We won't develop vector spaces in this more general setting because the additional abstraction can be a distraction. The ideas we want to bring out already appear when we stick to the reals.

The only exception is in Chapter Five. In that chapter we must factor polynomials, so we will switch to considering vector spaces over the field of complex numbers. We will discuss this more, including a brief review of complex arithmetic, when we get there.

Exercises

1 Show that the real numbers form a field.

2 Prove that these are fields.
  (a) The rational numbers $\mathbb{Q}$ **(b)** The complex numbers $\mathbb{C}$

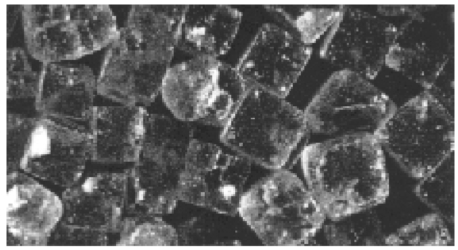
3 Give an example that shows that the integer number system is not a field.

4 Consider the set $\{0, 1\}$ subject to the operations given above. Show that it is a field.

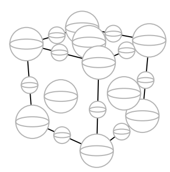
5 Give suitable operations to make the set $\{0, 1, 2\}$ a field.

Topic: Crystals

Everyone has noticed that table salt comes in little cubes.

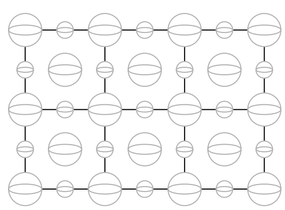


Remarkably, the explanation for the cubical external shape is the simplest one: the internal shape, the way the atoms lie, is also cubical. The internal structure is pictured below. Salt is sodium cloride, and the small spheres shown are sodium while the big ones are cloride. To simplify the view, it only shows the sodiums and clorides on the front, top, and right.



The specks of salt that we see when we spread a little out on the table consist of many repetitions of this fundamental unit. That is, these cubes of atoms stack up to make the larger cubical structure that we see. A solid, such as table salt, with a regular internal structure is a *crystal*.

We can restrict our attention to the front face. There, we have the square repeated many times.

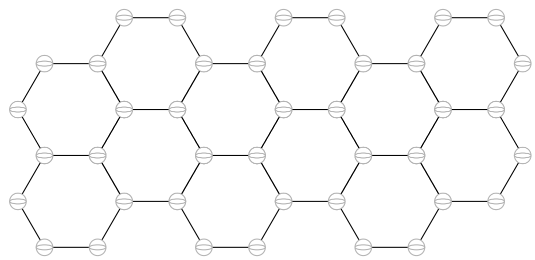


The distance between the corners of the square cell is about 3.34 Ångstroms (an Ångstrom is 10^{-10} meters). Obviously that unit is unwieldly. Instead, the thing to do is to take as a unit the length of each side of the square. That is, we naturally adopt this basis.

$$\langle \begin{pmatrix} 3.34 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3.34 \end{pmatrix} \rangle$$

Then we can describe, say, the corner in the upper right of the picture above as $3\vec{\beta}_1 + 2\vec{\beta}_2$.

Another crystal from everyday experience is pencil lead. It is graphite, formed from carbon atoms arranged in this shape.



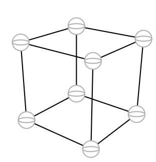
This is a single plane of graphite. A piece of graphite consists of many of these planes layered in a stack. (The chemical bonds between the planes are much weaker than the bonds inside the planes, which explains why pencils write — the graphite can be sheared so that the planes slide off and are left on the paper.) We can get a convienent unit of length by decomposing the hexagonal ring into three regions that are rotations of this *unit cell*.

Then a natural basis consists of the vectors that form the sides of that unit cell. The distance along the bottom and slant is 1.42 Ångstroms, so this

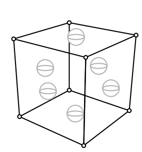
$$\langle \begin{pmatrix} 1.42 \\ 0 \end{pmatrix}, \begin{pmatrix} 1.23 \\ .71 \end{pmatrix} \rangle$$

is a good basis.

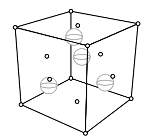
The selection of convienent bases extends to three dimensions. Another familiar crystal formed from carbon is diamond. Like table salt, it is built from cubes, but the structure inside each cube is more complicated than salt's. In addition to carbons at each corner,



there are carbons in the middle of each face.



(To show the added face carbons clearly, the corner carbons have been reduced to dots.) There are also four more carbons inside the cube, two that are a quarter of the way up from the bottom and two that are a quarter of the way down from the top.



(As before, carbons shown earlier have been reduced here to dots.) The distance along any edge of the cube is 2.18 Ångstroms. Thus, a natural basis for describing the locations of the carbons, and the bonds between them, is this.

$$\langle \begin{pmatrix} 2.18 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2.18 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2.18 \end{pmatrix} \rangle$$

Even the few examples given here show that the structures of crystals is complicated enough that some organized system to give the locations of the atoms, and how they are chemically bound, is needed. One tool for that organization is a convienent basis. This application of bases is simple, but it shows a context where the idea arises naturally. The work in this chapter just takes this simple idea and develops it.

Exercises

1 How many fundamental regions are there in one face of a speck of salt? (With a ruler, we can estimate that face is a square that is 0.1 cm on a side.)

2 In the graphite picture, imagine that we are interested in a point 5.67 Ångstroms over and 3.14 Ångstroms up from the origin.

(a) Express that point in terms of the basis given for graphite.

(b) How many hexagonal shapes away is this point from the origin?

(c) Express that point in terms of a second basis, where the first basis vector is the same, but the second is perpendicular to the first (going up the plane) and of the same length.

3 Give the locations of the atoms in the diamond cube both in terms of the basis, and in Ångstroms.

4 This illustrates how the dimensions of a unit cell could be computed from the shape in which a substance crystalizes ([Ebbing], p. 462).

(a) Recall that there are 6.022×10^{23} atoms in a mole (this is Avagadro's number). From that, and the fact that platinum has a mass of 195.08 grams per mole, calculate the mass of each atom.

(b) Platinum crystalizes in a face-centered cubic lattice with atoms at each lattice point, that is, it looks like the middle picture given above for the diamond crystal. Find the number of platinums per unit cell (hint: sum the fractions of platinums that are inside of a single cell).

(c) From that, find the mass of a unit cell.

(d) Platinum crystal has a density of 21.45 grams per cubic centimeter. From this, and the mass of a unit cell, calculate the volume of a unit cell.

(e) Find the length of each edge.
(f) Describe a natural three-dimensional basis.

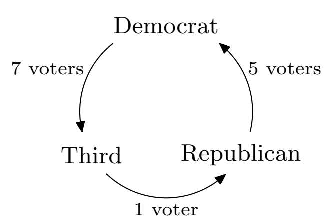
Topic: Voting Paradoxes

Imagine that a Political Science class studying the American presidential process holds a mock election. Members of the class are asked to rank, from most preferred to least preferred, the nominees from the Democratic Party, the Republican Party, and the Third Party, and this is the result ($>$ means 'is preferred to').

preference order	*number with that preference*
Democrat $>$ Republican $>$ Third	5
Democrat $>$ Third $>$ Republican	4
Republican $>$ Democrat $>$ Third	2
Republican $>$ Third $>$ Democrat	8
Third $>$ Democrat $>$ Republican	8
Third $>$ Republican $>$ Democrat	2
total	29

What is the preference of the group as a whole?

Overall, the group prefers the Democrat to the Republican by five votes; seventeen voters ranked the Democrat above the Republican versus twelve the other way. And, overall, the group prefers the Republican to the Third's nominee, fifteen to fourteen. But, strangely enough, the group also prefers the Third to the Democrat, eighteen to eleven.

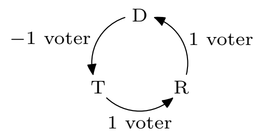


This is an example of a *voting paradox*, specifically, a *majority cycle*.

Voting paradoxes are studied in part because of their implications for practical politics. For instance, the instructor can manipulate the class into choosing the Democrat as the overall winner by first asking the class to choose between the Republican and the Third, and then asking the class to choose between the winner of that contest, the Republican, and the Democrat. By similar manipulations, any of the other two candidates can be made to come out as the winner. (In this Topic we will stick to three-candidate elections, but similar results apply to larger elections.)

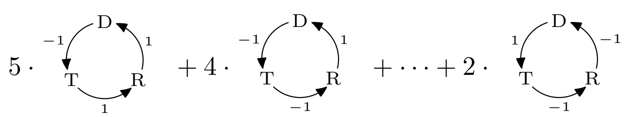
Voting paradoxes are also studied simply because they are mathematically interesting. One interesting aspect is that the group's overall majority cycle occurs despite that each single voters's preference list is *rational*—in a straight-line order. That is, the majority cycle seems to arise in the aggregate, without being present in the elements of that aggregate, the preference lists. Recently, however, linear algebra has been used [Zwicker] to argue that a tendency toward cyclic preference is actually present in each voter's list, and that it surfaces when there is more adding of the tendency than cancelling.

For this argument, abbreviating the choices as D, R, and T, we can describe how a voter with preference order $D > R > T$ contributes to the above cycle.



(The negative sign is here because the arrow describes T as preferred to D, but this voter likes them the other way.) The descriptions for the other preference lists are in the table on page 147.

Now, to conduct the election we linearly combine these descriptions; for instance, the Political Science mock election



yields the circular group preference shown earlier.

Of course, taking linear combinations is linear algebra. The above cycle notation is suggestive but inconvienent, so we temporarily switch to using column vectors by starting at the D and taking the numbers from the cycle in counterclockwise order. Thus, the mock election and a single $D > R > T$ vote are represented in this way.

$$\begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

We will decompose vote vectors into two parts, one cyclic and the other acyclic. For the first part, we say that a vector is *purely cyclic* if it is in this subspace of $\mathbb{R}^3$.

$$C = \{ \begin{pmatrix} k \\ k \\ k \end{pmatrix} \bigm| k \in \mathbb{R} \} = \{ k \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \bigm| k \in \mathbb{R} \}$$

For the second part, consider the subspace (see Exercise 6) of vectors that are perpendicular to all of the vectors in C.

$$\begin{aligned} C^{\perp} &= \{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \bigm| \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot \begin{pmatrix} k \\ k \\ k \end{pmatrix} = 0 \text{ for all } k \in \mathbb{R} \} \\ &= \{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \bigm| c_1 + c_2 + c_3 = 0 \} = \{ c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \bigm| c_2, c_3 \in \mathbb{R} \} \end{aligned}$$

(Read that aloud as "C perp".) So we are led to this basis for $\mathbb{R}^3$.

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

We can represent votes with respect to this basis, and thereby decompose them into a cyclic part and an acyclic part. *(Note for readers who have covered the optional section in this chapter: that is, the space is the direct sum of C and $C^{\perp}$.)*

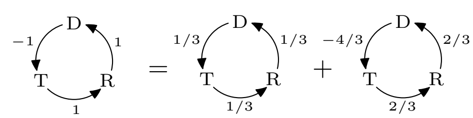
For example, consider the $D > R > T$ voter discussed above. The representation in terms of the basis is easily found,

$$\begin{array}{rcl} c_1 - c_2 - c_3 &=& -1 \\ c_1 + c_2 \phantom{{}-c_3} &=& 1 \\ c_1 \phantom{{}-c_2} + c_3 &=& 1 \end{array} \quad \xrightarrow[-\rho_1+\rho_3]{-\rho_1+\rho_2} \; \xrightarrow{(-1/2)\rho_2+\rho_3} \quad \begin{array}{rcl} c_1 - c_2 - c_3 &=& -1 \\ 2c_2 + c_3 &=& 2 \\ (3/2)c_3 &=& 1 \end{array}$$

so that $c_1 = 1/3$, $c_2 = 2/3$, and $c_3 = 2/3$. Then

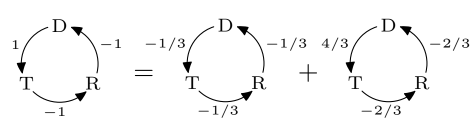
$$\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} + \begin{pmatrix} -4/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

gives the desired decomposition into a cyclic part and and an acyclic part.



Thus, this $D > R > T$ voter's rational preference list can indeed be seen to have a cyclic part.

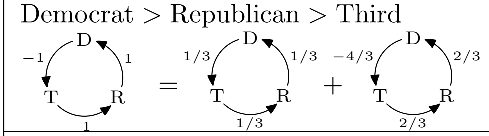
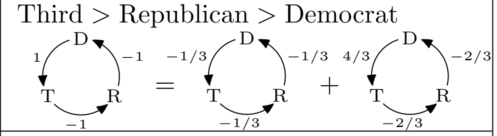
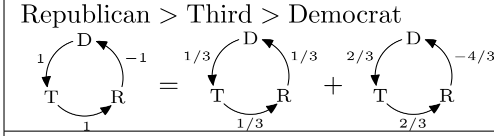
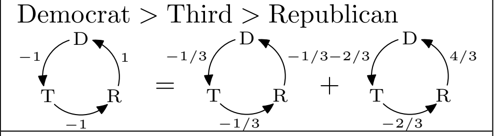
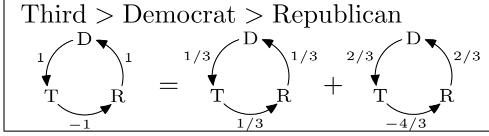
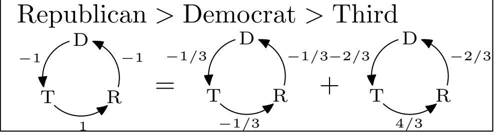
The $T > R > D$ voter is opposite to the one just considered in that the '$>$' symbols are reversed. This voter's decomposition



shows that these opposite preferences have decompositions that are opposite. We say that the first voter has positive *spin* since the cycle part is with the direction we have chosen for the arrows, while the second voter's spin is negative.

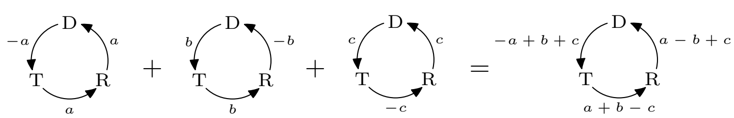
The fact that that these opposite voters cancel each other is reflected in the fact that their vote vectors add to zero. This suggests an alternate way to tally an election. We could first cancel as many opposite preference lists as possible, and then determine the outcome by adding the remaining lists.

The rows of the table below contain the three pairs of opposite preference lists. The columns group those pairs by spin. For instance, the first row contains the two voters just considered.

positive spin	*negative spin*
Democrat > Republican > Third 	Third > Republican > Democrat 
Republican > Third > Democrat 	Democrat > Third > Republican 
Third > Democrat > Republican 	Republican > Democrat > Third 

If we conduct the election as just described then after the cancellation of as many opposite pairs of voters as possible, there will be left three sets of preference lists, one set from the first row, one set from the second row, and one set from the third row. We will finish by proving that a voting paradox can happen only if the spins of these three sets are in the same direction. That is, for a voting paradox to occur, the three remaining sets must all come from the left of the table or all come from the right (see Exercise 3). This shows that there is some connection between the majority cycle and the decomposition that we are using—a voting paradox can happen only when the tendencies toward cyclic preference reinforce each other.

For the proof, assume that opposite preference orders have been cancelled, and we are left with one set of preference lists from each of the three rows. Consider the sum of these three (here, the numbers a, b, and c could be positive, negative, or zero).



A voting paradox occurs when the three numbers on the right, $a-b+c$ and $a+b-c$ and $-a+b+c$, are all nonnegative or all nonpositive. On the left, at least two of the three numbers, a and b and c, are both nonnegative or both nonpositive. We can assume that they are a and b. That makes four cases: the cycle is nonnegative and a and b are nonnegative, the cycle is nonpositive and a and b are nonpositive, etc. We will do only the first case, since the second is similar and the other two are also easy.

So assume that the cycle is nonnegative and that a and b are nonnegative. The conditions $0 \leq a-b+c$ and $0 \leq -a+b+c$ add to give that $0 \leq 2c$, which implies that c is also nonnegative, as desired. That ends the proof.

This result says only that having all three spin in the same direction is a necessary condition for a majority cycle. It is not sufficient; see Exercise 4.

Voting theory and associated topics are the subject of current research. There are many intriguing results, most notably the one produced by K. Arrow [Arrow], who won the Nobel Prize in part for this work, showing that no voting system is entirely fair (for a reasonable definition of "fair"). For more information, some good introductory articles are [Gardner, 1970], [Gardner, 1974], [Gardner, 1980], and [Neimi & Riker]. A quite readable recent book is [Taylor]. The long list of cases from recent American political history given in [Poundstone] show that manipulation of these paradoxes is routine in practice (and the author proposes a solution).

This Topic is largely drawn from [Zwicker]. *(Author's Note: I would like to thank Professor Zwicker for his kind and illuminating discussions.)*

Exercises

1 Here is a reasonable way in which a voter could have a cyclic preference. Suppose that this voter ranks each candidate on each of three criteria.

(a) Draw up a table with the rows labelled 'Democrat', 'Republican', and 'Third', and the columns labelled 'character', 'experience', and 'policies'. Inside each column, rank some candidate as most preferred, rank another as in the middle, and rank the remaining one as least preferred.

(b) In this ranking, is the Democrat preferred to the Republican in (at least) two out of three criteria, or vice versa? Is the Republican preferred to the Third?

(c) Does the table that was just constructed have a cyclic preference order? If not, make one that does.

So it is possible for a voter to have a cyclic preference among candidates. The paradox described above, however, is that even if each voter has a straight-line preference list, a cyclic preference can still arise for the entire group.

2 Compute the values in the table of decompositions.

3 Do the cancellations of opposite preference orders for the Political Science class's mock election. Are all the remaining preferences from the left three rows of the table or from the right?

4 The necessary condition that is proved above—a voting paradox can happen only if all three preference lists remaining after cancellation have the same spin—is not also sufficient.

(a) Continuing the positive cycle case considered in the proof, use the two inequalities $0 \leq a - b + c$ and $0 \leq -a + b + c$ to show that $|a - b| \leq c$.

(b) Also show that $c \leq a + b$, and hence that $|a - b| \leq c \leq a + b$.

(c) Give an example of a vote where there is a majority cycle, and addition of one more voter with the same spin causes the cycle to go away.

(d) Can the opposite happen; can addition of one voter with a "wrong" spin cause a cycle to appear?

(e) Give a condition that is both necessary and sufficient to get a majority cycle.

5 A one-voter election cannot have a majority cycle because of the requirement that we've imposed that the voter's list must be rational.

(a) Show that a two-voter election may have a majority cycle. (We consider the group preference a majority cycle if all three group totals are nonnegative or if all three are nonpositive—that is, we allow some zero's in the group preference.)

(b) Show that for any number of voters greater than one, there is an election involving that many voters that results in a majority cycle.

6 Let U be a subspace of $\mathbb{R}^3$. Prove that the set $U^{\perp} = \{\vec{v} \bigm| \vec{v} \cdot \vec{u} = 0 \text{ for all } \vec{u} \in U\}$ of vectors that are perpendicular to each vector in U is also subspace of $\mathbb{R}^3$. Does this hold if U is not a subspace?

Topic: Dimensional Analysis

"You can't add apples and oranges," the old saying goes. It reflects our experience that in applications the quantities have units and keeping track of those units is worthwhile. Everyone has done calculations such as this one that use the units as a check.

$$60\,\frac{\text{sec}}{\text{min}}\cdot 60\,\frac{\text{min}}{\text{hr}}\cdot 24\,\frac{\text{hr}}{\text{day}}\cdot 365\,\frac{\text{day}}{\text{year}}=31\,536\,000\,\frac{\text{sec}}{\text{year}}$$

However, the idea of including the units can be taken beyond bookkeeping. It can be used to draw conclusions about what relationships are possible among the physical quantities.

To start, consider the physics equation: distance $=16\cdot(\text{time})^2$. If the distance is in feet and the time is in seconds then this is a true statement about falling bodies. However it is not correct in other unit systems; for instance, it is not correct in the meter-second system. We can fix that by making the 16 a *dimensional constant*.

$$\text{dist}=16\,\frac{\text{ft}}{\text{sec}^2}\cdot(\text{time})^2$$

For instance, the above equation holds in the yard-second system.

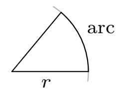
$$\text{distance in yards}=16\,\frac{(1/3)\,\text{yd}}{\text{sec}^2}\cdot(\text{time in sec})^2=\frac{16}{3}\,\frac{\text{yd}}{\text{sec}^2}\cdot(\text{time in sec})^2$$

So our first point is that by "including the units" we mean that we are restricting our attention to equations that use dimensional constants.

By using dimensional constants, we can be vague about units and say only that all quantities are measured in combinations of some units of length L, mass M, and time T. We shall refer to these three as *dimensions* (these are the only three dimensions that we shall need in this Topic). For instance, velocity could be measured in feet/second or fathoms/hour, but in all events it involves some unit of length divided by some unit of time so the *dimensional formula* of velocity is L/T. Similarly, the dimensional formula of density is M/L^3. We shall prefer using negative exponents over the fraction bars and we shall include the dimensions with a zero exponent, that is, we shall write the dimensional formula of velocity as $L^1M^0T^{-1}$ and that of density as $L^{-3}M^1T^0$.

In this context, "You can't add apples to oranges" becomes the advice to check that all of an equation's terms have the same dimensional formula. An example is this version of the falling body equation: $d-gt^2=0$. The dimensional formula of the d term is $L^1M^0T^0$. For the other term, the dimensional formula of g is $L^1M^0T^{-2}$ (g is the dimensional constant given above as $16\,\text{ft}/\text{sec}^2$) and the dimensional formula of t is $L^0M^0T^1$, so that of the entire gt^2 term is $L^1M^0T^{-2}(L^0M^0T^1)^2=L^1M^0T^0$. Thus the two terms have the same dimensional formula. An equation with this property is *dimensionally homogeneous*.

Quantities with dimensional formula $L^0M^0T^0$ are *dimensionless*. For example, we measure an angle by taking the ratio of the subtended arc to the radius



which is the ratio of a length to a length $L^1M^0T^0/L^1M^0T^0$ and thus angles have the dimensional formula $L^0M^0T^0$.

The classic example of using the units for more than bookkeeping, using them to draw conclusions, considers the formula for the period of a pendulum.

$$p = \text{–some expression involving the length of the string, etc.–}$$

The period is in units of time $L^0M^0T^1$. So the quantities on the other side of the equation must have dimensional formulas that combine in such a way that their L's and M's cancel and only a single T remains. The table on page 152 has the quantities that an experienced investigator would consider possibly relevant. The only dimensional formulas involving L are for the length of the string and the acceleration due to gravity. For the L's of these two to cancel, when they appear in the equation they must be in ratio, e.g., as $(\ell/g)^2$, or as $\cos(\ell/g)$, or as $(\ell/g)^{-1}$. Therefore the period is a function of ℓ/g.

This is a remarkable result: with a pencil and paper analysis, before we ever took out the pendulum and made measurements, we have determined something about the relationship among the quantities.

To do dimensional analysis systematically, we need to know two things (arguments for these are in [Bridgman], Chapter II and IV). The first is that each equation relating physical quantities that we shall see involves a sum of terms, where each term has the form

$$m_1^{p_1} m_2^{p_2} \cdots m_k^{p_k}$$

for numbers $m_1, \ldots, m_k$ that measure the quantities.

For the second, observe that an easy way to construct a dimensionally homogeneous expression is by taking a product of dimensionless quantities or by adding such dimensionless terms. Buckingham's Theorem states that any complete relationship among quantities with dimensional formulas can be algebraically manipulated into a form where there is some function f such that

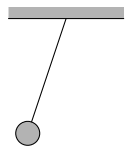
$$f(\Pi_1, \ldots, \Pi_n) = 0$$

for a complete set $\{\Pi_1, \ldots, \Pi_n\}$ of dimensionless products. (The first example below describes what makes a set of dimensionless products 'complete'.) We usually want to express one of the quantities, m_1 for instance, in terms of the others, and for that we will assume that the above equality can be rewritten

$$m_1 = m_2^{-p_2} \cdots m_k^{-p_k} \cdot \hat{f}(\Pi_2, \ldots, \Pi_n)$$

where $\Pi_1 = m_1 m_2^{p_2} \cdots m_k^{p_k}$ is dimensionless and the products $\Pi_2, \ldots, \Pi_n$ don't involve m_1 (as with f, here $\hat{f}$ is just some function, this time of $n-1$ arguments). Thus, to do dimensional analysis we should find which dimensionless products are possible.

For example, consider again the formula for a pendulum's period.



quantity	*dimensional formula*
period p	$L^0M^0T^1$
length of string ℓ	$L^1M^0T^0$
mass of bob m	$L^0M^1T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
arc of swing θ	$L^0M^0T^0$

By the first fact cited above, we expect the formula to have (possibly sums of terms of) the form $p^{p_1}\ell^{p_2}m^{p_3}g^{p_4}\theta^{p_5}$. To use the second fact, to find which combinations of the powers $p_1, \ldots, p_5$ yield dimensionless products, consider this equation.

$$(L^0M^0T^1)^{p_1}(L^1M^0T^0)^{p_2}(L^0M^1T^0)^{p_3}(L^1M^0T^{-2})^{p_4}(L^0M^0T^0)^{p_5} = L^0M^0T^0$$

It gives three conditions on the powers.

$$\begin{array}{rcrcrl} & p_2 & & + & p_4 & =0 \\ & & p_3 & & & =0 \\ p_1 & & & - & 2p_4 & =0 \end{array}$$

Note that p_3 is 0 and so the mass of the bob does not affect the period. Gaussian reduction and parametrization of that system gives this

$$\{\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} p_5 \bigm| p_1, p_5 \in \mathbb{R}\}$$

(we've taken p_1 as one of the parameters in order to express the period in terms of the other quantities).

Here is the linear algebra. The set of dimensionless products contains all terms $p^{p_1}\ell^{p_2}m^{p_3}a^{p_4}\theta^{p_5}$ subject to the conditions above. This set forms a vector space under the '+' operation of multiplying two such products and the '·' operation of raising such a product to the power of the scalar (see Exercise 5). The term 'complete set of dimensionless products' in Buckingham's Theorem means a basis for this vector space.

We can get a basis by first taking $p_1 = 1$, $p_5 = 0$ and then $p_1 = 0$, $p_5 = 1$. The associated dimensionless products are $\Pi_1 = p\ell^{-1/2}g^{1/2}$ and $\Pi_2 = \theta$. Because the set $\{\Pi_1, \Pi_2\}$ is complete, Buckingham's Theorem says that

$$p = \ell^{1/2}g^{-1/2} \cdot \hat{f}(\theta) = \sqrt{\ell/g} \cdot \hat{f}(\theta)$$

where $\hat{f}$ is a function that we cannot determine from this analysis (a first year physics text will show by other means that for small angles it is approximately the constant function $\hat{f}(\theta) = 2\pi$).

Thus, analysis of the relationships that are possible between the quantities with the given dimensional formulas has produced a fair amount of information: a pendulum's period does not depend on the mass of the bob, and it rises with the square root of the length of the string.

For the next example we try to determine the period of revolution of two bodies in space orbiting each other under mutual gravitational attraction. An experienced investigator could expect that these are the relevant quantities.

quantity	*dimensional formula*
period p	$L^0M^0T^1$
mean separation r	$L^1M^0T^0$
first mass m_1	$L^0M^1T^0$
second mass m_2	$L^0M^1T^0$
grav. constant G	$L^3M^{-1}T^{-2}$

To get the complete set of dimensionless products we consider the equation

$$(L^0M^0T^1)^{p_1}(L^1M^0T^0)^{p_2}(L^0M^1T^0)^{p_3}(L^0M^1T^0)^{p_4}(L^3M^{-1}T^{-2})^{p_5} = L^0M^0T^0$$

which results in a system

$$\begin{aligned} p_2 \qquad\qquad\qquad + 3p_5 &= 0 \\ p_3 + p_4 - \ p_5 &= 0 \\ p_1 \qquad\qquad\qquad\quad - 2p_5 &= 0 \end{aligned}$$

with this solution.

$$\{\begin{pmatrix} 1 \\ -3/2 \\ 1/2 \\ 0 \\ 1/2 \end{pmatrix} p_1 + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} p_4 \;\big|\; p_1, p_4 \in \mathbb{R}\}$$

As earlier, the linear algebra here is that the set of dimensionless products of these quantities forms a vector space, and we want to produce a basis for that space, a 'complete' set of dimensionless products. One such set, gotten from setting $p_1 = 1$ and $p_4 = 0$, and also setting $p_1 = 0$ and $p_4 = 1$ is $\{\Pi_1 = pr^{-3/2}m_1^{1/2}G^{1/2},\ \Pi_2 = m_1^{-1}m_2\}$. With that, Buckingham's Theorem says that any complete relationship among these quantities is stateable this form.

$$p = r^{3/2}m_1^{-1/2}G^{-1/2} \cdot \hat{f}(m_1^{-1}m_2) = \frac{r^{3/2}}{\sqrt{Gm_1}} \cdot \hat{f}(m_2/m_1)$$

Remark. An important application of the prior formula is when m_1 is the mass of the sun and m_2 is the mass of a planet. Because m_1 is very much greater than m_2, the argument to $\hat{f}$ is approximately 0, and we can wonder whether this part of the formula remains approximately constant as m_2 varies. One way to see that it does is this. The sun is so much larger than the planet that the mutual rotation is approximately about the sun's center. If we vary the planet's mass m_2 by a factor of x (e.g., Venus's mass is $x = 0.815$ times Earth's mass), then the force of attraction is multiplied by x, and x times the force acting on x times the mass gives, since $F = ma$, the same acceleration, about the same center (approximately). Hence, the orbit will be the same and so its period will be the same, and thus the right side of the above equation also remains unchanged (approximately). Therefore, $\hat{f}(m_2/m_1)$ is approximately constant as m_2 varies. This is Kepler's Third Law: the square of the period of a planet is proportional to the cube of the mean radius of its orbit about the sun.

The final example was one of the first explicit applications of dimensional analysis. Lord Raleigh considered the speed of a wave in deep water and suggested these as the relevant quantities.

quantity	*dimensional formula*
velocity of the wave v	$L^1M^0T^{-1}$
density of the water d	$L^{-3}M^1T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
wavelength λ	$L^1M^0T^0$

The equation

$$(L^1M^0T^{-1})^{p_1}(L^{-3}M^1T^0)^{p_2}(L^1M^0T^{-2})^{p_3}(L^1M^0T^0)^{p_4} = L^0M^0T^0$$

gives this system

$$\begin{array}{rcrcrcrcl} p_1 & - & 3p_2 & + & p_3 & + & p_4 & = & 0 \\ & & p_2 & & & & & = & 0 \\ -p_1 & & & - & 2p_3 & & & = & 0 \end{array}$$

with this solution space

$$\{\begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -1/2 \end{pmatrix} p_1 \mid p_1 \in \mathbb{R}\}$$

(as in the pendulum example, one of the quantities d turns out not to be involved in the relationship). There is one dimensionless product, $\Pi_1 = vg^{-1/2}\lambda^{-1/2}$, and so v is $\sqrt{\lambda g}$ times a constant ($\hat{f}$ is constant since it is a function of no arguments).

As the three examples above show, dimensional analysis can bring us far toward expressing the relationship among the quantities. For further reading, the classic reference is [Bridgman]—this brief book is delightful. Another source is [Giordano, Wells, Wilde]. A description of dimensional analysis's place in modeling is in [Giordano, Jaye, Weir].

Exercises

1 Consider a projectile, launched with initial velocity v_0, at an angle θ. An investigation of this motion might start with the guess that these are the relevant quantities. [de Mestre]

quantity	*dimensional formula*
horizontal position x	$L^1M^0T^0$
vertical position y	$L^1M^0T^0$
initial speed v_0	$L^1M^0T^{-1}$
angle of launch θ	$L^0M^0T^0$
acceleration due to gravity g	$L^1M^0T^{-2}$
time t	$L^0M^0T^1$

(a) Show that $\{gt/v_0, gx/v_0^2, gy/v_0^2, \theta\}$ is a complete set of dimensionless products. (*Hint.* This can be done by finding the appropriate free variables in the linear system that arises, but there is a shortcut that uses the properties of a basis.)

(b) These two equations of motion for projectiles are familiar: $x = v_0\cos(\theta)t$ and $y = v_0\sin(\theta)t - (g/2)t^2$. Manipulate each to rewrite it as a relationship among the dimensionless products of the prior item.

2 [Einstein] conjectured that the infrared characteristic frequencies of a solid may be determined by the same forces between atoms as determine the solid's ordanary elastic behavior. The relevant quantities are

quantity	*dimensional formula*
characteristic frequency ν	$L^0M^0T^{-1}$
compressibility k	$L^1M^{-1}T^2$
number of atoms per cubic cm N	$L^{-3}M^0T^0$
mass of an atom m	$L^0M^1T^0$

Show that there is one dimensionless product. Conclude that, in any complete relationship among quantities with these dimensional formulas, k is a constant times $\nu^{-2}N^{-1/3}m^{-1}$. This conclusion played an important role in the early study of quantum phenomena.

3 The torque produced by an engine has dimensional formula $L^2M^1T^{-2}$. We may first guess that it depends on the engine's rotation rate (with dimensional formula $L^0M^0T^{-1}$), and the volume of air displaced (with dimensional formula $L^3M^0T^0$). [Giordano, Wells, Wilde]

(a) Try to find a complete set of dimensionless products. What goes wrong?

(b) Adjust the guess by adding the density of the air (with dimensional formula $L^{-3}M^1T^0$). Now find a complete set of dimensionless products.

4 Dominoes falling make a wave. We may conjecture that the wave speed v depends on the the spacing d between the dominoes, the height h of each domino, and the acceleration due to gravity g. [Tilley]

(a) Find the dimensional formula for each of the four quantities.

(b) Show that $\{\Pi_1 = h/d, \Pi_2 = dg/v^2\}$ is a complete set of dimensionless products.

(c) Show that if h/d is fixed then the propagation speed is proportional to the square root of d.

5 Prove that the dimensionless products form a vector space under the $\vec{+}$ operation of multiplying two such products and the $\vec{\cdot}$ operation of raising such the product to the power of the scalar. (The vector arrows are a precaution against confusion.) That is, prove that, for any particular homogeneous system, this set of products of powers of $m_1, \ldots, m_k$

$$\{m_1^{p_1} \ldots m_k^{p_k} \bigm| p_1, \ldots, p_k \text{ satisfy the system}\}$$

is a vector space under:

$$m_1^{p_1} \ldots m_k^{p_k} \vec{+} m_1^{q_1} \ldots m_k^{q_k} = m_1^{p_1+q_1} \ldots m_k^{p_k+q_k}$$

and

$$r \vec{\cdot} (m_1^{p_1} \ldots m_k^{p_k}) = m_1^{rp_1} \ldots m_k^{rp_k}$$

(assume that all variables represent real numbers).

6 The advice about apples and oranges is not right. Consider the familiar equations for a circle $C = 2\pi r$ and $A = \pi r^2$.

(a) Check that C and A have different dimensional formulas.

(b) Produce an equation that is not dimensionally homogeneous (i.e., it adds apples and oranges) but is nonetheless true of any circle.

(c) The prior item asks for an equation that is complete but not dimensionally homogeneous. Produce an equation that is dimensionally homogeneous but not complete.

(Just because the old saying isn't strictly right, doesn't keep it from being a useful strategy. Dimensional homogeneity is often used as a check on the plausibility of equations used in models. For an argument that any complete equation can easily be made dimensionally homogeneous, see [Bridgman], Chapter I, especially page 15.)

Chapter Three

Maps Between Spaces

I Isomorphisms

In the examples following the definition of a vector space we developed the intuition that some spaces are "the same" as others. For instance, the space of two-tall column vectors and the space of two-wide row vectors are not equal because their elements — column vectors and row vectors — are not equal, but we have the idea that these spaces differ only in how their elements appear. We will now make this idea precise.

This section illustrates a common aspect of a mathematical investigation. With the help of some examples, we've gotten an idea. We will next give a formal definition, and then we will produce some results backing our contention that the definition captures the idea. We've seen this happen already, for instance, in the first section of the Vector Space chapter. There, the study of linear systems led us to consider collections closed under linear combinations. We defined such a collection as a vector space, and we followed it with some supporting results.

Of course, that definition wasn't an end point, instead it led to new insights such as the idea of a basis. Here too, after producing a definition, and supporting it, we will get two surprises (pleasant ones). First, we will find that the definition applies to some unforeseen, and interesting, cases. Second, the study of the definition will lead to new ideas. In this way, our investigation will build a momentum.

I.1 Definition and Examples

We start with two examples that suggest the right definition.

1.1 Example Consider the example mentioned above, the space of two-wide row vectors and the space of two-tall column vectors. They are "the same" in that if we associate the vectors that have the same components, e.g.,

$$\begin{pmatrix}1 & 2\end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix}1\\2\end{pmatrix}$$

then this correspondence preserves the operations, for instance this addition

$$\begin{pmatrix}1 & 2\end{pmatrix} + \begin{pmatrix}3 & 4\end{pmatrix} = \begin{pmatrix}4 & 6\end{pmatrix} \quad\longleftrightarrow\quad \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$$

and this scalar multiplication.

$$5\cdot\begin{pmatrix}1 & 2\end{pmatrix} = \begin{pmatrix}5 & 10\end{pmatrix} \quad\longleftrightarrow\quad 5\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}5\\10\end{pmatrix}$$

More generally stated, under the correspondence

$$\begin{pmatrix}a_0 & a_1\end{pmatrix} \quad\longleftrightarrow\quad \begin{pmatrix}a_0\\a_1\end{pmatrix}$$

both operations are preserved:

$$\begin{pmatrix}a_0 & a_1\end{pmatrix} + \begin{pmatrix}b_0 & b_1\end{pmatrix} = \begin{pmatrix}a_0+b_0 & a_1+b_1\end{pmatrix} \longleftrightarrow \begin{pmatrix}a_0\\a_1\end{pmatrix} + \begin{pmatrix}b_0\\b_1\end{pmatrix} = \begin{pmatrix}a_0+b_0\\a_1+b_1\end{pmatrix}$$

and

$$r\cdot\begin{pmatrix}a_0 & a_1\end{pmatrix} = \begin{pmatrix}ra_0 & ra_1\end{pmatrix} \quad\longleftrightarrow\quad r\cdot\begin{pmatrix}a_0\\a_1\end{pmatrix} = \begin{pmatrix}ra_0\\ra_1\end{pmatrix}$$

(all of the variables are real numbers).

1.2 Example Another two spaces we can think of as "the same" are $\mathcal{P}_2$, the space of quadratic polynomials, and $\mathbb{R}^3$. A natural correspondence is this.

$$a_0 + a_1x + a_2x^2 \quad\longleftrightarrow\quad \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix} \qquad (\text{e.g., } 1 + 2x + 3x^2 \longleftrightarrow \begin{pmatrix}1\\2\\3\end{pmatrix})$$

The structure is preserved: corresponding elements add in a corresponding way

$$\begin{array}{r} a_0 + a_1x + a_2x^2 \\ +\; b_0 + b_1x + b_2x^2 \\ \hline (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \end{array} \quad\longleftrightarrow\quad \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix} + \begin{pmatrix}b_0\\b_1\\b_2\end{pmatrix} = \begin{pmatrix}a_0+b_0\\a_1+b_1\\a_2+b_2\end{pmatrix}$$

and scalar multiplication corresponds also.

$$r\cdot(a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2 \quad\longleftrightarrow\quad r\cdot\begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix} = \begin{pmatrix}ra_0\\ra_1\\ra_2\end{pmatrix}$$

1.3 Definition An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

(1) is a correspondence: f is one-to-one and onto;*

(2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = r\,f(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

("Morphism" means map, so "isomorphism" means a map expressing sameness.)

1.4 Example The vector space $G = \{c_1 \cos\theta + c_2 \sin\theta \bigm| c_1, c_2 \in \mathbb{R}\}$ of functions of θ is isomorphic to the vector space $\mathbb{R}^2$ under this map.

$$c_1 \cos\theta + c_2 \sin\theta \overset{f}{\longmapsto} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

We will check this by going through the conditions in the definition.

We will first verify condition (1), that the map is a correspondence between the sets underlying the spaces.

To establish that f is one-to-one, we must prove that $f(\vec{a}) = f(\vec{b})$ only when $\vec{a} = \vec{b}$. If

$$f(a_1 \cos\theta + a_2 \sin\theta) = f(b_1 \cos\theta + b_2 \sin\theta)$$

then, by the definition of f,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

from which we can conclude that $a_1 = b_1$ and $a_2 = b_2$ because column vectors are equal only when they have equal components. We've proved that $f(\vec{a}) = f(\vec{b})$ implies that $\vec{a} = \vec{b}$, which shows that f is one-to-one.

To check that f is onto we must check that any member of the codomain $\mathbb{R}^2$ is the image of some member of the domain G. But that's clear — any

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

is the image under f of $x\cos\theta + y\sin\theta \in G$.

Next we will verify condition (2), that f preserves structure.

*More information on one-to-one and onto maps is in the appendix.

This computation shows that f preserves addition.

$$\begin{aligned}
f\bigl(\,(a_1\cos\theta+a_2\sin\theta)+(b_1\cos\theta+b_2\sin\theta)\,\bigr)&\\
&=f\bigl(\,(a_1+b_1)\cos\theta+(a_2+b_2)\sin\theta\,\bigr)\\
&=\begin{pmatrix}a_1+b_1\\ a_2+b_2\end{pmatrix}\\
&=\begin{pmatrix}a_1\\ a_2\end{pmatrix}+\begin{pmatrix}b_1\\ b_2\end{pmatrix}\\
&=f(a_1\cos\theta+a_2\sin\theta)+f(b_1\cos\theta+b_2\sin\theta)
\end{aligned}$$

A similar computation shows that f preserves scalar multiplication.

$$\begin{aligned}
f\bigl(\,r\cdot(a_1\cos\theta+a_2\sin\theta)\,\bigr)&=f(\,ra_1\cos\theta+ra_2\sin\theta\,)\\
&=\begin{pmatrix}ra_1\\ ra_2\end{pmatrix}\\
&=r\cdot\begin{pmatrix}a_1\\ a_2\end{pmatrix}\\
&=r\cdot\,f(a_1\cos\theta+a_2\sin\theta)
\end{aligned}$$

With that, conditions (1) and (2) are verified, so we know that f is an isomorphism and we can say that the spaces are isomorphic $G\cong\mathbb{R}^2$.

1.5 Example Let V be the space $\{c_1x+c_2y+c_3z \mid c_1,c_2,c_3\in\mathbb{R}\}$ of linear combinations of three variables x, y, and z, under the natural addition and scalar multiplication operations. Then V is isomorphic to $\mathcal{P}_2$, the space of quadratic polynomials.

To show this we will produce an isomorphism map. There is more than one possibility; for instance, here are four.

$$c_1x+c_2y+c_3z\quad\begin{array}{ll}
\stackrel{f_1}{\longmapsto} & c_1+c_2x+c_3x^2\\
\stackrel{f_2}{\longmapsto} & c_2+c_3x+c_1x^2\\
\stackrel{f_3}{\longmapsto} & -c_1-c_2x-c_3x^2\\
\stackrel{f_4}{\longmapsto} & c_1+(c_1+c_2)x+(c_1+c_3)x^2
\end{array}$$

The first map is the more natural correspondence in that it just carries the coefficients over. However, below we shall verify that the second one is an isomorphism, to underline that there are isomorphisms other than just the obvious one (showing that f_1 is an isomorphism is Exercise 12).

To show that f_2 is one-to-one, we will prove that if $f_2(c_1x+c_2y+c_3z)=f_2(d_1x+d_2y+d_3z)$ then $c_1x+c_2y+c_3z=d_1x+d_2y+d_3z$. The assumption that $f_2(c_1x+c_2y+c_3z)=f_2(d_1x+d_2y+d_3z)$ gives, by the definition of f_2, that $c_2+c_3x+c_1x^2=d_2+d_3x+d_1x^2$. Equal polynomials have equal coefficients, so $c_2=d_2$, $c_3=d_3$, and $c_1=d_1$. Thus $f_2(c_1x+c_2y+c_3z)=f_2(d_1x+d_2y+d_3z)$ implies that $c_1x+c_2y+c_3z=d_1x+d_2y+d_3z$ and therefore f_2 is one-to-one.

The map f_2 is onto because any member $a+bx+cx^2$ of the codomain is the image of some member of the domain, namely it is the image of $cx+ay+bz$. For instance, $2+3x-4x^2$ is $f_2(-4x+2y+3z)$.

The computations for structure preservation are like those in the prior example. This map preserves addition

$$\begin{aligned}
&f_2\big((c_1x+c_2y+c_3z)+(d_1x+d_2y+d_3z)\big)\\
&\qquad\qquad\qquad\qquad = f_2\big((c_1+d_1)x+(c_2+d_2)y+(c_3+d_3)z\big)\\
&\qquad\qquad\qquad\qquad = (c_2+d_2)+(c_3+d_3)x+(c_1+d_1)x^2\\
&\qquad\qquad\qquad\qquad = (c_2+c_3x+c_1x^2)+(d_2+d_3x+d_1x^2)\\
&\qquad\qquad\qquad\qquad = f_2(c_1x+c_2y+c_3z)+f_2(d_1x+d_2y+d_3z)
\end{aligned}$$

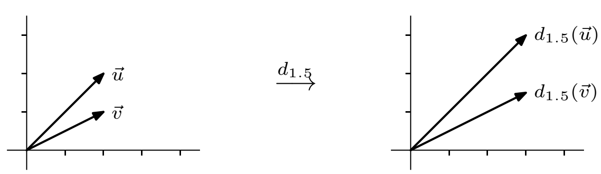
and scalar multiplication.

$$\begin{aligned}
f_2\big(r\cdot(c_1x+c_2y+c_3z)\big) &= f_2(rc_1x+rc_2y+rc_3z)\\
&= rc_2+rc_3x+rc_1x^2\\
&= r\cdot(c_2+c_3x+c_1x^2)\\
&= r\cdot\ f_2(c_1x+c_2y+c_3z)
\end{aligned}$$

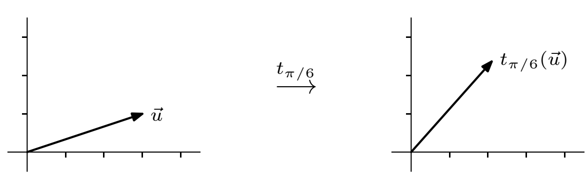
Thus f_2 is an isomorphism and we write $V\cong\mathcal{P}_2$.

We are sometimes interested in an isomorphism of a space with itself, called an *automorphism*. An identity map is an automorphism. The next two examples show that there are others.

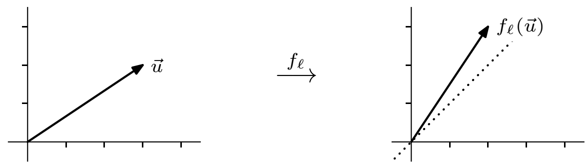
1.6 Example A *dilation* map $d_s\colon\mathbb{R}^2\to\mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



A *rotation* or *turning map* $t_\theta\colon\mathbb{R}^2\to\mathbb{R}^2$ that rotates all vectors through an angle θ is an automorphism.



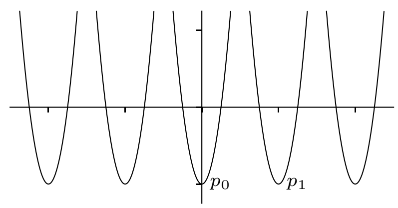
A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon\mathbb{R}^2\to\mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



See Exercise 29.

1.7 Example Consider the space $\mathcal{P}_5$ of polynomials of degree 5 or less and the map f that sends a polynomial $p(x)$ to $p(x-1)$. For instance, under this map $x^2 \mapsto (x-1)^2 = x^2-2x+1$ and $x^3+2x \mapsto (x-1)^3+2(x-1) = x^3-3x^2+5x-3$. This map is an automorphism of this space; the check is Exercise 21.

This isomorphism of $\mathcal{P}_5$ with itself does more than just tell us that the space is "the same" as itself. It gives us some insight into the space's structure. For instance, below is shown a family of parabolas, graphs of members of $\mathcal{P}_5$. Each has a vertex at $y=-1$, and the left-most one has zeroes at -2.25 and -1.75, the next one has zeroes at -1.25 and -0.75, etc.



Geometrically, the substitution of $x-1$ for x in any function's argument shifts its graph to the right by one. Thus, $f(p_0) = p_1$ and f's action is to shift all of the parabolas to the right by one. Notice that the picture before f is applied is the same as the picture after f is applied, because while each parabola moves to the right, another one comes in from the left to take its place. This also holds true for cubics, etc. So the automorphism f gives us the insight that P_5 has a certain horizontal-homogeneity; this space looks the same near $x=1$ as near $x=0$.

As described in the preamble to this section, we will next produce some results supporting the contention that the definition of isomorphism above captures our intuition of vector spaces being the same.

Of course the definition itself is persuasive: a vector space consists of two components, a set and some structure, and the definition simply requires that the sets correspond and that the structures correspond also. Also persuasive are the examples above. In particular, Example 1.1, which gives an isomorphism between the space of two-wide row vectors and the space of two-tall column vectors, dramatizes our intuition that isomorphic spaces are the same in all relevant respects. Sometimes people say, where $V \cong W$, that "W is just V painted green" — any differences are merely cosmetic.

Further support for the definition, in case it is needed, is provided by the following results that, taken together, suggest that all the things of interest in a vector space correspond under an isomorphism. Since we studied vector spaces to study linear combinations, "of interest" means "pertaining to linear combinations". Not of interest is the way that the vectors are presented typographically (or their color!).

As an example, although the definition of isomorphism doesn't explicitly say that the zero vectors must correspond, it is a consequence of that definition.

1.8 Lemma An isomorphism maps a zero vector to a zero vector.

PROOF. Where $f\colon V \to W$ is an isomorphism, fix any $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

The definition of isomorphism requires that sums of two vectors correspond and that so do scalar multiples. We can extend that to say that all linear combinations correspond.

1.9 Lemma For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

PROOF. Since the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear, we need only show that (1) $\Longrightarrow$ (3). Assume statement (1). We will prove statement (3) by induction on the number of summands n.

The one-summand base case, that $f(c\vec{v}_1) = c\, f(\vec{v}_1)$, is covered by the assumption of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands, that is, whenever $n = 1$, or $n = 2$, ..., or $n = k$. Consider the $k+1$-summand case. The first half of (1) gives

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

by breaking the sum along the final '+'. Then the inductive hypothesis lets us break up the k-term sum.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Finally, the second half of statement (1) gives

$$= c_1\, f(\vec{v}_1) + \cdots + c_k\, f(\vec{v}_k) + c_{k+1}\, f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED

In addition to adding to the intuition that the definition of isomorphism does indeed preserve the things of interest in a vector space, that lemma's second item is an especially handy way of checking that a map preserves structure.

We close with a summary. The material in this section augments the chapter on Vector Spaces. There, after giving the definition of a vector space, we informally looked at what different things can happen. Here, we defined the relation '$\cong$' between vector spaces and we have argued that it is the right way to split the collection of vector spaces into cases because it preserves the features of interest in a vector space — in particular, it preserves linear combinations. That is, we have now said precisely what we mean by 'the same', and by 'different', and so we have precisely classified the vector spaces.

Exercises

✓ **1.10** Verify, using Example 1.4 as a model, that the two correspondences given before the definition are isomorphisms.
(a) Example 1.1 **(b)** Example 1.2

✓ **1.11** For the map $f\colon \mathcal{P}_1 \to \mathbb{R}^2$ given by

$$a + bx \stackrel{f}{\longmapsto} \begin{pmatrix} a-b \\ b \end{pmatrix}$$

Find the image of each of these elements of the domain.
(a) $3 - 2x$ **(b)** $2 + 2x$ **(c)** x
Show that this map is an isomorphism.

1.12 Show that the natural map f_1 from Example 1.5 is an isomorphism.

✓ **1.13** Decide whether each map is an isomorphism (if it is an isomorphism then prove it and if it isn't then state a condition that it fails to satisfy).
(a) $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$$

(b) $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{pmatrix}$$

(c) $f\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a)x^2 + ax^3$$

(d) $f\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c + (d+c)x + (b+a+1)x^2 + ax^3$$

1.14 Show that the map $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ given by $f(x) = x^3$ is one-to-one and onto. Is it an isomorphism?

✓ **1.15** Refer to Example 1.1. Produce two more isomorphisms (of course, you must also verify that they satisfy the conditions in the definition of isomorphism).

1.16 Refer to Example 1.2. Produce two more isomorphisms (and verify that they satisfy the conditions).

✓ **1.17** Show that, although $\mathbb{R}^2$ is not itself a subspace of $\mathbb{R}^3$, it is isomorphic to the xy-plane subspace of $\mathbb{R}^3$.

1.18 Find two isomorphisms between $\mathbb{R}^{16}$ and $\mathcal{M}_{4\times4}$.

✓ **1.19** For what k is $\mathcal{M}_{m\times n}$ isomorphic to $\mathbb{R}^k$?

1.20 For what k is $\mathcal{P}_k$ isomorphic to $\mathbb{R}^n$?

1.21 Prove that the map in Example 1.7, from $\mathcal{P}_5$ to $\mathcal{P}_5$ given by $p(x) \mapsto p(x-1)$, is a vector space isomorphism.

1.22 Why, in Lemma 1.8, must there be a $\vec{v} \in V$? That is, why must V be nonempty?

1.23 Are any two trivial spaces isomorphic?

1.24 In the proof of Lemma 1.9, what about the zero-summands case (that is, if n is zero)?

1.25 Show that any isomorphism $f\colon \mathcal{P}_0 \to \mathbb{R}^1$ has the form $a \mapsto ka$ for some nonzero real number k.

✓ **1.26** These prove that isomorphism is an equivalence relation.
(a) Show that the identity map $\text{id}\colon V \to V$ is an isomorphism. Thus, any vector space is isomorphic to itself.
(b) Show that if $f\colon V \to W$ is an isomorphism then so is its inverse $f^{-1}\colon W \to V$. Thus, if V is isomorphic to W then also W is isomorphic to V.
(c) Show that a composition of isomorphisms is an isomorphism: if $f\colon V \to W$ is an isomorphism and $g\colon W \to U$ is an isomorphism then so also is $g \circ f\colon V \to U$. Thus, if V is isomorphic to W and W is isomorphic to U, then also V is isomorphic to U.

1.27 Suppose that $f\colon V \to W$ preserves structure. Show that f is one-to-one if and only if the unique member of V mapped by f to $\vec{0}_W$ is $\vec{0}_V$.

1.28 Suppose that $f\colon V \to W$ is an isomorphism. Prove that the set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subseteq V$ is linearly dependent if and only if the set of images $\{f(\vec{v}_1), \ldots, f(\vec{v}_k)\} \subseteq W$ is linearly dependent.

✓ **1.29** Show that each type of map from Example 1.6 is an automorphism.
(a) Dilation d_s by a nonzero scalar s.
(b) Rotation t_θ through an angle θ.
(c) Reflection f_ℓ over a line through the origin.
Hint. For the second and third items, polar coordinates are useful.

1.30 Produce an automorphism of $\mathcal{P}_2$ other than the identity map, and other than a shift map $p(x) \mapsto p(x-k)$.

1.31 (a) Show that a function $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ is an automorphism if and only if it has the form $x \mapsto kx$ for some $k \neq 0$.
(b) Let f be an automorphism of $\mathbb{R}^1$ such that $f(3) = 7$. Find $f(-2)$.
(c) Show that a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an automorphism if and only if it has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$. *Hint.* Exercises in prior subsections have shown that

$$\begin{pmatrix} b \\ d \end{pmatrix} \text{ is not a multiple of } \begin{pmatrix} a \\ c \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

(d) Let f be an automorphism of $\mathbb{R}^2$ with

$$f(\begin{pmatrix}1\\3\end{pmatrix})=\begin{pmatrix}2\\-1\end{pmatrix}\quad\text{and}\quad f(\begin{pmatrix}1\\4\end{pmatrix})=\begin{pmatrix}0\\1\end{pmatrix}.$$

Find

$$f(\begin{pmatrix}0\\-1\end{pmatrix}).$$

1.32 Refer to Lemma 1.8 and Lemma 1.9. Find two more things preserved by isomorphism.

1.33 We show that isomorphisms can be tailored to fit in that, sometimes, given vectors in the domain and in the range we can produce an isomorphism associating those vectors.

(a) Let $B=\langle\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_3\rangle$ be a basis for $\mathcal{P}_2$ so that any $\vec{p}\in\mathcal{P}_2$ has a unique representation as $\vec{p}=c_1\vec{\beta}_1+c_2\vec{\beta}_2+c_3\vec{\beta}_3$, which we denote in this way.

$$\operatorname{Rep}_B(\vec{p})=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$$

Show that the $\operatorname{Rep}_B(\cdot)$ operation is a function from $\mathcal{P}_2$ to $\mathbb{R}^3$ (this entails showing that with every domain vector $\vec{v}\in\mathcal{P}_2$ there is an associated image vector in $\mathbb{R}^3$, and further, that with every domain vector $\vec{v}\in\mathcal{P}_2$ there is at most one associated image vector).

(b) Show that this $\operatorname{Rep}_B(\cdot)$ function is one-to-one and onto.

(c) Show that it preserves structure.

(d) Produce an isomorphism from $\mathcal{P}_2$ to $\mathbb{R}^3$ that fits these specifications.

$$x+x^2\mapsto\begin{pmatrix}1\\0\\0\end{pmatrix}\quad\text{and}\quad 1-x\mapsto\begin{pmatrix}0\\1\\0\end{pmatrix}$$

1.34 Prove that a space is n-dimensional if and only if it is isomorphic to $\mathbb{R}^n$. *Hint.* Fix a basis B for the space and consider the map sending a vector over to its representation with respect to B.

1.35 *(Requires the subsection on Combining Subspaces, which is optional.)* Let U and W be vector spaces. Define a new vector space, consisting of the set $U\times W=\{(\vec{u},\vec{w})\bigm|\vec{u}\in U\text{ and }\vec{w}\in W\}$ along with these operations.

$$(\vec{u}_1,\vec{w}_1)+(\vec{u}_2,\vec{w}_2)=(\vec{u}_1+\vec{u}_2,\vec{w}_1+\vec{w}_2)\quad\text{and}\quad r\cdot(\vec{u},\vec{w})=(r\vec{u},r\vec{w})$$

This is a vector space, the *external direct sum* of U and W.

(a) Check that it is a vector space.

(b) Find a basis for, and the dimension of, the external direct sum $\mathcal{P}_2\times\mathbb{R}^2$.

(c) What is the relationship among $\dim(U)$, $\dim(W)$, and $\dim(U\times W)$?

(d) Suppose that U and W are subspaces of a vector space V such that $V=U\oplus W$ (in this case we say that V is the *internal direct sum* of U and W). Show that the map $f\colon U\times W\to V$ given by

$$(\vec{u},\vec{w})\overset{f}{\longmapsto}\vec{u}+\vec{w}$$

is an isomorphism. Thus if the internal direct sum is defined then the internal and external direct sums are isomorphic.

I.2 Dimension Characterizes Isomorphism

In the prior subsection, after stating the definition of an isomorphism, we gave some results supporting the intuition that such a map describes spaces as "the same". Here we will formalize this intuition. While two spaces that are isomorphic are not equal, we think of them as almost equal — as equivalent. In this subsection we shall show that the relationship 'is isomorphic to' is an equivalence relation.*

2.1 Theorem Isomorphism is an equivalence relation between vector spaces.

Proof. We must prove that this relation has the three properties of being symmetric, reflexive, and transitive. For each of the three we will use item (2) of Lemma 1.9 and show that the map preserves structure by showing that it preserves linear combinations of two members of the domain.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. The calculation showing that it preserves linear combinations is easy.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

To check symmetry, that if V is isomorphic to W via some map $f\colon V \to W$ then there is an isomorphism going the other way, consider the inverse map $f^{-1}\colon W \to V$. As stated in the appendix, such an inverse function exists and it is also a correspondence. Thus we have reduced the symmetry issue to checking that, because f preserves linear combinations, so also does f^{-1}. Assume that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$, i.e., that $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$.

$$\begin{aligned}
f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\
&= f^{-1}\big(\, f\big(c_1\vec{v}_1 + c_2\vec{v}_2\big)\,\big) \\
&= c_1\vec{v}_1 + c_2\vec{v}_2 \\
&= c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2)
\end{aligned}$$

Finally, we must check transitivity, that if V is isomorphic to W via some map f and if W is isomorphic to U via some map g then also V is isomorphic to U. Consider the composition $g \circ f\colon V \to U$. The appendix notes that the composition of two correspondences is a correspondence, so we need only check that the composition preserves linear combinations.

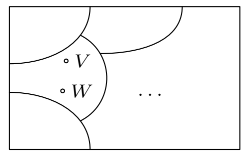
$$\begin{aligned}
g \circ f\,\big(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\big) &= g\big(\, f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2)\,\big) \\
&= g\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\
&= c_1 \cdot g\big(f(\vec{v}_1)\big) + c_2 \cdot g\big(f(\vec{v}_2)\big) \\
&= c_1 \cdot (g \circ f)\,(\vec{v}_1) + c_2 \cdot (g \circ f)\,(\vec{v}_2)
\end{aligned}$$

Thus $g \circ f\colon V \to U$ is an isomorphism. QED

* More information on equivalence relations and equivalence classes is in the appendix.

As a consequence of that result, we know that the universe of vector spaces is partitioned into classes: every space is in one and only one isomorphism class.

All finite dimensional vector spaces:



$V \cong W$

2.2 Theorem Vector spaces are isomorphic if and only if they have the same dimension.

This follows from the next two lemmas.

2.3 Lemma If spaces are isomorphic then they have the same dimension.

Proof. We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, where $f\colon V \to W$ is an isomorphism and a basis for the domain V is $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, then the image set $D = \langle f(\vec{\beta}_1), \ldots, f(\vec{\beta}_n) \rangle$ is a basis for the codomain W. (The other half of the correspondence — that for any basis of W the inverse image is a basis for V — follows on recalling that if f is an isomorphism then f^{-1} is also an isomorphism, and applying the prior sentence to f^{-1}.)

To see that D spans W, fix any $\vec{w} \in W$, note that f is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$, and expand $\vec{v}$ as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of D, if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$$

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have that $\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, implying that all of the c's are zero. QED

2.4 Lemma If spaces have the same dimension then they are isomorphic.

Proof. To show that any two spaces of dimension n are isomorphic, we can simply show that any one is isomorphic to $\mathbb{R}^n$. Then we will have shown that they are isomorphic to each other, by the transitivity of isomorphism (which was established in Theorem 2.1).

Let V be n-dimensional. Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain V. Consider the representation of the members of that domain with respect to the basis as a function from V to $\mathbb{R}^n$

$$\vec{v} = v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n \stackrel{\text{Rep}_B}{\longmapsto} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

(it is well-defined* since every $\vec{v}$ has one and only one such representation — see Remark 2.5 below).

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1+\cdots+u_n\vec{\beta}_n)=\mathrm{Rep}_B(v_1\vec{\beta}_1+\cdots+v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$$

and so $u_1=v_1,\ \ldots,\ u_n=v_n$, and therefore the original arguments $u_1\vec{\beta}_1+\cdots+u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1+\cdots+v_n\vec{\beta}_n$ are equal.

This function is onto; any n-tall vector

$$\vec{w}=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$$

is the image of some $\vec{v}\in V$, namely $\vec{w}=\mathrm{Rep}_B(w_1\vec{\beta}_1+\cdots+w_n\vec{\beta}_n)$.

Finally, this function preserves structure.

$$\begin{aligned}\mathrm{Rep}_B(r\cdot\vec{u}+s\cdot\vec{v})&=\mathrm{Rep}_B(\,(ru_1+sv_1)\vec{\beta}_1+\cdots+(ru_n+sv_n)\vec{\beta}_n\,)\\&=\begin{pmatrix}ru_1+sv_1\\ \vdots\\ ru_n+sv_n\end{pmatrix}\\&=r\cdot\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}+s\cdot\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\\&=r\cdot\mathrm{Rep}_B(\vec{u})+s\cdot\mathrm{Rep}_B(\vec{v})\end{aligned}$$

Thus the Rep_B function is an isomorphism and thus any n-dimensional space is isomorphic to the n-dimensional space $\mathbb{R}^n$. Consequently, any two spaces with the same dimension are isomorphic. QED

2.5 Remark The parenthetical comment in that proof about the role played by the 'one and only one representation' result requires some explanation. We need to show that (for a fixed B) each vector in the domain is associated by Rep_B with one and only one vector in the codomain.

A contrasting example, where an association doesn't have this property, is illuminating. Consider this subset of $\mathcal{P}_2$, which is not a basis.

$$A=\{1+0x+0x^2,0+1x+0x^2,0+0x+1x^2,1+1x+2x^2\}$$

* More information on well-definedness is in the appendix.

Call those four polynomials $\vec{\alpha}_1, \ldots, \vec{\alpha}_4$. If, mimicing above proof, we try to write the members of $\mathcal{P}_2$ as $\vec{p} = c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 + c_3\vec{\alpha}_3 + c_4\vec{\alpha}_4$, and associate $\vec{p}$ with the four-tall vector with components $c_1, \ldots, c_4$ then there is a problem. For, consider $\vec{p}(x) = 1 + x + x^2$. The set A spans the space $\mathcal{P}_2$, so there is at least one four-tall vector associated with $\vec{p}$. But A is not linearly independent and so vectors do not have unique decompositions. In this case, both

$$\vec{p}(x) = 1\vec{\alpha}_1 + 1\vec{\alpha}_2 + 1\vec{\alpha}_3 + 0\vec{\alpha}_4 \quad\text{and}\quad \vec{p}(x) = 0\vec{\alpha}_1 + 0\vec{\alpha}_2 - 1\vec{\alpha}_3 + 1\vec{\alpha}_4$$

and so there is more than one four-tall vector associated with $\vec{p}$.

$$\begin{pmatrix}1\\1\\1\\0\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$$

That is, with input $\vec{p}$ this association does not have a well-defined (i.e., single) output value.

Any map whose definition appears possibly ambiguous must be checked to see that it is well-defined. For Rep_B in the above proof that check is Exercise 18.

That ends the proof of Theorem 2.2. We say that the isomorphism classes are *characterized* by dimension because we can describe each class simply by giving the number that is the dimension of all of the spaces in that class.

This subsection's results give us a collection of representatives of the isomorphism classes.

2.6 Corollary A finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

The proofs above pack many ideas into a small space. Through the rest of this chapter we'll consider these ideas again, and fill them out. For a taste of this, we will expand here on the proof of Lemma 2.4.

2.7 Example The space $\mathcal{M}_{2\times2}$ of 2×2 matrices is isomorphic to $\mathbb{R}^4$. With this basis for the domain

$$B = \langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \rangle$$

the isomorphism given in the lemma, the representation map $f_1 = \mathrm{Rep}_B$, simply carries the entries over.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} \xmapsto{f_1} \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

One way to think of the map f_1 is: fix the basis B for the domain and the basis $\mathcal{E}_4$ for the codomain, and associate $\vec{\beta}_1$ with $\vec{e}_1$, and $\vec{\beta}_2$ with $\vec{e}_2$, etc. Then extend this association to all of the members of two spaces.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\vec{\beta}_1 + b\vec{\beta}_2 + c\vec{\beta}_3 + d\vec{\beta}_4 \quad \stackrel{f_1}{\longmapsto} \quad a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + d\vec{e}_4 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

We say that the map has been *extended linearly* from the bases to the spaces.

We can do the same thing with different bases, for instance, taking this basis for the domain.

$$A = \langle \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \rangle$$

Associating corresponding members of A and $\mathcal{E}_4$ and extending linearly

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (a/2)\vec{\alpha}_1 + (b/2)\vec{\alpha}_2 + (c/2)\vec{\alpha}_3 + (d/2)\vec{\alpha}_4 \\ &\stackrel{f_2}{\longmapsto} \quad (a/2)\vec{e}_1 + (b/2)\vec{e}_2 + (c/2)\vec{e}_3 + (d/2)\vec{e}_4 = \begin{pmatrix} a/2 \\ b/2 \\ c/2 \\ d/2 \end{pmatrix} \end{aligned}$$

gives rise to an isomorphism that is different than f_1.

The prior map arose by changing the basis for the domain. We can also change the basis for the codomain. Starting with

$$B \quad \text{and} \quad D = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rangle$$

associating $\vec{\beta}_1$ with $\vec{\delta}_1$, etc., and then linearly extending that correspondence to all of the two spaces

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\vec{\beta}_1 + b\vec{\beta}_2 + c\vec{\beta}_3 + d\vec{\beta}_4 \quad \stackrel{f_3}{\longmapsto} \quad a\vec{\delta}_1 + b\vec{\delta}_2 + c\vec{\delta}_3 + d\vec{\delta}_4 = \begin{pmatrix} a \\ b \\ d \\ c \end{pmatrix}$$

gives still another isomorphism.

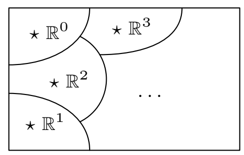
So there is a connection between the maps between spaces and bases for those spaces. Later sections will explore that connection.

We will close this section with a summary.

Recall that in the first chapter we defined two matrices as row equivalent if they can be derived from each other by elementary row operations (this was the meaning of same-ness that was of interest there). We showed that is an equivalence relation and so the collection of matrices is partitioned into classes, where all the matrices that are row equivalent fall together into a single class. Then, for insight into which matrices are in each class, we gave representatives for the classes, the reduced echelon form matrices.

In this section, except that the appropriate notion of same-ness here is vector space isomorphism, we have followed much the same outline. First we defined isomorphism, saw some examples, and established some properties. Then we showed that it is an equivalence relation, and now we have a set of class representatives, the real vector spaces $\mathbb{R}^1$, $\mathbb{R}^2$, etc.

All finite dimensional vector spaces:



One representative per class

As before, the list of representatives helps us to understand the partition. It is simply a classification of spaces by dimension.

In the second chapter, with the definition of vector spaces, we seemed to have opened up our studies to many examples of new structures besides the familiar $\mathbb{R}^n$'s. We now know that isn't the case. Any finite-dimensional vector space is actually "the same" as a real space. We are thus considering exactly the structures that we need to consider.

The rest of the chapter fills out the work in this section. In particular, in the next section we will consider maps that preserve structure, but are not necessarily correspondences.

Exercises

✓ **2.8** Decide if the spaces are isomorphic.
(a) $\mathbb{R}^2$, $\mathbb{R}^4$ **(b)** $\mathcal{P}_5$, $\mathbb{R}^5$ **(c)** $\mathcal{M}_{2\times3}$, $\mathbb{R}^6$ **(d)** $\mathcal{P}_5$, $\mathcal{M}_{2\times3}$ **(e)** $\mathcal{M}_{2\times k}$, $\mathbb{C}^k$

✓ **2.9** Consider the isomorphism $\text{Rep}_B(\cdot)\colon \mathcal{P}_1 \to \mathbb{R}^2$ where $B = \langle 1, 1+x\rangle$. Find the image of each of these elements of the domain.
(a) $3-2x$; **(b)** $2+2x$; **(c)** x

✓ **2.10** Show that if $m \neq n$ then $\mathbb{R}^m \ncong \mathbb{R}^n$.

✓ **2.11** Is $\mathcal{M}_{m\times n} \cong \mathcal{M}_{n\times m}$?

✓ **2.12** Are any two planes through the origin in $\mathbb{R}^3$ isomorphic?

2.13 Find a set of equivalence class representatives other than the set of $\mathbb{R}^n$'s.

2.14 True or false: between any n-dimensional space and $\mathbb{R}^n$ there is exactly one isomorphism.

2.15 Can a vector space be isomorphic to one of its (proper) subspaces?

✓ **2.16** This subsection shows that for any isomorphism, the inverse map is also an isomorphism. This subsection also shows that for a fixed basis B of an n-dimensional vector space V, the map $\text{Rep}_B\colon V \to \mathbb{R}^n$ is an isomorphism. Find the inverse of this map.

✓ **2.17** Prove these facts about matrices.
(a) The row space of a matrix is isomorphic to the column space of its transpose.
(b) The row space of a matrix is isomorphic to its column space.

2.18 Show that the function from Theorem 2.2 is well-defined.

2.19 Is the proof of Theorem 2.2 valid when $n = 0$?

2.20 For each, decide if it is a set of isomorphism class representatives.
(a) $\{\mathbb{C}^k \bigm| k \in \mathbb{N}\}$ **(b)** $\{\mathcal{P}_k \bigm| k \in \{-1, 0, 1, \ldots\}\}$ **(c)** $\{\mathcal{M}_{m\times n} \bigm| m, n \in \mathbb{N}\}$

2.21 Let f be a correspondence between vector spaces V and W (that is, a map that is one-to-one and onto). Show that the spaces V and W are isomorphic via f if and only if there are bases $B \subset V$ and $D \subset W$ such that corresponding vectors have the same coordinates: $\mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(f(\vec{v}))$.

2.22 Consider the isomorphism $\mathrm{Rep}_B \colon \mathcal{P}_3 \to \mathbb{R}^4$.
(a) Vectors in a real space are orthogonal if and only if their dot product is zero. Give a definition of orthogonality for polynomials.
(b) The derivative of a member of $\mathcal{P}_3$ is in $\mathcal{P}_3$. Give a definition of the derivative of a vector in $\mathbb{R}^4$.

✓ **2.23** Does every correspondence between bases, when extended to the spaces, give an isomorphism?

2.24 *(Requires the subsection on Combining Subspaces, which is optional.)* Suppose that $V = V_1 \oplus V_2$ and that V is isomorphic to the space U under the map f. Show that $U = f(V_1) \oplus f(U_2)$.

2.25 Show that this is not a well-defined function from the rational numbers to the integers: with each fraction, associate the value of its numerator.

II Homomorphisms

The definition of isomorphism has two conditions. In this section we will consider the second one, that the map must preserve the algebraic structure of the space. We will focus on this condition by studying maps that are required only to preserve structure; that is, maps that are not required to be correspondences.

Experience shows that this kind of map is tremendously useful in the study of vector spaces. For one thing, as we shall see in the second subsection below, while isomorphisms describe how spaces are the same, these maps describe how spaces can be thought of as alike.

II.1 Definition

1.1 Definition A function between vector spaces $h\colon V \to W$ that preserves the operations of addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

1.2 Example The projection map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a homomorphism. It preserves addition

$$\pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = \pi(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + \pi(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})$$

and scalar multiplication.

$$\pi(r \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) = \pi(\begin{pmatrix} rx_1 \\ ry_1 \\ rz_1 \end{pmatrix}) = \begin{pmatrix} rx_1 \\ ry_1 \end{pmatrix} = r \cdot \pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix})$$

This map is not an isomorphism since it is not one-to-one. For instance, both $\vec{0}$ and $\vec{e}_3$ in $\mathbb{R}^3$ are mapped to the zero vector in $\mathbb{R}^2$.

1.3 Example Of course, the domain and codomain might be other than spaces of column vectors. Both of these are homomorphisms; the verifications are straightforward.

(1) $f_1 \colon \mathcal{P}_2 \to \mathcal{P}_3$ given by

$$a_0 + a_1 x + a_2 x^2 \ \mapsto\ a_0 x + (a_1/2)x^2 + (a_2/3)x^3$$

(2) $f_2 \colon M_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

1.4 Example Between any two spaces there is a *zero homomorphism*, mapping every vector in the domain to the zero vector in the codomain.

1.5 Example These two suggest why we use the term 'linear map'.

(1) The map $g \colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{g}{\longmapsto} 3x + 2y - 4.5z$$

is linear (i.e., is a homomorphism). In contrast, the map $\hat{g} \colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\hat{g}}{\longmapsto} 3x + 2y - 4.5z + 1$$

is not; for instance,

$$\hat{g}(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 4 \quad \text{while} \quad \hat{g}(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) + \hat{g}(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 5$$

(to show that a map is not linear we need only produce one example of a linear combination that is not preserved).

(2) The first of these two maps $t_1, t_2 \colon \mathbb{R}^3 \to \mathbb{R}^2$ is linear while the second is not.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t_1}{\longmapsto} \begin{pmatrix} 5x - 2y \\ x + y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t_2}{\longmapsto} \begin{pmatrix} 5x - 2y \\ xy \end{pmatrix}$$

Finding an example that the second fails to preserve structure is easy.

What distinguishes the homomorphisms is that the coordinate functions are linear combinations of the arguments. See also Exercise 23.

Obviously, any isomorphism is a homomorphism — an isomorphism is a homomorphism that is also a correspondence. So, one way to think of the 'homomorphism' idea is that it is a generalization of 'isomorphism', motivated by the observation that many of the properties of isomorphisms have only to do with the map's structure preservation property and not to do with it being a correspondence. As examples, these two results from the prior section do not use one-to-one-ness or onto-ness in their proof, and therefore apply to any homomorphism.

1.6 Lemma A homomorphism sends a zero vector to a zero vector.

1.7 Lemma Each of these is a necessary and sufficient condition for $f\colon V \to W$ to be a homomorphism.

(1) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(2) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

Part (1) is often used to check that a function is linear.

1.8 Example The map $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x/2 \\ 0 \\ x+y \\ 3y \end{pmatrix}$$

satisfies (1) of the prior result

$$\begin{pmatrix} r_1(x_1/2) + r_2(x_2/2) \\ 0 \\ r_1(x_1+y_1) + r_2(x_2+y_2) \\ r_1(3y_1) + r_2(3y_2) \end{pmatrix} = r_1 \begin{pmatrix} x_1/2 \\ 0 \\ x_1+y_1 \\ 3y_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2/2 \\ 0 \\ x_2+y_2 \\ 3y_2 \end{pmatrix}$$

and so it is a homomorphism.

However, some of the results that we have seen for isomorphisms fail to hold for homomorphisms in general. Consider the theorem that an isomorphism between spaces gives a correspondence between their bases. Homomorphisms do not give any such correspondence; Example 1.2 shows that there is no such correspondence, and another example is the zero map between any two nontrivial spaces. Instead, for homomorphisms a weaker but still very useful result holds.

1.9 Theorem A homomorphism is determined by its action on a basis. That is, if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis of a vector space V and $\vec{w}_1, \ldots, \vec{w}_n$ are (perhaps not distinct) elements of a vector space W then there exists a homomorphism from V to W sending $\vec{\beta}_1$ to $\vec{w}_1$, …, and $\vec{\beta}_n$ to $\vec{w}_n$, and that homomorphism is unique.

Proof. We will define the map by associating $\vec{\beta}_1$ with $\vec{w}_1$, etc., and then extending linearly to all of the domain. That is, where $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the map $h\colon V \to W$ is given by $h(\vec{v}) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$. This is well-defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

This map is a homomorphism since it preserves linear combinations; where $\vec{v_1} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ and $\vec{v_2} = d_1\vec{\beta}_1 + \cdots + d_n\vec{\beta}_n$, we have this.

$$\begin{aligned}
h(r_1\vec{v}_1 + r_2\vec{v}_2) &= h((r_1c_1 + r_2d_1)\vec{\beta}_1 + \cdots + (r_1c_n + r_2d_n)\vec{\beta}_n) \\
&= (r_1c_1 + r_2d_1)\vec{w}_1 + \cdots + (r_1c_n + r_2d_n)\vec{w}_n \\
&= r_1h(\vec{v}_1) + r_2h(\vec{v}_2)
\end{aligned}$$

And, this map is unique since if $\hat{h}\colon V \to W$ is another homomorphism such that $\hat{h}(\vec{\beta}_i) = \vec{w}_i$ for each i then h and $\hat{h}$ agree on all of the vectors in the domain.

$$\begin{aligned}
\hat{h}(\vec{v}) &= \hat{h}(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) \\
&= c_1\hat{h}(\vec{\beta}_1) + \cdots + c_n\hat{h}(\vec{\beta}_n) \\
&= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n \\
&= h(\vec{v})
\end{aligned}$$

Thus, h and $\hat{h}$ are the same map. QED

1.10 Example This result says that we can construct a homomorphism by fixing a basis for the domain and specifying where the map sends those basis vectors. For instance, if we specify a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that acts on the standard basis $\mathcal{E}_2$ in this way

$$h(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}-1\\1\end{pmatrix} \quad\text{and}\quad h(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}-4\\4\end{pmatrix}$$

then the action of h on any other member of the domain is also specified. For instance, the value of h on this argument

$$h(\begin{pmatrix}3\\-2\end{pmatrix}) = h(3\cdot\begin{pmatrix}1\\0\end{pmatrix} - 2\cdot\begin{pmatrix}0\\1\end{pmatrix}) = 3\cdot h(\begin{pmatrix}1\\0\end{pmatrix}) - 2\cdot h(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}5\\-5\end{pmatrix}$$

is a direct consequence of the value of h on the basis vectors.

Later in this chapter we shall develop a scheme, using matrices, that is convienent for computations like this one.

Just as the isomorphisms of a space with itself are useful and interesting, so too are the homomorphisms of a space with itself.

1.11 Definition A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

1.12 Remark In this book we use 'linear transformation' only in the case where the codomain equals the domain, but it is widely used in other texts as a general synonym for 'homomorphism'.

1.13 Example The map on $\mathbb{R}^2$ that projects all vectors down to the x-axis

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

is a linear transformation.

1.14 Example The derivative map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$

$$a_0 + a_1x + \cdots + a_nx^n \overset{d/dx}{\longmapsto} a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

is a linear transformation, as this result from calculus notes: $d(c_1f + c_2g)/dx = c_1\,(df/dx) + c_2\,(dg/dx)$.

1.15 Example The matrix transpose map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is a linear transformation of $\mathcal{M}_{2\times 2}$. Note that this transformation is one-to-one and onto, and so in fact it is an automorphism.

We finish this subsection about maps by recalling that we can linearly combine maps. For instance, for these maps from $\mathbb{R}^2$ to itself

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} 2x \\ 3x - 2y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \overset{g}{\longmapsto} \begin{pmatrix} 0 \\ 5x \end{pmatrix}$$

the linear combination $5f - 2g$ is also a map from R^2 to itself.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{5f-2g}{\longmapsto} \begin{pmatrix} 10x \\ 5x - 10y \end{pmatrix}$$

1.16 Lemma For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W. It is denoted $\mathcal{L}(V, W)$.

Proof. This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under linear combinations. Let $f, g\colon V \to W$ be linear. Then their sum is linear

$$\begin{aligned}(f + g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1f(\vec{v}_1) + c_2f(\vec{v}_2) + c_1g(\vec{v}_1) + c_2g(\vec{v}_2) \\ &= c_1\big(f + g\big)(\vec{v}_1) + c_2\big(f + g\big)(\vec{v}_2)\end{aligned}$$

and any scalar multiple is also linear.

$$\begin{aligned}(r \cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1f(\vec{v}_1) + c_2f(\vec{v}_2)) \\ &= c_1(r \cdot f)(\vec{v}_1) + c_2(r \cdot f)(\vec{v}_2)\end{aligned}$$

Hence $\mathcal{L}(V, W)$ is a subspace. QED

We started this section by isolating the structure preservation property of isomorphisms. That is, we defined homomorphisms as a generalization of isomorphisms. Some of the properties that we studied for isomorphisms carried over unchanged, while others were adapted to this more general setting.

It would be a mistake, though, to view this new notion of homomorphism as derived from, or somehow secondary to, that of isomorphism. In the rest of this chapter we shall work mostly with homomorphisms, partly because any statement made about homomorphisms is automatically true about isomorphisms, but more because, while the isomorphism concept is perhaps more natural, experience shows that the homomorphism concept is actually more fruitful and more central to further progress.

Exercises

✓ **1.17** Decide if each $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ is linear.

(a) $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} x \\ x+y+z \end{pmatrix}$ **(b)** $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **(c)** $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) $h(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = \begin{pmatrix} 2x+y \\ 3y-4z \end{pmatrix}$

✓ **1.18** Decide if each map $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is linear.

(a) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + d$

(b) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = ad - bc$

(c) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = 2a + 3b + c - d$

(d) $h(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a^2 + b^2$

✓ **1.19** Show that these two maps are homomorphisms.

(a) $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $a_0 + a_1x + a_2x^2 + a_3x^3$ maps to $a_1 + 2a_2x + 3a_3x^2$

(b) $\int\colon \mathcal{P}_2 \to \mathcal{P}_3$ given by $b_0 + b_1x + b_2x^2$ maps to $b_0x + (b_1/2)x^2 + (b_2/3)x^3$

Are these maps inverse to each other?

1.20 Is (perpendicular) projection from $\mathbb{R}^3$ to the xz-plane a homomorphism? Projection to the yz-plane? To the x-axis? The y-axis? The z-axis? Projection to the origin?

1.21 Show that, while the maps from Example 1.3 preserve linear operations, they are not isomorphisms.

1.22 Is an identity map a linear transformation?

✓ **1.23** Stating that a function is 'linear' is different than stating that its graph is a line.

(a) The function $f_1\colon \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = 2x - 1$ has a graph that is a line. Show that it is not a linear function.

(b) The function $f_2\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + 2y$$

does not have a graph that is a line. Show that it is a linear function.

✓ **1.24** Part of the definition of a linear function is that it respects addition. Does a linear function respect subtraction?

1.25 Assume that h is a linear transformation of V and that $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ is a basis of V. Prove each statement.

(a) If $h(\vec{\beta}_i)=\vec{0}$ for each basis vector then h is the zero map.

(b) If $h(\vec{\beta}_i)=\vec{\beta}_i$ for each basis vector then h is the identity map.

(c) If there is a scalar r such that $h(\vec{\beta}_i)=r\cdot\vec{\beta}_i$ for each basis vector then $h(\vec{v})=r\cdot\vec{v}$ for all vectors in V.

✓ **1.26** Consider the vector space $\mathbb{R}^+$ where vector addition and scalar multiplication are not the ones inherited from $\mathbb{R}$ but rather are these: $a+b$ is the product of a and b, and $r\cdot a$ is the r-th power of a. (This was shown to be a vector space in an earlier exercise.) Verify that the natural logarithm map $\ln\colon\mathbb{R}^+\to\mathbb{R}$ is a homomorphism between these two spaces. Is it an isomorphism?

✓ **1.27** Consider this transformation of $\mathbb{R}^2$.

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x/2\\y/3\end{pmatrix}$$

Find the image under this map of this ellipse.

$$\{\begin{pmatrix}x\\y\end{pmatrix}\mid (x^2/4)+(y^2/9)=1\}$$

✓ **1.28** Imagine a rope wound around the earth's equator so that it fits snugly (suppose that the earth is a sphere). How much extra rope must be added to raise the circle to a constant six feet off the ground?

✓ **1.29** Verify that this map $h\colon\mathbb{R}^3\to\mathbb{R}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-1\end{pmatrix}=3x-y-z$$

is linear. Generalize.

1.30 Show that every homomorphism from $\mathbb{R}^1$ to $\mathbb{R}^1$ acts via multiplication by a scalar. Conclude that every nontrivial linear transformation of $\mathbb{R}^1$ is an isomorphism. Is that true for transformations of $\mathbb{R}^2$? $\mathbb{R}^n$?

1.31 **(a)** Show that for any scalars $a_{1,1},\ldots,a_{m,n}$ this map $h\colon\mathbb{R}^n\to\mathbb{R}^m$ is a homomorphism.

$$\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\mapsto\begin{pmatrix}a_{1,1}x_1+\cdots+a_{1,n}x_n\\\vdots\\a_{m,1}x_1+\cdots+a_{m,n}x_n\end{pmatrix}$$

(b) Show that for each i, the i-th derivative operator d^i/dx^i is a linear transformation of $\mathcal{P}_n$. Conclude that for any scalars $c_k,\ldots,c_0$ this map is a linear transformation of that space.

$$f\mapsto\frac{d^k}{dx^k}f+c_{k-1}\frac{d^{k-1}}{dx^{k-1}}f+\cdots+c_1\frac{d}{dx}f+c_0f$$

1.32 Lemma 1.16 shows that a sum of linear functions is linear and that a scalar multiple of a linear function is linear. Show also that a composition of linear functions is linear.

✓ **1.33** Where $f\colon V\to W$ is linear, suppose that $f(\vec{v}_1)=\vec{w}_1,\ \ldots,\ f(\vec{v}_n)=\vec{w}_n$ for some vectors $\vec{w}_1,\ \ldots,\ \vec{w}_n$ from W.

(a) If the set of $\vec{w}$'s is independent, must the set of $\vec{v}$'s also be independent?

(b) If the set of $\vec{v}\,$'s is independent, must the set of $\vec{w}\,$'s also be independent?
(c) If the set of $\vec{w}\,$'s spans W, must the set of $\vec{v}\,$'s span V?
(d) If the set of $\vec{v}\,$'s spans V, must the set of $\vec{w}\,$'s span W?

1.34 Generalize Example 1.15 by proving that the matrix transpose map is linear. What is the domain and codomain?

1.35 **(a)** Where $\vec{u},\vec{v}\in\mathbb{R}^n$, the line segment connecting them is defined to be the set $\ell=\{t\cdot\vec{u}+(1-t)\cdot\vec{v}\mid t\in[0..1]\}$. Show that the image, under a homomorphism h, of the segment between $\vec{u}$ and $\vec{v}$ is the segment between $h(\vec{u})$ and $h(\vec{v})$.
(b) A subset of $\mathbb{R}^n$ is *convex* if, for any two points in that set, the line segment joining them lies entirely in that set. (The inside of a sphere is convex while the skin of a sphere is not.) Prove that linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ preserve the property of set convexity.

✓ **1.36** Let $h\colon\mathbb{R}^n\to\mathbb{R}^m$ be a homomorphism.
(a) Show that the image under h of a line in $\mathbb{R}^n$ is a (possibly degenerate) line in $\mathbb{R}^m$.
(b) What happens to a k-dimensional linear surface?

1.37 Prove that the restriction of a homomorphism to a subspace of its domain is another homomorphism.

1.38 Assume that $h\colon V\to W$ is linear.
(a) Show that the *rangespace* of this map $\{h(\vec{v})\mid\vec{v}\in V\}$ is a subspace of the codomain W.
(b) Show that the *nullspace* of this map $\{\vec{v}\in V\mid h(\vec{v})=\vec{0}_W\}$ is a subspace of the domain V.
(c) Show that if U is a subspace of the domain V then its image $\{h(\vec{u})\mid\vec{u}\in U\}$ is a subspace of the codomain W. This generalizes the first item.
(d) Generalize the second item.

1.39 Consider the set of isomorphisms from a vector space to itself. Is this a subspace of the space $\mathcal{L}(V,V)$ of homomorphisms from the space to itself?

1.40 Does Theorem 1.9 need that $\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ is a basis? That is, can we still get a well-defined and unique homomorphism if we drop either the condition that the set of $\vec{\beta}$'s be linearly independent, or the condition that it span the domain?

1.41 Let V be a vector space and assume that the maps $f_1,f_2\colon V\to\mathbb{R}^1$ are linear.
(a) Define a map $F\colon V\to\mathbb{R}^2$ whose component functions are the given linear ones.

$$\vec{v}\mapsto\begin{pmatrix}f_1(\vec{v})\\ f_2(\vec{v})\end{pmatrix}$$

Show that F is linear.
(b) Does the converse hold — is any linear map from V to $\mathbb{R}^2$ made up of two linear component maps to $\mathbb{R}^1$?
(c) Generalize.

II.2 Rangespace and Nullspace

Isomorphisms and homomorphisms both preserve structure. The difference is that homomorphisms needn't be onto and needn't be one-to-one. This means that homomorphisms are a more general kind of map, subject to fewer restrictions than isomorphisms. We will examine what can happen with a homomorphism that is prevented by the extra restrictions satisfied by an isomorphism.

We first consider the effect of dropping the onto requirement, of not requiring as part of the definition that a homomorphism be onto its codomain. For instance, the injection map $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is not an isomorphism because it is not onto. Of course, being a function, a homomorphism is onto some set, namely its range; the map ι is onto the xy-plane subset of $\mathbb{R}^3$.

2.1 Lemma Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

Proof. Let $h\colon V \to W$ be linear and let S be a subspace of the domain V. The image $h(S)$ is a subset of the codomain W. It is nonempty because S is nonempty and thus to show that $h(S)$ is a subspace of W we need only show that it is closed under linear combinations of two vectors. If $h(\vec{s}_1)$ and $h(\vec{s}_2)$ are members of $h(S)$ then $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$ is also a member of $h(S)$ because it is the image of $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$ from S. QED

2.2 Definition The *rangespace* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the rangespace is the map's *rank*.

(We shall soon see the connection between the rank of a map and the rank of a matrix.)

2.3 Example Recall that the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + 2a_2x + 3a_3x^2$ is linear. The rangespace $\mathscr{R}(d/dx)$ is the set of quadratic polynomials $\{r + sx + tx^2 \mid r, s, t \in \mathbb{R}\}$. Thus, the rank of this map is three.

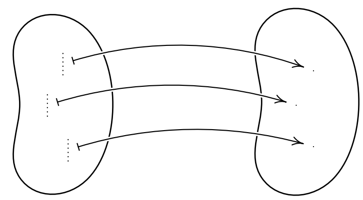
2.4 Example With this homomorphism $h\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + b + 2d) + 0x + cx^2 + cx^3$$

an image vector in the range can have any constant term, must have an x coefficient of zero, and must have the same coefficient of x^2 as of x^3. That is, the rangespace is $\mathscr{R}(h) = \{r + 0x + sx^2 + sx^3 \mid r, s \in \mathbb{R}\}$ and so the rank is two.

The prior result shows that, in passing from the definition of isomorphism to the more general definition of homomorphism, omitting the 'onto' requirement doesn't make an essential difference. Any homomorphism is onto its rangespace.

However, omitting the 'one-to-one' condition does make a difference. A homomorphism may have many elements of the domain that map to one element of the codomain. Below is a "bean" sketch of a many-to-one map between sets.* It shows three elements of the codomain that are each the image of many members of the domain.

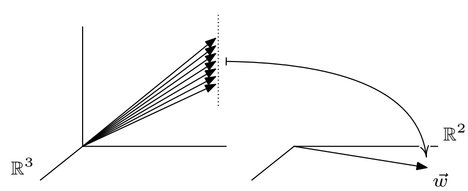


Recall that for any function $h\colon V\to W$, the set of elements of V that are mapped to $\vec{w}\in W$ is the *inverse image* $h^{-1}(\vec{w})=\{\vec{v}\in V \bigm| h(\vec{v})=\vec{w}\}$. Above, the three sets of many elements on the left are inverse images.

2.5 Example Consider the projection $\pi\colon\mathbb{R}^3\to\mathbb{R}^2$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}x\\y\end{pmatrix}$$

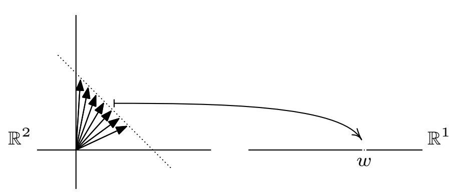
which is a homomorphism that is many-to-one. In this instance, an inverse image set is a vertical line of vectors in the domain.



2.6 Example This homomorphism $h\colon\mathbb{R}^2\to\mathbb{R}^1$

$$\begin{pmatrix}x\\y\end{pmatrix}\stackrel{h}{\longmapsto}x+y$$

is also many-to-one; for a fixed $w\in\mathbb{R}^1$, the inverse image $h^{-1}(w)$



* More information on many-to-one maps is in the appendix.

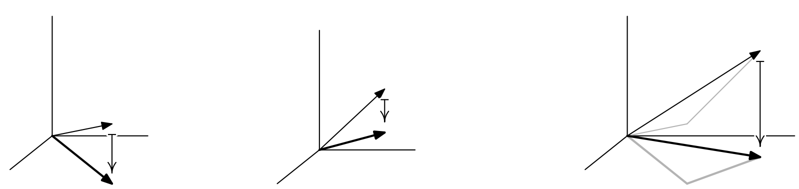
is the set of plane vectors whose components add to w.

The above examples have only to do with the fact that we are considering functions, specifically, many-to-one functions. They show the inverse images as sets of vectors that are related to the image vector $\vec{w}$. But these are more than just arbitrary functions, they are homomorphisms; what do the two preservation conditions say about the relationships?

In generalizing from isomorphisms to homomorphisms by dropping the one-to-one condition, we lose the property that we've stated intuitively as: the domain is "the same as" the range. That is, we lose that the domain corresponds perfectly to the range in a one-vector-by-one-vector way. What we shall keep, as the examples below illustrate, is that a homomorphism describes a way in which the domain is "like", or "analgous to", the range.

2.7 Example We think of $\mathbb{R}^3$ as being like $\mathbb{R}^2$, except that vectors have an extra component. That is, we think of the vector with components x, y, and z as like the vector with components x and y. In defining the projection map π, we make precise which members of the domain we are thinking of as related to which members of the codomain.

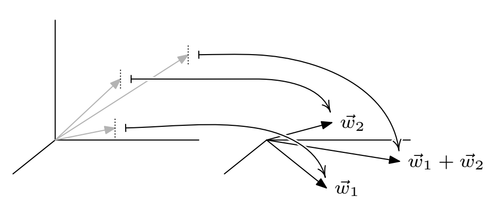
Understanding in what way the preservation conditions in the definition of homomorphism show that the domain elements are like the codomain elements is easiest if we draw $\mathbb{R}^2$ as the xy-plane inside of $\mathbb{R}^3$. (Of course, $\mathbb{R}^2$ is a set of two-tall vectors while the xy-plane is a set of three-tall vectors with a third component of zero, but there is an obvious correspondence.) Then, $\pi(\vec{v})$ is the "shadow" of $\vec{v}$ in the plane and the preservation of addition property says that



$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \text{ above } \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ plus } \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \text{ above } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ equals } \begin{pmatrix} x_1 + y_1 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \text{ above } \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Briefly, the shadow of a sum $\pi(\vec{v}_1 + \vec{v}_2)$ equals the sum of the shadows $\pi(\vec{v}_1) + \pi(\vec{v}_2)$. (Preservation of scalar multiplication has a similar interpretation.)

Redrawing by separating the two spaces, moving the codomain $\mathbb{R}^2$ to the right, gives an uglier picture but one that is more faithful to the "bean" sketch.



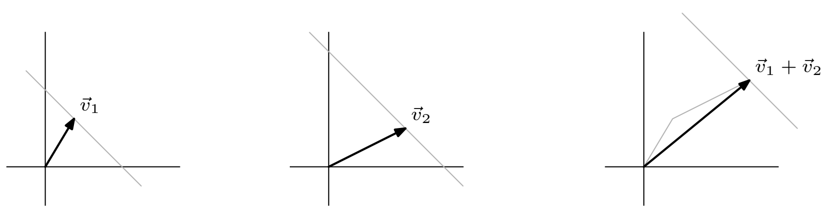
Again in this drawing, the vectors that map to $\vec{w}_1$ lie in the domain in a vertical line (only one such vector is shown, in gray). Call any such member of this inverse image a "$\vec{w}_1$ vector". Similarly, there is a vertical line of "$\vec{w}_2$ vectors" and a vertical line of "$\vec{w}_1 + \vec{w}_2$ vectors". Now, π has the property that if $\pi(\vec{v}_1) = \vec{w}_1$ and $\pi(\vec{v}_2) = \vec{w}_2$ then $\pi(\vec{v}_1 + \vec{v}_2) = \pi(\vec{v}_1) + \pi(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$. This says that the vector classes add, in the sense that any $\vec{w}_1$ vector plus any $\vec{w}_2$ vector equals a $\vec{w}_1 + \vec{w}_2$ vector, (A similar statement holds about the classes under scalar multiplication.)

Thus, although the two spaces $\mathbb{R}^3$ and $\mathbb{R}^2$ are not isomorphic, π describes a way in which they are alike: vectors in $\mathbb{R}^3$ add as do the associated vectors in $\mathbb{R}^2$ — vectors add as their shadows add.

2.8 Example A homomorphism can be used to express an analogy between spaces that is more subtle than the prior one. For the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{h}{\longmapsto} x + y$$

from Example 2.6 fix two numbers w_1, w_2 in the range $\mathbb{R}$. A $\vec{v}_1$ that maps to w_1 has components that add to w_1, that is, the inverse image $h^{-1}(w_1)$ is the set of vectors with endpoint on the diagonal line $x + y = w_1$. Call these the "w_1 vectors". Similarly, we have the "w_2 vectors" and the "$w_1 + w_2$ vectors". Then the addition preservation property says that



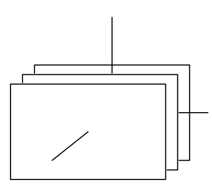
a "w_1 vector" plus a "w_2 vector" equals a "$w_1 + w_2$ vector".

Restated, if a w_1 vector is added to a w_2 vector then the result is mapped by h to a $w_1 + w_2$ vector. Briefly, the image of a sum is the sum of the images. Even more briefly, $h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$. (The preservation of scalar multiplication condition has a similar restatement.)

2.9 Example The inverse images can be structures other than lines. For the linear map $h\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ x \end{pmatrix}$$

the inverse image sets are planes $x = 0$, $x = 1$, etc., perpendicular to the x-axis.



We won't describe how every homomorphism that we will use is an analogy because the formal sense that we make of "alike in that ..." is 'a homomorphism exists such that ...'. Nonetheless, the idea that a homomorphism between two spaces expresses how the domain's vectors fall into classes that act like the range's vectors is a good way to view homomorphisms.

Another reason that we won't treat all of the homomorphisms that we see as above is that many vector spaces are hard to draw (e.g., a space of polynomials). However, there is nothing bad about gaining insights from those spaces that we are able to draw, especially when those insights extend to all vector spaces. We derive two such insights from the three examples 2.7, 2.8, and 2.9.

First, in all three examples, the inverse images are lines or planes, that is, linear surfaces. In particular, the inverse image of the range's zero vector is a line or plane through the origin — a subspace of the domain.

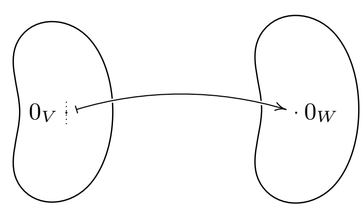
2.10 Lemma For any homomorphism, the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

Proof. Let $h\colon V \to W$ be a homomorphism and let S be a subspace of the rangespace h. Consider $h^{-1}(S) = \{\vec{v} \in V \mid h(\vec{v}) \in S\}$, the inverse image of the set S. It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$, which is an element S, as S is a subspace. To show that $h^{-1}(S)$ is closed under linear combinations, let $\vec{v}_1$ and $\vec{v}_2$ be elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S, and then $c_1\vec{v}_1 + c_2\vec{v}_2$ is also in the inverse image because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$ is a member of the subspace S. QED

2.11 Definition The *nullspace* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of 0_W

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}.$$

The dimension of the nullspace is the map's *nullity*.



2.12 Example The map from Example 2.3 has this nullspace $\mathscr{N}(d/dx) = \{a_0 + 0x + 0x^2 + 0x^3 \mid a_0 \in \mathbb{R}\}$.

2.13 Example The map from Example 2.4 has this nullspace.

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ 0 & -(a+b)/2 \end{pmatrix} \mid a, b \in \mathbb{R}\}$$

Now for the second insight from the above pictures. In Example 2.7, each of the vertical lines is squashed down to a single point — π, in passing from the domain to the range, takes all of these one-dimensional vertical lines and "zeroes them out", leaving the range one dimension smaller than the domain. Similarly, in Example 2.8, the two-dimensional domain is mapped to a one-dimensional range by breaking the domain into lines (here, they are diagonal lines), and compressing each of those lines to a single member of the range. Finally, in Example 2.9, the domain breaks into planes which get "zeroed out", and so the map starts with a three-dimensional domain but ends with a one-dimensional range — this map "subtracts" two from the dimension. (Notice that, in this third example, the codomain is two-dimensional but the range of the map is only one-dimensional, and it is the dimension of the range that is of interest.)

2.14 Theorem A linear map's rank plus its nullity equals the dimension of its domain.

Proof. Let $h\colon V \to W$ be linear and let $B_N = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ be a basis for the nullspace. Extend that to a basis $B_V = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$ for the entire domain. We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for the rangespace. Then counting the size of these bases gives the result.

To see that B_R is linearly independent, consider the equation $c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n) = \vec{0}_W$. This gives that $h(c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n) = \vec{0}_W$ and so $c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$ is in the nullspace of h. As B_N is a basis for this nullspace, there are scalars $c_1, \ldots, c_k \in \mathbb{R}$ satisfying this relationship.

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

But B_V is a basis for V so each scalar equals zero. Therefore B_R is linearly independent.

To show that B_R spans the rangespace, consider $h(\vec{v}) \in \mathscr{R}(h)$ and write $\vec{v}$ as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V. This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_k h(\vec{\beta}_k) + c_{k+1} h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$ and since $\vec{\beta}_1, \ldots, \vec{\beta}_k$ are in the nullspace, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1} h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R, and so B_R spans the space. QED

2.15 Example Where $h\colon \mathbb{R}^3 \to \mathbb{R}^4$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} x \\ 0 \\ y \\ 0 \end{pmatrix}$$

the rangespace and nullspace are

$$\mathscr{R}(h) = \{ \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \bigm| a, b \in \mathbb{R} \} \quad \text{and} \quad \mathscr{N}(h) = \{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \bigm| z \in \mathbb{R} \}$$

and so the rank of h is two while the nullity is one.

2.16 Example If $t\colon \mathbb{R} \to \mathbb{R}$ is the linear transformation $x \mapsto -4x$, then the range is $\mathscr{R}(t) = \mathbb{R}^1$, and so the rank of t is one and the nullity is zero.

2.17 Corollary The rank of a linear map is less than or equal to the dimension of the domain. Equality holds if and only if the nullity of the map is zero.

We know that an isomorphism exists between two spaces if and only if their dimensions are equal. Here we see that for a homomorphism to exist, the dimension of the range must be less than or equal to the dimension of the domain. For instance, there is no homomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^3$. There are many homomorphisms from $\mathbb{R}^2$ into $\mathbb{R}^3$, but none is onto all of three-space.

The rangespace of a linear map can be of dimension strictly less than the dimension of the domain (Example 2.3's derivative transformation on $\mathcal{P}_3$ has a domain of dimension four but a range of dimension three). Thus, under a homomorphism, linearly independent sets in the domain may map to linearly dependent sets in the range (for instance, the derivative sends $\{1, x, x^2, x^3\}$ to $\{0, 1, 2x, 3x^2\}$). That is, under a homomorphism, independence may be lost. In contrast, dependence stays.

2.18 Lemma Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof. Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$, with some c_i nonzero. Then, because $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n)$ and because $h(\vec{0}_V) = \vec{0}_W$, we have that $c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = \vec{0}_W$ with some nonzero c_i. QED

When is independence not lost? One obvious sufficient condition is when the homomorphism is an isomorphism. This condition is also necessary; see Exercise 35. We will finish this subsection comparing homomorphisms with isomorphisms by observing that a one-to-one homomorphism is an isomorphism from its domain onto its range.

2.19 Definition A linear map that is one-to-one is *nonsingular*.

(In the next section we will see the connection between this use of 'nonsingular' for maps and its familiar use for matrices.)

2.20 Example This nonsingular homomorphism $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

gives the obvious correspondence between $\mathbb{R}^2$ and the xy-plane inside of $\mathbb{R}^3$.

The prior observation allows us to adapt some results about isomorphisms to this setting.

2.21 Theorem In an n-dimensional vector space V, these:
(1) h is nonsingular, that is, one-to-one
(2) h has a linear inverse
(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$
(4) $\text{rank}(h) = n$
(5) if $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$
are equivalent statements about a linear map $h\colon V \to W$.

Proof. We will first show that (1) $\iff$ (2). We will then show that (1) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (2).

For (1) $\implies$ (2), suppose that the linear map h is one-to-one, and so has an inverse. The domain of that inverse is the range of h and so a linear combination of two members of that domain has the form $c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$. On that combination, the inverse h^{-1} gives this.

$$\begin{aligned} h^{-1}(c_1h(\vec{v}_1) + c_2h(\vec{v}_2)) &= h^{-1}(h(c_1\vec{v}_1 + c_2\vec{v}_2)) \\ &= h^{-1} \circ h\,(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1\vec{v}_1 + c_2\vec{v}_2 \\ &= c_1h^{-1} \circ h\,(\vec{v}_1) + c_2h^{-1} \circ h\,(\vec{v}_2) \\ &= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2)) \end{aligned}$$

Thus the inverse of a one-to-one linear map is automatically linear. But this also gives the (2) $\implies$ (1) implication, because the inverse itself must be one-to-one.

Of the remaining implications, (1) $\implies$ (3) holds because any homomorphism maps $\vec{0}_V$ to $\vec{0}_W$, but a one-to-one map sends at most one member of V to $\vec{0}_W$.

Next, (3) $\implies$ (4) is true since rank plus nullity equals the dimension of the domain.

For (4) $\implies$ (5), to show that $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for the rangespace we need only show that it is a spanning set, because by assumption the range has dimension n. Consider $h(\vec{v}) \in \mathscr{R}(h)$. Expressing $\vec{v}$ as a linear combination of basis elements produces $h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n)$, which gives that $h(\vec{v}) = c_1h(\vec{\beta}_1) + \cdots + c_nh(\vec{\beta}_n)$, as desired.

Finally, for the (5) $\implies$ (2) implication, assume that $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for V so that $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$. Then every $\vec{w} \in \mathscr{R}(h)$ a the unique representation $\vec{w} = c_1h(\vec{\beta}_1) + \cdots + c_nh(\vec{\beta}_n)$. Define a map from $\mathscr{R}(h)$ to V by

$$\vec{w} \;\mapsto\; c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$$

(uniqueness of the representation makes this well-defined). Checking that it is linear and that it is the inverse of h are easy. QED

We've now seen that a linear map shows how the structure of the domain is like that of the range. Such a map can be thought to organize the domain space into inverse images of points in the range. In the special case that the map is one-to-one, each inverse image is a single point and the map is an isomorphism between the domain and the range.

Exercises

✓ **2.22** Let $h\colon \mathcal{P}_3 \to \mathcal{P}_4$ be given by $p(x) \mapsto x \cdot p(x)$. Which of these are in the nullspace? Which are in the rangespace?

(a) x^3 **(b)** 0 **(c)** 7 **(d)** $12x - 0.5x^3$ **(e)** $1 + 3x^2 - x^3$

✓ **2.23** Find the nullspace, nullity, rangespace, and rank of each map.

(a) $h\colon \mathbb{R}^2 \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a + ax + ax^2$$

(b) $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

(c) $h\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + b + c + dx^2$$

(d) the zero map $Z\colon \mathbb{R}^3 \to \mathbb{R}^4$

✓ **2.24** Find the nullity of each map.

(a) $h\colon \mathbb{R}^5 \to \mathbb{R}^8$ of rank five **(b)** $h\colon \mathcal{P}_3 \to \mathcal{P}_3$ of rank one
(c) $h\colon \mathbb{R}^6 \to \mathbb{R}^3$, an onto map **(d)** $h\colon \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$, onto

✓ **2.25** What is the nullspace of the differentiation transformation $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$? What is the nullspace of the second derivative, as a transformation of $\mathcal{P}_n$? The k-th derivative?

2.26 Example 2.7 restates the first condition in the definition of homomorphism as 'the shadow of a sum is the sum of the shadows'. Restate the second condition in the same style.

2.27 For the homomorphism $h\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $h(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + (a_0 + a_1)x + (a_2 + a_3)x^3$ find these.

(a) $\mathscr{N}(h)$ **(b)** $h^{-1}(2 - x^3)$ **(c)** $h^{-1}(1 + x^2)$

✓ **2.28** For the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = 2x + y$$

sketch these inverse image sets: $f^{-1}(-3)$, $f^{-1}(0)$, and $f^{-1}(1)$.

✓ **2.29** Each of these transformations of $\mathcal{P}_3$ is nonsingular. Find the inverse function of each.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + 2a_2x^2 + 3a_3x^3$
(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$
(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + a_2x + a_3x^2 + a_0x^3$
(d) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3$

2.30 Describe the nullspace and rangespace of a transformation given by $\vec{v} \mapsto 2\vec{v}$.

2.31 List all pairs $(\text{rank}(h), \text{nullity}(h))$ that are possible for linear maps from $\mathbb{R}^5$ to $\mathbb{R}^3$.

2.32 Does the differentiation map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$ have an inverse?

✓ **2.33** Find the nullity of the map $h\colon \mathcal{P}_n \to \mathbb{R}$ given by

$$a_0 + a_1x + \cdots + a_nx^n \mapsto \int_{x=0}^{x=1} a_0 + a_1x + \cdots + a_nx^n \, dx.$$

2.34 **(a)** Prove that a homomorphism is onto if and only if its rank equals the dimension of its codomain.
(b) Conclude that a homomorphism between vector spaces with the same dimension is one-to-one if and only if it is onto.

2.35 Show that a linear map is nonsingular if and only if it preserves linear independence.

2.36 Corollary 2.17 says that for there to be an onto homomorphism from a vector space V to a vector space W, it is necessary that the dimension of W be less than or equal to the dimension of V. Prove that this condition is also sufficient; use Theorem 1.9 to show that if the dimension of W is less than or equal to the dimension of V, then there is a homomorphism from V to W that is onto.

✓ **2.37** Recall that the nullspace is a subset of the domain and the rangespace is a subset of the codomain. Are they necessarily distinct? Is there a homomorphism that has a nontrivial intersection of its nullspace and its rangespace?

2.38 Prove that the image of a span equals the span of the images. That is, where $h\colon V \to W$ is linear, prove that if S is a subset of V then $h([S])$ equals $[h(S)]$. This generalizes Lemma 2.1 since it shows that if U is any subspace of V then its image $\{h(\vec{u}) \bigm| \vec{u} \in U\}$ is a subspace of W, because the span of the set U is U.

✓ **2.39** **(a)** Prove that for any linear map $h\colon V \to W$ and any $\vec{w} \in W$, the set $h^{-1}(\vec{w})$ has the form

$$\{\vec{v} + \vec{n} \bigm| \vec{n} \in \mathscr{N}(h)\}$$

for $\vec{v} \in V$ with $h(\vec{v}) = \vec{w}$ (if h is not onto then this set may be empty). Such a set is a *coset* of $\mathscr{N}(h)$ and is denoted $\vec{v} + \mathscr{N}(h)$.
(b) Consider the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

for some scalars a, b, c, and d. Prove that t is linear.
(c) Conclude from the prior two items that for any linear system of the form

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

the solution set can be written (the vectors are members of $\mathbb{R}^2$)

$$\{\vec{p} + \vec{h} \bigm| \vec{h} \text{ satisfies the associated homogeneous system}\}$$

where $\vec{p}$ is a particular solution of that linear system (if there is no particular solution then the above set is empty).
(d) Show that this map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n \end{pmatrix}$$

for any scalars $a_{1,1}, \ldots, a_{m,n}$. Extend the conclusion made in the prior item.
(e) Show that the k-th derivative map is a linear transformation of $\mathcal{P}_n$ for each k. Prove that this map is a linear transformation of that space

$$f \mapsto \frac{d^k}{dx^k}f + c_{k-1}\frac{d^{k-1}}{dx^{k-1}}f + \cdots + c_1\frac{d}{dx}f + c_0 f$$

for any scalars $c_k, \ldots, c_0$. Draw a conclusion as above.

2.40 Prove that for any transformation $t\colon V \to V$ that is rank one, the map given by composing the operator with itself $t \circ t\colon V \to V$ satisfies $t \circ t = r \cdot t$ for some real number r.

2.41 Let $h\colon V \to \mathbb{R}$ be a homomorphism, but not the zero homomorphism. Prove that if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for the nullspace and if $\vec{v} \in V$ is not in the nullspace then $\langle \vec{v}, \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for the entire domain V.

2.42 Show that for any space V of dimension n, the *dual space*

$$\mathcal{L}(V, \mathbb{R}) = \{h\colon V \to \mathbb{R} \mid h \text{ is linear}\}$$

is isomorphic to $\mathbb{R}^n$. It is often denoted V^*. Conclude that $V^* \cong V$.

2.43 Show that any linear map is the sum of maps of rank one.

2.44 Is 'is homomorphic to' an equivalence relation? (*Hint:* the difficulty is to decide on an appropriate meaning for the quoted phrase.)

2.45 Show that the rangespaces and nullspaces of powers of linear maps $t\colon V \to V$ form descending

$$V \supseteq \mathscr{R}(t) \supseteq \mathscr{R}(t^2) \supseteq \ldots$$

and ascending

$$\{\vec{0}\} \subseteq \mathscr{N}(t) \subseteq \mathscr{N}(t^2) \subseteq \ldots$$

chains. Also show that if k is such that $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1})$ then all following rangespaces are equal: $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2}) \ldots$. Similarly, if $\mathscr{N}(t^k) = \mathscr{N}(t^{k+1})$ then $\mathscr{N}(t^k) = \mathscr{N}(t^{k+1}) = \mathscr{N}(t^{k+2}) = \ldots$.

III Computing Linear Maps

The prior section shows that a linear map is determined by its action on a basis. In fact, the equation

$$h(\vec{v}) = h(c_1 \cdot \vec{\beta}_1 + \cdots + c_n \cdot \vec{\beta}_n) = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n)$$

shows that, if we know the value of the map on the vectors in a basis, then we can compute the value of the map on any vector $\vec{v}$ at all. We just need to find the c's to express $\vec{v}$ with respect to the basis.

This section gives the scheme that computes, from the representation of a vector in the domain $\mathrm{Rep}_B(\vec{v})$, the representation of that vector's image in the codomain $\mathrm{Rep}_D(h(\vec{v}))$, using the representations of $h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)$.

III.1 Representing Linear Maps with Matrices

1.1 Example Consider a map h with domain $\mathbb{R}^2$ and codomain $\mathbb{R}^3$, fixing

$$B = \langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rangle \quad \text{and} \quad D = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

as the bases for these spaces, that is determined by this action on the vectors in the domain's basis.

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 4 \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

To compute the action of this map on any vector at all from the domain, we first express $h(\vec{\beta}_1)$ and $h(\vec{\beta}_2)$ with respect to the codomain's basis:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{so} \quad \mathrm{Rep}_D(h(\vec{\beta}_1)) = \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix}_D$$

and

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{so} \quad \mathrm{Rep}_D(h(\vec{\beta}_2)) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}_D$$

(these are easy to check). Then, as described in the preamble, for any member

$\vec{v}$ of the domain, we can express the image $h(\vec{v})$ in terms of the $h(\vec{\beta})$'s.

$$\begin{aligned}
h(\vec{v}) &= h(c_1\cdot\begin{pmatrix}2\\0\end{pmatrix}+c_2\cdot\begin{pmatrix}1\\4\end{pmatrix}) \\
&= c_1\cdot h(\begin{pmatrix}2\\0\end{pmatrix})+c_2\cdot h(\begin{pmatrix}1\\4\end{pmatrix}) \\
&= c_1\cdot(0\begin{pmatrix}1\\0\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}0\\-2\\0\end{pmatrix}+1\begin{pmatrix}1\\0\\1\end{pmatrix})+c_2\cdot(1\begin{pmatrix}1\\0\\0\end{pmatrix}-1\begin{pmatrix}0\\-2\\0\end{pmatrix}+0\begin{pmatrix}1\\0\\1\end{pmatrix}) \\
&= (0c_1+1c_2)\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}+(-\frac{1}{2}c_1-1c_2)\cdot\begin{pmatrix}0\\-2\\0\end{pmatrix}+(1c_1+0c_2)\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}
\end{aligned}$$

Thus,

$$\text{with }\operatorname{Rep}_B(\vec{v})=\begin{pmatrix}c_1\\c_2\end{pmatrix}\text{ then }\operatorname{Rep}_D(\,h(\vec{v})\,)=\begin{pmatrix}0c_1+1c_2\\-(1/2)c_1-1c_2\\1c_1+0c_2\end{pmatrix}.$$

For instance,

$$\text{with }\operatorname{Rep}_B(\begin{pmatrix}4\\8\end{pmatrix})=\begin{pmatrix}1\\2\end{pmatrix}_B\text{ then }\operatorname{Rep}_D(\,h(\begin{pmatrix}4\\8\end{pmatrix})\,)=\begin{pmatrix}2\\-5/2\\1\end{pmatrix}.$$

We will express computations like the one above with a matrix notation.

$$\begin{pmatrix}0&1\\-1/2&-1\\1&0\end{pmatrix}_{B,D}\begin{pmatrix}c_1\\c_2\end{pmatrix}_B=\begin{pmatrix}0c_1+1c_2\\(-1/2)c_1-1c_2\\1c_1+0c_2\end{pmatrix}_D$$

In the middle is the argument $\vec{v}$ to the map, represented with respect to the domain's basis B by a column vector with components c_1 and c_2. On the right is the value $h(\vec{v})$ of the map on that argument, represented with respect to the codomain's basis D by a column vector with components $0c_1+1c_2$, etc. The matrix on the left is the new thing. It consists of the coefficients from the vector on the right, 0 and 1 from the first row, $-1/2$ and -1 from the second row, and 1 and 0 from the third row.

This notation simply breaks the parts from the right, the coefficients and the c's, out separately on the left, into a vector that represents the map's argument and a matrix that we will take to represent the map itself.

1.2 Definition Suppose that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V\to W$ is a linear map. If

$$\mathrm{Rep}_D(h(\vec{\beta}_1))=\begin{pmatrix}h_{1,1}\\h_{2,1}\\\vdots\\h_{m,1}\end{pmatrix}_D\quad\ldots\quad\mathrm{Rep}_D(h(\vec{\beta}_n))=\begin{pmatrix}h_{1,n}\\h_{2,n}\\\vdots\\h_{m,n}\end{pmatrix}_D$$

then

$$\mathrm{Rep}_{B,D}(h)=\begin{pmatrix}h_{1,1}&h_{1,2}&\ldots&h_{1,n}\\h_{2,1}&h_{2,2}&\ldots&h_{2,n}\\&\vdots&&\\h_{m,1}&h_{m,2}&\ldots&h_{m,n}\end{pmatrix}_{B,D}$$

is the *matrix representation of h with respect to B,D.*

Briefly, the vectors representing the $h(\vec{\beta})$'s are adjoined to make the matrix representing the map.

$$\mathrm{Rep}_{B,D}(h)=\left(\begin{array}{c|c|c}\vdots&&\vdots\\\mathrm{Rep}_D(\,h(\vec{\beta}_1)\,)&\cdots&\mathrm{Rep}_D(\,h(\vec{\beta}_n)\,)\\\vdots&&\vdots\end{array}\right)$$

Observe that the number of columns n of the matrix is the dimension of the domain of the map and the number of rows m is the dimension of the codomain.

1.3 Example If $h\colon\mathbb{R}^3\to\mathcal{P}_1$ is given by

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\stackrel{h}{\longmapsto}(2a_1+a_2)+(-a_3)x$$

then where

$$B=\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}2\\0\\0\end{pmatrix}\rangle\quad\text{and}\quad D=\langle 1+x,-1+x\rangle$$

the action of h on B is given by

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\stackrel{h}{\longmapsto}-x\qquad\begin{pmatrix}0\\2\\0\end{pmatrix}\stackrel{h}{\longmapsto}2\qquad\begin{pmatrix}2\\0\\0\end{pmatrix}\stackrel{h}{\longmapsto}4$$

and a simple calculation gives

$$\mathrm{Rep}_D(-x)=\begin{pmatrix}-1/2\\-1/2\end{pmatrix}_D\quad\mathrm{Rep}_D(2)=\begin{pmatrix}1\\-1\end{pmatrix}_D\quad\mathrm{Rep}_D(4)=\begin{pmatrix}2\\-2\end{pmatrix}_D$$

showing that this is the matrix representing h with respect to the bases.

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} -1/2 & 1 & 2 \\ -1/2 & -1 & -2 \end{pmatrix}_{B,D}$$

We will use lower case letters for a map, upper case for the matrix, and lower case again for the entries of the matrix. Thus for the map h, the matrix representing it is H, with entries $h_{i,j}$.

1.4 Theorem Assume that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

and $\vec{v} \in V$ is represented by

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_D(\,h(\vec{v})\,) = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}_D$$

Proof. This formalizes Example 1.1; see Exercise 28. QED

1.5 Definition The *matrix-vector product* of a $m{\times}n$ matrix and a $n{\times}1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + a_{1,2}c_2 + \cdots + a_{1,n}c_n \\ a_{2,1}c_1 + a_{2,2}c_2 + \cdots + a_{2,n}c_n \\ \vdots \\ a_{m,1}c_1 + a_{m,2}c_2 + \cdots + a_{m,n}c_n \end{pmatrix}$$

The point of Definition 1.2 is to generalize Example 1.1. That is, the point of the definition is Theorem 1.4: the product of the matrix $\mathrm{Rep}_{B,D}(h)$ and the vector $\mathrm{Rep}_B(\vec{v})$ is the vector $\mathrm{Rep}_D(h(\vec{v}))$. Briefly, application of a linear map is represented by the matrix-vector product of the map's representative and the vector's representative.

1.6 Example With the matrix from Example 1.3 we can calculate where that map sends this vector.

$$\vec{v} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

This vector is represented, with respect to the domain basis B, by

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 0 \\ 1/2 \\ 2 \end{pmatrix}_B$$

and so this is the representation of the value $h(\vec{v})$ with respect to the codomain basis D.

$$\begin{aligned} \mathrm{Rep}_D(h(\vec{v})) &= \begin{pmatrix} -1/2 & 1 & 2 \\ -1/2 & -1 & -2 \end{pmatrix}_{B,D} \begin{pmatrix} 0 \\ 1/2 \\ 2 \end{pmatrix}_B \\ &= \begin{pmatrix} (-1/2)\cdot 0 + 1\cdot(1/2) + 2\cdot 2 \\ (-1/2)\cdot 0 - 1\cdot(1/2) - 2\cdot 2 \end{pmatrix}_D = \begin{pmatrix} 9/2 \\ -9/2 \end{pmatrix}_D \end{aligned}$$

To find $h(\vec{v})$ itself, not its representation, take $(9/2)(1+x) - (9/2)(-1+x) = 9$.

1.7 Example Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection onto the xy-plane. To give a matrix representing this map, we first fix bases.

$$B = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

For each vector in the domain's basis, we find its image under the map.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then we find the representation of each image with respect to the codomain's basis

$$\mathrm{Rep}_D(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} -1 \\ 0 \end{pmatrix}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(these are easily checked). Finally, adjoining these representations gives the matrix representing π with respect to B, D.

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}_{B,D}$$

We can illustrate Theorem 1.4 by computing the matrix-vector product representing the following statement about the projection map.

$$\pi(\begin{pmatrix}2\\2\\1\end{pmatrix}) = \begin{pmatrix}2\\2\end{pmatrix}$$

Representing this vector from the domain with respect to the domain's basis

$$\mathrm{Rep}_B(\begin{pmatrix}2\\2\\1\end{pmatrix}) = \begin{pmatrix}1\\2\\1\end{pmatrix}_B$$

gives this matrix-vector product.

$$\mathrm{Rep}_D(\,\pi(\begin{pmatrix}2\\1\\1\end{pmatrix})\,) = \begin{pmatrix}1&0&-1\\-1&1&1\end{pmatrix}_{B,D}\begin{pmatrix}1\\2\\1\end{pmatrix}_B = \begin{pmatrix}0\\2\end{pmatrix}_D$$

Expanding this representation into a linear combination of vectors from D

$$0\cdot\begin{pmatrix}2\\1\end{pmatrix} + 2\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix}$$

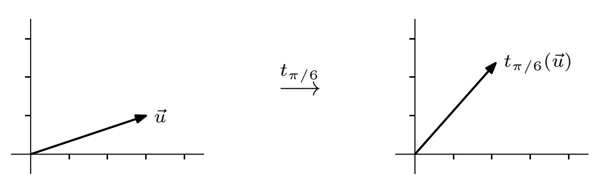
checks that the map's action is indeed reflected in the operation of the matrix. (We will sometimes compress these three displayed equations into one

$$\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}_B \overset{h}{\underset{H}{\longmapsto}} \begin{pmatrix}0\\2\end{pmatrix}_D = \begin{pmatrix}2\\2\end{pmatrix}$$

in the course of a calculation.)

We now have two ways to compute the effect of projection, the straightforward formula that drops each three-tall vector's third component to make a two-tall vector, and the above formula that uses representations and matrix-vector multiplication. Compared to the first way, the second way might seem complicated. However, it has advantages. The next example shows that giving a formula for some maps is simplified by this new scheme.

1.8 Example To represent a *rotation* map $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that turns all vectors in the plane counterclockwise through an angle θ



we start by fixing bases. Using $\mathcal{E}_2$ both as a domain basis and as a codomain basis is natural, Now, we find the image under the map of each vector in the domain's basis.

$$\begin{pmatrix}1\\0\end{pmatrix}\overset{t_\theta}{\longmapsto}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\qquad\begin{pmatrix}0\\1\end{pmatrix}\overset{t_\theta}{\longmapsto}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Then we represent these images with respect to the codomain's basis. Because this basis is $\mathcal{E}_2$, vectors are represented by themselves. Finally, adjoining the representations gives the matrix representing the map.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_\theta)=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}$$

The advantage of this scheme is that just by knowing how to represent the image of the two basis vectors, we get a formula that tells us the image of any vector at all; here a vector rotated by $\theta=\pi/6$.

$$\begin{pmatrix}3\\-2\end{pmatrix}\overset{t_{\pi/6}}{\longmapsto}\begin{pmatrix}\sqrt{3}/2&-1/2\\ 1/2&\sqrt{3}/2\end{pmatrix}\begin{pmatrix}3\\-2\end{pmatrix}\approx\begin{pmatrix}3.598\\-0.232\end{pmatrix}$$

(Again, we are using the fact that, with respect to $\mathcal{E}_2$, vectors represent themselves.)

We have already seen the addition and scalar multiplication operations of matrices and the dot product operation of vectors. Matrix-vector multiplication is a new operation in the arithmetic of vectors and matrices. Nothing in Definition 1.5 requires us to view it in terms of representations. We can get some insight into this operation by turning away from what is being represented, and instead focusing on how the entries combine.

1.9 Example In the definition the width of the matrix equals the height of the vector. Hence, the first product below is defined while the second is not.

$$\begin{pmatrix}1&0&0\\4&3&1\end{pmatrix}\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}\qquad\begin{pmatrix}1&0&0\\4&3&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

One reason that this product is not defined is purely formal: the definition requires that the sizes match, and these sizes don't match. Behind the formality, though, is a reason why we will leave it undefined — the matrix represents a map with a three-dimensional domain while the vector represents a member of a two-dimensional space.

A good way to view a matrix-vector product is as the dot products of the rows of the matrix with the column vector.

$$\begin{pmatrix}&\vdots&&\\ a_{i,1}&a_{i,2}&\ldots&a_{i,n}\\ &\vdots&&\end{pmatrix}\begin{pmatrix}c_1\\c_2\\ \vdots\\c_n\end{pmatrix}=\begin{pmatrix}\vdots\\ a_{i,1}c_1+a_{i,2}c_2+\ldots+a_{i,n}c_n\\ \vdots\end{pmatrix}$$

Looked at in this row-by-row way, this new operation generalizes dot product.

Matrix-vector product can also be viewed column-by-column.

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}$$
$$= c_1 \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}$$

1.10 Example

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

The result has the columns of the matrix weighted by the entries of the vector. This way of looking at it brings us back to the objective stated at the start of this section, to compute $h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$ as $c_1h(\vec{\beta}_1) + \cdots + c_nh(\vec{\beta}_n)$.

We began this section by noting that the equality of these two enables us to compute the action of h on any argument knowing only $h(\vec{\beta}_1)$, ..., $h(\vec{\beta}_n)$. We have developed this into a scheme to compute the action of the map by taking the matrix-vector product of the matrix representing the map and the vector representing the argument. In this way, any linear map is represented with respect to some bases by a matrix. In the next subsection, we will show the converse, that any matrix represents a linear map.

Exercises

✓ **1.11** Multiply the matrix

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

by each vector (or state "not defined").

(a) $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

1.12 Perform, if possible, each matrix-vector multiplication.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & -1/2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

✓ **1.13** Solve this matrix equation.

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$$

✓ **1.14** For a homomorphism from $\mathcal{P}_2$ to $\mathcal{P}_3$ that sends

$$1 \mapsto 1+x, \quad x \mapsto 1+2x, \quad \text{and} \quad x^2 \mapsto x - x^3$$

where does $1 - 3x + 2x^2$ go?

✓ **1.15** Assume that $h\colon \mathbb{R}^2 \to \mathbb{R}^3$ is determined by this action.

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}2\\2\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Using the standard bases, find

(a) the matrix representing this map;

(b) a general formula for $h(\vec{v})$.

✓ **1.16** Let $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ be the derivative transformation.

(a) Represent d/dx with respect to B, B where $B = \langle 1, x, x^2, x^3 \rangle$.

(b) Represent d/dx with respect to B, D where $D = \langle 1, 2x, 3x^2, 4x^3 \rangle$.

✓ **1.17** Represent each linear map with respect to each pair of bases.

(a) $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$ with respect to B, B where $B = \langle 1, x, \ldots, x^n \rangle$, given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

(b) $\int\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$ with respect to B_n, B_{n+1} where $B_i = \langle 1, x, \ldots, x^i \rangle$, given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1}$$

(c) $\int_0^1\colon \mathcal{P}_n \to \mathbb{R}$ with respect to $B, \mathcal{E}_1$ where $B = \langle 1, x, \ldots, x^n \rangle$ and $\mathcal{E}_1 = \langle 1 \rangle$, given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1}$$

(d) $\text{eval}_3\colon \mathcal{P}_n \to \mathbb{R}$ with respect to $B, \mathcal{E}_1$ where $B = \langle 1, x, \ldots, x^n \rangle$ and $\mathcal{E}_1 = \langle 1 \rangle$, given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 + \cdots + a_n \cdot 3^n$$

(e) $\text{slide}_{-1}\colon \mathcal{P}_n \to \mathcal{P}_n$ with respect to B, B where $B = \langle 1, x, \ldots, x^n \rangle$, given by

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mapsto a_0 + a_1 \cdot (x+1) + \cdots + a_n \cdot (x+1)^n$$

1.18 Represent the identity map on any nontrivial space with respect to B, B, where B is any basis.

1.19 Represent, with respect to the natural basis, the transpose transformation on the space $\mathcal{M}_{2\times 2}$ of 2×2 matrices.

1.20 Assume that $B = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3, \vec{\beta}_4 \rangle$ is a basis for a vector space. Represent with respect to B, B the transformation that is determined by each.

(a) $\vec{\beta}_1 \mapsto \vec{\beta}_2$, $\vec{\beta}_2 \mapsto \vec{\beta}_3$, $\vec{\beta}_3 \mapsto \vec{\beta}_4$, $\vec{\beta}_4 \mapsto \vec{0}$

(b) $\vec{\beta}_1 \mapsto \vec{\beta}_2$, $\vec{\beta}_2 \mapsto \vec{0}$, $\vec{\beta}_3 \mapsto \vec{\beta}_4$, $\vec{\beta}_4 \mapsto \vec{0}$

(c) $\vec{\beta}_1 \mapsto \vec{\beta}_2$, $\vec{\beta}_2 \mapsto \vec{\beta}_3$, $\vec{\beta}_3 \mapsto \vec{0}$, $\vec{\beta}_4 \mapsto \vec{0}$

1.21 Example 1.8 shows how to represent the rotation transformation of the plane with respect to the standard basis. Express these other transformations also with respect to the standard basis.

(a) the *dilation* map d_s, which multiplies all vectors by the same scalar s

(b) the *reflection* map f_ℓ, which reflects all all vectors across a line ℓ through the origin

✓ **1.22** Consider a linear transformation of $\mathbb{R}^2$ determined by these two.

$$\begin{pmatrix}1\\1\end{pmatrix} \mapsto \begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\end{pmatrix}$$

(a) Represent this transformation with respect to the standard bases.

(b) Where does the transformation send this vector?

$$\begin{pmatrix}0\\5\end{pmatrix}$$

(c) Represent this transformation with respect to these bases.

$$B=\langle\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\rangle \qquad D=\langle\begin{pmatrix}2\\2\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\rangle$$

(d) Using B from the prior item, represent the transformation with respect to B,B.

1.23 Suppose that $h\colon V\to W$ is nonsingular so that by Theorem 2.21, for any basis $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle\subset V$ the image $h(B)=\langle h(\vec{\beta}_1),\ldots,h(\vec{\beta}_n)\rangle$ is a basis for W.

(a) Represent the map h with respect to $B,h(B)$.

(b) For a member $\vec{v}$ of the domain, where the representation of $\vec{v}$ has components c_1, ..., c_n, represent the image vector $h(\vec{v})$ with respect to the image basis $h(B)$.

1.24 Give a formula for the product of a matrix and $\vec{e}_i$, the column vector that is all zeroes except for a single one in the i-th position.

✓ **1.25** For each vector space of functions of one real variable, represent the derivative transformation with respect to B,B.

(a) $\{a\cos x+b\sin x \mid a,b\in\mathbb{R}\}$, $B=\langle\cos x,\sin x\rangle$

(b) $\{ae^x+be^{2x} \mid a,b\in\mathbb{R}\}$, $B=\langle e^x,e^{2x}\rangle$

(c) $\{a+bx+ce^x+dxe^x \mid a,b,c,d\in\mathbb{R}\}$, $B=\langle 1,x,e^x,xe^x\rangle$

1.26 Find the range of the linear transformation of $\mathbb{R}^2$ represented with respect to the standard bases by each matrix.

(a) $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ **(b)** $\begin{pmatrix}0&0\\3&2\end{pmatrix}$ **(c)** a matrix of the form $\begin{pmatrix}a&b\\2a&2b\end{pmatrix}$

✓ **1.27** Can one matrix represent two different linear maps? That is, can $\text{Rep}_{B,D}(h)=\text{Rep}_{\hat{B},\hat{D}}(\hat{h})$?

1.28 Prove Theorem 1.4.

✓ **1.29** Example 1.8 shows how to represent rotation of all vectors in the plane through an angle θ about the origin, with respect to the standard bases.

(a) Rotation of all vectors in three-space through an angle θ about the x-axis is a transformation of $\mathbb{R}^3$. Represent it with respect to the standard bases. Arrange the rotation so that to someone whose feet are at the origin and whose head is at $(1,0,0)$, the movement appears clockwise.

(b) Repeat the prior item, only rotate about the y-axis instead. (Put the person's head at $\vec{e}_2$.)

(c) Repeat, about the z-axis.

(d) Extend the prior item to $\mathbb{R}^4$. (*Hint:* 'rotate about the z-axis' can be restated as 'rotate parallel to the xy-plane'.)

1.30 (Schur's Triangularization Lemma)

(a) Let U be a subspace of V and fix bases $B_U\subseteq B_V$. What is the relationship between the representation of a vector from U with respect to B_U and the representation of that vector (viewed as a member of V) with respect to B_V?

(b) What about maps?

(c) Fix a basis $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ for V and observe that the spans

$$[\{\vec{0}\}]=\{\vec{0}\}\subset[\{\vec{\beta}_1\}]\subset[\{\vec{\beta}_1,\vec{\beta}_2\}]\subset\quad\cdots\quad\subset[B]=V$$

form a strictly increasing chain of subspaces. Show that for any linear map $h\colon V \to W$ there is a chain $W_0 = \{\vec{0}\} \subseteq W_1 \subseteq \cdots \subseteq W_m = W$ of subspaces of W such that

$$h([\{\vec{\beta}_1, \ldots, \vec{\beta}_i\}]) \subset W_i$$

for each i.

(d) Conclude that for every linear map $h\colon V \to W$ there are bases B, D so the matrix representing h with respect to B, D is upper-triangular (that is, each entry $h_{i,j}$ with $i > j$ is zero).

(e) Is an upper-triangular representation unique?

III.2 Any Matrix Represents a Linear Map

The prior subsection shows that the action of a linear map h is described by a matrix H, with respect to appropriate bases, in this way.

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_B \quad \overset{h}{\underset{H}{\longmapsto}} \quad \begin{pmatrix} h_{1,1}v_1 + \cdots + h_{1,n}v_n \\ \vdots \\ h_{m,1}v_1 + \cdots + h_{m,n}v_n \end{pmatrix}_D = h(\vec{v})$$

In this subsection, we will show the converse, that each matrix represents a linear map.

Recall that, in the definition of the matrix representation of a linear map, the number of columns of the matrix is the dimension of the map's domain and the number of rows of the matrix is the dimension of the map's codomain. Thus, for instance, a 2×3 matrix cannot represent a map from $\mathbb{R}^5$ to $\mathbb{R}^4$. The next result says that, beyond this restriction on the dimensions, there are no other limitations: the 2×3 matrix represents a map from any three-dimensional space to any two-dimensional space.

2.1 Theorem Any matrix represents a homomorphism between vector spaces of appropriate dimensions, with respect to any pair of bases.

Proof. For the matrix

$$H = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}$$

fix any n-dimensional domain space V and any m-dimensional codomain space W. Also fix bases $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $D = \langle \vec{\delta}_1, \ldots, \vec{\delta}_m \rangle$ for those spaces. Define a function $h\colon V \to W$ by: where $\vec{v}$ in the domain is represented as

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_B$$

then its image $h(\vec{v})$ is the member the codomain represented by

$$\mathrm{Rep}_D(\,h(\vec{v})\,)=\begin{pmatrix}h_{1,1}v_1+\cdots+h_{1,n}v_n\\ \vdots\\ h_{m,1}v_1+\cdots+h_{m,n}v_n\end{pmatrix}_D$$

that is, $h(\vec{v})=h(v_1\vec{\beta}_1+\cdots+v_n\vec{\beta}_n)$ is defined to be $(h_{1,1}v_1+\cdots+h_{1,n}v_n)\cdot\vec{\delta}_1+\cdots+(h_{m,1}v_1+\cdots+h_{m,n}v_n)\cdot\vec{\delta}_m$. (This is well-defined by the uniqueness of the representation $\mathrm{Rep}_B(\vec{v})$.)

Observe that h has simply been defined to make it the map that is represented with respect to B,D by the matrix H. So to finish, we need only check that h is linear. If $\vec{v},\vec{u}\in V$ are such that

$$\mathrm{Rep}_B(\vec{v})=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\quad\text{and}\quad\mathrm{Rep}_B(\vec{u})=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$$

and $c,d\in\mathbb{R}$ then the calculation

$$\begin{aligned}h(c\vec{v}+d\vec{u})&=\bigl(h_{1,1}(cv_1+du_1)+\cdots+h_{1,n}(cv_n+du_n)\bigr)\cdot\vec{\delta}_1+\\ &\qquad\cdots+\bigl(h_{m,1}(cv_1+du_1)+\cdots+h_{m,n}(cv_n+du_n)\bigr)\cdot\vec{\delta}_m\\ &=c\cdot h(\vec{v})+d\cdot h(\vec{u})\end{aligned}$$

provides this verification. QED

2.2 Example Which map the matrix represents depends on which bases are used. If

$$H=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad B_1=D_1=\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\rangle,\quad\text{and}\quad B_2=D_2=\langle\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\rangle,$$

then $h_1\colon\mathbb{R}^2\to\mathbb{R}^2$ represented by H with respect to B_1,D_1 maps

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}_{B_1}\quad\mapsto\quad\begin{pmatrix}c_1\\0\end{pmatrix}_{D_1}=\begin{pmatrix}c_1\\0\end{pmatrix}$$

while $h_2\colon\mathbb{R}^2\to\mathbb{R}^2$ represented by H with respect to B_2,D_2 is this map.

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\begin{pmatrix}c_2\\c_1\end{pmatrix}_{B_2}\quad\mapsto\quad\begin{pmatrix}c_2\\0\end{pmatrix}_{D_2}=\begin{pmatrix}0\\c_2\end{pmatrix}$$

These two are different. The first is projection onto the x axis, while the second is projection onto the y axis.

So not only is any linear map described by a matrix but any matrix describes a linear map. This means that we can, when convenient, handle linear maps entirely as matrices, simply doing the computations, without have to worry that a matrix of interest does not represent a linear map on some pair of spaces of interest. (In practice, when we are working with a matrix but no spaces or bases have been specified, we will often take the domain and codomain to be $\mathbb{R}^n$ and $\mathbb{R}^m$ and use the standard bases. In this case, because the representation is transparent — the representation with respect to the standard basis of $\vec{v}$ is $\vec{v}$ — the column space of the matrix equals the range of the map. Consequently, the column space of H is often denoted by $\mathscr{R}(H)$.)

With the theorem, we have characterized linear maps as those maps that act in this matrix way. Each linear map is described by a matrix and each matrix describes a linear map. We finish this section by illustrating how a matrix can be used to tell things about its maps.

2.3 Theorem The rank of a matrix equals the rank of any map that it represents.

Proof. Suppose that the matrix H is $m\times n$. Fix domain and codomain spaces V and W of dimension n and m, with bases $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose rangespace

$$\begin{aligned}\{h(\vec{v})\bigm| \vec{v}\in V\} &= \{h(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)\bigm| c_1,\ldots,c_n\in\mathbb{R}\}\\ &= \{c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)\bigm| c_1,\ldots,c_n\in\mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1),\ldots,h(\vec{\beta}_n)\}]$. The rank of h is the dimension of this rangespace.

The rank of the matrix is its column rank (or its row rank; the two are equal). This is the dimension of the column space of the matrix, which is the span of the set of column vectors $[\{\mathrm{Rep}_D(h(\vec{\beta}_1)),\ldots,\mathrm{Rep}_D(h(\vec{\beta}_n))\}]$.

To see that the two spans have the same dimension, recall that a representation with respect to a basis gives an isomorphism $\mathrm{Rep}_D\colon W\to\mathbb{R}^m$. Under this isomorphism, there is a linear relationship among members of the rangespace if and only if the same relationship holds in the column space, e.g, $\vec{0}=c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)$ if and only if $\vec{0}=c_1\mathrm{Rep}_D(h(\vec{\beta}_1))+\cdots+c_n\mathrm{Rep}_D(h(\vec{\beta}_n))$. Hence, a subset of the rangespace is linearly independent if and only if the corresponding subset of the column space is linearly independent. This means that the size of the largest linearly independent subset of the rangespace equals the size of the largest linearly independent subset of the column space, and so the two spaces have the same dimension. QED

2.4 Example Any map represented by

$$\begin{pmatrix}1&2&2\\1&2&1\\0&0&3\\0&0&2\end{pmatrix}$$

must, by definition, be from a three-dimensional domain to a four-dimensional codomain. In addition, because the rank of this matrix is two (we can spot this by eye or get it with Gauss' method), any map represented by this matrix has a two-dimensional rangespace.

2.5 Corollary Let h be a linear map represented by a matrix H. Then h is onto if and only if the rank of H equals the number of its rows, and h is one-to-one if and only if the rank of H equals the number of its columns.

Proof. For the first half, the dimension of the rangespace of h is the rank of h, which equals the rank of H by the theorem. Since the dimension of the codomain of h is the number of rows in H, if the rank of H equals the number of rows, then the dimension of the rangespace equals the dimension of the codomain. But a subspace with the same dimension as its superspace must equal that superspace (a basis for the rangespace is a linearly independent subset of the codomain, whose size is equal to the dimension of the codomain, and so this set is a basis for the codomain).

For the second half, a linear map is one-to-one if and only if it is an isomorphism between its domain and its range, that is, if and only if its domain has the same dimension as its range. But the number of columns in h is the dimension of h's domain, and by the theorem the rank of H equals the dimension of h's range. QED

The above results end any confusion caused by our use of the word 'rank' to mean apparently different things when applied to matrices and when applied to maps. We can also justify the dual use of 'nonsingular'. We've defined a matrix to be nonsingular if it is square and is the matrix of coefficients of a linear system with a unique solution, and we've defined a linear map to be nonsingular if it is one-to-one.

2.6 Corollary A square matrix represents nonsingular maps if and only if it is a nonsingular matrix. Thus, a matrix represents an isomorphism if and only if it is square and nonsingular.

Proof. Immediate from the prior result. QED

2.7 Example Any map from $\mathbb{R}^2$ to $\mathcal{P}_1$ represented with respect to any pair of bases by

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

is nonsingular because this matrix has rank two.

2.8 Example Any map $g\colon V \to W$ represented by

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

is not nonsingular because this matrix is not nonsingular.

We've now seen that the relationship between maps and matrices goes both ways: for a particular pair of bases, any linear map is represented by a matrix and any matrix describes a linear map. That is, by fixing spaces and bases we get a correspondence between maps and matrices. In the rest of this chapter we will explore this correspondence. For instance, we've defined for linear maps the operations of addition and scalar multiplication and we shall see what the corresponding matrix operations are. We shall also see the matrix operation that represent the map operation of composition. And, we shall see how to find the matrix that represents a map's inverse.

Exercises

✓ **2.9** Decide if the vector is in the column space of the matrix.

(a) $\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 4 & -8 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

✓ **2.10** Decide if each vector lies in the range of the map from $\mathbb{R}^3$ to $\mathbb{R}^2$ represented with respect to the standard bases by the matrix.

(a) $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

✓ **2.11** Consider this matrix, representing a transformation of $\mathbb{R}^2$, and these bases for that space.

$$\frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \quad D = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

(a) To what vector in the codomain is the first member of B mapped?
(b) The second member?
(c) Where is a general vector from the domain (a vector with components x and y) mapped? That is, what transformation of $\mathbb{R}^2$ is represented with respect to B, D by this matrix?

2.12 What transformation of $F = \{a \cos\theta + b \sin\theta \bigm| a, b \in \mathbb{R}\}$ is represented with respect to $B = \langle \cos\theta - \sin\theta, \sin\theta \rangle$ and $D = \langle \cos\theta + \sin\theta, \cos\theta \rangle$ by this matrix?

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

✓ **2.13** Decide if $1 + 2x$ is in the range of the map from $\mathbb{R}^3$ to $\mathcal{P}_2$ represented with respect to $\mathcal{E}_3$ and $\langle 1, 1 + x^2, x \rangle$ by this matrix.

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

2.14 Example 2.8 gives a matrix that is nonsingular, and is therefore associated with maps that are nonsingular.

(a) Find the set of column vectors representing the members of the nullspace of any map represented by this matrix.
(b) Find the nullity of any such map.
(c) Find the set of column vectors representing the members of the rangespace of any map represented by this matrix.
(d) Find the rank of any such map.
(e) Check that rank plus nullity equals the dimension of the domain.

✓ **2.15** Because the rank of a matrix equals the rank of any map it represents, if one matrix represents two different maps $H = \mathrm{Rep}_{B,D}(h) = \mathrm{Rep}_{\hat{B},\hat{D}}(\hat{h})$ (where $h, \hat{h}\colon V \to W$) then the dimension of the rangespace of h equals the dimension of the rangespace of $\hat{h}$. Must these equal-dimensioned rangespaces actually be the same?

✓ **2.16** Let V be an n-dimensional space with bases B and D. Consider a map that sends, for $\vec{v} \in V$, the column vector representing $\vec{v}$ with respect to B to the column vector representing $\vec{v}$ with respect to D. Show that map is a linear transformation of $\mathbb{R}^n$.

2.17 Example 2.2 shows that changing the pair of bases can change the map that a matrix represents, even though the domain and codomain remain the same. Could the map ever not change? Is there a matrix H, vector spaces V and W, and associated pairs of bases B_1, D_1 and B_2, D_2 (with $B_1 \neq B_2$ or $D_1 \neq D_2$ or both) such that the map represented by H with respect to B_1, D_1 equals the map represented by H with respect to B_2, D_2?

✓ **2.18** A square matrix is a *diagonal* matrix if it is all zeroes except possibly for the entries on its upper-left to lower-right diagonal — its $1,1$ entry, its $2,2$ entry, etc. Show that a linear map is an isomorphism if there are bases such that, with respect to those bases, the map is represented by a diagonal matrix with no zeroes on the diagonal.

2.19 Describe geometrically the action on $\mathbb{R}^2$ of the map represented with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_2$ by this matrix.

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Do the same for these.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

2.20 The fact that for any linear map the rank plus the nullity equals the dimension of the domain shows that a necessary condition for the existence of a homomorphism between two spaces, onto the second space, is that there be no gain in dimension. That is, where $h\colon V \to W$ is onto, the dimension of W must be less than or equal to the dimension of V.

(a) Show that this (strong) converse holds: no gain in dimension implies that there is a homomorphism and, further, any matrix with the correct size and correct rank represents such a map.

(b) Are there bases for $\mathbb{R}^3$ such that this matrix

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

represents a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose range is the xy plane subspace of $\mathbb{R}^3$?

2.21 Let V be an n-dimensional space and suppose that $\vec{x} \in \mathbb{R}^n$. Fix a basis B for V and consider the map $h_{\vec{x}}\colon V \to \mathbb{R}$ given $\vec{v} \mapsto \vec{x} \cdot \mathrm{Rep}_B(\vec{v})$ by the dot product.

(a) Show that this map is linear.

(b) Show that for any linear map $g\colon V \to \mathbb{R}$ there is an $\vec{x} \in \mathbb{R}^n$ such that $g = h_{\vec{x}}$.

(c) In the prior item we fixed the basis and varied the $\vec{x}$ to get all possible linear maps. Can we get all possible linear maps by fixing an $\vec{x}$ and varying the basis?

2.22 Let V, W, X be vector spaces with bases B, C, D.

(a) Suppose that $h\colon V \to W$ is represented with respect to B, C by the matrix H. Give the matrix representing the scalar multiple rh (where $r \in \mathbb{R}$) with respect to B, C by expressing it in terms of H.

(b) Suppose that $h, g\colon V \to W$ are represented with respect to B, C by H and G. Give the matrix representing $h + g$ with respect to B, C by expressing it in terms of H and G.

(c) Suppose that $h\colon V \to W$ is represented with respect to B, C by H and $g\colon W \to X$ is represented with respect to C, D by G. Give the matrix representing $g \circ h$ with respect to B, D by expressing it in terms of H and G.

IV Matrix Operations

The prior section shows how matrices represent linear maps. A good strategy, on seeing a new idea, is to explore how it interacts with some already-established ideas. In the first subsection we will ask how the representation of the sum of two maps $f+g$ is related to the representations of the two maps, and how the representation of a scalar product $r\cdot h$ of a map is related to the representation of that map. In later subsections we will see how to represent map composition and map inverse.

IV.1 Sums and Scalar Products

Recall that for two maps f and g with the same domain and codomain, the map sum $f+g$ has this definition.

$$\vec{v} \overset{f+g}{\longmapsto} f(\vec{v})+g(\vec{v})$$

The easiest way to see how the representations of the maps combine to represent the map sum is with an example.

1.1 Example Suppose that $f,g\colon\mathbb{R}^2\to\mathbb{R}^3$ are represented with respect to the bases B and D by these matrices.

$$F=\operatorname{Rep}_{B,D}(f)=\begin{pmatrix}1&3\\2&0\\1&0\end{pmatrix}_{B,D}\qquad G=\operatorname{Rep}_{B,D}(g)=\begin{pmatrix}0&0\\-1&-2\\2&4\end{pmatrix}_{B,D}$$

Then, for any $\vec{v}\in V$ represented with respect to B, computation of the representation of $f(\vec{v})+g(\vec{v})$

$$\begin{pmatrix}1&3\\2&0\\1&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}+\begin{pmatrix}0&0\\-1&-2\\2&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}1v_1+3v_2\\2v_1+0v_2\\1v_1+0v_2\end{pmatrix}+\begin{pmatrix}0v_1+0v_2\\-1v_1-2v_2\\2v_1+4v_2\end{pmatrix}$$

gives this representation of $f+g\,(\vec{v})$.

$$\begin{pmatrix}(1+0)v_1+(3+0)v_2\\(2-1)v_1+(0-2)v_2\\(1+2)v_1+(0+4)v_2\end{pmatrix}=\begin{pmatrix}1v_1+3v_2\\1v_1-2v_2\\3v_1+4v_2\end{pmatrix}$$

Thus, the action of $f+g$ is described by this matrix-vector product.

$$\begin{pmatrix}1&3\\1&-2\\3&4\end{pmatrix}_{B,D}\begin{pmatrix}v_1\\v_2\end{pmatrix}_B=\begin{pmatrix}1v_1+3v_2\\1v_1-2v_2\\3v_1+4v_2\end{pmatrix}_D$$

This matrix is the entry-by-entry sum of original matrices, e.g., the $1,1$ entry of $\operatorname{Rep}_{B,D}(f+g)$ is the sum of the $1,1$ entry of F and the $1,1$ entry of G.

Representing a scalar multiple of a map works the same way.

1.2 Example If t is a transformation represented by

$$\text{Rep}_{B,D}(t) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_{B,D} \quad \text{so that} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_B \mapsto \begin{pmatrix} v_1 \\ v_1 + v_2 \end{pmatrix}_D = t(\vec{v})$$

then the scalar multiple map $5t$ acts in this way.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_B \longmapsto \begin{pmatrix} 5v_1 \\ 5v_1 + 5v_2 \end{pmatrix}_D = 5 \cdot t(\vec{v})$$

Therefore, this is the matrix representing $5t$.

$$\text{Rep}_{B,D}(5t) = \begin{pmatrix} 5 & 0 \\ 5 & 5 \end{pmatrix}_{B,D}$$

1.3 Definition The *sum* of two same-sized matrices is their entry-by-entry sum. The *scalar multiple* of a matrix is the result of entry-by-entry scalar multiplication.

1.4 Remark These extend the vector addition and scalar multiplication operations that we defined in the first chapter.

1.5 Theorem Let $h, g\colon V \to W$ be linear maps represented with respect to bases B, D by the matrices H and G, and let r be a scalar. Then the map $h + g\colon V \to W$ is represented with respect to B, D by $H + G$, and the map $r \cdot h\colon V \to W$ is represented with respect to B, D by rH.

Proof. Exercise 9; generalize the examples above. QED

A special case of scalar multiplication is multiplication by zero. For any map $0 \cdot h$ is the zero homomorphism and for any matrix $0 \cdot H$ is the matrix with all entries zero.

1.6 Definition A *zero matrix*has all entries 0. We write $Z_{n\times m}$, or simply Z (another, very common, notation is to use $0_{n\times m}$ or just 0).

1.7 Example The zero map from any three-dimensional space to any two-dimensional space is represented by the 2×3 zero matrix

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

no matter which domain and codomain bases are used.

Exercises

✓ **1.8** Perform the indicated operations, if defined.

(a) $\begin{pmatrix}5&-1&2\\6&1&1\end{pmatrix}+\begin{pmatrix}2&1&4\\3&0&5\end{pmatrix}$

(b) $6\cdot\begin{pmatrix}2&-1&-1\\1&2&3\end{pmatrix}$

(c) $\begin{pmatrix}2&1\\0&3\end{pmatrix}+\begin{pmatrix}2&1\\0&3\end{pmatrix}$

(d) $4\begin{pmatrix}1&2\\3&-1\end{pmatrix}+5\begin{pmatrix}-1&4\\-2&1\end{pmatrix}$

(e) $3\begin{pmatrix}2&1\\3&0\end{pmatrix}+2\begin{pmatrix}1&1&4\\3&0&5\end{pmatrix}$

1.9 Prove Theorem 1.5.

(a) Prove that matrix addition represents addition of linear maps.

(b) Prove that matrix scalar multiplication represents scalar multiplication of linear maps.

✓ **1.10** Prove each, where the operations are defined, where G, H, and J are matrices, where Z is the zero matrix, and where r and s are scalars.

(a) Matrix addition is commutative $G+H=H+G$.

(b) Matrix addition is associative $G+(H+J)=(G+H)+J$.

(c) The zero matrix is an additive identity $G+Z=G$.

(d) $0\cdot G=Z$

(e) $(r+s)G=rG+sG$

(f) Matrices have an additive inverse $G+(-1)\cdot G=Z$.

(g) $r(G+H)=rG+rH$

(h) $(rs)G=r(sG)$

1.11 Fix domain and codomain spaces. In general, one matrix can represent many different maps with respect to different bases. However, prove that a zero matrix represents only a zero map. Are there other such matrices?

✓ **1.12** Let V and W be vector spaces of dimensions n and m. Show that the space $\mathcal{L}(V,W)$ of linear maps from V to W is isomorphic to $\mathcal{M}_{m\times n}$.

✓ **1.13** Show that it follows from the prior questions that for any six transformations $t_1,\ldots,t_6\colon\mathbb{R}^2\to\mathbb{R}^2$ there are scalars $c_1,\ldots,c_6\in\mathbb{R}$ such that $c_1t_1+\cdots+c_6t_6$ is the zero map. (*Hint:* this is a bit of a misleading question.)

1.14 The *trace* of a square matrix is the sum of the entries on the main diagonal (the $1,1$ entry plus the $2,2$ entry, etc.; we will see the significance of the trace in Chapter Five). Show that $\text{trace}(H+G)=\text{trace}(H)+\text{trace}(G)$. Is there a similar result for scalar multiplication?

1.15 Recall that the *transpose* of a matrix M is another matrix, whose i,j entry is the j,i entry of M. Verifiy these identities.

(a) $(G+H)^{\text{trans}}=G^{\text{trans}}+H^{\text{trans}}$

(b) $(r\cdot H)^{\text{trans}}=r\cdot H^{\text{trans}}$

✓ **1.16** A square matrix is *symmetric* if each i,j entry equals the j,i entry, that is, if the matrix equals its transpose.

(a) Prove that for any H, the matrix $H+H^{\text{trans}}$ is symmetric. Does every symmetric matrix have this form?

(b) Prove that the set of $n\times n$ symmetric matrices is a subspace of $\mathcal{M}_{n\times n}$.

✓ **1.17** **(a)** How does matrix rank interact with scalar multiplication — can a scalar product of a rank n matrix have rank less than n? Greater?

(b) How does matrix rank interact with matrix addition — can a sum of rank n matrices have rank less than n? Greater?

IV.2 Matrix Multiplication

After representing addition and scalar multiplication of linear maps in the prior subsection, the natural next map operation to consider is composition.

2.1 Lemma A composition of linear maps is linear.

Proof. *(This argument has appeared earlier, as part of the proof that isomorphism is an equivalence relation between spaces.)* Let $h\colon V\to W$ and $g\colon W\to U$ be linear. The calculation

$$\begin{aligned}g\circ h\,\bigl(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2\bigr)&=g\bigl(\,h(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2)\,\bigr)=g\bigl(\,c_1\cdot h(\vec{v}_1)+c_2\cdot h(\vec{v}_2)\,\bigr)\\&=c_1\cdot g\bigl(h(\vec{v}_1)\bigr)+c_2\cdot g\bigl(h(\vec{v}_2)\bigr)=c_1\cdot(g\circ h)(\vec{v}_1)+c_2\cdot(g\circ h)(\vec{v}_2)\end{aligned}$$

shows that $g\circ h\colon V\to U$ preserves linear combinations. QED

To see how the representation of the composite arises out of the representations of the two compositors, consider an example.

2.2 Example Let $h\colon\mathbb{R}^4\to\mathbb{R}^2$ and $g\colon\mathbb{R}^2\to\mathbb{R}^3$, fix bases $B\subset\mathbb{R}^4$, $C\subset\mathbb{R}^2$, $D\subset\mathbb{R}^3$, and let these be the representations.

$$H=\mathrm{Rep}_{B,C}(h)=\begin{pmatrix}4&6&8&2\\5&7&9&3\end{pmatrix}_{B,C}\qquad G=\mathrm{Rep}_{C,D}(g)=\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}_{C,D}$$

To represent the composition $g\circ h\colon\mathbb{R}^4\to\mathbb{R}^3$ we fix a $\vec{v}$, represent h of $\vec{v}$, and then represent g of that. The representation of $h(\vec{v})$ is the product of h's matrix and $\vec{v}$'s vector.

$$\mathrm{Rep}_C(\,h(\vec{v})\,)=\begin{pmatrix}4&6&8&2\\5&7&9&3\end{pmatrix}_{B,C}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}_B=\begin{pmatrix}4v_1+6v_2+8v_3+2v_4\\5v_1+7v_2+9v_3+3v_4\end{pmatrix}_C$$

The representation of $g(\,h(\vec{v})\,)$ is the product of g's matrix and $h(\vec{v})$'s vector.

$$\begin{aligned}\mathrm{Rep}_D(\,g(h(\vec{v}))\,)&=\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}_{C,D}\begin{pmatrix}4v_1+6v_2+8v_3+2v_4\\5v_1+7v_2+9v_3+3v_4\end{pmatrix}_C\\&=\begin{pmatrix}1\cdot(4v_1+6v_2+8v_3+2v_4)+1\cdot(5v_1+7v_2+9v_3+3v_4)\\0\cdot(4v_1+6v_2+8v_3+2v_4)+1\cdot(5v_1+7v_2+9v_3+3v_4)\\1\cdot(4v_1+6v_2+8v_3+2v_4)+0\cdot(5v_1+7v_2+9v_3+3v_4)\end{pmatrix}_D\end{aligned}$$

Distributing and regrouping on the v's gives

$$= \begin{pmatrix} (1\cdot 4+1\cdot 5)v_1+(1\cdot 6+1\cdot 7)v_2+(1\cdot 8+1\cdot 9)v_3+(1\cdot 2+1\cdot 3)v_4 \\ (0\cdot 4+1\cdot 5)v_1+(0\cdot 6+1\cdot 7)v_2+(0\cdot 8+1\cdot 9)v_3+(0\cdot 2+1\cdot 3)v_4 \\ (1\cdot 4+0\cdot 5)v_1+(1\cdot 6+0\cdot 7)v_2+(1\cdot 8+0\cdot 9)v_3+(1\cdot 2+0\cdot 3)v_4 \end{pmatrix}_D$$

which we recognizing as the result of this matrix-vector product.

$$= \begin{pmatrix} 1\cdot 4+1\cdot 5 & 1\cdot 6+1\cdot 7 & 1\cdot 8+1\cdot 9 & 1\cdot 2+1\cdot 3 \\ 0\cdot 4+1\cdot 5 & 0\cdot 6+1\cdot 7 & 0\cdot 8+1\cdot 9 & 0\cdot 2+1\cdot 3 \\ 1\cdot 4+0\cdot 5 & 1\cdot 6+0\cdot 7 & 1\cdot 8+0\cdot 9 & 1\cdot 2+0\cdot 3 \end{pmatrix}_{B,D} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}_D$$

Thus, the matrix representing $g\circ h$ has the rows of G combined with the columns of H.

2.3 Definition The *matrix-multiplicative product* of the $m\times r$ matrix G and the $r\times n$ matrix H is the $m\times n$ matrix P, where

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

that is, the i,j-th entry of the product is the dot product of the i-th row and the j-th column.

$$GH = \begin{pmatrix} & \vdots & & \\ g_{i,1} & g_{i,2} & \ldots & g_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & h_{1,j} & \\ \ldots & h_{2,j} & \ldots \\ & \vdots & \\ & h_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \ldots & p_{i,j} & \ldots \\ & \vdots & \end{pmatrix}$$

2.4 Example The matrices from Example 2.2 combine in this way.

$$\begin{pmatrix} 1\cdot 4+1\cdot 5 & 1\cdot 6+1\cdot 7 & 1\cdot 8+1\cdot 9 & 1\cdot 2+1\cdot 3 \\ 0\cdot 4+1\cdot 5 & 0\cdot 6+1\cdot 7 & 0\cdot 8+1\cdot 9 & 0\cdot 2+1\cdot 3 \\ 1\cdot 4+0\cdot 5 & 1\cdot 6+0\cdot 7 & 1\cdot 8+0\cdot 9 & 1\cdot 2+0\cdot 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & 9 & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

2.5 Example

$$\begin{pmatrix} 2 & 0 \\ 4 & 6 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2\cdot 1+0\cdot 5 & 2\cdot 3+0\cdot 7 \\ 4\cdot 1+6\cdot 5 & 4\cdot 3+6\cdot 7 \\ 8\cdot 1+2\cdot 5 & 8\cdot 3+2\cdot 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 34 & 54 \\ 18 & 38 \end{pmatrix}$$

2.6 Theorem A composition of linear maps is represented by the matrix product of the representatives.

PROOF. *(This argument parallels Example 2.2.)* Let $h\colon V\to W$ and $g\colon W\to X$ be represented by H and G with respect to bases $B\subset V$, $C\subset W$, and $D\subset X$, of sizes n, r, and m. For any $\vec{v}\in V$, the k-th component of $\operatorname{Rep}_C(\,h(\vec{v})\,)$ is

$$h_{k,1}v_1 + \cdots + h_{k,n}v_n$$

and so the i-th component of $\mathrm{Rep}_D(\, g \circ h\,(\vec{v})\,)$ is this.

$$g_{i,1} \cdot (h_{1,1}v_1 + \cdots + h_{1,n}v_n) + g_{i,2} \cdot (h_{2,1}v_1 + \cdots + h_{2,n}v_n) \\ + \cdots + g_{i,r} \cdot (h_{r,1}v_1 + \cdots + h_{r,n}v_n)$$

Distribute and regroup on the v's.

$$= (g_{i,1}h_{1,1} + g_{i,2}h_{2,1} + \cdots + g_{i,r}h_{r,1}) \cdot v_1 \\ + \cdots + (g_{i,1}h_{1,n} + g_{i,2}h_{2,n} + \cdots + g_{i,r}h_{r,n}) \cdot v_n$$

Finish by recognizing that the coefficient of each v_j

$$g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

matches the definition of the i, j entry of the product GH. QED

The theorem is an example of a result that supports a definition. We can picture what the definition and theorem together say with this *arrow diagram* ('wrt' abbreviates 'with respect to').

$$\begin{tikzcd} & W_{wrt\ C} \arrow[dr, "g", "G"'] & \\ V_{wrt\ B} \arrow[ur, "h", "H"'] \arrow[rr, "g \circ h", "GH"'] & & X_{wrt\ D} \end{tikzcd}$$

Above the arrows, the maps show that the two ways of going from V to X, straight over via the composition or else by way of W, have the same effect

$$\vec{v} \stackrel{g\circ h}{\longmapsto} g(h(\vec{v})) \qquad \vec{v} \stackrel{h}{\longmapsto} h(\vec{v}) \stackrel{g}{\longmapsto} g(h(\vec{v}))$$

(this is just the definition of composition). Below the arrows, the matrices indicate that the product does the same thing — multiplying GH into the column vector $\mathrm{Rep}_B(\vec{v})$ has the same effect as multiplying the column first by H and then multiplying the result by G.

$$\mathrm{Rep}_{B,D}(g \circ h) = GH = \mathrm{Rep}_{C,D}(g)\,\mathrm{Rep}_{B,C}(h)$$

The definition of the matrix-matrix product operation does not restrict us to view it as a representation of a linear map composition. We can get insight into this operation by studying it as a mechanical procedure. The striking thing is the way that rows and columns combine.

One aspect of that combination is that the sizes of the matrices involved is significant. Briefly, $m \times r$ times $r \times n$ equals $m \times n$.

2.7 Example This product is not defined

$$\begin{pmatrix} -1 & 2 & 0 \\ 0 & 10 & 1.1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

because the number of columns on the left does not equal the number of rows on the right.

In terms of the underlying maps, the fact that the sizes must match up reflects the fact that matrix multiplication is defined only when a corresponding function composition

$$\text{dimension } n \text{ space} \xrightarrow{h} \text{dimension } r \text{ space} \xrightarrow{g} \text{dimension } m \text{ space}$$

is possible.

2.8 Remark The order in which these things are written can be confusing. In the '$m \times r$ times $r \times n$ equals $m \times n$' equation, the number written first m is the dimension of g's codomain and is thus the number that appears last in the map dimension description above. The explanation is that while f is done first and then g is applied, that composition is written $g \circ f$, from the notation '$g(f(\vec{v}))$'. (Some people try to lessen confusion by reading '$g \circ f$' aloud as "g following f".) That order then carries over to matrices: $g \circ f$ is represented by GF.

Another aspect of the way that rows and columns combine in the matrix product operation is that in the definition of the i, j entry

$$p_{i,j} = g_{i,\boxed{1}} h_{\boxed{1},j} + g_{i,\boxed{2}} h_{\boxed{2},j} + \cdots + g_{i,\boxed{r}} h_{\boxed{r},j}$$

the boxed subscripts on the g's are column indicators while those on the h's indicate rows. That is, summation takes place over the columns of G but over the rows of H; left is treated differently than right, so GH may be unequal to HG. Matrix multiplication is not commutative.

2.9 Example Matrix multiplication hardly ever commutes. Test that by multiplying randomly chosen matrices both ways.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} \qquad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$

2.10 Example Commutativity can fail more dramatically:

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 23 & 34 & 0 \\ 31 & 46 & 0 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

isn't even defined.

2.11 Remark The fact that matrix multiplication is not commutative may be puzzling at first sight, perhaps just because most algebraic operations in elementary mathematics are commutative. But on further reflection, it isn't so surprising. After all, matrix multiplication represents function composition, which is not commutative — if $f(x) = 2x$ and $g(x) = x + 1$ then $g \circ f(x) = 2x + 1$ while $f \circ g(x) = 2(x + 1) = 2x + 2$. True, this g is not linear and we might have hoped that linear functions commute, but this perspective shows that the failure of commutativity for matrix multiplication fits into a larger context.

Except for the lack of commutativity, matrix multiplication is algebraically well-behaved. Below are some nice properties and more are in Exercise 23 and Exercise 24.

2.12 Theorem If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G+H) = FG+FH$ and $(G+H)F = GF+HF$.

PROOF. Associativity holds because matrix multiplication represents function composition, which is associative: the maps $(f\circ g)\circ h$ and $f\circ(g\circ h)$ are equal as both send $\vec{v}$ to $f(g(h(\vec{v})))$.

Distributivity is similar. For instance, the first one goes $f\circ(g+h)\,(\vec{v}) = f\big((g+h)(\vec{v})\big) = f\big(g(\vec{v})+h(\vec{v})\big) = f(g(\vec{v}))+f(h(\vec{v})) = f\circ g(\vec{v})+f\circ h(\vec{v})$ (the third equality uses the linearity of f). QED

2.13 Remark We could alternatively prove that result by slogging through the indices. For example, associativity goes: the i,j-th entry of $(FG)H$ is

$$\begin{aligned}&(f_{i,1}g_{1,1}+f_{i,2}g_{2,1}+\cdots+f_{i,r}g_{r,1})h_{1,j}\\&\quad+(f_{i,1}g_{1,2}+f_{i,2}g_{2,2}+\cdots+f_{i,r}g_{r,2})h_{2,j}\\&\quad\ \vdots\\&\quad+(f_{i,1}g_{1,s}+f_{i,2}g_{2,s}+\cdots+f_{i,r}g_{r,s})h_{s,j}\end{aligned}$$

(where F, G, and H are $m\times r$, $r\times s$, and $s\times n$ matrices), distribute

$$\begin{aligned}&f_{i,1}g_{1,1}h_{1,j}+f_{i,2}g_{2,1}h_{1,j}+\cdots+f_{i,r}g_{r,1}h_{1,j}\\&\quad+f_{i,1}g_{1,2}h_{2,j}+f_{i,2}g_{2,2}h_{2,j}+\cdots+f_{i,r}g_{r,2}h_{2,j}\\&\quad\ \vdots\\&\quad+f_{i,1}g_{1,s}h_{s,j}+f_{i,2}g_{2,s}h_{s,j}+\cdots+f_{i,r}g_{r,s}h_{s,j}\end{aligned}$$

and regroup around the f's

$$\begin{aligned}&f_{i,1}(g_{1,1}h_{1,j}+g_{1,2}h_{2,j}+\cdots+g_{1,s}h_{s,j})\\&\quad+f_{i,2}(g_{2,1}h_{1,j}+g_{2,2}h_{2,j}+\cdots+g_{2,s}h_{s,j})\\&\quad\ \vdots\\&\quad+f_{i,r}(g_{r,1}h_{1,j}+g_{r,2}h_{2,j}+\cdots+g_{r,s}h_{s,j})\end{aligned}$$

to get the i,j entry of $F(GH)$.

Contrast these two ways of verifying associativity, the one in the proof and the one just above. The argument just above is hard to understand in the sense that, while the calculations are easy to check, the arithmetic seems unconnected to any idea (it also essentially repeats the proof of Theorem 2.6 and so is inefficient). The argument in the proof is shorter, clearer, and says why this property "really" holds. This illustrates the comments made in the preamble to the chapter on vector spaces — at least some of the time an argument from higher-level constructs is clearer.

We have now seen how the representation of the composition of two linear maps is derived from the representations of the two maps. We have called the combination the product of the two matrices. This operation is extremely important. Before we go on to study how to represent the inverse of a linear map, we will explore it some more in the next subsection.

Exercises

✓ **2.14** Compute, or state "not defined".

(a) $\begin{pmatrix}3&1\\-4&2\end{pmatrix}\begin{pmatrix}0&5\\0&0.5\end{pmatrix}$ **(b)** $\begin{pmatrix}1&1&-1\\4&0&3\end{pmatrix}\begin{pmatrix}2&-1&-1\\3&1&1\\3&1&1\end{pmatrix}$

(c) $\begin{pmatrix}2&-7\\7&4\end{pmatrix}\begin{pmatrix}1&0&5\\-1&1&1\\3&8&4\end{pmatrix}$ **(d)** $\begin{pmatrix}5&2\\3&1\end{pmatrix}\begin{pmatrix}-1&2\\3&-5\end{pmatrix}$

✓ **2.15** Where

$$A=\begin{pmatrix}1&-1\\2&0\end{pmatrix}\quad B=\begin{pmatrix}5&2\\4&4\end{pmatrix}\quad C=\begin{pmatrix}-2&3\\-4&1\end{pmatrix}$$

compute or state 'not defined'.

(a) AB **(b)** $(AB)C$ **(c)** BC **(d)** $A(BC)$

2.16 Which products are defined?

(a) 3×2 times 2×3 **(b)** 2×3 times 3×2 **(c)** 2×2 times 3×3
(d) 3×3 times 2×2

✓ **2.17** Give the size of the product or state "not defined".

(a) a 2×3 matrix times a 3×1 matrix
(b) a 1×12 matrix times a 12×1 matrix
(c) a 2×3 matrix times a 2×1 matrix
(d) a 2×2 matrix times a 2×2 matrix

✓ **2.18** Find the system of equations resulting from starting with

$$\begin{aligned}h_{1,1}x_1+h_{1,2}x_2+h_{1,3}x_3&=d_1\\h_{2,1}x_1+h_{2,2}x_2+h_{2,3}x_3&=d_2\end{aligned}$$

and making this change of variable (i.e., substitution).

$$\begin{aligned}x_1&=g_{1,1}y_1+g_{1,2}y_2\\x_2&=g_{2,1}y_1+g_{2,2}y_2\\x_3&=g_{3,1}y_1+g_{3,2}y_2\end{aligned}$$

2.19 As Definition 2.3 points out, the matrix product operation generalizes the dot product. Is the dot product of a $1\times n$ row vector and a $n\times1$ column vector the same as their matrix-multiplicative product?

✓ **2.20** Represent the derivative map on $\mathcal{P}_n$ with respect to B,B where B is the natural basis $\langle 1,x,\ldots,x^n\rangle$. Show that the product of this matrix with itself is defined; what the map does it represent?

2.21 Show that composition of linear transformations on $\mathbb{R}^1$ is commutative. Is this true for any one-dimensional space?

2.22 Why is matrix multiplication not defined as entry-wise multiplication? That would be easier, and commutative too.

✓ **2.23** **(a)** Prove that $H^pH^q=H^{p+q}$ and $(H^p)^q=H^{pq}$ for positive integers p,q.
(b) Prove that $(rH)^p=r^p\cdot H^p$ for any positive integer p and scalar $r\in\mathbb{R}$.

✓ **2.24** **(a)** How does matrix multiplication interact with scalar multiplication: is $r(GH)=(rG)H$? Is $G(rH)=r(GH)$?

(b) How does matrix multiplication interact with linear combinations: is $F(rG+sH)=r(FG)+s(FH)$? Is $(rF+sG)H=rFH+sGH$?

2.25 We can ask how the matrix product operation interacts with the transpose operation.

(a) Show that $(GH)^{\text{trans}}=H^{\text{trans}}G^{\text{trans}}$.

(b) A square matrix is *symmetric* if each i,j entry equals the j,i entry, that is, if the matrix equals its own transpose. Show that the matrices HH^{trans} and $H^{\text{trans}}H$ are symmetric.

✓ **2.26** Rotation of vectors in $\mathbb{R}^3$ about an axis is a linear map. Show that linear maps do not commute by showing geometrically that rotations do not commute.

2.27 In the proof of Theorem 2.12 some maps are used. What are the domains and codomains?

2.28 How does matrix rank interact with matrix multiplication?

(a) Can the product of rank n matrices have rank less than n? Greater?

(b) Show that the rank of the product of two matrices is less than or equal to the minimum of the rank of each factor.

2.29 Is 'commutes with' an equivalence relation among $n\times n$ matrices?

✓ **2.30** *(This will be used in the Matrix Inverses exercises.)* Here is another property of matrix multiplication that might be puzzling at first sight.

(a) Prove that the composition of the projections $\pi_x,\pi_y\colon\mathbb{R}^3\to\mathbb{R}^3$ onto the x and y axes is the zero map despite that neither one is itself the zero map.

(b) Prove that the composition of the derivatives d^2/dx^2, $d^3/dx^3\colon\mathcal{P}_4\to\mathcal{P}_4$ is the zero map despite that neither is the zero map.

(c) Give a matrix equation representing the first fact.

(d) Give a matrix equation representing the second.

When two things multiply to give zero despite that neither is zero, each is said to be a *zero divisor*.

2.31 Show that, for square matrices, $(S+T)(S-T)$ need not equal S^2-T^2.

✓ **2.32** Represent the identity transformation $\mathrm{id}\colon V\to V$ with respect to B,B for any basis B. This is the *identity matrix* I. Show that this matrix plays the role in matrix multiplication that the number 1 plays in real number multiplication: $HI=IH=H$ (for all matrices H for which the product is defined).

2.33 In real number algebra, quadratic equations have at most two solutions. That is not so with matrix algebra. Show that the 2×2 matrix equation $T^2=I$ has more than two solutions, where I is the identity matrix (this matrix has ones in its $1,1$ and $2,2$ entries and zeroes elsewhere; see Exercise 32).

2.34 **(a)** Prove that for any 2×2 matrix T there are scalars $c_0,\dots,c_4$ that are not all 0 such that the combination $c_4T^4+c_3T^3+c_2T^2+c_1T+c_0I$ is the zero matrix (where I is the 2×2 identity matrix, with 1's in its $1,1$ and $2,2$ entries and zeroes elsewhere; see Exercise 32).

(b) Let $p(x)$ be a polynomial $p(x)=c_nx^n+\cdots+c_1x+c_0$. If T is a square matrix we define $p(T)$ to be the matrix $c_nT^n+\cdots+c_1T+I$ (where I is the appropriately-sized identity matrix). Prove that for any square matrix there is a polynomial such that $p(T)$ is the zero matrix.

(c) The *minimal polynomial* $m(x)$ of a square matrix is the polynomial of least degree, and with leading coefficient 1, such that $m(T)$ is the zero matrix. Find the minimal polynomial of this matrix.

$$\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

(This is the representation with respect to $\mathcal{E}_2, \mathcal{E}_2$, the standard basis, of a rotation through $\pi/6$ radians counterclockwise.)

2.35 The infinite-dimensional space $\mathcal{P}$ of all finite-degree polynomials gives a memorable example of the non-commutativity of linear maps. Let $d/dx\colon \mathcal{P} \to \mathcal{P}$ be the usual derivative and let $s\colon \mathcal{P} \to \mathcal{P}$ be the *shift* map.

$$a_0 + a_1x + \cdots + a_nx^n \overset{s}{\longmapsto} 0 + a_0x + a_1x^2 + \cdots + a_nx^{n+1}$$

Show that the two maps don't commute $d/dx \circ s \neq s \circ d/dx$; in fact, not only is $(d/dx \circ s) - (s \circ d/dx)$ not the zero map, it is the identity map.

2.36 Recall the notation for the sum of the sequence of numbers $a_1, a_2, \ldots, a_n$.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

In this notation, the i, j entry of the product of G and H is this.

$$p_{i,j} = \sum_{k=1}^{r} g_{i,k} h_{k,j}$$

Using this notation,

(a) reprove that matrix multiplication is associative;

(b) reprove Theorem 2.6.

IV.3 Mechanics of Matrix Multiplication

In this subsection we consider matrix multiplication as a mechanical process, putting aside for the moment any implications about the underlying maps. As described earlier, the striking thing about matrix multiplication is the way rows and columns combine. The i, j entry of the matrix product is the dot product of row i of the left matrix with column j of the right one. For instance, here a second row and a third column combine to make a $2, 3$ entry.

$$\begin{pmatrix} 1 & 1 \\ \boxed{0 \quad 1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & \boxed{\begin{matrix} 8 \\ 9 \end{matrix}} & 2 \\ 5 & 7 & & 3 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 & 5 \\ 5 & 7 & \boxed{9} & 3 \\ 4 & 6 & 8 & 2 \end{pmatrix}$$

We can view this as the left matrix acting by multiplying its rows, one at a time, into the columns of the right matrix. Of course, another perspective is that the right matrix uses its columns to act on the left matrix's rows. Below, we will examine actions from the left and from the right for some simple matrices.

The first case, the action of a zero matrix, is very easy.

3.1 Example Multiplying by an appropriately-sized zero matrix from the left or from the right

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

results in a zero matrix.

After zero matrices, the matrices whose actions are easiest to understand are the ones with a single nonzero entry.

3.2 Definition A matrix with all zeroes except for a one in the i,j entry is an i,j *unit* matrix.

3.3 Example This is the $1,2$ unit matrix with three rows and two columns, multiplying from the left.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Acting from the left, an i,j unit matrix copies row j of the multiplicand into row i of the result. From the right an i,j unit matrix copies column i of the multiplicand into column j of the result.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 4 \\ 0 & 7 \end{pmatrix}$$

3.4 Example Rescaling these matrices simply rescales the result. This is the action from the left of the matrix that is twice the one in the prior example.

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 14 & 16 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And this is the action of the matrix that is minus three times the one from the prior example.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 0 & -12 \\ 0 & -21 \end{pmatrix}$$

Next in complication are matrices with two nonzero entries. There are two cases. If a left-multiplier has entries in different rows then their actions don't interact.

3.5 Example

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= (\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 14 & 16 & 18 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 14 & 16 & 18 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

But if the left-multiplier's nonzero entries are in the same row then that row of the result is a combination.

3.6 Example

$$\begin{aligned}
\begin{pmatrix}1&0&2\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} &= (\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix})\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \\
&= \begin{pmatrix}1&2&3\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}14&16&18\\0&0&0\\0&0&0\end{pmatrix} \\
&= \begin{pmatrix}15&18&21\\0&0&0\\0&0&0\end{pmatrix}
\end{aligned}$$

Right-multiplication acts in the same way, with columns.

These observations about matrices that are mostly zeroes extend to arbitrary matrices.

3.7 Lemma In a product of two matrices G and H, the columns of GH are formed by taking G times the columns of H

$$G\cdot\left(\begin{array}{c|c|c}\vdots&&\vdots\\ \vec{h}_1&\cdots&\vec{h}_n\\ \vdots&&\vdots\end{array}\right)=\left(\begin{array}{c|c|c}\vdots&&\vdots\\ G\cdot\vec{h}_1&\cdots&G\cdot\vec{h}_n\\ \vdots&&\vdots\end{array}\right)$$

and the rows of GH are formed by taking the rows of G times H

$$\left(\begin{array}{c}\cdots\ \vec{g}_1\ \cdots\\ \hline \vdots\\ \hline \cdots\ \vec{g}_r\ \cdots\end{array}\right)\cdot H=\left(\begin{array}{c}\cdots\ \vec{g}_1\cdot H\ \cdots\\ \hline \vdots\\ \hline \cdots\ \vec{g}_r\cdot H\ \cdots\end{array}\right)$$

(ignoring the extra parentheses).

Proof. We will show the 2×2 case and leave the general case as an exercise.

$$GH=\begin{pmatrix}g_{1,1}&g_{1,2}\\g_{2,1}&g_{2,2}\end{pmatrix}\begin{pmatrix}h_{1,1}&h_{1,2}\\h_{2,1}&h_{2,2}\end{pmatrix}=\begin{pmatrix}g_{1,1}h_{1,1}+g_{1,2}h_{2,1}&g_{1,1}h_{1,2}+g_{1,2}h_{2,2}\\g_{2,1}h_{1,1}+g_{2,2}h_{2,1}&g_{2,1}h_{1,2}+g_{2,2}h_{2,2}\end{pmatrix}$$

The right side of the first equation in the result

$$\left(G\begin{pmatrix}h_{1,1}\\h_{2,1}\end{pmatrix}\;\middle|\;G\begin{pmatrix}h_{1,2}\\h_{2,2}\end{pmatrix}\right)=\left(\begin{pmatrix}g_{1,1}h_{1,1}+g_{1,2}h_{2,1}\\g_{2,1}h_{1,1}+g_{2,2}h_{2,1}\end{pmatrix}\;\middle|\;\begin{pmatrix}g_{1,1}h_{1,2}+g_{1,2}h_{2,2}\\g_{2,1}h_{1,2}+g_{2,2}h_{2,2}\end{pmatrix}\right)$$

is indeed the same as the right side of GH, except for the extra parentheses (the ones marking the columns as column vectors). The other equation is similarly easy to recognize. QED

An application of those observations is that there is a matrix that just copies out the rows and columns.

3.8 Definition The *main diagonal* (or *principle diagonal* or *diagonal*) of a square matrix goes from the upper left to the lower right.

3.9 Definition An *identity matrix* is square and has with all entries zero except for ones in the main diagonal.

$$I_{n\times n}=\begin{pmatrix}1&0&\ldots&0\\0&1&\ldots&0\\&\vdots&&\\0&0&\ldots&1\end{pmatrix}$$

3.10 Example Here is the 2×2 identity matrix leaving its multiplicand unchaged when it acts from the right.

$$\begin{pmatrix}1&-2\\0&-2\\1&-1\\4&3\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&-2\\0&-2\\1&-1\\4&3\end{pmatrix}$$

3.11 Example Here the 3×3 identity leaves its multiplicand unchanged both from the left

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}=\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}$$

and from the right.

$$\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}2&3&6\\1&3&8\\-7&1&0\end{pmatrix}$$

In short, an identity matrix is the identity element of the set of $n\times n$ matrices with respect to the operation of matrix multiplication.

We next see two ways to generalize the identity matrix.

The first is that if the ones are relaxed to arbitrary reals, the resulting matrix will rescale whole rows or columns.

3.12 Definition A *diagonal matrix* is square and has zeros off the main diagonal.

$$\begin{pmatrix}a_{1,1}&0&\ldots&0\\0&a_{2,2}&\ldots&0\\&\vdots&&\\0&0&\ldots&a_{n,n}\end{pmatrix}$$

3.13 Example From the left, the action of multiplication by a diagonal matrix is to rescales the rows.

$$\begin{pmatrix}2&0\\0&-1\end{pmatrix}\begin{pmatrix}2&1&4&-1\\-1&3&4&4\end{pmatrix}=\begin{pmatrix}4&2&8&-2\\1&-3&-4&-4\end{pmatrix}$$

From the right such a matrix rescales the columns.

$$\begin{pmatrix}1&2&1\\2&2&2\end{pmatrix}\begin{pmatrix}3&0&0\\0&2&0\\0&0&-2\end{pmatrix}=\begin{pmatrix}3&4&-2\\6&4&-4\end{pmatrix}$$

The second generalization of identity matrices is that we can put a single one in each row and column in ways other than putting them down the diagonal.

3.14 Definition A *permutation matrix* is square and is all zeros except for a single one in each row and column.

3.15 Example From the left these matrices permute rows.

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}=\begin{pmatrix}7&8&9\\1&2&3\\4&5&6\end{pmatrix}$$

From the right they permute columns.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}=\begin{pmatrix}2&3&1\\5&6&4\\8&9&7\end{pmatrix}$$

We finish this subsection by applying these observations to get matrices that perform Gauss' method and Gauss-Jordan reduction.

3.16 Example We have seen how to produce a matrix that will rescale rows. Multiplying by this diagonal matrix rescales the second row of the other by a factor of three.

$$\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}0&2&1&1\\0&1/3&1&-1\\1&0&2&0\end{pmatrix}=\begin{pmatrix}0&2&1&1\\0&1&3&-3\\1&0&2&0\end{pmatrix}$$

We have seen how to produce a matrix that will swap rows. Multiplying by this permutation matrix swaps the first and third rows.

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}0&2&1&1\\0&1&3&-3\\1&0&2&0\end{pmatrix}=\begin{pmatrix}1&0&2&0\\0&1&3&-3\\0&2&1&1\end{pmatrix}$$

To see how to perform a row combination, we observe something about those two examples. The matrix that rescales the second row by a factor of three arises in this way from the identity.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{3\rho_2}\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}$$

Similarly, the matrix that swaps first and third rows arises in this way.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{\rho_1\leftrightarrow\rho_3}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

3.17 Example The 3×3 matrix that arises as

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{-2\rho_2+\rho_3}\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}$$

will, when it acts from the left, perform the combination operation $-2\rho_2+\rho_3$.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&2&0\\0&1&3&-3\\0&2&1&1\end{pmatrix}=\begin{pmatrix}1&0&2&0\\0&1&3&-3\\0&0&-5&7\end{pmatrix}$$

3.18 Definition The *elementary reduction matrices* are obtained from identity matrices with one Gaussian operation. We denote them:

(1) $I\xrightarrow{k\rho_i}M_i(k)$ for $k\neq 0$;

(2) $I\xrightarrow{\rho_i\leftrightarrow\rho_j}P_{i,j}$ for $i\neq j$;

(3) $I\xrightarrow{k\rho_i+\rho_j}C_{i,j}(k)$ for $i\neq j$.

3.19 Lemma Gaussian reduction can be done through matrix multiplication.

(1) If $H\xrightarrow{k\rho_i}G$ then $M_i(k)H=G$.

(2) If $H\xrightarrow{\rho_i\leftrightarrow\rho_j}G$ then $P_{i,j}H=G$.

(3) If $H\xrightarrow{k\rho_i+\rho_j}G$ then $C_{i,j}(k)H=G$.

Proof. Clear. QED

3.20 Example This is the first system, from the first chapter, on which we performed Gauss' method.

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ (1/3)x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

It can be reduced with matrix multiplication. Swap the first and third rows,

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

triple the first row,

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

and then add -1 times the first row to the second.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right)$$

Now back substitution will give the solution.

3.21 Example Gauss-Jordan reduction works the same way. For the matrix ending the prior example, first adjust the leading entries

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

and to finish, clear the third column and then the second column.

$$\begin{pmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We have observed the following result, which we shall use in the next subsection.

3.22 Corollary For any matrix H there are elementary reduction matrices $R_1, \ldots, R_r$ such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

Until now we have taken the point of view that our primary objects of study are vector spaces and the maps between them, and have adopted matrices only for computational convenience. This subsection show that this point of view isn't the whole story. Matrix theory is a fascinating and fruitful area.

In the rest of this book we shall continue to focus on maps as the primary objects, but we will be pragmatic — if the matrix point of view gives some clearer idea then we shall use it.

Exercises

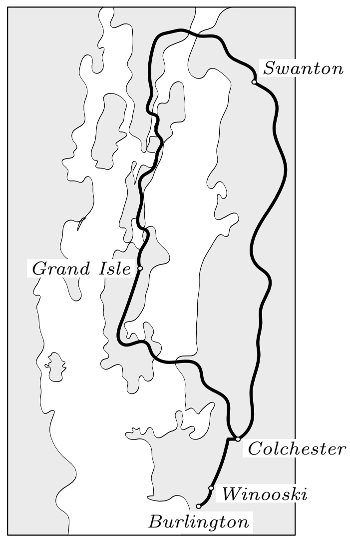
✓ **3.23** Predict the result of each multiplication by an elementary reduction matrix, and then check by multiplying it out.

(a) $\begin{pmatrix}3&0\\0&0\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}$ **(b)** $\begin{pmatrix}4&0\\0&2\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}$ **(c)** $\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\3&4\end{pmatrix}$

(d) $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}$ **(e)** $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

✓ **3.24** The need to take linear combinations of rows and columns in tables of numbers arises often in practice. For instance, this is a map of part of Vermont and New York.

In part because of Lake Champlain, there are no roads directly connecting some pairs of towns. For instance, there is no way to go from Winooski to Grand Isle without going through Colchester. (Of course, many other roads and towns have been left off to simplify the graph. From top to bottom of this map is about forty miles.)



(a) The *incidence matrix* of a map is the square matrix whose i, j entry is the number of roads from city i to city j. Produce the incidence matrix of this map (take the cities in alphabetical order).

(b) A matrix is *symmetric* if it equals its transpose. Show that an incidence matrix is symmetric. (These are all two-way streets. Vermont doesn't have many one-way streets.)

(c) What is the significance of the square of the incidence matrix? The cube?

✓ **3.25** This table gives the number of hours of each type done by each worker, and the associated pay rates. Use matrices to compute the wages due.

	regular	*overtime*
Alan	40	12
Betty	35	6
Catherine	40	18
Donald	28	0

	wage
regular	\$25.00
overtime	\$45.00

(*Remark.* This illustrates, as did the prior problem, that in practice we often want to compute linear combinations of rows and columns in a context where we really aren't interested in any associated linear maps.)

3.26 Find the product of this matrix with its transpose.

$$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$$

✓ **3.27** Prove that the diagonal matrices form a subspace of $\mathcal{M}_{n\times n}$. What is its dimension?

3.28 Does the identity matrix represent the identity map if the bases are unequal?

3.29 Show that every multiple of the identity commutes with every square matrix. Are there other matrices that commute with all square matrices?

3.30 Prove or disprove: nonsingular matrices commute.

✓ **3.31** Show that the product of a permutation matrix and its transpose is an identity matrix.

3.32 Show that if the first and second rows of G are equal then so are the first and second rows of GH. Generalize.

3.33 Describe the product of two diagonal matrices.

3.34 Write

$$\begin{pmatrix} 1 & 0 \\ -3 & 3 \end{pmatrix}$$

as the product of two elementary reduction matrices.

✓ **3.35** Show that if G has a row of zeros then GH (if defined) has a row of zeros. Does that work for columns?

3.36 Show that the set of unit matrices forms a basis for $\mathcal{M}_{n\times m}$.

3.37 Find the formula for the n-th power of this matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

✓ **3.38** The *trace* of a square matrix is the sum of the entries on its diagonal (its significance appears in Chapter Five). Show that $\text{trace}(GH) = \text{trace}(HG)$.

✓ **3.39** A square matrix is *upper triangular* if its only nonzero entries lie above, or on, the diagonal. Show that the product of two upper triangular matrices is upper triangular. Does this hold for lower triangular also?

3.40 A square matrix is a *Markov matrix* if each entry is between zero and one and the sum along each row is one. Prove that a product of Markov matrices is Markov.

✓ **3.41** Give an example of two matrices of the same rank with squares of differing rank.

3.42 Combine the two generalizations of the identity matrix, the one allowing entires to be other than ones, and the one allowing the single one in each row and column to be off the diagonal. What is the action of this type of matrix?

3.43 On a computer multiplications are more costly than additions, so people are interested in reducing the number of multiplications used to compute a matrix product.

(a) How many real number multiplications are needed in formula we gave for the product of a $m\times r$ matrix and a $r\times n$ matrix?

(b) Matrix multiplication is associative, so all associations yield the same result. The cost in number of multiplications, however, varies. Find the association requiring the fewest real number multiplications to compute the matrix product of a 5×10 matrix, a 10×20 matrix, a 20×5 matrix, and a 5×1 matrix.

(c) *(Very hard.)* Find a way to multiply two 2×2 matrices using only seven multiplications instead of the eight suggested by the naive approach.

? **3.44** If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$? [Putnam, 1990, A-5]

3.45 Demonstrate these four assertions to get an alternate proof that column rank equals row rank. [Am. Math. Mon., Dec. 1966]

(a) $\vec{y}\cdot\vec{y}=\vec{0}$ iff $\vec{y}=\vec{0}$.
(b) $A\vec{x}=\vec{0}$ iff $A^{\text{trans}}A\vec{x}=\vec{0}$.
(c) $\dim(\mathscr{R}(A))=\dim(\mathscr{R}(A^{\text{trans}}A))$.
(d) $\text{col rank}(A)=\text{col rank}(A^{\text{trans}})=\text{row rank}(A)$.

3.46 Prove (where A is an $n\times n$ matrix and so defines a transformation of any n-dimensional space V with respect to B,B where B is a basis) that $\dim(\mathscr{R}(A)\cap\mathscr{N}(A))=\dim(\mathscr{R}(A))-\dim(\mathscr{R}(A^2))$. Conclude

(a) $\mathscr{N}(A)\subset\mathscr{R}(A)$ iff $\dim(\mathscr{N}(A))=\dim(\mathscr{R}(A))-\dim(\mathscr{R}(A^2))$;
(b) $\mathscr{R}(A)\subseteq\mathscr{N}(A)$ iff $A^2=0$;
(c) $\mathscr{R}(A)=\mathscr{N}(A)$ iff $A^2=0$ and $\dim(\mathscr{N}(A))=\dim(\mathscr{R}(A))$;
(d) $\dim(\mathscr{R}(A)\cap\mathscr{N}(A))=0$ iff $\dim(\mathscr{R}(A))=\dim(\mathscr{R}(A^2))$;
(e) *(Requires the Direct Sum subsection, which is optional.)* $V=\mathscr{R}(A)\oplus\mathscr{N}(A)$ iff $\dim(\mathscr{R}(A))=\dim(\mathscr{R}(A^2))$.

[Ackerson]

IV.4 Inverses

We now consider how to represent the inverse of a linear map.

We start by recalling some facts about function inverses.* Some functions have no inverse, or have an inverse on the left side or right side only.

4.1 Example Where $\pi\colon\mathbb{R}^3\to\mathbb{R}^2$ is the projection map

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x\\y\end{pmatrix}$$

and $\eta\colon\mathbb{R}^2\to\mathbb{R}^3$ is the embedding

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\y\\0\end{pmatrix}$$

the composition $\pi\circ\eta$ is the identity map on $\mathbb{R}^2$.

$$\begin{pmatrix}x\\y\end{pmatrix}\stackrel{\eta}{\longmapsto}\begin{pmatrix}x\\y\\0\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}x\\y\end{pmatrix}$$

We say π is a *left inverse map* of η or, what is the same thing, that η is a *right inverse map* of π. However, composition in the other order $\eta\circ\pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\eta\circ\pi$.

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}0\\0\end{pmatrix}\stackrel{\eta}{\longmapsto}\begin{pmatrix}0\\0\\0\end{pmatrix}$$

* More information on function inverses is in the appendix.

In fact, the projection π has no left inverse at all. For, if f were to be a left inverse of π then we would have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all of the infinitely many z's. But no function f can send a single argument to more than one value.

(An example of a function with no inverse on either side is the zero transformation on $\mathbb{R}^2$.) Some functions have a *two-sided inverse map*, another function that is the inverse of the first, both from the left and from the right. For instance, the map given by $\vec{v} \mapsto 2 \cdot \vec{v}$ has the two-sided inverse $\vec{v} \mapsto (1/2) \cdot \vec{v}$. In this subsection we will focus on two-sided inverses. The appendix shows that a function has a two-sided inverse if and only if it is both one-to-one and onto. The appendix also shows that if a function f has a two-sided inverse then it is unique, and so it is called 'the' inverse, and is denoted f^{-1}. So our purpose in this subsection is, where a linear map h has an inverse, to find the relationship between $\mathrm{Rep}_{B,D}(h)$ and $\mathrm{Rep}_{D,B}(h^{-1})$ (recall that we have shown, in Theorem 2.21 of Section II of this chapter, that if a linear map has an inverse then the inverse is a linear map also).

4.2 Definition A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse matrix* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix*. That two-sided inverse is called *the inverse matrix* and is denoted H^{-1}.

Because of the correspondence between linear maps and matrices, statements about map inverses translate into statements about matrix inverses.

4.3 Lemma If a matrix has both a left inverse and a right inverse then the two are equal.

4.4 Theorem A matrix is invertible if and only if it is nonsingular.

Proof. *(For both results.)* Given a matrix H, fix spaces of appropriate dimension for the domain and codomain. Fix bases for these spaces. With respect to these bases, H represents a map h. The statements are true about the map and therefore they are true about the matrix. QED

4.5 Lemma A product of invertible matrices is invertible — if G and H are invertible and if GH is defined then GH is invertible and $(GH)^{-1} = H^{-1}G^{-1}$.

Proof. *(This is just like the prior proof except that it requires two maps.)* Fix appropriate spaces and bases and consider the represented maps h and g. Note that $h^{-1}g^{-1}$ is a two-sided map inverse of gh since $(h^{-1}g^{-1})(gh) = h^{-1}(\mathrm{id})h = h^{-1}h = \mathrm{id}$ and $(gh)(h^{-1}g^{-1}) = g(\mathrm{id})g^{-1} = gg^{-1} = \mathrm{id}$. This equality is reflected in the matrices representing the maps, as required. QED

Here is the arrow diagram giving the relationship between map inverses and matrix inverses. It is a special case of the diagram for function composition and matrix multiplication.

$$\begin{tikzcd}
& W_{wrt\ C} \arrow[dr, "h^{-1}", "H^{-1}"'] & \\
V_{wrt\ B} \arrow[ur, "h", "H"'] \arrow[rr, "\mathrm{id}", "I"'] & & V_{wrt\ B}
\end{tikzcd}$$

Beyond its place in our general program of seeing how to represent map operations, another reason for our interest in inverses comes from solving linear systems. A linear system is equivalent to a matrix equation, as here.

$$\begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 2 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad (*)$$

By fixing spaces and bases (e.g., $\mathbb{R}^2, \mathbb{R}^2$ and $\mathcal{E}_2, \mathcal{E}_2$), we take the matrix H to represent some map h. Then solving the system is the same as asking: what domain vector $\vec{x}$ is mapped by h to the result $\vec{d}$? If we could invert h then we could solve the system by multiplying $\mathrm{Rep}_{D,B}(h^{-1}) \cdot \mathrm{Rep}_D(\vec{d})$ to get $\mathrm{Rep}_B(\vec{x})$.

4.6 Example We can find a left inverse for the matrix just given

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by using Gauss' method to solve the resulting linear system.

$$\begin{array}{rcrcl} m + 2n & & & = & 1 \\ m - n & & & = & 0 \\ & & p + 2q & = & 0 \\ & & p - q & = & 1 \end{array}$$

Answer: $m = 1/3$, $n = 1/3$, $p = 2/3$, and $q = -1/3$. This matrix is actually the two-sided inverse of H, as can easily be checked. With it we can solve the system $(*)$ above by applying the inverse.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 4/3 \end{pmatrix}$$

4.7 Remark Why solve systems this way, when Gauss' method takes less arithmetic (this assertion can be made precise by counting the number of arithmetic operations, as computer algorithm designers do)? Beyond its conceptual appeal of fitting into our program of discovering how to represent the various map operations, solving linear systems by using the matrix inverse has at least two advantages.

First, once the work of finding an inverse has been done, solving a system with the same coefficients but different constants is easy and fast: if we change the entries on the right of the system $(*)$ then we get a related problem

$$\begin{pmatrix}1&1\\2&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}$$

with a related solution method.

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/3&1/3\\2/3&-1/3\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}$$

In applications, solving many systems having the same matrix of coefficients is common.

Another advantage of inverses is that we can explore a system's sensitivity to changes in the constants. For example, tweaking the 3 on the right of the system $(*)$ to

$$\begin{pmatrix}1&1\\2&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}3.01\\2\end{pmatrix}$$

can be solved with the inverse.

$$\begin{pmatrix}1/3&1/3\\2/3&-1/3\end{pmatrix}\begin{pmatrix}3.01\\2\end{pmatrix}=\begin{pmatrix}(1/3)(3.01)+(1/3)(2)\\(2/3)(3.01)-(1/3)(2)\end{pmatrix}$$

to show that x_1 changes by $1/3$ of the tweak while x_2 moves by $2/3$ of that tweak. This sort of analysis is used, for example, to decide how accurately data must be specified in a linear model to ensure that the solution has a desired accuracy.

We finish by describing the computational procedure usually used to find the inverse matrix.

4.8 Lemma A matrix is invertible if and only if it can be written as the product of elementary reduction matrices. The inverse can be computed by applying to the identity matrix the same row steps, in the same order, as are used to Gauss-Jordan reduce the invertible matrix.

Proof. A matrix H is invertible if and only if it is nonsingular and thus Gauss-Jordan reduces to the identity. By Corollary 3.22 this reduction can be done with elementary matrices $R_r \cdot R_{r-1} \ldots R_1 \cdot H = I$. This equation gives the two halves of the result.

First, elementary matrices are invertible and their inverses are also elementary. Applying R_r^{-1} to the left of both sides of that equation, then R_{r-1}^{-1}, etc., gives H as the product of elementary matrices $H = R_1^{-1} \cdots R_r^{-1} \cdot I$ (the I is here to cover the trivial $r = 0$ case).

Second, matrix inverses are unique and so comparison of the above equation with $H^{-1}H = I$ shows that $H^{-1} = R_r \cdot R_{r-1} \ldots R_1 \cdot I$. Therefore, applying R_1 to the identity, followed by R_2, etc., yields the inverse of H. QED

4.9 Example To find the inverse of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

we do Gauss-Jordan reduction, meanwhile performing the same operations on the identity. For clerical convenience we write the matrix and the identity side-by-side, and do the reduction steps together.

$$\begin{aligned}
\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) &\xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \\
&\xrightarrow{-1/3\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \\
&\xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)
\end{aligned}$$

This calculation has found the inverse.

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{pmatrix}$$

4.10 Example This one happens to start with a row swap.

$$\begin{aligned}
\left(\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) &\xrightarrow{\rho_1\leftrightarrow\rho_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \\
&\xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 \end{array}\right) \\
&\quad\vdots \\
&\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & 3/4 \\ 0 & 1 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & 3/4 & -3/4 \end{array}\right)
\end{aligned}$$

4.11 Example A non-invertible matrix is detected by the fact that the left half won't reduce to the identity.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

This procedure will find the inverse of a general $n\times n$ matrix. The 2×2 case is handy.

4.12 Corollary The inverse for a 2×2 matrix exists and equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

PROOF. This computation is Exercise 22. QED

We have seen here, as in the Mechanics of Matrix Multiplication subsection, that we can exploit the correspondence between linear maps and matrices. So we can fruitfully study both maps and matrices, translating back and forth to whichever helps us the most.

Over the entire four subsections of this section we have developed an algebra system for matrices. We can compare it with the familiar algebra system for the real numbers. Here we are working not with numbers but with matrices. We have matrix addition and subtraction operations, and they work in much the same way as the real number operations, except that they only combine same-sized matrices. We also have a matrix multiplication operation and an operation inverse to multiplication. These are somewhat like the familiar real number operations (associativity, and distributivity over addition, for example), but there are differences (failure of commutativity, for example). And, we have scalar multiplication, which is in some ways another extension of real number multiplication. This matrix system provides an example that algebra systems other than the elementary one can be interesting and useful.

Exercises

4.13 Supply the intermediate steps in Example 4.10.

✓ **4.14** Use Corollary 4.12 to decide if each matrix has an inverse.

(a) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 4 \\ 1 & -3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$

✓ **4.15** For each invertible matrix in the prior problem, use Corollary 4.12 to find its inverse.

✓ **4.16** Find the inverse, if it exists, by using the Gauss-Jordan method. Check the answers for the 2×2 matrices with Corollary 4.12.

(a) $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 1/2 \\ 3 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 1 & 5 \\ 0 & -2 & 4 \\ 2 & 3 & -2 \end{pmatrix}$ **(f)** $\begin{pmatrix} 2 & 2 & 3 \\ 1 & -2 & -3 \\ 4 & -2 & -3 \end{pmatrix}$

✓ **4.17** What matrix has this one for its inverse?

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

4.18 How does the inverse operation interact with scalar multiplication and addition of matrices?

(a) What is the inverse of rH?

(b) Is $(H+G)^{-1} = H^{-1} + G^{-1}$?

✓ **4.19** Is $(T^k)^{-1} = (T^{-1})^k$?

4.20 Is H^{-1} invertible?

4.21 For each real number θ let $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be represented with respect to the standard bases by this matrix.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Show that $t_{\theta_1+\theta_2} = t_{\theta_1} \cdot t_{\theta_2}$. Show also that ${t_\theta}^{-1} = t_{-\theta}$.

4.22 Do the calculations for the proof of Corollary 4.12.

4.23 Show that this matrix

$$H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

has infinitely many right inverses. Show also that it has no left inverse.

4.24 In Example 4.1, how many left inverses has η?

4.25 If a matrix has infinitely many right-inverses, can it have infinitely many left-inverses? Must it have?

✓ **4.26** Assume that H is invertible and that HG is the zero matrix. Show that G is a zero matrix.

4.27 Prove that if H is invertible then the inverse commutes with a matrix $GH^{-1} = H^{-1}G$ if and only if H itself commutes with that matrix $GH = HG$.

✓ **4.28** Show that if T is square and if T^4 is the zero matrix then $(I - T)^{-1} = I + T + T^2 + T^3$. Generalize.

✓ **4.29** Let D be diagonal. Describe D^2, D^3, ... , etc. Describe D^{-1}, D^{-2}, ... , etc. Define D^0 appropriately.

4.30 Prove that any matrix row-equivalent to an invertible matrix is also invertible.

4.31 *The first question below appeared as Exercise 28.*

(a) Show that the rank of the product of two matrices is less than or equal to the minimum of the rank of each.

(b) Show that if T and S are square then $TS = I$ if and only if $ST = I$.

4.32 Show that the inverse of a permutation matrix is its transpose.

4.33 *The first two parts of this question appeared as Exercise 25.*

(a) Show that $(GH)^{\text{trans}} = H^{\text{trans}}G^{\text{trans}}$.

(b) A square matrix is *symmetric* if each i, j entry equals the j, i entry (that is, if the matrix equals its transpose). Show that the matrices HH^{trans} and $H^{\text{trans}}H$ are symmetric.

(c) Show that the inverse of the transpose is the transpose of the inverse.

(d) Show that the inverse of a symmetric matrix is symmetric.

✓ **4.34** *The items starting this question appeared as Exercise 30.*

(a) Prove that the composition of the projections $\pi_x, \pi_y \colon \mathbb{R}^3 \to \mathbb{R}^3$ is the zero map despite that neither is the zero map.

(b) Prove that the composition of the derivatives d^2/dx^2, $d^3/dx^3 \colon \mathcal{P}_4 \to \mathcal{P}_4$ is the zero map despite that neither map is the zero map.

(c) Give matrix equations representing each of the prior two items.

When two things multiply to give zero despite that neither is zero, each is said to be a *zero divisor*. Prove that no zero divisor is invertible.

4.35 In real number algebra, there are exactly two numbers, 1 and -1, that are their own multiplicative inverse. Does $H^2 = I$ have exactly two solutions for 2×2 matrices?

4.36 Is the relation 'is a two-sided inverse of' transitive? Reflexive? Symmetric?

4.37 Prove: if the sum of the elements of a square matrix is k, then the sum of the elements in each row of the inverse matrix is $1/k$. [Am. Math. Mon., Nov. 1951]

V Change of Basis

Representations, whether of vectors or of maps, vary with the bases. For instance, with respect to the two bases $\mathcal{E}_2$ and

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

for $\mathbb{R}^2$, the vector $\vec{e}_1$ has two different representations.

$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{e}_1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Similarly, with respect to $\mathcal{E}_2, \mathcal{E}_2$ and $\mathcal{E}_2, B$, the identity map has two different representations.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id}) = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

With our point of view that the objects of our studies are vectors and maps, in fixing bases we are adopting a scheme of tags or names for these objects, that are convienent for computation. We will now see how to translate among these names — we will see exactly how representations vary as the bases vary.

V.1 Changing Representations of Vectors

In converting $\mathrm{Rep}_B(\vec{v})$ to $\mathrm{Rep}_D(\vec{v})$ the underlying vector $\vec{v}$ doesn't change. Thus, this translation is accomplished by the identity map on the space, described so that the domain space vectors are represented with respect to B and the codomain space vectors are represented with respect to D.

$$\begin{array}{c} V_{\text{w.r.t. } B} \\ \mathrm{id}\big\downarrow \\ V_{\text{w.r.t. } D} \end{array}$$

(The diagram is vertical to fit with the ones in the next subsection.)

1.1 Definition The *change of basis matrix* for bases $B, D \subset V$ is the representation of the identity map $\mathrm{id}\colon V \to V$ with respect to those bases.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \left(\begin{array}{c|c|c} \vdots & & \vdots \\ \mathrm{Rep}_D(\vec{\beta}_1) & \cdots & \mathrm{Rep}_D(\vec{\beta}_n) \\ \vdots & & \vdots \end{array}\right)$$

1.2 Lemma Left-multiplication by the change of basis matrix for B, D converts a representation with respect to B to one with respect to D. Conversely, if left-multiplication by a matrix changes bases $M \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\vec{v})$ then M is a change of basis matrix.

Proof. For the first sentence, for each $\vec{v}$, as matrix-vector multiplication represents a map application, $\mathrm{Rep}_{B,D}(\mathrm{id}) \cdot \mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_D(\,\mathrm{id}(\vec{v})\,) = \mathrm{Rep}_D(\vec{v})$. For the second sentence, with respect to B, D the matrix M represents some linear map, whose action is $\vec{v} \mapsto \vec{v}$, and is therefore the identity map. QED

1.3 Example With these bases for $\mathbb{R}^2$,

$$B = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

because

$$\mathrm{Rep}_D(\,\mathrm{id}(\begin{pmatrix} 2 \\ 1 \end{pmatrix})) = \begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}_D \qquad \mathrm{Rep}_D(\,\mathrm{id}(\begin{pmatrix} 1 \\ 0 \end{pmatrix})) = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}_D$$

the change of basis matrix is this.

$$\mathrm{Rep}_{B,D}(\mathrm{id}) = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix}$$

We can see this matrix at work by finding the two representations of $\vec{e}_2$

$$\mathrm{Rep}_B(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \mathrm{Rep}_D(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

and checking that the conversion goes as expected.

$$\begin{pmatrix} -1/2 & -1/2 \\ 3/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

We finish this subsection by recognizing that the change of basis matrices are familiar.

1.4 Lemma A matrix changes bases if and only if it is nonsingular.

Proof. For one direction, if left-multiplication by a matrix changes bases then the matrix represents an invertible function, simply because the function is inverted by changing the bases back. Such a matrix is itself invertible, and so nonsingular.

To finish, we will show that any nonsingular matrix M performs a change of basis operation from any given starting basis B to some ending basis. Because the matrix is nonsingular, it will Gauss-Jordan reduce to the identity, so there are elementatry reduction matrices such that $R_r \cdots R_1 \cdot M = I$. Elementary matrices are invertible and their inverses are also elementary, so multiplying from the left first by ${R_r}^{-1}$, then by ${R_{r-1}}^{-1}$, etc., gives M as a product of elementary matrices $M = R_1{}^{-1} \cdots R_r{}^{-1}$. Thus, we will be done if we show that elementary matrices change a given basis to another basis, for then $R_r{}^{-1}$ changes B to some other basis B_r, and $R_{r-1}{}^{-1}$ changes B_r to some B_{r-1}, ..., and the net effect is that M changes B to B_1. We will prove this about elementary matrices by covering the three types as separate cases.

Applying a row-multiplication matrix

$$M_i(k)\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ kc_i \\ \vdots \\ c_n \end{pmatrix}$$

changes a representation with respect to $\langle\vec{\beta}_1, \ldots, \vec{\beta}_i, \ldots, \vec{\beta}_n\rangle$ to one with respect to $\langle\vec{\beta}_1, \ldots, (1/k)\vec{\beta}_i, \ldots, \vec{\beta}_n\rangle$ in this way.

$$\begin{aligned} \vec{v} = c_1 \cdot \vec{\beta}_1 + \cdots + c_i \cdot \vec{\beta}_i + \cdots + c_n \cdot \vec{\beta}_n& \\ \mapsto\; c_1 \cdot \vec{\beta}_1 + \cdots + kc_i \cdot (1/k)\vec{\beta}_i + \cdots + c_n \cdot \vec{\beta}_n = \vec{v}& \end{aligned}$$

Similarly, left-multiplication by a row-swap matrix $P_{i,j}$ changes a representation with respect to the basis $\langle\vec{\beta}_1, \ldots, \vec{\beta}_i, \ldots, \vec{\beta}_j, \ldots, \vec{\beta}_n\rangle$ into one with respect to the basis $\langle\vec{\beta}_1, \ldots, \vec{\beta}_j, \ldots, \vec{\beta}_i, \ldots, \vec{\beta}_n\rangle$ in this way.

$$\begin{aligned} \vec{v} = c_1 \cdot \vec{\beta}_1 + \cdots + c_i \cdot \vec{\beta}_i + \cdots + c_j\vec{\beta}_j + \cdots + c_n \cdot \vec{\beta}_n& \\ \mapsto\; c_1 \cdot \vec{\beta}_1 + \cdots + c_j \cdot \vec{\beta}_j + \cdots + c_i \cdot \vec{\beta}_i + \cdots + c_n \cdot \vec{\beta}_n = \vec{v}& \end{aligned}$$

And, a representation with respect to $\langle\vec{\beta}_1, \ldots, \vec{\beta}_i, \ldots, \vec{\beta}_j, \ldots, \vec{\beta}_n\rangle$ changes via left-multiplication by a row-combination matrix $C_{i,j}(k)$ into a representation with respect to $\langle\vec{\beta}_1, \ldots, \vec{\beta}_i - k\vec{\beta}_j, \ldots, \vec{\beta}_j, \ldots, \vec{\beta}_n\rangle$

$$\begin{aligned} \vec{v} = c_1 \cdot \vec{\beta}_1 + \cdots + c_i \cdot \vec{\beta}_i + c_j\vec{\beta}_j + \cdots + c_n \cdot \vec{\beta}_n& \\ \mapsto\; c_1 \cdot \vec{\beta}_1 + \cdots + c_i \cdot (\vec{\beta}_i - k\vec{\beta}_j) + \cdots + (kc_i + c_j) \cdot \vec{\beta}_j + \cdots + c_n \cdot \vec{\beta}_n = \vec{v}& \end{aligned}$$

(the definition of reduction matrices specifies that $i \neq k$ and $k \neq 0$ and so this last one is a basis). QED

1.5 Corollary A matrix is nonsingular if and only if it represents the identity map with respect to some pair of bases.

In the next subsection we will see how to translate among representations of maps, that is, how to change $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$. The above corollary is a special case of this, where the domain and range are the same space, and where the map is the identity map.

Exercises

✓ **1.6** In $\mathbb{R}^2$, where

$$D = \langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \rangle$$

find the change of basis matrices from D to $\mathcal{E}_2$ and from $\mathcal{E}_2$ to D. Multiply the two.

✓ **1.7** Find the change of basis matrix for $B, D \subseteq \mathbb{R}^2$.

(a) $B = \mathcal{E}_2$, $D = \langle \vec{e}_2, \vec{e}_1 \rangle$ **(b)** $B = \mathcal{E}_2$, $D = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rangle$

(c) $B = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \rangle$, $D = \mathcal{E}_2$ **(d)** $B = \langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$, $D = \langle \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$

1.8 For the bases in Exercise 7, find the change of basis matrix in the other direction, from D to B.

✓ **1.9** Find the change of basis matrix for each $B, D \subseteq \mathcal{P}_2$.

(a) $B = \langle 1, x, x^2 \rangle, D = \langle x^2, 1, x \rangle$ **(b)** $B = \langle 1, x, x^2 \rangle, D = \langle 1, 1+x, 1+x+x^2 \rangle$

(c) $B = \langle 2, 2x, x^2 \rangle, D = \langle 1 + x^2, 1 - x^2, x + x^2 \rangle$

✓ **1.10** Decide if each changes bases on $\mathbb{R}^2$. To what basis is $\mathcal{E}_2$ changed?

(a) $\begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 & 4 \\ 2 & -8 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

1.11 Find bases such that this matrix represents the identity map with respect to those bases.

$$\begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

1.12 Conside the vector space of real-valued functions with basis $\langle \sin(x), \cos(x) \rangle$. Show that $\langle 2\sin(x)+\cos(x), 3\cos(x) \rangle$ is also a basis for this space. Find the change of basis matrix in each direction.

1.13 Where does this matrix

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

send the standard basis for $\mathbb{R}^2$? Any other bases? *Hint.* Consider the inverse.

✓ **1.14** What is the change of basis matrix with respect to B, B?

1.15 Prove that a matrix changes bases if and only if it is invertible.

1.16 Finish the proof of Lemma 1.4.

✓ **1.17** Let H be a $n \times n$ nonsingular matrix. What basis of $\mathbb{R}^n$ does H change to the standard basis?

✓ **1.18** **(a)** In $\mathcal{P}_3$ with basis $B = \langle 1 + x, 1 - x, x^2 + x^3, x^2 - x^3 \rangle$ we have this represenatation.

$$\text{Rep}_B(1 - x + 3x^2 - x^3) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}_B$$

Find a basis D giving this different representation for the same polynomial.

$$\text{Rep}_D(1 - x + 3x^2 - x^3) = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}_D$$

(b) State and prove that any nonzero vector representation can be changed to any other.

Hint. The proof of Lemma 1.4 is constructive — it not only says the bases change, it shows how they change.

1.19 Let V, W be vector spaces, and let $B, \hat{B}$ be bases for V and $D, \hat{D}$ be bases for W. Where $h\colon V \to W$ is linear, find a formula relating $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$.

✓ **1.20** Show that the columns of an $n \times n$ change of basis matrix form a basis for $\mathbb{R}^n$. Do all bases appear in that way: can the vectors from any $\mathbb{R}^n$ basis make the columns of a change of basis matrix?

✓ **1.21** Find a matrix having this effect.

$$\begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}4\\-1\end{pmatrix}$$

That is, find a M that left-multiplies the starting vector to yield the ending vector. Is there a matrix having these two effects?

(a) $\begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}2\\-1\end{pmatrix} \mapsto \begin{pmatrix}-1\\-1\end{pmatrix}$ **(b)** $\begin{pmatrix}1\\3\end{pmatrix} \mapsto \begin{pmatrix}1\\1\end{pmatrix}$ $\begin{pmatrix}2\\6\end{pmatrix} \mapsto \begin{pmatrix}-1\\-1\end{pmatrix}$

Give a necessary and sufficient condition for there to be a matrix such that $\vec{v}_1 \mapsto \vec{w}_1$ and $\vec{v}_2 \mapsto \vec{w}_2$.

V.2 Changing Map Representations

The first subsection shows how to convert the representation of a vector with respect to one basis to the representation of that same vector with respect to another basis. Here we will see how to convert the representation of a map with respect to one pair of bases to the representation of that map with respect to a different pair, how to change $\mathrm{Rep}_{B,D}(h)$ to $\mathrm{Rep}_{\hat{B},\hat{D}}(h)$.

That is, we want the relationship between the matrices in this arrow diagram.

$$\begin{array}{ccc}
V_{\text{w.r.t. } B} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D} \\
\text{id}\big\downarrow & & \text{id}\big\downarrow \\
V_{\text{w.r.t. } \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } \hat{D}}
\end{array}$$

To move from the lower-left of this diagram to the lower-right we can either go straight over, or else up to V_B then over to W_D and then down. So we can calculate $\hat{H} = \mathrm{Rep}_{\hat{B},\hat{D}}(h)$ either by simply using $\hat{B}$ and $\hat{D}$, or else by first changing bases with $\mathrm{Rep}_{\hat{B},B}(\mathrm{id})$ then multiplying by $H = \mathrm{Rep}_{B,D}(h)$ and then changing bases with $\mathrm{Rep}_{D,\hat{D}}(\mathrm{id})$.

This equation summarizes.

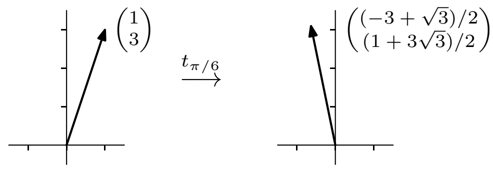
$$\hat{H} = \mathrm{Rep}_{D,\hat{D}}(\mathrm{id}) \cdot H \cdot \mathrm{Rep}_{\hat{B},B}(\mathrm{id}) \qquad (*)$$

(To compare this equation with the sentence before it, remember that the equation is read from right to left because function composition is read right to left and matrix multiplication represent the composition.)

2.1 Example The matrix

$$T = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

represents, with respect to $\mathcal{E}_2, \mathcal{E}_2$, the transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors $\pi/6$ radians counterclockwise.



We can translate that representation with respect to $\mathcal{E}_2, \mathcal{E}_2$ to one with respect to

$$\hat{B} = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle \qquad \hat{D} = \langle \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rangle$$

by using the arrow diagram and formula $(*)$ above.

$$\begin{tikzcd} \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \arrow[r, "t", "T"'] \arrow[d, "\text{id}"'] & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \arrow[d, "\text{id}"'] \\ \mathbb{R}^2_{\text{w.r.t. } \hat{B}} \arrow[r, "t", "\hat{T}"'] & \mathbb{R}^2_{\text{w.r.t. } \hat{D}} \end{tikzcd} \qquad \hat{T} = \text{Rep}_{\mathcal{E}_2, \hat{D}}(\text{id}) \cdot T \cdot \text{Rep}_{\hat{B}, \mathcal{E}_2}(\text{id})$$

Note that $\text{Rep}_{\mathcal{E}_2, \hat{D}}(\text{id})$ can be calculated as the matrix inverse of $\text{Rep}_{\hat{D}, \mathcal{E}_2}(\text{id})$.

$$\begin{aligned} \text{Rep}_{\hat{B}, \hat{D}}(t) &= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (5-\sqrt{3})/6 & (3+2\sqrt{3})/3 \\ (1+\sqrt{3})/6 & \sqrt{3}/3 \end{pmatrix} \end{aligned}$$

Although the new matrix is messier-appearing, the map that it represents is the same. For instance, to replicate the effect of t in the picture, start with $\hat{B}$,

$$\text{Rep}_{\hat{B}}(\begin{pmatrix} 1 \\ 3 \end{pmatrix}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\hat{B}}$$

apply $\hat{T}$,

$$\begin{pmatrix} (5-\sqrt{3})/6 & (3+2\sqrt{3})/3 \\ (1+\sqrt{3})/6 & \sqrt{3}/3 \end{pmatrix}_{\hat{B}, \hat{D}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\hat{B}} = \begin{pmatrix} (11+3\sqrt{3})/6 \\ (1+3\sqrt{3})/6 \end{pmatrix}_{\hat{D}}$$

and check it against $\hat{D}$

$$\frac{11+3\sqrt{3}}{6} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{1+3\sqrt{3}}{6} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (-3+\sqrt{3})/2 \\ (1+3\sqrt{3})/2 \end{pmatrix}$$

to see that it is the same result as above.

2.2 Example One reason to change bases is that the matrix may be simpler. On $\mathbb{R}^3$ the map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$$

that is represented with respect to the standard basis in this way

$$\text{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

can also be represented with respect to another basis

$$\text{if } B = \langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle \qquad \text{then } \text{Rep}_{B,B}(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in a way that is simpler, in that the action of a diagonal matrix is easy to understand.

Naturally, we usually prefer basis changes that make the representation easier to understand. When the representation with respect to equal starting and ending bases is a diagonal matrix we say the map or matrix has been *diagonalized*. In Chaper Five we shall see which maps and matrices are diagonalizable, and where one is not, we shall see how to get a representation that is nearly diagonal.

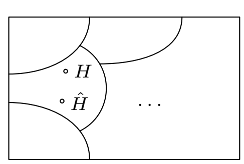
We finish this subsection by considering the easier case where representations are with respect to possibly different starting and ending bases. Recall that the prior subsection shows that a matrix changes bases if and only if it is nonsingular. That gives us another version of the above arrow diagram and equation $(*)$.

2.3 Definition Same-sized matrices H and $\hat{H}$ are *matrix equivalent* if there are nonsingular matrices P and Q such that $\hat{H} = PHQ$.

2.4 Corollary Matrix equivalent matrices represent the same map, with respect to appropriate pairs of bases.

Exercise 19 checks that matrix equivalence is an equivalence relation. Thus it partitions the set of matrices into matrix equivalence classes.

All matrices:



H matrix equivalent to $\hat{H}$

We can get some insight into the classes by comparing matrix equivalence with row equivalence (recall that matrices are row equivalent when they can be reduced to each other by row operations). In $\hat{H} = PHQ$, the matrices P and Q are nonsingular and thus each can be written as a product of elementary reduction matrices (Lemma 4.8). Left-multiplication by the reduction matrices making up P has the effect of performing row operations. Right-multiplication by the reduction matrices making up Q performs column operations. Therefore, matrix equivalence is a generalization of row equivalence — two matrices are row equivalent if one can be converted to the other by a sequence of row reduction steps, while two matrices are matrix equivalent if one can be converted to the other by a sequence of row reduction steps followed by a sequence of column reduction steps.

Thus, if matrices are row equivalent then they are also matrix equivalent (since we can take Q to be the identity matrix and so perform no column operations). The converse, however, does not hold: two matrices can be matrix equivalent but not row equivalent.

2.5 Example These two

$$\begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad \begin{pmatrix}1&1\\0&0\end{pmatrix}$$

are matrix equivalent because the second can be reduced to the first by the column operation of taking -1 times the first column and adding to the second. They are not row equivalent because they have different reduced echelon forms (in fact, both are already in reduced form).

We will close this section by finding a set of representatives for the matrix equivalence classes.*

2.6 Theorem Any $m\times n$ matrix of rank k is matrix equivalent to the $m\times n$ matrix that is all zeros except that the first k diagonal entries are ones.

$$\begin{pmatrix}1&0&\ldots&0&0&\ldots&0\\0&1&\ldots&0&0&\ldots&0\\&\vdots\\0&0&\ldots&1&0&\ldots&0\\0&0&\ldots&0&0&\ldots&0\\&\vdots\\0&0&\ldots&0&0&\ldots&0\end{pmatrix}$$

Sometimes this is described as a *block partial-identity* form.

$$\left(\begin{array}{c|c}I&Z\\\hline Z&Z\end{array}\right)$$

* More information on class representatives is in the appendix.

Proof. As discussed above, Gauss-Jordan reduce the given matrix and combine all the reduction matrices used there to make P. Then use the leading entries to do column reduction and finish by swapping columns to put the leading ones on the diagonal. Combine the reduction matrices used for those column operations into Q. QED

2.7 Example We illustrate the proof by finding the P and Q for this matrix.

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix}$$

First Gauss-Jordan row-reduce.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then column-reduce, which involves right-multiplication.

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finish by swapping columns.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally, combine the left-multipliers together as P and the right-multipliers together as Q to get the PHQ equation.

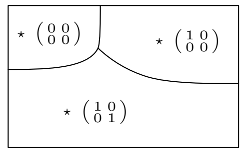
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 2 & 4 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.8 Corollary Two same-sized matrices are matrix equivalent if and only if they have the same rank. That is, the matrix equivalence classes are characterized by rank.

Proof. Two same-sized matrices with the same rank are equivalent to the same block partial-identity matrix. QED

2.9 Example The 2×2 matrices have only three possible ranks: zero, one, or two. Thus there are three matrix-equivalence classes.

All 2×2 matrices:



Three equivalence classes

Each class consists of all of the 2×2 matrices with the same rank. There is only one rank zero matrix, so that class has only one member, but the other two classes each have infinitely many members.

In this subsection we have seen how to change the representation of a map with respect to a first pair of bases to one with respect to a second pair. That led to a definition describing when matrices are equivalent in this way. Finally we noted that, with the proper choice of (possibly different) starting and ending bases, any map can be represented in block partial-identity form.

One of the nice things about this representation is that, in some sense, we can completely understand the map when it is expressed in this way: if the bases are $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ and $D = \langle \vec{\delta}_1, \dots, \vec{\delta}_m \rangle$ then the map sends

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k + c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n \longmapsto c_1\vec{\delta}_1 + \cdots + c_k\vec{\delta}_k + \vec{0} + \cdots + \vec{0}$$

where k is the map's rank. Thus, we can understand any linear map as a kind of projection.

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix}_B \mapsto \begin{pmatrix} c_1 \\ \vdots \\ c_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}_D$$

Of course, "understanding" a map expressed in this way requires that we understand the relationship between B and D. However, despite that difficulty, this is a good classification of linear maps.

Exercises

✓ **2.10** Decide if these matrices are matrix equivalent.

(a) $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 2 & -6 \end{pmatrix}$

✓ **2.11** Find the canonical representative of the matrix-equivalence class of each matrix.

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 4 \\ 3 & 3 & 3 & -1 \end{pmatrix}$

2.12 Suppose that, with respect to

$$B = \mathcal{E}_2 \qquad D = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

the transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ is represented by this matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Use change of basis matrices to represent t with respect to each pair.

(a) $\hat{B} = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$, $\hat{D} = \langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle$

(b) $\hat{B} = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$, $\hat{D} = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle$

✓ **2.13** What sizes are P and Q in the equation $\hat{H} = PHQ$?

✓ **2.14** Use Theorem 2.6 to show that a square matrix is nonsingular if and only if it is equivalent to an identity matrix.

✓ **2.15** Show that, where A is a nonsingular square matrix, if P and Q are nonsingular square matrices such that $PAQ = I$ then $QP = A^{-1}$.

✓ **2.16** Why does Theorem 2.6 not show that every matrix is diagonalizable (see Example 2.2)?

2.17 Must matrix equivalent matrices have matrix equivalent transposes?

2.18 What happens in Theorem 2.6 if $k = 0$?

✓ **2.19** Show that matrix-equivalence is an equivalence relation.

✓ **2.20** Show that a zero matrix is alone in its matrix equivalence class. Are there other matrices like that?

2.21 What are the matrix equivalence classes of matrices of transformations on $\mathbb{R}^1$? $\mathbb{R}^3$?

2.22 How many matrix equivalence classes are there?

2.23 Are matrix equivalence classes closed under scalar multiplication? Addition?

2.24 Let $t\colon \mathbb{R}^n \to \mathbb{R}^n$ represented by T with respect to $\mathcal{E}_n, \mathcal{E}_n$.

(a) Find $\text{Rep}_{B,B}(t)$ in this specific case.

$$T = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \rangle$$

(b) Describe $\text{Rep}_{B,B}(t)$ in the general case where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$.

2.25 **(a)** Let V have bases B_1 and B_2 and suppose that W has the basis D. Where $h\colon V \to W$, find the formula that computes $\text{Rep}_{B_2,D}(h)$ from $\text{Rep}_{B_1,D}(h)$.

(b) Repeat the prior question with one basis for V and two bases for W.

2.26 **(a)** If two matrices are matrix-equivalent and invertible, must their inverses be matrix-equivalent?

(b) If two matrices have matrix-equivalent inverses, must the two be matrix-equivalent?

(c) If two matrices are square and matrix-equivalent, must their squares be matrix-equivalent?

(d) If two matrices are square and have matrix-equivalent squares, must they be matrix-equivalent?

✓ **2.27** Square matrices are *similar* if they represent the same transformation, but each with respect to the same ending as starting basis. That is, $\mathrm{Rep}_{B_1,B_1}(t)$ is similar to $\mathrm{Rep}_{B_2,B_2}(t)$.

(a) Give a definition of matrix similarity like that of Definition 2.3.

(b) Prove that similar matrices are matrix equivalent.

(c) Show that similarity is an equivalence relation.

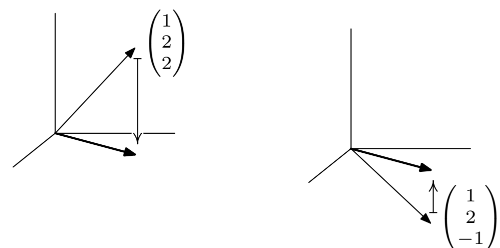
(d) Show that if T is similar to $\hat{T}$ then T^2 is similar to $\hat{T}^2$, the cubes are similar, etc. *Contrast with the prior exercise.*

(e) Prove that there are matrix equivalent matrices that are not similar.

VI Projection

This section is optional; only the last two sections of Chapter Five require this material.

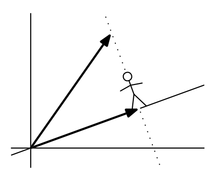
We have described the projection π from $\mathbb{R}^3$ into its xy plane subspace as a 'shadow map'. This shows why, but it also shows that some shadows fall upward.



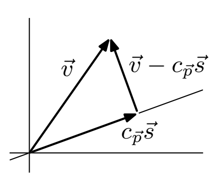
So perhaps a better description is: the projection of $\vec{v}$ is the $\vec{p}$ in the plane with the property that someone standing on $\vec{p}$ and looking straight up or down sees $\vec{v}$. In this section we will generalize this to other projections, both orthogonal (i.e., 'straight up and down') and nonorthogonal.

VI.1 Orthogonal Projection Into a Line

We first consider orthogonal projection into a line. To orthogonally project a vector $\vec{v}$ into a line ℓ, darken a point on the line if someone on that line and looking straight up or down (from that person's point of view) sees $\vec{v}$.



The picture shows someone who has walked out on the line until the tip of $\vec{v}$ is straight overhead. That is, where the line is described as the span of some nonzero vector $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$, the person has walked out to find the coefficient $c_{\vec{p}}$ with the property that $\vec{v} - c_{\vec{p}} \cdot \vec{s}$ is orthogonal to $c_{\vec{p}} \cdot \vec{s}$.



We can solve for this coefficient by noting that because $\vec{v} - c_{\vec{p}}\vec{s}$ is orthogonal to a scalar multiple of $\vec{s}$ it must be orthogonal to $\vec{s}$ itself, and then the consequent fact that the dot product $(\vec{v} - c_{\vec{p}}\vec{s}) \cdot \vec{s}$ is zero gives that $c_{\vec{p}} = \vec{v} \cdot \vec{s} / \vec{s} \cdot \vec{s}$.

1.1 Definition The *orthogonal projection of $\vec{v}$ into the line spanned by a nonzero $\vec{s}$* is this vector.

$$\operatorname{proj}_{[\vec{s}\,]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

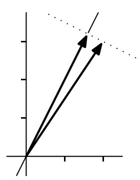
Exercise 19 checks that the outcome of the calculation depends only on the line and not on which vector $\vec{s}$ happens to be used to describe that line.

1.2 Remark The wording of that definition says 'spanned by $\vec{s}$' instead the more formal 'the span of the set $\{\vec{s}\,\}$'. This casual first phrase is common.

1.3 Example To orthogonally project the vector $\binom{2}{3}$ into the line $y = 2x$, we first pick a direction vector for the line. For instance,

$$\vec{s} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

will do. Then the calculation is routine.



$$\frac{\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{8}{5} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8/5 \\ 16/5 \end{pmatrix}$$

1.4 Example In $\mathbb{R}^3$, the orthogonal projection of a general vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

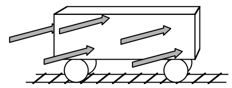
into the y-axis is

$$\frac{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$$

which matches our intuitive expectation.

The picture above with the stick figure walking out on the line until $\vec{v}$'s tip is overhead is one way to think of the orthogonal projection of a vector into a line. We finish this subsection with two other ways.

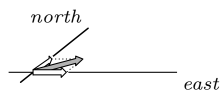
1.5 Example A railroad car left on an east-west track without its brake is pushed by a wind blowing toward the northeast at fifteen miles per hour; what speed will the car reach?



For the wind we use a vector of length 15 that points toward the northeast.

$$\vec{v} = \begin{pmatrix} 15\sqrt{1/2} \\ 15\sqrt{1/2} \end{pmatrix}$$

The car can only be affected by the part of the wind blowing in the east-west direction — the part of $\vec{v}$ in the direction of the x-axis is this (the picture has the same perspective as the railroad car picture above).

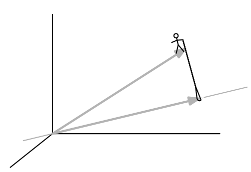


$$\vec{p} = \begin{pmatrix} 15\sqrt{1/2} \\ 0 \end{pmatrix}$$

So the car will reach a velocity of $15\sqrt{1/2}$ miles per hour toward the east.

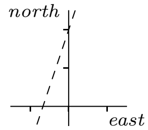
Thus, another way to think of the picture that precedes the definition is that it shows $\vec{v}$ as decomposed into two parts, the part with the line (here, the part with the tracks, $\vec{p}$), and the part that is orthogonal to the line (shown here lying on the north-south axis). These two are "not interacting" or "independent", in the sense that the east-west car is not at all affected by the north-south part of the wind (see Exercise 11). So the orthogonal projection of $\vec{v}$ into the line spanned by $\vec{s}$ can be thought of as the part of $\vec{v}$ that lies in the direction of $\vec{s}$.

Finally, another useful way to think of the orthogonal projection is to have the person stand not on the line, but on the vector that is to be projected to the line. This person has a rope over the line and pulls it tight, naturally making the rope orthogonal to the line.



That is, we can think of the projection $\vec{p}$ as being the vector in the line that is closest to $\vec{v}$ (see Exercise 17).

1.6 Example A submarine is tracking a ship moving along the line $y = 3x + 2$. Torpedo range is one-half mile. Can the sub stay where it is, at the origin on the chart below, or must it move to reach a place where the ship will pass within range?



The formula for projection into a line does not immediately apply because the line doesn't pass through the origin, and so isn't the span of any $\vec{s}$. To adjust for this, we start by shifting the entire map down two units. Now the line is $y = 3x$, which is a subspace, and we can project to get the point $\vec{p}$ of closest approach, the point on the line through the origin closest to

$$\vec{v} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

the sub's shifted position.

$$\vec{p} = \frac{\begin{pmatrix} 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -9/5 \end{pmatrix}$$

The distance between $\vec{v}$ and $\vec{p}$ is approximately 0.63 miles and so the sub must move to get in range.

This subsection has developed a natural projection map: orthogonal projection into a line. As suggested by the examples, it is often called for in applications. The next subsection shows how the definition of orthogonal projection into a line gives us a way to calculate especially convienent bases for vector spaces, again something that is common in applications. The final subsection completely generalizes projection, orthogonal or not, into any subspace at all.

Exercises

✓ **1.7** Project the first vector orthogonally into the line spanned by the second vector.

(a) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 12 \end{pmatrix}$

✓ **1.8** Project the vector orthogonally into the line.

(a) $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \{c \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix} \mid c \in \mathbb{R}\}$ **(b)** $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$, the line $y = 3x$

1.9 Although the development of Definition 1.1 is guided by the pictures, we are not restricted to spaces that we can draw. In $\mathbb{R}^4$ project this vector into this line.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} \qquad \ell = \{c \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

✓ **1.10** Definition 1.1 uses two vectors $\vec{s}$ and $\vec{v}$. Consider the transformation of $\mathbb{R}^2$ resulting from fixing

$$\vec{s} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

and projecting $\vec{v}$ into the line that is the span of $\vec{s}$. Apply it to these vectors.

(a) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

Show that in general the projection tranformation is this.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} (x_1 + 3x_2)/10 \\ (3x_1 + 9x_2)/10 \end{pmatrix}$$

Express the action of this transformation with a matrix.

1.11 Example 1.5 suggests that projection breaks $\vec{v}$ into two parts, $\operatorname{proj}_{[\vec{s}\,]}(\vec{v}\,)$ and $\vec{v} - \operatorname{proj}_{[\vec{s}\,]}(\vec{v}\,)$, that are "not interacting". Recall that the two are orthogonal. Show that any two nonzero orthogonal vectors make up a linearly independent set.

1.12 **(a)** What is the orthogonal projection of $\vec{v}$ into a line if $\vec{v}$ is a member of that line?

(b) Show that if $\vec{v}$ is not a member of the line then the set $\{\vec{v}, \vec{v} - \operatorname{proj}_{[\vec{s}\,]}(\vec{v}\,)\}$ is linearly independent.

1.13 Definition 1.1 requires that $\vec{s}$ be nonzero. Why? What is the right definition of the orthogonal projection of a vector into the (degenerate) line spanned by the zero vector?

1.14 Are all vectors the projection of some other vector into some line?

✓ **1.15** Show that the projection of $\vec{v}$ into the line spanned by $\vec{s}$ has length equal to the absolute value of the number $\vec{v} \cdot \vec{s}$ divided by the length of the vector $\vec{s}$.

1.16 Find the formula for the distance from a point to a line.

1.17 Find the scalar c such that (cs_1, cs_2) is a minimum distance from the point (v_1, v_2) by using calculus (i.e., consider the distance function, set the first derivative equal to zero, and solve). Generalize to $\mathbb{R}^n$.

✓ **1.18** Prove that the orthogonal projection of a vector into a line is shorter than the vector.

✓ **1.19** Show that the definition of orthogonal projection into a line does not depend on the spanning vector: if $\vec{s}$ is a nonzero multiple of $\vec{q}$ then $(\vec{v} \cdot \vec{s}/\vec{s} \cdot \vec{s}\,) \cdot \vec{s}$ equals $(\vec{v} \cdot \vec{q}/\vec{q} \cdot \vec{q}\,) \cdot \vec{q}$.

✓ **1.20** Consider the function mapping to plane to itself that takes a vector to its projection into the line $y = x$. These two each show that the map is linear, the first one in a way that is bound to the coordinates (that is, it fixes a basis and then computes) and the second in a way that is more conceptual.

(a) Produce a matrix that describes the function's action.

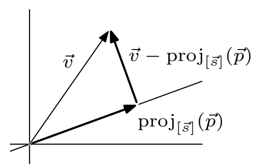
(b) Show also that this map can be obtained by first rotating everything in the plane $\pi/4$ radians clockwise, then projecting into the x-axis, and then rotating $\pi/4$ radians counterclockwise.

1.21 For $\vec{a}, \vec{b} \in \mathbb{R}^n$ let $\vec{v}_1$ be the projection of $\vec{a}$ into the line spanned by $\vec{b}$, let $\vec{v}_2$ be the projection of $\vec{v}_1$ into the line spanned by $\vec{a}$, let $\vec{v}_3$ be the projection of $\vec{v}_2$ into the line spanned by $\vec{b}$, etc., back and forth between the spans of $\vec{a}$ and $\vec{b}$. That is, $\vec{v}_{i+1}$ is the projection of $\vec{v}_i$ into the span of $\vec{a}$ if $i+1$ is even, and into the span of $\vec{b}$ if $i+1$ is odd. Must that sequence of vectors eventually settle down — must there be a sufficiently large i such that $\vec{v}_{i+2}$ equals $\vec{v}_i$ and $\vec{v}_{i+3}$ equals $\vec{v}_{i+1}$? If so, what is the earliest such i?

VI.2 Gram-Schmidt Orthogonalization

This subsection is optional. It requires material from the prior, also optional, subsection. The work done here will only be needed in the final two sections of Chapter Five.

The prior subsection suggests that projecting into the line spanned by $\vec{s}$ decomposes a vector $\vec{v}$ into two parts



$$\vec{v} = \mathrm{proj}_{[\vec{s}\,]}(\vec{v}) + \Big(\vec{v} - \mathrm{proj}_{[\vec{s}\,]}(\vec{v})\Big)$$

that are orthogonal and so are “not interacting”. We will now develop that suggestion.

2.1 Definition Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$ are *mutually orthogonal* when any two are orthogonal: if $i \neq j$ then the dot product $\vec{v}_i \cdot \vec{v}_j$ is zero.

2.2 Theorem If the vectors in a set $\{\vec{v}_1, \ldots, \vec{v}_k\} \subset \mathbb{R}^n$ are mutually orthogonal and nonzero then that set is linearly independent.

Proof. Consider a linear relationship $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$. If $i \in [1..k]$ then taking the dot product of $\vec{v}_i$ with both sides of the equation

$$\begin{aligned} \vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) &= \vec{v}_i \cdot \vec{0} \\ c_i \cdot (\vec{v}_i \cdot \vec{v}_i) &= 0 \end{aligned}$$

shows, since $\vec{v}_i$ is nonzero, that c_i is zero. QED

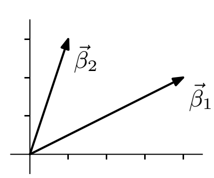
2.3 Corollary If the vectors in a size k subset of a k dimensional space are mutually orthogonal and nonzero then that set is a basis for the space.

Proof. Any linearly independent size k subset of a k dimensional space is a basis. QED

Of course, the converse of Corollary 2.3 does not hold — not every basis of every subspace of $\mathbb{R}^n$ is made of mutually orthogonal vectors. However, we can get the partial converse that for every subspace of $\mathbb{R}^n$ there is at least one basis consisting of mutually orthogonal vectors.

2.4 Example The members $\vec{\beta}_1$ and $\vec{\beta}_2$ of this basis for $\mathbb{R}^2$ are not orthogonal.

$$B = \langle \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rangle$$

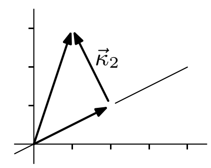


However, we can derive from B a new basis for the same space that does have mutually orthogonal members. For the first member of the new basis we simply use $\vec{\beta}_1$.

$$\vec{\kappa}_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

For the second member of the new basis, we take away from $\vec{\beta}_2$ its part in the direction of $\vec{\kappa}_1$,

$$\vec{\kappa}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \mathrm{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix} 1 \\ 3 \end{pmatrix}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$



which leaves the part, $\vec{\kappa}_2$ pictured above, of $\vec{\beta}_2$ that is orthogonal to $\vec{\kappa}_1$ (it is orthogonal by the definition of the projection into the span of $\vec{\kappa}_1$). Note that, by the corollary, $\{\vec{\kappa}_1, \vec{\kappa}_2\}$ is a basis for $\mathbb{R}^2$.

2.5 Definition An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

The next result gives a way to produce an orthogonal basis from any given starting basis. We first see an example.

2.6 Example To turn this basis for $\mathbb{R}^3$

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \rangle$$

into an orthogonal basis, we take the first vector as it is given.

$$\vec{\kappa}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We get $\vec{\kappa}_2$ by starting with the given second vector $\vec{\beta}_2$ and subtracting away the part of it in the direction of $\vec{\kappa}_1$.

$$\vec{\kappa}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \mathrm{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2/3 \\ 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix}$$

Finally, we get $\vec{\kappa}_3$ by taking the third given vector and subtracting the part of it in the direction of $\vec{\kappa}_1$, and also the part of it in the direction of $\vec{\kappa}_2$.

$$\vec{\kappa}_3 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \mathrm{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}) - \mathrm{proj}_{[\vec{\kappa}_2]}(\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Again the corollary gives that

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 4/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rangle$$

is a basis for the space.

The next result verifies that the process used in those examples works with any basis for any subspace of an $\mathbb{R}^n$ (we are restricted to $\mathbb{R}^n$ only because we have not given a definition of orthogonality for other vector spaces).

2.7 Theorem (Gram-Schmidt orthogonalization) If $\langle \vec{\beta}_1, \ldots \vec{\beta}_k \rangle$ is a basis for a subspace of $\mathbb{R}^n$ then, where

$$\begin{aligned}
\vec{\kappa}_1 &= \vec{\beta}_1 \\
\vec{\kappa}_2 &= \vec{\beta}_2 - \operatorname{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) \\
\vec{\kappa}_3 &= \vec{\beta}_3 - \operatorname{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \operatorname{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\
&\vdots \\
\vec{\kappa}_k &= \vec{\beta}_k - \operatorname{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_k) - \cdots - \operatorname{proj}_{[\vec{\kappa}_{k-1}]}(\vec{\beta}_k)
\end{aligned}$$

the $\vec{\kappa}$'s form an orthogonal basis for the same subspace.

Proof. We will use induction to check that each $\vec{\kappa}_i$ is nonzero, is in the span of $\langle \vec{\beta}_1, \ldots \vec{\beta}_i \rangle$ and is orthogonal to all preceding vectors: $\vec{\kappa}_1 \cdot \vec{\kappa}_i = \cdots = \vec{\kappa}_{i-1} \cdot \vec{\kappa}_i = 0$. With those, and with Corollary 2.3, we will have that $\langle \vec{\kappa}_1, \ldots \vec{\kappa}_k \rangle$ is a basis for the same space as $\langle \vec{\beta}_1, \ldots \vec{\beta}_k \rangle$.

We shall cover the cases up to $i = 3$, which give the sense of the argument. Completing the details is Exercise 23.

The $i = 1$ case is trivial — setting $\vec{\kappa}_1$ equal to $\vec{\beta}_1$ makes it a nonzero vector since $\vec{\beta}_1$ is a member of a basis, it is obviously in the desired span, and the 'orthogonal to all preceding vectors' condition is vacuously met.

For the $i = 2$ case, expand the definition of $\vec{\kappa}_2$.

$$\vec{\kappa}_2 = \vec{\beta}_2 - \operatorname{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1$$

This expansion shows that $\vec{\kappa}_2$ is nonzero or else this would be a non-trivial linear dependence among the $\vec{\beta}$'s (it is nontrivial because the coefficient of $\vec{\beta}_2$ is 1) and also shows that $\vec{\kappa}_2$ is in the desired span. Finally, $\vec{\kappa}_2$ is orthogonal to the only preceding vector

$$\vec{\kappa}_1 \cdot \vec{\kappa}_2 = \vec{\kappa}_1 \cdot (\vec{\beta}_2 - \operatorname{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)) = 0$$

because this projection is orthogonal.

The $i = 3$ case is the same as the $i = 2$ case except for one detail. As in the $i = 2$ case, expanding the definition

$$\begin{aligned}\vec{\kappa}_3 &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \vec{\kappa}_2 \\ &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \bigl(\vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1\bigr)\end{aligned}$$

shows that $\vec{\kappa}_3$ is nonzero and is in the span. A calculation shows that $\vec{\kappa}_3$ is orthogonal to the preceding vector $\vec{\kappa}_1$.

$$\begin{aligned}\vec{\kappa}_1 \cdot \vec{\kappa}_3 &= \vec{\kappa}_1 \cdot \bigl(\vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)\bigr) \\ &= \vec{\kappa}_1 \cdot \bigl(\vec{\beta}_3 - \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3)\bigr) - \vec{\kappa}_1 \cdot \mathrm{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\ &= 0\end{aligned}$$

(Here's the difference from the $i = 2$ case — the second line has two kinds of terms. The first term is zero because this projection is orthogonal, as in the $i = 2$ case. The second term is zero because $\vec{\kappa}_1$ is orthogonal to $\vec{\kappa}_2$ and so is orthogonal to any vector in the line spanned by $\vec{\kappa}_2$.) The check that $\vec{\kappa}_3$ is also orthogonal to the other preceding vector $\vec{\kappa}_2$ is similar. QED

Beyond having the vectors in the basis be orthogonal, we can do more; we can arrange for each vector to have length one by dividing each by its own length (we can *normalize* the lengths).

2.8 Example Normalizing the length of each vector in the orthogonal basis of Example 2.6 produces this *orthonormal basis*.

$$\langle \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \rangle$$

Besides its intuitive appeal, and its analogy with the standard basis $\mathcal{E}_n$ for $\mathbb{R}^n$, an orthonormal basis also simplifies some computations. See Exercise 17, for example.

Exercises

2.9 Perform the Gram-Schmidt process on each of these bases for $\mathbb{R}^2$.

(a) $\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle$ **(b)** $\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \rangle$ **(c)** $\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \rangle$

Then turn those orthogonal bases into orthonormal bases.

✓ **2.10** Perform the Gram-Schmidt process on each of these bases for $\mathbb{R}^3$.

(a) $\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \rangle$ **(b)** $\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \rangle$

Then turn those orthogonal bases into orthonormal bases.

✓ **2.11** Find an orthonormal basis for this subspace of $\mathbb{R}^3$: the plane $x - y + z = 0$.

2.12 Find an orthonormal basis for this subspace of $\mathbb{R}^4$.

$$\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \mid x - y - z + w = 0 \text{ and } x + z = 0 \}$$

2.13 Show that any linearly independent subset of $\mathbb{R}^n$ can be orthogonalized without changing its span.

✓ **2.14** What happens if we apply the Gram-Schmidt process to a basis that is already orthogonal?

2.15 Let $\langle \vec{\kappa}_1, \ldots, \vec{\kappa}_k \rangle$ be a set of mutually orthogonal vectors in $\mathbb{R}^n$.

(a) Prove that for any $\vec{v}$ in the space, the vector $\vec{v} - (\text{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \text{proj}_{[\vec{v}_k]}(\vec{v}\,))$ is orthogonal to each of $\vec{\kappa}_1, \ldots, \vec{\kappa}_k$.

(b) Illustrate the prior item in $\mathbb{R}^3$ by using $\vec{e}_1$ as $\vec{\kappa}_1$, using $\vec{e}_2$ as $\vec{\kappa}_2$, and taking $\vec{v}$ to have components 1, 2, and 3.

(c) Show that $\text{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \text{proj}_{[\vec{v}_k]}(\vec{v}\,)$ is the vector in the span of the set of $\vec{\kappa}$'s that is closest to $\vec{v}$. *Hint.* To the illustration done for the prior part, add a vector $d_1\vec{\kappa}_1 + d_2\vec{\kappa}_2$ and apply the Pythagorean Theorem to the resulting triangle.

2.16 Find a vector in $\mathbb{R}^3$ that is orthogonal to both of these.

$$\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

✓ **2.17** One advantage of orthogonal bases is that they simplify finding the representation of a vector with respect to that basis.

(a) For this vector and this non-orthogonal basis for $\mathbb{R}^2$

$$\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$$

first represent the vector with respect to the basis. Then project the vector into the span of each basis vector $[\vec{\beta}_1]$ and $[\vec{\beta}_2]$.

(b) With this orthogonal basis for $\mathbb{R}^2$

$$K = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle$$

represent the same vector $\vec{v}$ with respect to the basis. Then project the vector into the span of each basis vector. Note that the coefficients in the representation and the projection are the same.

(c) Let $K = \langle \vec{\kappa}_1, \ldots, \vec{\kappa}_k \rangle$ be an orthogonal basis for some subspace of $\mathbb{R}^n$. Prove that for any $\vec{v}$ in the subspace, the i-th component of the representation $\text{Rep}_K(\vec{v}\,)$ is the scalar coefficient $(\vec{v} \cdot \vec{\kappa}_i)/(\vec{\kappa}_i \cdot \vec{\kappa}_i)$ from $\text{proj}_{[\vec{\kappa}_i]}(\vec{v}\,)$.

(d) Prove that $\vec{v} = \text{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \text{proj}_{[\vec{\kappa}_k]}(\vec{v}\,)$.

2.18 *Bessel's Inequality.* Consider these orthonormal sets

$$B_1 = \{\vec{e}_1\} \quad B_2 = \{\vec{e}_1, \vec{e}_2\} \quad B_3 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \quad B_4 = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$$

along with the vector $\vec{v} \in \mathbb{R}^4$ whose components are 4, 3, 2, and 1.

(a) Find the coefficient c_1 for the projection of $\vec{v}$ into the span of the vector in B_1. Check that $\|\vec{v}\,\|^2 \geq |c_1|^2$.

(b) Find the coefficients c_1 and c_2 for the projection of $\vec{v}$ into the spans of the two vectors in B_2. Check that $\|\vec{v}\|^2 \geq |c_1|^2 + |c_2|^2$.

(c) Find c_1, c_2, and c_3 associated with the vectors in B_3, and c_1, c_2, c_3, and c_4 for the vectors in B_4. Check that $\|\vec{v}\|^2 \geq |c_1|^2 + \cdots + |c_3|^2$ and that $\|\vec{v}\|^2 \geq |c_1|^2 + \cdots + |c_4|^2$.

Show that this holds in general: where $\{\vec{\kappa}_1, \ldots, \vec{\kappa}_k\}$ is an orthonormal set and c_i is coefficient of the projection of a vector $\vec{v}$ from the space then $\|\vec{v}\|^2 \geq |c_1|^2 + \cdots + |c_k|^2$. *Hint.* One way is to look at the inequality $0 \leq \|\vec{v} - (c_1\vec{\kappa}_1 + \cdots + c_k\vec{\kappa}_k)\|^2$ and expand the c's.

2.19 Prove or disprove: every vector in $\mathbb{R}^n$ is in some orthogonal basis.

2.20 Show that the columns of an $n\times n$ matrix form an orthonormal set if and only if the inverse of the matrix is its transpose. Produce such a matrix.

2.21 Does the proof of Theorem 2.2 fail to consider the possibility that the set of vectors is empty (i.e., that $k = 0$)?

2.22 Theorem 2.7 describes a change of basis from any basis $B = \langle\vec{\beta}_1, \ldots, \vec{\beta}_k\rangle$ to one that is orthogonal $K = \langle\vec{\kappa}_1, \ldots, \vec{\kappa}_k\rangle$. Consider the change of basis matrix $\mathrm{Rep}_{B,K}(\mathrm{id})$.

(a) Prove that the matrix $\mathrm{Rep}_{K,B}(\mathrm{id})$ changing bases in the direction opposite to that of the theorem has an upper triangular shape — all of its entries below the main diagonal are zeros.

(b) Prove that the inverse of an upper triangular matrix is also upper triangular (if the matrix is invertible, that is). This shows that the matrix $\mathrm{Rep}_{B,K}(\mathrm{id})$ changing bases in the direction described in the theorem is upper triangular.

2.23 Complete the induction argument in the proof of Theorem 2.7.

VI.3 Projection Into a Subspace

This subsection, like the others in this section, is optional. It also requires material from the optional earlier subsection on Combining Subspaces.

The prior subsections project a vector into a line by decomposing it into two parts: the part in the line $\mathrm{proj}_{[\vec{s}\,]}(\vec{v}\,)$ and the rest $\vec{v} - \mathrm{proj}_{[\vec{s}\,]}(\vec{v}\,)$. To generalize projection to arbitrary subspaces, we follow this idea.

3.1 Definition For any direct sum $V = M \oplus N$ and any $\vec{v} \in V$, the *projection of $\vec{v}$ into M along N* is

$$\mathrm{proj}_{M,N}(\vec{v}\,) = \vec{m}$$

where $\vec{v} = \vec{m} + \vec{n}$ with $\vec{m} \in M$, $\vec{n} \in N$.

This definition doesn't involve a sense of 'orthogonal' so we can apply it to spaces other than subspaces of an $\mathbb{R}^n$. (Definitions of orthogonality for other spaces are perfectly possible, but we haven't seen any in this book.)

3.2 Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is the direct sum of these two.

$$M = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R}\} \qquad N = \{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in \mathbb{R}\}$$

To project

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$$

into M along N, we first fix bases for the two subspaces.

$$B_M = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle \qquad B_N = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

The concatenation of these

$$B = B_M {}^\frown B_N = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle$$

is a basis for the entire space, because the space is the direct sum, so we can use it to represent A.

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now the projection of A into M along N is found by keeping the M part of this sum and dropping the N part.

$$\mathrm{proj}_{M,N}(\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}) = 3 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$$

3.3 Example Both subscripts on $\mathrm{proj}_{M,N}(\vec{v}\,)$ are significant. The first subscript M matters because the result of the projection is an $\vec{m} \in M$, and changing this subspace would change the possible results. For an example showing that the second subscript matters, fix this plane subspace of $\mathbb{R}^3$ and its basis

$$M = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| y - 2z = 0 \} \qquad B_M = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \rangle$$

and compare the projections along two different subspaces.

$$N = \{ k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bigm| k \in \mathbb{R} \} \qquad \hat{N} = \{ k \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \bigm| k \in \mathbb{R} \}$$

(Verification that $\mathbb{R}^3 = M \oplus N$ and $\mathbb{R}^3 = M \oplus \hat{N}$ is routine.) We will check that these projections are different by checking that they have different effects on this vector.

$$\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

For the first one we find a basis for N

$$B_N = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

and represent $\vec{v}$ with respect to the concatenation $B_M \frown B_N$.

$$\begin{pmatrix}2\\2\\5\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} + 4\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

The projection of $\vec{v}$ into M along N is found by keeping the M part and dropping the N part.

$$\mathrm{proj}_{M,N}(\vec{v}\,) = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

For the other subspace $\hat{N}$, this basis is natural.

$$B_{\hat{N}} = \langle\begin{pmatrix}0\\1\\-2\end{pmatrix}\rangle$$

Representing $\vec{v}$ with respect to the concatenation

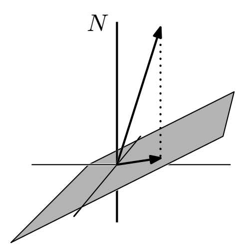
$$\begin{pmatrix}2\\2\\5\end{pmatrix} = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + (9/5)\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} - (8/5)\cdot\begin{pmatrix}0\\1\\-2\end{pmatrix}$$

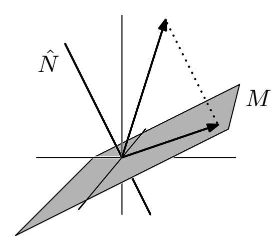
and then keeping only the M part gives this.

$$\mathrm{proj}_{M,\hat{N}}(\vec{v}\,) = 2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + (9/5)\cdot\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}2\\18/5\\9/5\end{pmatrix}$$

Therefore projection along different subspaces may yield different results.

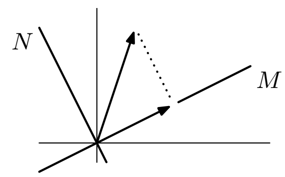
These pictures compare the two maps. Both show that the projection is indeed 'into' the plane and 'along' the line.





Notice that the projection along N is not orthogonal — there are members of the plane M that are not orthogonal to the dotted line. But the projection along $\hat{N}$ is orthogonal.

A natural question is: what is the relationship between the projection operation defined above, and the operation of orthogonal projection into a line? The second picture above suggests the answer — orthogonal projection into a line is a special case of the projection defined above; it is just projection along a subspace perpendicular to the line.



In addition to pointing out that projection along a subspace is a generalization, this scheme shows how to define orthogonal projection into any subspace of $\mathbb{R}^n$, of any dimension.

3.4 Definition The *orthogonal complement* of a subspace M of $\mathbb{R}^n$ is

$$M^{\perp} = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \text{ is perpendicular to all vectors in } M\}$$

(read "M perp"). The *orthogonal projection* $\mathrm{proj}_M(\vec{v}\,)$ of a vector is its projection into M along $M^{\perp}$.

3.5 Example In $\mathbb{R}^3$, to find the orthogonal complement of the plane

$$P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 3x + 2y - z = 0\}$$

we start with a basis for P.

$$B = \langle \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \rangle$$

Any $\vec{v}$ perpendicular to every vector in B is perpendicular to every vector in the span of B (the proof of this assertion is Exercise 19). Therefore, the subspace $P^{\perp}$ consists of the vectors that satisfy these two conditions.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \qquad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

We can express those conditions more compactly as a linear system.

$$P^{\perp} = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

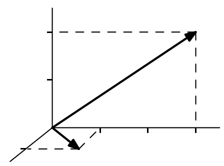
We are thus left with finding the nullspace of the map represented by the matrix, that is, with calculating the solution set of a homogeneous linear system.

$$P^{\perp} = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mid \begin{array}{rl} v_1 \phantom{{}+v_2} + 3v_3 &= 0 \\ v_2 + 2v_3 &= 0 \end{array}\} = \{k \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \mid k \in \mathbb{R}\}$$

Instead of the term orthogonal complement, in some contexts this is called the line *normal* to the plane.

3.6 Example Where M is the xy-plane subspace of $\mathbb{R}^3$, what is $M^\perp$? A common first reaction is that $M^\perp$ is the yz-plane, but that's not right. Some vectors from the yz-plane are not perpendicular to every vector in the xy-plane.

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \not\perp \begin{pmatrix}0\\3\\2\end{pmatrix}$$



$$\theta = \arccos(\frac{1\cdot 0+1\cdot 3+0\cdot 2}{\sqrt{2}\cdot\sqrt{13}}) \approx 0.94 \text{ rad}$$

Instead $M^\perp$ is the z-axis, since proceeding as in the prior example and taking the natural basis for the xy-plane gives this.

$$M^\perp = \{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\} = \{\begin{pmatrix}x\\y\\z\end{pmatrix} \mid x=0 \text{ and } y=0\}$$

The two examples that we've seen since Definition 3.4 illustrate the first sentence in that definition. The next result justifies the second sentence.

3.7 Lemma Let M be a subspace of $\mathbb{R}^n$. The orthogonal complement of M is also a subspace. The space is the direct sum of the two $\mathbb{R}^n = M \oplus M^\perp$. And, for any $\vec{v} \in \mathbb{R}^n$, the vector $\vec{v} - \mbox{proj}_M(\vec{v}\,)$ is perpendicular to every vector in M.

Proof. First, the orthogonal complement $M^\perp$ is a subspace of $\mathbb{R}^n$ because, as noted in the prior two examples, it is a nullspace.

Next, we can start with any basis $B_M = \langle\vec{\mu}_1,\ldots,\vec{\mu}_k\rangle$ for M and expand it to a basis for the entire space. Apply the Gram-Schmidt process to get an orthogonal basis $K = \langle\vec{\kappa}_1,\ldots,\vec{\kappa}_n\rangle$ for $\mathbb{R}^n$. This K is the concatenation of two bases $\langle\vec{\kappa}_1,\ldots,\vec{\kappa}_k\rangle$ (with the same number of members as B_M) and $\langle\vec{\kappa}_{k+1},\ldots,\vec{\kappa}_n\rangle$. The first is a basis for M, so if we show that the second is a basis for $M^\perp$ then we will have that the entire space is the direct sum of the two subspaces.

Exercise 17 from the prior subsection proves this about any orthogonal basis: each vector $\vec{v}$ in the space is the sum of its orthogonal projections onto the lines spanned by the basis vectors.

$$\vec{v} = \mbox{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \mbox{proj}_{[\vec{\kappa}_n]}(\vec{v}\,) \qquad (*)$$

To check this, represent the vector $\vec{v} = r_1\vec{\kappa}_1 + \cdots + r_n\vec{\kappa}_n$, apply $\vec{\kappa}_i$ to both sides $\vec{v}\cdot\vec{\kappa}_i = (r_1\vec{\kappa}_1+\cdots+r_n\vec{\kappa}_n)\cdot\vec{\kappa}_i = r_1\cdot 0+\cdots+r_i\cdot(\vec{\kappa}_i\cdot\vec{\kappa}_i)+\cdots+r_n\cdot 0$, and solve to get $r_i = (\vec{v}\cdot\vec{\kappa}_i)/(\vec{\kappa}_i\cdot\vec{\kappa}_i)$, as desired.

Since obviously any member of the span of $\langle\vec{\kappa}_{k+1},\ldots,\vec{\kappa}_n\rangle$ is orthogonal to any vector in M, to show that this is a basis for $M^\perp$ we need only show the other containment — that any $\vec{w} \in M^\perp$ is in the span of this basis. The prior paragraph does this. On projections into basis vectors from M, any $\vec{w} \in M^\perp$ gives $\mbox{proj}_{[\vec{\kappa}_1]}(\vec{w}\,) = \vec{0},\ldots,$ $\mbox{proj}_{[\vec{\kappa}_k]}(\vec{w}\,) = \vec{0}$ and therefore $(*)$ gives that $\vec{w}$ is a linear combination of $\vec{\kappa}_{k+1},\ldots,\vec{\kappa}_n$. Thus this is a basis for $M^\perp$ and $\mathbb{R}^n$ is the direct sum of the two.

The final sentence is proved in much the same way. Write $\vec{v} = \mathrm{proj}_{[\vec{\kappa}_1]}(\vec{v}\,) + \cdots + \mathrm{proj}_{[\vec{\kappa}_n]}(\vec{v}\,)$. Then $\mathrm{proj}_M(\vec{v}\,)$ is gotten by keeping only the M part and dropping the $M^\perp$ part $\mathrm{proj}_M(\vec{v}\,) = \mathrm{proj}_{[\vec{\kappa}_{k+1}]}(\vec{v}\,) + \cdots + \mathrm{proj}_{[\vec{\kappa}_k]}(\vec{v}\,)$. Therefore $\vec{v} - \mathrm{proj}_M(\vec{v}\,)$ consists of a linear combination of elements of $M^\perp$ and so is perpendicular to every vector in M. QED

We can find the orthogonal projection into a subspace by following the steps of the proof, but the next result gives a formula.

3.8 Theorem Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and let M be a subspace of $\mathbb{R}^n$ with basis $\langle\vec{\beta}_1, \ldots, \vec{\beta}_k\rangle$. If A is the matrix whose columns are the $\vec{\beta}$'s then $\mathrm{proj}_M(\vec{v}\,) = c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k$ where the coefficients c_i are the entries of the vector $(A^{\mathrm{trans}}A)^{-1}A^{\mathrm{trans}} \cdot \vec{v}$. That is, $\mathrm{proj}_M(\vec{v}\,) = A(A^{\mathrm{trans}}A)^{-1}A^{\mathrm{trans}} \cdot \vec{v}$.

Proof. The vector $\mathrm{proj}_M(\vec{v})$ is a member of M and so it is a linear combination of basis vectors $c_1 \cdot \vec{\beta}_1 + \cdots + c_k \cdot \vec{\beta}_k$. Since A's columns are the $\vec{\beta}$'s, that can be expressed as: there is a $\vec{c} \in \mathbb{R}^k$ such that $\mathrm{proj}_M(\vec{v}\,) = A\vec{c}$ (this is expressed compactly with matrix multiplication as in Example 3.5 and 3.6). Because $\vec{v} - \mathrm{proj}_M(\vec{v}\,)$ is perpendicular to each member of the basis, we have this (again, expressed compactly).

$$\vec{0} = A^{\mathrm{trans}}\bigl(\vec{v} - A\vec{c}\bigr) = A^{\mathrm{trans}}\vec{v} - A^{\mathrm{trans}}A\vec{c}$$

Solving for $\vec{c}$ (showing that $A^{\mathrm{trans}}A$ is invertible is an exercise)

$$\vec{c} = \bigl(A^{\mathrm{trans}}A\bigr)^{-1}A^{\mathrm{trans}} \cdot \vec{v}$$

gives the formula for the projection matrix as $\mathrm{proj}_M(\vec{v}\,) = A \cdot \vec{c}$. QED

3.9 Example To orthogonally project this vector into this subspace

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad P = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| x + z = 0\}$$

first make a matrix whose columns are a basis for the subspace

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and then compute.

$$\begin{aligned} A\bigl(A^{\mathrm{trans}}A\bigr)^{-1}A^{\mathrm{trans}} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \end{aligned}$$

With the matrix, calculating the orthogonal projection of any vector into P is easy.

$$\operatorname{proj}_P(\vec{v}) = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

Note, as a check, that this result is indeed in P.

Exercises

✓ **3.10** Project the vectors into M along N.

(a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\quad M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 0\}$, $\quad N = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid -x - 2y = 0\}$

(b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\quad M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x - y = 0\}$, $\quad N = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid 2x + y = 0\}$

(c) $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\quad M = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y = 0\}$, $\quad N = \{c \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$

✓ **3.11** Find $M^\perp$.

(a) $M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 0\}$ **(b)** $M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid -2x + 3y = 0\}$

(c) $M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x - y = 0\}$ **(d)** $M = \{\vec{0}\}$ **(e)** $M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x = 0\}$

(f) $M = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid -x + 3y + z = 0\}$ **(g)** $M = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 0 \text{ and } y + z = 0\}$

3.12 This subsection shows how to project orthogonally in two ways, the method of Example 3.2 and 3.3, and the method of Theorem 3.8. To compare them, consider the plane P specified by $3x + 2y - z = 0$ in $\mathbb{R}^3$.

(a) Find a basis for P.

(b) Find $P^\perp$ and a basis for $P^\perp$.

(c) Represent this vector with respect to the concatenation of the two bases from the prior item.

$$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

(d) Find the orthogonal projection of $\vec{v}$ into P by keeping only the P part from the prior item.

(e) Check that against the result from applying Theorem 3.8.

✓ **3.13** We have three ways to find the orthogonal projection of a vector into a line, the Definition 1.1 way from the first subsection of this section, the Example 3.2 and 3.3 way of representing the vector with respect to a basis for the space and then keeping the M part, and the way of Theorem 3.8. For these cases, do all three ways.

(a) $\vec{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\quad M = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x + y = 0\}$

(b) $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\quad M = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + z = 0 \text{ and } y = 0\}$

3.14 Check that the operation of Definition 3.1 is well-defined. That is, in Example 3.2 and 3.3, doesn't the answer depend on the choice of bases?

3.15 What is the orthogonal projection into the trivial subspace?

3.16 What is the projection of $\vec{v}$ into M along N if $\vec{v} \in M$?

3.17 Show that if $M \subseteq \mathbb{R}^n$ is a subspace with orthonormal basis $\langle \vec{\kappa}_1, \ldots, \vec{\kappa}_n \rangle$ then the orthogonal projection of $\vec{v}$ into M is this.

$$(\vec{v} \cdot \vec{\kappa}_1) \cdot \vec{\kappa}_1 + \cdots + (\vec{v} \cdot \vec{\kappa}_n) \cdot \vec{\kappa}_n$$

✓ **3.18** Prove that the map $p\colon V \to V$ is the projection into M along N if and only if the map $\mathrm{id} - p$ is the projection into N along M. (Recall the definition of the difference of two maps: $(\mathrm{id} - p)\,(\vec{v}) = \mathrm{id}(\vec{v}) - p(\vec{v}) = \vec{v} - p(\vec{v})$.)

✓ **3.19** Show that if a vector is perpendicular to every vector in a set then it is perpendicular to every vector in the span of that set.

3.20 True or false: the intersection of a subspace and its orthogonal complement is trivial.

3.21 Show that the dimensions of orthogonal complements add to the dimension of the entire space.

✓ **3.22** Suppose that $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ are such that for all complements $M, N \subseteq \mathbb{R}^n$, the projections of $\vec{v}_1$ and $\vec{v}_2$ into M along N are equal. Must $\vec{v}_1$ equal $\vec{v}_2$? (If so, what if we relax the condition to: all orthogonal projections of the two are equal?)

✓ **3.23** Let M, N be subspaces of $\mathbb{R}^n$. The perp operator acts on subspaces; we can ask how it interacts with other such operations.

(a) Show that two perps cancel: $(M^{\perp})^{\perp} = M$.
(b) Prove that $M \subseteq N$ implies that $N^{\perp} \subseteq M^{\perp}$.
(c) Show that $(M + N)^{\perp} = M^{\perp} \cap N^{\perp}$.

✓ **3.24** The material in this subsection allows us to express a geometric relationship that we have not yet seen between the rangespace and the nullspace of a linear map.

(a) Represent $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto 1v_1 + 2v_2 + 3v_3$$

with respect to the standard bases and show that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is a member of the perp of the nullspace. Prove that $\mathscr{N}(f)^{\perp}$ is equal to the span of this vector.

(b) Generalize that to apply to any $f\colon \mathbb{R}^n \to \mathbb{R}$.

(c) Represent $f\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \begin{pmatrix} 1v_1 + 2v_2 + 3v_3 \\ 4v_1 + 5v_2 + 6v_3 \end{pmatrix}$$

with respect to the standard bases and show that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

are both members of the perp of the nullspace. Prove that $\mathscr{N}(f)^{\perp}$ is the span of these two. (*Hint.* See the third item of Exercise 23.)

(d) Generalize that to apply to any $f\colon \mathbb{R}^n \to \mathbb{R}^m$.

This, and related results, is called the *Fundamental Theorem of Linear Algebra* in [Strang 93].

3.25 Define a *projection* to be a linear transformation $t\colon V \to V$ with the property that repeating the projection does nothing more than does the projection alone: $(t \circ t)\,(\vec{v}) = t(\vec{v})$ for all $\vec{v} \in V$.

(a) Show that orthogonal projection into a line has that property.

(b) Show that projection along a subspace has that property.

(c) Show that for any such t there is a basis $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ for V such that

$$t(\vec{\beta}_i) = \begin{cases} \vec{\beta}_i & i = 1, 2, \dots, r \\ \vec{0} & i = r+1, r+2, \dots, n \end{cases}$$

where r is the rank of t.

(d) Conclude that every projection is a projection along a subspace.

(e) Also conclude that every projection has a representation

$$\text{Rep}_{B,B}(t) = \left(\begin{array}{c|c} I & Z \\ \hline Z & Z \end{array}\right)$$

in block partial-identity form.

3.26 A square matrix is *symmetric* if each i, j entry equals the j, i entry (i.e., if the matrix equals its transpose). Show that the projection matrix $A(A^{\text{trans}}A)^{-1}A^{\text{trans}}$ is symmetric. [Strang 80] *Hint.* Find properties of transposes by looking in the index under ‘transpose’.

Topic: Line of Best Fit

This Topic requires the formulas from the subsections on Orthogonal Projection Into a Line, and Projection Into a Subspace.

Scientists are often presented with a system that has no solution and they must find an answer anyway. More precisely stated, they must find a best answer.

For instance, this is the result of flipping a penny, including some intermediate numbers.

number of flips	30	60	90
number of heads	16	34	51

In an experiment we can expect that samples will vary — here, sometimes the experimental ratio of heads to flips overestimates this penny's long-term ratio and sometimes it underestimates. So we expect that the system derived from the experiment has no solution.

$$\begin{aligned} 30m &= 16 \\ 60m &= 34 \\ 90m &= 51 \end{aligned}$$

That is, the vector of experimental data is not in the subspace of solutions.

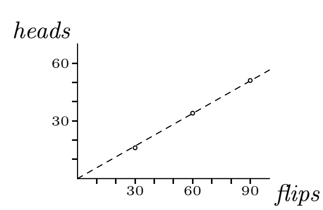
$$\begin{pmatrix} 16 \\ 34 \\ 51 \end{pmatrix} \notin \{m \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix} \bigm| m \in \mathbb{R}\}$$

However, we want to find the m that most nearly works. An orthogonal projection of the data vector into the line subspace gives our best guess.

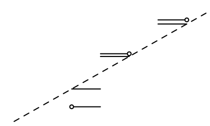
$$\frac{\begin{pmatrix} 16 \\ 34 \\ 51 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix}}{\begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix}} \cdot \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix} = \frac{7110}{12600} \cdot \begin{pmatrix} 30 \\ 60 \\ 90 \end{pmatrix}$$

The estimate ($m = 7110/12600 \approx 0.56$) is a bit high but not much, so probably the penny is fair enough.

The line with the slope $m \approx 0.56$ is the *line of best fit* for this data.



Minimizing the distance between the given vector and the vector used as the right-hand side minimizes the total of these vertical lengths, and consequently we say that the line has been obtained through *fitting by least-squares*



(the vertical scale here has been exaggerated ten times to make the lengths visible).

We arranged the equation above so that the line must pass through $(0, 0)$ because we take it to be the line whose slope is this coin's true proportion of heads to flips. We can also handle cases where the line need not pass through the origin.

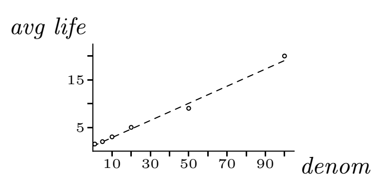
For example, the different denominations of U.S. money have different average times in circulation (the \$2 bill is left off as a special case). How long should we expect a \$25 bill to last?

denomination	1	5	10	20	50	100
average life (years)	1.5	2	3	5	9	20

The plot (see below) looks roughly linear. It isn't a perfect line, i.e., the linear system with equations $b + 1m = 1.5$, $\ldots$, $b + 100m = 20$ has no solution, but we can again use orthogonal projection to find a best approximation. Consider the matrix of coefficients of that linear system and also its vector of constants, the experimentally-determined values.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 5 \\ 1 & 10 \\ 1 & 20 \\ 1 & 50 \\ 1 & 100 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1.5 \\ 2 \\ 3 \\ 5 \\ 9 \\ 20 \end{pmatrix}$$

The ending result in the subsection on Projection into a Subspace says that coefficients b and m so that the linear combination of the columns of A is as close as possible to the vector $\vec{v}$ are the entries of $(A^{\text{trans}}A)^{-1}A^{\text{trans}} \cdot \vec{v}$. Some calculation gives an intercept of $b = 1.05$ and a slope of $m = 0.18$.



Plugging $x = 25$ into the equation of the line shows that such a bill should last between five and six years.

We close by considering the progression of world record times for the men's mile race.[Oakley & Baker] In the early 1900's many people wondered when this record would fall below the four minute mark. Here are the times that were in force on January first of each decade through the first half of that century. (Restricting ourselves to the times at the start of each decade reduces the data entry burden and gives much the same result. There are a number of different sequences of times from competing standards bodies but these are from [WikipediaMensMile].)

year	1870	1880	1890	1900	1910	1920	1930	1940	1950
secs	268.8	264.5	258.4	255.6	255.6	252.6	250.4	246.4	241.4

We can use this data to predict the date for 240 seconds, and we can then compare to the actual date.

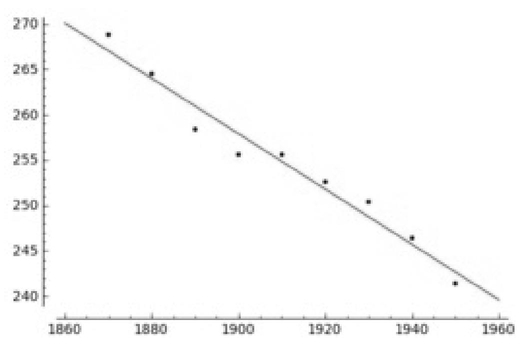
A few minutes in *Sage* gives the slope and intercept.

```
sage: data=[[1870,268.8], [1880,264.5], [1890,258.4], [1900,255.6],
....:    [1910,255.6], [1920,252.6], [1930,250.4], [1940,246.4],
....:    [1950,241.4]]
sage: var('slope,intercept')
(slope, intercept)
sage: model(x) = slope*x+intercept
sage: find_fit(data,model)
[intercept == 837.0872267857003, slope == -0.30483333572258886]
```

Plotting the data along with the line of best fit

```
sage: points(data)+plot(model(intercept=find_fit(data,model)[0].rhs(),
....:   slope=find_fit(data,model)[1].rhs()),(x,1860,1960),color='red')
```

gives this graph.



Note that the progression is surprisingly linear. Our prediction is 1958.73; the actual date of Roger Bannister's record was 1954-May-06.

Exercises

The calculations here are best done on a computer. Some of the problems require more data that is available in your library, on the Internet, or in the Answers to the Exercises.

1 Use least-squares to judge if the coin in this experiment is fair.

flips	8	16	24	32	40
heads	4	9	13	17	20

2 For the men's mile record, rather than give each of the many records and its exact date, we've "smoothed" the data somewhat by taking a periodic sample. Do the longer calculation and compare the conclusions.

3 Find the line of best fit for the men's 1500 meter run. How does the slope compare with that for the men's mile? (The distances are close; a mile is about 1609 meters.)

4 Find the line of best fit for the records for women's mile.

5 Do the lines of best fit for the men's and women's miles cross?

6 When the space shuttle Challenger exploded in 1986, one of the criticisms made of NASA's decision to launch was in the way the analysis of number of O-ring failures versus temperature was made (of course, O-ring failure caused the explosion). Four O-ring failures will cause the rocket to explode. NASA had data from 24 previous flights.

temp $^\circ F$	53	75	57	58	63	70	70	66	67	67	67
failures	3	2	1	1	1	1	1	0	0	0	0

68	69	70	70	72	73	75	76	76	78	79	80	81
0	0	0	0	0	0	0	0	0	0	0	0	0

The temperature that day was forecast to be 31°F.

(a) NASA based the decision to launch partially on a chart showing only the flights that had at least one O-ring failure. Find the line that best fits these seven flights. On the basis of this data, predict the number of O-ring failures when the temperature is 31, and when the number of failures will exceed four.

(b) Find the line that best fits all 24 flights. On the basis of this extra data, predict the number of O-ring failures when the temperature is 31, and when the number of failures will exceed four.

Which do you think is the more accurate method of predicting? (An excellent discussion appears in [Dalal, et. al.].)

7 This table lists the average distance from the sun to each of the first seven planets, using earth's average as a unit.

Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus
0.39	0.72	1.00	1.52	5.20	9.54	19.2

(a) Plot the number of the planet (Mercury is 1, etc.) versus the distance. Note that it does not look like a line, and so finding the line of best fit is not fruitful.

(b) It does, however look like an exponential curve. Therefore, plot the number of the planet versus the logarithm of the distance. Does this look like a line?

(c) The asteroid belt between Mars and Jupiter is thought to be what is left of a planet that broke apart. Renumber so that Jupiter is 6, Saturn is 7, and Uranus is 8, and plot against the log again. Does this look better?

(d) Use least squares on that data to predict the location of Neptune.

(e) Repeat to predict where Pluto is.

(f) Is the formula accurate for Neptune and Pluto?

This method was used to help discover Neptune (although the second item is misleading about the history; actually, the discovery of Neptune in position 9 prompted people to look for the "missing planet" in position 5). See [Gardner, 1970]

8 William Bennett has proposed an Index of Leading Cultural Indicators for the US ([Bennett], in 1993). Among the statistics cited are the average daily hours spent watching TV, and the average combined SAT scores.

	1960	1965	1970	1975	1980	1985	1990	1992
TV	5:06	5:29	5:56	6:07	6:36	7:07	6:55	7:04
SAT	975	969	948	910	890	906	900	899

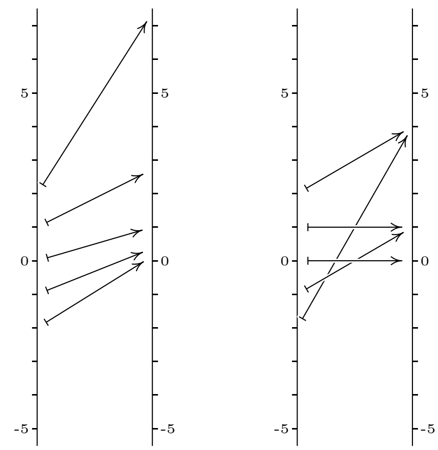
Suppose that a cause and effect relationship is proposed between the time spent watching TV and the decline in SAT scores (in this article, Mr. Bennett does not argue that there is a direct connection).

(a) Find the line of best fit relating the independent variable of average daily TV hours to the dependent variable of SAT scores.

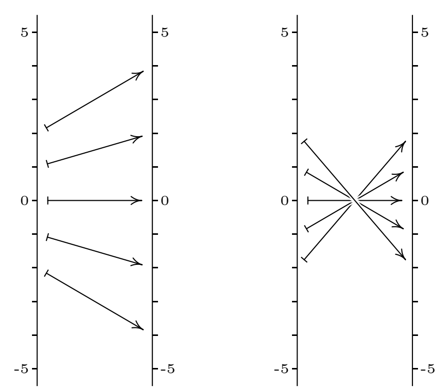
(b) Find the most recent estimate of the average daily TV hours (Bennett's cites Neilsen Media Research as the source of these estimates). Estimate the associated SAT score. How close is your estimate to the actual average? (Warning: a change has been made recently in the SAT, so you should investigate whether some adjustment needs to be made to the reported average to make a valid comparison.)

Topic: Geometry of Linear Maps

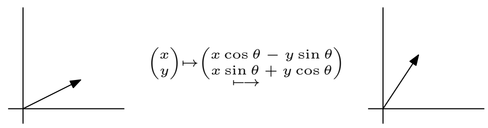
The pictures below contrast $f_1(x) = e^x$ and $f_2(x) = x^2$, which are nonlinear, with $h_1(x) = 2x$ and $h_2(x) = -x$, which are linear. Each of the four pictures shows the domain $\mathbb{R}^1$ on the left mapped to the codomain $\mathbb{R}^1$ on the right. Arrows trace out where each map sends $x = 0$, $x = 1$, $x = 2$, $x = -1$, and $x = -2$. Note how the nonlinear maps distort the domain in transforming it into the range. For instance, $f_1(1)$ is further from $f_1(2)$ than it is from $f_1(0)$ — the map is spreading the domain out unevenly so that an interval near $x = 2$ is spread apart more than is an interval near $x = 0$ when they are carried over to the range.



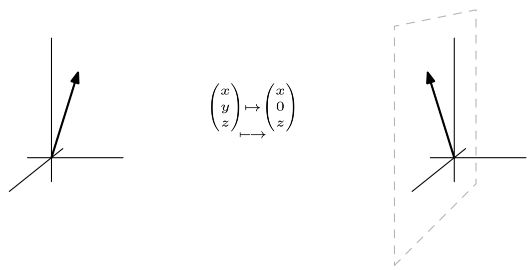
The linear maps are nicer, more regular, in that for each map all of the domain is spread by the same factor.



The only linear maps from $\mathbb{R}^1$ to $\mathbb{R}^1$ are multiplications by a scalar. In higher dimensions more can happen. For instance, this linear transformation of $\mathbb{R}^2$, rotates vectors counterclockwise, and is not just a scalar multiplication.



The transformation of $\mathbb{R}^3$ which projects vectors into the xz-plane is also not just a rescaling.



Nonetheless, even in higher dimensions the situation isn't too complicated.

Below, we use the standard bases to represent each linear map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ by a matrix H. Recall that any H can be factored $H = PBQ$, where P and Q are nonsingular and B is a partial-identity matrix. Further, recall that nonsingular matrices factor into elementary matrices $PBQ = T_nT_{n-1}\cdots T_jBT_{j-1}\cdots T_1$, which are matrices that are obtained from the identity I with one Gaussian step

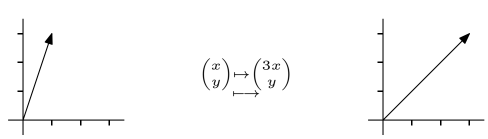
$$I \xrightarrow{k\rho_i} M_i(k) \qquad I \xrightarrow{\rho_i \leftrightarrow \rho_j} P_{i,j} \qquad I \xrightarrow{k\rho_i + \rho_j} C_{i,j}(k)$$

($i \neq j$, $k \neq 0$). So if we understand the effect of a linear map described by a partial-identity matrix, and the effect of linear mapss described by the elementary matrices, then we will in some sense understand the effect of any linear map. (The pictures below stick to transformations of $\mathbb{R}^2$ for ease of drawing, but the statements hold for maps from any $\mathbb{R}^n$ to any $\mathbb{R}^m$.)

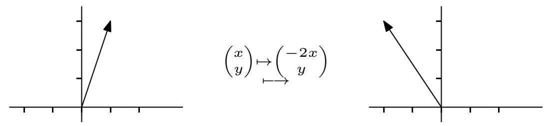
The geometric effect of the linear transformation represented by a partial-identity matrix is projection.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\mathcal{E}_3,\,\mathcal{E}_3}} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

For the $M_i(k)$ matrices, the geometric action of a transformation represented by such a matrix (with respect to the standard basis) is to stretch vectors by a factor of k along the i-th axis. This map stretches by a factor of 3 along the x-axis.

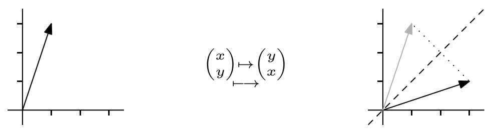


Note that if $0 \leq k < 1$ or if $k < 0$ then the i-th component goes the other way; here, toward the left.



Either of these is a *dilation*.

The action of a transformation represented by a $P_{i,j}$ permutation matrix is to interchange the i-th and j-th axes; this is a particular kind of reflection.

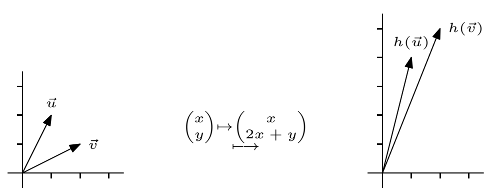


In higher dimensions, permutations involving many axes can be decomposed into a combination of swaps of pairs of axes — see Exercise 5.

The remaining case is that of matrices of the form $C_{i,j}(k)$. Recall that, for instance, that $C_{1,2}(2)$ performs $2\rho_1 + \rho_2$.

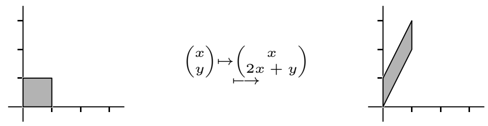
$$\begin{pmatrix}x\\y\end{pmatrix} \xrightarrow{\begin{pmatrix}1&0\\2&1\end{pmatrix}_{\varepsilon_2,\varepsilon_2}} \begin{pmatrix}x\\2x+y\end{pmatrix}$$

In the picture below, the vector $\vec{u}$ with the first component of 1 is affected less than the vector $\vec{v}$ with the first component of 2 — $h(\vec{u})$ is only 2 higher than $\vec{u}$ while $h(\vec{v})$ is 4 higher than $\vec{v}$.

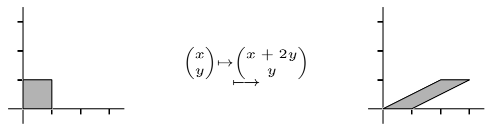


Any vector with a first component of 1 would be affected as is $\vec{u}$; it would be slid up by 2. And any vector with a first component of 2 would be slid up 4, as was $\vec{v}$. That is, the transformation represented by $C_{i,j}(k)$ affects vectors depending on their i-th component.

Another way to see this same point is to consider the action of this map on the unit square. In the next picture, vectors with a first component of 0, like the origin, are not pushed vertically at all but vectors with a positive first component are slid up. Here, all vectors with a first component of 1 — the entire right side of the square — is affected to the same extent. More generally, vectors on the same vertical line are slid up the same amount, namely, they are slid up by twice their first component. The resulting shape, a rhombus, has the same base and height as the square (and thus the same area) but the right angles are gone.



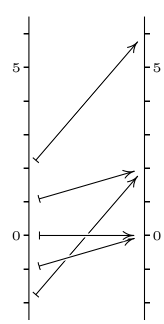
For contrast the next picture shows the effect of the map represented by $C_{2,1}(1)$. In this case, vectors are affected according to their second component. The vector $\binom{x}{y}$ is slid horozontally by twice y.



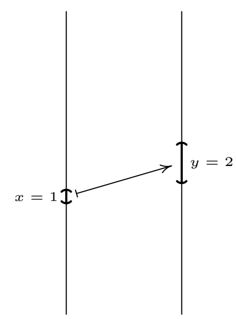
Because of this action, this kind of map is called a *skew*.

With that, we have covered the geometric effect of the four types of components in the expansion $H = T_n T_{n-1} \cdots T_j B T_{j-1} \cdots T_1$, the partial-identity projection B and the elementary T_i's. Since we understand its components, we in some sense understand the action of any H. As an illustration of this assertion, recall that under a linear map, the image of a subspace is a subspace and thus the linear transformation h represented by H maps lines through the origin to lines through the origin. (The dimension of the image space cannot be greater than the dimension of the domain space, so a line can't map onto, say, a plane.) We will extend that to show that any line, not just those through the origin, is mapped by h to a line. The proof is simply that the partial-identity projection B and the elementary T_i's each turn a line input into a line output (verifying the four cases is Exercise 6), and therefore their composition also preserves lines. Thus, by understanding its components we can understand arbitrary square matrices H, in the sense that we can prove things about them.

An understanding of the geometric effect of linear transformations on $\mathbb{R}^n$ is very important in mathematics. Here is a familiar application from calculus. On the left is a picture of the action of the nonlinear function $y(x) = x^2 + x$. As at that start of this Topic, overall the geometric effect of this map is irregular in that at different domain points it has different effects (e.g., as the domain point x goes from 2 to -2, the associated range point $f(x)$ at first decreases, then pauses instantaneously, and then increases).



But in calculus we don't focus on the map overall, we focus instead on the local effect of the map. At $x = 1$ the derivative is $y'(1) = 3$, so that near $x = 1$ we have $\Delta y \approx 3 \cdot \Delta x$. That is, in a neighborhood of $x = 1$, in carrying the domain to the codomain this map causes it to grow by a factor of 3 — it is, locally, approximately, a dilation. The picture below shows a small interval in the domain $(x - \Delta x \,..\, x + \Delta x)$ carried over to an interval in the codomain $(y - \Delta y \,..\, y + \Delta y)$ that is three times as wide: $\Delta y \approx 3 \cdot \Delta x$.



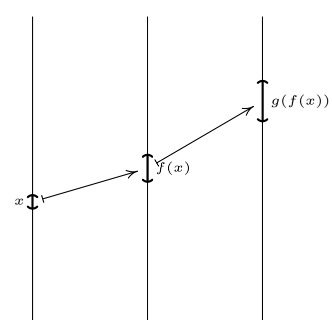
(When the above picture is drawn in the traditional cartesian way then the prior sentence about the rate of growth of $y(x)$ is usually stated: the derivative $y'(1) = 3$ gives the slope of the line tangent to the graph at the point $(1, 2)$.)

In higher dimensions, the idea is the same but the approximation is not just the $\mathbb{R}^1$-to-$\mathbb{R}^1$ scalar multiplication case. Instead, for a function $y\colon \mathbb{R}^n \to \mathbb{R}^m$ and a point $\vec{x} \in \mathbb{R}^n$, the derivative is defined to be the linear map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ best approximating how y changes near $y(\vec{x})$. So the geometry studied above applies.

We will close this Topic by remarking how this point of view makes clear an often-misunderstood, but very important, result about derivatives: the derivative of the composition of two functions is computed by using the Chain Rule for combining their derivatives. Recall that (with suitable conditions on the two functions)

$$\frac{d\,(g \circ f)}{dx}(x) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x)$$

so that, for instance, the derivative of $\sin(x^2 + 3x)$ is $\cos(x^2 + 3x) \cdot (2x + 3)$. How does this combination arise? From this picture of the action of the composition.



The first map f dilates the neighborhood of x by a factor of

$$\frac{df}{dx}(x)$$

and the second map g dilates some more, this time dilating a neighborhood of $f(x)$ by a factor of

$$\frac{dg}{dx}(\,f(x)\,)$$

and as a result, the composition dilates by the product of these two.

In higher dimensions the map expressing how a function changes near a point is a linear map, and is expressed as a matrix. (So we understand the basic geometry of higher-dimensional derivatives; they are compositions of dilations, interchanges of axes, shears, and a projection). And, the Chain Rule just multiplies the matrices.

Thus, the geometry of linear maps $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is appealing both for its simplicity and for its usefulness.

Exercises

1 Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates vectors clockwise by $\pi/4$ radians.

(a) Find the matrix H representing h with respect to the standard bases. Use Gauss' method to reduce H to the identity.

(b) Translate the row reduction to to a matrix equation $T_jT_{j-1}\cdots T_1H = I$ (the prior item shows both that H is similar to I, and that no column operations are needed to derive I from H).

(c) Solve this matrix equation for H.

(d) Sketch the geometric effect matrix, that is, sketch how H is expressed as a combination of dilations, flips, skews, and projections (the identity is a trivial projection).

2 What combination of dilations, flips, skews, and projections produces a rotation counterclockwise by $2\pi/3$ radians?

3 What combination of dilations, flips, skews, and projections produces the map $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ represented with respect to the standard bases by this matrix?

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

4 Show that any linear transformation of $\mathbb{R}^1$ is the map that multiplies by a scalar $x \mapsto kx$.

5 Show that for any permutation (that is, reordering) p of the numbers $1, \ldots, n$, the map

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_{p(1)} \\ x_{p(2)} \\ \vdots \\ x_{p(n)} \end{pmatrix}$$

can be accomplished with a composition of maps, each of which only swaps a single pair of coordinates. *Hint:* it can be done by induction on n. (*Remark:* in the fourth chapter we will show this and we will also show that the parity of the number of swaps used is determined by p. That is, although a particular permutation could be accomplished in two different ways with two different numbers of swaps, either both ways use an even number of swaps, or both use an odd number.)

6 Show that linear maps preserve the linear structures of a space.

(a) Show that for any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, the image of any line is a line. The image may be a degenerate line, that is, a single point.

(b) Show that the image of any linear surface is a linear surface. This generalizes the result that under a linear map the image of a subspace is a subspace.

(c) Linear maps preserve other linear ideas. Show that linear maps preserve "betweeness": if the point B is between A and C then the image of B is between the image of A and the image of C.

7 Use a picture like the one that appears in the discussion of the Chain Rule to answer: if a function $f\colon \mathbb{R} \to \mathbb{R}$ has an inverse, what's the relationship between how the function — locally, approximately — dilates space, and how its inverse dilates space (assuming, of course, that it has an inverse)?

Topic: Markov Chains

Here is a simple game: a player bets on coin tosses, a dollar each time, and the game ends either when the player has no money left or is up to five dollars. If the player starts with three dollars, what is the chance that the game takes at least five flips? Twenty-five flips?

At any point, this player has either \$0, or \$1, ..., or \$5. We say that the player is in the *state* s_0, s_1, ..., or s_5. A game consists of moving from state to state. For instance, a player now in state s_3 has on the next flip a .5 chance of moving to state s_2 and a .5 chance of moving to s_4. The boundary states are a bit different; once in state s_0 or state s_5, the player never leaves.

Let $p_i(n)$ be the probability that the player is in state s_i after n flips. Then, for instance, we have that the probability of being in state s_0 after flip $n+1$ is $p_0(n+1) = p_0(n) + 0.5 \cdot p_1(n)$. This matrix equation sumarizes.

$$\begin{pmatrix} 1 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 1 \end{pmatrix} \begin{pmatrix} p_0(n) \\ p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{pmatrix} = \begin{pmatrix} p_0(n+1) \\ p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \\ p_5(n+1) \end{pmatrix}$$

With the initial condition that the player starts with three dollars, calculation gives this.

$$\begin{array}{c|cccccc} n=0 & n=1 & n=2 & n=3 & n=4 & \cdots & n=24 \\ \hline \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ .5 \\ 0 \\ .5 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ .25 \\ 0 \\ .5 \\ 0 \\ .25 \end{pmatrix} & \begin{pmatrix} .125 \\ 0 \\ .375 \\ 0 \\ .25 \\ .25 \end{pmatrix} & \begin{pmatrix} .125 \\ .1875 \\ 0 \\ .3125 \\ 0 \\ .375 \end{pmatrix} & \cdots & \begin{pmatrix} .39600 \\ .00276 \\ 0 \\ .00447 \\ 0 \\ .59676 \end{pmatrix} \end{array}$$

As this computational exploration suggests, the game is not likely to go on for long, with the player quickly ending in either state s_0 or state s_5. For instance, after the fourth flip there is a probability of 0.50 that the game is already over. (Because a player who enters either of the boundary states never leaves, they are said to be *absorbtive*.)

This game is an example of a *Markov chain*, named for A.A. Markov, who worked in the first half of the 1900's. Each vector of p's is a *probability vector* and the matrix is a *transition matrix*. The notable feature of a Markov chain model is that it is *historyless* in that with a fixed transition matrix, the next state depends only on the current state, not on any prior states. Thus a player, say, who arrives at s_2 by starting in state s_3, then going to state s_2, then to s_1, and then to s_2 has at this point exactly the same chance of moving next to state s_3 as does a player whose history was to start in s_3, then go to s_4, and to s_3, and then to s_2.

Here is a Markov chain from sociology. A study ([Macdonald & Ridge], p. 202) divided occupations in the United Kingdom into upper level (executives and professionals), middle level (supervisors and skilled manual workers), and lower level (unskilled). To determine the mobility across these levels in a generation, about two thousand men were asked, “At which level are you, and at which level was your father when you were fourteen years old?” This equation summarizes the results.

$$\begin{pmatrix} .60 & .29 & .16 \\ .26 & .37 & .27 \\ .14 & .34 & .57 \end{pmatrix} \begin{pmatrix} p_U(n) \\ p_M(n) \\ p_L(n) \end{pmatrix} = \begin{pmatrix} p_U(n+1) \\ p_M(n+1) \\ p_L(n+1) \end{pmatrix}$$

For instance, a child of a lower class worker has a .27 probability of growing up to be middle class. Notice that the Markov model assumption about history seems reasonable — we expect that while a parent’s occupation has a direct influence on the occupation of the child, the grandparent’s occupation has no such direct influence. With the initial distribution of the respondents’s fathers given below, this table lists the distributions for the next five generations.

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$\begin{pmatrix} .12 \\ .32 \\ .56 \end{pmatrix}$	$\begin{pmatrix} .23 \\ .34 \\ .42 \end{pmatrix}$	$\begin{pmatrix} .29 \\ .34 \\ .37 \end{pmatrix}$	$\begin{pmatrix} .31 \\ .34 \\ .35 \end{pmatrix}$	$\begin{pmatrix} .32 \\ .33 \\ .34 \end{pmatrix}$	$\begin{pmatrix} .33 \\ .33 \\ .34 \end{pmatrix}$

One more example, from a very important subject, indeed. The World Series of American baseball is played between the team winning the American League and the team winning the National League (we follow [Brunner] but see also [Woodside]). The series is won by the first team to win four games. That means that a series is in one of twenty-four states: 0-0 (no games won yet by either team), 1-0 (one game won for the American League team and no games for the National League team), etc. If we assume that there is a probability p that the American League team wins each game then we have the following transition matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ p & 0 & 0 & 0 & \ldots \\ 1-p & 0 & 0 & 0 & \ldots \\ 0 & p & 0 & 0 & \ldots \\ 0 & 1-p & p & 0 & \ldots \\ 0 & 0 & 1-p & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} p_{\text{0-0}}(n) \\ p_{\text{1-0}}(n) \\ p_{\text{0-1}}(n) \\ p_{\text{2-0}}(n) \\ p_{\text{1-1}}(n) \\ p_{\text{0-2}}(n) \\ \vdots \end{pmatrix} = \begin{pmatrix} p_{\text{0-0}}(n+1) \\ p_{\text{1-0}}(n+1) \\ p_{\text{0-1}}(n+1) \\ p_{\text{2-0}}(n+1) \\ p_{\text{1-1}}(n+1) \\ p_{\text{0-2}}(n+1) \\ \vdots \end{pmatrix}$$

An especially interesting special case is $p = 0.50$; this table lists the resulting components of the $n = 0$ through $n = 7$ vectors. (The code to generate this table in the computer algebra system Octave follows the exercises.)

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$0-0$	1	0	0	0	0	0	0	0
$1-0$	0	0.5	0	0	0	0	0	0
$0-1$	0	0.5	0	0	0	0	0	0
$2-0$	0	0	0.25	0	0	0	0	0
$1-1$	0	0	0.5	0	0	0	0	0
$0-2$	0	0	0.25	0	0	0	0	0
$3-0$	0	0	0	0.125	0	0	0	0
$2-1$	0	0	0	0.375	0	0	0	0
$1-2$	0	0	0	0.375	0	0	0	0
$0-3$	0	0	0	0.125	0	0	0	0
$4-0$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$3-1$	0	0	0	0	0.25	0	0	0
$2-2$	0	0	0	0	0.375	0	0	0
$1-3$	0	0	0	0	0.25	0	0	0
$0-4$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$4-1$	0	0	0	0	0	0.125	0.125	0.125
$3-2$	0	0	0	0	0	0.3125	0	0
$2-3$	0	0	0	0	0	0.3125	0	0
$1-4$	0	0	0	0	0	0.125	0.125	0.125
$4-2$	0	0	0	0	0	0	0.15625	0.15625
$3-3$	0	0	0	0	0	0	0.3125	0
$2-4$	0	0	0	0	0	0	0.15625	0.15625
$4-3$	0	0	0	0	0	0	0	0.15625
$3-4$	0	0	0	0	0	0	0	0.15625

Note that evenly-matched teams are likely to have a long series — there is a probability of 0.625 that the series goes at least six games.

One reason for the inclusion of this Topic is that Markov chains are one of the most widely-used applications of matrix operations. Another reason is that it provides an example of the use of matrices where we do not consider the significance of the maps represented by the matrices. For more on Markov chains, there are many sources such as [Kemeny & Snell] and [Iosifescu].

Exercises

Use a computer for these problems. You can, for instance, adapt the Octave script given below.

1 These questions refer to the coin-flipping game.

(a) Check the computations in the table at the end of the first paragraph.

(b) Consider the second row of the vector table. Note that this row has alternating 0's. Must $p_1(j)$ be 0 when j is odd? Prove that it must be, or produce a counterexample.

(c) Perform a computational experiment to estimate the chance that the player ends at five dollars, starting with one dollar, two dollars, and four dollars.

2 We consider throws of a die, and say the system is in state s_i if the largest number yet appearing on the die was i.

(a) Give the transition matrix.

(b) Start the system in state s_1, and run it for five throws. What is the vector at the end?

[Feller], p. 424

3 There has been much interest in whether industries in the United States are moving from the Northeast and North Central regions to the South and West, motivated by the warmer climate, by lower wages, and by less unionization. Here is the transition matrix for large firms in Electric and Electronic Equipment ([Kelton], p. 43)

	NE	*NC*	*S*	*W*	*Z*
NE	0.787	0	0	0.111	0.102
NC	0	0.966	0.034	0	0
S	0	0.063	0.937	0	0
W	0	0	0.074	0.612	0.314
Z	0.021	0.009	0.005	0.010	0.954

For example, a firm in the Northeast region will be in the West region next year with probability 0.111. (The Z entry is a "birth-death" state. For instance, with probability 0.102 a large Electric and Electronic Equipment firm from the Northeast will move out of this system next year: go out of business, move abroad, or move to another category of firm. There is a 0.021 probability that a firm in the *National Census of Manufacturers* will move into Electronics, or be created, or move in from abroad, into the Northeast. Finally, with probability 0.954 a firm out of the categories will stay out, according to this research.)

(a) Does the Markov model assumption of lack of history seem justified?

(b) Assume that the initial distribution is even, except that the value at Z is 0.9. Compute the vectors for $n = 1$ through $n = 4$.

(c) Suppose that the initial distribution is this.

NE	*NC*	*S*	*W*	*Z*
0.0000	0.6522	0.3478	0.0000	0.0000

Calculate the distributions for $n = 1$ through $n = 4$.

(d) Find the distribution for $n = 50$ and $n = 51$. Has the system settled down to an equilibrium?

4 This model has been suggested for some kinds of learning ([Wickens], p. 41). The learner starts in an undecided state s_U. Eventually the learner has to decide to do either response A (that is, end in state s_A) or response B (ending in s_B). However, the learner doesn't jump right from being undecided to being sure A is the correct thing to do (or B). Instead, the learner spends some time in a "tentative-A" state, or a "tentative-B" state, trying the response out (denoted here t_A and t_B). Imagine that once the learner has decided, it is final, so once s_A or s_B is entered it is never left. For the other state changes, imagine a transition is made with probability p in either direction.

(a) Construct the transition matrix.

(b) Take $p = 0.25$ and take the initial vector to be 1 at s_U. Run this for five steps. What is the chance of ending up at s_A?

(c) Do the same for $p = 0.20$.

(d) Graph p versus the chance of ending at s_A. Is there a threshold value for p, above which the learner is almost sure not to take longer than five steps?

5 A certain town is in a certain country (this is a hypothetical problem). Each year ten percent of the town dwellers move to other parts of the country. Each year one percent of the people from elsewhere move to the town. Assume that there are two states s_T, living in town, and s_C, living elsewhere.

(a) Construct the transistion matrix.

(b) Starting with an initial distribution $s_T = 0.3$ and $s_C = 0.7$, get the results for the first ten years.
(c) Do the same for $s_T = 0.2$.
(d) Are the two outcomes alike or different?

6 For the World Series application, use a computer to generate the seven vectors for $p = 0.55$ and $p = 0.6$.
(a) What is the chance of the National League team winning it all, even though they have only a probability of 0.45 or 0.40 of winning any one game?
(b) Graph the probability p against the chance that the American League team wins it all. Is there a threshold value — a p above which the better team is essentially ensured of winning?
(Some sample code is included below.)

7 A *Markov matrix* has each entry positive and each column sums to 1.
(a) Check that the three transistion matrices shown in this Topic meet these two conditions. Must any transition matrix do so?
(b) Observe that if $A\vec{v}_0 = \vec{v}_1$ and $A\vec{v}_1 = \vec{v}_2$ then A^2 is a transition matrix from $\vec{v}_0$ to $\vec{v}_2$. Show that a power of a Markov matrix is also a Markov matrix.
(c) Generalize the prior item by proving that the product of two appropriately-sized Markov matrices is a Markov matrix.

Computer Code

This script *markov.m* for the computer algebra system Octave was used to generate the table of World Series outcomes. (The sharp character # marks the rest of a line as a comment.)

```
# Octave script file to compute chance of World Series outcomes.
function w = markov(p,v)
  q = 1-p;
  A=[0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 0-0
     p,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 1-0
     q,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 0-1_
     0,p,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 2-0
     0,q,p,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 1-1
     0,0,q,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 0-2__
     0,0,0,p,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 3-0
     0,0,0,q,p,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 2-1
     0,0,0,0,q,p, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 1-2_
     0,0,0,0,0,q, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 0-3
     0,0,0,0,0,0, p,0,0,0,1,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 4-0
     0,0,0,0,0,0, q,p,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 3-1__
     0,0,0,0,0,0, 0,q,p,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 2-2
     0,0,0,0,0,0, 0,0,q,p,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0;  # 1-3
     0,0,0,0,0,0, 0,0,0,q,0,0, 0,0,1,0,0,0, 0,0,0,0,0,0;  # 0-4_
     0,0,0,0,0,0, 0,0,0,0,0,p, 0,0,0,1,0,0, 0,0,0,0,0,0;  # 4-1
     0,0,0,0,0,0, 0,0,0,0,0,q, p,0,0,0,0,0, 0,0,0,0,0,0;  # 3-2
     0,0,0,0,0,0, 0,0,0,0,0,0, q,p,0,0,0,0, 0,0,0,0,0,0;  # 2-3__
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,q,0,0,0,0, 1,0,0,0,0,0;  # 1-4
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,p,0, 0,1,0,0,0,0;  # 4-2
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,q,p, 0,0,0,0,0,0;  # 3-3_
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,q, 0,0,0,1,0,0;  # 2-4
```

```
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,p,0,1,0;  # 4-3
     0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,0,0,0,0, 0,0,q,0,0,1]; # 3-4
  w = A * v;
endfunction
```

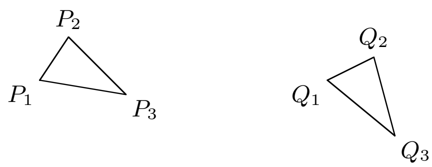
Then the Octave session was this.

```
> v0=[1;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0;0]
> p=.5
> v1=markov(p,v0)
> v2=markov(p,v1)
  ...
```

Translating to another computer algebra system should be easy — all have commands similar to these.

Topic: Orthonormal Matrices

In *The Elements*, Euclid considers two figures to be the same if they have the same size and shape. That is, the triangles below are not equal because they are not the same set of points. But they are *congruent* — essentially indistinguishable for Euclid's purposes — because we can imagine picking the plane up, sliding it over and rotating it a bit, although not warping or stretching it, and then putting it back down, to superimpose the first figure on the second. (Euclid never explicitly states this principle but he uses it often [Casey].)



In modern terminology, "picking the plane up ..." means considering a map from the plane to itself. Euclid has limited consideration to only certain transformations of the plane, ones that may possibly slide or turn the plane but not bend or stretch it. Accordingly, we define a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be *distance-preserving* or a *rigid motion* or an *isometry*, if for all points $P_1, P_2 \in \mathbb{R}^2$, the distance from $f(P_1)$ to $f(P_2)$ equals the distance from P_1 to P_2. We also define a plane *figure* to be a set of points in the plane and we say that two figures are *congruent* if there is a distance-preserving map from the plane to itself that carries one figure onto the other.

Many statements from Euclidean geometry follow easily from these definitions. Some are: (i) collinearity is invariant under any distance-preserving map (that is, if P_1, P_2, and P_3 are collinear then so are $f(P_1)$, $f(P_2)$, and $f(P_3)$), (ii) betweeness is invariant under any distance-preserving map (if P_2 is between P_1 and P_3 then so is $f(P_2)$ between $f(P_1)$ and $f(P_3)$), (iii) the property of being a triangle is invariant under any distance-preserving map (if a figure is a triangle then the image of that figure is also a triangle), (iv) and the property of being a circle is invariant under any distance-preserving map. In 1872, F. Klein suggested that Euclidean geometry can be characterized as the study of properties that are invariant under these maps. (This forms part of Klein's Erlanger Program, which proposes the organizing principle that each kind of geometry — Euclidean, projective, etc. — can be described as the study of the properties that are invariant under some group of transformations. The word 'group' here means more than just 'collection', but that lies outside of our scope.)

We can use linear algebra to characterize the distance-preserving maps of the plane.

First, there are distance-preserving transformations of the plane that are not linear. The obvious example is this *translation*.

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad\mapsto\quad \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$$

However, this example turns out to be the only example, in the sense that if f is distance-preserving and sends $\vec{0}$ to $\vec{v}_0$ then the map $\vec{v} \mapsto f(\vec{v})-\vec{v}_0$ is linear. That will follow immediately from this statement: a map t that is distance-preserving and sends $\vec{0}$ to itself is linear. To prove this equivalent statement, let

$$t(\vec{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix} \qquad t(\vec{e}_2) = \begin{pmatrix} c \\ d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. Then to show that t is linear, we can show that it can be represented by a matrix, that is, that t acts in this way for all $x, y \in \mathbb{R}$.

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix} \tag{$*$}$$

Recall that if we fix three non-collinear points then any point in the plane can be described by giving its distance from those three. So any point $\vec{v}$ in the domain is determined by its distance from the three fixed points $\vec{0}$, $\vec{e}_1$, and $\vec{e}_2$. Similarly, any point $t(\vec{v})$ in the codomain is determined by its distance from the three fixed points $t(\vec{0})$, $t(\vec{e}_1)$, and $t(\vec{e}_2)$ (these three are not collinear because, as mentioned above, collinearity is invariant and $\vec{0}$, $\vec{e}_1$, and $\vec{e}_2$ are not collinear). In fact, because t is distance-preserving, we can say more: for the point $\vec{v}$ in the plane that is determined by being the distance d_0 from $\vec{0}$, the distance d_1 from $\vec{e}_1$, and the distance d_2 from $\vec{e}_2$, its image $t(\vec{v})$ must be the unique point in the codomain that is determined by being d_0 from $t(\vec{0})$, d_1 from $t(\vec{e}_1)$, and d_2 from $t(\vec{e}_2)$. Because of the uniqueness, checking that the action in $(*)$ works in the d_0, d_1, and d_2 cases

$$\mathrm{dist}(\begin{pmatrix} x \\ y \end{pmatrix}, \vec{0}) = \mathrm{dist}(t(\begin{pmatrix} x \\ y \end{pmatrix}), t(\vec{0})) = \mathrm{dist}(\begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}, \vec{0})$$

(t is assumed to send $\vec{0}$ to itself)

$$\mathrm{dist}(\begin{pmatrix} x \\ y \end{pmatrix}, \vec{e}_1) = \mathrm{dist}(t(\begin{pmatrix} x \\ y \end{pmatrix}), t(\vec{e}_1)) = \mathrm{dist}(\begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix})$$

and

$$\mathrm{dist}(\begin{pmatrix} x \\ y \end{pmatrix}, \vec{e}_2) = \mathrm{dist}(t(\begin{pmatrix} x \\ y \end{pmatrix}), t(\vec{e}_2)) = \mathrm{dist}(\begin{pmatrix} ax + cy \\ bx + dy \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix})$$

suffices to show that $(*)$ describes t. Those checks are routine.

Thus, any distance-preserving $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ can be written $f(\vec{v}) = t(\vec{v}) + \vec{v}_0$ for some constant vector $\vec{v}_0$ and linear map t that is distance-preserving.

Not every linear map is distance-preserving, for example, $\vec{v} \mapsto 2\vec{v}$ does not preserve distances. But there is a neat characterization: a linear transformation t of the plane is distance-preserving if and only if both $\|t(\vec{e}_1)\| = \|t(\vec{e}_2)\| = 1$ and $t(\vec{e}_1)$ is orthogonal to $t(\vec{e}_2)$. The 'only if' half of that statement is easy — because t is distance-preserving it must preserve the lengths of vectors, and because t is distance-preserving the Pythagorean theorem shows that it must preserve orthogonality. For the 'if' half, it suffices to check that the map preserves lengths of vectors, because then for all $\vec{p}$ and $\vec{q}$ the distance between the two is preserved $\|t(\vec{p}-\vec{q})\| = \|t(\vec{p})-t(\vec{q})\| = \|\vec{p}-\vec{q}\|$. For that check, let

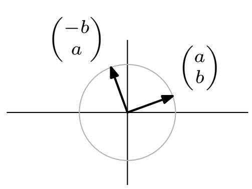
$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} \quad t(\vec{e}_1) = \begin{pmatrix} a \\ b \end{pmatrix} \quad t(\vec{e}_2) = \begin{pmatrix} c \\ d \end{pmatrix}$$

and, with the 'if' assumptions that $a^2+b^2=c^2+d^2=1$ and $ac+bd=0$ we have this.

$$\begin{aligned}
\|t(\vec{v})\|^2 &= (ax+cy)^2+(bx+dy)^2 \\
&= a^2x^2+2acxy+c^2y^2+b^2x^2+2bdxy+d^2y^2 \\
&= x^2(a^2+b^2)+y^2(c^2+d^2)+2xy(ac+bd) \\
&= x^2+y^2 \\
&= \|\vec{v}\|^2
\end{aligned}$$

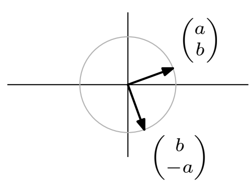
One thing that is neat about this characterization is that we can easily recognize matrices that represent such a map with respect to the standard bases. Those matrices have that when the columns are written as vectors then they are of length one and are mutually orthogonal. Such a matrix is called an *orthonormal matrix* or *orthogonal matrix* (the second term is commonly used to mean not just that the columns are orthogonal, but also that they have length one).

We can use this insight to delimit the geometric actions possible in distance-preserving maps. Because $\|t(\vec{v})\| = \|\vec{v}\|$, any $\vec{v}$ is mapped by t to lie somewhere on the circle about the origin that has radius equal to the length of $\vec{v}$. In particular, $\vec{e}_1$ and $\vec{e}_2$ are mapped to the unit circle. What's more, once we fix the unit vector $\vec{e}_1$ as mapped to the vector with components a and b then there are only two places where $\vec{e}_2$ can be mapped if that image is to be perpendicular to the first vector: one where $\vec{e}_2$ maintains its position a quarter circle clockwise from $\vec{e}_1$



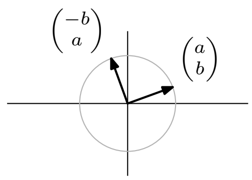
$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and one where is is mapped a quarter circle counterclockwise.



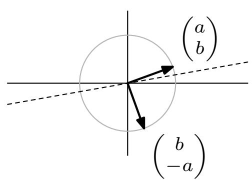
$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

We can geometrically describe these two cases. Let θ be the angle between the x-axis and the image of $\vec{e}_1$, measured counterclockwise. The first matrix above represents, with respect to the standard bases, a *rotation* of the plane by θ radians.



$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

The second matrix above represents a *reflection* of the plane through the line bisecting the angle between $\vec{e}_1$ and $t(\vec{e}_1)$.



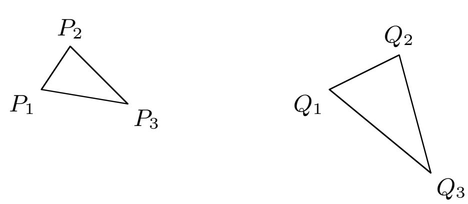
$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} x\cos\theta + y\sin\theta \\ x\sin\theta - y\cos\theta \end{pmatrix}$$

(This picture shows $\vec{e}_1$ reflected up into the first quadrant and $\vec{e}_2$ reflected down into the fourth quadrant.)

Note again: the angle between $\vec{e}_1$ and $\vec{e}_2$ runs counterclockwise, and in the first map above the angle from $t(\vec{e}_1)$ to $t(\vec{e}_2)$ is also counterclockwise, so the orientation of the angle is preserved. But in the second map the orientation is reversed. A distance-preserving map is *direct* if it preserves orientations and *opposite* if it reverses orientation.

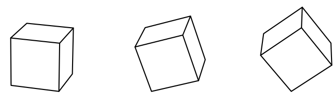
So, we have characterized the Euclidean study of congruence: it considers, for plane figures, the properties that are invariant under combinations of (i) a rotation followed by a translation, or (ii) a reflection followed by a translation (a reflection followed by a non-trivial translation is a *glide reflection*).

Another idea, besides congruence of figures, encountered in elementary geometry is that figures are *similar* if they are congruent after a change of scale. These two triangles are similar since the second is the same shape as the first, but 3/2-ths the size.



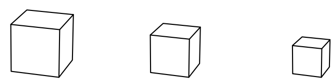
From the above work, we have that figures are similar if there is an orthonormal matrix T such that the points $\vec{q}$ on one are derived from the points $\vec{p}$ by $\vec{q} = (kT)\vec{v} + \vec{p}_0$ for some nonzero real number k and constant vector $\vec{p}_0$.

Although many of these ideas were first explored by Euclid, mathematics is timeless and they are very much in use today. One application of the maps studied above is in computer graphics. We can, for example, animate this top view of a cube by putting together film frames of it rotating; that's a rigid motion.



Frame 1 Frame 2 Frame 3

We could also make the cube appear to be moving away from us by producing film frames of it shrinking, which gives us figures that are similar.



Frame 1: Frame 2: Frame 3:

Computer graphics incorporates techniques from linear algebra in many other ways (see Exercise 4).

So the analysis above of distance-preserving maps is useful as well as interesting. A beautiful book that explores some of this area is [Weyl]. More on groups, of transformations and otherwise, can be found in any book on Modern Algebra, for instance [Birkhoff & MacLane]. More on Klein and the Erlanger Program is in [Yaglom].

Exercises

1 Decide if each of these is an orthonormal matrix.

(a) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

(b) $\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$

(c) $\begin{pmatrix} 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \\ -\sqrt{2}/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$

2 Write down the formula for each of these distance-preserving maps.

(a) the map that rotates $\pi/6$ radians, and then translates by $\vec{e}_2$

(b) the map that reflects about the line $y = 2x$

(c) the map that reflects about $y = -2x$ and translates over 1 and up 1

3 **(a)** The proof that a map that is distance-preserving and sends the zero vector to itself incidentally shows that such a map is one-to-one and onto (the point in the domain determined by d_0, d_1, and d_2 corresponds to the point in the codomain determined by those three). Therefore any distance-preserving map has an inverse. Show that the inverse is also distance-preserving.

(b) Prove that congruence is an equivalence relation between plane figures.

4 In practice the matrix for the distance-preserving linear transformation and the translation are often combined into one. Check that these two computations yield the same first two components.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \qquad \begin{pmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

(These are *homogeneous coordinates*; see the Topic on Projective Geometry).

5 **(a)** Verify that the properties described in the second paragraph of this Topic as invariant under distance-preserving maps are indeed so.

(b) Give two more properties that are of interest in Euclidean geometry from your experience in studying that subject that are also invariant under distance-preserving maps.

(c) Give a property that is not of interest in Euclidean geometry and is not invariant under distance-preserving maps.

Chapter Four

Determinants

In the first chapter of this book we considered linear systems and we picked out the special case of systems with the same number of equations as unknowns, those of the form $T\vec{x}=\vec{b}$ where T is a square matrix. We noted a distinction between two classes of T's. While such systems may have a unique solution or no solutions or infinitely many solutions, if a particular T is associated with a unique solution in any system, such as the homogeneous system $T\vec{x}=\vec{0}$, then T is associated with a unique solution for every $\vec{b}$. We call such a matrix of coefficients 'nonsingular'. The other kind of T, where every linear system for which it is the matrix of coefficients has either no solution or infinitely many solutions, we call 'singular'.

Through the second and third chapters the value of this distinction has been a theme. For instance, we now know that nonsingularity of an $n\times n$ matrix T is equivalent to each of these:

- a system $T\vec{x}=\vec{b}$ has a solution, and that solution is unique;
- Gauss-Jordan reduction of T yields an identity matrix;
- the rows of T form a linearly independent set;
- the columns of T form a basis for $\mathbb{R}^n$;
- any map that T represents is an isomorphism;
- an inverse matrix T^{-1} exists.

So when we look at a particular square matrix, the question of whether it is nonsingular is one of the first things that we ask. This chapter develops a formula to determine this. (Since we will restrict the discussion to square matrices, in this chapter we will usually simply say 'matrix' in place of 'square matrix'.)

More precisely, we will develop infinitely many formulas, one for 1×1 matrices, one for 2×2 matrices, etc. Of course, these formulas are related — that is, we will develop a family of formulas, a scheme that describes the formula for each size.

I Definition

For 1×1 matrices, determining nonsingularity is trivial.

$$\begin{pmatrix} a \end{pmatrix} \text{ is nonsingular iff } a \neq 0$$

The 2×2 formula came out in the course of developing the inverse.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is nonsingular iff } ad - bc \neq 0$$

The 3×3 formula can be produced similarly (see Exercise 9).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ is nonsingular iff } aei + bfg + cdh - hfa - idb - gec \neq 0$$

With these cases in mind, we posit a family of formulas, a, $ad-bc$, etc. For each n the formula gives rise to a *determinant* function $\det_{n\times n} \colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that an $n\times n$ matrix T is nonsingular if and only if $\det_{n\times n}(T) \neq 0$. (We usually omit the subscript because if T is $n\times n$ then '$\det(T)$' could only mean '$\det_{n\times n}(T)$'.)

I.1 Exploration

This subsection is optional. It briefly describes how an investigator might come to a good general definition, which is given in the next subsection.

The three cases above don't show an evident pattern to use for the general $n\times n$ formula. We may spot that the 1×1 term a has one letter, that the 2×2 terms ad and bc have two letters, and that the 3×3 terms aei, etc., have three letters. We may also observe that in those terms there is a letter from each row and column of the matrix, e.g., the letters in the cdh term

$$\begin{pmatrix} & & c \\ d & & \\ & h & \end{pmatrix}$$

come one from each row and one from each column. But these observations perhaps seem more puzzling than enlightening. For instance, we might wonder why some of the terms are added while others are subtracted.

A good problem solving strategy is to see what properties a solution must have and then search for something with those properties. So we shall start by asking what properties we require of the formulas.

At this point, our primary way to decide whether a matrix is singular is to do Gaussian reduction and then check whether the diagonal of resulting echelon form matrix has any zeroes (that is, to check whether the product down the diagonal is zero). So, we may expect that the proof that a formula determines singularity will involve applying Gauss' method to the matrix, to show that in the end the product down the diagonal is zero if and only if the determinant formula gives zero. This suggests our initial plan: we will look for a family of functions with the property of being unaffected by row operations and with the property that a determinant of an echelon form matrix is the product of its diagonal entries. Under this plan, a proof that the functions determine singularity would go, "Where $T \rightarrow \cdots \rightarrow \hat{T}$ is the Gaussian reduction, the determinant of T equals the determinant of $\hat{T}$ (because the determinant is unchanged by row operations), which is the product down the diagonal, which is zero if and only if the matrix is singular". In the rest of this subsection we will test this plan on the 2×2 and 3×3 determinants that we know. We will end up modifying the "unaffected by row operations" part, but not by much.

The first step in checking the plan is to test whether the 2×2 and 3×3 formulas are unaffected by the row operation of combining: if

$$T \xrightarrow{k\rho_i+\rho_j} \hat{T}$$

then is $\det(\hat{T}) = \det(T)$? This check of the 2×2 determinant after the $k\rho_1 + \rho_2$ operation

$$\det(\begin{pmatrix} a & b \\ ka+c & kb+d \end{pmatrix}) = a(kb+d) - (ka+c)b = ad - bc$$

shows that it is indeed unchanged, and the other 2×2 combination $k\rho_2 + \rho_1$ gives the same result. The 3×3 combination $k\rho_3 + \rho_2$ leaves the determinant unchanged

$$\begin{aligned} \det(\begin{pmatrix} a & b & c \\ kg+d & kh+e & ki+f \\ g & h & i \end{pmatrix}) &= a(kh+e)i + b(ki+f)g + c(kg+d)h \\ &\quad - h(ki+f)a - i(kg+d)b - g(kh+e)c \\ &= aei + bfg + cdh - hfa - idb - gec \end{aligned}$$

as do the other 3×3 row combination operations.

So there seems to be promise in the plan. Of course, perhaps the 4×4 determinant formula is affected by row combinations. We are exploring a possibility here and we do not yet have all the facts. Nonetheless, so far, so good.

The next step is to compare $\det(\hat{T})$ with $\det(T)$ for the operation

$$T \xrightarrow{\rho_i\leftrightarrow\rho_j} \hat{T}$$

of swapping two rows. The 2×2 row swap $\rho_1 \leftrightarrow \rho_2$

$$\det(\begin{pmatrix} c & d \\ a & b \end{pmatrix}) = cb - ad$$

does not yield $ad - bc$. This $\rho_1 \leftrightarrow \rho_3$ swap inside of a 3×3 matrix

$$\det(\begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}) = gec + hfa + idb - bfg - cdh - aei$$

also does not give the same determinant as before the swap — again there is a sign change. Trying a different 3×3 swap $\rho_1 \leftrightarrow \rho_2$

$$\det(\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}) = dbi + ecg + fah - hcd - iae - gbf$$

also gives a change of sign.

Thus, row swaps appear to change the sign of a determinant. This modifies our plan, but does not wreck it. We intend to decide nonsingularity by considering only whether the determinant is zero, not by considering its sign. Therefore, instead of expecting determinants to be entirely unaffected by row operations, will look for them to change sign on a swap.

To finish, we compare $\det(\hat{T})$ to $\det(T)$ for the operation

$$T \xrightarrow{k\rho_i} \hat{T}$$

of multiplying a row by a scalar $k \neq 0$. One of the 2×2 cases is

$$\det(\begin{pmatrix} a & b \\ kc & kd \end{pmatrix}) = a(kd) - (kc)b = k \cdot (ad - bc)$$

and the other case has the same result. Here is one 3×3 case

$$\begin{aligned} \det(\begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix}) &= ae(ki) + bf(kg) + cd(kh) \\ &\qquad -(kh)fa - (ki)db - (kg)ec \\ &= k \cdot (aei + bfg + cdh - hfa - idb - gec) \end{aligned}$$

and the other two are similar. These lead us to suspect that multiplying a row by k multiplies the determinant by k. This fits with our modified plan because we are asking only that the zeroness of the determinant be unchanged and we are not focusing on the determinant's sign or magnitude.

In summary, to develop the scheme for the formulas to compute determinants, we look for determinant functions that remain unchanged under the operation of row combination, that change sign on a row swap, and that rescale on the rescaling of a row. In the next two subsections we will find that for each n such a function exists and is unique.

For the next subsection, note that, as above, scalars come out of each row without affecting other rows. For instance, in this equality

$$\det(\begin{pmatrix} 3 & 3 & 9 \\ 2 & 1 & 1 \\ 5 & 10 & -5 \end{pmatrix}) = 3 \cdot \det(\begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 5 & 10 & -5 \end{pmatrix})$$

the 3 isn't factored out of all three rows, only out of the top row. The determinant acts on each row of independently of the other rows. When we want to use this property of determinants, we shall write the determinant as a function of the rows: '$\det(\vec{\rho}_1, \vec{\rho}_2, \dots \vec{\rho}_n)$', instead of as '$\det(T)$' or '$\det(t_{1,1}, \dots, t_{n,n})$'. The definition of the determinant that starts the next subsection is written in this way.

Exercises

✓ **1.1** Evaluate the determinant of each.

(a) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

1.2 Evaluate the determinant of each.

(a) $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{pmatrix}$

✓ **1.3** Verify that the determinant of an upper-triangular 3×3 matrix is the product down the diagonal.

$$\det(\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}) = aei$$

Do lower-triangular matrices work the same way?

✓ **1.4** Use the determinant to decide if each is singular or nonsingular.

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

1.5 Singular or nonsingular? Use the determinant to decide.

(a) $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & 1 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

✓ **1.6** Each pair of matrices differ by one row operation. Use this operation to compare $\det(A)$ with $\det(B)$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 2 & -6 \\ 1 & 0 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}$

1.7 Show this.

$$\det(\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}) = (b-a)(c-a)(c-b)$$

✓ **1.8** Which real numbers x make this matrix singular?

$$\begin{pmatrix} 12-x & 4 \\ 8 & 8-x \end{pmatrix}$$

1.9 Do the Gaussian reduction to check the formula for 3×3 matrices stated in the preamble to this section.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ is nonsingular iff } aei + bfg + cdh - hfa - idb - gec \neq 0$$

1.10 Show that the equation of a line in $\mathbb{R}^2$ thru (x_1, y_1) and (x_2, y_2) is expressed by this determinant.

$$\det(\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}) = 0 \qquad x_1 \neq x_2$$

✓ **1.11** Many people know this mnemonic for the determinant of a 3×3 matrix: first repeat the first two columns and then sum the products on the forward diagonals and subtract the products on the backward diagonals. That is, first write

$$\left(\begin{array}{ccc|cc} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} & h_{2,3} & h_{2,1} & h_{2,2} \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{3,1} & h_{3,2} \end{array}\right)$$

and then calculate this.

$$\begin{array}{l} h_{1,1}h_{2,2}h_{3,3} + h_{1,2}h_{2,3}h_{3,1} + h_{1,3}h_{2,1}h_{3,2} \\ -h_{3,1}h_{2,2}h_{1,3} - h_{3,2}h_{2,3}h_{1,1} - h_{3,3}h_{2,1}h_{1,2} \end{array}$$

(a) Check that this agrees with the formula given in the preamble to this section.
(b) Does it extend to other-sized determinants?

1.12 The *cross product* of the vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

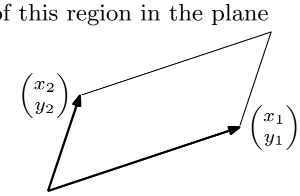
is the vector computed as this determinant.

$$\vec{x} \times \vec{y} = \det(\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix})$$

Note that the first row is composed of vectors, the vectors from the standard basis for $\mathbb{R}^3$. Show that the cross product of two vectors is perpendicular to each vector.

1.13 Prove that each statement holds for 2×2 matrices.
(a) The determinant of a product is the product of the determinants $\det(ST) = \det(S) \cdot \det(T)$.
(b) If T is invertible then the determinant of the inverse is the inverse of the determinant $\det(T^{-1}) = (\det(T))^{-1}$.
Matrices T and T' are *similar* if there is a nonsingular matrix P such that $T' = PTP^{-1}$. (This definition is in Chapter Five.) Show that similar 2×2 matrices have the same determinant.

✓ **1.14** Prove that the area of this region in the plane



is equal to the value of this determinant.

$$\det(\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix})$$

Compare with this.

$$\det(\begin{pmatrix} x_2 & x_1 \\ y_2 & y_1 \end{pmatrix})$$

1.15 Prove that for 2×2 matrices, the determinant of a matrix equals the determinant of its transpose. Does that also hold for 3×3 matrices?

✓ **1.16** Is the determinant function linear — is $\det(x \cdot T + y \cdot S) = x \cdot \det(T) + y \cdot \det(S)$?

1.17 Show that if A is 3×3 then $\det(c \cdot A) = c^3 \cdot \det(A)$ for any scalar c.

1.18 Which real numbers θ make

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

singular? Explain geometrically.

? 1.19 If a third order determinant has elements 1, 2, ..., 9, what is the maximum value it may have? [Am. Math. Mon., Apr. 1955]

I.2 Properties of Determinants

As described above, we want a formula to determine whether an $n\times n$ matrix is nonsingular. We will not begin by stating such a formula. Instead, we will begin by considering the function that such a formula calculates. We will define the function by its properties, then prove that the function with these properties exist and is unique and also describe formulas that compute this function. (Because we will show that the function exists and is unique, from the start we will say '$\det(T)$' instead of 'if there is a determinant function then $\det(T)$' and 'the determinant' instead of 'any determinant'.)

2.1 Definition A $n\times n$ *determinant* is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that

(1) $\det(\vec{\rho}_1,\ldots,k\cdot\vec{\rho}_i+\vec{\rho}_j,\ldots,\vec{\rho}_n)=\det(\vec{\rho}_1,\ldots,\vec{\rho}_j,\ldots,\vec{\rho}_n)$ for $i\neq j$

(2) $\det(\vec{\rho}_1,\ldots,\vec{\rho}_j,\ldots,\vec{\rho}_i,\ldots,\vec{\rho}_n)=-\det(\vec{\rho}_1,\ldots,\vec{\rho}_i,\ldots,\vec{\rho}_j,\ldots,\vec{\rho}_n)$ for $i\neq j$

(3) $\det(\vec{\rho}_1,\ldots,k\vec{\rho}_i,\ldots,\vec{\rho}_n)=k\cdot\det(\vec{\rho}_1,\ldots,\vec{\rho}_i,\ldots,\vec{\rho}_n)$ for $k\neq 0$

(4) $\det(I)=1$ where I is an identity matrix

(the $\vec{\rho}$'s are the rows of the matrix). We often write $|T|$ for $\det(T)$.

2.2 Remark Property (2) is redundant since

$$T \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_j+\rho_i} \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_i} \hat{T}$$

swaps rows i and j. It is listed only for convenience.

The first result shows that a function satisfying these conditions gives a criterion for nonsingularity. (Its last sentence is that, in the context of the first three conditions, (4) is equivalent to the condition that the determinant of an echelon form matrix is the product down the diagonal.)

2.3 Lemma A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

Proof. To verify the first sentence, swap the two equal rows. The sign of the determinant changes, but the matrix is unchanged and so its determinant is unchanged. Thus the determinant is zero.

The second sentence is clearly true if the matrix is 1×1. If it has at least two rows then apply property (1) of the definition with the zero row as row j and with $k=1$.

$$\det(\ldots,\vec{\rho}_i,\ldots,\vec{0},\ldots)=\det(\ldots,\vec{\rho}_i,\ldots,\vec{\rho}_i+\vec{0},\ldots)$$

The first sentence of this lemma gives that the determinant is zero.

For the third sentence, where $T\rightarrow\cdots\rightarrow\hat{T}$ is the Gauss-Jordan reduction, by the definition the determinant of T is zero if and only if the determinant of $\hat{T}$ is zero (although they could differ in sign or magnitude). A nonsingular T Gauss-Jordan reduces to an identity matrix and so has a nonzero determinant. A singular T reduces to a $\hat{T}$ with a zero row; by the second sentence of this lemma its determinant is zero.

Finally, for the fourth sentence, if an echelon form matrix is singular then it has a zero on its diagonal, that is, the product down its diagonal is zero. The third sentence says that if a matrix is singular then its determinant is zero. So if the echelon form matrix is singular then its determinant equals the product down its diagonal.

If an echelon form matrix is nonsingular then none of its diagonal entries is zero so we can use property (3) of the definition to factor them out (again, the vertical bars $|\cdots|$ indicate the determinant operation).

$$\begin{vmatrix} t_{1,1} & t_{1,2} & & t_{1,n} \\ 0 & t_{2,2} & & t_{2,n} \\ & & \ddots & \\ 0 & & & t_{n,n} \end{vmatrix} = t_{1,1}\cdot t_{2,2}\cdots t_{n,n}\cdot \begin{vmatrix} 1 & t_{1,2}/t_{1,1} & & t_{1,n}/t_{1,1} \\ 0 & 1 & & t_{2,n}/t_{2,2} \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix}$$

Next, the Jordan half of Gauss-Jordan elimination, using property (1) of the definition, leaves the identity matrix.

$$= t_{1,1}\cdot t_{2,2}\cdots t_{n,n}\cdot \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix} = t_{1,1}\cdot t_{2,2}\cdots t_{n,n}\cdot 1$$

Therefore, if an echelon form matrix is nonsingular then its determinant is the product down its diagonal. QED

That result gives us a way to compute the value of a determinant function on a matrix: do Gaussian reduction, keeping track of any changes of sign caused by row swaps and any scalars that are factored out, and then finish by multiplying down the diagonal of the echelon form result. This takes the same amount of time as Gauss' method and so is fast enugh to be practical on the matrices that we see in this book.

2.4 Example Doing 2×2 determinants

$$\begin{vmatrix}2&4\\-1&3\end{vmatrix}=\begin{vmatrix}2&4\\0&5\end{vmatrix}=10$$

with Gauss' method won't give a big savings because the 2×2 determinant formula is so easy. However, a 3×3 determinant is usually easier to calculate with Gauss' method than with the formula given earlier.

$$\begin{vmatrix}2&2&6\\4&4&3\\0&-3&5\end{vmatrix}=\begin{vmatrix}2&2&6\\0&0&-9\\0&-3&5\end{vmatrix}=-\begin{vmatrix}2&2&6\\0&-3&5\\0&0&-9\end{vmatrix}=-54$$

2.5 Example Determinants of matrices any bigger than 3×3 are almost always most quickly done with this Gauss' method procedure.

$$\begin{vmatrix}1&0&1&3\\0&1&1&4\\0&0&0&5\\0&1&0&1\end{vmatrix}=\begin{vmatrix}1&0&1&3\\0&1&1&4\\0&0&0&5\\0&0&-1&-3\end{vmatrix}=-\begin{vmatrix}1&0&1&3\\0&1&1&4\\0&0&-1&-3\\0&0&0&5\end{vmatrix}=-(-5)=5$$

The prior example illustrates an important point. Although we have not yet found a 4×4 determinant formula, if one exists then we know what value it gives to the matrix — if there is a function with properties (1)-(4) then on the above matrix the function must return 5.

2.6 Lemma For each n, if there is an $n\times n$ determinant function then it is unique.

PROOF. For any $n\times n$ matrix we can perform Gauss' method on the matrix, keeping track of how the sign alternates on row swaps, and then multiply down the diagonal of the echelon form result. By the definition and the lemma, all $n{\times}n$ determinant functions must return this value on this matrix. Thus all $n{\times}n$ determinant functions are equal, that is, there is only one input argument/output value relationship satisfying the four conditions. QED

The 'if there is an $n{\times}n$ determinant function' emphasizes that, although we can use Gauss' method to compute the only value that a determinant function could possibly return, we haven't yet shown that such a determinant function exists for all n. In the rest of the section we will produce determinant functions.

Exercises

For these, assume that an $n\times n$ determinant function exists for all n.

✓ **2.7** Use Gauss' method to find each determinant.

(a) $\begin{vmatrix}3&1&2\\3&1&0\\0&1&4\end{vmatrix}$ **(b)** $\begin{vmatrix}1&0&0&1\\2&1&1&0\\-1&0&1&0\\1&1&1&0\end{vmatrix}$

2.8 Use Gauss' method to find each.

(a) $\begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 5 & 2 & 2 \end{vmatrix}$

2.9 For which values of k does this system have a unique solution?

$$\begin{aligned} x \quad + \quad z - w &= 2 \\ y - 2z \qquad &= 3 \\ x \quad + kz \qquad &= 4 \\ z - w &= 2 \end{aligned}$$

✓ **2.10** Express each of these in terms of $|H|$.

(a) $\begin{vmatrix} h_{3,1} & h_{3,2} & h_{3,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{1,1} & h_{1,2} & h_{1,3} \end{vmatrix}$

(b) $\begin{vmatrix} -h_{1,1} & -h_{1,2} & -h_{1,3} \\ -2h_{2,1} & -2h_{2,2} & -2h_{2,3} \\ -3h_{3,1} & -3h_{3,2} & -3h_{3,3} \end{vmatrix}$

(c) $\begin{vmatrix} h_{1,1}+h_{3,1} & h_{1,2}+h_{3,2} & h_{1,3}+h_{3,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ 5h_{3,1} & 5h_{3,2} & 5h_{3,3} \end{vmatrix}$

✓ **2.11** Find the determinant of a diagonal matrix.

2.12 Describe the solution set of a homogeneous linear system if the determinant of the matrix of coefficients is nonzero.

✓ **2.13** Show that this determinant is zero.

$$\begin{vmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

2.14 **(a)** Find the 1×1, 2×2, and 3×3 matrices with i,j entry given by $(-1)^{i+j}$.
(b) Find the determinant of the square matrix with i,j entry $(-1)^{i+j}$.

2.15 **(a)** Find the 1×1, 2×2, and 3×3 matrices with i,j entry given by $i+j$.
(b) Find the determinant of the square matrix with i,j entry $i+j$.

✓ **2.16** Show that determinant functions are not linear by giving a case where $|A+B| \neq |A|+|B|$.

2.17 The second condition in the definition, that row swaps change the sign of a determinant, is somewhat annoying. It means we have to keep track of the number of swaps, to compute how the sign alternates. Can we get rid of it? Can we replace it with the condition that row swaps leave the determinant unchanged? (If so then we would need new 1×1, 2×2, and 3×3 formulas, but that would be a minor matter.)

2.18 Prove that the determinant of any triangular matrix, upper or lower, is the product down its diagonal.

2.19 Refer to the definition of elementary matrices in the Mechanics of Matrix Multiplication subsection.
(a) What is the determinant of each kind of elementary matrix?
(b) Prove that if E is any elementary matrix then $|ES| = |E||S|$ for any appropriately sized S.
(c) *(This question doesn't involve determinants.)* Prove that if T is singular then a product TS is also singular.
(d) Show that $|TS| = |T||S|$.
(e) Show that if T is nonsingular then $|T^{-1}| = |T|^{-1}$.

2.20 Prove that the determinant of a product is the product of the determinants $|TS| = |T|\,|S|$ in this way. Fix the $n \times n$ matrix S and consider the function $d\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ given by $T \mapsto |TS|/|S|$.

(a) Check that d satisfies property (1) in the definition of a determinant function.
(b) Check property (2).
(c) Check property (3).
(d) Check property (4).
(e) Conclude the determinant of a product is the product of the determinants.

2.21 A *submatrix* of a given matrix A is one that can be obtained by deleting some of the rows and columns of A. Thus, the first matrix here is a submatrix of the second.

$$\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & 4 & 1 \\ 0 & 9 & -2 \\ 2 & -1 & 5 \end{pmatrix}$$

Prove that for any square matrix, the rank of the matrix is r if and only if r is the largest integer such that there is an $r\times r$ submatrix with a nonzero determinant.

✓ **2.22** Prove that a matrix with rational entries has a rational determinant.

? **2.23** Find the element of likeness in (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}.$$

[Am. Math. Mon., Feb. 1953]

I.3 The Permutation Expansion

The prior subsection defines a function to be a determinant if it satisfies four conditions and shows that there is at most one $n\times n$ determinant function for each n. What is left is to show that for each n such a function exists.

How could such a function not exist? After all, we have done computations that start with a square matrix, follow the conditions, and end with a number.

The difficulty is that, as far as we know, the computation might not give a well-defined result. To illustrate this possibility, suppose that we were to change the second condition in the definition of determinant to be that the value of a determinant does not change on a row swap. By Remark 2.2 we know that this conflicts with the first and third conditions. Here is an instance of the conflict: here are two Gauss' method reductions of the same matrix, the first without any row swap

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

and the second with a swap.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \xrightarrow{-(1/3)\rho_1 + \rho_2} \begin{pmatrix} 3 & 4 \\ 0 & 2/3 \end{pmatrix}$$

Following Definition 2.1 gives that both calculations yield the determinant -2 since in the second one we keep track of the fact that the row swap changes the sign of the result of multiplying down the diagonal. But if we follow the supposition and change the second condition then the two calculations yield different values, -2 and 2. That is, under the supposition the outcome would not be well-defined — no function exists that satisfies the changed second condition along with the other three.

Of course, observing that Definition 2.1 does the right thing in this one instance is not enough; what we will do in the rest of this section is to show that there is never a conflict. The natural way to try this would be to define the determinant function with: "The value of the function is the result of doing Gauss' method, keeping track of row swaps, and finishing by multiplying down the diagonal". (Since Gauss' method allows for some variation, such as a choice of which row to use when swapping, we would have to fix an explicit algorithm.) Then we would be done if we verified that this way of computing the determinant satisfies the four properties. For instance, if T and $\hat{T}$ are related by a row swap then we would need to show that this algorithm returns determinants that are negatives of each other. However, how to verify this is not evident. So the development below will not proceed in this way. Instead, in this subsection we will define a different way to compute the value of a determinant, a formula, and we will use this way to prove that the conditions are satisfied.

The formula that we shall use is based on an insight gotten from property (3) of the definition of determinants. This property shows that determinants are not linear.

3.1 Example For this matrix $\det(2A) \neq 2 \cdot \det(A)$.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

Instead, the scalar comes out of each of the two rows.

$$\begin{vmatrix} 4 & 2 \\ -2 & 6 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 \\ -2 & 6 \end{vmatrix} = 4 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

Since scalars come out a row at a time, we might guess that determinants are linear a row at a time.

3.2 Definition Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

(1) $f(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) = f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + f(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$

(2) $f(\vec{\rho}_1, \ldots, k\vec{v}, \ldots, \vec{\rho}_n) = k \cdot f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n)$

for $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

3.3 Lemma Determinants are multilinear.

Proof. The definition of determinants gives property (2) (Lemma 2.3 following that definition covers the $k = 0$ case) so we need only check property (1).

$$\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) = \det(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + \det(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$$

If the set $\{\vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n\}$ is linearly dependent then all three matrices are singular and so all three determinants are zero and the equality is trivial. Therefore assume that the set is linearly independent. This set of n-wide row vectors has $n - 1$ members, so we can make a basis by adding one more vector $\langle \vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\beta}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n \rangle$. Express $\vec{v}$ and $\vec{w}$ with respect to this basis

$$\begin{aligned}
\vec{v} &= v_1\vec{\rho}_1 + \cdots + v_{i-1}\vec{\rho}_{i-1} + v_i\vec{\beta} + v_{i+1}\vec{\rho}_{i+1} + \cdots + v_n\vec{\rho}_n \\
\vec{w} &= w_1\vec{\rho}_1 + \cdots + w_{i-1}\vec{\rho}_{i-1} + w_i\vec{\beta} + w_{i+1}\vec{\rho}_{i+1} + \cdots + w_n\vec{\rho}_n
\end{aligned}$$

giving this.

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{\rho}_1 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

By the definition of determinant, the value of $\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n)$ is unchanged by the operation of adding $-(v_1 + w_1)\vec{\rho}_1$ to $\vec{v} + \vec{w}$.

$$\vec{v} + \vec{w} - (v_1 + w_1)\vec{\rho}_1 = (v_2 + w_2)\vec{\rho}_2 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

Then, to the result, we can add $-(v_2 + w_2)\vec{\rho}_2$, etc. Thus

$$\begin{aligned}
&\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) \\
&\qquad = \det(\vec{\rho}_1, \ldots, (v_i + w_i) \cdot \vec{\beta}, \ldots, \vec{\rho}_n) \\
&\qquad = (v_i + w_i) \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) \\
&\qquad = v_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) + w_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n)
\end{aligned}$$

(using (2) for the second equality). To finish, bring v_i and w_i back inside in front of $\vec{\beta}$ and use row combination again, this time to reconstruct the expressions of $\vec{v}$ and $\vec{w}$ in terms of the basis, e.g., start with the operations of adding $v_1\vec{\rho}_1$ to $v_i\vec{\beta}$ and $w_1\vec{\rho}_1$ to $w_i\vec{\rho}_1$, etc. QED

Multilinearity allows us to expand a determinant into a sum of determinants, each of which involves a simple matrix.

3.4 Example We can use multilinearity to split this determinant into two, first breaking up the first row

$$\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix}$$

and then separating each of those two, breaking along the second rows.

$$= \begin{vmatrix} 2 & 0 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix}$$

We are left with four determinants, such that in each row of each matrix there is a single entry from the original matrix.

3.5 Example In the same way, a 3×3 determinant separates into a sum of many simpler determinants. We start by splitting along the first row, producing three determinants (the zero in the $1,3$ position is underlined to set it off visually from the zeroes that appear in the splitting).

$$\begin{vmatrix}2&1&-1\\4&3&\underline{0}\\2&1&5\end{vmatrix}=\begin{vmatrix}2&0&0\\4&3&\underline{0}\\2&1&5\end{vmatrix}+\begin{vmatrix}0&1&0\\4&3&\underline{0}\\2&1&5\end{vmatrix}+\begin{vmatrix}0&0&-1\\4&3&\underline{0}\\2&1&5\end{vmatrix}$$

Each of these three will itself split in three along the second row. Each of the resulting nine splits in three along the third row, resulting in twenty seven determinants

$$=\begin{vmatrix}2&0&0\\4&0&0\\2&0&0\end{vmatrix}+\begin{vmatrix}2&0&0\\4&0&0\\0&1&0\end{vmatrix}+\begin{vmatrix}2&0&0\\4&0&0\\0&0&5\end{vmatrix}+\begin{vmatrix}2&0&0\\0&3&0\\2&0&0\end{vmatrix}+\cdots+\begin{vmatrix}0&0&-1\\0&0&\underline{0}\\0&0&5\end{vmatrix}$$

such that each row contains a single entry from the starting matrix.

So an $n\times n$ determinant expands into a sum of n^n determinants where each row of each summands contains a single entry from the starting matrix. However, many of these summand determinants are zero.

3.6 Example In each of these three matrices from the above expansion, two of the rows have their entry from the starting matrix in the same column, e.g., in the first matrix, the 2 and the 4 both come from the first column.

$$\begin{vmatrix}2&0&0\\4&0&0\\0&1&0\end{vmatrix}\qquad\begin{vmatrix}0&0&-1\\0&3&0\\0&0&5\end{vmatrix}\qquad\begin{vmatrix}0&1&0\\0&0&\underline{0}\\0&0&5\end{vmatrix}$$

Any such matrix is singular, because in each, one row is a multiple of the other (or is a zero row). Thus, any such determinant is zero, by Lemma 2.3.

Therefore, the above expansion of the 3×3 determinant into the sum of the twenty seven determinants simplifies to the sum of these six.

$$\begin{aligned}\begin{vmatrix}2&1&-1\\4&3&\underline{0}\\2&1&5\end{vmatrix}&=\begin{vmatrix}2&0&0\\0&3&0\\0&0&5\end{vmatrix}+\begin{vmatrix}2&0&0\\0&0&\underline{0}\\0&1&0\end{vmatrix}\\&\quad+\begin{vmatrix}0&1&0\\4&0&0\\0&0&5\end{vmatrix}+\begin{vmatrix}0&1&0\\0&0&\underline{0}\\2&0&0\end{vmatrix}\\&\quad+\begin{vmatrix}0&0&-1\\4&0&0\\0&1&0\end{vmatrix}+\begin{vmatrix}0&0&-1\\0&3&0\\2&0&0\end{vmatrix}\end{aligned}$$

We can bring out the scalars.

$$\begin{aligned}
&= (2)(3)(5)\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} + (2)(\underline{0})(1)\begin{vmatrix}1&0&0\\0&0&1\\0&1&0\end{vmatrix}\\
&\quad + (1)(4)(5)\begin{vmatrix}0&1&0\\1&0&0\\0&0&1\end{vmatrix} + (1)(\underline{0})(2)\begin{vmatrix}0&1&0\\0&0&1\\1&0&0\end{vmatrix}\\
&\quad + (-1)(4)(1)\begin{vmatrix}0&0&1\\1&0&0\\0&1&0\end{vmatrix} + (-1)(3)(2)\begin{vmatrix}0&0&1\\0&1&0\\1&0&0\end{vmatrix}
\end{aligned}$$

To finish, we evaluate those six determinants by row-swapping them to the identity matrix, keeping track of the resulting sign changes.

$$\begin{aligned}
&= 30\cdot(+1) + 0\cdot(-1)\\
&\quad + 20\cdot(-1) + 0\cdot(+1)\\
&\quad - 4\cdot(+1) - 6\cdot(-1) = 12
\end{aligned}$$

That example illustrates the key idea. We've applied multilinearity to a 3×3 determinant to get 3^3 separate determinants, each with one distinguished entry per row. We can drop most of these new determinants because the matrices are singular, with one row a multiple of another. We are left with the one-entry-per-row determinants also having only one entry per column (one entry from the original determinant, that is). And, since we can factor scalars out, we can further reduce to only considering determinants of one-entry-per-row-and-column matrices where the entries are ones.

These are permutation matrices. Thus, the determinant can be computed in this three-step way *(Step 1)* for each permutation matrix, multiply together the entries from the original matrix where that permutation matrix has ones, *(Step 2)* multiply that by the determinant of the permutation matrix and *(Step 3)* do that for all permutation matrices and sum the results together.

To state this as a formula, we introduce a notation for permutation matrices. Let ι_j be the row vector that is all zeroes except for a one in its j-th entry, so that the four-wide ι_2 is $\begin{pmatrix}0&1&0&0\end{pmatrix}$. We can construct permutation matrices by permuting — that is, scrambling — the numbers 1, 2, …, n, and using them as indices on the ι's. For instance, to get a 4×4 permutation matrix matrix, we can scramble the numbers from 1 to 4 into this sequence $\langle 3,2,1,4\rangle$ and take the corresponding row vector ι's.

$$\begin{pmatrix}\iota_3\\\iota_2\\\iota_1\\\iota_4\end{pmatrix} = \begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$$

3.7 Definition An *n-permutation* is a sequence consisting of an arrangement of the numbers 1, 2, …, n.

3.8 Example The 2-permutations are $\phi_1 = \langle 1,2\rangle$ and $\phi_2 = \langle 2,1\rangle$. These are the associated permutation matrices.

$$P_{\phi_1} = \begin{pmatrix} \iota_1 \\ \iota_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{\phi_2} = \begin{pmatrix} \iota_2 \\ \iota_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We sometimes write permutations as functions, e.g., $\phi_2(1) = 2$, and $\phi_2(2) = 1$. Then the rows of P_{ϕ_2} are $\iota_{\phi_2(1)} = \iota_2$ and $\iota_{\phi_2(2)} = \iota_1$.

The 3-permutations are $\phi_1 = \langle 1,2,3\rangle$, $\phi_2 = \langle 1,3,2\rangle$, $\phi_3 = \langle 2,1,3\rangle$, $\phi_4 = \langle 2,3,1\rangle$, $\phi_5 = \langle 3,1,2\rangle$, and $\phi_6 = \langle 3,2,1\rangle$. Here are two of the associated permutation matrices.

$$P_{\phi_2} = \begin{pmatrix} \iota_1 \\ \iota_3 \\ \iota_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad P_{\phi_5} = \begin{pmatrix} \iota_3 \\ \iota_1 \\ \iota_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For instance, the rows of P_{ϕ_5} are $\iota_{\phi_5(1)} = \iota_3$, $\iota_{\phi_5(2)} = \iota_1$, and $\iota_{\phi_5(3)} = \iota_2$.

3.9 Definition The *permutation expansion* for determinants is

$$\begin{vmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{vmatrix} \begin{aligned} = \;& t_{1,\phi_1(1)}t_{2,\phi_1(2)}\cdots t_{n,\phi_1(n)}|P_{\phi_1}| \\ & + t_{1,\phi_2(1)}t_{2,\phi_2(2)}\cdots t_{n,\phi_2(n)}|P_{\phi_2}| \\ & \vdots \\ & + t_{1,\phi_k(1)}t_{2,\phi_k(2)}\cdots t_{n,\phi_k(n)}|P_{\phi_k}| \end{aligned}$$

where $\phi_1, \ldots, \phi_k$ are all of the n-permutations.

This formula is often written in *summation notation*

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)}t_{2,\phi(2)}\cdots t_{n,\phi(n)}\,|P_\phi|$$

read aloud as "the sum, over all permutations ϕ, of terms having the form $t_{1,\phi(1)}t_{2,\phi(2)}\cdots t_{n,\phi(n)}|P_\phi|$". This phrase is just a restating of the three-step process *(Step 1)* for each permutation matrix, compute $t_{1,\phi(1)}t_{2,\phi(2)}\cdots t_{n,\phi(n)}$ *(Step 2)* multiply that by $|P_\phi|$ and *(Step 3)* sum all such terms together.

3.10 Example The familiar formula for the determinant of a 2×2 matrix can be derived in this way.

$$\begin{aligned} \begin{vmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{vmatrix} &= t_{1,1}t_{2,2}\cdot|P_{\phi_1}| + t_{1,2}t_{2,1}\cdot|P_{\phi_2}| \\ &= t_{1,1}t_{2,2}\cdot\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + t_{1,2}t_{2,1}\cdot\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= t_{1,1}t_{2,2} - t_{1,2}t_{2,1} \end{aligned}$$

(the second permutation matrix takes one row swap to pass to the identity). Similarly, the formula for the determinant of a 3×3 matrix is this.

$$\begin{aligned}
\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} &= t_{1,1}t_{2,2}t_{3,3}\,|P_{\phi_1}| + t_{1,1}t_{2,3}t_{3,2}\,|P_{\phi_2}| + t_{1,2}t_{2,1}t_{3,3}\,|P_{\phi_3}| \\
&\qquad + t_{1,2}t_{2,3}t_{3,1}\,|P_{\phi_4}| + t_{1,3}t_{2,1}t_{3,2}\,|P_{\phi_5}| + t_{1,3}t_{2,2}t_{3,1}\,|P_{\phi_6}| \\
&= t_{1,1}t_{2,2}t_{3,3} - t_{1,1}t_{2,3}t_{3,2} - t_{1,2}t_{2,1}t_{3,3} \\
&\qquad + t_{1,2}t_{2,3}t_{3,1} + t_{1,3}t_{2,1}t_{3,2} - t_{1,3}t_{2,2}t_{3,1}
\end{aligned}$$

Computing a determinant by permutation expansion usually takes longer than Gauss' method. However, here we are not trying to do the computation efficiently, we are instead trying to give a determinant formula that we can prove to be well-defined. While the permutation expansion is impractical for computations, we will find it useful in the proofs below.

3.11 Theorem For each n there is a $n\times n$ determinant function.

The proof is deferred to the following subsection. Also there is the proof of the next result (they share some features).

3.12 Theorem The determinant of a matrix equals the determinant of its transpose.

The consequence of this theorem is that, while we have so far stated results in terms of rows (e.g., determinants are multilinear in their rows, row swaps change the sign, etc.), all of the results also hold in terms of columns. The final result gives examples.

3.13 Corollary A matrix with two equal columns is singular. Column swaps change the sign of a determinant. Determinants are multilinear in their columns.

Proof. For the first statement, transposing the matrix results in a matrix with the same determinant, and with two equal rows, and hence a determinant of zero. The other two are proved in the same way. QED

We finish with a summary (although the final subsection contains the unfinished business of proving the two theorems). Determinant functions exist, are unique, and we know how to compute them. As for what determinants are about, perhaps these lines [Kemp] help make it memorable.

Determinant none,
Solution: lots or none.
Determinant some,
Solution: just one.

Exercises

These summarize the notation used in this book for the 2- and 3- permutations.

i	1	2
$\phi_1(i)$	1	2
$\phi_2(i)$	2	1

i	1	2	3
$\phi_1(i)$	1	2	3
$\phi_2(i)$	1	3	2
$\phi_3(i)$	2	1	3
$\phi_4(i)$	2	3	1
$\phi_5(i)$	3	1	2
$\phi_6(i)$	3	2	1

✓ **3.14** Compute the determinant by using the permutation expansion.

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ **(b)** $\begin{vmatrix} 2 & 2 & 1 \\ 3 & -1 & 0 \\ -2 & 0 & 5 \end{vmatrix}$

✓ **3.15** Compute these both with Gauss' method and with the permutation expansion formula.

(a) $\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$ **(b)** $\begin{vmatrix} 0 & 1 & 4 \\ 0 & 2 & 3 \\ 1 & 5 & 1 \end{vmatrix}$

✓ **3.16** Use the permutation expansion formula to derive the formula for 3×3 determinants.

3.17 List all of the 4-permutations.

3.18 A permutation, regarded as a function from the set $\{1,..,n\}$ to itself, is one-to-one and onto. Therefore, each permutation has an inverse.
(a) Find the inverse of each 2-permutation.
(b) Find the inverse of each 3-permutation.

3.19 Prove that f is multilinear if and only if for all $\vec{v},\vec{w}\in V$ and $k_1,k_2\in\mathbb{R}$, this holds.

$$f(\vec{\rho}_1,\ldots,k_1\vec{v}_1+k_2\vec{v}_2,\ldots,\vec{\rho}_n)=k_1f(\vec{\rho}_1,\ldots,\vec{v}_1,\ldots,\vec{\rho}_n)+k_2f(\vec{\rho}_1,\ldots,\vec{v}_2,\ldots,\vec{\rho}_n)$$

3.20 How would determinants change if we changed property (4) of the definition to read that $|I|=2$?

3.21 Verify the second and third statements in Corollary 3.13.

✓ **3.22** Show that if an $n\times n$ matrix has a nonzero determinant then any column vector $\vec{v}\in\mathbb{R}^n$ can be expressed as a linear combination of the columns of the matrix.

3.23 True or false: a matrix whose entries are only zeros or ones has a determinant equal to zero, one, or negative one. [Strang 80]

3.24 **(a)** Show that there are 120 terms in the permutation expansion formula of a 5×5 matrix.
(b) How many are sure to be zero if the $1,2$ entry is zero?

3.25 How many n-permutations are there?

3.26 A matrix A is *skew-symmetric* if $A^{\text{trans}}=-A$, as in this matrix.

$$A=\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$$

Show that $n\times n$ skew-symmetric matrices with nonzero determinants exist only for even n.

✓ **3.27** What is the smallest number of zeros, and the placement of those zeros, needed to ensure that a 4×4 matrix has a determinant of zero?

✓ **3.28** If we have n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and want to find a polynomial $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ passing through those points then we can plug in the points to get an n equation/n unknown linear system. The matrix of coefficients for that system is called the *Vandermonde matrix*. Prove that the determinant of the transpose of that matrix of coefficients

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ {x_1}^2 & {x_2}^2 & \ldots & {x_n}^2 \\ & \vdots & & \\ {x_1}^{n-1} & {x_2}^{n-1} & \ldots & {x_n}^{n-1} \end{vmatrix}$$

equals the product, over all indices $i, j \in \{1, \ldots, n\}$ with $i < j$, of terms of the form $x_j - x_i$. (This shows that the determinant is zero, and the linear system has no solution, if and only if the x_i's in the data are not distinct.)

3.29 A matrix can be divided into *blocks*, as here,

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & -2 \end{array}\right)$$

which shows four blocks, the square 2×2 and 1×1 ones in the upper left and lower right, and the zero blocks in the upper right and lower left. Show that if a matrix can be partitioned as

$$T = \left(\begin{array}{c|c} J & Z_2 \\ \hline Z_1 & K \end{array}\right)$$

where J and K are square, and Z_1 and Z_2 are all zeroes, then $|T| = |J| \cdot |K|$.

✓ **3.30** Prove that for any $n \times n$ matrix T there are at most n distinct reals r such that the matrix $T - rI$ has determinant zero (we shall use this result in Chapter Five).

? **3.31** The nine positive digits can be arranged into 3×3 arrays in $9!$ ways. Find the sum of the determinants of these arrays. [Math. Mag., Jan. 1963, Q307]

3.32 Show that

$$\begin{vmatrix} x-2 & x-3 & x-4 \\ x+1 & x-1 & x-3 \\ x-4 & x-7 & x-10 \end{vmatrix} = 0.$$

[Math. Mag., Jan. 1963, Q237]

? **3.33** Let S be the sum of the integer elements of a magic square of order three and let D be the value of the square considered as a determinant. Show that D/S is an integer. [Am. Math. Mon., Jan. 1949]

? **3.34** Show that the determinant of the n^2 elements in the upper left corner of the Pascal triangle

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & . & . \\ 1 & 2 & 3 & . & . & \\ 1 & 3 & . & . & & \\ 1 & . & . & & & \\ . & & & & & \\ . & & & & & \end{array}$$

has the value unity. [Am. Math. Mon., Jun. 1931]

I.4 Determinants Exist

This subsection is optional. It consists of proofs of two results from the prior subsection. These proofs involve the properties of permutations, which will not be used later, except in the optional Jordan Canonical Form subsection.

The prior subsection attacks the problem of showing that for any size there is a determinant function on the set of square matrices of that size by using multilinearity to develop the permutation expansion.

$$\begin{aligned}
\begin{vmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{vmatrix}
&= \; t_{1,\phi_1(1)}t_{2,\phi_1(2)}\cdots t_{n,\phi_1(n)}|P_{\phi_1}| \\
&\quad + t_{1,\phi_2(1)}t_{2,\phi_2(2)}\cdots t_{n,\phi_2(n)}|P_{\phi_2}| \\
&\quad \vdots \\
&\quad + t_{1,\phi_k(1)}t_{2,\phi_k(2)}\cdots t_{n,\phi_k(n)}|P_{\phi_k}| \\
&= \sum_{\text{permutations } \phi} t_{1,\phi(1)}t_{2,\phi(2)}\cdots t_{n,\phi(n)}\,|P_\phi|
\end{aligned}$$

This reduces the problem to showing that there is a determinant function on the set of permutation matrices of that size.

Of course, a permutation matrix can be row-swapped to the identity matrix and to calculate its determinant we can keep track of the number of row swaps. However, the problem is still not solved. We still have not shown that the result is well-defined. For instance, the determinant of

$$P_\phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

could be computed with one swap

$$P_\phi \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or with three.

$$P_\phi \xrightarrow{\rho_3 \leftrightarrow \rho_1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Both reductions have an odd number of swaps so we figure that $|P_\phi| = -1$ but how do we know that there isn't some way to do it with an even number of swaps? Corollary 4.6 below proves that there is no permutation matrix that can be row-swapped to an identity matrix in two ways, one with an even number of swaps and the other with an odd number of swaps.

4.1 Definition Two rows of a permutation matrix

$$\begin{pmatrix} \vdots \\ \iota_k \\ \vdots \\ \iota_j \\ \vdots \end{pmatrix}$$

such that $k > j$ are in an *inversion* of their natural order.

4.2 Example This permutation matrix

$$\begin{pmatrix} \iota_3 \\ \iota_2 \\ \iota_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has three inversions: ι_3 precedes ι_1, ι_3 precedes ι_2, and ι_2 precedes ι_1.

4.3 Lemma A row-swap in a permutation matrix changes the number of inversions from even to odd, or from odd to even.

Proof. Consider a swap of rows j and k, where $k > j$. If the two rows are adjacent

$$P_\phi = \begin{pmatrix} \vdots \\ \iota_{\phi(j)} \\ \iota_{\phi(k)} \\ \vdots \end{pmatrix} \xrightarrow{\rho_k \leftrightarrow \rho_j} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j)} \\ \vdots \end{pmatrix}$$

then the swap changes the total number of inversions by one — either removing or producing one inversion, depending on whether $\phi(j) > \phi(k)$ or not, since inversions involving rows not in this pair are not affected. Consequently, the total number of inversions changes from odd to even or from even to odd.

If the rows are not adjacent then they can be swapped via a sequence of adjacent swaps, first bringing row k up

$$\begin{pmatrix} \vdots \\ \iota_{\phi(j)} \\ \iota_{\phi(j+1)} \\ \iota_{\phi(j+2)} \\ \vdots \\ \iota_{\phi(k)} \\ \vdots \end{pmatrix} \xrightarrow{\rho_k \leftrightarrow \rho_{k-1}} \xrightarrow{\rho_{k-1} \leftrightarrow \rho_{k-2}} \cdots \xrightarrow{\rho_{j+1} \leftrightarrow \rho_j} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j)} \\ \iota_{\phi(j+1)} \\ \vdots \\ \iota_{\phi(k-1)} \\ \vdots \end{pmatrix}$$

and then bringing row j down.

$$\xrightarrow{\rho_{j+1}\leftrightarrow\rho_{j+2}}\ \xrightarrow{\rho_{j+2}\leftrightarrow\rho_{j+3}}\ \ldots\ \xrightarrow{\rho_{k-1}\leftrightarrow\rho_{k}} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j+1)} \\ \iota_{\phi(j+2)} \\ \vdots \\ \iota_{\phi(j)} \\ \vdots \end{pmatrix}$$

Each of these adjacent swaps changes the number of inversions from odd to even or from even to odd. There are an odd number $(k-j)+(k-j-1)$ of them. The total change in the number of inversions is from even to odd or from odd to even. QED

4.4 Definition The *signum* of a permutation $\operatorname{sgn}(\phi)$ is $+1$ if the number of inversions in P_ϕ is even, and is -1 if the number of inversions is odd.

4.5 Example With the subscripts from Example 3.8 for the 3-permutations, $\operatorname{sgn}(\phi_1)=1$ while $\operatorname{sgn}(\phi_2)=-1$.

4.6 Corollary If a permutation matrix has an odd number of inversions then swapping it to the identity takes an odd number of swaps. If it has an even number of inversions then swapping to the identity takes an even number of swaps.

Proof. The identity matrix has zero inversions. To change an odd number to zero requires an odd number of swaps, and to change an even number to zero requires an even number of swaps. QED

We still have not shown that the permutation expansion is well-defined because we have not considered row operations on permutation matrices other than row swaps. We will finesse this problem: we will define a function $d\colon \mathcal{M}_{n\times n}\to\mathbb{R}$ by altering the permutation expansion formula, replacing $|P_\phi|$ with $\operatorname{sgn}(\phi)$

$$d(T)=\sum_{\text{permutations }\phi} t_{1,\phi(1)}t_{2,\phi(2)}\ldots t_{n,\phi(n)}\operatorname{sgn}(\phi)$$

(this gives the same value as the permutation expansion because the prior result shows that $\det(P_\phi)=\operatorname{sgn}(\phi)$). This formula's advantage is that the number of inversions is clearly well-defined — just count them. Therefore, we will show that a determinant function exists for all sizes by showing that d is it, that is, that d satisfies the four conditions.

4.7 Lemma The function d is a determinant. Hence determinants exist for every n.

PROOF. We'll must check that it has the four properties from the definition.

Property (4) is easy; in

$$d(I) = \sum_{\text{perms } \phi} \iota_{1,\phi(1)}\iota_{2,\phi(2)} \cdots \iota_{n,\phi(n)} \operatorname{sgn}(\phi)$$

all of the summands are zero except for the product down the diagonal, which is one.

For property (3) consider $d(\hat{T})$ where $T \xrightarrow{k\rho_i} \hat{T}$.

$$\sum_{\text{perms } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi) = \sum_{\phi} t_{1,\phi(1)} \cdots kt_{i,\phi(i)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

Factor the k out of each term to get the desired equality.

$$= k \cdot \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) = k \cdot d(T)$$

For (2), let $T \xrightarrow{\rho_i \leftrightarrow \rho_j} \hat{T}$.

$$d(\hat{T}) = \sum_{\text{perms } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi)$$

To convert to unhatted t's, for each ϕ consider the permutation σ that equals ϕ except that the i-th and j-th numbers are interchanged, $\sigma(i) = \phi(j)$ and $\sigma(j) = \phi(i)$. Replacing the ϕ in $\hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)}$ with this σ gives $t_{1,\sigma(1)} \cdots t_{j,\sigma(j)} \cdots t_{i,\sigma(i)} \cdots t_{n,\sigma(n)}$. Now $\operatorname{sgn}(\phi) = -\operatorname{sgn}(\sigma)$ (by Lemma 4.3) and so we get

$$\begin{aligned} &= \sum_{\sigma} t_{1,\sigma(1)} \cdots t_{j,\sigma(j)} \cdots t_{i,\sigma(i)} \cdots t_{n,\sigma(n)} \cdot \bigl(-\operatorname{sgn}(\sigma)\bigr) \\ &= -\sum_{\sigma} t_{1,\sigma(1)} \cdots t_{j,\sigma(j)} \cdots t_{i,\sigma(i)} \cdots t_{n,\sigma(n)} \cdot \operatorname{sgn}(\sigma) \end{aligned}$$

where the sum is over all permutations σ derived from another permutation ϕ by a swap of the i-th and j-th numbers. But any permutation can be derived from some other permutation by such a swap, in one and only one way, so this summation is in fact a sum over all permutations, taken once and only once. Thus $d(\hat{T}) = -d(T)$.

To do property (1) let $T \xrightarrow{k\rho_i + \rho_j} \hat{T}$ and consider

$$\begin{aligned} d(\hat{T}) &= \sum_{\text{perms } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi) \\ &= \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots (kt_{i,\phi(j)} + t_{j,\phi(j)}) \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \end{aligned}$$

(notice: that's $kt_{i,\phi(j)}$, not $kt_{j,\phi(j)}$). Distribute, commute, and factor.

$$\begin{aligned}
&= \sum_{\phi} \big[t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots kt_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
&\qquad\quad + t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{j,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)\big] \\
&= \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots kt_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
&\qquad + \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{j,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
&= k \cdot \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
&\qquad + d(T)
\end{aligned}$$

We finish by showing that the terms $t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{i,\phi(j)} \ldots t_{n,\phi(n)} \operatorname{sgn}(\phi)$ add to zero. This sum represents $d(S)$ where S is a matrix equal to T except that row j of S is a copy of row i of T (because the factor is $t_{i,\phi(j)}$, not $t_{j,\phi(j)}$). Thus, S has two equal rows, rows i and j. Since we have already shown that d changes sign on row swaps, as in Lemma 2.3 we conclude that $d(S) = 0$. QED

We have now shown that determinant functions exist for each size. We already know that for each size there is at most one determinant. Therefore, the permutation expansion computes the one and only determinant value of a square matrix.

We end this subsection by proving the other result remaining from the prior subsection, that the determinant of a matrix equals the determinant of its transpose.

4.8 Example Writing out the permutation expansion of the general 3×3 matrix and of its transpose, and comparing corresponding terms

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \cdots + cdh \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + \cdots$$

(terms with the same letters)

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \cdots + dhc \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + \cdots$$

shows that the corresponding permutation matrices are transposes. That is, there is a relationship between these corresponding permutations. Exercise 16 shows that they are inverses.

4.9 Theorem The determinant of a matrix equals the determinant of its transpose.

Proof. Call the matrix T and denote the entries of T^{trans} with s's so that $t_{i,j} = s_{j,i}$. Substitution gives this

$$|T| = \sum_{\text{perms } \phi} t_{1,\phi(1)} \dots t_{n,\phi(n)} \operatorname{sgn}(\phi) = \sum_{\phi} s_{\phi(1),1} \dots s_{\phi(n),n} \operatorname{sgn}(\phi)$$

and we can finish the argument by manipulating the expression on the right to be recognizable as the determinant of the transpose. We have written all permutation expansions (as in the middle expression above) with the row indices ascending. To rewrite the expression on the right in this way, note that because ϕ is a permutation, the row indices in the term on the right $\phi(1), \dots, \phi(n)$ are just the numbers $1, \dots, n$, rearranged. We can thus commute to have these ascend, giving $s_{1,\phi^{-1}(1)} \cdots s_{n,\phi^{-1}(n)}$ (if the column index is j and the row index is $\phi(j)$ then, where the row index is i, the column index is $\phi^{-1}(i)$). Substituting on the right gives

$$= \sum_{\phi^{-1}} s_{1,\phi^{-1}(1)} \cdots s_{n,\phi^{-1}(n)} \operatorname{sgn}(\phi^{-1})$$

(Exercise 15 shows that $\operatorname{sgn}(\phi^{-1}) = \operatorname{sgn}(\phi)$). Since every permutation is the inverse of another, a sum over all ϕ^{-1} is a sum over all permutations ϕ

$$= \sum_{\text{perms } \sigma} s_{1,\sigma(1)} \dots s_{n,\sigma(n)} \operatorname{sgn}(\sigma) = \left|T^{\text{trans}}\right|$$

as required. QED

Exercises

These summarize the notation used in this book for the 2- and 3- permutations.

i	1	2
$\phi_1(i)$	1	2
$\phi_2(i)$	2	1

i	1	2	3
$\phi_1(i)$	1	2	3
$\phi_2(i)$	1	3	2
$\phi_3(i)$	2	1	3
$\phi_4(i)$	2	3	1
$\phi_5(i)$	3	1	2
$\phi_6(i)$	3	2	1

4.10 Give the permutation expansion of a general 2×2 matrix and its transpose.

✓ **4.11** *This problem appears also in the prior subsection.*

(a) Find the inverse of each 2-permutation.

(b) Find the inverse of each 3-permutation.

✓ **4.12** **(a)** Find the signum of each 2-permutation.

(b) Find the signum of each 3-permutation.

4.13 Find the only nonzero term in the permutation expansion of this matrix.

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

Compute that determinant by finding the signum of the associated permutation.

4.14 What is the signum of the n-permutation $\phi = \langle n, n-1, \dots, 2, 1 \rangle$? [Strang 80]

4.15 Prove these.

(a) Every permutation has an inverse.

(b) $\operatorname{sgn}(\phi^{-1}) = \operatorname{sgn}(\phi)$

(c) Every permutation is the inverse of another.

4.16 Prove that the matrix of the permutation inverse is the transpose of the matrix of the permutation $P_{\phi^{-1}} = {P_{\phi}}^{\text{trans}}$, for any permutation ϕ.

✓ **4.17** Show that a permutation matrix with m inversions can be row swapped to the identity in m steps. Contrast this with Corollary 4.6.

✓ **4.18** For any permutation ϕ let $g(\phi)$ be the integer defined in this way.

$$g(\phi) = \prod_{i<j} [\phi(j) - \phi(i)]$$

(This is the product, over all indices i and j with $i < j$, of terms of the given form.)

(a) Compute the value of g on all 2-permutations.

(b) Compute the value of g on all 3-permutations.

(c) Prove that $g(\phi)$ is not 0.

(d) Prove this.

$$\operatorname{sgn}(\phi) = \frac{g(\phi)}{|g(\phi)|}$$

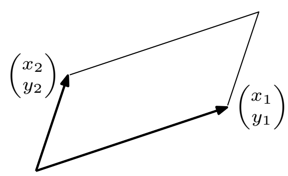
Many authors give this formula as the definition of the signum function.

II Geometry of Determinants

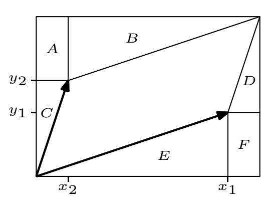
The prior section develops the determinant algebraically, by considering what formulas satisfy certain properties. This section complements that with a geometric approach. One advantage of this approach is that, while we have so far only considered whether or not a determinant is zero, here we shall give a meaning to the value of that determinant. (The prior section handles determinants as functions of the rows, but in this section columns are more convenient. The final result of the prior section says that we can make the switch.)

II.1 Determinants as Size Functions

This parallelogram picture



is familiar from the construction of the sum of the two vectors. One way to compute the area that it encloses is to draw this rectangle and subtract the area of each subregion.



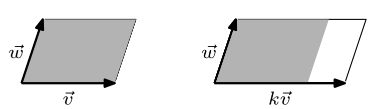
$$\begin{aligned}\text{area of parallelogram}&\\ &= \text{area of rectangle} - \text{area of } A - \text{area of } B\\ &\quad - \cdots - \text{area of } F\\ &= (x_1+x_2)(y_1+y_2) - x_2y_1 - x_1y_1/2\\ &\quad - x_2y_2/2 - x_2y_2/2 - x_1y_1/2 - x_2y_1\\ &= x_1y_2 - x_2y_1\end{aligned}$$

The fact that the area equals the value of the determinant

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

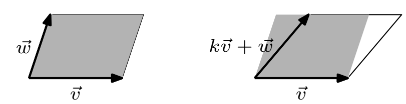
is no coincidence. The properties in the definition of determinants make reasonable postulates for a function that measures the size of the region enclosed by the vectors in the matrix.

For instance, this shows the effect of multiplying one of the box-defining vectors by a scalar (the scalar used is $k = 1.4$).



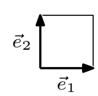
The region formed by $k\vec{v}$ and $\vec{w}$ is bigger, by a factor of k, than the shaded region enclosed by $\vec{v}$ and $\vec{w}$. That is, $\text{size}(k\vec{v},\vec{w}) = k\cdot\text{size}(\vec{v},\vec{w})$ and in general we expect of the size measure that $\text{size}(\ldots,k\vec{v},\ldots) = k\cdot\text{size}(\ldots,\vec{v},\ldots)$. Of course, this postulate is already familiar as one of the properties in the defintion of determinants.

Another property of determinants is that they are unaffected by combining rows. Here are before-combining and after-combining boxes (the scalar used is $k = 0.35$).



Although the region on the right, the box formed by v and $k\vec{v}+\vec{w}$, is more slanted than the shaded region, the two have the same base and the same height and hence the same area. This illustrates that $\text{size}(\vec{v},k\vec{v}+\vec{w}) = \text{size}(\vec{v},\vec{w})$. Generalized, $\text{size}(\ldots,\vec{v},\ldots,\vec{w},\ldots) = \text{size}(\ldots,\vec{v},\ldots,k\vec{v}+\vec{w},\ldots)$, which is a restatement of the determinant postulate.

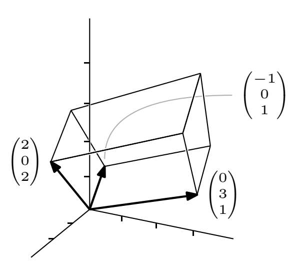
Of course, this picture



shows that $\text{size}(\vec{e}_1,\vec{e}_2) = 1$, and we naturally extend that to any number of dimensions $\text{size}(\vec{e}_1,\ldots,\vec{e}_n) = 1$, which is a restatement of the property that the determinant of the identity matrix is one.

With that, because property (2) of determinants is redundant (as remarked right after the definition), we have that all of the properties of determinants are reasonable to expect of a function that gives the size of boxes. We can now cite the work done in the prior section to show that the determinant exists and is unique to be assured that these postulates are consistent and sufficient (that is, we do not need any more postulates). So we've got an intuitive justification to interpret $\det(\vec{v}_1,\ldots,\vec{v}_n)$ as the size of the box formed by the vectors. (*Comment.* An even more basic approach, which also leads to the definition below, is in [Weston].)

1.1 Example The volume of this parallelepiped, which can be found by the usual formula from high school geometry, is 12.



$$\begin{vmatrix}2&0&-1\\0&3&0\\2&1&1\end{vmatrix} = 12$$

1.2 Remark Although property (2) of the definition of determinants is redundant, it raises an important point. Consider these two.

$$\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix} = 10 \qquad \begin{vmatrix}1 & 4\\ 3 & 2\end{vmatrix} = -10$$

The only difference between them is in the order in which the vectors are taken. If we take $\vec{u}$ first and then go to $\vec{v}$, follow the counterclockwise arc shown, then the sign is positive. Following a clockwise arc gives a negative sign. The sign returned by the size function reflects the *orientation* or *sense* of the box. We see the same thing if we picture the effect of scalar multiplication by a negative scalar.

Although it is both interesting and important, we don't need the idea of orientation for the development below and so we will pass it by. (See Exercise 27.)

1.3 Definition In $\mathbb{R}^n$ the *box* (or *parallelepiped*) formed by $\langle\vec{v}_1,\ldots,\vec{v}_n\rangle$ includes all of the set $\{t_1\vec{v}_1+\cdots+t_n\vec{v}_n \bigm| t_1,\ldots,t_n\in[0..1]\}$. The *volume* of a box is the absolute value of the determinant of the matrix with those vectors as columns.

1.4 Example Volume, because it is an absolute value, does not depend on the order in which the vectors are given. The volume of the parallelepiped in Exercise 1.1, can also be computed as the absolute value of this determinant.

$$\begin{vmatrix}0 & 2 & 0\\ 3 & 0 & 3\\ 1 & 2 & 1\end{vmatrix} = -12$$

The definition of volume gives a geometric interpretation to something in the space, boxes made from vectors. The next result relates the geometry to the functions that operate on spaces.

1.5 Theorem A transformation $t\colon\mathbb{R}^n\to\mathbb{R}^n$ changes the size of all boxes by the same factor, namely the size of the image of a box $|t(S)|$ is $|T|$ times the size of the box $|S|$, where T is the matrix representing t with respect to the standard basis. That is, for all $n{\times}n$ matrices, the determinant of a product is the product of the determinants $|TS|=|T|\cdot|S|$.

The two sentences state the same idea, first in map terms and then in matrix terms. Although we tend to prefer a map point of view, the second sentence, the matrix version, is more convienent for the proof and is also the way that we shall use this result later. (Alternate proofs are given as Exercise 23 and Exercise 28.)

Proof. The two statements are equivalent because $|t(S)| = |TS|$, as both give the size of the box that is the image of the unit box $\mathcal{E}_n$ under the composition $t \circ s$ (where s is the map represented by S with respect to the standard basis).

First consider the case that $|T| = 0$. A matrix has a zero determinant if and only if it is not invertible. Observe that if TS is invertible, so that there is an M such that $(TS)M = I$, then the associative property of matrix multiplication $T(SM) = I$ shows that T is also invertible (with inverse SM). Therefore, if T is not invertible then neither is TS — if $|T| = 0$ then $|TS| = 0$, and the result holds in this case.

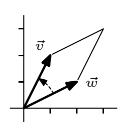
Now consider the case that $|T| \neq 0$, that T is nonsingular. Recall that any nonsingular matrix can be factored into a product of elementary matrices, so that $TS = E_1E_2 \cdots E_rS$. In the rest of this argument, we will verify that if E is an elementary matrix then $|ES| = |E| \cdot |S|$. The result will follow because then $|TS| = |E_1 \cdots E_rS| = |E_1| \cdots |E_r| \cdot |S| = |E_1 \cdots E_r| \cdot |S| = |T| \cdot |S|$.

If the elementary matrix E is $M_i(k)$ then $M_i(k)S$ equals S except that row i has been multiplied by k. The third property of determinant functions then gives that $|M_i(k)S| = k \cdot |S|$. But $|M_i(k)| = k$, again by the third property because $M_i(k)$ is derived from the identity by multiplication of row i by k, and so $|ES| = |E| \cdot |S|$ holds for $E = M_i(k)$. The $E = P_{i,j} = -1$ and $E = C_{i,j}(k)$ checks are similar. QED

1.6 Example Application of the map t represented with respect to the standard bases by

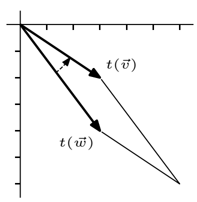
$$\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$$

will double sizes of boxes, e.g., from this



$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

to this



$$\begin{vmatrix} 3 & 3 \\ -4 & -2 \end{vmatrix} = 6$$

1.7 Corollary If a matrix is invertible then the determinant of its inverse is the inverse of its determinant $|T^{-1}| = 1/|T|$.

Proof. $1 = |I| = |TT^{-1}| = |T| \cdot |T^{-1}|$ QED

Recall that determinants are not additive homomorphisms, $\det(A+B)$ need not equal $\det(A) + \det(B)$. The above theorem says, in contrast, that determinants are multiplicative homomorphisms: $\det(AB)$ does equal $\det(A) \cdot \det(B)$.

Exercises

1.8 Find the volume of the region formed.

(a) $\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \end{pmatrix} \rangle$

(b) $\langle \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ -3 \\ 8 \end{pmatrix} \rangle$

(c) $\langle \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 7 \end{pmatrix} \rangle$

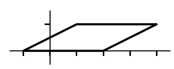
✓ **1.9** Is

$$\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

inside of the box formed by these three?

$$\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

✓ **1.10** Find the volume of this region.



✓ **1.11** Suppose that $|A| = 3$. By what factor do these change volumes?
(a) A **(b)** A^2 **(c)** A^{-2}

✓ **1.12** By what factor does each transformation change the size of boxes?

(a) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ 3y \end{pmatrix}$ **(b)** $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3x - y \\ -2x + y \end{pmatrix}$ **(c)** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ x + y + z \\ y - 2z \end{pmatrix}$

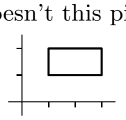
1.13 What is the area of the image of the rectangle $[2..4] \times [2..5]$ under the action of this matrix?

$$\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

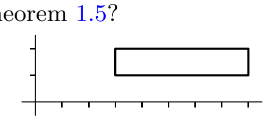
1.14 If $t\colon \mathbb{R}^3 \to \mathbb{R}^3$ changes volumes by a factor of 7 and $s\colon \mathbb{R}^3 \to \mathbb{R}^3$ changes volumes by a factor of $3/2$ then by what factor will their composition changes volumes?

1.15 In what way does the definition of a box differ from the defintion of a span?

✓ **1.16** Why doesn't this picture contradict Theorem 1.5?



$\xrightarrow{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}$



area is 2 determinant is 2 area is 5

✓ **1.17** Does $|TS| = |ST|$? $|T(SP)| = |(TS)P|$?

1.18 **(a)** Suppose that $|A| = 3$ and that $|B| = 2$. Find $|A^2 \cdot B^{\mathrm{trans}} \cdot B^{-2} \cdot A^{\mathrm{trans}}|$.
(b) Assume that $|A| = 0$. Prove that $|6A^3 + 5A^2 + 2A| = 0$.

✓ **1.19** Let T be the matrix representing (with respect to the standard bases) the map that rotates plane vectors counterclockwise thru θ radians. By what factor does T change sizes?

✓ **1.20** Must a transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ that preserves areas also preserve lengths?

✓ **1.21** What is the volume of a parallelepiped in $\mathbb{R}^3$ bounded by a linearly dependent set?

✓ **1.22** Find the area of the triangle in $\mathbb{R}^3$ with endpoints $(1,2,1)$, $(3,-1,4)$, and $(2,2,2)$. (Area, not volume. The triangle defines a plane — what is the area of the triangle in that plane?)

✓ **1.23** An alternate proof of Theorem 1.5 uses the definition of determinant functions.

(a) Note that the vectors forming S make a linearly dependent set if and only if $|S|=0$, and check that the result holds in this case.

(b) For the $|S|\neq 0$ case, to show that $|TS|/|S|=|T|$ for all transformations, consider the function $d\colon \mathcal{M}_{n\times n}\to\mathbb{R}$ given by $T\mapsto |TS|/|S|$. Show that d has the first property of a determinant.

(c) Show that d has the remaining three properties of a determinant function.

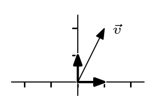
(d) Conclude that $|TS|=|T|\cdot|S|$.

1.24 Give a non-identity matrix with the property that $A^{\text{trans}}=A^{-1}$. Show that if $A^{\text{trans}}=A^{-1}$ then $|A|=\pm 1$. Does the converse hold?

1.25 The algebraic property of determinants that factoring a scalar out of a single row will multiply the determinant by that scalar shows that where H is 3×3, the determinant of cH is c^3 times the determinant of H. Explain this geometrically, that is, using Theorem 1.5. (The observation that increasing the linear size of a three-dimensional object by a factor of c will increase its volume by a factor of c^3 (while only increasing its surface area by an amount proportional to a factor of c^2) is the *Square-cube law* [Wikipedia].)

✓ **1.26** Matrices H and G are said to be *similar* if there is a nonsingular matrix P such that $H=P^{-1}GP$ (we will study this relation in Chapter Five). Show that similar matrices have the same determinant.

1.27 We usually represent vectors in $\mathbb{R}^2$ with respect to the standard basis so vectors in the first quadrant have both coordinates positive.



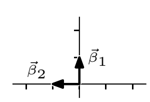
$$\mathrm{Rep}_{\mathcal{E}_2}(\vec{v})=\begin{pmatrix}+3\\+2\end{pmatrix}$$

Moving counterclockwise around the origin, we cycle thru four regions:

$$\cdots\longrightarrow\begin{pmatrix}+\\+\end{pmatrix}\longrightarrow\begin{pmatrix}-\\+\end{pmatrix}\longrightarrow\begin{pmatrix}-\\-\end{pmatrix}\longrightarrow\begin{pmatrix}+\\-\end{pmatrix}\longrightarrow\cdots.$$

Using this basis

$$B=\langle\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\rangle$$



gives the same counterclockwise cycle. We say these two bases have the same *orientation*.

(a) Why do they give the same cycle?

(b) What other configurations of unit vectors on the axes give the same cycle?

(c) Find the determinants of the matrices formed from those (ordered) bases.

(d) What other counterclockwise cycles are possible, and what are the associated determinants?

(e) What happens in $\mathbb{R}^1$?

(f) What happens in $\mathbb{R}^3$?

A fascinating general-audience discussion of orientations is in [Gardner].

1.28 *This question uses material from the optional Determinant Functions Exist subsection.* Prove Theorem 1.5 by using the permutation expansion formula for the determinant.

✓ **1.29** **(a)** Show that this gives the equation of a line in $\mathbb{R}^2$ thru (x_2, y_2) and (x_3, y_3).

$$\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(b) [Petersen] Prove that the area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

(c) [Math. Mag., Jan. 1973] Prove that the area of a triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3) whose coordinates are integers has an area of N or $N/2$ for some positive integer N.

III Other Formulas

(This section is optional. Later sections do not depend on this material.)

Determinants are a fount of interesting and amusing formulas. Here is one that is often seen in calculus classes and used to compute determinants by hand.

III.1 Laplace's Expansion

1.1 Example In this permutation expansion

$$\begin{aligned}\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} &= t_{1,1}t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &\quad + t_{1,2}t_{2,1}t_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,2}t_{2,3}t_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ &\quad + t_{1,3}t_{2,1}t_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{1,3}t_{2,2}t_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\end{aligned}$$

we can, for instance, factor out the entries from the first row

$$\begin{aligned}&= t_{1,1}\cdot\left[t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}\right] \\ &\quad + t_{1,2}\cdot\left[t_{2,1}t_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}\right] \\ &\quad + t_{1,3}\cdot\left[t_{2,1}t_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{2,2}t_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\right]\end{aligned}$$

and swap rows in the permutation matrices to get this.

$$\begin{aligned}&= t_{1,1}\cdot\left[t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}\right] \\ &\quad - t_{1,2}\cdot\left[t_{2,1}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,1}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}\right] \\ &\quad + t_{1,3}\cdot\left[t_{2,1}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,2}t_{3,1}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}\right]\end{aligned}$$

The point of the swapping (one swap to each of the permutation matrices on the second line and two swaps to each on the third line) is that the three lines simplify to three terms.

$$= t_{1,1}\cdot\begin{vmatrix} t_{2,2} & t_{2,3} \\ t_{3,2} & t_{3,3}\end{vmatrix} - t_{1,2}\cdot\begin{vmatrix} t_{2,1} & t_{2,3} \\ t_{3,1} & t_{3,3}\end{vmatrix} + t_{1,3}\cdot\begin{vmatrix} t_{2,1} & t_{2,2} \\ t_{3,1} & t_{3,2}\end{vmatrix}$$

The formula given in Theorem 1.5, which generalizes this example, is a *recurrence* — the determinant is expressed as a combination of determinants. This formula isn't circular because, as here, the determinant is expressed in terms of determinants of matrices of smaller size.

1.2 Definition For any $n\times n$ matrix T, the $(n-1)\times(n-1)$ matrix formed by deleting row i and column j of T is the i,j *minor* of T. The i,j *cofactor* $T_{i,j}$ of T is $(-1)^{i+j}$ times the determinant of the i,j minor of T.

1.3 Example The $1,2$ cofactor of the matrix from Example 1.1 is the negative of the second 2×2 determinant.

$$T_{1,2} = -1\cdot\begin{vmatrix} t_{2,1} & t_{2,3} \\ t_{3,1} & t_{3,3}\end{vmatrix}$$

1.4 Example Where

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{pmatrix}$$

these are the $1,2$ and $2,2$ cofactors.

$$T_{1,2} = (-1)^{1+2}\cdot\begin{vmatrix} 4 & 6 \\ 7 & 9\end{vmatrix} = 6 \qquad T_{2,2} = (-1)^{2+2}\cdot\begin{vmatrix} 1 & 3 \\ 7 & 9\end{vmatrix} = -12$$

1.5 Theorem (Laplace Expansion of Determinants) Where T is an $n\times n$ matrix, the determinant can be found by expanding by cofactors on row i or column j.

$$\begin{aligned}|T| &= t_{i,1}\cdot T_{i,1} + t_{i,2}\cdot T_{i,2} + \cdots + t_{i,n}\cdot T_{i,n} \\ &= t_{1,j}\cdot T_{1,j} + t_{2,j}\cdot T_{2,j} + \cdots + t_{n,j}\cdot T_{n,j}\end{aligned}$$

Proof. Exercise 27. QED

1.6 Example We can compute the determinant

$$|T| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{vmatrix}$$

by expanding along the first row, as in Example 1.1.

$$|T| = 1\cdot(+1)\begin{vmatrix} 5 & 6 \\ 8 & 9\end{vmatrix} + 2\cdot(-1)\begin{vmatrix} 4 & 6 \\ 7 & 9\end{vmatrix} + 3\cdot(+1)\begin{vmatrix} 4 & 5 \\ 7 & 8\end{vmatrix} = -3 + 12 - 9 = 0$$

Alternatively, we can expand down the second column.

$$|T| = 2\cdot(-1)\begin{vmatrix}4&6\\7&9\end{vmatrix} + 5\cdot(+1)\begin{vmatrix}1&3\\7&9\end{vmatrix} + 8\cdot(-1)\begin{vmatrix}1&3\\4&6\end{vmatrix} = 12-60+48=0$$

1.7 Example A row or column with many zeroes suggests a Laplace expansion.

$$\begin{vmatrix}1&5&0\\2&1&1\\3&-1&0\end{vmatrix} = 0\cdot(+1)\begin{vmatrix}2&1\\3&-1\end{vmatrix} + 1\cdot(-1)\begin{vmatrix}1&5\\3&-1\end{vmatrix} + 0\cdot(+1)\begin{vmatrix}1&5\\2&1\end{vmatrix} = 16$$

We finish by applying this result to derive a new formula for the inverse of a matrix. With Theorem 1.5, the determinant of an $n\times n$ matrix T can be calculated by taking linear combinations of entries from a row and their associated cofactors.

$$t_{i,1}\cdot T_{i,1} + t_{i,2}\cdot T_{i,2} + \cdots + t_{i,n}\cdot T_{i,n} = |T| \qquad (*)$$

Recall that a matrix with two identical rows has a zero determinant. Thus, for any matrix T, weighing the cofactors by entries from the "wrong" row — row k with $k\neq i$ — gives zero

$$t_{i,1}\cdot T_{k,1} + t_{i,2}\cdot T_{k,2} + \cdots + t_{i,n}\cdot T_{k,n} = 0 \qquad (**)$$

because it represents the expansion along the row k of a matrix with row i equal to row k. This equation summarizes $(*)$ and $(**)$.

$$\begin{pmatrix}t_{1,1}&t_{1,2}&\ldots&t_{1,n}\\t_{2,1}&t_{2,2}&\ldots&t_{2,n}\\&\vdots&&\\t_{n,1}&t_{n,2}&\ldots&t_{n,n}\end{pmatrix}\begin{pmatrix}T_{1,1}&T_{2,1}&\ldots&T_{n,1}\\T_{1,2}&T_{2,2}&\ldots&T_{n,2}\\&\vdots&&\\T_{1,n}&T_{2,n}&\ldots&T_{n,n}\end{pmatrix} = \begin{pmatrix}|T|&0&\ldots&0\\0&|T|&\ldots&0\\&\vdots&&\\0&0&\ldots&|T|\end{pmatrix}$$

Note that the order of the subscripts in the matrix of cofactors is opposite to the order of subscripts in the other matrix; e.g., along the first row of the matrix of cofactors the subscripts are $1,1$ then $2,1$, etc.

1.8 Definition The matrix *adjoint* to the square matrix T is

$$\operatorname{adj}(T) = \begin{pmatrix}T_{1,1}&T_{2,1}&\ldots&T_{n,1}\\T_{1,2}&T_{2,2}&\ldots&T_{n,2}\\&\vdots&&\\T_{1,n}&T_{2,n}&\ldots&T_{n,n}\end{pmatrix}$$

where $T_{j,i}$ is the j,i cofactor.

1.9 Theorem Where T is a square matrix, $T\cdot\operatorname{adj}(T) = \operatorname{adj}(T)\cdot T = |T|\cdot I$.

Proof. Equations $(*)$ and $(**)$. QED

1.10 Example If

$$T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

then the adjoint $\operatorname{adj}(T)$ is

$$\begin{pmatrix} T_{1,1} & T_{2,1} & T_{3,1} \\ T_{1,2} & T_{2,2} & T_{3,2} \\ T_{1,3} & T_{2,3} & T_{3,3} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4 \\ -3 & -3 & 9 \\ -1 & 0 & 1 \end{pmatrix}$$

and taking the product with T gives the diagonal matrix $|T| \cdot I$.

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4 \\ -3 & -3 & 9 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

1.11 Corollary If $|T| \neq 0$ then $T^{-1} = (1/|T|) \cdot \operatorname{adj}(T)$.

1.12 Example The inverse of the matrix from Example 1.10 is $(1/-3) \cdot \operatorname{adj}(T)$.

$$T^{-1} = \begin{pmatrix} 1/-3 & 0/-3 & -4/-3 \\ -3/-3 & -3/-3 & 9/-3 \\ -1/-3 & 0/-3 & 1/-3 \end{pmatrix} = \begin{pmatrix} -1/3 & 0 & 4/3 \\ 1 & 1 & -3 \\ 1/3 & 0 & -1/3 \end{pmatrix}$$

The formulas from this section are often used for by-hand calculation and are sometimes useful with special types of matrices. However, they are not the best choice for computation with arbitrary matrices because they require more arithmetic than, for instance, the Gauss-Jordan method.

Exercises

✓ **1.13** Find the cofactor.

$$T = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \\ 0 & 2 & -1 \end{pmatrix}$$

(a) $T_{2,3}$ **(b)** $T_{3,2}$ **(c)** $T_{1,3}$

✓ **1.14** Find the determinant by expanding

$$\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 3 & 0 \end{vmatrix}$$

(a) on the first row **(b)** on the second row **(c)** on the third column.

1.15 Find the adjoint of the matrix in Example 1.6.

✓ **1.16** Find the matrix adjoint to each.

(a) $\begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$ **(c)** $\begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 1 & 8 & 9 \end{pmatrix}$

✓ **1.17** Find the inverse of each matrix in the prior question with Theorem 1.9.

1.18 Find the matrix adjoint to this one.

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

✓ **1.19** Expand across the first row to derive the formula for the determinant of a $2{\times}2$ matrix.

✓ **1.20** Expand across the first row to derive the formula for the determinant of a $3{\times}3$ matrix.

✓ **1.21** **(a)** Give a formula for the adjoint of a $2{\times}2$ matrix.
(b) Use it to derive the formula for the inverse.

✓ **1.22** Can we compute a determinant by expanding down the diagonal?

1.23 Give a formula for the adjoint of a diagonal matrix.

✓ **1.24** Prove that the transpose of the adjoint is the adjoint of the transpose.

1.25 Prove or disprove: $\operatorname{adj}(\operatorname{adj}(T)) = T$.

1.26 A square matrix is *upper triangular* if each i, j entry is zero in the part above the diagonal, that is, when $i > j$.
(a) Must the adjoint of an upper triangular matrix be upper triangular? Lower triangular?
(b) Prove that the inverse of a upper triangular matrix is upper triangular, if an inverse exists.

1.27 *This question requires material from the optional Determinants Exist subsection.* Prove Theorem 1.5 by using the permutation expansion.

1.28 Prove that the determinant of a matrix equals the determinant of its transpose using Laplace's expansion and induction on the size of the matrix.

? 1.29 Show that

$$F_n = \begin{vmatrix} 1 & -1 & 1 & -1 & 1 & -1 & \ldots \\ 1 & 1 & 0 & 1 & 0 & 1 & \ldots \\ 0 & 1 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 0 & 1 & \ldots \\ . & . & . & . & . & . & \ldots \end{vmatrix}$$

where F_n is the n-th term of $1, 1, 2, 3, 5, \ldots, x, y, x+y, \ldots$, the Fibonacci sequence, and the determinant is of order $n-1$. [Am. Math. Mon., Jun. 1949]

Topic: Cramer's Rule

We have introduced determinant functions algebraically by looking for a formula to decide whether a matrix is nonsingular. After that introduction we saw a geometric interpretation, that the determinant function gives the size of the box with sides formed by the columns of the matrix. This Topic makes a connection between the two views.

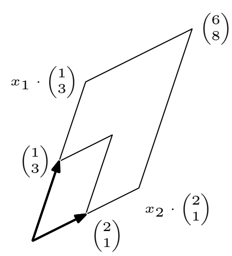
First, a linear system

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 3x_1 + x_2 &= 8 \end{aligned}$$

is equivalent to a linear relationship among vectors.

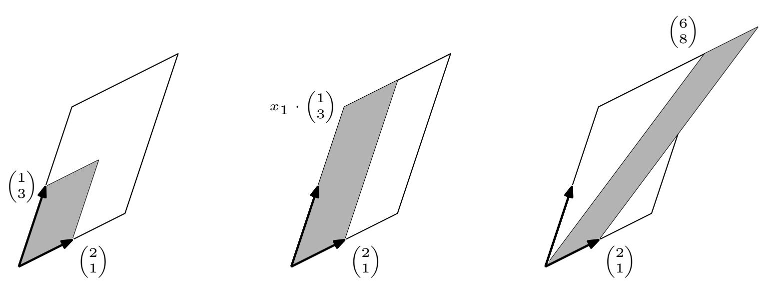
$$x_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

The picture below shows a parallelogram with sides formed from $\binom{1}{3}$ and $\binom{2}{1}$ nested inside a parallelogram with sides formed from $x_1\binom{1}{3}$ and $x_2\binom{2}{1}$.



So even without determinants we can state the algebraic issue that opened this book, finding the solution of a linear system, in geometric terms: by what factors x_1 and x_2 must we dilate the vectors to expand the small parallegram to fill the larger one?

However, by employing the geometric significance of determinants we can get something that is not just a restatement, but also gives us a new insight and sometimes allows us to compute answers quickly. Compare the sizes of these shaded boxes.



The second is formed from $x_1\binom{1}{3}$ and $\binom{2}{1}$, and one of the properties of the size function — the determinant — is that its size is therefore x_1 times the size of the first box. Since the third box is formed from $x_1\binom{1}{3}+x_2\binom{2}{1}=\binom{6}{8}$ and $\binom{2}{1}$, and the determinant is unchanged by adding x_2 times the second column to the first column, the size of the third box equals that of the second. We have this.

$$\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix} = \begin{vmatrix} x_1\cdot 1 & 2 \\ x_1\cdot 3 & 1 \end{vmatrix} = x_1\cdot\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

Solving gives the value of one of the variables.

$$x_1 = \frac{\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

The theorem that generalizes this example, *Cramer's Rule*, is: if $|A|\neq 0$ then the system $A\vec{x}=\vec{b}$ has the unique solution $x_i=|B_i|/|A|$ where the matrix B_i is formed from A by replacing column i with the vector $\vec{b}$. Exercise 3 asks for a proof.

For instance, to solve this system for x_2

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

we do this computation.

$$x_2 = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 4 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix}} = \frac{-18}{-3}$$

Cramer's Rule allows us to solve many two equations/two unknowns systems by eye. It is also sometimes used for three equations/three unknowns systems. But computing large determinants takes a long time, so solving large systems by Cramer's Rule is not practical.

Exercises

1 Use Cramer's Rule to solve each for each of the variables.

(a) $\begin{aligned} x-\ \ y&=4 \\ -x+2y&=-7 \end{aligned}$ **(b)** $\begin{aligned} -2x+\ \ y&=-2 \\ x-2y&=-2 \end{aligned}$

2 Use Cramer's Rule to solve this system for z.

$$\begin{aligned} 2x+y+z&=1 \\ 3x+z&=4 \\ x-y-z&=2 \end{aligned}$$

3 Prove Cramer's Rule.

4 Suppose that a linear system has as many equations as unknowns, that all of its coefficients and constants are integers, and that its matrix of coefficients has determinant 1. Prove that the entries in the solution are all integers. (*Remark.* This is often used to invent linear systems for exercises. If an instructor makes the linear system with this property then the solution is not some disagreeable fraction.)

5 Use Cramer's Rule to give a formula for the solution of a two equations/two unknowns linear system.

6 Can Cramer's Rule tell the difference between a system with no solutions and one with infinitely many?

7 The first picture in this Topic (the one that doesn't use determinants) shows a unique solution case. Produce a similar picture for the case of infintely many solutions, and the case of no solutions.

Topic: Speed of Calculating Determinants

The permutation expansion formula for computing determinants is useful for proving theorems, but the method of using row operations is a much better for finding the determinants of a large matrix. We can make this statement precise by considering, as computer algorithm designers do, the number of arithmetic operations that each method uses.

The speed of an algorithm is measured by finding how the time taken by the computer grows as the size of its input data set grows. For instance, how much longer will the algorithm take if we increase the size of the input data by a factor of ten, from a 1000 row matrix to a $10,000$ row matrix or from $10,000$ to $100,000$? Does the time taken grow by a factor of ten, or by a factor of a hundred, or by a factor of a thousand? That is, is the time taken by the algorithm proportional to the size of the data set, or to the square of that size, or to the cube of that size, etc.?

Recall the permutation expansion formula for determinants.

$$\begin{aligned}
\begin{vmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{vmatrix} &= \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \left|P_\phi\right| \\
&= t_{1,\phi_1(1)} \cdot t_{2,\phi_1(2)} \cdots t_{n,\phi_1(n)} \left|P_{\phi_1}\right| \\
&\quad + t_{1,\phi_2(1)} \cdot t_{2,\phi_2(2)} \cdots t_{n,\phi_2(n)} \left|P_{\phi_2}\right| \\
&\quad \vdots \\
&\quad + t_{1,\phi_k(1)} \cdot t_{2,\phi_k(2)} \cdots t_{n,\phi_k(n)} \left|P_{\phi_k}\right|
\end{aligned}$$

There are $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ different n-permutations. For numbers n of any size at all, this is a large value; for instance, even if n is only 10 then the expansion has $10! = 3,628,800$ terms, all of which are obtained by multiplying n entries together. This is a very large number of multiplications (for instance, [Knuth] suggests $10!$ steps as a rough boundary for the limit of practical calculation). The factorial function grows faster than the square function. It grows faster than the cube function, the fourth power function, or any polynomial function. (One way to see that the factorial function grows faster than the square is to note that multiplying the first two factors in $n!$ gives $n \cdot (n-1)$, which for large n is approximately n^2, and then multiplying in more factors will make it even larger. The same argument works for the cube function, etc.) So a computer that is programmed to use the permutation expansion formula, and thus to perform a number of operations that is greater than or equal to the factorial of the number of rows, would take very long times as its input data set grows.

In contrast, the time taken by the row reduction method does not grow so fast. This fragment of row-reduction code is in the computer language FORTRAN, which is widely used for numeric code. The matrix is stored in the $N \times N$ array A. For each ROW between 1 and N parts of the program not shown here have already found the leading entry $A(ROW, COL)$. Now the program does a row combination.

$$-PIVINV \cdot \rho_{ROW} + \rho_i$$

(This code fragment is for illustration only and is incomplete. Still, analysis of a finished version that includes all of the tests and subcases is messier but gives essentially the same conclusion.)

```
PIVINV=1.0/A(ROW,COL)
DO 10 I=ROW+1, N
   DO 20 J=I, N
      A(I,J)=A(I,J)-PIVINV*A(ROW,J)
   20 CONTINUE
10 CONTINUE
```

The outermost loop (not shown) runs through $N-1$ rows. For each row, the nested I and J loops shown perform arithmetic on the entries in A that are below and to the right of the leading entry. Assume that this entry is found in the expected place, that is, that $COL = ROW$. Then there are $(N - ROW)^2$ entries below and to the right of it. On average, ROW will be $N/2$. Thus, we estimate that the arithmetic will be performed about $(N/2)^2$ times, that is, will run in a time proportional to the square of the number of equations. Taking into account the outer loop that is not shown, we get the estimate that the running time of the algorithm is proportional to the cube of the number of equations.

Finding the fastest algorithm to compute the determinant is a topic of current research. Algorithms are known that run in time between the second and third power.

Speed estimates like these help us to understand how quickly or slowly an algorithm will run. Algorithms that run in time proportional to the size of the data set are fast, algorithms that run in time proportional to the square of the size of the data set are less fast, but typically quite usable, and algorithms that run in time proportional to the cube of the size of the data set are still reasonable in speed for not-too-big input data. However, algorithms that run in time (greater than or equal to) the factorial of the size of the data set are not practical for input of any appreciable size.

There are other methods besides the two discussed here that are also used for computation of determinants. Those lie outside of our scope. Nonetheless, this contrast of the two methods for computing determinants makes the point that although in principle they give the same answer, in practice the idea is to select the one that is fast.

Exercises

Most of these problems presume access to a computer.

1 Computer systems generate random numbers (of course, these are only pseudo-random, in that they are generated by an algorithm, but they pass a number of reasonable statistical tests for randomness).

(a) Fill a 5×5 array with random numbers (say, in the range $[0..1)$). See if it is singular. Repeat that experiment a few times. Are singular matrices frequent or rare (in this sense)?

(b) Time your computer algebra system at finding the determinant of ten 5×5 arrays of random numbers. Find the average time per array. Repeat the prior item for 15×15 arrays, 25×25 arrays, 35×35 arrays, etc. You may find that you need to get above a certain size to get a timing that you can use. (Notice that, when an array is singular, it can sometimes be found to be so quite quickly, for instance if the first row equals the second. In the light of your answer to the first part, do you expect that singular systems play a large role in your average?)

(c) Graph the input size versus the average time.

2 Compute the determinant of each of these by hand using the two methods discussed above.

(a) $\begin{vmatrix} 2 & 1 \\ 5 & -3 \end{vmatrix}$ **(b)** $\begin{vmatrix} 3 & 1 & 1 \\ -1 & 0 & 5 \\ -1 & 2 & -2 \end{vmatrix}$ **(c)** $\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix}$

Count the number of multiplications and divisions used in each case, for each of the methods. (On a computer, multiplications and divisions take much longer than additions and subtractions, so algorithm designers worry about them more.)

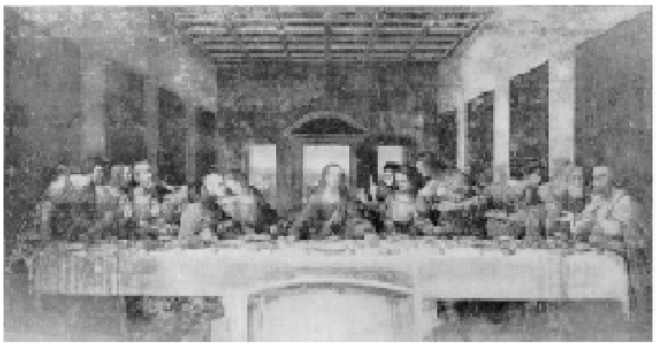
3 What 10×10 array can you invent that takes your computer system the longest to reduce? The shortest?

4 Write the rest of the FORTRAN program to do a straightforward implementation of calculating determinants via Gauss' method. (Don't test for a zero leading entry.) Compare the speed of your code to that used in your computer algebra system.

5 The FORTRAN language specification requires that arrays be stored "by column", that is, the entire first column is stored contiguously, then the second column, etc. Does the code fragment given take advantage of this, or can it be rewritten to make it faster, by taking advantage of the fact that computer fetches are faster from contiguous locations?

Topic: Projective Geometry

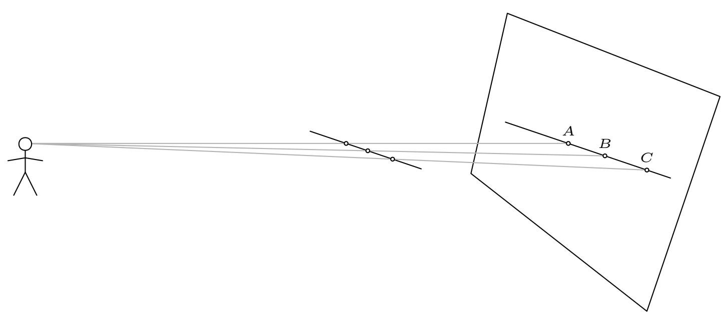
There are geometries other than the familiar Euclidean one. One such geometry arose in art, where it was observed that what a viewer sees is not necessarily what is there. This is Leonardo da Vinci's *The Last Supper*.



What is there in the room, for instance where the ceiling meets the left and right walls, are lines that are parallel. However, what a viewer sees is lines that, if extended, would intersect. The intersection point is called the *vanishing point*. This aspect of perspective is also familiar as the image of a long stretch of railroad tracks that appear to converge at the horizon.

To depict the room, da Vinci has adopted a model of how we see, of how we project the three dimensional scene to a two dimensional image. This model is only a first approximation — it does not take into account that our retina is curved and our lens bends the light, that we have binocular vision, or that our brain's processing greatly affects what we see — but nonetheless it is interesting, both artistically and mathematically.

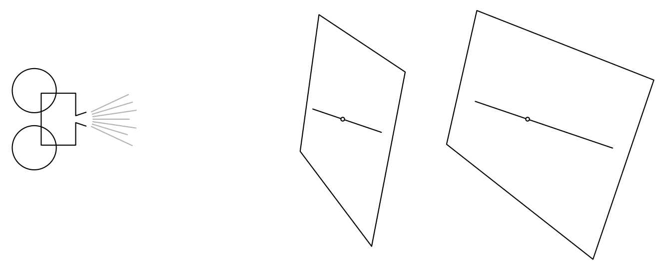
The projection is not orthogonal, it is a *central projection* from a single point, to the plane of the canvas.



(It is not an orthogonal projection since the line from the viewer to C is not orthogonal to the image plane.) As the picture suggests, the operation of central projection preserves some geometric properties — lines project to lines. However, it fails to preserve some others — equal length segments can project to segments of unequal length; the length of AB is greater than the length of

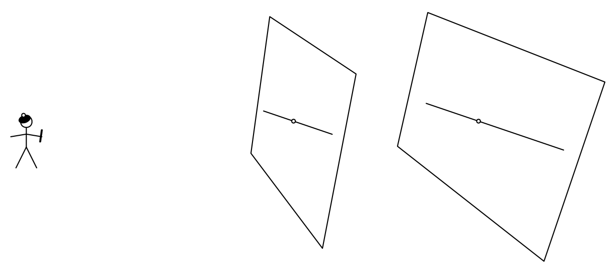
BC because the segment projected to AB is closer to the viewer and closer things look bigger. The study of the effects of central projections is projective geometry. We will see how linear algebra can be used in this study.

There are three cases of central projection. The first is the projection done by a movie projector.



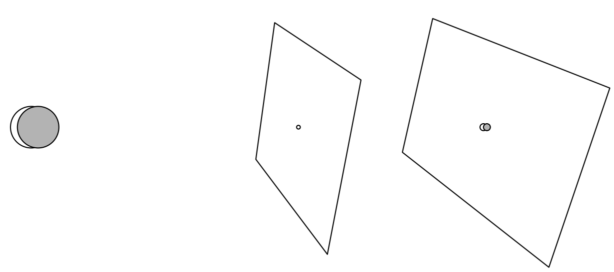
projector P *source S* *image I*

We can think that each source point is "pushed" from the domain plane outward to the image point in the codomain plane. This case of projection has a somewhat different character than the second case, that of the artist "pulling" the source back to the canvas.



painter P *image I* *source S*

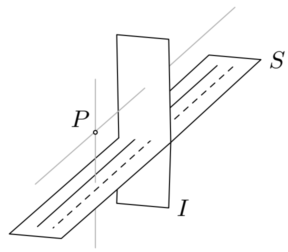
In the first case S is in the middle while in the second case I is in the middle. One more configuration is possible, with P in the middle. An example of this is when we use a pinhole to shine the image of a solar eclipse onto a piece of paper.



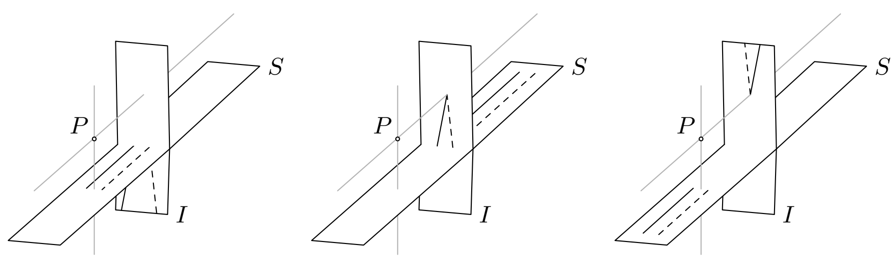
source S *pinhole P* *image I*

We shall take each of the three to be a central projection by P of S to I.

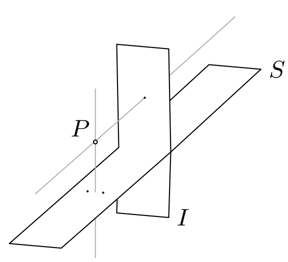
Consider again the effect of railroad tracks that appear to converge to a point. We model this with parallel lines in a domain plane S and a projection via a P to a codomain plane I. (The gray lines are parallel to S and I.)



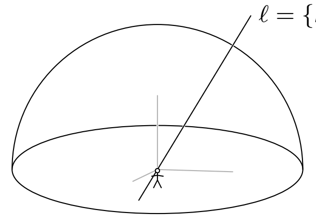
All three projection cases appear here. The first picture below shows P acting like a movie projector by pushing points from part of S out to image points on the lower half of I. The middle picture shows P acting like the artist by pulling points from another part of S back to image points in the middle of I. In the third picture, P acts like the pinhole, projecting points from S to the upper part of I. This picture is the trickiest — the points that are projected near to the vanishing point are the ones that are far out on the bottom left of S. Points in S that are near to the vertical gray line are sent high up on I.



There are two awkward things about this situation. The first is that neither of the two points in the domain nearest to the vertical gray line (see below) has an image because a projection from those two is along the gray line that is parallel to the codomain plane (we sometimes say that these two are projected "to infinity"). The second awkward thing is that the vanishing point in I isn't the image of any point from S because a projection to this point would be along the gray line that is parallel to the domain plane (we sometimes say that the vanishing point is the image of a projection "from infinity").



For a better model, put the projector P at the origin. Imagine that P is covered by a glass hemispheric dome. As P looks outward, anything in the line of vision is projected to the same spot on the dome. This includes things on the line between P and the dome, as in the case of projection by the movie projector. It includes things on the line further from P than the dome, as in the case of projection by the painter. It also includes things on the line that lie behind P, as in the case of projection by a pinhole.



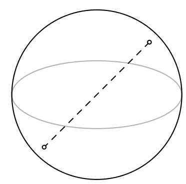
From this perspective P, all of the spots on the line are seen as the same point. Accordingly, for any nonzero vector $\vec{v} \in \mathbb{R}^3$, we define the associated *point v in the projective plane* to be the set $\{k\vec{v} \bigm| k \in \mathbb{R} \text{ and } k \neq 0\}$ of nonzero vectors lying on the same line through the origin as $\vec{v}$. To describe a projective point we can give any representative member of the line, so that the projective point shown above can be represented in any of these three ways.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 1/3 \\ 2/3 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$$

Each of these is a *homogeneous coordinate vector* for v.

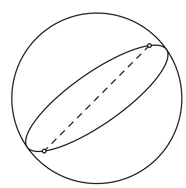
This picture, and the above definition that arises from it, clarifies the description of central projection but there is something awkward about the dome model: what if the viewer looks down? If we draw P's line of sight so that the part coming toward us, out of the page, goes down below the dome then we can trace the line of sight backward, up past P and toward the part of the hemisphere that is behind the page. So in the dome model, looking down gives a projective point that is behind the viewer. Therefore, if the viewer in the picture above drops the line of sight toward the bottom of the dome then the projective point drops also and as the line of sight continues down past the equator, the projective point suddenly shifts from the front of the dome to the back of the dome. This discontinuity in the drawing means that we often have to treat equatorial points as a separate case. That is, while the railroad track discussion of central projection has three cases, the dome model has two.

We can do better than this. Consider a sphere centered at the origin. Any line through the origin intersects the sphere in two spots, which are said to be *antipodal*. Because we associate each line through the origin with a point in the projective plane, we can draw such a point as a pair of antipodal spots on the sphere. Below, the two antipodal spots are shown connected by a dashed line to emphasize that they are not two different points, the pair of spots together make one projective point.



While drawing a point as a pair of antipodal spots is not as natural as the one-spot-per-point dome mode, on the other hand the awkwardness of the dome model is gone, in that if as a line of view slides from north to south, no sudden changes happen on the picture. This model of central projection is uniform — the three cases are reduced to one.

So far we have described points in projective geometry. What about lines? What a viewer at the origin sees as a line is shown below as a great circle, the intersection of the model sphere with a plane through the origin.



(One of the projective points on this line is shown to bring out a subtlety. Because two antipodal spots together make up a single projective point, the great circle's behind-the-paper part is the same set of projective points as its in-front-of-the-paper part.) Just as we did with each projective point, we will also describe a projective line with a triple of reals. For instance, the members of this plane through the origin in $\mathbb{R}^3$

$$\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bigm| x+y-z=0\}$$

project to a line that we can described with the triple $\begin{pmatrix}1 & 1 & -1\end{pmatrix}$ (we use row vectors to typographically distinguish lines from points). In general, for any nonzero three-wide row vector $\vec{L}$ we define the associated *line in the projective plane*, to be the set $L=\{k\vec{L} \bigm| k\in\mathbb{R} \text{ and } k\neq 0\}$ of nonzero multiples of $\vec{L}$.

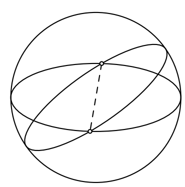
The reason that this description of a line as a triple is convienent is that in the projective plane, a point v and a line L are *incident* — the point lies on the line, the line passes throught the point — if and only if a dot product of their representatives $v_1L_1+v_2L_2+v_3L_3$ is zero (Exercise 4 shows that this is independent of the choice of representatives $\vec{v}$ and $\vec{L}$). For instance, the projective point described above by the column vector with components 1, 2, and 3 lies in the projective line described by $\begin{pmatrix}1 & 1 & -1\end{pmatrix}$, simply because any vector in $\mathbb{R}^3$ whose components are in ratio $1:2:3$ lies in the plane through the origin whose equation is of the form $1k\cdot x + 1k\cdot y - 1k\cdot z = 0$ for any nonzero k. That is, the incidence formula is inherited from the three-space lines and planes of which v and L are projections.

Thus, we can do analytic projective geometry. For instance, the projective line $L = \begin{pmatrix}1 & 1 & -1\end{pmatrix}$ has the equation $1v_1 + 1v_2 - 1v_3 = 0$, because points incident on the line are characterized by having the property that their representatives satisfy this equation. One difference from familiar Euclidean anlaytic geometry is that in projective geometry we talk about the equation of a point. For a fixed point like

$$v = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

the property that characterizes lines through this point (that is, lines incident on this point) is that the components of any representatives satisfy $1L_1 + 2L_2 + 3L_3 = 0$ and so this is the equation of v.

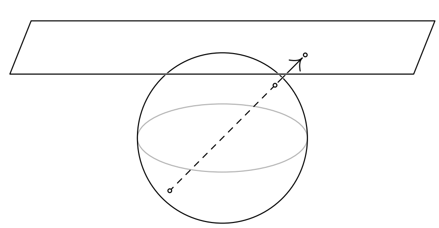
This symmetry of the statements about lines and points brings up the *Duality Principle* of projective geometry: in any true statement, interchanging 'point' with 'line' results in another true statement. For example, just as two distinct points determine one and only one line, in the projective plane, two distinct lines determine one and only one point. Here is a picture showing two lines that cross in antipodal spots and thus cross at one projective point.



$(*)$

Contrast this with Euclidean geometry, where two distinct lines may have a unique intersection or may be parallel. In this way, projective geometry is simpler, more uniform, than Euclidean geometry.

That simplicity is relevant because there is a relationship between the two spaces: the projective plane can be viewed as an extension of the Euclidean plane. Take the sphere model of the projective plane to be the unit sphere in $\mathbb{R}^3$ and take Euclidean space to be the plane $z = 1$. This gives us a way of viewing some points in projective space as corresponding to points in Euclidean space, because all of the points on the plane are projections of antipodal spots from the sphere.



$(**)$

Note though that projective points on the equator don't project up to the plane. Instead, these project 'out to infinity'. We can thus think of projective space as consisting of the Euclidean plane with some extra points adjoined — the Euclidean plane is embedded in the projective plane. These extra points, the equatorial points, are the *ideal points* or *points at infinity* and the equator is the *ideal line* or *line at infinity* (note that it is not a Euclidean line, it is a projective line).

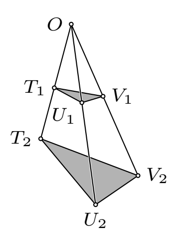
The advantage of the extension to the projective plane is that some of the awkwardness of Euclidean geometry disappears. For instance, the projective lines shown above in ($*$) cross at antipodal spots, a single projective point, on the sphere's equator. If we put those lines into ($**$) then they correspond to Euclidean lines that are parallel. That is, in moving from the Euclidean plane to the projective plane, we move from having two cases, that lines either intersect or are parallel, to having only one case, that lines intersect (possibly at a point at infinity).

The projective case is nicer in many ways than the Euclidean case but has the problem that we don't have the same experience or intuitions with it. That's one advantage of doing analytic geometry, where the equations can lead us to the right conclusions. Analytic projective geometry uses linear algebra. For instance, for three points of the projective plane t, u, and v, setting up the equations for those points by fixing vectors representing each, shows that the three are collinear — incident in a single line — if and only if the resulting three-equation system has infinitely many row vector solutions representing that line. That, in turn, holds if and only if this determinant is zero.

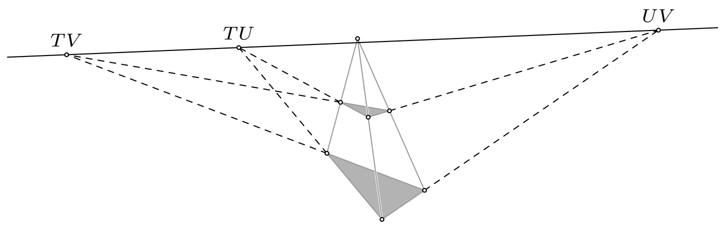
$$\begin{vmatrix} t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \\ t_3 & u_3 & v_3 \end{vmatrix}$$

Thus, three points in the projective plane are collinear if and only if any three representative column vectors are linearly dependent. Similarly (and illustrating the Duality Principle), three lines in the projective plane are incident on a single point if and only if any three row vectors representing them are linearly dependent.

The following result is more evidence of the 'niceness' of the geometry of the projective plane, compared to the Euclidean case. These two triangles are said to be *in perspective* from P because their corresponding vertices are collinear.



Consider the pairs of corresponding sides: the sides T_1U_1 and T_2U_2, the sides T_1V_1 and T_2V_2, and the sides U_1V_1 and U_2V_2. Desargue's Theorem is that when the three pairs of corresponding sides are extended to lines, they intersect (shown here as the point TU, the point TV, and the point UV), and further, those three intersection points are collinear.



We will prove this theorem, using projective geometry. (These are drawn as Euclidean figures because it is the more familiar image. To consider them as projective figures, we can imagine that, although the line segments shown are parts of great circles and so are curved, the model has such a large radius compared to the size of the figures that the sides appear in this sketch to be straight.)

For this proof, we need a preliminary lemma [Coxeter]: if W, X, Y, Z are four points in the projective plane (no three of which are collinear) then there are homogeneous coordinate vectors $\vec{w}$, $\vec{x}$, $\vec{y}$, and $\vec{z}$ for the projective points, and a basis B for $\mathbb{R}^3$, satisfying this.

$$\mathrm{Rep}_B(\vec{w}) = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \mathrm{Rep}_B(\vec{x}) = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \mathrm{Rep}_B(\vec{y}) = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \mathrm{Rep}_B(\vec{z}) = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

The proof is straightforward. Because W, X, Y are not on the same projective line, any homogeneous coordinate vectors $\vec{w}_0$, $\vec{x}_0$, $\vec{y}_0$ do not line on the same plane through the origin in $\mathbb{R}^3$ and so form a spanning set for $\mathbb{R}^3$. Thus any homogeneous coordinate vector for Z can be written as a combination $\vec{z}_0 = a\cdot\vec{w}_0 + b\cdot\vec{x}_0 + c\cdot\vec{y}_0$. Then, we can take $\vec{w} = a\cdot\vec{w}_0$, $\vec{x} = b\cdot\vec{x}_0$, $\vec{y} = c\cdot\vec{y}_0$, and $\vec{z} = \vec{z}_0$, where the basis is $B = \langle\vec{w},\vec{x},\vec{y}\rangle$.

Now, to prove of Desargue's Theorem, use the lemma to fix homogeneous coordinate vectors and a basis.

$$\mathrm{Rep}_B(\vec{t}_1) = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \mathrm{Rep}_B(\vec{u}_1) = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \mathrm{Rep}_B(\vec{o}) = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Because the projective point T_2 is incident on the projective line OT_1, any homogeneous coordinate vector for T_2 lies in the plane through the origin in $\mathbb{R}^3$ that is spanned by homogeneous coordinate vectors of O and T_1:

$$\mathrm{Rep}_B(\vec{t}_2) = a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\\0\end{pmatrix}$$

for some scalars a and b. That is, the homogenous coordinate vectors of members T_2 of the line OT_1 are of the form on the left below, and the forms for U_2 and V_2 are similar.

$$\mathrm{Rep}_B(\vec{t}_2) = \begin{pmatrix} t_2 \\ 1 \\ 1 \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{u}_2) = \begin{pmatrix} 1 \\ u_2 \\ 1 \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} 1 \\ 1 \\ v_2 \end{pmatrix}$$

The projective line T_1U_1 is the image of a plane through the origin in $\mathbb{R}^3$. A quick way to get its equation is to note that any vector in it is linearly dependent on the vectors for T_1 and U_1 and so this determinant is zero.

$$\begin{vmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{vmatrix} = 0 \qquad \Longrightarrow \qquad z = 0$$

The equation of the plane in $\mathbb{R}^3$ whose image is the projective line T_2U_2 is this.

$$\begin{vmatrix} t_2 & 1 & x \\ 1 & u_2 & y \\ 1 & 1 & z \end{vmatrix} = 0 \qquad \Longrightarrow \qquad (1-u_2)\cdot x + (1-t_2)\cdot y + (t_2u_2 - 1)\cdot z = 0$$

Finding the intersection of the two is routine.

$$T_1U_1 \,\cap\, T_2U_2 = \begin{pmatrix} t_2 - 1 \\ 1 - u_2 \\ 0 \end{pmatrix}$$

(This is, of course, the homogeneous coordinate vector of a projective point.) The other two intersections are similar.

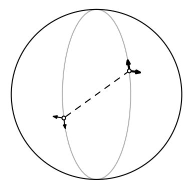
$$T_1V_1 \,\cap\, T_2V_2 = \begin{pmatrix} 1 - t_2 \\ 0 \\ v_2 - 1 \end{pmatrix} \qquad U_1V_1 \,\cap\, U_2V_2 = \begin{pmatrix} 0 \\ u_2 - 1 \\ 1 - v_2 \end{pmatrix}$$

The proof is finished by noting that these projective points are on one projective line because the sum of the three homogeneous coordinate vectors is zero.

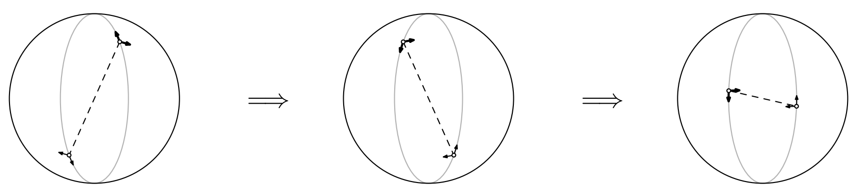
Every projective theorem has a translation to a Euclidean version, although the Euclidean result is often messier to state and prove. Desargue's theorem illustrates this. In the translation to Euclidean space, the case where O lies on the ideal line must be treated separately for then the lines T_1T_2, U_1U_2, and V_1V_2 are parallel.

The parenthetical remark following the statement of Desargue's Theorem suggests thinking of the Euclidean pictures as figures from projective geometry for a model of very large radius. That is, just as a small area of the earth appears flat to people living there, the projective plane is also 'locally Euclidean'.

Although its local properties are the familiar Euclidean ones, there is a global property of the projective plane that is quite different. The picture below shows a projective point. At that point is drawn an xy-axis. There is something interesting about the way this axis appears at the antipodal ends of the sphere. In the northern hemisphere, where the axis are drawn in black, a right hand put down with fingers on the x-axis will have the thumb point along the y-axis. But the antipodal axis has just the opposite: a right hand placed with its fingers on the x-axis will have the thumb point in the wrong way, instead, it is a left hand that works. Briefly, the projective plane is not orientable: in this geometry, left and right handedness are not fixed properties of figures.



The sequence of pictures below dramatizes this non-orientability. They sketch a trip around this space in the direction of the y part of the xy-axis. (Warning: the trip shown is not halfway around, it is a full circuit. True, if we made this into a movie then we could watch the northern hemisphere spots in the drawing above gradually rotate about halfway around the sphere to the last picture below. And we could watch the southern hemisphere spots in the picture above slide through the south pole and up through the equator to the last picture. But: the spots at either end of the dashed line are the same projective point. We don't need to continue on much further; we are pretty much back to the projective point where we started by the last picture.)



At the end of the circuit, the x part of the xy-axes sticks out in the other direction. Thus, in the projective plane we cannot describe a figure as right- or left-handed (another way to make this point is that we cannot describe a spiral as clockwise or counterclockwise).

This exhibition of the existence of a non-orientable space raises the question of whether our universe is orientable: is is possible for an astronaut to leave right-handed and return left-handed? An excellent nontechnical reference is [Gardner]. An classic science fiction story about orientation reversal is [Clarke].

So projective geometry is mathematically interesting, in addition to the natural way in which it arises in art. It is more than just a technical device to shorten some proofs. For an overview, see [Courant & Robbins]. The approach we've taken here, the analytic approach, leads to quick theorems and — most importantly for us — illustrates the power of linear algebra (see [Hanes], [Ryan], and [Eggar]). But another approach, the synthetic approach of deriving the results from an axiom system, is both extraordinarily beautiful and is also the historical route of development. Two fine sources for this approach are [Coxeter] or [Seidenberg]. An interesting and easy application is [Davies]

Exercises

1 What is the equation of this point?

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2 **(a)** Find the line incident on these points in the projective plane.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

(b) Find the point incident on both of these projective lines.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

3 Find the formula for the line incident on two projective points. Find the formula for the point incident on two projective lines.

4 Prove that the definition of incidence is independent of the choice of the representatives of p and L. That is, if p_1, p_2, p_3, and q_1, q_2, q_3 are two triples of homogeneous coordinates for p, and L_1, L_2, L_3, and M_1, M_2, M_3 are two triples of homogeneous coordinates for L, prove that $p_1L_1 + p_2L_2 + p_3L_3 = 0$ if and only if $q_1M_1 + q_2M_2 + q_3M_3 = 0$.

5 Give a drawing to show that central projection does not preserve circles, that a circle may project to an ellipse. Can a (non-circular) ellipse project to a circle?

6 Give the formula for the correspondence between the non-equatorial part of the antipodal modal of the projective plane, and the plane $z = 1$.

7 (Pappus's Theorem) Assume that T_0, U_0, and V_0 are collinear and that T_1, U_1, and V_1 are collinear. Consider these three points: (i) the intersection V_2 of the lines T_0U_1 and T_1U_0, (ii) the intersection U_2 of the lines T_0V_1 and T_1V_0, and (iii) the intersection T_2 of U_0V_1 and U_1V_0.

(a) Draw a (Euclidean) picture.

(b) Apply the lemma used in Desargue's Theorem to get simple homogeneous coordinate vectors for the T's and V_0.

(c) Find the resulting homogeneous coordinate vectors for U's (these must each involve a parameter as, e.g., U_0 could be anywhere on the T_0V_0 line).

(d) Find the resulting homogeneous coordinate vectors for V_1. (*Hint:* it involves two parameters.)

(e) Find the resulting homogeneous coordinate vectors for V_2. (It also involves two parameters.)

(f) Show that the product of the three parameters is 1.

(g) Verify that V_2 is on the T_2U_2 line.

Chapter Five

Similarity

While studying matrix equivalence, we have shown that for any homomorphism there are bases B and D such that the representation matrix has a block partial-identity form.

$$\mathrm{Rep}_{B,D}(h) = \left(\begin{array}{c|c} \mathit{Identity} & \mathit{Zero} \\ \hline \mathit{Zero} & \mathit{Zero} \end{array}\right)$$

This representation describes the map as sending $c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ to $c_1\vec{\delta}_1 + \cdots + c_k\vec{\delta}_k + \vec{0} + \cdots + \vec{0}$, where n is the dimension of the domain and k is the dimension of the range. So, under this representation the action of the map is easy to understand because most of the matrix entries are zero.

This chapter considers the special case where the domain and the codomain are equal, that is, where the homomorphism is a transformation. In this case we naturally ask to find a single basis B so that $\mathrm{Rep}_{B,B}(t)$ is as simple as possible (we will take 'simple' to mean that it has many zeroes). A matrix having the above block partial-identity form is not always possible here. But we will develop a form that comes close, a representation that is nearly diagonal.

I Complex Vector Spaces

This chapter requires that we factor polynomials. Of course, many polynomials do not factor over the real numbers; for instance, $x^2 + 1$ does not factor into the product of two linear polynomials with real coefficients. For that reason, we shall from now on take our scalars from the complex numbers.

That is, we are shifting from studying vector spaces over the real numbers to vector spaces over the complex numbers — in this chapter vector and matrix entries are complex.

Any real number is a complex number and a glance through this chapter shows that most of the examples use only real numbers. Nonetheless, the critical theorems require that the scalars be complex numbers, so the first section below is a quick review of complex numbers.

In this book we are moving to the more general context of taking scalars to be complex only for the pragmatic reason that we must do so in order to develop the representation. We will not go into using other sets of scalars in more detail because it could distract from our goal. However, the idea of taking scalars from a structure other than the real numbers is an interesting one. Delightful presentations taking this approach are in [Halmos] and [Hoffman & Kunze].

I.1 Factoring and Complex Numbers; A Review

This subsection is a review only and we take the main results as known. For proofs, see [Birkhoff & MacLane] or [Ebbinghaus].

Just as integers have a division operation — e.g., '4 goes 5 times into 21 with remainder 1' — so do polynomials.

1.1 Theorem (Division Theorem for Polynomials) Let $c(x)$ be a polynomial. If $m(x)$ is a non-zero polynomial then there are *quotient* and *remainder* polynomials $q(x)$ and $r(x)$ such that

$$c(x) = m(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is strictly less than the degree of $m(x)$.

In this book constant polynomials, including the zero polynomial, are said to have degree 0. (This is not the standard definition, but it is convienent here.)

The point of the integer division statement '4 goes 5 times into 21 with remainder 1' is that the remainder is less than 4 — while 4 goes 5 times, it does not go 6 times. In the same way, the point of the polynomial division statement is its final clause.

1.2 Example If $c(x) = 2x^3 - 3x^2 + 4x$ and $m(x) = x^2 + 1$ then $q(x) = 2x - 3$ and $r(x) = 2x + 3$. Note that $r(x)$ has a lower degree than $m(x)$.

1.3 Corollary The remainder when $c(x)$ is divided by $x - \lambda$ is the constant polynomial $r(x) = c(\lambda)$.

Proof. The remainder must be a constant polynomial because it is of degree less than the divisor $x - \lambda$, To determine the constant, take $m(x)$ from the theorem to be $x - \lambda$ and substitute λ for x to get $c(\lambda) = (\lambda - \lambda) \cdot q(\lambda) + r(x)$. QED

If a divisor $m(x)$ goes into a dividend $c(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $m(x)$ is a *factor* of $c(x)$. Any *root* of the factor (any $\lambda \in \mathbb{R}$ such that $m(\lambda) = 0$) is a root of $c(x)$ since $c(\lambda) = m(\lambda) \cdot q(\lambda) = 0$. The prior corollary immediately yields the following converse.

1.4 Corollary If λ is a root of the polynomial $c(x)$ then $x - \lambda$ divides $c(x)$ evenly, that is, $x - \lambda$ is a factor of $c(x)$.

Finding the roots and factors of a high-degree polynomial can be hard. But for second-degree polynomials we have the quadratic formula: the roots of $ax^2 + bx + c$ are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(if the discriminant $b^2 - 4ac$ is negative then the polynomial has no real number roots). A polynomial that cannot be factored into two lower-degree polynomials with real number coefficients is *irreducible over the reals*.

1.5 Theorem Any constant or linear polynomial is irreducible over the reals. A quadratic polynomial is irreducible over the reals if and only if its discriminant is negative. No cubic or higher-degree polynomial is irreducible over the reals.

1.6 Corollary Any polynomial with real coefficients can be factored into linear and irreducible quadratic polynomials. That factorization is unique; any two factorizations have the same powers of the same factors.

Note the analogy with the prime factorization of integers. In both cases, the uniqueness clause is very useful.

1.7 Example Because of uniqueness we know, without multiplying them out, that $(x + 3)^2(x^2 + 1)^3$ does not equal $(x + 3)^4(x^2 + x + 1)^2$.

1.8 Example By uniqueness, if $c(x) = m(x) \cdot q(x)$ then where $c(x) = (x - 3)^2(x + 2)^3$ and $m(x) = (x - 3)(x + 2)^2$, we know that $q(x) = (x - 3)(x + 2)$.

While $x^2 + 1$ has no real roots and so doesn't factor over the real numbers, if we imagine a root — traditionally denoted i so that $i^2 + 1 = 0$ — then $x^2 + 1$ factors into a product of linears $(x - i)(x + i)$.

So we adjoin this root i to the reals and close the new system with respect to addition, multiplication, etc. (i.e., we also add $3 + i$, and $2i$, and $3 + 2i$, etc., putting in all linear combinations of 1 and i). We then get a new structure, the *complex numbers*, denoted $\mathbb{C}$.

In $\mathbb{C}$ we can factor (obviously, at least some) quadratics that would be irreducible if we were to stick to the real numbers. Surprisingly, in $\mathbb{C}$ we can not only factor $x^2 + 1$ and its close relatives, we can factor any quadratic.

$$ax^2 + bx + c = a \cdot \big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\big) \cdot \big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\big)$$

1.9 Example The second degree polynomial x^2+x+1 factors over the complex numbers into the product of two first degree polynomials.

$$\big(x - \frac{-1 + \sqrt{-3}}{2}\big)\big(x - \frac{-1 - \sqrt{-3}}{2}\big) = \big(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\big)\big(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)\big)$$

1.10 Corollary (Fundamental Theorem of Algebra) Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

I.2 Complex Representations

Recall the definitions of the complex number addition

$$(a+bi)\ +\ (c+di) = (a+c)+(b+d)i$$

and multiplication.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bd(-1)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

2.1 Example For instance, $(1-2i)\ +\ (5+4i) = 6+2i$ and $(2-3i)(4-0.5i) = 6.5-13i$.

Handling scalar operations with those rules, all of the operations that we've covered for real vector spaces carry over unchanged.

2.2 Example Matrix multiplication is the same, although the scalar arithmetic involves more bookkeeping.

$$\begin{aligned}&\begin{pmatrix}1+1i & 2-0i\\ i & -2+3i\end{pmatrix}\begin{pmatrix}1+0i & 1-0i\\ 3i & -i\end{pmatrix}\\ &=\begin{pmatrix}(1+1i)\cdot(1+0i)+(2-0i)\cdot(3i) & (1+1i)\cdot(1-0i)+(2-0i)\cdot(-i)\\ (i)\cdot(1+0i)+(-2+3i)\cdot(3i) & (i)\cdot(1-0i)+(-2+3i)\cdot(-i)\end{pmatrix}\\ &=\begin{pmatrix}1+7i & 1-1i\\ -9-5i & 3+3i\end{pmatrix}\end{aligned}$$

Everything else from prior chapters that we can, we shall also carry over unchanged. For instance, we shall call this

$$\langle\begin{pmatrix}1+0i\\ 0+0i\\ \vdots\\ 0+0i\end{pmatrix},\dots,\begin{pmatrix}0+0i\\ 0+0i\\ \vdots\\ 1+0i\end{pmatrix}\rangle$$

the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and again denote it $\mathcal{E}_n$.

II Similarity

II.1 Definition and Examples

We've defined H and $\hat{H}$ to be matrix-equivalent if there are nonsingular matrices P and Q such that $\hat{H} = PHQ$. That definition is motivated by this diagram

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ V_{\text{w.r.t. } \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } \hat{D}} \end{array}$$

showing that H and $\hat{H}$ both represent h but with respect to different pairs of bases. We now specialize that setup to the case where the codomain equals the domain, and where the codomain's basis equals the domain's basis.

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow{t} & V_{\text{w.r.t. } B} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ V_{\text{w.r.t. } D} & \xrightarrow{t} & V_{\text{w.r.t. } D} \end{array}$$

To move from the lower left to the lower right we can either go straight over, or up, over, and then down. In matrix terms,

$$\text{Rep}_{D,D}(t) = \text{Rep}_{B,D}(\text{id})\ \text{Rep}_{B,B}(t)\ \bigl(\text{Rep}_{B,D}(\text{id})\bigr)^{-1}$$

(recall that a representation of composition like this one reads right to left).

1.1 Definition The matrices T and S are *similar* if there is a nonsingular P such that $T = PSP^{-1}$.

Since nonsingular matrices are square, the similar matrices T and S must be square and of the same size.

1.2 Example With these two,

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

calculation gives that S is similar to this matrix.

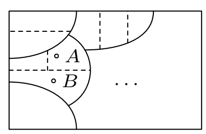
$$T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

1.3 Example The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The only matrix similar to the identity matrix is itself: $PIP^{-1} = PP^{-1} = I$.

Since matrix similarity is a special case of matrix equivalence, if two matrices are similar then they are equivalent. What about the converse: must matrix equivalent square matrices be similar? The answer is no. The prior example shows that the similarity classes are different from the matrix equivalence classes, because the matrix equivalence class of the identity consists of all nonsingular matrices of that size. Thus, for instance, these two are matrix equivalent but not similar.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

So some matrix equivalence classes split into two or more similarity classes — similarity gives a finer partition than does equivalence. This picture shows some matrix equivalence classes subdivided into similarity classes.



To understand the similarity relation we shall study the similarity classes. We approach this question in the same way that we've studied both the row equivalence and matrix equivalence relations, by finding a canonical form for representatives* of the similarity classes, called Jordan form. With this canonical form, we can decide if two matrices are similar by checking whether they reduce to the same representative. We've also seen with both row equivalence and matrix equivalence that a canonical form gives us insight into the ways in which members of the same class are alike (e.g., two identically-sized matrices are matrix equivalent if and only if they have the same rank).

Exercises

1.4 For

$$S = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 \\ -11/2 & -5 \end{pmatrix} \quad P = \begin{pmatrix} 4 & 2 \\ -3 & 2 \end{pmatrix}$$

check that $T = PSP^{-1}$.

✓ **1.5** Example 1.3 shows that the only matrix similar to a zero matrix is itself and that the only matrix similar to the identity is itself.

(a) Show that the 1×1 matrix (2), also, is similar only to itself.

(b) Is a matrix of the form cI for some scalar c similar only to itself?

(c) Is a diagonal matrix similar only to itself?

1.6 Show that these matrices are not similar.

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

---

* More information on representatives is in the appendix.

1.7 Consider the transformation $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ described by $x^2 \mapsto x+1$, $x \mapsto x^2-1$, and $1 \mapsto 3$.

(a) Find $T = \mathrm{Rep}_{B,B}(t)$ where $B = \langle x^2, x, 1\rangle$.

(b) Find $S = \mathrm{Rep}_{D,D}(t)$ where $D = \langle 1, 1+x, 1+x+x^2\rangle$.

(c) Find the matrix P such that $T = PSP^{-1}$.

✓ **1.8** Exhibit an nontrivial similarity relationship in this way: let $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ act by

$$\begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}3\\0\end{pmatrix} \qquad \begin{pmatrix}-1\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\2\end{pmatrix}$$

and pick two bases, and represent t with respect to then $T = \mathrm{Rep}_{B,B}(t)$ and $S = \mathrm{Rep}_{D,D}(t)$. Then compute the P and P^{-1} to change bases from B to D and back again.

1.9 Explain Example 1.3 in terms of maps.

✓ **1.10** Are there two matrices A and B that are similar while A^2 and B^2 are not similar? [Halmos]

✓ **1.11** Prove that if two matrices are similar and one is invertible then so is the other.

✓ **1.12** Show that similarity is an equivalence relation.

1.13 Consider a matrix representing, with respect to some B, B, reflection across the x-axis in $\mathbb{R}^2$. Consider also a matrix representing, with respect to some D, D, reflection across the y-axis. Must they be similar?

1.14 Prove that similarity preserves determinants and rank. Does the converse hold?

1.15 Is there a matrix equivalence class with only one matrix similarity class inside? One with infinitely many similarity classes?

1.16 Can two different diagonal matrices be in the same similarity class?

✓ **1.17** Prove that if two matrices are similar then their k-th powers are similar when $k > 0$. What if $k \leq 0$?

✓ **1.18** Let $p(x)$ be the polynomial $c_n x^n + \cdots + c_1 x + c_0$. Show that if T is similar to S then $p(T) = c_n T^n + \cdots + c_1 T + c_0 I$ is similar to $p(S) = c_n S^n + \cdots + c_1 S + c_0 I$.

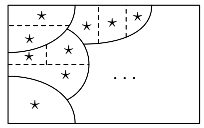
1.19 List all of the matrix equivalence classes of 1×1 matrices. Also list the similarity classes, and describe which similarity classes are contained inside of each matrix equivalence class.

1.20 Does similarity preserve sums?

1.21 Show that if $T - \lambda I$ and N are similar matrices then T and $N + \lambda I$ are also similar.

II.2 Diagonalizability

The prior subsection defines the relation of similarity and shows that, although similar matrices are necessarily matrix equivalent, the converse does not hold. Some matrix-equivalence classes break into two or more similarity classes (the nonsingular $n \times n$ matrices, for instance). This means that the canonical form for matrix equivalence, a block partial-identity, cannot be used as a canonical form for matrix similarity because the partial-identities cannot be in more than one similarity class, so there are similarity classes without one. This picture illustrates. As earlier in this book, class representatives are shown with stars.



We are developing a canonical form for representatives of the similarity classes. We naturally try to build on our previous work, meaning first that the partial identity matrices should represent the similarity classes into which they fall, and beyond that, that the representatives should be as simple as possible. The simplest extension of the partial-identity form is a diagonal form.

2.1 Definition A transformation is *diagonalizable* if it has a diagonal representation with respect to the same basis for the codomain as for the domain. A *diagonalizable matrix* is one that is similar to a diagonal matrix: T is diagonalizable if there is a nonsingular P such that PTP^{-1} is diagonal.

2.2 Example The matrix

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

is diagonalizable.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

2.3 Example Not every matrix is diagonalizable. The square of

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is the zero matrix. Thus, for any map n that N represents (with respect to the same basis for the domain as for the codomain), the composition $n \circ n$ is the zero map. This implies that no such map n can be diagonally represented (with respect to any B, B) because no power of a nonzero diagonal matrix is zero. That is, there is no diagonal matrix in N's similarity class.

That example shows that a diagonal form will not do for a canonical form — we cannot find a diagonal matrix in each matrix similarity class. However, the canonical form that we are developing has the property that if a matrix can be diagonalized then the diagonal matrix is the canonical representative of the similarity class. The next result characterizes which maps can be diagonalized.

2.4 Corollary A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \dots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

Proof. This follows from the definition by considering a diagonal representation matrix.

$$\operatorname{Rep}_{B,B}(t) = \left(\begin{array}{c|c|c} \vdots & & \vdots \\ \operatorname{Rep}_B(t(\vec{\beta}_1)) & \cdots & \operatorname{Rep}_B(t(\vec{\beta}_n)) \\ \vdots & & \vdots \end{array}\right) = \left(\begin{array}{c|c|c} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{array}\right)$$

This representation is equivalent to the existence of a basis satisfying the stated conditions simply by the definition of matrix representation. QED

2.5 Example To diagonalize

$$T = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

we take it as the representation of a transformation with respect to the standard basis $T = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ and we look for a basis $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ such that

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

that is, such that $t(\vec{\beta}_1) = \lambda_1 \vec{\beta}_1$ and $t(\vec{\beta}_2) = \lambda_2 \vec{\beta}_2$.

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \vec{\beta}_1 = \lambda_1 \cdot \vec{\beta}_1 \qquad \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \vec{\beta}_2 = \lambda_2 \cdot \vec{\beta}_2$$

We are looking for scalars x such that this equation

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions b_1 and b_2, which are not both zero. Rewrite that as a linear system.

$$\begin{aligned} (3-x) \cdot b_1 + \qquad 2 \cdot b_2 &= 0 \\ (1-x) \cdot b_2 &= 0 \end{aligned} \qquad (*)$$

In the bottom equation the two numbers multiply to give zero only if at least one of them is zero so there are two possibilities, $b_2 = 0$ and $x = 1$. In the $b_2 = 0$ possibility, the first equation gives that either $b_1 = 0$ or $x = 3$. Since the case of both $b_1 = 0$ and $b_2 = 0$ is disallowed, we are left looking at the possibility of $x = 3$. With it, the first equation in $(*)$ is $0 \cdot b_1 + 2 \cdot b_2 = 0$ and so associated with 3 are vectors with a second component of zero and a first component that is free.

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$$

That is, one solution to $(*)$ is $\lambda_1 = 3$, and we have a first basis vector.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In the $x = 1$ possibility, the first equation in $(*)$ is $2 \cdot b_1 + 2 \cdot b_2 = 0$, and so associated with 1 are vectors whose second component is the negative of their first component.

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix} = 1 \cdot \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix}$$

Thus, another solution is $\lambda_2 = 1$ and a second basis vector is this.

$$\vec{\beta}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To finish, drawing the similarity diagram

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ \mathbb{R}^2_{\text{w.r.t. } B} & \xrightarrow[D]{t} & \mathbb{R}^2_{\text{w.r.t. } B} \end{array}$$

and noting that the matrix $\text{Rep}_{B,\mathcal{E}_2}(\text{id})$ is easy leads to this diagonalization.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

In the next subsection, we will expand on that example by considering more closely the property of Corollary 2.4. This includes seeing another way, the way that we will routinely use, to find the λ's.

Exercises

✓ **2.6** Repeat Example 2.5 for the matrix from Example 2.2.

2.7 Diagonalize these upper triangular matrices.

(a) $\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$

✓ **2.8** What form do the powers of a diagonal matrix have?

2.9 Give two same-sized diagonal matrices that are not similar. Must any two different diagonal matrices come from different similarity classes?

2.10 Give a nonsingular diagonal matrix. Can a diagonal matrix ever be singular?

✓ **2.11** Show that the inverse of a diagonal matrix is the diagonal of the the inverses, if no element on that diagonal is zero. What happens when a diagonal entry is zero?

2.12 The equation ending Example 2.5

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

is a bit jarring because for P we must take the first matrix, which is shown as an inverse, and for P^{-1} we take the inverse of the first matrix, so that the two -1 powers cancel and this matrix is shown without a superscript -1.

(a) Check that this nicer-appearing equation holds.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}$$

(b) Is the previous item a coincidence? Or can we always switch the P and the P^{-1}?

2.13 Show that the P used to diagonalize in Example 2.5 is not unique.

2.14 Find a formula for the powers of this matrix *Hint*: see Exercise 8.

$$\begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix}$$

✓ **2.15** Diagonalize these.

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2.16 We can ask how diagonalization interacts with the matrix operations. Assume that $t, s\colon V \to V$ are each diagonalizable. Is ct diagonalizable for all scalars c? What about $t+s$? $t \circ s$?

✓ 2.17 Show that matrices of this form are not diagonalizable.

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad c \neq 0$$

2.18 Show that each of these is diagonalizable.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ $\quad x, y, z$ scalars

II.3 Eigenvalues and Eigenvectors

In this subsection we will focus on the property of Corollary 2.4.

3.1 Definition A transformation $t\colon V \to V$ has a scalar *eigenvalue* λ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

("Eigen" is German for "characteristic of" or "peculiar to"; some authors call these *characteristic* values and vectors. No authors call them "peculiar".)

3.2 Example The projection map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad x, y, z \in \mathbb{C}$$

has an eigenvalue of 1 associated with any eigenvector of the form

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

where x and y are non-0 scalars. On the other hand, 2 is not an eigenvalue of π since no non-$\vec{0}$ vector is doubled.

That example shows why the 'non-$\vec{0}$' appears in the definition. Disallowing $\vec{0}$ as an eigenvector eliminates trivial eigenvalues. (Note, however, that a matrix can have an eigenvalue $\lambda = 0$.)

3.3 Example The only transformation on the trivial space $\{\vec{0}\,\}$ is $\vec{0} \mapsto \vec{0}$. This map has no eigenvalues because there are no non-$\vec{0}$ vectors $\vec{v}$ mapped to a scalar multiple $\lambda \cdot \vec{v}$ of themselves.

3.4 Example Consider the homomorphism $t\colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1x \mapsto (c_0+c_1)+(c_0+c_1)x$. The range of t is one-dimensional. Thus an application of t to a vector in the range will simply rescale that vector: $c+cx \mapsto (2c)+(2c)x$. That is, t has an eigenvalue of 2 associated with eigenvectors of the form $c+cx$ where $c \neq 0$.

This map also has an eigenvalue of 0 associated with eigenvectors of the form $c-cx$ where $c \neq 0$.

3.5 Definition A square matrix T has a scalar *eigenvalue* λ associated with the non-$\vec{0}$ *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

3.6 Remark Although this extension from maps to matrices is obvious, there is a point that must be made. Eigenvalues of a map are also the eigenvalues of matrices representing that map, and so similar matrices have the same eigenvalues. But the eigenvectors are different — similar matrices need not have the same eigenvectors.

For instance, consider again the transformation $t\colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1x \mapsto (c_0+c_1)+(c_0+c_1)x$. It has an eigenvalue of 2 associated with eigenvectors of the form $c+cx$ where $c \neq 0$. If we represent t with respect to $B = \langle 1+1x, 1-1x\rangle$

$$T = \text{Rep}_{B,B}(t) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

then 2 is an eigenvalue of T, associated with these eigenvectors.

$$\{\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \mid \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2c_0 \\ 2c_1 \end{pmatrix}\} = \{\begin{pmatrix} c_0 \\ 0 \end{pmatrix} \mid c_0 \in \mathbb{C},\ c_0 \neq 0\}$$

On the other hand, representing t with respect to $D = \langle 2+1x, 1+0x\rangle$ gives

$$S = \text{Rep}_{D,D}(t) = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}$$

and the eigenvectors of S associated with the eigenvalue 2 are these.

$$\{\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \mid \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2c_0 \\ 2c_1 \end{pmatrix}\} = \{\begin{pmatrix} 0 \\ c_1 \end{pmatrix} \mid c_1 \in \mathbb{C},\ c_1 \neq 0\}$$

Thus similar matrices can have different eigenvectors.

Here is an informal description of what's happening. The underlying transformation doubles the eigenvectors $\vec{v} \mapsto 2 \cdot \vec{v}$. But when the matrix representing the transformation is $T = \text{Rep}_{B,B}(t)$ then it "assumes" that column vectors are representations with respect to B. In contrast, $S = \text{Rep}_{D,D}(t)$ "assumes" that column vectors are representations with respect to D. So the vectors that get doubled by each matrix look different.

The next example illustrates the basic tool for finding eigenvectors and eigenvalues.

3.7 Example What are the eigenvalues and eigenvectors of this matrix?

$$T=\begin{pmatrix}1&2&1\\2&0&-2\\-1&2&3\end{pmatrix}$$

To find the scalars x such that $T\vec{\zeta}=x\vec{\zeta}$ for non-$\vec{0}$ eigenvectors $\vec{\zeta}$, bring everything to the left-hand side

$$\begin{pmatrix}1&2&1\\2&0&-2\\-1&2&3\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}-x\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\vec{0}$$

and factor $(T-xI)\vec{\zeta}=\vec{0}$. (Note that it says $T-xI$; the expression $T-x$ doesn't make sense because T is a matrix while x is a scalar.) This homogeneous linear system

$$\begin{pmatrix}1-x&2&1\\2&0-x&-2\\-1&2&3-x\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a non-$\vec{0}$ solution if and only if the matrix is singular. We can determine when that happens.

$$\begin{aligned}0&=|T-xI|\\&=\begin{vmatrix}1-x&2&1\\2&0-x&-2\\-1&2&3-x\end{vmatrix}\\&=x^3-4x^2+4x\\&=x(x-2)^2\end{aligned}$$

The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$. To find the associated eigenvectors, plug in each eigenvalue. Plugging in $\lambda_1=0$ gives

$$\begin{pmatrix}1-0&2&1\\2&0-0&-2\\-1&2&3-0\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad\Longrightarrow\qquad\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}a\\-a\\a\end{pmatrix}$$

for a scalar parameter $a\neq 0$ (a is non-0 because eigenvectors must be non-$\vec{0}$). In the same way, plugging in $\lambda_2=2$ gives

$$\begin{pmatrix}1-2&2&1\\2&0-2&-2\\-1&2&3-2\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad\Longrightarrow\qquad\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}b\\0\\b\end{pmatrix}$$

with $b\neq 0$.

3.8 Example If

$$S = \begin{pmatrix} \pi & 1 \\ 0 & 3 \end{pmatrix}$$

(here π is not a projection map, it is the number $3.14\ldots$) then

$$\left| \begin{pmatrix} \pi - x & 1 \\ 0 & 3 - x \end{pmatrix} \right| = (x - \pi)(x - 3)$$

so S has eigenvalues of $\lambda_1 = \pi$ and $\lambda_2 = 3$. To find associated eigenvectors, first plug in λ_1 for x:

$$\begin{pmatrix} \pi - \pi & 1 \\ 0 & 3 - \pi \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

for a scalar $a \neq 0$, and then plug in λ_2:

$$\begin{pmatrix} \pi - 3 & 1 \\ 0 & 3 - 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -b/(\pi - 3) \\ b \end{pmatrix}$$

where $b \neq 0$.

3.9 Definition The *characteristic polynomial* of a square matrix T is the determinant of the matrix $T - xI$, where x is a variable. The *characteristic equation* is $|T - xI| = 0$. The characteristic polynomial of a transformation t is the polynomial of any $\mathrm{Rep}_{B,B}(t)$.

Exercise 30 checks that the characteristic polynomial of a transformation is well-defined, that is, any choice of basis yields the same polynomial.

3.10 Lemma A linear transformation on a nontrivial vector space has at least one eigenvalue.

Proof. Any root of the characteristic polynomial is an eigenvalue. Over the complex numbers, any polynomial of degree one or greater has a root. (This is the reason that in this chapter we've gone to scalars that are complex.) QED

Notice the familiar form of the sets of eigenvectors in the above examples.

3.11 Definition The *eigenspace* of a transformation t associated with the eigenvalue λ is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}\,) = \lambda\vec{\zeta}\,\}$. The eigenspace of a matrix is defined analogously.

3.12 Lemma An eigenspace is a subspace.

Proof. An eigenspace must be nonempty — for one thing it contains the zero vector since a. linear transformation maps the zero vector to the zero vector.

Thus we need only check closure. Take vectors $\vec{\zeta}_1, \ldots, \vec{\zeta}_n$ from V_λ, to show that any linear combination is in V_λ

$$\begin{aligned} t(c_1\vec{\zeta}_1 + c_2\vec{\zeta}_2 + \cdots + c_n\vec{\zeta}_n) &= c_1 t(\vec{\zeta}_1) + \cdots + c_n t(\vec{\zeta}_n) \\ &= c_1\lambda\vec{\zeta}_1 + \cdots + c_n\lambda\vec{\zeta}_n \\ &= \lambda(c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n) \end{aligned}$$

(the second equality holds even if any $\vec{\zeta}_i$ is $\vec{0}$ since $t(\vec{0}) = \lambda \cdot \vec{0} = \vec{0}$). QED

3.13 Example In Example 3.8 the eigenspace associated with the eigenvalue π and the eigenspace associated with the eigenvalue 3 are these.

$$V_\pi = \{\begin{pmatrix} a \\ 0 \end{pmatrix} \bigm| a \in \mathbb{R}\} \qquad V_3 = \{\begin{pmatrix} -b/\pi - 3 \\ b \end{pmatrix} \bigm| b \in \mathbb{R}\}$$

3.14 Example In Example 3.7, these are the eigenspaces associated with the eigenvalues 0 and 2.

$$V_0 = \{\begin{pmatrix} a \\ -a \\ a \end{pmatrix} \bigm| a \in \mathbb{R}\}, \qquad V_2 = \{\begin{pmatrix} b \\ 0 \\ b \end{pmatrix} \bigm| b \in \mathbb{R}\}.$$

3.15 Remark The characteristic equation is $0 = x(x-2)^2$ so in some sense 2 is an eigenvalue "twice". However there are not "twice" as many eigenvectors, in that the dimension of the eigenspace is one, not two. The next example shows a case where a number, 1, is a double root of the characteristic equation and the dimension of the associated eigenspace is two.

3.16 Example With respect to the standard bases, this matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

represents projection.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad x, y, z \in \mathbb{C}$$

Its eigenspace associated with the eigenvalue 0 and its eigenspace associated with the eigenvalue 1 are easy to find.

$$V_0 = \{\begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \bigm| c_3 \in \mathbb{C}\} \qquad V_1 = \{\begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \bigm| c_1, c_2 \in \mathbb{C}\}$$

By the lemma, if two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are associated with the same eigenvalue then any linear combination of those two is also an eigenvector associated with that same eigenvalue. But, if two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are associated with different eigenvalues then the sum $\vec{v}_1 + \vec{v}_2$ need not be related to the eigenvalue of either one. In fact, just the opposite. If the eigenvalues are different then the eigenvectors are not linearly related.

3.17 Theorem For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

PROOF. We will use induction on the number of eigenvalues. If there is no eigenvalue or only one eigenvalue then the set of associated eigenvectors is empty or is a singleton set with a non-$\vec{0}$ member, and in either case is linearly independent.

For induction, assume that the theorem is true for any set of k distinct eigenvalues, suppose that $\lambda_1, \ldots, \lambda_{k+1}$ are distinct eigenvalues, and let $\vec{v}_1, \ldots, \vec{v}_{k+1}$ be associated eigenvectors. If $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} = \vec{0}$ then after multiplying both sides of the displayed equation by λ_{k+1}, applying the map or matrix to both sides of the displayed equation, and subtracting the first result from the second, we have this.

$$c_1(\lambda_{k+1} - \lambda_1)\vec{v}_1 + \cdots + c_k(\lambda_{k+1} - \lambda_k)\vec{v}_k + c_{k+1}(\lambda_{k+1} - \lambda_{k+1})\vec{v}_{k+1} = \vec{0}$$

The induction hypothesis now applies: $c_1(\lambda_{k+1}-\lambda_1) = 0, \ldots, c_k(\lambda_{k+1}-\lambda_k) = 0$. Thus, as all the eigenvalues are distinct, $c_1, \ldots, c_k$ are all 0. Finally, now c_{k+1} must be 0 because we are left with the equation $\vec{v}_{k+1} \neq \vec{0}$. QED

3.18 Example The eigenvalues of

$$\begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{pmatrix}$$

are distinct: $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. A set of associated eigenvectors like

$$\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\}$$

is linearly independent.

3.19 Corollary An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

PROOF. Form a basis of eigenvectors. Apply Corollary 2.4. QED

Exercises

3.20 For each, find the characteristic polynomial and the eigenvalues.

(a) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 & 3 \\ 7 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

✓ **3.21** For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

(a) $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

3.22 Find the characteristic equation, and the eigenvalues and associated eigenvectors for this matrix. *Hint.* The eigenvalues are complex.

$$\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$$

3.23 Find the characteristic polynomial, the eigenvalues, and the associated eigenvectors of this matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

✓ **3.24** For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

(a) $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

✓ **3.25** Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be

$$a_0 + a_1x + a_2x^2 \mapsto (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find its eigenvalues and the associated eigenvectors.

3.26 Find the eigenvalues and eigenvectors of this map $t\colon \mathcal{M}_2 \to \mathcal{M}_2$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 2c & a+c \\ b-2c & d \end{pmatrix}$$

✓ **3.27** Find the eigenvalues and associated eigenvectors of the differentiation operator $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$.

3.28 Prove that the eigenvalues of a triangular matrix (upper or lower triangular) are the entries on the diagonal.

✓ **3.29** Find the formula for the characteristic polynomial of a 2×2 matrix.

3.30 Prove that the characteristic polynomial of a transformation is well-defined.

3.31 Prove or disprove: if all the eigenvalues of a matrix are 0 then it must be the zero matrix.

✓ **3.32** **(a)** Show that any non-$\vec{0}$ vector in any nontrivial vector space can be a eigenvector. That is, given a $\vec{v} \neq \vec{0}$ from a nontrivial V, show that there is a transformation $t\colon V \to V$ having a scalar eigenvalue $\lambda \in \mathbb{R}$ such that $\vec{v} \in V_\lambda$.

(b) What if we are given a a scalar λ? Can any non-$\vec{0}$ member of any nontrivial vector space be an eigenvector associated with λ?

✓ **3.33** Suppose that $t\colon V \to V$ and $T = \text{Rep}_{B,B}(t)$. Prove that the eigenvectors of T associated with λ are the non-$\vec{0}$ vectors in the kernel of the map represented (with respect to the same bases) by $T - \lambda I$.

3.34 Prove that if $a, \ldots, d$ are all integers and $a + b = c + d$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integral eigenvalues, namely $a + b$ and $a - c$.

✓ **3.35** Prove that if T is nonsingular and has eigenvalues $\lambda_1, \ldots, \lambda_n$ then T^{-1} has eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Is the converse true?

✓ **3.36** Suppose that T is $n \times n$ and c, d are scalars.

(a) Prove that if T has the eigenvalue λ with an associated eigenvector $\vec{v}$ then $\vec{v}$ is an eigenvector of $cT + dI$ associated with eigenvalue $c\lambda + d$.

(b) Prove that if T is diagonalizable then so is $cT + dI$.

✓ **3.37** Show that λ is an eigenvalue of T if and only if the map represented by $T - \lambda I$ is not an isomorphism.

3.38 [Strang 80]

(a) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k.

(b) What is wrong with this proof generalizing that? "If λ is an eigenvalue of A and μ is an eigenvalue for B, then $\lambda\mu$ is an eigenvalue for AB, for, if $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$ then $AB\vec{x} = A\mu\vec{x} = \mu A\vec{x} = \mu\lambda\vec{x}$"?

3.39 Do matrix-equivalent matrices have the same eigenvalues?

3.40 Show that a square matrix with real entries and an odd number of rows has at least one real eigenvalue.

3.41 Diagonalize.

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$$

3.42 Suppose that P is a nonsingular $n \times n$ matrix. Show that the *similarity transformation* map $t_P \colon \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ sending $T \mapsto PTP^{-1}$ is an isomorphism.

? **3.43** Show that if A is an n square matrix and each row (column) sums to c then c is a characteristic root of A. [Math. Mag., Nov. 1967]

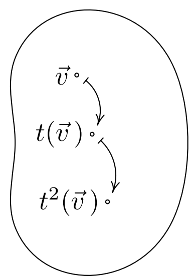
III Nilpotence

The goal of this chapter is to show that every square matrix is similar to one that is a sum of two kinds of simple matrices. The prior section focused on the first simple kind, diagonal matrices. We now consider the other kind.

III.1 Self-Composition

This subsection is optional, although it is necessary for later material in this section and in the next one.

A linear transformations $t\colon V \to V$, because it has the same domain and codomain, can be iterated.* That is, compositions of t with itself such as $t^2 = t \circ t$ and $t^3 = t \circ t \circ t$ are defined.



Note that this power notation for the linear transformation functions dovetails with the notation that we've used earlier for their square matrix representations because if $\text{Rep}_{B,B}(t) = T$ then $\text{Rep}_{B,B}(t^j) = T^j$.

1.1 Example For the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$a + bx + cx^2 + dx^3 \overset{d/dx}{\longmapsto} b + 2cx + 3dx^2$$

the second power is the second derivative

$$a + bx + cx^2 + dx^3 \overset{d^2/dx^2}{\longmapsto} 2c + 6dx$$

the third power is the third derivative

$$a + bx + cx^2 + dx^3 \overset{d^3/dx^3}{\longmapsto} 6d$$

and any higher power is the zero map.

1.2 Example This transformation of the space of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} b & a \\ d & 0 \end{pmatrix}$$

* More information on function interation is in the appendix.

has this second power

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{t^2}{\longmapsto} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

and this third power.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{t^3}{\longmapsto} \begin{pmatrix} b & a \\ 0 & 0 \end{pmatrix}$$

After that, $t^4 = t^2$ and $t^5 = t^3$, etc.

These examples suggest that on iteration more and more zeros appear until there is a settling down. The next result makes this precise.

1.3 Lemma For any transformation $t\colon V \to V$, the rangespaces of the powers form a descending chain

$$V \supseteq \mathscr{R}(t) \supseteq \mathscr{R}(t^2) \supseteq \cdots$$

and the nullspaces form an ascending chain.

$$\{\vec{0}\,\} \subseteq \mathscr{N}(t) \subseteq \mathscr{N}(t^2) \subseteq \cdots$$

Further, there is a k such that for powers less than k the subsets are proper (if $j < k$ then $\mathscr{R}(t^j) \supset \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) \subset \mathscr{N}(t^{j+1})$), while for powers greater than k the sets are equal (if $j \geq k$ then $\mathscr{R}(t^j) = \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) = \mathscr{N}(t^{j+1})$).

Proof. We will do the rangespace half and leave the rest for Exercise 13. Recall, however, that for any map the dimension of its rangespace plus the dimension of its nullspace equals the dimension of its domain. So if the rangespaces shrink then the nullspaces must grow.

That the rangespaces form chains is clear because if $\vec{w} \in \mathscr{R}(t^{j+1})$, so that $\vec{w} = t^{j+1}(\vec{v})$, then $\vec{w} = t^j(\,t(\vec{v})\,)$ and so $\vec{w} \in \mathscr{R}(t^j)$. To verify the "further" property, first observe that if any pair of rangespaces in the chain are equal $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1})$ then all subsequent ones are also equal $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$, etc. This is because if $t\colon \mathscr{R}(t^{k+1}) \to \mathscr{R}(t^{k+2})$ is the same map, with the same domain, as $t\colon \mathscr{R}(t^k) \to \mathscr{R}(t^{k+1})$ and it therefore has the same range: $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$ (and induction shows that it holds for all higher powers). So if the chain of rangespaces ever stops being strictly decreasing then it is stable from that point onward.

But the chain must stop decreasing. Each rangespace is a subspace of the one before it. For it to be a proper subspace it must be of strictly lower dimension (see Exercise 11). These spaces are finite-dimensional and so the chain can fall for only finitely-many steps, that is, the power k is at most the dimension of V. QED

1.4 Example The derivative map $a + bx + cx^2 + dx^3 \stackrel{d/dx}{\longmapsto} b + 2cx + 3dx^2$ of Example 1.1 has this chain of rangespaces

$$\mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1 \supset \mathcal{P}_0 \supset \{\vec{0}\,\} = \{\vec{0}\,\} = \cdots$$

and this chain of nullspaces.

$$\{\vec{0}\,\} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 = \mathcal{P}_3 = \cdots$$

1.5 Example The transformation $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

has $\mathbb{C}^3 \supset \mathscr{R}(\pi) = \mathscr{R}(\pi^2) = \cdots$ and $\{\vec{0}\,\} \subset \mathscr{N}(\pi) = \mathscr{N}(\pi^2) = \cdots$.

1.6 Example Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the map $c_0 + c_1 x + c_2 x^2 \mapsto 2c_0 + c_2 x$. As the lemma describes, on iteration the rangespace shrinks

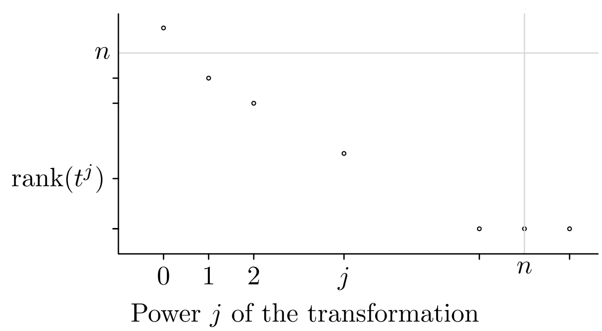
$$\mathscr{R}(t^0) = \mathcal{P}_2 \quad \mathscr{R}(t) = \{a + bx \mid a, b \in \mathbb{C}\} \quad \mathscr{R}(t^2) = \{a \mid a \in \mathbb{C}\}$$

and then stabilizes $\mathscr{R}(t^2) = \mathscr{R}(t^3) = \cdots$, while the nullspace grows

$$\mathscr{N}(t^0) = \{0\} \quad \mathscr{N}(t) = \{cx \mid c \in \mathbb{C}\} \quad \mathscr{N}(t^2) = \{cx + d \mid c, d \in \mathbb{C}\}$$

and then stabilizes $\mathscr{N}(t^2) = \mathscr{N}(t^3) = \cdots$.

This graph illustrates Lemma 1.3. The horizontal axis gives the power j of a transformation. The vertical axis gives the dimension of the rangespace of t^j as the distance above zero — and thus also shows the dimension of the nullspace as the distance below the gray horizontal line, because the two add to the dimension n of the domain.



As sketched, on iteration the rank falls and with it the nullity grows until the two reach a steady state. This state must be reached by the n-th iterate. The steady state's distance above zero is the dimension of the generalized rangespace and its distance below n is the dimension of the generalized nullspace.

1.7 Definition Let t be a transformation on an n-dimensional space. The *generalized rangespace* (or the *closure of the rangespace*) is $\mathscr{R}_\infty(t) = \mathscr{R}(t^n)$ The *generalized nullspace* (or the *closure of the nullspace*) is $\mathscr{N}_\infty(t) = \mathscr{N}(t^n)$.

Exercises

1.8 Give the chains of rangespaces and nullspaces for the zero and identity transformations.

1.9 For each map, give the chain of rangespaces and the chain of nullspaces, and the generalized rangespace and the generalized nullspace.

(a) $t_0\colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx^2$

(b) $t_1\colon \mathbb{R}^2 \to \mathbb{R}^2$,
$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ a \end{pmatrix}$$

(c) $t_2\colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx + ax^2$

(d) $t_3\colon \mathbb{R}^3 \to \mathbb{R}^3$,
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ a \\ b \end{pmatrix}$$

1.10 Prove that function composition is associative $(t \circ t) \circ t = t \circ (t \circ t)$ and so we can write t^3 without specifying a grouping.

1.11 Check that a subspace must be of dimension less than or equal to the dimension of its superspace. Check that if the subspace is proper (the subspace does not equal the superspace) then the dimension is strictly less. *(This is used in the proof of Lemma 1.3.)*

1.12 Prove that the generalized rangespace $\mathscr{R}_\infty(t)$ is the entire space, and the generalized nullspace $\mathscr{N}_\infty(t)$ is trivial, if the transformation t is nonsingular. Is this 'only if' also?

1.13 Verify the nullspace half of Lemma 1.3.

1.14 Give an example of a transformation on a three dimensional space whose range has dimension two. What is its nullspace? Iterate your example until the rangespace and nullspace stabilize.

1.15 Show that the rangespace and nullspace of a linear transformation need not be disjoint. Are they ever disjoint?

III.2 Strings

This subsection is optional, and requires material from the optional Direct Sum subsection.

The prior subsection shows that as j increases, the dimensions of the $\mathscr{R}(t^j)$'s fall while the dimensions of the $\mathscr{N}(t^j)$'s rise, in such a way that this rank and nullity split the dimension of V. Can we say more; do the two split a basis — is $V = \mathscr{R}(t^j) \oplus \mathscr{N}(t^j)$?

The answer is yes for the smallest power $j = 0$ since $V = \mathscr{R}(t^0) \oplus \mathscr{N}(t^0) = V \oplus \{\vec{0}\}$. The answer is also yes at the other extreme.

2.1 Lemma Where $t\colon V \to V$ is a linear transformation, the space is the direct sum $V = \mathscr{R}_\infty(t) \oplus \mathscr{N}_\infty(t)$. That is, both $\dim(V) = \dim(\mathscr{R}_\infty(t)) + \dim(\mathscr{N}_\infty(t))$ and $\mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t) = \{\vec{0}\,\}$.

Proof. We will verify the second sentence, which is equivalent to the first. The first clause, that the dimension n of the domain of t^n equals the rank of t^n plus the nullity of t^n, holds for any transformation and so we need only verify the second clause.

Assume that $\vec{v} \in \mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t) = \mathscr{R}(t^n) \cap \mathscr{N}(t^n)$, to prove that $\vec{v}$ is $\vec{0}$. Because $\vec{v}$ is in the nullspace, $t^n(\vec{v}) = \vec{0}$. On the other hand, because $\mathscr{R}(t^n) = \mathscr{R}(t^{n+1})$, the map $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is a dimension-preserving homomorphism and therefore is one-to-one. A composition of one-to-one maps is one-to-one, and so $t^n\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one. But now — because only $\vec{0}$ is sent by a one-to-one linear map to $\vec{0}$ — the fact that $t^n(\vec{v}) = \vec{0}$ implies that $\vec{v} = \vec{0}$. QED

2.2 Note Technically we should distinguish the map $t\colon V \to V$ from the map $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ because the domains or codomains might differ. The second one is said to be the *restriction*[*] of t to $\mathscr{R}(t^k)$. We shall use later a point from that proof about the restriction map, namely that it is nonsingular.

In contrast to the $j = 0$ and $j = n$ cases, for intermediate powers the space V might not be the direct sum of $\mathscr{R}(t^j)$ and $\mathscr{N}(t^j)$. The next example shows that the two can have a nontrivial intersection.

2.3 Example Consider the transformation of $\mathbb{C}^2$ defined by this action on the elements of the standard basis.

$$\begin{pmatrix}1\\0\end{pmatrix} \overset{n}{\longmapsto} \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \overset{n}{\longmapsto} \begin{pmatrix}0\\0\end{pmatrix} \qquad N = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(n) = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

The vector

$$\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

is in both the rangespace and nullspace. Another way to depict this map's action is with a *string*.

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{0}$$

2.4 Example A map $\hat{n}\colon \mathbb{C}^4 \to \mathbb{C}^4$ whose action on $\mathcal{E}_4$ is given by the string

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{e}_3 \mapsto \vec{e}_4 \mapsto \vec{0}$$

has $\mathscr{R}(\hat{n}) \cap \mathscr{N}(\hat{n})$ equal to the span $[\{\vec{e}_4\}]$, has $\mathscr{R}(\hat{n}^2) \cap \mathscr{N}(\hat{n}^2) = [\{\vec{e}_3, \vec{e}_4\}]$, and has $\mathscr{R}(\hat{n}^3) \cap \mathscr{N}(\hat{n}^3) = [\{\vec{e}_4\}]$. The matrix representation is all zeros except for some subdiagonal ones.

$$\hat{N} = \mathrm{Rep}_{\mathcal{E}_4,\mathcal{E}_4}(\hat{n}) = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

[*] More information on map restrictions is in the appendix.

2.5 Example Transformations can act via more than one string. A transformation t acting on a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_5 \rangle$ by

$$\begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_4 \mapsto \vec{\beta}_5 \mapsto \vec{0} \end{array}$$

is represented by a matrix that is all zeros except for blocks of subdiagonal ones

$$\text{Rep}_{B,B}(t) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

(the lines just visually organize the blocks).

In those three examples all vectors are eventually transformed to zero.

2.6 Definition A *nilpotent* transformation is one with a power that is the zero map. A *nilpotent matrix* is one with a power that is the zero matrix. In either case, the least such power is the *index of nilpotency*.

2.7 Example In Example 2.3 the index of nilpotency is two. In Example 2.4 it is four. In Example 2.5 it is three.

2.8 Example The differentiation map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$ is nilpotent of index three since the third derivative of any quadratic polynomial is zero. This map's action is described by the string $x^2 \mapsto 2x \mapsto 2 \mapsto 0$ and taking the basis $B = \langle x^2, 2x, 2 \rangle$ gives this representation.

$$\text{Rep}_{B,B}(d/dx) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Not all nilpotent matrices are all zeros except for blocks of subdiagonal ones.

2.9 Example With the matrix $\hat{N}$ from Example 2.4, and this four-vector basis

$$D = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

a change of basis operation produces this representation with respect to D, D.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -3 & -2 & 5 & 0 \\ -2 & -1 & 3 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}$$

The new matrix is nilpotent; it's fourth power is the zero matrix since

$$(P\hat{N}P^{-1})^4 = P\hat{N}P^{-1}\cdot P\hat{N}P^{-1}\cdot P\hat{N}P^{-1}\cdot P\hat{N}P^{-1} = P\hat{N}^4P^{-1}$$

and $\hat{N}^4$ is the zero matrix.

The goal of this subsection is Theorem 2.13, which shows that the prior example is prototypical in that every nilpotent matrix is similar to one that is all zeros except for blocks of subdiagonal ones.

2.10 Definition Let t be a nilpotent transformation on V. A *t-string generated by $\vec{v}\in V$* is a sequence $\langle\vec{v},t(\vec{v}),\ldots,t^{k-1}(\vec{v})\rangle$. This sequence has *length k*. A *t-string basis* is a basis that is a concatenation of t-strings.

2.11 Example In Example 2.5, the t-strings $\langle\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_3\rangle$ and $\langle\vec{\beta}_4,\vec{\beta}_5\rangle$, of length three and two, can be concatenated to make a basis for the domain of t.

2.12 Lemma If a space has a t-string basis then the longest string in it has length equal to the index of nilpotency of t.

Proof. Suppose not. Those strings cannot be longer; if the index is k then t^k sends any vector — including those starting the string — to $\vec{0}$. So suppose instead that there is a transformation t of index k on some space, such that the space has a t-string basis where all of the strings are shorter than length k. Because t has index k, there is a vector $\vec{v}$ such that $t^{k-1}(\vec{v})\neq\vec{0}$. Represent $\vec{v}$ as a linear combination of basis elements and apply t^{k-1}. We are supposing that t^{k-1} sends each basis element to $\vec{0}$ but that it does not send $\vec{v}$ to $\vec{0}$. That is impossible. QED

We shall show that every nilpotent map has an associated string basis. Then our goal theorem, that every nilpotent matrix is similar to one that is all zeros except for blocks of subdiagonal ones, is immediate, as in Example 2.5.

Looking for a counterexample, a nilpotent map without an associated string basis that is disjoint, will suggest the idea for the proof. Consider the map $t\colon\mathbb{C}^5\to\mathbb{C}^5$ with this action.

$$\begin{tikzcd}[row sep=small] \vec{e}_1 \arrow[dr, mapsto] & & \\ & \vec{e}_3 \arrow[r, mapsto] & \vec{0} \\ \vec{e}_2 \arrow[ur, mapsto] & & \\ \vec{e}_4 \arrow[r, mapsto] & \vec{e}_5 \arrow[r, mapsto] & \vec{0} \end{tikzcd} \qquad \mathrm{Rep}_{\mathcal{E}_5,\mathcal{E}_5}(t)=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 1&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0 \end{pmatrix}$$

Even after ommitting the zero vector, these three strings aren't disjoint, but that doesn't end hope of finding a t-string basis. It only means that $\mathcal{E}_5$ will not do for the string basis.

To find a basis that will do, we first find the number and lengths of its strings. Since t's index of nilpotency is two, Lemma 2.12 says that at least one string in the basis has length two. Thus the map must act on a string basis in one of these two ways.

$$\begin{array}{lll} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} & \qquad & \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \\ \vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0} & & \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_5 \mapsto \vec{0} & & \vec{\beta}_4 \mapsto \vec{0} \\ & & \vec{\beta}_5 \mapsto \vec{0} \end{array}$$

Now, the key point. A transformation with the left-hand action has a nullspace of dimension three since that's how many basis vectors are sent to zero. A transformation with the right-hand action has a nullspace of dimension four. Using the matrix representation above, calculation of t's nullspace

$$\mathscr{N}(t) = \{ \begin{pmatrix} x \\ -x \\ z \\ 0 \\ r \end{pmatrix} \bigm| x, z, r \in \mathbb{C} \}$$

shows that it is three-dimensional, meaning that we want the left-hand action.

To produce a string basis, first pick $\vec{\beta}_2$ and $\vec{\beta}_4$ from $\mathscr{R}(t) \cap \mathscr{N}(t)$

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(other choices are possible, just be sure that $\{\vec{\beta}_2, \vec{\beta}_4\}$ is linearly independent). For $\vec{\beta}_5$ pick a vector from $\mathscr{N}(t)$ that is not in the span of $\{\vec{\beta}_2, \vec{\beta}_4\}$.

$$\vec{\beta}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Finally, take $\vec{\beta}_1$ and $\vec{\beta}_3$ such that $t(\vec{\beta}_1) = \vec{\beta}_2$ and $t(\vec{\beta}_3) = \vec{\beta}_4$.

$$\vec{\beta}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, with respect to $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_5 \rangle$, the matrix of t is as desired.

$$\operatorname{Rep}_{B,B}(t) = \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2.13 Theorem Any nilpotent transformation t is associated with a t-string basis. While the basis is not unique, the number and the length of the strings is determined by t.

Proof. This illustrates the argument below, which describes three kinds of basis vectors (these basis vectors are shown as squares or circles, according to whether they are in the nullspace or not).

$$\begin{array}{lllllllllll}
3 & \mapsto & 1 & \mapsto & \cdots & & \cdots & \mapsto & 1 & \mapsto & \boxed{1} \mapsto \vec{0} \\
3 & \mapsto & 1 & \mapsto & \cdots & & \cdots & \mapsto & 1 & \mapsto & \boxed{1} \mapsto \vec{0} \\
 & \vdots & & & & & & & & & \\
3 & \mapsto & 1 & \mapsto & \cdots & \mapsto & 1 & \mapsto & \boxed{1} & \mapsto & \vec{0} \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & & \\
 & \vdots & & & & & & & & & \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & &
\end{array}$$

Fix a vector space V; we will argue by induction on the index of nilpotency of $t\colon V \to V$. If that index is 1 then t is the zero map and any basis is a string basis $\vec{\beta}_1 \mapsto \vec{0}$, ..., $\vec{\beta}_n \mapsto \vec{0}$. For the inductive step, assume that the theorem holds for any transformation with an index of nilpotency between 1 and $k-1$ and consider the index k case.

First observe that the restriction to the rangespace $t\colon \mathscr{R}(t) \to \mathscr{R}(t)$ is also nilpotent, of index $k-1$. Apply the inductive hypothesis to get a string basis for $\mathscr{R}(t)$, where the number and length of the strings is determined by t.

$$B = \langle \vec{\beta}_1, t(\vec{\beta}_1), \dots, t^{h_1}(\vec{\beta}_1) \rangle^\frown \langle \vec{\beta}_2, \dots, t^{h_2}(\vec{\beta}_2) \rangle^\frown \cdots ^\frown \langle \vec{\beta}_i, \dots, t^{h_i}(\vec{\beta}_i) \rangle$$

(In the illustration these are the basis vectors of kind 1, so there are i strings shown with this kind of basis vector.)

Second, note that taking the final nonzero vector in each string gives a basis $C = \langle t^{h_1}(\vec{\beta}_1), \dots, t^{h_i}(\vec{\beta}_i) \rangle$ for $\mathscr{R}(t) \cap \mathscr{N}(t)$. (These are illustrated with 1's in squares.) For, a member of $\mathscr{R}(t)$ is mapped to zero if and only if it is a linear combination of those basis vectors that are mapped to zero. Extend C to a basis for all of $\mathscr{N}(t)$.

$$\hat{C} = C^\frown \langle \vec{\xi}_1, \dots, \vec{\xi}_p \rangle$$

(The $\vec{\xi}$'s are the vectors of kind 2 so that $\hat{C}$ is the set of squares.) While many choices are possible for the $\vec{\xi}$'s, their number p is determined by the map t as it is the dimension of $\mathscr{N}(t)$ minus the dimension of $\mathscr{R}(t) \cap \mathscr{N}(t)$.

Finally, $B^\frown\hat{C}$ is a basis for $\mathscr{R}(t)+\mathscr{N}(t)$ because any sum of something in the rangespace with something in the nullspace can be represented using elements of B for the rangespace part and elements of $\hat{C}$ for the part from the nullspace. Note that

$$\begin{aligned}\dim\bigl(\mathscr{R}(t)+\mathscr{N}(t)\bigr) &= \dim(\mathscr{R}(t))+\dim(\mathscr{N}(t))-\dim(\mathscr{R}(t)\cap\mathscr{N}(t)) \\ &= \operatorname{rank}(t)+\operatorname{nullity}(t)-i \\ &= \dim(V)-i\end{aligned}$$

and so $B^\frown\hat{C}$ can be extended to a basis for all of V by the addition of i more vectors. Specifically, remember that each of $\vec{\beta}_1,\dots,\vec{\beta}_i$ is in $\mathscr{R}(t)$, and extend $B^\frown\hat{C}$ with vectors $\vec{v}_1,\dots,\vec{v}_i$ such that $t(\vec{v}_1)=\vec{\beta}_1,\dots,t(\vec{v}_i)=\vec{\beta}_i$. (In the illustration, these are the 3's.) The check that linear independence is preserved by this extension is Exercise 29. QED

2.14 Corollary Every nilpotent matrix is similar to a matrix that is all zeros except for blocks of subdiagonal ones. That is, every nilpotent map is represented with respect to some basis by such a matrix.

This form is unique in the sense that if a nilpotent matrix is similar to two such matrices then those two simply have their blocks ordered differently. Thus this is a canonical form for the similarity classes of nilpotent matrices provided that we order the blocks, say, from longest to shortest.

2.15 Example The matrix

$$M=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$$

has an index of nilpotency of two, as this calculation shows.

p	M^p	$\mathscr{N}(M^p)$
1	$M=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$	$\{\begin{pmatrix}x\\x\end{pmatrix}\mid x\in\mathbb{C}\}$
2	$M^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\mathbb{C}^2$

The calculation also describes how a map m represented by M must act on any string basis. With one map application the nullspace has dimension one and so one vector of the basis is sent to zero. On a second application, the nullspace has dimension two and so the other basis vector is sent to zero. Thus, the action of the map is $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$ and the canonical form of the matrix is this.

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

We can exhibit such a m-string basis and the change of basis matrices witnessing the matrix similarity. For the basis, take M to represent m with respect to the standard bases, pick a $\vec{\beta}_2 \in \mathscr{N}(m)$ and also pick a $\vec{\beta}_1$ so that $m(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(If we take M to be a representative with respect to some nonstandard bases then this picking step is just more messy.) Recall the similarity diagram.

$$\begin{tikzcd}
\mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} \arrow[r, "m", "M"'] \arrow[d, "\mathrm{id}"', "P"] & \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} \arrow[d, "\mathrm{id}"', "P"] \\
C^2_{\text{w.r.t. } B} \arrow[r, "m"] & \mathbb{C}^2_{\text{w.r.t. } B}
\end{tikzcd}$$

The canonical form equals $\mathrm{Rep}_{B,B}(m) = PMP^{-1}$, where

$$P^{-1} = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad P = (P^{-1})^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and the verification of the matrix calculation is routine.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

2.16 Example The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \end{pmatrix}$$

is nilpotent. These calculations show the nullspaces growing.

p	N^p	$\mathscr{N}(N^p)$
1	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \end{pmatrix}$	$\{\begin{pmatrix} 0 \\ 0 \\ u-v \\ u \\ v \end{pmatrix} \bigm\vert u, v \in \mathbb{C}\}$
2	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{\begin{pmatrix} 0 \\ y \\ z \\ u \\ v \end{pmatrix} \bigm\vert y, z, u, v \in \mathbb{C}\}$
3	*–zero matrix–*	$\mathbb{C}^5$

That table shows that any string basis must satisfy: the nullspace after one map application has dimension two so two basis vectors are sent directly to zero, the nullspace after the second application has dimension four so two additional basis vectors are sent to zero by the second iteration, and the nullspace after three applications is of dimension five so the final basis vector is sent to zero in three hops.

$$\begin{array}{lllllll} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{\beta}_3 & \mapsto & \vec{0} \\ \vec{\beta}_4 & \mapsto & \vec{\beta}_5 & \mapsto & \vec{0} & & \end{array}$$

To produce such a basis, first pick two independent vectors from $\mathscr{N}(n)$

$$\vec{\beta}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{\beta}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

then add $\vec{\beta}_2, \vec{\beta}_4 \in \mathscr{N}(n^2)$ such that $n(\vec{\beta}_2) = \vec{\beta}_3$ and $n(\vec{\beta}_4) = \vec{\beta}_5$

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and finish by adding $\vec{\beta}_1 \in \mathscr{N}(n^3) = \mathbb{C}^5)$ such that $n(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Exercises

✓ **2.17** What is the index of nilpotency of the *left-shift* operator, here acting on the space of triples of reals?

$$(x, y, z) \mapsto (0, x, y)$$

✓ **2.18** For each string basis state the index of nilpotency and give the dimension of the rangespace and nullspace of each iteration of the nilpotent map.

(a) $\begin{array}{lllll} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{0} \\ \vec{\beta}_3 & \mapsto & \vec{\beta}_4 & \mapsto & \vec{0} \end{array}$

(b) $\begin{array}{lllllll} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{\beta}_3 & \mapsto & \vec{0} \\ \vec{\beta}_4 & \mapsto & \vec{0} & & & & \\ \vec{\beta}_5 & \mapsto & \vec{0} & & & & \\ \vec{\beta}_6 & \mapsto & \vec{0} & & & & \end{array}$

(c) $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$

Also give the canonical form of the matrix.

2.19 Decide which of these matrices are nilpotent.

(a) $\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} -3 & 2 & 1 \\ -3 & 2 & 1 \\ -3 & 2 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 & 4 \\ 3 & 0 & -1 \\ 5 & 2 & 7 \end{pmatrix}$

(e) $\begin{pmatrix} 45 & -22 & -19 \\ 33 & -16 & -14 \\ 69 & -34 & -29 \end{pmatrix}$

✓ **2.20** Find the canonical form of this matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

✓ **2.21** Consider the matrix from Example 2.16.
(a) Use the action of the map on the string basis to give the canonical form.
(b) Find the change of basis matrices that bring the matrix to canonical form.
(c) Use the answer in the prior item to check the answer in the first item.

✓ **2.22** Each of these matrices is nilpotent.

(a) $\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ **(c)** $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

Put each in canonical form.

2.23 Describe the effect of left or right multiplication by a matrix that is in the canonical form for nilpotent matrices.

2.24 Is nilpotence invariant under similarity? That is, must a matrix similar to a nilpotent matrix also be nilpotent? If so, with the same index?

✓ **2.25** Show that the only eigenvalue of a nilpotent matrix is zero.

2.26 Is there a nilpotent transformation of index three on a two-dimensional space?

2.27 In the proof of Theorem 2.13, why isn't the proof's base case that the index of nilpotency is zero?

✓ **2.28** Let $t\colon V \to V$ be a linear transformation and suppose $\vec{v} \in V$ is such that $t^k(\vec{v}) = \vec{0}$ but $t^{k-1}(\vec{v}) \neq \vec{0}$. Consider the t-string $\langle \vec{v}, t(\vec{v}), \ldots, t^{k-1}(\vec{v}) \rangle$.
(a) Prove that t is a transformation on the span of the set of vectors in the string, that is, prove that t restricted to the span has a range that is a subset of the span. We say that the span is a *t-invariant* subspace.
(b) Prove that the restriction is nilpotent.
(c) Prove that the t-string is linearly independent and so is a basis for its span.
(d) Represent the restriction map with respect to the t-string basis.

2.29 Finish the proof of Theorem 2.13.

2.30 Show that the terms 'nilpotent transformation' and 'nilpotent matrix', as given in Definition 2.6, fit with each other: a map is nilpotent if and only if it is represented by a nilpotent matrix. (Is it that a transformation is nilpotent if an only if there is a basis such that the map's representation with respect to that basis is a nilpotent matrix, or that any representation is a nilpotent matrix?)

2.31 Let T be nilpotent of index four. How big can the rangespace of T^3 be?

2.32 Recall that similar matrices have the same eigenvalues. Show that the converse does not hold.

2.33 Prove a nilpotent matrix is similar to one that is all zeros except for blocks of super-diagonal ones.

✓ **2.34** Prove that if a transformation has the same rangespace as nullspace. then the dimension of its domain is even.

2.35 Prove that if two nilpotent matrices commute then their product and sum are also nilpotent.

2.36 Consider the transformation of $\mathcal{M}_{n\times n}$ given by $t_S(T) = ST - TS$ where S is an $n\times n$ matrix. Prove that if S is nilpotent then so is t_S.

2.37 Show that if N is nilpotent then $I - N$ is invertible. Is that 'only if' also?

IV Jordan Form

This section uses material from three optional subsections: Direct Sum, Determinants Exist, and Other Formulas for the Determinant.

The chapter on linear maps shows that every $h\colon V \to W$ can be represented by a partial-identity matrix with respect to some bases $B \subset V$ and $D \subset W$. This chapter revisits this issue in the special case that the map is a linear transformation $t\colon V \to V$. Of course, the general result still applies but with the codomain and domain equal we naturally ask about having the two bases also be equal. That is, we want a canonical form to represent transformations as $\mathrm{Rep}_{B,B}(t)$.

After a brief review section, we began by noting that a block partial identity form matrix is not always obtainable in this B, B case. We therefore considered the natural generalization, diagonal matrices, and showed that if its eigenvalues are distinct then a map or matrix can be diagonalized. But we also gave an example of a matrix that cannot be diagonalized and in the section prior to this one we developed that example. We showed that a linear map is nilpotent — if we take higher and higher powers of the map or matrix then we eventually get the zero map or matrix — if and only if there is a basis on which it acts via disjoint strings. That led to a canonical form for nilpotent matrices.

Now, this section concludes the chapter. We will show that the two cases we've studied are exhaustive in that for any linear transformation there is a basis such that the matrix representation $\mathrm{Rep}_{B,B}(t)$ is the sum of a diagonal matrix and a nilpotent matrix in its canonical form.

IV.1 Polynomials of Maps and Matrices

Recall that the set of square matrices is a vector space under entry-by-entry addition and scalar multiplication and that this space $\mathcal{M}_{n\times n}$ has dimension n^2. Thus, for any $n\times n$ matrix T the n^2+1-member set $\{I, T, T^2, \ldots, T^{n^2}\}$ is linearly dependent and so there are scalars $c_0, \ldots, c_{n^2}$ such that $c_{n^2}T^{n^2} + \cdots + c_1T + c_0I$ is the zero matrix.

1.1 Remark This observation is small but important. It says that every transformation exhibits a generalized nilpotency: the powers of a square matrix cannot climb forever without a "repeat".

1.2 Example Rotation of plane vectors $\pi/6$ radians counterclockwise is represented with respect to the standard basis by

$$T = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

and verifying that $0T^4 + 0T^3 + 1T^2 - 2T - 1I$ equals the zero matrix is easy.

1.3 Definition For any polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$, where t is a linear transformation then $f(t)$ is the transformation $c_n t^n + \cdots + c_1 t + c_0(\mathrm{id})$ on the same space and where T is a square matrix then $f(T)$ is the matrix $c_n T^n + \cdots + c_1 T + c_0 I$.

1.4 Remark If, for instance, $f(x) = x - 3$, then most authors write in the identity matrix: $f(T) = T - 3I$. But most authors don't write in the identity map: $f(t) = t - 3$. In this book we shall also observe this convention.

Of course, if $T = \mathrm{Rep}_{B,B}(t)$ then $f(T) = \mathrm{Rep}_{B,B}(f(t))$, which follows from the relationships $T^j = \mathrm{Rep}_{B,B}(t^j)$, and $cT = \mathrm{Rep}_{B,B}(ct)$, and $T_1 + T_2 = \mathrm{Rep}_{B,B}(t_1 + t_2)$.

As Example 1.2 shows, there may be polynomials of degree smaller than n^2 that zero the map or matrix.

1.5 Definition The *minimal polynomial* $m(x)$ of a transformation t or a square matrix T is the polynomial of least degree and with leading coefficient 1 such that $m(t)$ is the zero map or $m(T)$ is the zero matrix.

A minimal polynomial always exists by the observation opening this subsection. A minimal polynomial is unique by the 'with leading coefficient 1' clause. This is because if there are two polynomials $m(x)$ and $\hat{m}(x)$ that are both of the minimal degree to make the map or matrix zero (and thus are of equal degree), and both have leading 1's, then their difference $m(x) - \hat{m}(x)$ has a smaller degree than either and still sends the map or matrix to zero. Thus $m(x) - \hat{m}(x)$ is the zero polynomial and the two are equal. (The leading coefficient requirement also prevents a minimal polynomial from being the zero polynomial.)

1.6 Example We can see that $m(x) = x^2 - 2x - 1$ is minimal for the matrix of Example 1.2 by computing the powers of T up to the power $n^2 = 4$.

$$T^2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T^4 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Next, put $c_4 T^4 + c_3 T^3 + c_2 T^2 + c_1 T + c_0 I$ equal to the zero matrix

$$\begin{array}{rl} -(1/2)c_4 \qquad\quad + \quad (1/2)c_2 + (\sqrt{3}/2)c_1 + c_0 &= 0 \\ -(\sqrt{3}/2)c_4 - c_3 - (\sqrt{3}/2)c_2 - \quad (1/2)c_1 \qquad &= 0 \\ (\sqrt{3}/2)c_4 + c_3 + (\sqrt{3}/2)c_2 + \quad (1/2)c_1 \qquad &= 0 \\ -(1/2)c_4 \qquad\quad + \quad (1/2)c_2 + (\sqrt{3}/2)c_1 + c_0 &= 0 \end{array}$$

and use Gauss' method.

$$\begin{array}{rl} c_4 \qquad - \qquad c_2 - \sqrt{3}c_1 - \quad 2c_0 &= 0 \\ c_3 + \sqrt{3}c_2 + \quad 2c_1 + \sqrt{3}c_0 &= 0 \end{array}$$

Setting c_4, c_3, and c_2 to zero forces c_1 and c_0 to also come out as zero. To get a leading one, the most we can do is to set c_4 and c_3 to zero. Thus the minimal polynomial is quadratic.

Using the method of that example to find the minimal polynomial of a 3×3 matrix would mean doing Gaussian reduction on a system with nine equations in ten unknowns. We shall develop an alternative. To begin, note that we can break a polynomial of a map or a matrix into its components. (For this lemma, recall that we are using complex numbers in this chapter so all polynomials break completely into linear factors.)

1.7 Lemma Suppose that the polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ factors as $k(x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$. If t is a linear transformation then these two are equal maps.

$$c_n t^n + \cdots + c_1 t + c_0 = k\cdot(t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$$

Consequently, if T is a square matrix then $f(T)$ and $k\cdot(T-\lambda_1 I)^{q_1}\cdots(T-\lambda_\ell I)^{q_\ell}$ are equal matrices.

Proof. This argument is by induction on the degree of the polynomial. The cases where the polynomial is of degree 0 and 1 are clear. The full induction argument is Exercise 1.7 but the degree two case gives its sense.

A quadratic polynomial factors into two linear terms $f(x) = k(x-\lambda_1)\cdot(x-\lambda_2) = k(x^2 + (\lambda_1+\lambda_2)x + \lambda_1\lambda_2)$ (the roots λ_1 and λ_2 might be equal). We can check that substituting t for x in the factored and unfactored versions gives the same map.

$$\begin{aligned}
\big(k\cdot(t-\lambda_1)\circ(t-\lambda_2)\big)\,(\vec{v}) &= \big(k\cdot(t-\lambda_1)\big)\,(t(\vec{v})-\lambda_2\vec{v}) \\
&= k\cdot\big(t(t(\vec{v})) - t(\lambda_2\vec{v}) - \lambda_1 t(\vec{v}) - \lambda_1\lambda_2\vec{v}\big) \\
&= k\cdot\big(t\circ t\,(\vec{v}) - (\lambda_1+\lambda_2)t(\vec{v}) + \lambda_1\lambda_2\vec{v}\big) \\
&= k\cdot(t^2 - (\lambda_1+\lambda_2)t + \lambda_1\lambda_2)\,(\vec{v})
\end{aligned}$$

The third equality holds because the scalar λ_2 comes out of the second term, as t is linear. QED

In particular, if a minimial polynomial $m(x)$ for a transformation t factors as $m(x) = (x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$ then $m(t) = (t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t-\lambda_i$ sends some nonzero vectors to zero. So, too, in the matrix case—if m is minimal for T then $m(T) = (T-\lambda_1 I)^{q_1}\cdots(T-\lambda_\ell I)^{q_\ell}$ is the zero matrix and at least one of the matrices $T-\lambda_i I$ sends some nonzero vectors to zero. Rewording both cases: at least some of the λ_i are eigenvalues. (See Exercise 29.)

Recall how we have earlier found eigenvalues. We have looked for λ such that $T\vec{v} = \lambda\vec{v}$ by considering the equation $\vec{0} = T\vec{v} - x\vec{v} = (T-xI)\vec{v}$ and computing the determinant of the matrix $T-xI$. That determinant is a polynomial in x, the characteristic polynomial, whose roots are the eigenvalues. The major result of this subsection, the next result, is that there is a connection between this characteristic polynomial and the minimal polynomial. This results expands on the prior paragraph's insight that some roots of the minimal polynomial are eigenvalues by asserting that every root of the minimal polynomial is an eigenvalue and further that every eigenvalue is a root of the minimal polynomial (this is because it says '$1 \leq q_i$' and not just '$0 \leq q_i$').

1.8 Theorem (Cayley-Hamilton) If the characteristic polynomial of a transformation or square matrix factors into

$$k \cdot (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2}\cdots(x-\lambda_\ell)^{p_\ell}$$

then its minimal polynomial factors into

$$(x-\lambda_1)^{q_1}(x-\lambda_2)^{q_2}\cdots(x-\lambda_\ell)^{q_\ell}$$

where $1 \leq q_i \leq p_i$ for each i between 1 and ℓ.

The proof takes up the next three lemmas. Although they are stated only in matrix terms, they apply equally well to maps. We give the matrix version only because it is convenient for the first proof.

The first result is the key — some authors call it the Cayley-Hamilton Theorem and call Theorem 1.8 above a corollary. For the proof, observe that a matrix of polynomials can be thought of as a polynomial with matrix coefficients.

$$\begin{pmatrix} 2x^2+3x-1 & x^2+2 \\ 3x^2+4x+1 & 4x^2+x+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} x^2 + \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

1.9 Lemma If T is a square matrix with characteristic polynomial $c(x)$ then $c(T)$ is the zero matrix.

Proof. Let C be $T - xI$, the matrix whose determinant is the characteristic polynomial $c(x) = c_n x^n + \cdots + c_1 x + c_0$.

$$C = \begin{pmatrix} t_{1,1}-x & t_{1,2} & \ldots & \\ t_{2,1} & t_{2,2}-x & & \\ \vdots & & \ddots & \\ & & & t_{n,n}-x \end{pmatrix}$$

Recall that the product of the adjoint of a matrix with the matrix itself is the determinant of that matrix times the identity.

$$c(x)\cdot I = \operatorname{adj}(C)C = \operatorname{adj}(C)(T-xI) = \operatorname{adj}(C)T - \operatorname{adj}(C)\cdot x \qquad (*)$$

The entries of $\operatorname{adj}(C)$ are polynomials, each of degree at most $n-1$ since the minors of a matrix drop a row and column. Rewrite it, as suggested above, as $\operatorname{adj}(C) = C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$ where each C_i is a matrix of scalars. The left and right ends of equation $(*)$ above give this.

$$\begin{aligned} c_n I x^n + c_{n-1} I x^{n-1} + \cdots + c_1 I x + c_0 I = (C_{n-1}T)x^{n-1} &+ \cdots + (C_1 T)x + C_0 T \\ - C_{n-1}x^n - C_{n-2}x^{n-1} &- \cdots - C_0 x \end{aligned}$$

Equate the coefficients of x^n, the coefficients of x^{n-1}, etc.

$$\begin{aligned}
c_n I &= -C_{n-1} \\
c_{n-1} I &= -C_{n-2} + C_{n-1} T \\
&\vdots \\
c_1 I &= -C_0 + C_1 T \\
c_0 I &= C_0 T
\end{aligned}$$

Multiply (from the right) both sides of the first equation by T^n, both sides of the second equation by T^{n-1}, etc. Add. The result on the left is $c_n T^n + c_{n-1} T^{n-1} + \cdots + c_0 I$, and the result on the right is the zero matrix. QED

We sometimes refer to that lemma by saying that a matrix or map *satisfies* its characteristic polynomial.

1.10 Lemma Where $f(x)$ is a polynomial, if $f(T)$ is the zero matrix then $f(x)$ is divisible by the minimal polynomial of T. That is, any polynomial satisfied by T is divisable by T's minimal polynomial.

Proof. Let $m(x)$ be minimal for T. The Division Theorem for Polynomials gives $f(x) = q(x)m(x) + r(x)$ where the degree of r is strictly less than the degree of m. Plugging T in shows that $r(T)$ is the zero matrix, because T satisfies both f and m. That contradicts the minimality of m unless r is the zero polynomial. QED

Combining the prior two lemmas gives that the minimal polynomial divides the characteristic polynomial. Thus, any root of the minimal polynomial is also a root of the characteristic polynomial. That is, so far we have that if $m(x) = (x - \lambda_1)^{q_1} \ldots (x - \lambda_i)^{q_i}$ then $c(x)$ must has the form $(x - \lambda_1)^{p_1} \ldots (x - \lambda_i)^{p_i} (x - \lambda_{i+1})^{p_{i+1}} \ldots (x - \lambda_\ell)^{p_\ell}$ where each q_j is less than or equal to p_j. The proof of the Cayley-Hamilton Theorem is finished by showing that in fact the characteristic polynomial has no extra roots λ_{i+1}, etc.

1.11 Lemma Each linear factor of the characteristic polynomial of a square matrix is also a linear factor of the minimal polynomial.

Proof. Let T be a square matrix with minimal polynomial $m(x)$ and assume that $x - \lambda$ is a factor of the characteristic polynomial of T, that is, assume that λ is an eigenvalue of T. We must show that $x - \lambda$ is a factor of m, that is, that $m(\lambda) = 0$.

In general, where λ is associated with the eigenvector $\vec{v}$, for any polynomial function $f(x)$, application of the matrix $f(T)$ to $\vec{v}$ equals the result of multiplying $\vec{v}$ by the scalar $f(\lambda)$. (For instance, if T has eigenvalue λ associated with the eigenvector $\vec{v}$ and $f(x) = x^2 + 2x + 3$ then $(T^2 + 2T + 3)(\vec{v}) = T^2(\vec{v}) + 2T(\vec{v}) + 3\vec{v} = \lambda^2 \cdot \vec{v} + 2\lambda \cdot \vec{v} + 3 \cdot \vec{v} = (\lambda^2 + 2\lambda + 3) \cdot \vec{v}$.) Now, as $m(T)$ is the zero matrix, $\vec{0} = m(T)(\vec{v}) = m(\lambda) \cdot \vec{v}$ and therefore $m(\lambda) = 0$. QED

1.12 Example We can use the Cayley-Hamilton Theorem to help find the minimal polynomial of this matrix.

$$T = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

First, its characteristic polynomial $c(x) = (x-1)(x-2)^3$ can be found with the usual determinant. Now, the Cayley-Hamilton Theorem says that T's minimal polynomial is either $(x-1)(x-2)$ or $(x-1)(x-2)^2$ or $(x-1)(x-2)^3$. We can decide among the choices just by computing:

$$(T - 1I)(T - 2I) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(T - 1I)(T - 2I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so $m(x) = (x-1)(x-2)^2$.

Exercises

✓ **1.13** What are the possible minimal polynomials if a matrix has the given characteristic polynomial?

(a) $8 \cdot (x-3)^4$ **(b)** $(1/3) \cdot (x+1)^3(x-4)$ **(c)** $-1 \cdot (x-2)^2(x-5)^2$
(d) $5 \cdot (x+3)^2(x-1)(x-2)^2$

What is the degree of each possibility?

✓ **1.14** Find the minimal polynomial of each matrix.

(a) $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ **(f)** $\begin{pmatrix} -1 & 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 \\ 3 & -9 & -4 & 2 & -1 \\ 1 & 5 & 4 & 1 & 4 \end{pmatrix}$

1.15 Find the minimal polynomial of this matrix.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

✓ **1.16** What is the minimal polynomial of the differentiation operator d/dx on $\mathcal{P}_n$?

✓ **1.17** Find the minimal polynomial of matrices of this form

$$\begin{pmatrix} \lambda & 0 & 0 & \ldots & & 0 \\ 1 & \lambda & 0 & & & 0 \\ 0 & 1 & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & 0 \\ 0 & 0 & \ldots & & 1 & \lambda \end{pmatrix}$$

where the scalar λ is fixed (i.e., is not a variable).

1.18 What is the minimal polynomial of the transformation of $\mathcal{P}_n$ that sends $p(x)$ to $p(x+1)$?

1.19 What is the minimal polynomial of the map $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates?

1.20 Find a 3×3 matrix whose minimal polynomial is x^2.

1.21 What is wrong with this claimed proof of Lemma 1.9: "if $c(x) = |T - xI|$ then $c(T) = |T - TI| = 0$"? [Cullen]

1.22 Verify Lemma 1.9 for 2×2 matrices by direct calculation.

✓ **1.23** Prove that the minimal polynomial of an $n\times n$ matrix has degree at most n (not n^2 as might be guessed from this subsection's opening). Verify that this maximum, n, can happen.

✓ **1.24** The only eigenvalue of a nilpotent map is zero. Show that the converse statement holds.

1.25 What is the minimal polynomial of a zero map or matrix? Of an identity map or matrix?

✓ **1.26** Interpret the minimal polynomial of Example 1.2 geometrically.

1.27 What is the minimal polynomial of a diagonal matrix?

✓ **1.28** A *projection* is any transformation t such that $t^2 = t$. (For instance, the transformation of the plane $\mathbb{R}^2$ projecting each vector onto its first coordinate will, if done twice, result in the same value as if it is done just once.) What is the minimal polynomial of a projection?

1.29 *The first two items of this question are review.*

(a) Prove that the composition of one-to-one maps is one-to-one.

(b) Prove that if a linear map is not one-to-one then at least one nonzero vector from the domain is sent to the zero vector in the codomain.

(c) Verify the statement, excerpted here, that preceeds Theorem 1.8.

> ... if a minimial polynomial $m(x)$ for a transformation t factors as $m(x) = (x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$ then $m(t) = (t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t-\lambda_i$ sends some nonzero vectors to zero. ... Rewording ...: at least some of the λ_i are eigenvalues.

1.30 True or false: for a transformation on an n dimensional space, if the minimal polynomial has degree n then the map is diagonalizable.

1.31 Let $f(x)$ be a polynomial. Prove that if A and B are similar matrices then $f(A)$ is similar to $f(B)$.

(a) Now show that similar matrices have the same characteristic polynomial.

(b) Show that similar matrices have the same minimal polynomial.

(c) Decide if these are similar.

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$$

1.32 **(a)** Show that a matrix is invertible if and only if the constant term in its minimal polynomial is not 0.

(b) Show that if a square matrix T is not invertible then there is a nonzero matrix S such that ST and TS both equal the zero matrix.

✓ **1.33** **(a)** Finish the proof of Lemma 1.7.

(b) Give an example to show that the result does not hold if t is not linear.

1.34 Any transformation or square matrix has a minimal polynomial. Does the converse hold?

IV.2 Jordan Canonical Form

This subsection moves from the canonical form for nilpotent matrices to the one for all matrices.

We have shown that if a map is nilpotent then all of its eigenvalues are zero. We can now prove the converse.

2.1 Lemma A linear transformation whose only eigenvalue is zero is nilpotent.

Proof. If a transformation t on an n-dimensional space has only the single eigenvalue of zero then its characteristic polynomial is x^n. The Cayley-Hamilton Theorem says that a map satisfies its characteristic polynimial so t^n is the zero map. Thus t is nilpotent. QED

We have a canonical form for nilpotent matrices, that is, for each matrix whose single eigenvalue is zero: each such matrix is similar to one that is all zeroes except for blocks of subdiagonal ones. (To make this representation unique we can fix some arrangement of the blocks, say, from longest to shortest.) We next extend this to all single-eigenvalue matrices.

Observe that if t's only eigenvalue is λ then $t-\lambda$'s only eigenvalue is 0 because $t(\vec{v}) = \lambda\vec{v}$ if and only if $(t-\lambda)\,(\vec{v}) = 0 \cdot \vec{v}$. The natural way to extend the results for nilpotent matrices is to represent $t-\lambda$ in the canonical form N, and try to use that to get a simple representation T for t. The next result says that this try works.

2.2 Lemma If the matrices $T - \lambda I$ and N are similar then T and $N + \lambda I$ are also similar, via the same change of basis matrices.

Proof. With $N = P(T - \lambda I)P^{-1} = PTP^{-1} - P(\lambda I)P^{-1}$ we have $N = PTP^{-1} - PP^{-1}(\lambda I)$ since the diagonal matrix λI commutes with anything, and so $N = PTP^{-1} - \lambda I$. Therefore $N + \lambda I = PTP^{-1}$, as required. QED

2.3 Example The characteristic polynomial of

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

is $(x-3)^2$ and so T has only the single eigenvalue 3. Thus for

$$T - 3I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

the only eigenvalue is 0, and $T - 3I$ is nilpotent. The null spaces are routine to find; to ease this computation we take T to represent the transformation $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis (we shall maintain this convention for the rest of the chapter).

$$\mathscr{N}(t-3) = \{\begin{pmatrix} -y \\ y \end{pmatrix} \mid y \in \mathbb{C}\} \qquad \mathscr{N}((t-3)^2) = \mathbb{C}^2$$

The dimensions of these null spaces show that the action of an associated map $t-3$ on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. Thus, the canonical form for $t-3$ with one choice for a string basis is

$$\mathrm{Rep}_{B,B}(t-3) = N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \rangle$$

and by Lemma 2.2, T is similar to this matrix.

$$\mathrm{Rep}_t(B,B) = N + 3I = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

We can produce the similarity computation. Recall from the Nilpotence section how to find the change of basis matrices P and P^{-1} to express N as $P(T-3I)P^{-1}$. The similarity diagram

$$\begin{array}{ccc} \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T-3I]{t-3} & \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow P & & \text{id}\big\downarrow P \\ C^2_{\text{w.r.t. } B} & \xrightarrow[N]{t-3} & \mathbb{C}^2_{\text{w.r.t. } B} \end{array}$$

describes that to move from the lower left to the upper left we multiply by

$$P^{-1} = \big(\mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id})\big)^{-1} = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

and to move from the upper right to the lower right we multiply by this matrix.

$$P = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix}$$

So the similarity is expressed by

$$\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

which is easily checked.

2.4 Example This matrix has characteristic polynomial $(x-4)^4$

$$T = \begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$$

and so has the single eigenvalue 4. The nullities of $t-4$ are: the null space of $t-4$ has dimension two, the null space of $(t-4)^2$ has dimension three, and the null space of $(t-4)^3$ has dimension four. Thus, $t-4$ has the action on a string basis of $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$ and $\vec{\beta}_4 \mapsto \vec{0}$. This gives the canonical form N for $t-4$, which in turn gives the form for t.

$$N + 4I = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

An array that is all zeroes, except for some number λ down the diagonal and blocks of subdiagonal ones, is a *Jordan block*. We have shown that Jordan block matrices are canonical representatives of the similarity classes of single-eigenvalue matrices.

2.5 Example The 3×3 matrices whose only eigenvalue is $1/2$ separate into three similarity classes. The three classes have these canonical representatives.

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

In particular, this matrix

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

belongs to the similarity class represented by the middle one, because we have adopted the convention of ordering the blocks of subdiagonal ones from the longest block to the shortest.

We will now finish the program of this chapter by extending this work to cover maps and matrices with multiple eigenvalues. The best possibility for general maps and matrices would be if we could break them into a part involving their first eigenvalue λ_1 (which we represent using its Jordan block), a part with λ_2, etc.

This ideal is in fact what happens. For any transformation $t\colon V \to V$, we shall break the space V into the direct sum of a part on which $t-\lambda_1$ is nilpotent, plus a part on which $t-\lambda_2$ is nilpotent, etc. More precisely, we shall take three steps to get to this section's major theorem and the third step shows that $V = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_\ell)$ where $\lambda_1, \ldots, \lambda_\ell$ are t's eigenvalues.

Suppose that $t\colon V \to V$ is a linear transformation. Note that the restriction* of t to a subspace M need not be a linear transformation on M because there may be an $\vec{m} \in M$ with $t(\vec{m}) \notin M$. To ensure that the restriction of a transformation to a 'part' of a space is a transformation on the partwe need the next condition.

2.6 Definition Let $t\colon V \to V$ be a transformation. A subspace M is t *invariant* if whenever $\vec{m} \in M$ then $t(\vec{m}) \in M$ (shorter: $t(M) \subseteq M$).

Two examples are that the generalized null space $\mathscr{N}_\infty(t)$ and the generalized range space $\mathscr{R}_\infty(t)$ of any transformation t are invariant. For the generalized null space, if $\vec{v} \in \mathscr{N}_\infty(t)$ then $t^n(\vec{v}) = \vec{0}$ where n is the dimension of the underlying space and so $t(\vec{v}) \in \mathscr{N}_\infty(t)$ because $t^n(\,t(\vec{v})\,)$ is zero also. For the generalized range space, if $\vec{v} \in \mathscr{R}_\infty(t)$ then $\vec{v} = t^n(\vec{w})$ for some $\vec{w}$ and then $t(\vec{v}) = t^{n+1}(\vec{w}) = t^n(\,t(\vec{w})\,)$ shows that $t(\vec{v})$ is also a member of $\mathscr{R}_\infty(t)$.

Thus the spaces $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_i$ invariant. Observe also that $t-\lambda_i$ is nilpotent on $\mathscr{N}_\infty(t-\lambda_i)$ because, simply, if $\vec{v}$ has the property that some power of $t-\lambda_i$ maps it to zero — that is, if it is in the generalized null space — then some power of $t-\lambda_i$ maps it to zero. The generalized null space $\mathscr{N}_\infty(t-\lambda_i)$ is a 'part' of the space on which the action of $t-\lambda_i$ is easy to understand.

The next result is the first of our three steps. It establishes that $t-\lambda_j$ leaves $t-\lambda_i$'s part unchanged.

2.7 Lemma A subspace is t invariant if and only if it is $t-\lambda$ invariant for any scalar λ. In particular, where λ_i is an eigenvalue of a linear transformation t, then for any other eigenvalue λ_j, the spaces $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_j$ invariant.

Proof. For the first sentence we check the two implications of the 'if and only if' separately. One of them is easy: if the subspace is $t-\lambda$ invariant for any λ then taking $\lambda = 0$ shows that it is t invariant. For the other implication suppose that the subspace is t invariant, so that if $\vec{m} \in M$ then $t(\vec{m}) \in M$, and let λ be any scalar. The subspace M is closed under linear combinations and so if $t(\vec{m}) \in M$ then $t(\vec{m}) - \lambda\vec{m} \in M$. Thus if $\vec{m} \in M$ then $(t-\lambda)\,(\vec{m}) \in M$, as required.

The second sentence follows straight from the first. Because the two spaces are $t-\lambda_i$ invariant, they are therefore t invariant. From this, applying the first sentence again, we conclude that they are also $t-\lambda_j$ invariant. QED

* More information on restrictions of functions is in the appendix.

The second step of the three that we will take to prove this section's major result makes use of an additional property of $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$, that they are complementary. Recall that if a space is the direct sum of two others $V = \mathscr{N} \oplus \mathscr{R}$ then any vector $\vec{v}$ in the space breaks into two parts $\vec{v} = \vec{n} + \vec{r}$ where $\vec{n} \in \mathscr{N}$ and $\vec{r} \in \mathscr{R}$, and recall also that if $B_\mathscr{N}$ and $B_\mathscr{R}$ are bases for $\mathscr{N}$ and $\mathscr{R}$ then the concatenation $B_\mathscr{N} {}^\frown B_\mathscr{R}$ is linearly independent (and so the two parts of $\vec{v}$ do not "overlap"). The next result says that for any subspaces $\mathscr{N}$ and $\mathscr{R}$ that are complementary as well as t invariant, the action of t on $\vec{v}$ breaks into the "non-overlapping" actions of t on $\vec{n}$ and on $\vec{r}$.

2.8 Lemma Let $t\colon V \to V$ be a transformation and let $\mathscr{N}$ and $\mathscr{R}$ be t invariant complementary subspaces of V. Then t can be represented by a matrix with blocks of square submatrices T_1 and T_2

$$\left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right) \begin{array}{l} \}\dim(\mathscr{N})\text{-many rows} \\ \}\dim(\mathscr{R})\text{-many rows} \end{array}$$

where Z_1 and Z_2 are blocks of zeroes.

Proof. Since the two subspaces are complementary, the concatenation of a basis for $\mathscr{N}$ and a basis for $\mathscr{R}$ makes a basis $B = \langle \vec{\nu}_1, \dots, \vec{\nu}_p, \vec{\mu}_1, \dots, \vec{\mu}_q \rangle$ for V. We shall show that the matrix

$$\operatorname{Rep}_{B,B}(t) = \left(\begin{array}{c|c|c} \vdots & & \vdots \\ \operatorname{Rep}_B(t(\vec{\nu}_1)) & \cdots & \operatorname{Rep}_B(t(\vec{\mu}_q)) \\ \vdots & & \vdots \end{array}\right)$$

has the desired form.

Any vector $\vec{v} \in V$ is in $\mathscr{N}$ if and only if its final q components are zeroes when it is represented with respect to B. As $\mathscr{N}$ is t invariant, each of the vectors $\operatorname{Rep}_B(t(\vec{\nu}_1))$, ..., $\operatorname{Rep}_B(t(\vec{\nu}_p))$ has that form. Hence the lower left of $\operatorname{Rep}_{B,B}(t)$ is all zeroes.

The argument for the upper right is similar. QED

To see that t has been decomposed into its action on the parts, observe that the restrictions of t to the subspaces $\mathscr{N}$ and $\mathscr{R}$ are represented, with respect to the obvious bases, by the matrices T_1 and T_2. So, with subspaces that are invariant and complementary, we can split the problem of examining a linear transformation into two lower-dimensional subproblems. The next result illustrates this decomposition into blocks.

2.9 Lemma If T is a matrices with square submatrices T_1 and T_2

$$T = \left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right)$$

where the Z's are blocks of zeroes, then $|T| = |T_1| \cdot |T_2|$.

Proof. Suppose that T is $n \times n$, that T_1 is $p \times p$, and that T_2 is $q \times q$. In the permutation formula for the determinant

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

each term comes from a rearrangement of the column numbers $1, \ldots, n$ into a new order $\phi(1), \ldots, \phi(n)$. The upper right block Z_2 is all zeroes, so if a ϕ has at least one of $p+1, \ldots, n$ among its first p column numbers $\phi(1), \ldots, \phi(p)$ then the term arising from ϕ is zero, e.g., if $\phi(1) = n$ then $t_{1,\phi(1)} t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0 \cdot t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0$.

So the above formula reduces to a sum over all permutations with two halves: any significant ϕ is the composition of a ϕ_1 that rearranges only $1, \ldots, p$ and a ϕ_2 that rearranges only $p+1, \ldots, p+q$. Now, the distributive law (and the fact that the signum of a composition is the product of the signums) gives that this

$$\begin{aligned} |T_1| \cdot |T_2| = \Bigg(\sum_{\substack{\text{perms } \phi_1 \\ \text{of } 1,\ldots,p}} t_{1,\phi_1(1)} \cdots t_{p,\phi_1(p)} \operatorname{sgn}(\phi_1) \Bigg) \\ \cdot \Bigg(\sum_{\substack{\text{perms } \phi_2 \\ \text{of } p+1,\ldots,p+q}} t_{p+1,\phi_2(p+1)} \cdots t_{p+q,\phi_2(p+q)} \operatorname{sgn}(\phi_2) \Bigg) \end{aligned}$$

equals $|T| = \sum_{\text{significant } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$. QED

2.10 Example

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 36$$

From Lemma 2.9 we conclude that if two subspaces are complementary and t invariant then t is nonsingular if and only if its restrictions to both subspaces are nonsingular.

Now for the promised third, final, step to the main result.

2.11 Lemma If a linear transformation $t\colon V \to V$ has the characteristic polynomial $(x-\lambda_1)^{p_1} \ldots (x-\lambda_\ell)^{p_\ell}$ then (1) $V = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_\ell)$ and (2) $\dim(\mathscr{N}_\infty(t-\lambda_i)) = p_i$.

Proof. Because $\dim(V)$ is the degree $p_1 + \cdots + p_\ell$ of the characteristic polynomial, to establish statement (1) we need only show that statement (2) holds and that $\mathscr{N}_\infty(t-\lambda_i) \cap \mathscr{N}_\infty(t-\lambda_j)$ is trivial whenever $i \neq j$.

For the latter, by Lemma 2.7, both $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{N}_\infty(t-\lambda_j)$ are t invariant. Notice that an intersection of t invariant subspaces is t invariant and so the restriction of t to $\mathscr{N}_\infty(t-\lambda_i) \cap \mathscr{N}_\infty(t-\lambda_j)$ is a linear transformation. But both $t-\lambda_i$ and $t-\lambda_j$ are nilpotent on this subspace and so if t has any eigenvalues on the intersection then its "only" eigenvalue is both λ_i and λ_j. That cannot be, so this restriction has no eigenvalues: $\mathscr{N}_\infty(t-\lambda_i) \cap \mathscr{N}_\infty(t-\lambda_j)$ is trivial (Lemma 3.10 shows that the only transformation without any eigenvalues is on the trivial space).

To prove statement (2), fix the index i. Decompose V as $\mathscr{N}_\infty(t-\lambda_i) \oplus \mathscr{R}_\infty(t-\lambda_i)$ and apply Lemma 2.8.

$$T = \left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right) \begin{array}{l} \}\dim(\,\mathscr{N}_\infty(t-\lambda_i)\,)\text{-many rows} \\ \}\dim(\,\mathscr{R}_\infty(t-\lambda_i)\,)\text{-many rows} \end{array}$$

By Lemma 2.9, $|T - xI| = |T_1 - xI| \cdot |T_2 - xI|$. By the uniqueness clause of the Fundamental Theorem of Arithmetic, the determinants of the blocks have the same factors as the characteristic polynomial $|T_1 - xI| = (x-\lambda_1)^{q_1} \ldots (x-\lambda_\ell)^{q_\ell}$ and $|T_2 - xI| = (x-\lambda_1)^{r_1} \ldots (x-\lambda_\ell)^{r_\ell}$, and the sum of the powers of these factors is the power of the factor in the characteristic polynomial: $q_1 + r_1 = p_1$, $\ldots$, $q_\ell + r_\ell = p_\ell$. Statement (2) will be proved if we will show that $q_i = p_i$ and that $q_j = 0$ for all $j \neq i$, because then the degree of the polynomial $|T_1 - xI|$ — which equals the dimension of the generalized null space — is as required.

For that, first, as the restriction of $t - \lambda_i$ to $\mathscr{N}_\infty(t-\lambda_i)$ is nilpotent on that space, the only eigenvalue of t on it is λ_i. Thus the characteristic equation of t on $\mathscr{N}_\infty(t-\lambda_i)$ is $|T_1 - xI| = (x-\lambda_i)^{q_i}$. And thus $q_j = 0$ for all $j \neq i$.

Now consider the restriction of t to $\mathscr{R}_\infty(t-\lambda_i)$. By Note II.2.2, the map $t - \lambda_i$ is nonsingular on $\mathscr{R}_\infty(t-\lambda_i)$ and so λ_i is not an eigenvalue of t on that subspace. Therefore, $x - \lambda_i$ is not a factor of $|T_2 - xI|$, and so $q_i = p_i$. QED

Our major result just translates those steps into matrix terms.

2.12 Theorem Any square matrix is similar to one in *Jordan form*

$$\begin{pmatrix} J_{\lambda_1} & & \text{–}\textit{zeroes}\text{–} & & \\ & J_{\lambda_2} & & & \\ & & \ddots & & \\ & & & J_{\lambda_{\ell-1}} & \\ & & \text{–}\textit{zeroes}\text{–} & & J_{\lambda_\ell} \end{pmatrix}$$

where each J_λ is the Jordan block associated with the eigenvalue λ of the original matrix (that is, is all zeroes except for λ's down the diagonal and some subdiagonal ones).

Proof. Given an $n \times n$ matrix T, consider the linear map $t\colon \mathbb{C}^n \to \mathbb{C}^n$ that it represents with respect to the standard bases. Use the prior lemma to write $\mathbb{C}^n = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_\ell)$ where $\lambda_1, \ldots, \lambda_\ell$ are the eigenvalues of t. Because each $\mathscr{N}_\infty(t-\lambda_i)$ is t invariant, Lemma 2.8 and the prior lemma show that t is represented by a matrix that is all zeroes except for square blocks along the diagonal. To make those blocks into Jordan blocks, pick each B_{λ_i} to be a string basis for the action of $t - \lambda_i$ on $\mathscr{N}_\infty(t-\lambda_i)$. QED

Jordan form is a canonical form for similarity classes of square matrices, provided that we make it unique by arranging the Jordan blocks from least eigenvalue to greatest and then arranging the subdiagonal 1 blocks inside each Jordan block from longest to shortest.

2.13 Example This matrix has the characteristic polynomial $(x-2)^2(x-6)$.

$$T=\begin{pmatrix}2&0&1\\0&6&2\\0&0&2\end{pmatrix}$$

We will handle the eigenvalues 2 and 6 separately.

Computation of the powers, and the null spaces and nullities, of $T-2I$ is routine. (Recall from Example 2.3 the convention of taking T to represent a transformation, here $t\colon\mathbb{C}^3\to\mathbb{C}^3$, with respect to the standard basis.)

power p	$(T-2I)^p$	$\mathscr{N}((t-2)^p)$	*nullity*
1	$\begin{pmatrix}0&0&1\\0&4&2\\0&0&0\end{pmatrix}$	$\{\begin{pmatrix}x\\0\\0\end{pmatrix}\bigm\vert x\in\mathbb{C}\}$	1
2	$\begin{pmatrix}0&0&0\\0&16&8\\0&0&0\end{pmatrix}$	$\{\begin{pmatrix}x\\-z/2\\z\end{pmatrix}\bigm\vert x,z\in\mathbb{C}\}$	2
3	$\begin{pmatrix}0&0&0\\0&64&32\\0&0&0\end{pmatrix}$	*–same–*	—

So the generalized null space $\mathscr{N}_\infty(t-2)$ has dimension two. We've noted that the restriction of $t-2$ is nilpotent on this subspace. From the way that the nullities grow we know that the action of $t-2$ on a string basis $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$. Thus the restriction can be represented in the canonical form

$$N_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}=\mathrm{Rep}_{B,B}(t-2)\qquad B_2=\langle\begin{pmatrix}1\\1\\-2\end{pmatrix},\begin{pmatrix}-2\\0\\0\end{pmatrix}\rangle$$

where many choices of basis are possible. Consequently, the action of the restriction of t to $\mathscr{N}_\infty(t-2)$ is represented by this matrix.

$$J_2=N_2+2I=\mathrm{Rep}_{B_2,B_2}(t)=\begin{pmatrix}2&0\\1&2\end{pmatrix}$$

The second eigenvalue's computations are easier. Because the power of $x-6$ in the characteristic polynomial is one, the restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$ must be nilpotent of index one. Its action on a string basis must be $\vec{\beta}_3\mapsto\vec{0}$ and since it is the zero map, its canonical form N_6 is the 1×1 zero matrix. Consequently, the canonical form J_6 for the action of t on $\mathscr{N}_\infty(t-6)$ is the 1×1 matrix with the single entry 6. For the basis we can use any nonzero vector from the generalized null space.

$$B_6 = \langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle$$

Taken together, these two give that the Jordan form of T is

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix}2&0&0\\1&2&0\\0&0&6\end{pmatrix}$$

where B is the concatenation of B_2 and B_6.

2.14 Example Contrast the prior example with

$$T = \begin{pmatrix}2&2&1\\0&6&2\\0&0&2\end{pmatrix}$$

which has the same characteristic polynomial $(x-2)^2(x-6)$.

While the characteristic polynomial is the same,

power p	$(T-2I)^p$	$\mathscr{N}((t-2)^p)$	*nullity*
1	$\begin{pmatrix}0&2&1\\0&4&2\\0&0&0\end{pmatrix}$	$\{\begin{pmatrix}x\\-z/2\\z\end{pmatrix} \bigm\vert x,z\in\mathbb{C}\}$	2
2	$\begin{pmatrix}0&8&4\\0&16&8\\0&0&0\end{pmatrix}$	*–same–*	—

here the action of $t-2$ is stable after only one application — the restriction of of $t-2$ to $\mathscr{N}_\infty(t-2)$ is nilpotent of index only one. (So the contrast with the prior example is that while the characteristic polynomial tells us to look at the action of the $t-2$ on its generalized null space, the characteristic polynomial does not describe completely its action and we must do some computations to find, in this example, that the minimal polynomial is $(x-2)(x-6)$.) The restriction of $t-2$ to the generalized null space acts on a string basis as $\vec{\beta}_1 \mapsto \vec{0}$ and $\vec{\beta}_2 \mapsto \vec{0}$, and we get this Jordan block associated with the eigenvalue 2.

$$J_2 = \begin{pmatrix}2&0\\0&2\end{pmatrix}$$

For the other eigenvalue, the arguments for the second eigenvalue of the prior example apply again. The restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$ is nilpotent of index one (it can't be of index less than one, and since $x-6$ is a factor of the characteristic polynomial to the power one it can't be of index more than one either). Thus $t-6$'s canonical form N_6 is the 1×1 zero matrix, and the associated Jordan block J_6 is the 1×1 matrix with entry 6.

Therefore, T is diagonalizable.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}2&0&0\\0&2&0\\0&0&6\end{pmatrix}\qquad B=B_2{}^\frown B_6=\langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix},\begin{pmatrix}3\\4\\0\end{pmatrix}\rangle$$

(Checking that the third vector in B is in the nullspace of $t-6$ is routine.)

2.15 Example A bit of computing with

$$T=\begin{pmatrix}-1&4&0&0&0\\0&3&0&0&0\\0&-4&-1&0&0\\3&-9&-4&2&-1\\1&5&4&1&4\end{pmatrix}$$

shows that its characteristic polynomial is $(x-3)^3(x+1)^2$. This table

power p	$(T-3I)^p$	$\mathscr{N}((t-3)^p)$	*nullity*
1	$\begin{pmatrix}-4&4&0&0&0\\0&0&0&0&0\\0&-4&-4&0&0\\3&-9&-4&-1&-1\\1&5&4&1&1\end{pmatrix}$	$\{\begin{pmatrix}-(u+v)/2\\-(u+v)/2\\(u+v)/2\\u\\v\end{pmatrix}\bigm\vert u,v\in\mathbb{C}\}$	2
2	$\begin{pmatrix}16&-16&0&0&0\\0&0&0&0&0\\0&16&16&0&0\\-16&32&16&0&0\\0&-16&-16&0&0\end{pmatrix}$	$\{\begin{pmatrix}-z\\-z\\z\\u\\v\end{pmatrix}\bigm\vert z,u,v\in\mathbb{C}\}$	3
3	$\begin{pmatrix}-64&64&0&0&0\\0&0&0&0&0\\0&-64&-64&0&0\\64&-128&-64&0&0\\0&64&64&0&0\end{pmatrix}$	*–same–*	—

shows that the restriction of $t-3$ to $\mathscr{N}_\infty(t-3)$ acts on a string basis via the two strings $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$ and $\vec{\beta}_3\mapsto\vec{0}$.

A similar calculation for the other eigenvalue

power p	$(T+1I)^p$	$\mathscr{N}((t+1)^p)$	*nullity*
1	$\begin{pmatrix} 0&4&0&0&0 \\ 0&4&0&0&0 \\ 0&-4&0&0&0 \\ 3&-9&-4&3&-1 \\ 1&5&4&1&5 \end{pmatrix}$	$\{\begin{pmatrix} -(u+v) \\ 0 \\ -v \\ u \\ v \end{pmatrix} \bigm\vert u,v\in\mathbb{C}\}$	2
2	$\begin{pmatrix} 0&16&0&0&0 \\ 0&16&0&0&0 \\ 0&-16&0&0&0 \\ 8&-40&-16&8&-8 \\ 8&24&16&8&24 \end{pmatrix}$	–*same*–	—

shows that the restriction of $t+1$ to its generalized null space acts on a string basis via the two separate strings $\vec{\beta}_4 \mapsto \vec{0}$ and $\vec{\beta}_5 \mapsto \vec{0}$.

Therefore T is similar to this Jordan form matrix.

$$\begin{pmatrix} -1&0&0&0&0 \\ 0&-1&0&0&0 \\ 0&0&3&0&0 \\ 0&0&1&3&0 \\ 0&0&0&0&3 \end{pmatrix}$$

We close with the statement that the subjects considered earlier in this Chapter are indeed, in this sense, exhaustive.

2.16 Corollary Every square matrix is similar to the sum of a diagonal matrix and a nilpotent matrix.

Exercises

2.17 Do the check for Example 2.3.

2.18 Each matrix is in Jordan form. State its characteristic polynomial and its minimal polynomial.

(a) $\begin{pmatrix} 3&0 \\ 1&3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1&0 \\ 0&-1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2&0&0 \\ 1&2&0 \\ 0&0&-1/2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3&0&0 \\ 1&3&0 \\ 0&1&3 \end{pmatrix}$

(e) $\begin{pmatrix} 3&0&0&0 \\ 1&3&0&0 \\ 0&0&3&0 \\ 0&0&1&3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 4&0&0&0 \\ 1&4&0&0 \\ 0&0&-4&0 \\ 0&0&1&-4 \end{pmatrix}$ **(g)** $\begin{pmatrix} 5&0&0 \\ 0&2&0 \\ 0&0&3 \end{pmatrix}$

(h) $\begin{pmatrix} 5&0&0&0 \\ 0&2&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{pmatrix}$ **(i)** $\begin{pmatrix} 5&0&0&0 \\ 0&2&0&0 \\ 0&1&2&0 \\ 0&0&0&3 \end{pmatrix}$

✓ **2.19** Find the Jordan form from the given data.

(a) The matrix T is 5×5 with the single eigenvalue 3. The nullities of the powers are: $T-3I$ has nullity two, $(T-3I)^2$ has nullity three, $(T-3I)^3$ has nullity four, and $(T-3I)^4$ has nullity five.

(b) The matrix S is $5{\times}5$ with two eigenvalues. For the eigenvalue 2 the nullities are: $S-2I$ has nullity two, and $(S-2I)^2$ has nullity four. For the eigenvalue -1 the nullities are: $S+1I$ has nullity one.

2.20 Find the change of basis matrices for each example.
(a) Example 2.13 **(b)** Example 2.14 **(c)** Example 2.15

✓ **2.21** Find the Jordan form and a Jordan basis for each matrix.

(a) $\begin{pmatrix}-10&4\\-25&10\end{pmatrix}$

(b) $\begin{pmatrix}5&-4\\9&-7\end{pmatrix}$

(c) $\begin{pmatrix}4&0&0\\2&1&3\\5&0&4\end{pmatrix}$

(d) $\begin{pmatrix}5&4&3\\-1&0&-3\\1&-2&1\end{pmatrix}$

(e) $\begin{pmatrix}9&7&3\\-9&-7&-4\\4&4&4\end{pmatrix}$

(f) $\begin{pmatrix}2&2&-1\\-1&-1&1\\-1&-2&2\end{pmatrix}$

(g) $\begin{pmatrix}7&1&2&2\\1&4&-1&-1\\-2&1&5&-1\\1&1&2&8\end{pmatrix}$

✓ **2.22** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^2(x+2)^2$.

2.23 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^3(x+2)$.

✓ **2.24** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^3(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

2.25 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^4(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

✓ **2.26** Diagonalize these.

(a) $\begin{pmatrix}1&1\\0&0\end{pmatrix}$ **(b)** $\begin{pmatrix}0&1\\1&0\end{pmatrix}$

✓ **2.27** Find the Jordan matrix representing the differentiation operator on $\mathcal{P}_3$.

✓ **2.28** Decide if these two are similar.

$$\begin{pmatrix}1&-1\\4&-3\end{pmatrix}\qquad\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$$

2.29 Find the Jordan form of this matrix.

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

Also give a Jordan basis.

2.30 How many similarity classes are there for $3{\times}3$ matrices whose only eigenvalues are -3 and 4?

✓ **2.31** Prove that a matrix is diagonalizable if and only if its minimal polynomial has only linear factors.

2.32 Give an example of a linear transformation on a vector space that has no non-trivial invariant subspaces.

2.33 Show that a subspace is $t - \lambda_1$ invariant if and only if it is $t - \lambda_2$ invariant.

2.34 Prove or disprove: two $n \times n$ matrices are similar if and only if they have the same characteristic and minimal polynomials.

2.35 The *trace* of a square matrix is the sum of its diagonal entries.

(a) Find the formula for the characteristic polynomial of a 2×2 matrix.

(b) Show that trace is invariant under similarity, and so we can sensibly speak of the 'trace of a map'. (*Hint:* see the prior item.)

(c) Is trace invariant under matrix equivalence?

(d) Show that the trace of a map is the sum of its eigenvalues (counting multiplicities).

(e) Show that the trace of a nilpotent map is zero. Does the converse hold?

2.36 To use Definition 2.6 to check whether a subspace is t invariant, we seemingly have to check all of the infinitely many vectors in a (nontrivial) subspace to see if they satisfy the condition. Prove that a subspace is t invariant if and only if its subbasis has the property that for all of its elements, $t(\vec{\beta})$ is in the subspace.

✓ **2.37** Is t invariance preserved under intersection? Under union? Complementation? Sums of subspaces?

2.38 Give a way to order the Jordan blocks if some of the eigenvalues are complex numbers. That is, suggest a reasonable ordering for the complex numbers.

2.39 Let $\mathcal{P}_j(\mathbb{R})$ be the vector space over the reals of degree j polynomials. Show that if $j \leq k$ then $\mathcal{P}_j(\mathbb{R})$ is an invariant subspace of $\mathcal{P}_k(\mathbb{R})$ under the differentiation operator. In $\mathcal{P}_7(\mathbb{R})$, does any of $\mathcal{P}_0(\mathbb{R}), \ldots, \mathcal{P}_6(\mathbb{R})$ have an invariant complement?

2.40 In $\mathcal{P}_n(\mathbb{R})$, the vector space (over the reals) of degree n polynomials,

$$\mathcal{E} = \{p(x) \in \mathcal{P}_n(\mathbb{R}) \mid p(-x) = p(x) \text{ for all } x\}$$

and

$$\mathcal{O} = \{p(x) \in \mathcal{P}_n(\mathbb{R}) \mid p(-x) = -p(x) \text{ for all } x\}$$

are the *even* and the *odd* polynomials; $p(x) = x^2$ is even while $p(x) = x^3$ is odd. Show that they are subspaces. Are they complementary? Are they invariant under the differentiation transformation?

2.41 Lemma 2.8 says that if M and N are invariant complements then t has a representation in the given block form (with respect to the same ending as starting basis, of course). Does the implication reverse?

2.42 A matrix S is the *square root* of another T if $S^2 = T$. Show that any nonsingular matrix has a square root.

Topic: Method of Powers

In practice, calculating eigenvalues and eigenvectors is a difficult problem. Finding, and solving, the characteristic polynomial of the large matrices often encountered in applications is too slow and too hard. Other techniques, indirect ones that avoid the characteristic polynomial, are used. Here we shall see such a method that is suitable for large matrices that are 'sparse' (the great majority of the entries are zero).

Suppose that the $n{\times}n$ matrix T has the n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\mathbb{R}^n$ has a basis that is composed of the associated eigenvectors $\langle\vec{\zeta}_1, \ldots, \vec{\zeta}_n\rangle$. For any $\vec{v} \in \mathbb{R}^n$, writing $\vec{v} = c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n$ and iterating T on $\vec{v}$ gives these.

$$\begin{aligned}
T\vec{v} &= c_1\lambda_1\vec{\zeta}_1 + c_2\lambda_2\vec{\zeta}_2 + \cdots + c_n\lambda_n\vec{\zeta}_n \\
T^2\vec{v} &= c_1\lambda_1^2\vec{\zeta}_1 + c_2\lambda_2^2\vec{\zeta}_2 + \cdots + c_n\lambda_n^2\vec{\zeta}_n \\
T^3\vec{v} &= c_1\lambda_1^3\vec{\zeta}_1 + c_2\lambda_2^3\vec{\zeta}_2 + \cdots + c_n\lambda_n^3\vec{\zeta}_n \\
&\vdots \\
T^k\vec{v} &= c_1\lambda_1^k\vec{\zeta}_1 + c_2\lambda_2^k\vec{\zeta}_2 + \cdots + c_n\lambda_n^k\vec{\zeta}_n
\end{aligned}$$

If one of the eigenvalues has a larger absolute value than any of the other eigenvalues then its term will dominate the above expression. Put another way, assuming that the absolute value of λ_1 is the largest and dividing through

$$\frac{T^k\vec{v}}{\lambda_1^k} = c_1\vec{\zeta}_1 + c_2\frac{\lambda_2^k}{\lambda_1^k}\vec{\zeta}_2 + \cdots + c_n\frac{\lambda_n^k}{\lambda_1^k}\vec{\zeta}_n$$

shows that as k gets larger the fractions go to zero. Thus, the entire expression goes to $c_1\vec{\zeta}_1$.

That is (as long as c_1 is not zero), as k increases, the vectors $T^k\vec{v}$ will tend toward the direction of the eigenvectors associated with the dominant eigenvalue, and, consequently, the ratios of the lengths $\|\,T^k\vec{v}\,\|/\|\,T^{k-1}\vec{v}\,\|$ will tend toward that dominant eigenvalue.

For example (sample computer code for this follows the exercises), because the matrix

$$T = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

is triangular, its eigenvalues are just the entries on the diagonal, 3 and -1. Arbitrarily taking $\vec{v}$ to have the components 1 and 1 gives

$\vec{v}$	$T\vec{v}$	$T^2\vec{v}$	$\cdots$	$T^9\vec{v}$	$T^{10}\vec{v}$
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 7 \end{pmatrix}$	$\begin{pmatrix} 9 \\ 17 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} 19\,683 \\ 39\,367 \end{pmatrix}$	$\begin{pmatrix} 59\,049 \\ 118\,097 \end{pmatrix}$

and the ratio between the lengths of the last two is 2.999 9.

Two implementation issues must be addressed. The first issue is that, instead of finding the powers of T and applying them to $\vec{v}$, we will compute $\vec{v}_1$ as $T\vec{v}$ and then compute $\vec{v}_2$ as $T\vec{v}_1$, etc. (i.e., we never separately calculate T^2, T^3, etc.).

These matrix-vector products can be done quickly even if T is large, provided that it is sparse. The second issue is that, to avoid generating numbers that are so large that they overflow our computer's capability, we can normalize the $\vec{v}_i$'s at each step. For instance, we can divide each $\vec{v}_i$ by its length (other possibilities are to divide it by its largest component, or simply by its first component). We thus implement this method by generating

$$\begin{aligned}
\vec{w}_0 &= \vec{v}_0/\|\vec{v}_0\| \\
\vec{v}_1 &= T\vec{w}_0 \\
\vec{w}_1 &= \vec{v}_1/\|\vec{v}_1\| \\
\vec{v}_2 &= T\vec{w}_2 \\
&\vdots \\
\vec{w}_{k-1} &= \vec{v}_{k-1}/\|\vec{v}_{k-1}\| \\
\vec{v}_k &= T\vec{w}_k
\end{aligned}$$

until we are satisfied. Then the vector $\vec{v}_k$ is an approximation of an eigenvector, and the approximation of the dominant eigenvalue is the ratio $\|\vec{v}_k\|/\|\vec{w}_{k-1}\|$.

One way we could be 'satisfied' is to iterate until our approximation of the eigenvalue settles down. We could decide, for instance, to stop the iteration process not after some fixed number of steps, but instead when $\|\vec{v}_k\|$ differs from $\|\vec{v}_{k-1}\|$ by less than one percent, or when they agree up to the second significant digit.

The rate of convergence is determined by the rate at which the powers of $\|\lambda_2/\lambda_1\|$ go to zero, where λ_2 is the eigenvalue of second largest norm. If that ratio is much less than one then convergence is fast, but if it is only slightly less than one then convergence can be quite slow. Consequently, the method of powers is not the most commonly used way of finding eigenvalues (although it is the simplest one, which is why it is here as the illustration of the possibility of computing eigenvalues without solving the characteristic polynomial). Instead, there are a variety of methods that generally work by first replacing the given matrix T with another that is similar to it and so has the same eigenvalues, but is in some reduced form such as *tridiagonal form*: the only nonzero entries are on the diagonal, or just above or below it. Then special techniques can be used to find the eigenvalues. Once the eigenvalues are known, the eigenvectors of T can be easily computed. These other methods are outside of our scope. A good reference is [Goult, *et al.*]

Exercises

1 Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1 and 2. Compare the answer with the one obtained by solving the characteristic equation.

(a) $\begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

2 Redo the prior exercise by iterating until $\|\vec{v}_k\| - \|\vec{v}_{k-1}\|$ has absolute value less than 0.01 At each step, normalize by dividing each vector by its length. How many iterations are required? Are the answers significantly different?

3 Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1, 2, and 3. Compare the answer with the one obtained by solving the characteristic equation.

(a) $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$

4 Redo the prior exercise by iterating until $\|\vec{v}_k\| - \|\vec{v}_{k-1}\|$ has absolute value less than 0.01. At each step, normalize by dividing each vector by its length. How many iterations does it take? Are the answers significantly different?

5 What happens if $c_1 = 0$? That is, what happens if the initial vector does not to have any component in the direction of the relevant eigenvector?

6 How can the method of powers be adopted to find the smallest eigenvalue?

Computer Code

This is the code for the computer algebra system Octave that was used to do the calculation above. (It has been lightly edited to remove blank lines, etc.)

```
>T=[3, 0;
    8, -1]
T=
   3   0
   8  -1
>v0=[1; 2]
v0=
   1
   1
>v1=T*v0
v1=
   3
   7
>v2=T*v1
v2=
   9
  17
>T9=T**9
T9=
  19683  0
  39368 -1
>T10=T**10
T10=
  59049  0
 118096  1
>v9=T9*v0
v9=
  19683
  39367
>v10=T10*v0
v10=
  59049
 118096
```

```
>norm(v10)/norm(v9)
ans=2.9999
```

Remark: we are ignoring the power of Octave here; there are built-in functions to automatically apply quite sophisticated methods to find eigenvalues and eigenvectors. Instead, we are using just the system as a calculator.

Topic: Stable Populations

Imagine a reserve park with animals from a species that we are trying to protect. The park doesn't have a fence and so animals cross the boundary, both from the inside out and in the other direction. Every year, 10% of the animals from inside of the park leave, and 1% of the animals from the outside find their way in. We can ask if we can find a stable level of population for this park: is there a population that, once established, will stay constant over time, with the number of animals leaving equal to the number of animals entering?

To answer that question, we must first establish the equations. Let the year n population in the park be p_n and in the rest of the world be r_n.

$$\begin{aligned} p_{n+1} &= .90p_n + .01r_n \\ r_{n+1} &= .10p_n + .99r_n \end{aligned}$$

We can set this system up as a matrix equation (see the Markov Chain topic).

$$\begin{pmatrix} p_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .90 & .01 \\ .10 & .99 \end{pmatrix} \begin{pmatrix} p_n \\ r_n \end{pmatrix}$$

Now, "stable level" means that $p_{n+1} = p_n$ and $r_{n+1} = r_n$, so that the matrix equation $\vec{v}_{n+1} = T\vec{v}_n$ becomes $\vec{v} = T\vec{v}$. We are therefore looking for eigenvectors for T that are associated with the eigenvalue 1. The equation $(I - T)\vec{v} = \vec{0}$ is

$$\begin{pmatrix} .10 & .01 \\ .10 & .01 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenspace: vectors with the restriction that $p = .1r$. Coupled with additional information, that the total world population of this species is is $p + r = 110\,000$, we find that the stable state is $p = 10,000$ and $r = 100,000$.

If we start with a park population of ten thousand animals, so that the rest of the world has one hundred thousand, then every year ten percent (a thousand animals) of those inside will leave the park, and every year one percent (a thousand) of those from the rest of the world will enter the park. It is stable, self-sustaining.

Now imagine that we are trying to gradually build up the total world population of this species. We can try, for instance, to have the world population grow at a rate of 1% per year. In this case, we can take a "stable" state for the park's population to be that it also grows at 1% per year. The equation $\vec{v}_{n+1} = 1.01 \cdot \vec{v}_n = T\vec{v}_n$ leads to $((1.01 \cdot I) - T)\vec{v} = \vec{0}$, which gives this system.

$$\begin{pmatrix} .11 & .01 \\ .10 & .02 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix is nonsingular, and so the only solution is $p = 0$ and $r = 0$. Thus, there is no (usable) initial population that we can establish at the park and expect that it will grow at the same rate as the rest of the world.

Knowing that an annual world population growth rate of 1% forces an unstable park population, we can ask which growth rates there are that would allow an initial population for the park that will be self-sustaining. We consider $\lambda\vec{v} = T\vec{v}$ and solve for λ.

$$0 = \begin{vmatrix} \lambda - .9 & .01 \\ .10 & \lambda - .99 \end{vmatrix} = (\lambda - .9)(\lambda - .99) - (.10)(.01) = \lambda^2 - 1.89\lambda + .89$$

A shortcut to factoring that quadratic is our knowledge that $\lambda = 1$ is an eigenvalue of T, so the other eigenvalue is .89. Thus there are two ways to have a stable park population (a population that grows at the same rate as the population of the rest of the world, despite the leaky park boundaries): have a world population that is does not grow or shrink, and have a world population that shrinks by 11% every year.

So this is one meaning of eigenvalues and eigenvectors — they give a stable state for a system. If the eigenvalue is 1 then the system is static. If the eigenvalue isn't 1 then the system is either growing or shrinking, but in a dynamically-stable way.

Exercises

1 What initial population for the park discussed above should be set up in the case where world populations are allowed to decline by 11% every year?

2 What will happen to the population of the park in the event of a growth in world population of 1% per year? Will it lag the world growth, or lead it? Assume that the inital park population is ten thousand, and the world population is one hunderd thousand, and calculate over a ten year span.

3 The park discussed above is partially fenced so that now, every year, only 5% of the animals from inside of the park leave (still, about 1% of the animals from the outside find their way in). Under what conditions can the park maintain a stable population now?

4 Suppose that a species of bird only lives in Canada, the United States, or in Mexico. Every year, 4% of the Canadian birds travel to the US, and 1% of them travel to Mexico. Every year, 6% of the US birds travel to Canada, and 4% go to Mexico. From Mexico, every year 10% travel to the US, and 0% go to Canada.

(a) Give the transition matrix.
(b) Is there a way for the three countries to have constant populations?
(c) Find all stable situations.

Topic: Linear Recurrences

In 1202 Leonardo of Pisa, also known as Fibonacci, posed this problem.

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

This moves past an elementary exponential growth model for population increase to include the fact that there is an initial period where newborns are not fertile. However, it retains other simplyfing assumptions, such as that there is no gestation period and no mortality.

To get the total number of pairs we will have next month, we add this month's total to the number of pairs that will be newborn next month. The number of pairs that will be productive next month, that will then be in their "second month on," is the number that were alive last month.

$$f(n+1) = f(n) + f(n-1) \qquad \text{where } f(0)=0,\ f(1)=1$$

The is an example of a *recurrence relation*, because f recurs in its own defining equation. From it, we can easily answer Fibonacci's twelve-month question.

month	0	1	2	3	4	5	6	7	8	9	10	11	12
pairs	1	1	2	3	5	8	13	21	34	55	89	144	233

The sequence of numbers defined by the above equation (of which the first few are listed) is the *Fibonacci sequence*. The material of this chapter can be used to give a formula with which we can can calculate $f(n+1)$ without having to first find $f(n)$, $f(n-1)$, etc.

For that, observe that the recurrence is a linear relationship and so we can give a suitable matrix formulation of it.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(n) \\ f(n-1) \end{pmatrix} = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} \qquad \text{where } \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then, where we write T for the matrix and $\vec{v}_n$ for the vector with components $f(n+1)$ and $f(n)$, we have that $\vec{v}_n = T^n\vec{v}_0$. The advantage of this matrix formulation is that by diagonalizing T we get a fast way to compute its powers: where $T = PDP^{-1}$ we have $T^n = PD^nP^{-1}$, and the n-th power of the diagonal matrix D is the diagonal matrix whose entries that are the n-th powers of the entries of D.

The characteristic equation of T is $\lambda^2 - \lambda - 1$. The quadratic formula gives its roots as $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. Diagonalizing gives this.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

Introducing the vectors and taking the n-th power, we have

$$\begin{aligned}\begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

The calculation is ugly but not hard.

$$\begin{aligned}\begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \\ -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n} \end{pmatrix}\end{aligned}$$

We want the second component of that equation.

$$f(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

Notice that $(1-\sqrt{5})/2 \approx 0.618$ has absolute value less than one and so its powers go to zero. Thus the expression is dominated by its first term. Although we have extended the elementary model of population growth by adding a delay period before the onset of fertility, we nonetheless still get an asmyptotically exponential function.

In general, a *linear recurrence relation* has the form

$$f(n+1) = a_n f(n) + a_{n-1} f(n-1) + \cdots + a_{n-k} f(n-k)$$

(it is also called a *difference equation*). This recurrence relation is *homogeneous* because there is no constant term; i.e, it can be put into the form $0 = -f(n+1) + a_n f(n) + a_{n-1} f(n-1) + \cdots + a_{n-k} f(n-k)$. This is said to be a relation of *order* k. The relation, along with the *initial conditions* $f(0), \ldots, f(k)$ completely determine a sequence. For instance, the Fibonacci relation is of order 2 and it, along with the two initial conditions $f(0) = 1$ and $f(1) = 1$, determines the Fibonacci sequence simply because we can compute any $f(n)$ by first computing $f(2)$, $f(3)$, etc. In this Topic, we shall see how linear algebra can be used to solve linear recurrence relations.

First, we define the vector space in which we are working. Let V be the set of functions f from the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ to the real numbers.

(Below we shall have functions with domain $\{1, 2, \ldots\}$, that is, without 0, but it is not an important distinction.)

Putting the initial conditions aside for a moment, for any recurrence, we can consider the subset S of V of solutions. For example, without initial conditions, in addition to the function f given above, the Fibonacci relation is also solved by the function g whose first few values are $g(0) = 1$, $g(1) = 2$, $g(2) = 3$, $g(3) = 4$, and $g(4) = 7$.

The subset S is a subspace of V. It is nonempty because the zero function is a solution. It is closed under addition since if f_1 and f_2 are solutions, then

$$\begin{aligned}
&a_{n+1}(f_1+f_2)(n+1)+\cdots+a_{n-k}(f_1+f_2)(n-k) \\
&\qquad\qquad = (a_{n+1}f_1(n+1)+\cdots+a_{n-k}f_1(n-k)) \\
&\qquad\qquad\quad + (a_{n+1}f_2(n+1)+\cdots+a_{n-k}f_2(n-k)) \\
&\qquad\qquad = 0.
\end{aligned}$$

And, it is closed under scalar multiplication since

$$\begin{aligned}
&a_{n+1}(rf_1)(n+1)+\cdots+a_{n-k}(rf_1)(n-k) \\
&\qquad\qquad = r(a_{n+1}f_1(n+1)+\cdots+a_{n-k}f_1(n-k)) \\
&\qquad\qquad = r\cdot 0 \\
&\qquad\qquad = 0.
\end{aligned}$$

We can give the dimension of S. Consider this map from the set of functions S to the set of vectors $\mathbb{R}^k$.

$$f \mapsto \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(k) \end{pmatrix}$$

Exercise 3 shows that this map is linear. Because, as noted above, any solution of the recurrence is uniquely determined by the k initial conditions, this map is one-to-one and onto. Thus it is an isomorphism, and thus S has dimension k, the order of the recurrence.

So (again, without any initial conditions), we can describe the set of solutions of any linear homogeneous recurrence relation of degree k by taking linear combinations of only k linearly independent functions. It remains to produce those functions.

For that, we express the recurrence $f(n+1) = a_n f(n) + \cdots + a_{n-k} f(n-k)$ with a matrix equation.

$$\begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \ldots & a_{n-k+1} & a_{n-k} \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & & & \\
0 & 0 & 1 & & & \\
\vdots & \vdots & & \ddots & & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\begin{pmatrix} f(n) \\ f(n-1) \\ \vdots \\ f(n-k) \end{pmatrix}
=
\begin{pmatrix} f(n+1) \\ f(n) \\ \vdots \\ f(n-k+1) \end{pmatrix}$$

In trying to find the characteristic function of the matrix, we can see the pattern in the 2×2 case

$$\begin{pmatrix} a_n-\lambda & a_{n-1} \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - a_n\lambda - a_{n-1}$$

and 3×3 case.

$$\begin{pmatrix} a_n-\lambda & a_{n-1} & a_{n-2} \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + a_n\lambda^2 + a_{n-1}\lambda + a_{n-2}$$

Exercise 4 shows that the characteristic equation is this.

$$\begin{vmatrix} a_n-\lambda & a_{n-1} & a_{n-2} & \ldots & a_{n-k+1} & a_{n-k} \\ 1 & -\lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\lambda & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -\lambda \end{vmatrix}$$
$$= \pm(-\lambda^k + a_n\lambda^{k-1} + a_{n-1}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k})$$

We call that the polynomial 'associated' with the recurrence relation. (We will be finding the roots of this polynomial and so we can drop the $\pm$ as irrelevant.)

If $-\lambda^k + a_n\lambda^{k-1} + a_{n-1}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k}$ has no repeated roots then the matrix is diagonalizable and we can, in theory, get a formula for $f(n)$ as in the Fibonacci case. But, because we know that the subspace of solutions has dimension k, we do not need to do the diagonalization calculation, provided that we can exhibit k linearly independent functions satisfying the relation.

Where $r_1, r_2, \ldots, r_k$ are the distinct roots, consider the functions $f_{r_1}(n) = r_1^n$ through $f_{r_k}(n) = r_k^n$ of powers of those roots. Exercise 5 shows that each is a solution of the recurrence and that the k of them form a linearly independent set. So, given the homogeneous linear recurrence $f(n+1) = a_nf(n) + \cdots + a_{n-k}f(n-k)$ (that is, $0 = -f(n+1) + a_nf(n) + \cdots + a_{n-k}f(n-k)$) we consider the associated equation $0 = -\lambda^k + a_n\lambda^{k-1} + \cdots + a_{n-k+1}\lambda + a_{n-k}$. We find its roots $r_1, \ldots, r_k$, and if those roots are distinct then any solution of the relation has the form $f(n) = c_1r_1^n + c_2r_2^n + \cdots + c_kr_k^n$ for $c_1, \ldots, c_n \in \mathbb{R}$. (The case of repeated roots is also easily done, but we won't cover it here — see any text on Discrete Mathematics.)

Now, given some initial conditions, so that we are interested in a particular solution, we can solve for $c_1, \ldots, c_n$. For instance, the polynomial associated with the Fibonacci relation is $-\lambda^2 + \lambda + 1$, whose roots are $(1 \pm \sqrt{5})/2$ and so any solution of the Fibonacci equation has the form $f(n) = c_1((1+\sqrt{5})/2)^n + c_2((1-\sqrt{5})/2)^n$. Including the initial conditions for the cases $n = 0$ and $n = 1$ gives

$$\begin{aligned} c_1 + \qquad\qquad c_2 &= 1 \\ (1+\sqrt{5}/2)c_1 + (1-\sqrt{5}/2)c_2 &= 1 \end{aligned}$$

which yields $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, as was calculated above.

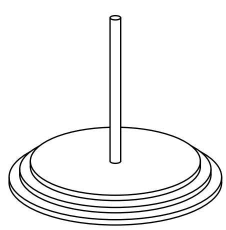
We close by considering the nonhomogeneous case, where the relation has the form $f(n+1) = a_n f(n) + a_{n-1} f(n-1) + \cdots + a_{n-k} f(n-k) + b$ for some nonzero b. As in the first chapter of this book, only a small adjustment is needed to make the transition from the homogeneous case. This classic example illustrates.

In 1883, Edouard Lucas posed the following problem.

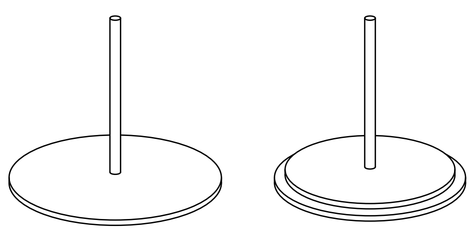
> In the great temple at Benares, beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty four disks of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dusk, and with a thunderclap the world will vanish. (Translation of [De Parville] from [Ball & Coxeter].)

How many disk moves will it take? Instead of tackling the sixty four disk problem right away, we will consider the problem for smaller numbers of disks, starting with three.

To begin, all three disks are on the same needle.



After moving the small disk to the far needle, the mid-sized disk to the middle needle, and then moving the small disk to the middle needle we have this.



Now we can move the big disk over. Then, to finish, we repeat the process of moving the smaller disks, this time so that they end up on the third needle, on top of the big disk.

So the thing to see is that to move the very largest disk, the bottom disk, at a minimum we must: first move the smaller disks to the middle needle, then move the big one, and then move all the smaller ones from the middle needle to the ending needle. Those three steps give us this recurence.

$$T(n+1) = T(n) + 1 + T(n) = 2T(n) + 1 \quad \text{where } T(1) = 1$$

We can easily get the first few values of T.

n	1	2	3	4	5	6	7	8	9	10
$T(n)$	1	3	7	15	31	63	127	255	511	1023

We recognize those as being simply one less than a power of two.

To derive this equation instead of just guessing at it, we write the original relation as $-1 = -T(n+1) + 2T(n)$, consider the homogeneous relation $0 = -T(n) + 2T(n-1)$, get its associated polynomial $-\lambda + 2$, which obviously has the single, unique, root of $r_1 = 2$, and conclude that functions satisfying the homogeneous relation take the form $T(n) = c_1 2^n$.

That's the homogeneous solution. Now we need a particular solution.

Because the nonhomogeneous relation $-1 = -T(n+1) + 2T(n)$ is so simple, in a few minutes (or by remembering the table) we can spot the particular solution $T(n) = -1$ (there are other particular solutions, but this one is easily spotted). So we have that — without yet considering the initial condition — any solution of $T(n+1) = 2T(n) + 1$ is the sum of the homogeneous solution and this particular solution: $T(n) = c_1 2^n - 1$.

The initial condition $T(1) = 1$ now gives that $c_1 = 1$, and we've gotten the formula that generates the table: the n-disk Tower of Hanoi problem requires a minimum of $2^n - 1$ moves.

Finding a particular solution in more complicated cases is, naturally, more complicated. A delightful and rewarding, but challenging, source on recurrence relations is [Graham, Knuth, Patashnik]., For more on the Tower of Hanoi, [Ball & Coxeter] or [Gardner 1957] are good starting points. So is [Hofstadter]. Some computer code for trying some recurrence relations follows the exercises.

Exercises

1 Solve each homogeneous linear recurrence relations.

(a) $f(n+1) = 5f(n) - 6f(n-1)$

(b) $f(n+1) = 4f(n-1)$

(c) $f(n+1) = 5f(n) - 2f(n-1) - 8f(n-2)$

2 Give a formula for the relations of the prior exercise, with these initial conditions.

(a) $f(0) = 1,\ f(1) = 1$

(b) $f(0) = 0,\ f(1) = 1$

(c) $f(0) = 1,\ f(1) = 1,\ f(2) = 3$.

3 Check that the isomorphism given betwween S and $\mathbb{R}^k$ is a linear map. It is argued above that this map is one-to-one. What is its inverse?

4 Show that the characteristic equation of the matrix is as stated, that is, is the polynomial associated with the relation. (Hint: expanding down the final column, and using induction will work.)

5 Given a homogeneous linear recurrence relation $f(n+1) = a_n f(n) + \cdots + a_{n-k} f(n-k)$, let $r_1, \ldots, r_k$ be the roots of the associated polynomial.

(a) Prove that each function $f_{r_i}(n) = r_k^n$ satisfies the recurrence (without initial conditions).

(b) Prove that no r_i is 0.

(c) Prove that the set $\{f_{r_1}, \ldots, f_{r_k}\}$ is linearly independent.

6 (This refers to the value $T(64) = 18,446,744,073,709,551,615$ given in the computer code below.) Transferring one disk per second, how many years would it take the priests at the Tower of Hanoi to finish the job?

Computer Code

This code allows the generation of the first few values of a function defined by a recurrence and initial conditions. It is in the Scheme dialect of LISP (specifically, it was written for A. Jaffer's free scheme interpreter SCM, although it should run in any Scheme implementation).

First, the Tower of Hanoi code is a straightforward implementation of the recurrence.

```
(define (tower-of-hanoi-moves n)
    (if (= n 1)
       1
       (+ (* (tower-of-hanoi-moves (- n 1))
             2)
          1) )  )
```

(Note for readers unused to recursive code: to compute $T(64)$, the computer is told to compute $2*T(63)-1$, which requires, of course, computing $T(63)$. The computer puts the 'times 2' and the 'plus 1' aside for a moment to do that. It computes $T(63)$ by using this same piece of code (that's what 'recursive' means), and to do that is told to compute $2*T(62)-1$. This keeps up (the next step is to try to do $T(62)$ while the other arithmetic is held in waiting), until, after 63 steps, the computer tries to compute $T(1)$. It then returns $T(1) = 1$, which now means that the computation of $T(2)$ can proceed, etc., up until the original computation of $T(64)$ finishes.)

The next routine calculates a table of the first few values. (Some language notes: `'()` is the empty list, that is, the empty sequence, and `cons` pushes something onto the start of a list. Note that, in the last line, the procedure `proc` is called on argument `n`.)

```
(define (first-few-outputs proc n)
    (first-few-outputs-helper proc n '()) )
;
(define (first-few-outputs-aux proc n lst)
    (if (< n 1)
```

```
    lst
    (first-few-outputs-aux proc (- n 1) (cons (proc n) lst)) ) )
```

The session at the SCM prompt went like this.

```
>(first-few-outputs tower-of-hanoi-moves 64)
Evaluation took 120 mSec
(1 3 7 15 31 63 127 255 511 1023 2047 4095 8191 16383 32767
65535 131071 262143 524287 1048575 2097151 4194303 8388607
16777215 33554431 67108863 134217727 268435455 536870911
1073741823 2147483647 4294967295 8589934591 17179869183
34359738367 68719476735 137438953471 274877906943 549755813887
1099511627775 2199023255551 4398046511103 8796093022207
17592186044415 35184372088831 70368744177663 140737488355327
281474976710655 562949953421311 1125899906842623
2251799813685247 4503599627370495 9007199254740991
18014398509481983 36028797018963967 72057594037927935
144115188075855871 288230376151711743 576460752303423487
1152921504606846975 2305843009213693951 4611686018427387903
9223372036854775807 18446744073709551615)
```

This is a list of $T(1)$ through $T(64)$. (The 120 mSec came on a 50 mHz '486 running in an XTerm of XWindow under Linux. The session was edited to put line breaks between numbers.)

Appendix

Mathematics is made of arguments (reasoned discourse that is, not crockery-throwing). This section is a reference to the most used techniques. A reader having trouble with, say, proof by contradiction, can turn here for an outline of that method.

But this section gives only a sketch. For more, these are classics: *Methods of Logic* by Quine, *Induction and Analogy in Mathematics* by Pólya, and *Naive Set Theory* by Halmos.

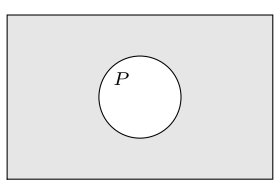
IV.3 Propositions

The point at issue in an argument is the *proposition*. Mathematicians usually write the point in full before the proof and label it either *Theorem* for major points, *Corollary* for points that follow immediately from a prior one, or *Lemma* for results chiefly used to prove other results.

The statements expressing propositions can be complex, with many subparts. The truth or falsity of the entire proposition depends both on the truth value of the parts, and on the words used to assemble the statement from its parts.

Not. For example, where P is a proposition, 'it is not the case that P' is true provided that P is false. Thus, 'n is not prime' is true only when n is the product of smaller integers.

We can picture the 'not' operation with a *Venn diagram.*

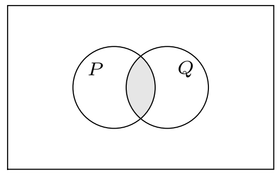


Where the box encloses all natural numbers, and inside the circle are the primes, the shaded area holds numbers satisfying 'not P'.

To prove that a 'not P' statement holds, show that P is false.

And. Consider the statement form 'P and Q'. For the statement to be true both halves must hold: '7 is prime and so is 3' is true, while '7 is prime and 3 is not' is false.

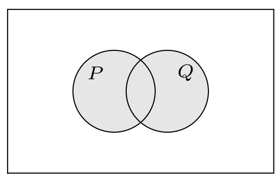
Here is the Venn diagram for 'P and Q'.



To prove 'P and Q', prove that each half holds.

Or. A 'P or Q' is true when either half holds: '7 is prime or 4 is prime' is true, while '7 is not prime or 4 is prime' is false. We take 'or' inclusively so that if both halves are true '7 is prime or 4 is not' then the statement as a whole is true. (In everyday speech, sometimes 'or' is meant in an exclusive way — "Eat your vegetables or no dessert" does not intend both halves to hold — but we will not use 'or' in that way.)

The Venn diagram for 'or' includes all of both circles.

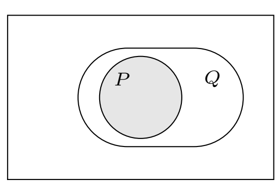


To prove 'P or Q', show that in all cases at least one half holds (perhaps sometimes one half and sometimes the other, but always at least one).

If-then. An 'if P then Q' statement (sometimes written 'P materially implies Q' or just 'P implies Q' or '$P \implies Q$') is true unless P is true while Q is false. Thus 'if 7 is prime then 4 is not' is true while 'if 7 is prime then 4 is also prime' is false. (Contrary to its use in casual speech, in mathematics 'if P then Q' does not connote that P precedes Q or causes Q.)

More subtly, in mathematics 'if P then Q' is true when P is false: 'if 4 is prime then 7 is prime' and 'if 4 is prime then 7 is not' are both true statements, sometimes said to be *vacuously true*. We adopt this convention because we want statements like 'if a number is a perfect square then it is not prime' to be true, for instance when the number is 5 or when the number is 6.

The diagram



shows that Q holds whenever P does (another phrasing is 'P is sufficient to give Q'). Notice again that if P does not hold, Q may or may not be in force.

There are two main ways to establish an implication. The first way is direct: assume that P is true and, using that assumption, prove Q. For instance, to show 'if a number is divisible by 5 then twice that number is divisible by 10', assume that the number is $5n$ and deduce that $2(5n) = 10n$. The second way is indirect: prove the *contrapositive* statement: 'if Q is false then P is false' (rephrased, 'Q can only be false when P is also false'). As an example, to show 'if a number is prime then it is not a perfect square', argue that if it were a square $p = n^2$ then it could be factored $p = n \cdot n$ where $n < p$ and so wouldn't be prime (of course $p = 0$ or $p = 1$ don't give $n < p$ but they are nonprime by definition).

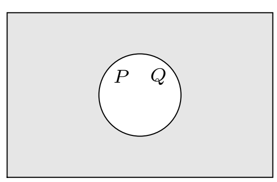
Note two things about this statement form.

First, an 'if P then Q' result can sometimes be improved by weakening P or strengthening Q. Thus, 'if a number is divisible by p^2 then its square is also divisible by p^2' could be upgraded either by relaxing its hypothesis: 'if a number is divisible by p then its square is divisible by p^2', or by tightening its conclusion: 'if a number is divisible by p^2 then its square is divisible by p^4'.

Second, after showing 'if P then Q', a good next step is to look into whether there are cases where Q holds but P does not. The idea is to better understand the relationship between P and Q, with an eye toward strengthening the proposition.

Equivalence. An if-then statement cannot be improved when not only does P imply Q, but also Q implies P. Some ways to say this are: 'P if and only if Q', 'P iff Q', 'P and Q are logically equivalent', 'P is necessary and sufficient to give Q', '$P \iff Q$'. For example, 'a number is divisible by a prime if and only if that number squared is divisible by the prime squared'.

The picture here shows that P and Q hold in exactly the same cases.



Although in simple arguments a chain like "P if and only if R, which holds if and only if S ..." may be practical, typically we show equivalence by showing the 'if P then Q' and 'if Q then P' halves separately.

IV.4 Quantifiers

Compare these two statements about natural numbers: 'there is an x such that x is divisible by x^2' is true, while 'for all numbers x, that x is divisible by x^2' is false. We call the 'there is' and 'for all' prefixes *quantifiers*.

For all. The 'for all' prefix is the *universal quantifier*, symbolized $\forall$.

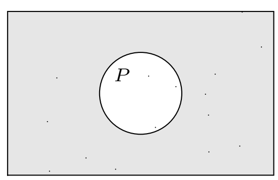
Venn diagrams aren't very helpful with quantifiers, but in a sense the box we draw to border the diagram shows the universal quantifier since it dilineates the universe of possible members.

To prove that a statement holds in all cases, we must show that it holds in each case. Thus, to prove 'every number divisible by p has its square divisible by p^2', take a single number of the form pn and square it $(pn)^2 = p^2n^2$. This is a "typical element" or "generic element" proof.

This kind of argument requires that we are careful to not assume properties for that element other than those in the hypothesis — for instance, this type of wrong argument is a common mistake: "if n is divisible by a prime, say 2, so that $n = 2k$ then $n^2 = (2k)^2 = 4k^2$ and the square of the number is divisible by the square of the prime". That is an argument about the case $p = 2$, but it isn't a proof for general p.

There exists. We will also use the *existential quantifier*, symbolized $\exists$ and read 'there exists'.

As noted above, Venn diagrams are not much help with quantifiers, but a picture of 'there is a number such that P' would show both that there can be more than one and that not all numbers need satisfy P.



An existence proposition can be proved by producing something satisfying the property: once, to settle the question of primality of $2^{2^5}+1$, Euler produced its divisor 641. But there are proofs showing that something exists without saying how to find it; Euclid's argument given in the next subsection shows there are infinitely many primes without naming them. In general, while demonstrating existence is better than nothing, giving an example is better, and an exhaustive list of all instances is great. Still, mathematicians take what they can get.

Finally, along with "Are there any?" we often ask "How many?" That is why the issue of uniqueness often arises in conjunction with questions of existence. Many times the two arguments are simpler if separated, so note that just as proving something exists does not show it is unique, neither does proving something is unique show that it exists. (Obviously 'the natural number with more factors than any other' would be unique, but in fact no such number exists.)

IV.5 Techniques of Proof

Induction. Many proofs are iterative, "Here's why the statement is true for for the case of the number 1, it then follows for 2, and from there to 3, and so on ...". These are called proofs by *induction.* Such a proof has two steps. In the *base step* the proposition is established for some first number, often 0 or 1. Then in the *inductive step* we assume that the proposition holds for numbers up to some k and deduce that it then holds for the next number $k+1$.

Here is an example.

> We will prove that $1+2+3+\cdots+n = n(n+1)/2$.
>
> For the base step we must show that the formula holds when $n = 1$. That's easy, the sum of the first 1 number does indeed equal $1(1+1)/2$.
>
> For the inductive step, assume that the formula holds for the numbers $1, 2, \ldots, k$. That is, assume all of these instances of the formula.
>
> $$\begin{aligned} 1 &= 1(1+1)/2 \\ \text{and}\quad 1+2 &= 2(2+1)/2 \\ \text{and}\quad 1+2+3 &= 3(3+1)/2 \\ &\vdots \\ \text{and}\quad 1+\cdots+k &= k(k+1)/2 \end{aligned}$$
>
> From this assumption we will deduce that the formula therefore also holds in the $k+1$ next case. The deduction is straightforward algebra.
>
> $$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

We've shown in the base case that the above proposition holds for 1. We've shown in the inductive step that if it holds for the case of 1 then it also holds for 2; therefore it does hold for 2. We've also shown in the inductive step that if the statement holds for the cases of 1 and 2 then it also holds for the next case 3, etc. Thus it holds for any natural number greater than or equal to 1.

Here is another example.

> We will prove that every integer greater than 1 is a product of primes.
>
> The base step is easy: 2 is the product of a single prime.
>
> For the inductive step assume that each of $2, 3, \ldots, k$ is a product of primes, aiming to show $k+1$ is also a product of primes. There are two

> possibilities: (i) if $k+1$ is not divisible by a number smaller than itself then it is a prime and so is the product of primes, and (ii) if $k+1$ is divisible then its factors can be written as a product of primes (by the inductive hypothesis) and so $k+1$ can be rewritten as a product of primes. That ends the proof.
>
> (*Remark.* The Prime Factorization Theorem of Number Theory says that not only does a factorization exist, but that it is unique. We've shown the easy half.)

There are two things to note about the 'next number' in an induction argument.

For one thing, while induction works on the integers, it's no good on the reals. There is no 'next' real.

The other thing is that we sometimes use induction to go down, say, from 10 to 9 to 8, etc., down to 0. So 'next number' could mean 'next lowest number'. Of course, at the end we have not shown the fact for all natural numbers, only for those less than or equal to 10.

Contradiction. Another technique of proof is to show something is true by showing it can't be false.

> The classic example is Euclid's, that there are infinitely many primes.
>
> Suppose there are only finitely many primes $p_1, \ldots, p_k$. Consider $p_1 \cdot p_2 \ldots p_k + 1$. None of the primes on this supposedly exhaustive list divides that number evenly, each leaves a remainder of 1. But every number is a product of primes so this can't be. Thus there cannot be only finitely many primes.

Every proof by contradiction has the same form: assume that the proposition is false and derive some contradiction to known facts.

> Another example is this proof that $\sqrt{2}$ is not a rational number.
>
> Suppose that $\sqrt{2} = m/n$.
>
> $$2n^2 = m^2$$
>
> Factor out any 2's: $n = 2^{k_n} \cdot \hat{n}$ and $m = 2^{k_m} \cdot \hat{m}$ and rewrite.
>
> $$2 \cdot (2^{k_n} \cdot \hat{n})^2 = (2^{k_m} \cdot \hat{m})^2$$
>
> The Prime Factorization Theorem says that there must be the same number of factors of 2 on both sides, but there are an odd number $1 + 2k_n$ on the left and an even number $2k_m$ on the right. That's a contradiction, so a rational with a square of 2 cannot be.

Both of these examples aimed to prove something doesn't exist. A negative proposition often suggests a proof by contradiction.

IV.6 Sets, Functions, and Relations

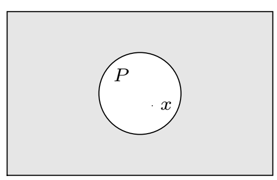
Sets. Mathematicians work with collections, called *sets*. A set can be given as a listing between curly braces as in $\{1, 4, 9, 16\}$, or, if that's unwieldy, by using set-builder notation as in $\{x \mid x^5 - 3x^3 + 2 = 0\}$ (read "the set of all x such that ..."). We name sets with capital roman letters as with the primes $P = \{2, 3, 5, 7, 11, \ldots\}$, except for a few special sets such as the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. To denote that something is an *element* (or *member*) of a set we use '$\in$', so that $7 \in \{3, 5, 7\}$ while $8 \notin \{3, 5, 7\}$.

What distinguishes a set from any other type of collection is the Principle of Extensionality, that two sets with the same elements are equal. Because of this principle, in a set repeats collapse $\{7, 7\} = \{7\}$ and order doesn't matter $\{2, \pi\} = \{\pi, 2\}$.

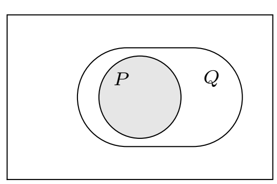
We use '$\subset$' for the *proper subset*relationship: A is a subset of B, so that any element of A is an element of B, but $A \neq B$. An example is $\{2, \pi\} \subset \{2, \pi, 7\}$. We use '$\subseteq$' if either $A \subset B$ or two sets are equal. These symbols may be flipped, for instance $\{2, \pi, 5\} \supset \{2, 5\}$.

Because of Extensionality, to prove that two sets are equal $A = B$, just show that they have the same members. Usually we show mutual inclusion, that both $A \subseteq B$ and $A \supseteq B$.

Set operations. Venn diagrams are handy here. For instance, $x \in P$ can be pictured

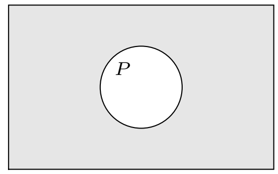


and '$P \subseteq Q$' looks like this.

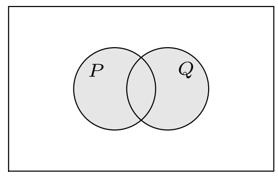


Note that this is a repeat of the diagram for 'if ... then ...' propositions. That's because '$P \subseteq Q$' means 'if $x \in P$ then $x \in Q$'.

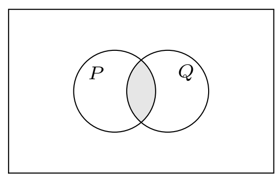
In general, for every propositional logic operator there is an associated set operator. For instance, the *complement* of P is $P^{\text{comp}} = \{x \mid \text{not}(x \in P)\}$



the *union* is $P \cup Q = \{x \mid (x \in P) \text{ or } (x \in Q)\}$



and the *intersection* is $P \cap Q = \{x \mid (x \in P) \text{ and } (x \in Q)\}$.



When two sets share no members their intersection is the *empty set* $\{\}$, symbolized $\varnothing$. Any set has the empty set for a subset, by the 'vacuously true' property of the definition of implication.

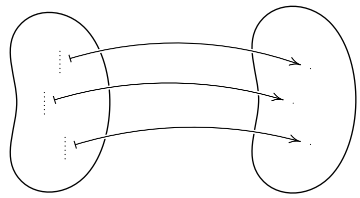
Sequences. We shall also use collections where order does matter and where repeats do not collapse. These are *sequences*, denoted with angle brackets: $\langle 2, 3, 7\rangle \neq \langle 2, 7, 3\rangle$. A sequence of length 2 is sometimes called an *ordered pair* and written with parentheses: $(\pi, 3)$. We also sometimes say 'ordered triple', 'ordered 4-tuple', etc. The set of ordered n-tuples of elements of a set A is denoted A^n. Thus the set of pairs of reals is $\mathbb{R}^2$.

Functions. We first see functions in elementary Algebra, where they are presented as formulas (e.g., $f(x) = 16x^2 - 100$), but progressing to more advanced Mathematics reveals more general functions — trigonometric ones, exponential and logarithmic ones, and even constructs like absolute value that involve piecing together parts — and we see that functions aren't formulas, instead the key idea is that a function associates with its input x a single output $f(x)$.

Consequently, a *function* or *map* is defined to be a set of ordered pairs $(x, f(x))$ such that x suffices to determine $f(x)$, that is: if $x_1 = x_2$ then $f(x_1) = f(x_2)$ (this requirement is referred to by saying a function is *well-defined*).*

Each input x is one of the function's *arguments* and each output $f(x)$ is a *value*. The set of all arguments is f's *domain* and the set of output values is its *range*. Usually we don't need know what is and is not in the range and we instead work with a superset of the range, the *codomain*. The notation for a function f with domain X and codomain Y is $f\colon X \to Y$.

*More on this is in the section on isomorphisms



We sometimes instead use the notation $x \stackrel{f}{\longmapsto} 16x^2 - 100$, read '$x$ maps under f to $16x^2 - 100$', or '$16x^2 - 100$ is the *image* of x'.

Some maps, like $x \mapsto \sin(1/x)$, can be thought of as combinations of simple maps, here, $g(y) = \sin(y)$ applied to the image of $f(x) = 1/x$. The *composition* of $g\colon Y \to Z$ with $f\colon X \to Y$, is the map sending $x \in X$ to $g(\,f(x)\,) \in Z$. It is denoted $g \circ f\colon X \to Z$. This definition only makes sense if the range of f is a subset of the domain of g.

Observe that the *identity map* $\mathrm{id}\colon Y \to Y$ defined by $\mathrm{id}(y) = y$ has the property that for any $f\colon X \to Y$, the composition $\mathrm{id} \circ f$ is equal to f. So an identity map plays the same role with respect to function composition that the number 0 plays in real number addition, or that the number 1 plays in multiplication.

In line with that analogy, define a *left inverse* of a map $f\colon X \to Y$ to be a function $g\colon \mathrm{range}(f) \to X$ such that $g \circ f$ is the identity map on X. Of course, a *right inverse* of f is a $h\colon Y \to X$ such that $f \circ h$ is the identity.

A map that is both a left and right inverse of f is called simply an *inverse*. An inverse, if one exists, is unique because if both g_1 and g_2 are inverses of f then $g_1(x) = g_1 \circ (f \circ g_2)(x) = (g_1 \circ f) \circ g_2(x) = g_2(x)$ (the middle equality comes from the associativity of function composition), so we often call it "the" inverse, written f^{-1}. For instance, the inverse of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 3$ is the function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = (x+3)/2$.

The superscript 'f^{-1}' notation for function inverse can be confusing — it doesn't mean $1/f(x)$. It is used because it fits into a larger scheme. Functions that have the same codomain as domain can be iterated, so that where $f\colon X \to X$, we can consider the composition of f with itself: $f \circ f$, and $f \circ f \circ f$, etc. Naturally enough, we write $f \circ f$ as f^2 and $f \circ f \circ f$ as f^3, etc. Note that the familiar exponent rules for real numbers obviously hold: $f^i \circ f^j = f^{i+j}$ and $(f^i)^j = f^{i \cdot j}$. The relationship with the prior paragraph is that, where f is invertible, writing f^{-1} for the inverse and f^{-2} for the inverse of f^2, etc., gives that these familiar exponent rules continue to hold, once f^0 is defined to be the identity map.

If the codomain Y equals the range of f then we say that the function is *onto*. A function has a right inverse if and only if it is onto (this is not hard to check). If no two arguments share an image, if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, then the function is *one-to-one*. A function has a left inverse if and only if it is one-to-one (this is also not hard to check).

By the prior paragraph, a map has an inverse if and only if it is both onto and one-to-one; such a function is a *correspondence*. It associates one and only one element of the domain with each element of the range (for example, finite sets must have the same number of elements to be matched up in this way). Because a composition of one-to-one maps is one-to-one, and a composition of onto maps is onto, a composition of correspondences is a correspondence.

We sometimes want to shrink the domain of a function. For instance, we may take the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and, in order to have an inverse, limit input arguments to nonnegative reals $\hat{f}\colon \mathbb{R}^+ \to \mathbb{R}$. Technically, $\hat{f}$ is a different function than f; we call it the *restriction* of f to the smaller domain.

A final point on functions: neither x nor $f(x)$ need be a number. As an example, we can think of $f(x,y) = x + y$ as a function that takes the ordered pair (x,y) as its argument.

Relations. Some familiar operations are obviously functions: addition maps $(5,3)$ to 8. But what of '$<$' or '$=$'? We here take the approach of rephrasing '$3 < 5$' to '$(3,5)$ is in the relation $<$'. That is, define a *binary relation* on a set A to be a set of ordered pairs of elements of A. For example, the $<$ relation is the set $\{(a,b) \bigm| a < b\}$; some elements of that set are $(3,5)$, $(3,7)$, and $(1,100)$.

Another binary relation on the natural numbers is equality; this relation is formally written as the set $\{\ldots,(-1,-1),(0,0),(1,1),\ldots\}$.

Still another example is 'closer than 10', the set $\{(x,y) \bigm| |x-y| < 10\}$. Some members of that relation are $(1,10)$, $(10,1)$, and $(42,44)$. Neither $(11,1)$ nor $(1,11)$ is a member.

Those examples illustrate the generality of the definition. All kinds of relationships (e.g., 'both numbers even' or 'first number is the second with the digits reversed') are covered under the definition.

Equivalence Relations. We shall need to say, formally, that two objects are alike in some way. While these alike things aren't identical, they are related (e.g., two integers that 'give the same remainder when divided by 2').

A binary relation $\{(a,b),\ldots\}$ is an *equivalence relation* when it satisfies

(1) *reflexivity*: any object is related to itself;

(2) *symmetry*: if a is related to b then b is related to a;

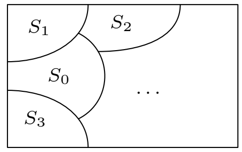
(3) *transitivity*: if a is related to b and b is related to c then a is related to c.

(To see that these conditions formalize being the same, read them again, replacing 'is related to' with 'is like'.)

Some examples (on the integers): '$=$' is an equivalence relation, '$<$' does not satisfy symmetry, 'same sign' is a equivalence, while 'nearer than 10' fails transitivity.

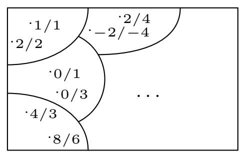
Partitions. In 'same sign' $\{(1,3),(-5,-7),(-1,-1),\ldots\}$ there are two kinds of pairs, the first with both numbers positive and the second with both negative. So integers fall into exactly one of two classes, positive or negative.

A *partition* of a set S is a collection of subsets $\{S_1, S_2, \ldots\}$ such that every element of S is in one and only one S_i: $S_1 \cup S_2 \cup \ldots = S$, and if i is not equal to j then $S_i \cap S_j = \varnothing$. Picture S being decomposed into distinct parts.



Thus, the first paragraph says 'same sign' partitions the integers into the positives and the negatives. Similarly, the equivalence relation '=' partitions the integers into one-element sets.

Another example is the fractions. Of course, $2/3$ and $4/6$ are equivalent fractions. That is, for the set $S = \{n/d \bigm| n, d \in \mathbb{Z} \text{ and } d \neq 0\}$, we define two elements n_1/d_1 and n_2/d_2 to be equivalent if $n_1 d_2 = n_2 d_1$. We can check that this is an equivalence relation, that is, that it satisfies the above three conditions. With that, S is divided up into parts.



Before we show that equivalence relations always give rise to partitions, we first illustrate the argument. Consider the relationship between two integers of 'same parity', the set $\{(-1,3),(2,4),(0,0),\dots\}$ (i.e., 'give the same remainder when divided by 2'). We want to say that the natural numbers split into two pieces, the evens and the odds, and inside a piece each member has the same parity as each other. So for each x we define the set of numbers associated with it: $S_x = \{y \bigm| (x,y) \in \text{'same parity'}\}$. Some examples are $S_1 = \{\dots,-3,-1,1,3,\dots\}$, and $S_4 = \{\dots,-2,0,2,4,\dots\}$, and $S_{-1} = \{\dots,-3,-1,1,3,\dots\}$. These are the parts, e.g., S_1 is the odds.

Theorem. An equivalence relation induces a partition on the underlying set.

Proof. Call the set S and the relation R. In line with the illustration in the paragraph above, for each $x \in S$ define $S_x = \{y \bigm| (x,y) \in R\}$.

Observe that, as x is a member if S_x, the union of all these sets is S. So we will be done if we show that distinct parts are disjoint: if $S_x \neq S_y$ then $S_x \cap S_y = \varnothing$. We will verify this through the contrapositive, that is, we wlll assume that $S_x \cap S_y \neq \varnothing$ in order to deduce that $S_x = S_y$.

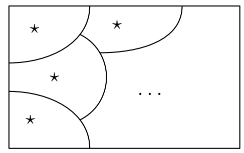
Let p be an element of the intersection. Then by definition of S_x and S_y, the two (x,p) and (y,p) are members of R, and by symmetry of this relation (p,x) and (p,y) are also members of R. To show that $S_x = S_y$ we will show each is a subset of the other.

Assume that $q \in S_x$ so that $(q,x) \in R$. Use transitivity along with $(x,p) \in R$ to conclude that (q,p) is also an element of R. But $(p,y) \in R$ so another use of transitivity gives that $(q,y) \in R$. Thus $q \in S_y$. Therefore $q \in S_x$ implies $q \in S_y$, and so $S_x \subseteq S_y$.

The same argument in the other direction gives the other inclusion, and so the two sets are equal, completing the contrapositive argument. QED

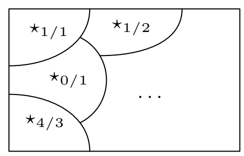
We call each part of a partition an *equivalence class* (or informally, ‘part’).

We somtimes pick a single element of each equivalence class to be the *class representative*.



Usually when we pick representatives we have some natural scheme in mind. In that case we call them the *canonical* representatives.

An example is the simplest form of a fraction. We’ve defined $3/5$ and $9/15$ to be equivalent fractions. In everyday work we often use the ‘simplest form’ or ‘reduced form’ fraction as the class representatives.



Bibliography

[Abbot] Edwin Abbott, *Flatland*, Dover.

[Ackerson] R. H. Ackerson, *A Note on Vector Spaces*, American Mathematical Monthly, volume 62 number 10 (Dec. 1955), p. 721.

[Agnew] Jeanne Agnew, *Explorations in Number Theory*, Brooks/Cole, 1972.

[Am. Math. Mon., Jun. 1931] C. A. Rupp (proposer), H. T. R. Aude (solver), problem 3468, American Mathematical Monthly, volume 37 number 6 (June-July 1931), p. 355.

[Am. Math. Mon., Feb. 1933] V. F. Ivanoff (proposer), T. C. Esty (solver), problem 3529, American Mathematical Monthly, volume 39 number 2 (Feb. 1933), p. 118.

[Am. Math. Mon., Jan. 1935] W. R. Ransom (proposer), Hansraj Gupta (solver), Elementary problem 105, American Mathematical Monthly, volume 42 number 1 (Jan. 1935), p. 47.

[Am. Math. Mon., Jan. 1949] C. W. Trigg (proposer), R. J. Walker (solver), Elementary problem 813, American Mathematical Monthly, volume 56 number 1 (Jan. 1949), p. 33.

[Am. Math. Mon., Jun. 1949] Don Walter (proposer), Alex Tytun (solver), Elementary problem 834, American Mathematical Monthly, volume 56 number 6 (June-July 1949), p. 409.

[Am. Math. Mon., Nov. 1951] Albert Wilansky, *The Row-Sums of the Inverse Matrix*, American Mathematical Monthly, volume 58 number 9 (Nov. 1951), p. 614.

[Am. Math. Mon., Feb. 1953] Norman Anning (proposer), C. W. Trigg (solver), Elementary problem 1016, American Mathematical Monthly, volume 60 number 2 (Feb. 1953), p. 115.

[Am. Math. Mon., Apr. 1955] Vern Haggett (proposer), F. W. Saunders (solver), Elementary problem 1135, American Mathematical Monthly, volume 62 number 4 (Apr. 1955), p. 257.

[Am. Math. Mon., Jan. 1963] Underwood Dudley, Arnold Lebow (proposers), David Rothman (solver), Elemantary problem 1151, American Mathematical Monthly, volume 70 number 1 (Jan. 1963), p. 93.

[Am. Math. Mon., Dec. 1966] Hans Liebeck, *A Proof of the Equality of Column Rank and Row Rank of a Matrix* American Mathematical Monthly, volume 73 number 10 (Dec. 1966), p. 1114.

[Anton] Howard Anton, *Elementary Linear Algebra*, John Wiley & Sons, 1987.

[Arrow] Kenneth J. Arrow, *Social Choice and Individual Values*, Wiley, 1963.

[Ball & Coxeter] W.W. Rouse Ball, *Mathematical Recreations and Essays*, revised by H.S.M. Coxeter, MacMillan, 1962.

[Bennett] William J. Bennett, *Quantifying America's Decline*, in *Wall Street Journal*, 15 March, 1993.

[Birkhoff & MacLane] Garrett Birkhoff, Saunders MacLane, *Survey of Modern Algebra*, third edition, Macmillan, 1965.

[Blass 1984] A. Blass, *Existence of Bases Implies the Axiom of Choice*, pp. 31 – 33, Axiomatic Set Theory, J. E. Baumgartner, ed., American Mathematical Society, Providence RI, 1984.

[Bridgman] P.W. Bridgman, *Dimensional Analysis*, Yale University Press, 1931.

[Casey] John Casey, *The Elements of Euclid, Books I to VI and XI*, ninth edition, Hodges, Figgis, and Co., Dublin, 1890.

[Clark & Coupe] David H. Clark, John D. Coupe, *The Bangor Area Economy Its Present and Future*, report to the city of Bangor ME, Mar. 1967.

[Clarke] Arthur C. Clarke, *Technical Error*, Fantasy, December 1946, reprinted in Great SF Stories 8 (1946), DAW Books, 1982.

[Con. Prob. 1955] *The Contest Problem Book*, 1955 number 38.

[Coxeter] H.S.M. Coxeter, *Projective Geometry*, second edition, Springer-Verlag, 1974.

[Courant & Robbins] Richard Courant, Herbert Robbins, *What is Mathematics?*, Oxford University Press, 1978.

[Cullen] Charles G. Cullen, *Matrices and Linear Transformations*, second edition, Dover, 1990.

[Dalal, et. al.] Siddhartha R. Dalal, Edward B. Fowlkes, & Bruce Hoadley, *Lesson Learned from Challenger: A Statistical Perspective*, Stats: the Magazine for Students of Statistics, Fall 1989, p. 3.

[Davies] Thomas D. Davies, *New Evidence Places Peary at the Pole*, National Geographic Magazine, vol. 177 no. 1 (Jan. 1990), p. 44.

[de Mestre] Neville de Mestre, *The Mathematics of Projectiles in Sport*, Cambridge University Press, 1990.

[De Parville] De Parville, *La Nature*, Paris, 1884, part I, p. 285 – 286 (citation from [Ball & Coxeter]).

[Duncan] W.J. Duncan, *Method of Dimensions*, in *Encyclopaedic Dictionary of Physics* Macmillan, 1962.

[Ebbing] Darrell D. Ebbing, *General Chemistry*, fourth edition, Houghton Mifflin, 1993.

[Ebbinghaus] H. D. Ebbinghaus, *Numbers*, Springer-Verlag, 1990.

[Einstein] A. Einstein, Annals of Physics, v. 35, 1911, p. 686.

[Eggar] M.H. Eggar, *Pinhole Cameras, Perspecitve, and Projective Geometry*, American Mathematicsl Monthly, August-September 1998, p. 618 – 630.

[Feller] William Feller, *An Introduction to Probability Theory and Its Applications* (vol. 1, 3rd ed.), John Wiley, 1968.

[Finkbeiner] Daniel T. Finkbeiner III, *Introduction to Matrices and Linear Transformations*, third edition, W. H. Freeman and Company, 1978.

[Fraleigh & Beauregard] Fraleigh & Beauregard, *Linear Algebra*.

[Gardner] Martin Gardner, *The New Ambidextrous Universe*, third revised edition, W. H. Freeman and Company, 1990.

[Gardner 1957] Martin Gardner, *Mathematical Games: About the remarkable similarity between the Icosian Game and the Tower of Hanoi*, Scientific American, May 1957, p. 150 – 154.

[Gardner, 1970] Martin Gardner, *Mathematical Games, Some mathematical curiosities embedded in the solar system*, Scientific American, April 1970, p. 108 – 112.

[Gardner, 1980] Martin Gardner, *Mathematical Games, From counting votes to making votes count: the mathematics of elections*, Scientific American, October 1980.

[Gardner, 1974] Martin Gardner, *Mathematical Games, On the paradoxical situations that arise from nontransitive relations*, Scientific American, October 1974.

[Giordano, Wells, Wilde] Frank R. Giordano, Michael E. Wells, Carroll O. Wilde, *Dimensional Analysis*, UMAP Unit 526, in *UMAP Modules, 1987*, COMAP, 1987.

[Giordano, Jaye, Weir] Frank R. Giordano, Michael J. Jaye, Maurice D. Weir, *The Use of Dimensional Analysis in Mathematical Modeling*, UMAP Unit 632, in *UMAP Modules, 1986*, COMAP, 1986.

[Glazer] A.M. Glazer, *The Structure of Crystals*, Adam Hilger, 1987.

[Goult, *et al.*] R.J. Goult, R.F. Hoskins, J.A. Milner, M.J. Pratt, *Computational Methods in Linear Algebra*, Wiley, 1975.

[Graham, Knuth, Patashnik] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, 1988.

[Halmos] Paul P. Halmos, *Finite Dimensional Vector Spaces*, second edition, Van Nostrand, 1958.

[Hamming] Richard W. Hamming, *Introduction to Applied Numerical Analysis*, Hemisphere Publishing, 1971.

[Hanes] Kit Hanes, *Analytic Projective Geometry and its Applications*, UMAP Unit 710, UMAP Modules, 1990, p. 111.

[Heath] T. Heath, *Euclid's Elements*, volume 1, Dover, 1956.

[Hoffman & Kunze] Kenneth Hoffman, Ray Kunze, *Linear Algebra*, second edition, Prentice-Hall, 1971.

[Hofstadter] Douglas R. Hofstadter, *Metamagical Themas: Questing for the Essence of Mind and Pattern*, Basic Books, 1985.

[Iosifescu] Marius Iofescu, *Finite Markov Processes and Their Applications*, John Wiley, 1980.

[Kelton] Christina M.L. Kelton, *Trends on the Relocation of U.S. Manufacturing*, UMI Research Press, 1983.

[Kemeny & Snell] John G. Kemeny, J. Laurie Snell, *Finite Markov Chains*, D. Van Nostrand, 1960.

[Kemp] Franklin Kemp *Linear Equations*, American Mathematical Monthly, volume 89 number 8 (Oct. 1982), p. 608.

[Knuth] Donald E. Knuth, *The Art of Computer Programming*, Addison Wesley, 1988.

[Lang] Serge Lang, *Linear Algebra*, Addison-Wesley, 1966.

[Leontief 1951] Wassily W. Leontief, *Input-Output Economics*, Scientific American, volume 185 number 4 (Oct. 1951), p. 15.

[Leontief 1965] Wassily W. Leontief, *The Structure of the U.S. Economy*, Scientific American, volume 212 number 4 (Apr. 1965), p. 25.

[Lieberman] David Lieberman, *The Big Bucks Ad Battles Over TV's Most Expensive Minutes*, TV Guide, Jan. 26 1991, p. 11.

[Macdonald & Ridge] Kenneth Macdonald, John Ridge, *Social Mobility*, in *British Social Trends Since 1900*, A.H. Halsey, Macmillian, 1988.

[Macmillan Dictionary] William D. Halsey, Macmillan, 1979.

[Math. Mag., Sept. 1952] Dewey Duncan (proposer), W. H. Quelch (solver), Mathematics Magazine, volume 26 number 1 (Sept-Oct. 1952), p. 48.

[Math. Mag., Jan. 1957] M. S. Klamkin (proposer), Trickie T-27, Mathematics Magazine, volume 30 number 3 (Jan-Feb. 1957), p. 173.

[Math. Mag., Jan. 1963, Q237] D. L. Silverman (proposer), C. W. Trigg (solver), Quickie 237, Mathematics Magazine, volume 36 number 1 (Jan. 1963).

[Math. Mag., Jan. 1963, Q307] C. W. Trigg (proposer). Quickie 307, Mathematics Magazine, volume 36 number 1 (Jan. 1963), p. 77.

[Math. Mag., Nov. 1967] Clarence C. Morrison (proposer), Quickie, Mathematics Magazine, volume 40 number 4 (Nov. 1967), p. 232.

[Math. Mag., Jan. 1973] Marvin Bittinger (proposer), Quickie 578, Mathematics Magazine, volume 46 number 5 (Jan. 1973), p. 286, 296.

[Munkres] James R. Munkres, *Elementary Linear Algebra*, Addison-Wesley, 1964.

[Neimi & Riker] Richard G. Neimi, William H. Riker, *The Choice of Voting Systems*, Scientific American, June 1976, p. 21 – 27.

[Nering] Evar D. Nering, *Linear Algebra and Matrix Theory*, second edition, John Wiley, 1970.

[Niven & Zuckerman] I. Niven, H. Zuckerman, *An Introduction to the Theory of Numbers*, third edition, John Wiley, 1972.

[Oakley & Baker] Cletus O. Oakley, Justine C. Baker, *Least Squares and the* $3:40$ *Mile*, Mathematics Teacher, Apr. 1977.

[Ohanian] Hans O'Hanian, *Physics*, volume one, W. W. Norton, 1985.

[Onan] Onan, *Linear Algebra*.

[Petersen] G. M. Petersen, *Area of a Triangle*, American Mathematical Monthly, volume 62 number 4 (Apr. 1955), p. 249.

[Polya] G. Polya, *Patterns of Plausible Inference*, [Poundstone] W. Poundstone, *Gaming the Vote*, Hill and Wang, 2008. ISBN-13: 978-0-8090-4893-9

[Programmer's Ref.] *Microsoft Programmers Reference* Microsoft Press.

[Putnam, 1988, A-6] William Lowell Putnam Mathematical Competition, Problem A-6, 1988, solution: American Mathematical Monthly, volume 96 number 8 (Oct. 1989), p. 688.

[Putnam, 1990, A-5] William Lowell Putnam Mathematical Competition, Problem A-5, 1990.

[Rice] John R. Rice, *Numerical Mathods, Software, and Analysis*, second edition, Academic Press, 1993.

[Quine] W. V. Quine, *Methods of Logic*, fourth edition, Harvard University Press, 1982.

[Rucker] Rudy Rucker, *Infinity and the Mind*, Birkhauser, 1982.

[Ryan] Patrick J. Ryan, *Euclidean and Non-Euclidean Geometry: an Analytic Approach*, Cambridge University Press, 1986.

[Shepelev] Anton Shepelev, private communication, Feb 19, 2011.

[Seidenberg] A. Seidenberg, *Lectures in Projective Geometry*, Van Nostrand, 1962.

[Smith] Larry Smith, *Linear Algebra*, second edition, Springer-Verlag, 1984.

[Strang 93] Gilbert Strang *The Fundamental Theorem of Linear Algebra*, American Mathematical Monthly, Nov. 1993, p. 848 – 855.

[Strang 80] Gilbert Strang, *Linear Algebra and its Applications*, second edition, Harcourt Brace Jovanovich, 1980.

[Taylor] Alan D. Taylor, *Mathematics and Politics: Strategy, Voting, Power, and Proof*, Springer-Verlag, 1995.

[Tilley] Burt Tilley, private communication, 1996.

[Trono] Tony Trono, compiler, *University of Vermont Mathematics Department High School Prize Examinations 1958-1991*, mimeographed printing, 1991.

[USSR Olympiad no. 174] *The USSR Mathematics Olympiad*, number 174.

[Weston] J. D. Weston, *Volume in Vector Spaces*, American Mathematical Monthly, volume 66 number 7 (Aug./Sept. 1959), p. 575 – 577.

[Weyl] Hermann Weyl, *Symmetry*, Princeton University Press, 1952.

[Wickens] Thomas D. Wickens, *Models for Behavior*, W.H. Freeman, 1982.

[Wilkinson 1965] Wilkinson, 1965

[Wikipedia] *The Square-cube law*, http://en.wikipedia.org/wiki/Square-cube_law, 2011-Jan-17.

[WikipediaMensMile] *Mile run world record progression*, http://en.wikipedia.org/wiki/Mile_run_world_record_progression, 2011-Apr-09.

[Wohascum no. 2] *The Wohascum County Problem Book* problem number 2.

[Wohascum no. 47] *The Wohascum County Problem Book* problem number 47.

[Yaglom] I.M. Yaglom, *Felix Klein and Sophus Lie: Evolution of the Idea of Symmetry in the Nineteenth Century*, translated by Sergei Sossinsky, Birkhäuser, 1988.

[Zwicker] William S. Zwicker, *The Voters' Paradox, Spin, and the Borda Count*, Mathematical Social Sciences, vol. 22 (1991), p. 187 – 227.