Two.I Definition of Vector Space

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition and examples

Vector space

1.1 *Definition* A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$+$' and '$\cdot$' subject to these conditions.

Vector space

1.1 *Definition* A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '$\cdot$' subject to these conditions.

Where $\vec{v}, \vec{w} \in V$, (1) their *vector sum* $\vec{v} + \vec{w}$ is an element of V. If $\vec{u}, \vec{v}, \vec{w} \in V$ then (2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ and (3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$. (4) There is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$. (5) Each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$.

Vector space

1.1 *Definition* A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '+' and '$\cdot$' subject to these conditions.

Where $\vec{v}, \vec{w} \in V$, (1) their *vector sum* $\vec{v} + \vec{w}$ is an element of V. If $\vec{u}, \vec{v}, \vec{w} \in V$ then (2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ and (3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$. (4) There is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$. (5) Each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$.

If r, s are *scalars*, members of $\mathbb{R}$, and $\vec{v}, \vec{w} \in V$ then (6) each *scalar multiple* $r \cdot \vec{v}$ is in V. If $r, s \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$ then (7) $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$, and (8) $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$, and (9) $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$, and (10) $1 \cdot \vec{v} = \vec{v}$.

Example Consider this reduced echelon form matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

In any matrix row-equivalent to A each row must be a multiple of the vector $\begin{pmatrix} 1 & 2 \end{pmatrix}$. We will verify that this set of row vectors

$$V = \{ \begin{pmatrix} a & 2a \end{pmatrix} \bigm| a \in \mathbb{R} \}$$

is a vector space under the natural operations of addition

$$\begin{pmatrix} a_1 & 2a_1 \end{pmatrix} + \begin{pmatrix} a_2 & 2a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 2a_1 + a_2 \end{pmatrix}$$

and scalar multiplication.

$$r\begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} ra_1 & ra_2 \end{pmatrix}$$

For that we will check each of the conditions. (This first time through, we verify these at length.)

We first check *closure under addition* (1), that the sum of two members of V is also a member of V. Take $\vec{v}$ and $\vec{w}$ to be members of V so that

$$\vec{v} = \begin{pmatrix} v_1 & 2v_1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 & 2w_1 \end{pmatrix}$$

and note that their sum

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 & 2v_1 + 2w_1 \end{pmatrix}$$

is also a member of V, because its second entry is twice its first.

We first check *closure under addition* (1), that the sum of two members of V is also a member of V. Take $\vec{v}$ and $\vec{w}$ to be members of V so that

$$\vec{v} = \begin{pmatrix} v_1 & 2v_1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 & 2w_1 \end{pmatrix}$$

and note that their sum

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 & 2v_1 + 2w_1 \end{pmatrix}$$

is also a member of V, because its second entry is twice its first.

Condition (2), commutativity of addition, is easy. The first sum is

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 & 2(v_1 + w_1) \end{pmatrix}$$

and the second sum is

$$\vec{w} + \vec{v} = \begin{pmatrix} w_1 + v_1 & 2(w_1 + v_1) \end{pmatrix}$$

and the two are equal because the sum of real numbers $v_1 + w_1$ equals the sum of real numbers $w_1 + v_1$.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v}+\vec{w})+\vec{u}=((v_1+w_1)+u_1 \quad (2v_1+2w_1)+2u_1)$$

while the right side is this.

$$\vec{v}+(\vec{w}+\vec{u})=(v_1+(w_1+u_1) \quad 2v_1+(2w_1+2u_1))$$

The two are equal because real number addition is associative $(v_1+w_1)+u_1=v_1+(w_1+u_1)$.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v}+\vec{w})+\vec{u} = ((v_1+w_1)+u_1 \quad (2v_1+2w_1)+2u_1)$$

while the right side is this.

$$\vec{v}+(\vec{w}+\vec{u}) = (v_1+(w_1+u_1) \quad 2v_1+(2w_1+2u_1))$$

The two are equal because real number addition is associative $(v_1+w_1)+u_1 = v_1+(w_1+u_1)$.

For condition (4) we just produce the member of V with the desired property. So consider the vector of 0's. Note that it is a member of V since its second component is twice its first, and note that it is the *identity element* with respect to addition.

$$\begin{aligned}\vec{v}+\vec{0} &= (v_1 \quad 2v_1)+(0 \quad 0)\\ &= (v_1 \quad 2v_1)\\ &= \vec{v}\end{aligned}$$

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \quad 2v_1)$ of V, consider $\vec{w} = (w_1 \quad 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels $\vec{v}$.

$$\vec{w} + \vec{v} = (-v_1 \quad -2v_1) + (v_1 \quad 2v_1) = \vec{0}$$

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \quad 2v_1)$ of V, consider $\vec{w} = (w_1 \quad 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels $\vec{v}$.

$$\vec{w} + \vec{v} = (-v_1 \quad -2v_1) + (v_1 \quad 2v_1) = \vec{0}$$

Next we verify the conditions for scalar multiplication. First, condition (6) is *closure under scalar multiplication*. We consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} = (v_1 \quad 2v_2) \in V$. The scalar multiple $r\vec{v} = (rv_1 \quad r2v_1)$ is also a member of V because the second component is twice the first.

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \quad 2v_1)$ of V, consider $\vec{w} = (w_1 \quad 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels $\vec{v}$.

$$\vec{w} + \vec{v} = (-v_1 \quad -2v_1) + (v_1 \quad 2v_1) = \vec{0}$$

Next we verify the conditions for scalar multiplication. First, condition (6) is *closure under scalar multiplication*. We consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} = (v_1 \quad 2v_2) \in V$. The scalar multiple $r\vec{v} = (rv_1 \quad r2v_1)$ is also a member of V because the second component is twice the first.

Condition (7) is that *real number addition distributes over scalar multiplication*. Let the scalars be $r, s \in \mathbb{R}$ and let the vector be $\vec{v} = (v_1 \quad 2v_1) \in V$. Then $(r+s)\vec{v} = ((r+s)v_1 \quad (r+2)2v_1)$, which equals $(rv_1 \quad 2rv_1) + (sv_1 \quad 2sv_1) = r\vec{v} + s\vec{v}$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v} + \vec{w}) = (rv_1 \quad 2rv_1) + (rw_1 \quad 2rw_1)$, which equals $(rv_1 + rw_1 \quad 2rv_1 + 2rw_1)$, which equals $r(v_1 \quad 2v_1) + r(w_1 \quad 2w_1) = r\vec{v} + r\vec{w}$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v} + \vec{w}) = (rv_1 \quad 2rv_1) + (rw_1 \quad 2rw_1)$, which equals $(rv_1 + rw_1 \quad 2rv_1 + 2rw_1)$, which equals $r(v_1 \quad 2v_1) + r(w_1 \quad 2w_1) = r\vec{v} + r\vec{w}$.

For condition (9) take $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \quad 2v_1) \in V$. The left side is $(rs)(v_1 \quad 2v_1) = ((rs)v_1 \quad (rs)2v_1)$, while the right side is $r(s(v_1 \quad 2v_1)) = r(sv_1 \quad s2v_1) = (r(sv_1) \quad r(s2v_1))$. The two are equal because, as they are real number multiplications, $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v} + \vec{w}) = (rv_1 \quad 2rv_1) + (rw_1 \quad 2rw_1)$, which equals $(rv_1 + rw_1 \quad 2rv_1 + 2rw_1)$, which equals $r(v_1 \quad 2v_1) + r(w_1 \quad 2w_1) = r\vec{v} + r\vec{w}$.

For condition (9) take $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \quad 2v_1) \in V$. The left side is $(rs)(v_1 \quad 2v_1) = ((rs)v_1 \quad (rs)2v_1)$, while the right side is $r(s(v_1 \quad 2v_1)) = r(sv_1 \quad s2v_1) = (r(sv_1) \quad r(s2v_1))$. The two are equal because, as they are real number multiplications, $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$.

The final condition is straightforward: for any $\vec{v} \in V$ we have $1\vec{v} = 1(v_1 \quad 2v_1) = (1 \cdot v_1 \quad 1 \cdot 2v_1) = \vec{v}$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v}+\vec{w}) = (rv_1 \quad 2rv_1) + (rw_1 \quad 2rw_1)$, which equals $(rv_1 + rw_1 \quad 2rv_1 + 2rw_1)$, which equals $r(v_1 \quad 2v_1) + r(w_1 \quad 2w_1) = r\vec{v} + r\vec{w}$.

For condition (9) take $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \quad 2v_1) \in V$. The left side is $(rs)(v_1 \quad 2v_1) = ((rs)v_1 \quad (rs)2v_1)$, while the right side is $r(s(v_1 \quad 2v_1)) = r(sv_1 \quad s2v_1) = (r(sv_1) \quad r(s2v_1))$. The two are equal because, as they are real number multiplications, $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$.

The final condition is straightforward: for any $\vec{v} \in V$ we have $1\vec{v} = 1(v_1 \quad 2v_1) = (1 \cdot v_1 \quad 1 \cdot 2v_1) = \vec{v}$.

Thus the set $V = \{(a \quad 2a) \bigm| a \in \mathbb{R}\}$ is a vector space under the natural addition and scalar multiplication operations.

Example The set $\mathbb{R}^3$ is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that we will check the conditions (more briefly than for the prior example).

Example The set $\mathbb{R}^3$ is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that we will check the conditions (more briefly than for the prior example).

Condition (1) is closure under addition. This is clear because the only condition for membership in the set $\mathbb{R}^3$ is to be a three-tall vector of reals, and the sum of two three-tall vectors of reals is also a three-tall vector of reals.

Example The set $\mathbb{R}^3$ is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that we will check the conditions (more briefly than for the prior example).

Condition (1) is closure under addition. This is clear because the only condition for membership in the set $\mathbb{R}^3$ is to be a three-tall vector of reals, and the sum of two three-tall vectors of reals is also a three-tall vector of reals.

Condition (2) is routine

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{w} + \vec{v}$$

Condition (3) is also a direct consequence of the related real number property.

$$(\vec{v}+\vec{w})+\vec{u} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1+w_1+u_1 \\ v_2+w_2+u_2 \\ v_3+w_3+u_3 \end{pmatrix}$$
$$= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1+u_1 \\ w_2+u_2 \\ w_3+u_3 \end{pmatrix} = \vec{v}+(\vec{w}+\vec{u})$$

Condition (3) is also a direct consequence of the related real number property.

$$(\vec{v}+\vec{w})+\vec{u} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1+w_1+u_1 \\ v_2+w_2+u_2 \\ v_3+w_3+u_3 \end{pmatrix}$$

$$= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1+u_1 \\ w_2+u_2 \\ w_3+u_3 \end{pmatrix} = \vec{v}+(\vec{w}+\vec{u})$$

For condition (4) take the vector of 0's.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

For condition (5), given $\vec{v} \in \mathbb{R}^3$, use $\vec{w} = -1\vec{v}$ as the additive inverse.

$$\begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Condition (6) is closure under scalar multiplication. Let the scalar be $r \in \mathbb{R}$ and the vector be $\vec{v} \in \mathbb{R}^3$. Then $r\vec{v}$ is a three-tall vector of reals, so $r\vec{v} \in \mathbb{R}^3$.

Condition (6) is closure under scalar multiplication. Let the scalar be $r \in \mathbb{R}$ and the vector be $\vec{v} \in \mathbb{R}^3$. Then $r\vec{v}$ is a three-tall vector of reals, so $r\vec{v} \in \mathbb{R}^3$.

Conditions (7)

$$(r+s)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \\ (r+s)v_3 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \\ rv_3 + sv_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix} + \begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}+s\vec{v}$$

and (8)

$$r(\vec{v}+\vec{w}) = r(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}) = r\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \\ rv_3 + rw_3 \end{pmatrix} = r\vec{v}+r\vec{w}$$

are straightforward.

Condition (9) is similar.

$$(rs)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \\ (rs)v_3 \end{pmatrix} = r\begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}(s\vec{v})$$

And (10) is also easy.

$$1\vec{v} = 1\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ 1 \cdot v_2 \\ 1 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$

Condition (9) is similar.

$$(rs)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \\ (rs)v_3 \end{pmatrix} = r\begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}(s\vec{v})$$

And (10) is also easy.

$$1\vec{v} = 1\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ 1 \cdot v_2 \\ 1 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$

So the set $\mathbb{R}^3$ is a vector space under the usual operations of vector addition and scalar-vector multiplication.

Example The set $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

Example The set $\mathcal{P}_2 = \{ a_0 + a_1x + a_2x^2 \bigm| a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$$

and scalar multiplication.

$$r\cdot(a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

Example The set of 3×3 matrices

$$\mathcal{M}_{3\times 3} = \{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \bigm| a_{i,j} \in \mathbb{R}\}$$

is a vector space under the usual matrix addition and scalar multiplication.

The empty set cannot be made a vector space, regardless of which operations we use, because it has no element to be the zero element.

The empty set cannot be made a vector space, regardless of which operations we use, because it has no element to be the zero element.

Example The set consisting only of the two-tall vector of 0's

$$V = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is a vector space (under the usual vector addition and scalar multiplication operations).

1.7 *Definition* A one-element vector space is a *trivial* space.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1), note that $\vec{v} = (1 + 0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1), note that $\vec{v} = (1 + 0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1 + 1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$ shows that we can write '$-\vec{v}$' for the additive inverse of $\vec{v}$ without worrying about possible confusion with $(-1) \cdot \vec{v}$.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

Proof For (1), note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1 + 1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$ shows that we can write '$-\vec{v}$' for the additive inverse of $\vec{v}$ without worrying about possible confusion with $(-1) \cdot \vec{v}$.

For (3) $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do. QED

Subspaces and spanning sets

Subspace

2.1 *Definition* For any vector space, a ***subspace*** is a subset that is itself a vector space, under the inherited operations.

Subspace

2.1 *Definition* For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

Subspace

2.1 *Definition* For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

Example In the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{\begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R}\}$$

is a subspace. The operations, as required by the definition, are the ones from $\mathbb{R}^2$. We can check all the conditions to show it is a vector space, but the next result gives an easier way.

Subspace

2.1 *Definition* For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

Any vector space has a trivial subspace $\{\vec{0}\,\}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

Example In the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{\begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R}\}$$

is a subspace. The operations, as required by the definition, are the ones from $\mathbb{R}^2$. We can check all the conditions to show it is a vector space, but the next result gives an easier way.

Example This subset of $\mathcal{M}_{2\times 2}$ is a subspace.

$$S = \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R}\} = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} b \mid a, b \in \mathbb{R}\}$$

As above, addition and scalar multiplication are the same as in $\mathcal{M}_{2\times 2}$.

Example This is not a subspace of $\mathbb{R}^3$.

$$T = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$$

It is a subset of $\mathbb{R}^3$ but it is not a vector space. One condition that it violates is that it is not closed under vector addition: here are two elements of T that sum to a vector that is not an element of T.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(Another reason that it is not a vector space is that it does not contain the zero vector.)

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements.

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is in S.

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements.

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is in S.

'The following are equivalent' means that each pair of statements are equivalent.

$$(1) \iff (2) \qquad (2) \iff (3) \qquad (3) \iff (1)$$

We will prove the equivalence by establishing that $(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by the observation that $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue the single implication $(2) \implies (1)$.

2.9 *Proof* Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The first item in the vector space definition has five conditions. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V, and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that (where $\vec{s}$ is any member of the nonempty set S) $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is in S; showing that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S; showing that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

2.9 *Proof* Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The first item in the vector space definition has five conditions. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V, and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that (where $\vec{s}$ is any member of the nonempty set S) $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is in S; showing that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S; showing that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

The conditions for scalar multiplication are similar. QED

Example The vector space of quadratic polynomials $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \bigm| a_0, a_1, a_2 \in \mathbb{R}\}$ has a subspace comprised of the linear polynomials $L = \{b_0 + b_1x \bigm| b_0, b_1 \in \mathbb{R}\}$. To verify that, take scalars $r, s \in \mathbb{R}$ and consider a linear combination.

$$r(b_0 + b_1x) + s(c_0 + c_1x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

The right side is a linear polynomial with real coefficients, and so is a member of L. Thus L is closed under linear combinations.

Example The vector space of quadratic polynomials $\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ has a subspace comprised of the linear polynomials $L = \{b_0 + b_1 x \mid b_0, b_1 \in \mathbb{R}\}$. To verify that, take scalars $r, s \in \mathbb{R}$ and consider a linear combination.

$$r(b_0 + b_1 x) + s(c_0 + c_1 x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

The right side is a linear polynomial with real coefficients, and so is a member of L. Thus L is closed under linear combinations.

Example Another subspace of $\mathcal{P}_2$ is the set of quadratic polynomials with all three coefficients equal.

$$M = \{a + ax + ax^2 \mid a \in \mathbb{R}\} = \{(1 + x + x^2)a \mid a \in \mathbb{R}\}$$

Verify that it is a subspace by taking two scalars $r, s \in \mathbb{R}$ and considering a linear combination of polynomials with all three coefficients the same.

$$r(a+ax+ax^2)+s(b+bx+bx^2) = (ra+sb)+(ra+sb)x+(ra+sb)x^2$$

The result is a quadratic polynomial with all three coefficents the same, and so M is closed under linear combinations.

The above examples of subspace paramatrize the description.

Example This set is a plane inside of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0 \}$$

We could verify that it is a subspace by checking that it is closed under linear combination as above.

The above examples of subspace paramatrize the description.

Example This set is a plane inside of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0 \}$$

We could verify that it is a subspace by checking that it is closed under linear combination as above.

That's easier if we first paramatrize the one-equation linear system $2x - y + z = 0$ using the free variables y and z.

$$P = \{ \begin{pmatrix} (1/2)y - (1/2)z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

The above examples of subspace paramatrize the description.

Example This set is a plane inside of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0 \}$$

We could verify that it is a subspace by checking that it is closed under linear combination as above.

That's easier if we first paramatrize the one-equation linear system $2x - y + z = 0$ using the free variables y and z.

$$P = \{ \begin{pmatrix} (1/2)y - (1/2)z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

Now members of P are described as a linear combination of those two vectors. Verifying that P is closed then involves taking a linear combination of linear combinations, which makes a linear combination.

Span

2.13 *Definition* The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are ‘span(S)’ and ‘sp(S)’.

Span

2.13 *Definition* The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

Example Inside the vector space of all two-wide row vectors, the span of this one-element set

$$S = \{\begin{pmatrix}1 & 2\end{pmatrix}\}$$

is this.

$$[S] = \{\begin{pmatrix}a & 2a\end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix}1 & 2\end{pmatrix}a \mid a \in \mathbb{R}\}$$

Example This is a subset of $\mathbb{R}^3$.

$$S = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Any vector in the xy-plane is a member of the span $[S]$; for instance, this system has a solution.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

Example This is a subset of $\mathbb{R}^3$.

$$S = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Any vector in the xy-plane is a member of the span $[S]$; for instance, this system has a solution.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

But vectors not in the xy-plane are not in the span; for instance, this system does not have a solution.

$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

(just consider the third components).

2.15 *Lemma* In a vector space, the span of any subset is a subspace.

2.15 *Lemma* In a vector space, the span of any subset is a subspace.

Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

(p, r scalars) is a linear combination of elements of S and so is in $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$, but it does not matter). QED