Four.I Determinants; Definition

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Properties of Determinants

Nonsingular matrices

An $n \times n$ matrix T is nonsingular if and only if each of these holds:

- any system $T\vec{x} = \vec{b}$ has a solution and that solution is unique;
- Gauss-Jordan reduction of T yields an identity matrix;
- the rows of T form a linearly independent set;
- the columns of T form a linearly independent set, a basis for $\mathbb{R}^n$;
- any map that T represents is an isomorphism;
- an inverse matrix T^{-1} exists.

In this chapter we will give a formula that determines whether a matrix is nonsingular.

Determining nonsingularity is trivial for $1{\times}1$ matrices.

$$\begin{pmatrix} a \end{pmatrix} \quad \text{is nonsingular iff} \quad a \neq 0$$

Corollary Three.IV.4.11 gives the $2{\times}2$ formula.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is nonsingular iff} \quad ad - bc \neq 0$$

We can produce the $3{\times}3$ formula as we did the prior one, although the computation is intricate (see Exercise 9).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{is nonsingular iff} \quad aei + bfg + cdh - hfa - idb - gec \neq 0$$

With these cases in mind, we posit a family of formulas: a, $ad - bc$, etc. For each n the formula defines a *determinant* function $\det_{n\times n} \colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that an $n{\times}n$ matrix T is nonsingular if and only if $\det_{n\times n}(T) \neq 0$.

We will define the determinant function by listing some of its properties. We are interested in these properties because they are convenient for computing the value of the determinant on an input square matrix. Then we will show that only one function with those properties exists.

We will define the determinant function by listing some of its properties. We are interested in these properties because they are convenient for computing the value of the determinant on an input square matrix. Then we will show that only one function with those properties exists.

Note The first section of the text algebraically motivates the determinant definition. The section after this will give a geometric motivation. In this section, beyond defining the determinant, we will show how to compute it and give some key results.

Definition of determinant

2.1 *Definition* A $n \times n$ *determinant* is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that

1) $\det(\vec{\rho}_1, \ldots, k\cdot\vec{\rho}_i + \vec{\rho}_j, \ldots, \vec{\rho}_n) = \det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
2) $\det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n) = -\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
3) $\det(\vec{\rho}_1, \ldots, k\vec{\rho}_i, \ldots, \vec{\rho}_n) = k\cdot\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)$ for any scalar k
4) $\det(I) = 1$ where I is an identity matrix

(the $\vec{\rho}$'s are the rows of the matrix). We often write $|T|$ for $\det(T)$.

Definition of determinant

2.1 *Definition* A $n \times n$ *determinant* is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that

1) $\det(\vec{\rho}_1, \ldots, k \cdot \vec{\rho}_i + \vec{\rho}_j, \ldots, \vec{\rho}_n) = \det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
2) $\det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n) = -\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
3) $\det(\vec{\rho}_1, \ldots, k\vec{\rho}_i, \ldots, \vec{\rho}_n) = k \cdot \det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)$ for any scalar k
4) $\det(I) = 1$ where I is an identity matrix

(the $\vec{\rho}$'s are the rows of the matrix). We often write $|T|$ for $\det(T)$.

2.2 *Remark* Condition (2) is redundant since

$T \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_j+\rho_i} \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_i} \hat{T}$ swaps rows i and j. We have only listed it for convenience.

Consequences of the definition

2.4 *Lemma* A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

Consequences of the definition

2.4 *Lemma* A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

Proof To verify the first sentence swap the two equal rows. The sign of the determinant changes but the matrix is the same and so its determinant is the same. Thus the determinant is zero.

Consequences of the definition

2.4 *Lemma* A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

Proof To verify the first sentence swap the two equal rows. The sign of the determinant changes but the matrix is the same and so its determinant is the same. Thus the determinant is zero.

For the second sentence multiply the zero row by two. That doubles the determinant but it also leaves the row unchanged, and hence leaves the determinant unchanged. Thus the determinant must be zero.

Do Gauss-Jordan reduction for the third sentence, $T \to \cdots \to \hat{T}$. By the first three properties the determinant of T is zero if and only if the determinant of $\hat{T}$ is zero (although the two could differ in sign or magnitude). A nonsingular matrix T Gauss-Jordan reduces to an identity matrix and so has a nonzero determinant. A singular T reduces to a $\hat{T}$ with a zero row; by the second sentence of this lemma its determinant is zero.

Do Gauss-Jordan reduction for the third sentence, $T \rightarrow \cdots \rightarrow \hat{T}$. By the first three properties the determinant of T is zero if and only if the determinant of $\hat{T}$ is zero (although the two could differ in sign or magnitude). A nonsingular matrix T Gauss-Jordan reduces to an identity matrix and so has a nonzero determinant. A singular T reduces to a $\hat{T}$ with a zero row; by the second sentence of this lemma its determinant is zero.

The fourth sentence has two cases. If the echelon form matrix is singular then it has a zero row. Thus it has a zero on its diagonal and the product down its diagonal is zero. By the third sentence of this result the determinant is zero and therefore this matrix's determinant equals the product down its diagonal.

If the echelon form matrix is nonsingular then none of its diagonal entries is zero so we can use condition (3) to get 1's on the diagonal.

$$\begin{vmatrix} t_{1,1} & t_{1,2} & & t_{1,n} \\ 0 & t_{2,2} & & t_{2,n} \\ & & \ddots & \\ 0 & & & t_{n,n} \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & t_{1,2}/t_{1,1} & & t_{1,n}/t_{1,1} \\ 0 & 1 & & t_{2,n}/t_{2,2} \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix}$$

(We need that diagonal entries are nonzero to write, e.g., $t_{1,2}/t_{1,1}$.) Then the Jordan half of Gauss-Jordan elimination leaves the identity matrix.

$$= t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot 1$$

So in this case also, the determinant is the product down the diagonal. QED

We can compute the determinant of a matrix using Gauss's Method (presuming that the determinant function exists, which we will cover later).

Example On this matrix the Gauss's Method reduces the first column with $-2\rho_1 + \rho_2$ and $-3\rho_1 + \rho_3$. Property (1) says that these row operations leave the determinant unchanged.

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & -10 & -11 \end{vmatrix}$$

We can compute the determinant of a matrix using Gauss's Method (presuming that the determinant function exists, which we will cover later).

Example On this matrix the Gauss's Method reduces the first column with $-2\rho_1+\rho_2$ and $-3\rho_1+\rho_3$. Property (1) says that these row operations leave the determinant unchanged.

$$\begin{vmatrix}1 & 3 & -2\\ 2 & 0 & 4\\ 3 & -1 & 5\end{vmatrix}=\begin{vmatrix}1 & 3 & -2\\ 0 & -6 & -8\\ 0 & -10 & -11\end{vmatrix}$$

Reduce the second column with $-(5/3)\rho_2+\rho_3$. Again, by property (1) the determinant stays the same.

$$=\begin{vmatrix}1 & 3 & -2\\ 0 & -6 & -8\\ 0 & 0 & -7/3\end{vmatrix}$$

We can compute the determinant of a matrix using Gauss's Method (presuming that the determinant function exists, which we will cover later).

Example On this matrix the Gauss's Method reduces the first column with $-2\rho_1 + \rho_2$ and $-3\rho_1 + \rho_3$. Property (1) says that these row operations leave the determinant unchanged.

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & -10 & -11 \end{vmatrix}$$

Reduce the second column with $-(5/3)\rho_2 + \rho_3$. Again, by property (1) the determinant stays the same.

$$= \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & 0 & -7/3 \end{vmatrix}$$

By the prior lemma we can now find the determinant by taking the product down the diagonal.

$$= 1 \cdot (-6) \cdot (-7/3) = 14$$

Example This matrix requires a row swap, which changes the sign of the determinant.

$$\begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 5 & 2 \end{vmatrix}$$

Performing $-\rho_1 + \rho_3$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

and $-\rho_2 + \rho_3$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

and then multiplying down the diagonal gives that the determinant of the original matrix is -3.

The $n \times n$ determinant is unique

2.7 *Lemma* For each n, if there is an $n \times n$ determinant function then it is unique.

The $n \times n$ determinant is unique

2.7 *Lemma* For each n, if there is an $n \times n$ determinant function then it is unique.

Proof Perform Gauss's Method on the matrix, keeping track of how the sign alternates on row swaps and any row-scaling factors, and then multiply down the diagonal of the echelon form result. By the definition and the lemma, all $n \times n$ determinant functions must return this value on the matrix. QED

The $n \times n$ determinant is unique

2.7 *Lemma* For each n, if there is an $n \times n$ determinant function then it is unique.

Proof Perform Gauss's Method on the matrix, keeping track of how the sign alternates on row swaps and any row-scaling factors, and then multiply down the diagonal of the echelon form result. By the definition and the lemma, all $n \times n$ determinant functions must return this value on the matrix. QED

So if there is a function mapping $\mathcal{M}_{n \times n}$ to $\mathbb{R}$ with the four properties of the definition then there is only one such function. The next two subsections show that for each n a determinant function exists.

The Permutation Expansion

The prior subsection defines a function to be a determinant if it satisfies four conditions and shows that there is at most one $n \times n$ determinant function for each n. What is left is to show that for each n such a function exists.

But, we easily compute determinants: we use Gauss's Method, keeping track of the sign changes from row swaps, and end by multiplying down the diagonal. How could they not exist?

The difficulty is to show that the computation gives a well-defined—that is, unique—result. Consider these two Gauss's Method reductions of the same matrix, the first without any row swap

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

and the second with one.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \xrightarrow{-(1/3)\rho_1+\rho_2} \begin{pmatrix} 3 & 4 \\ 0 & 2/3 \end{pmatrix}$$

Both yield the determinant -2 since in the second one we note that the row swap changes the sign of the result we get by multiplying down the diagonal.

That the above computation gives a consistent result for these two ways to do a reduction on one matrix does not ensure that determinants always give a well-defined value. Our algorithm for computing determinant values does not plainly eliminate the possibility that there might be, say, two reductions of some 7×7 matrix that lead to different determinant outputs. In that case there would exist no determinant function, since functions must have that for each input there is exactly one output.

That the above computation gives a consistent result for these two ways to do a reduction on one matrix does not ensure that determinants always give a well-defined value. Our algorithm for computing determinant values does not plainly eliminate the possibility that there might be, say, two reductions of some 7×7 matrix that lead to different determinant outputs. In that case there would exist no determinant function, since functions must have that for each input there is exactly one output.

To show that determinants are well-defined we will give an alternative way to compute the value of a determinant. This new way is less useful in practice since it makes the computations awkward and slow, which is why we didn't start with it. But it is useful for theory since it makes the proof that we need easier.

The determinant function is not linear

Example The second matrix is twice the first but the determinant does not double.

$$\begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} = -72 \qquad \begin{vmatrix} 6 & -6 & 18 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} = -576$$

Instead, by property (3) of Definition 2.1 the determinant scales one row at a time:

$$\begin{aligned} \begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} &= 3 \cdot \begin{vmatrix} 1 & -1 & 3 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} \\ &= 6 \cdot \begin{vmatrix} 1 & -1 & 3 \\ 1 & -1 & 7 \\ 1 & 2 & 0 \end{vmatrix} \end{aligned}$$

Multilinear

3.2 *Definition* Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

1) $f(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) =$
$f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + f(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$
2) $f(\vec{\rho}_1, \ldots, k\vec{v}, \ldots, \vec{\rho}_n) = k \cdot f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n)$

for $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

3.3 *Lemma* Determinants are multilinear.

Multilinear

3.2 *Definition* Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

1) $f(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) =$
   $f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + f(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$
2) $f(\vec{\rho}_1, \ldots, k\vec{v}, \ldots, \vec{\rho}_n) = k \cdot f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n)$

for $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

3.3 *Lemma* Determinants are multilinear.

Proof Property (2) here is just Definition 2.1 's condition (3) so we need only verify property (1).

Multilinear

3.2 *Definition* Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

1) $f(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n) =$
   $f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + f(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$
2) $f(\vec{\rho}_1, \ldots, k\vec{v}, \ldots, \vec{\rho}_n) = k \cdot f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n)$

for $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

3.3 *Lemma* Determinants are multilinear.

Proof Property (2) here is just Definition 2.1 's condition (3) so we need only verify property (1).

There are two cases. If the set of other rows $\{\vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n\}$ is linearly dependent then all three matrices are singular and so all three determinants are zero and the equality is trivial.

Therefore assume that the set of other rows is linearly independent. We can make a basis by adding one more vector $\langle \vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\beta}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n \rangle$. Express $\vec{v}$ and $\vec{w}$ with respect to this basis

$$\vec{v} = v_1\vec{\rho}_1 + \cdots + v_{i-1}\vec{\rho}_{i-1} + v_i\vec{\beta} + v_{i+1}\vec{\rho}_{i+1} + \cdots + v_n\vec{\rho}_n$$
$$\vec{w} = w_1\vec{\rho}_1 + \cdots + w_{i-1}\vec{\rho}_{i-1} + w_i\vec{\beta} + w_{i+1}\vec{\rho}_{i+1} + \cdots + w_n\vec{\rho}_n$$

and add.

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{\rho}_1 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

Consider the left side of (1) and expand $\vec{v} + \vec{w}$.

$$\det(\vec{\rho}_1, \ldots, (v_1 + w_1)\vec{\rho}_1 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n, \ldots, \vec{\rho}_n) \tag{*}$$

By the definition of determinant's condition (1), the value of $(*)$ is unchanged by the operation of adding $-(v_1 + w_1)\vec{\rho}_1$ to the i-th row $\vec{v} + \vec{w}$. The i-th row becomes this.

$$\vec{v} + \vec{w} - (v_1 + w_1)\vec{\rho}_1 = (v_2 + w_2)\vec{\rho}_2 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

Next add $-(v_2 + w_2)\vec{\rho}_2$, etc., to eliminate all of the terms from the other rows. Apply condition (3) from the definition of determinant.

$$
\begin{aligned}
\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n)& \\
&= \det(\vec{\rho}_1, \ldots, (v_i + w_i) \cdot \vec{\beta}, \ldots, \vec{\rho}_n) \\
&= (v_i + w_i) \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) \\
&= v_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) + w_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n)
\end{aligned}
$$

Now this is a sum of two determinants. To finish, bring v_i and w_i back inside in front of the $\vec{\beta}$'s and use row combinations again, this time to reconstruct the expressions of $\vec{v}$ and $\vec{w}$ in terms of the basis. That is, start with the operations of adding $v_1\vec{\rho}_1$ to $v_i\vec{\beta}$ and $w_1\vec{\rho}_1$ to $w_i\vec{\rho}_1$, etc., to get the expansions of $\vec{v}$ and $\vec{w}$. QED

Use multilinearity to break a determinant into a sum of simple determinants.

Example We can expand this determinant

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

along the first row

$$= \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$$

and the second row.

$$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix}$$

We have four matrices, each with a single nonzero entry in each row.

Use multilinearity to break a determinant into a sum of simple determinants.

Example We can expand this determinant

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

along the first row

$$= \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$$

and the second row.

$$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix}$$

We have four matrices, each with a single nonzero entry in each row.

The first and last determinants are 0 because the matrices are nonsingular (since one row is a multiple of the other). We are left with the two matrices in which there is one entry from each row and column from the starting matrix.

Example Similarly we can start to evaluate this determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by breaking it into a sum of determinants of matrices having one entry in each row from the starting matrix.

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

This gives a number of matrices, each all 0's except that each row has a single entry from the original matrix.

Example Similarly we can start to evaluate this determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by breaking it into a sum of determinants of matrices having one entry in each row from the starting matrix.

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

This gives a number of matrices, each all 0's except that each row has a single entry from the original matrix.

For any of these determinants, if two rows have their original matrix entry in the same column then the determinant is 0, since if either entry is 0 then the matrix has a zero row while if neither is 0 then each row is a multiple of the other.

Example Similarly we can start to evaluate this determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by breaking it into a sum of determinants of matrices having one entry in each row from the starting matrix.

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

This gives a number of matrices, each all 0's except that each row has a single entry from the original matrix.

For any of these determinants, if two rows have their original matrix entry in the same column then the determinant is 0, since if either entry is 0 then the matrix has a zero row while if neither is 0 then each row is a multiple of the other. Therefore, the above reduces to a sum of determinants, each all 0's but for a single entry in each row and column from the original matrix.

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{vmatrix} \\
&= 45 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 48 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&\quad + 72 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 84 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&\quad + 96 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + 105 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}
\end{aligned}
$$