Three.II Homomorphisms

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition

Homomorphism

1.1 *Definition* A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

Example The function $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ given by

$$h(a + bx + cx^2) = \begin{pmatrix} a + c \\ 0 \end{pmatrix}$$

is a homomorphism (it happens to be neither one-to-one nor onto). We will verify that it respects the addition and scalar multiplication operations.

Example The function $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ given by

$$h(a + bx + cx^2) = \begin{pmatrix} a + c \\ 0 \end{pmatrix}$$

is a homomorphism (it happens to be neither one-to-one nor onto). We will verify that it respects the addition and scalar multiplication operations.

Addition is routine.

$$\begin{aligned}
h(\,(a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)\,) \\
&= h(\,(a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\,) \\
&= \begin{pmatrix} a_1 + a_2 + c_1 + c_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} a_1 + c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 + c_2 \\ 0 \end{pmatrix} \\
&= h(a_1 + b_1x + c_1x^2) + h(a_2 + b_2x + c_2x^2)
\end{aligned}$$

Example The function $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ given by

$$h(a+bx+cx^2) = \begin{pmatrix} a+c \\ 0 \end{pmatrix}$$

is a homomorphism (it happens to be neither one-to-one nor onto). We will verify that it respects the addition and scalar multiplication operations.

Addition is routine.

$$\begin{aligned}
h(\,(a_1+b_1x+c_1x^2)+(a_2+b_2x+c_2x^2)\,) \\
&= h(\,(a_1+a_2)+(b_1+b_2)x+(c_1+c_2)x^2\,) \\
&= \begin{pmatrix} a_1+a_2+c_1+c_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} a_1+c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2+c_2 \\ 0 \end{pmatrix} \\
&= h(a_1+b_1x+c_1x^2)+h(a_2+b_2x+c_2x^2)
\end{aligned}$$

So is scalar multiplication.

$$r\cdot h(a+bx+cx^2) = r\cdot\begin{pmatrix} a+c \\ 0 \end{pmatrix} = \begin{pmatrix} ra+rc \\ 0 \end{pmatrix} = h(\,r(a+bx+cx^2)\,)$$

Example Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$ the first is linear while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{h}{\longmapsto} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{g}{\longmapsto} 2x - 3y + 1$$

Example Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$ the first is linear while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{h} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{g} 2x - 3y + 1$$

The map h respects addition

$$\begin{aligned} h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= h(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) = 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + h(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \end{aligned}$$

and scalar multiplication.

$$r \cdot h(\begin{pmatrix} x \\ y \end{pmatrix}) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

Example Of these two maps $h, g\colon \mathbb{R}^2 \to \mathbb{R}$ the first is linear while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{h}{\longmapsto} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{g}{\longmapsto} 2x - 3y + 1$$

The map h respects addition

$$\begin{aligned} h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = h(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}) &= 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + h(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \end{aligned}$$

and scalar multiplication.

$$r \cdot h(\begin{pmatrix} x \\ y \end{pmatrix}) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h(r \cdot \begin{pmatrix} x \\ y \end{pmatrix})$$

This example shows that g does not respect addition.

$$g(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -17 \quad \text{while} \quad g(\begin{pmatrix} 1 \\ 4 \end{pmatrix}) + g(\begin{pmatrix} 5 \\ 6 \end{pmatrix}) = -16$$

We proved these two in the context of studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) f is a homomorphism

(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$

(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

We proved these two in the context of studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f\colon V \to W$ between vector spaces.

(1) f is a homomorphism
(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$
(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

Example Between any two vector spaces the zero map $Z\colon V \to W$, defined by $Z(\vec{v}) = \vec{0}_W$ is a homomorphism. The check is: $Z(c_1\vec{v}_1 + c_2\vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2)$.

Example The *inclusion map* $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\iota(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism. Here is the verification.

$$\begin{aligned}
\iota(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) &= \iota(\begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix}) \\
&= \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c_1x_1 \\ c_1y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2x_2 \\ c_2y_2 \\ 0 \end{pmatrix} \\
&= c_1 \cdot \iota(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}) + c_2 \cdot \iota(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix})
\end{aligned}$$

Example Consider this function $h\colon \mathcal{P}_1 \to \mathcal{P}_1$.

$$h(a + bx) = b + bx$$

Here are two examples of the action of this function: $h(1 + 2x) = 2 + 2x$ and $h(3 - x) = -1 - x$.

Example Consider this function $h\colon \mathcal{P}_1 \to \mathcal{P}_1$.

$$h(a + bx) = b + bx$$

Here are two examples of the action of this function:
$h(1 + 2x) = 2 + 2x$ and $h(3 - x) = -1 - x$.

This function is linear.

$$\begin{aligned}
&h(\,c_1 \cdot (a_1 + b_1x) + c_2 \cdot (a_2 + b_2x)\,) \\
&\qquad = h(\,(c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)x\,) \\
&\qquad = (c_1b_1 + c_2b_2) + (c_1b_1 + c_2b_2)x \\
&\qquad = (c_1b_1 + c_1b_1x) + (c_2b_2 + c_2b_2x) \\
&\qquad = c_1 \cdot h(a_1 + b_1x) + c_2 \cdot h(a_2 + b_2x)
\end{aligned}$$

Example The derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ is given by $d/dx\,(ax^2 + bx + c) = 2ax + b$. For instance, $d/dx\,(3x^2 - 2x + 4) = 6x - 2$ and $d/dx\,(x^2 + 1) = 2x$.

Example The derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ is given by $d/dx\,(ax^2 + bx + c) = 2ax + b$. For instance, $d/dx\,(3x^2 - 2x + 4) = 6x - 2$ and $d/dx\,(x^2 + 1) = 2x$. It is a homomorphism.

$$\begin{aligned}
&d/dx\,\bigl(r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2)\bigr)\\
&\qquad = d/dx\,\bigl((r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2)\bigr)\\
&\qquad = 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2)\\
&\qquad = (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2)\\
&\qquad = r_1 \cdot d/dx\,(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx\,(a_2x^2 + b_2x + c_2)
\end{aligned}$$

Example The *trace* of a square matrix is the sum down the uppper-left to lower-right diagonal. Thus $\mathrm{Tr}\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is this.

$$\mathrm{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a + b$$

It is linear.

$$\begin{aligned}
\mathrm{Tr}(\,r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\,) \\
&= \mathrm{Tr}(\begin{pmatrix} r_1a_1 + r_2a_2 & r_1b_1 + r_2b_2 \\ r_1c_1 + r_2c_2 & r_1d_1 + r_2d_2 \end{pmatrix}) \\
&= (r_1a_1 + r_2a_2) + (r_1d_1 + r_2d_2) \\
&= r_1(a_1 + d_1) + r_2(a_2 + d_2) \\
&= r_1 \cdot \mathrm{Tr}(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}) + r_2 \cdot \mathrm{Tr}(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix})
\end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ and W is a vector space with elements $\vec{w}_1, \ldots, \vec{w}_n$ (perhaps not distinct elements) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and W is a vector space with elements $\vec{w}_1, \ldots, \vec{w}_n$ (perhaps not distinct elements) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

Proof For any input $\vec{v} \in V$ let its expression with respect to the basis be $\vec{v} = c_1 \vec{\beta}_1 + \cdots + c_n \vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v}) = c_1 \vec{w}_1 + \cdots + c_n \vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and W is a vector space with elements $\vec{w}_1, \ldots, \vec{w}_n$ (perhaps not distinct elements) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

Proof For any input $\vec{v} \in V$ let its expression with respect to the basis be $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v}) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

This map is a homomorphism because it preserves linear combinations: where $\vec{v_1} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ and $\vec{v_2} = d_1\vec{\beta}_1 + \cdots + d_n\vec{\beta}_n$, here is the calculation.

$$\begin{aligned}
h(r_1\vec{v}_1 + r_2\vec{v}_2) &= h(\,(r_1c_1 + r_2d_1)\vec{\beta}_1 + \cdots + (r_1c_n + r_2d_n)\vec{\beta}_n\,) \\
&= (r_1c_1 + r_2d_1)\vec{w}_1 + \cdots + (r_1c_n + r_2d_n)\vec{w}_n \\
&= r_1h(\vec{v}_1) + r_2h(\vec{v}_2)
\end{aligned}$$

This map is unique because if $\hat{h}\colon V \to W$ is another homomorphism satisfying that $h'(\vec{\beta}_i) = \vec{w}_i$ for each i then h and h' agree on all of the vectors in the domain.

$$\begin{aligned} h'(\vec{v}) = h'(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) &= c_1h'(\vec{\beta}_1) + \cdots + c_nh'(\vec{\beta}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = h(\vec{v}) \end{aligned}$$

They have the same action so they are the same function. QED

Example One basis of the space of quadratic polynomials $\mathcal{P}_2$ is $B = \langle x^2, x, 1 \rangle$. We can define a map $\mathrm{eval}_3 \colon \mathcal{P}_2 \to \mathbb{R}$ by specifying its action on that basis

$$x^2 \xmapsto{\mathrm{eval}_3} 9 \quad x \xmapsto{\mathrm{eval}_3} 3 \quad 1 \xmapsto{\mathrm{eval}_3} 1$$

and then extending linearly.

$$\mathrm{eval}_3(ax^2+bx+c) = a\cdot\mathrm{eval}_3(x^2)+b\cdot\mathrm{eval}_3(x)+c\cdot\mathrm{eval}_3(1) = 9a+3b+c$$

Example One basis of the space of quadratic polynomials $\mathcal{P}_2$ is $B = \langle x^2, x, 1\rangle$. We can define a map $\text{eval}_3\colon \mathcal{P}_2 \to \mathbb{R}$ by specifying its action on that basis

$$x^2 \xrightarrow{\text{eval}_3} 9 \quad x \xrightarrow{\text{eval}_3} 3 \quad 1 \xrightarrow{\text{eval}_3} 1$$

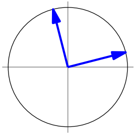
and then extending linearly.

$$\text{eval}_3(ax^2+bx+c) = a\cdot\text{eval}_3(x^2)+b\cdot\text{eval}_3(x)+c\cdot\text{eval}_3(1) = 9a+3b+c$$

The action of this map on the basis elements is to plug the value 3 in for x. That remains true when we extend linearly, so $\text{eval}_3(\,p(x)\,) = p(3)$.

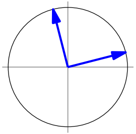
Example Consider the standard basis $\mathcal{E}_2$ for the vector space $\mathbb{R}^2$. We can specify a rotation of the two basis vectors as here.

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$



Example Consider the standard basis $\mathcal{E}_2$ for the vector space $\mathbb{R}^2$. We can specify a rotation of the two basis vectors as here.

$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$



Extend that linearly to get a homomorphism $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$.

$$\begin{aligned}
t_\theta(\begin{pmatrix}x\\y\end{pmatrix}) &= t_\theta(x\cdot\begin{pmatrix}1\\0\end{pmatrix} + y\cdot\begin{pmatrix}0\\1\end{pmatrix}) \\
&= x\cdot t_\theta(\begin{pmatrix}1\\0\end{pmatrix}) + y\cdot t_\theta(\begin{pmatrix}0\\1\end{pmatrix}) \\
&= x\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + y\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \\
&= \begin{pmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{pmatrix}
\end{aligned}$$

1.12 *Definition* A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

1.12 *Definition* A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

Example For any vector space V the *identity* map $\mathrm{id}\colon V \to V$ given by $\vec{v} \mapsto \vec{v}$ is a linear transformation. The check is easy.

1.12 *Definition* A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

Example For any vector space V the *identity* map $\mathrm{id}\colon V \to V$ given by $\vec{v} \mapsto \vec{v}$ is a linear transformation. The check is easy.

Example In $\mathbb{R}^3$ the function f_{yz}

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{f_{yz}}{\longmapsto} \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

that reflects vectors over the yz-plane is a linear transformation.

$$\begin{aligned} f_{yz}(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) &= f_{yz}(\begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix}) = \begin{pmatrix} -(r_1x_1 + r_2x_2) \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix} \\ &= r_1 \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} -x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 f_{yz}(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + r_2 f_{yz}(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) \end{aligned}$$

1.17 *Lemma* For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

1.17 *Lemma* For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

Proof This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under the operations. Let $f, g \colon V \to W$ be linear. Then the operation of function addition is preserved

$$\begin{aligned}
(f+g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2) \\
&= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + c_1 g(\vec{v}_1) + c_2 g(\vec{v}_2) \\
&= c_1 \bigl(f+g\bigr)(\vec{v}_1) + c_2 \bigl(f+g\bigr)(\vec{v}_2)
\end{aligned}$$

as is the operation of scalar multiplication of a function.

$$\begin{aligned}
(r \cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)) \\
&= c_1 (r \cdot f)(\vec{v}_1) + c_2 (r \cdot f)(\vec{v}_2)
\end{aligned}$$

Hence $\mathcal{L}(V, W)$ is a subspace. QED

Range space and null space

2.1 *Lemma* Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

2.1 *Lemma* Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

Proof Let $h\colon V \to W$ be linear and let S be a subspace of the domain V. The image $h(S)$ is a subset of the codomain W, which is nonempty because S is nonempty. Thus, to show that $h(S)$ is a subspace of W we need only show that it is closed under linear combinations of two vectors. If $h(\vec{s}_1)$ and $h(\vec{s}_2)$ are members of $h(S)$ then $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$ is also a member of $h(S)$ because it is the image of $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$ from S. QED

2.1 *Lemma* Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

Proof Let $h\colon V \to W$ be linear and let S be a subspace of the domain V. The image $h(S)$ is a subset of the codomain W, which is nonempty because S is nonempty. Thus, to show that $h(S)$ is a subspace of W we need only show that it is closed under linear combinations of two vectors. If $h(\vec{s}_1)$ and $h(\vec{s}_2)$ are members of $h(S)$ then $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$ is also a member of $h(S)$ because it is the image of $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$ from S. QED

Example For any angle θ, the function $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates vectors counterclockwise by θ is a homomorphism. In the domain $\mathbb{R}^2$ each line through the origin is a subspace. The image of that line under this map is another line through the origin and thus is a subspace of the codomain $\mathbb{R}^2$.

Range space

2.2 *Definition* The *range space* of a homomorphism $\mathrm{h}\colon V \to W$ is

$$\mathscr{R}(\mathrm{h}) = \{\mathrm{h}(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $\mathrm{h}(V)$. The dimension of the range space is the map's *rank*.

Range space

2.2 *Definition* The *range space* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathscr{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any vector with first component a and second component b. Thus the rank of π is 2.

Example The derivative map $d/dx\colon \mathbb{R}^4 \to \mathbb{R}^4$ is linear. Its range is $\mathscr{R}(d/dx) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$. (Verifying that every member of that space is the derivative of a fourth degree polynomial is easy.) The rank of the derivative function is 3.

Example The derivative map $d/dx\colon \mathbb{R}^4 \to \mathbb{R}^4$ is linear. Its range is $\mathscr{R}(d/dx) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$. (Verifying that every member of that space is the derivative of a fourth degree polynomial is easy.) The rank of the derivative function is 3.

Example This map from $\mathcal{M}_{2\times 2}$ to $\mathbb{R}^2$ is linear; the check is routine.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{h}{\longmapsto} \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The rangespace is this line through the origin

$$\{\begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R}\}$$

(every member of that set is the image

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix})$$

of a 2×2 matrix). The rank of this map is 1.

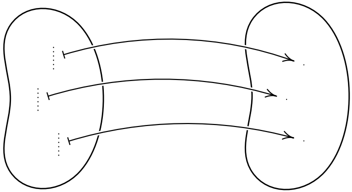
Homomorphisms organize the domain

When we moved from studying isomorphisms to studying homomorphisms we dropped the requirements that the maps be onto and one-to-one. We've seen that dropping the onto condition has no effect in the sense that any homomorphism $h\colon V \to W$ is onto some vector space, namely $\mathscr{R}(h)$.

Homomorphisms organize the domain

When we moved from studying isomorphisms to studying homomorphisms we dropped the requirements that the maps be onto and one-to-one. We've seen that dropping the onto condition has no effect in the sense that any homomorphism $\mathrm{h}\colon V \to W$ is onto some vector space, namely $\mathscr{R}(\mathrm{h})$.

We next consider the effect of dropping the one-to-one condition, so that for some vector $\vec{w} \in W$ in the range there may be many vectors $\vec{v} \in V$ mapped to $\vec{w}$.



Recall that for any function $\mathrm{h}\colon V \to W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $\mathrm{h}^{-1}(\vec{w}) = \{\vec{v} \in V \mid \mathrm{h}(\vec{v}) = \vec{w}\}$. Above, the left side shows three inverse image sets.

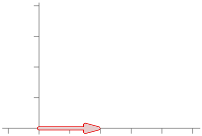
Example The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ is linear.

$$\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = x$$

Example The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ is linear.

$$\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = x$$

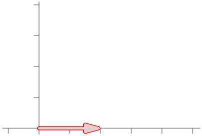
We can identify the codomain $\mathbb{R}$ with the x-axis in $\mathbb{R}^2$. Here is a member of the x-axis, drawn in red.



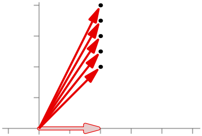
Example The projection map $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ is linear.

$$\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = x$$

We can identify the codomain $\mathbb{R}$ with the x-axis in $\mathbb{R}^2$. Here is a member of the x-axis, drawn in red.

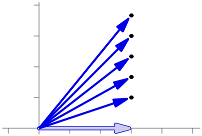


Next are some elements of $\pi^{-1}(2)$, shown both as dots as in the bean diagram and as vectors (these are also in red because they are associated by π with 2).

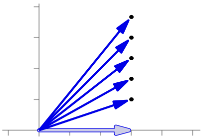


As an alternative to using colors we can refer to these as "2 vectors."

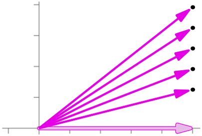
These are some “3-vectors,” inverse images of 3.



These are some “3-vectors,” inverse images of 3.

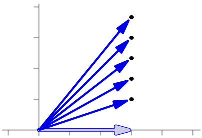


The definition of addition preservation is that $\pi(\vec{u}+\vec{v}) = \pi(\vec{u})+\pi(\vec{v})$. Therefore where $\pi(\vec{u}) = 2$ and $\pi(\vec{v}) = 3$, the vector sum $\vec{u}+\vec{v}$ will be mapped by π to 5.

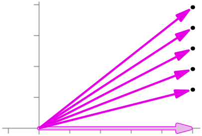


Thus, we can understand the definition of addition preservation as: red plus blue makes purple — a “2 vector” plus a “3 vector” sums to a “5 vector.”

These are some “3-vectors,” inverse images of 3.



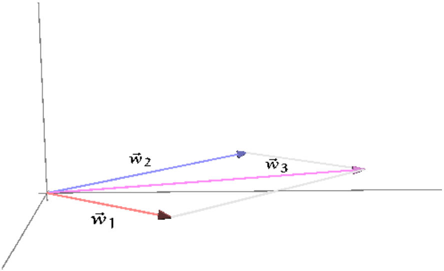
The definition of addition preservation is that $\pi(\vec{u}+\vec{v})=\pi(\vec{u})+\pi(\vec{v})$. Therefore where $\pi(\vec{u})=2$ and $\pi(\vec{v})=3$, the vector sum $\vec{u}+\vec{v}$ will be mapped by π to 5.



Thus, we can understand the definition of addition preservation as: red plus blue makes purple — a “2 vector” plus a “3 vector” sums to a “5 vector.” Preservation of scalar multiplication has a similar interpretation.

The same analysis holds for any homomorphism.

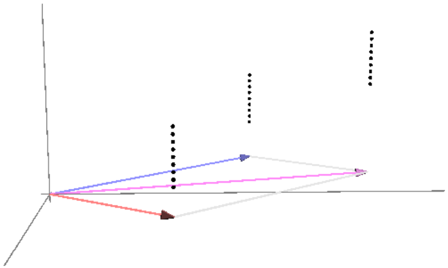
Example The projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map. As above we can identify the codomain with the xy-plane inside of $\mathbb{R}^3$.



In the xy-plane, red plus blue makes purple as shown by the parallelogram.

The same analysis holds for any homomorphism.

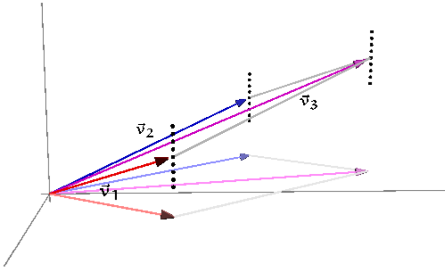
Example The projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map. As above we can identify the codomain with the xy-plane inside of $\mathbb{R}^3$.



In the xy-plane, red plus blue makes purple as shown by the parallelogram. Consider the inverse image sets; the diagram shows some of the infinitely many points in each $\pi^{-1}(\vec{w}_i)$.

The same analysis holds for any homomorphism.

Example The projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map. As above we can identify the codomain with the xy-plane inside of $\mathbb{R}^3$.



In the xy-plane, red plus blue makes purple as shown by the parallelogram. Consider the inverse image sets; the diagram shows some of the infinitely many points in each $\pi^{-1}(\vec{w}_i)$. If we take a $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$ and a $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$ then they sum to a $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$.

This also holds when the spaces are not ones that we can conveniently draw.

Example Consider $h\colon \mathcal{P}_2 \to \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This also holds when the spaces are not ones that we can conveniently draw.

Example Consider $h\colon \mathcal{P}_2 \to \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \; \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Think of these as "$\vec{w}_1$ vectors." Some examples are $3x^2 + x + 1$, $3x^2 + x - 4$, and $-2x^2 + x$.

This also holds when the spaces are not ones that we can conveniently draw.

Example Consider $h\colon \mathcal{P}_2 \to \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Think of these as "$\vec{w}_1$ vectors." Some examples are $3x^2 + x + 1$, $3x^2 + x - 4$, and $-2x^2 + x$. The inverse image of $\vec{w}_2$ is $h^{-1}(\vec{w}_2) = \{a_2x^2 - x + c_2 \mid a_2, c_2 \in \mathbb{R}^2\}$; these are "$\vec{w}_2$ vectors." The "$\vec{w}_3$ vectors" are members of the set $h^{-1}(\vec{w}_3) = \{a_3x^2 + c_3 \mid a_3, c_3 \in \mathbb{R}^2\}$.

This also holds when the spaces are not ones that we can conveniently draw.

Example Consider $h\colon \mathcal{P}_2 \to \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ b \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},\ \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The inverse image of $\vec{w}_1$ is $h^{-1}(\vec{w}_1) = \{a_1x^2 + x + c_1 \mid a_1, c_1 \in \mathbb{R}^2\}$. Think of these as "$\vec{w}_1$ vectors." Some examples are $3x^2 + x + 1$, $3x^2 + x - 4$, and $-2x^2 + x$. The inverse image of $\vec{w}_2$ is $h^{-1}(\vec{w}_2) = \{a_2x^2 - x + c_2 \mid a_2, c_2 \in \mathbb{R}^2\}$; these are "$\vec{w}_2$ vectors." The "$\vec{w}_3$ vectors" are members of the set $h^{-1}(\vec{w}_3) = \{a_3x^2 + c_3 \mid a_3, c_3 \in \mathbb{R}^2\}$.

As above, any $\vec{v}_1 \in h^{-1}(\vec{w}_1)$ plus any $\vec{v}_2 \in h^{-1}(\vec{w}_2)$ equals a $\vec{v}_3 \in h^{-1}(\vec{w}_3)$: a quadratic with an x coefficient of 1 plus a quadratic with an x coefficent of -1 equals a quadratic with an x coefficient of 0. That is, a "$\vec{w}_1$ vector" plus a "$\vec{w}_2$ vector" is a "$\vec{w}_3$ vector."

In each of those examples, because there is a homomorphism $h\colon V \to W$ we can view the domain V as organized into the inverse images $h^{-1}(\vec{w})$ for each $\vec{w} \in \mathscr{R}(h)$.

It is "organized" because these inverse image sets reflect the structure of the range in that a "$\vec{w}_1$ vector" plus a "$\vec{w}_2$ vector" equals a "$\vec{w}_1 + \vec{w}_2$ vector."

In each of those examples, because there is a homomorphism $h\colon V \to W$ we can view the domain V as organized into the inverse images $h^{-1}(\vec{w})$ for each $\vec{w} \in \mathscr{R}(h)$.

It is "organized" because these inverse image sets reflect the structure of the range in that a "$\vec{w}_1$ vector" plus a "$\vec{w}_2$ vector" equals a "$\vec{w}_1 + \vec{w}_2$ vector."

Vector spaces have a distinguished element, namely $\vec{0}$. So we next consider the inverse image of that element $h^{-1}(\vec{0})$.

2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

Proof Let $h\colon V \to W$ be a homomorphism and let S be a subspace of the range space of h. Consider the inverse image of S. It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$ and $\vec{0}_W$ is an element of S as S is a subspace. To finish we show that $h^{-1}(S)$ is closed under linear combinations. Let $\vec{v}_1$ and $\vec{v}_2$ be two of its elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S. Then $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of the inverse image $h^{-1}(S)$ because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$ is a member of S. QED

2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

Proof Let $h\colon V \to W$ be a homomorphism and let S be a subspace of the range space of h. Consider the inverse image of S. It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$ and $\vec{0}_W$ is an element of S as S is a subspace. To finish we show that $h^{-1}(S)$ is closed under linear combinations. Let $\vec{v}_1$ and $\vec{v}_2$ be two of its elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S. Then $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of the inverse image $h^{-1}(S)$ because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$ is a member of S. QED

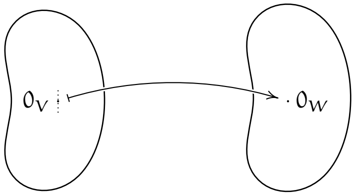
Note This result complements Lemma 2.1 .

Null space

2.11 *Definition* The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.

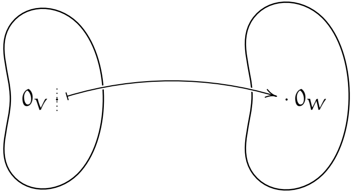


Null space

2.11 *Definition* The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.



Note Strictly, the nullspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$. Thus we should perhaps write the nullspace as $h^{-1}(\{\vec{0}_W\})$.

But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and writing it the first way is easier.

Example The derivative function $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ is linear.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

The polynomial $2ax + b$ equals the zero polynomial if an only if they have the same constant coefficient (which implies that $b = 0$), the same coefficient of x (which implies that $a = 0$), and the same coefficient of x^2 (which gives no restriction). Thus the nullspace is this, and the nullity is 1.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid a = 0,\ b = 0,\ c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

Example The derivative function $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ is linear.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

The polynomial $2ax + b$ equals the zero polynomial if an only if they have the same constant coefficient (which implies that $b = 0$), the same coefficient of x (which implies that $a = 0$), and the same coefficient of x^2 (which gives no restriction). Thus the nullspace is this, and the nullity is 1.

$$\mathscr{N}(d/dx) = \{ax^2 + bx + c \mid a = 0,\ b = 0,\ c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

Example The function $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this null space.

$$\mathscr{N}(h) = \{\begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0\} = \{\begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}$$

Its nullity is 1.

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has this null space

$$\mathscr{N}(f) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=0 \text{ and } c+d=0 \} = \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \}$$

and a nullity of 2.

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{f} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has this null space

$$\mathscr{N}(f) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+b=0 \text{ and } c+d=0 \} = \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \}$$

and a nullity of 2.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has a trivial null space $\mathscr{N}(d_3) = \{\vec{0}\}$ and its nullity is 0.

Rank plus nullity

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

Rank plus nullity

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

Proof Let $h\colon V \to W$ be linear and let $B_N = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ be a basis for the null space. Expand that to a basis $B_V = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$ for the entire domain, using Corollary Two.III.2.13 . We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for the range space. Then counting the size of the bases gives the result.

Rank plus nullity

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

Proof Let $h\colon V \to W$ be linear and let $B_N = \langle \vec{\beta}_1, \dots, \vec{\beta}_k \rangle$ be a basis for the null space. Expand that to a basis $B_V = \langle \vec{\beta}_1, \dots, \vec{\beta}_k, \vec{\beta}_{k+1}, \dots, \vec{\beta}_n \rangle$ for the entire domain, using Corollary Two.III.2.13 . We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \dots, h(\vec{\beta}_n) \rangle$ is a basis for the range space. Then counting the size of the bases gives the result.

To see that B_R is linearly independent, consider $\vec{0}_W = c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. We have $\vec{0}_W = h(c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n)$ and so $c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$ is in the null space of h. As B_N is a basis for the null space there are scalars $c_1, \dots, c_k$ satisfying this relationship.

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

But this is an equation among members of B_V, which is a basis for V, so each c_i equals 0. Therefore B_R is linearly independent.

To show that B_R spans the range space consider a member of the range space $h(\vec{v})$. Express $\vec{v}$ as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V. This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h(\vec{\beta}_1) + \cdots + c_kh(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$ and since $\vec{\beta}_1, \ldots, \vec{\beta}_k$ are in the null space, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R, and so B_R spans the range space. QED

To show that B_R spans the range space consider a member of the range space $h(\vec{v})$. Express $\vec{v}$ as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V. This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h(\vec{\beta}_1) + \cdots + c_kh(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$ and since $\vec{\beta}_1, \ldots, \vec{\beta}_k$ are in the null space, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R, and so B_R spans the range space. QED

Example The derivative function $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_1$ has this range space

$$\mathscr{R}(d/dx) = \{2ax + b \mid a, b \in \mathbb{R}\} = \mathcal{P}_1$$

(any $cx + d \in \mathcal{P}_1$ is the image of $ax^2 + bx + c$ where $a = c/2$, $b = d$, and c can be any real) and this null space (calculated above).

$$\mathscr{N}(d/dx) = \{c \mid c \in \mathbb{R}\}$$

The rank is 2 while the nullity is 1, and they add to the dimension of the domain $\mathcal{P}_2$.

Example The function $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this range space

$$\mathscr{R}(h) = \{2a + b \mid a, b \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

and this null space (calculated earlier).

$$\mathscr{N}(h) = \{\begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}$$

Its rank is 1 and its nullity is 1. Its domain $\mathbb{R}^2$ has dimension 2.

Example The function $h\colon \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this range space

$$\mathscr{R}(h) = \{2a + b \mid a, b \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

and this null space (calculated earlier).

$$\mathscr{N}(h) = \{\begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}$$

Its rank is 1 and its nullity is 1. Its domain $\mathbb{R}^2$ has dimension 2.

Example The dilation function $d_3\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has range space $\mathbb{R}^2$ and a trivial nullspace $\mathscr{N}(d_3) = \{\vec{0}\}$. So its rank is 2 and its nullity is 0.

Example The homomorphism $f\colon \mathcal{M}_{2\times 2} \to \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

has range space equal to $\mathbb{R}^2$ (to get a vector with a first component of x and a second component of y we can take $a = x$, $b = 0$, $c = y$, and $d = 0$). Thus f's rank is 2. We found its null space earlier

$$\mathscr{N}(f) = \{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \}$$

and its nullity is 2.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a three-dimensional domain to a 2-dimensional range, with a null space of the z-axis and so a nullity of 1.

We can step through the proof by taking the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a three-dimensional domain to a 2-dimensional range, with a null space of the z-axis and so a nullity of 1.

We can step through the proof by taking the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $\mathcal{E}_3$ for the entire domain.

Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a three-dimensional domain to a 2-dimensional range, with a null space of the z-axis and so a nullity of 1.

We can step through the proof by taking the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $\mathcal{E}_3$ for the entire domain. So the action of the map is that the third dimension collapses: a linear combinations in the domain $\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$ is sent to the combination $\vec{w} = c_1\vec{e}_1 + c_2\vec{e}_2 + \vec{0}$ in the range.

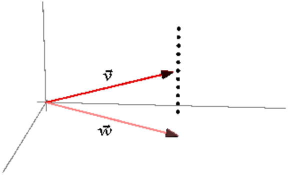
Example Projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

takes a three-dimensional domain to a 2-dimensional range, with a null space of the z-axis and so a nullity of 1.

We can step through the proof by taking the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis $\mathcal{E}_3$ for the entire domain. So the action of the map is that the third dimension collapses: a linear combinations in the domain $\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3$ is sent to the combination $\vec{w} = c_1\vec{e}_1 + c_2\vec{e}_2 + \vec{0}$ in the range.

Geometrically, all of the inverse images are vertical lines, just like the null space. The action of π is to zero them out.



2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

Example The trace function $\mathrm{Tr}\colon \mathcal{M}_{2\times2} \to \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

is linear. This set of matrices is dependent.

$$S = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\}$$

The three matrices map to 1, 0, and 2 respectively. The set $\{1, 0, 2\}$ is linearly dependent in $\mathbb{R}$.

2.20 *Theorem* In an n-dimensional vector space V, these are equivalent statements about a linear map $h\colon V \to W$.

(1) h is one-to-one

(2) h has an inverse from its range to its domain that is a linear map

(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$

(4) $\text{rank}(h) = n$

(5) if $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$

2.20 *Theorem* In an n-dimensional vector space V, these are equivalent statements about a linear map $h\colon V \to W$.

1. (1) h is one-to-one
2. (2) h has an inverse from its range to its domain that is a linear map
3. (3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$
4. (4) $\text{rank}(h) = n$
5. (5) if $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$ is a basis for $\mathscr{R}(h)$

Proof We will first show that (1) $\Longleftrightarrow$ (2). We will then show that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (2).

For $(1) \implies (2)$, suppose that the linear map h is one-to-one, and therefore has an inverse $h^{-1}\colon \mathscr{R}(h) \to V$. The domain of that inverse is the range of h and thus a linear combination of two members of it has the form $c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$. On that combination, the inverse h^{-1} gives this.

$$\begin{aligned}
h^{-1}(c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)) &= h^{-1}(h(c_1\vec{v}_1 + c_2\vec{v}_2)) \\
&= h^{-1} \circ h \,(c_1\vec{v}_1 + c_2\vec{v}_2) \\
&= c_1\vec{v}_1 + c_2\vec{v}_2 \\
&= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2))
\end{aligned}$$

Thus if a linear map has an inverse then the inverse must be linear. But this also gives the $(2) \implies (1)$ implication, because the inverse itself must be one-to-one.

For (1) $\implies$ (2), suppose that the linear map h is one-to-one, and therefore has an inverse $h^{-1}\colon \mathscr{R}(h) \to V$. The domain of that inverse is the range of h and thus a linear combination of two members of it has the form $c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$. On that combination, the inverse h^{-1} gives this.

$$\begin{aligned}
h^{-1}(c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)) &= h^{-1}(h(c_1\vec{v}_1 + c_2\vec{v}_2)) \\
&= h^{-1} \circ h\ (c_1\vec{v}_1 + c_2\vec{v}_2) \\
&= c_1\vec{v}_1 + c_2\vec{v}_2 \\
&= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2))
\end{aligned}$$

Thus if a linear map has an inverse then the inverse must be linear. But this also gives the (2) $\implies$ (1) implication, because the inverse itself must be one-to-one.

Of the remaining implications, (1) $\implies$ (3) holds because any homomorphism maps $\vec{0}_V$ to $\vec{0}_W$, but a one-to-one map sends at most one member of V to $\vec{0}_W$.

Next, (3) $\implies$ (4) is true since rank plus nullity equals the dimension of the domain.

For (4) $\Longrightarrow$ (5), to show that $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$ is a basis for the range space we need only show that it is a spanning set, because by assumption the range has dimension n. Consider $h(\vec{v}) \in \mathscr{R}(h)$. Expressing $\vec{v}$ as a linear combination of basis elements produces $h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n)$, which gives that $h(\vec{v}) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n)$, as desired. QED