Four.I Determinants; Definition

Linear Algebra

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Properties of Determinants

Nonsingular matrices

An $n \times n$ matrix T is nonsingular if and only if each of these holds:

- any system $T\vec{x} = \vec{b}$ has a solution and that solution is unique;
- Gauss-Jordan reduction of T yields an identity matrix;
- the rows of T form a linearly independent set;
- the columns of T form a linearly independent set, a basis for $\mathbb{R}^n$;
- any map that T represents is an isomorphism;
- an inverse matrix T^{-1} exists.

This chapter develops a formula to determine whether a matrix is nonsingular.

Determining nonsingularity is trivial for 1×1 matrices.

$$\begin{pmatrix} a \end{pmatrix} \quad \text{is nonsingular iff} \quad a \neq 0$$

Corollary Three.IV.4.11 gives the 2×2 formula.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is nonsingular iff} \quad ad - bc \neq 0$$

We can produce the 3×3 formula as we did the prior one, although the computation is intricate (see Exercise 9).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{is nonsingular iff} \quad aei+bfg+cdh-hfa-idb-gec \neq 0$$

With these cases in mind, we posit a family of formulas: a, $ad - bc$, etc. For each n the formula defines a *determinant* function $\det_{n \times n} \colon \mathcal{M}_{n \times n} \to \mathbb{R}$ such that an $n \times n$ matrix T is nonsingular if and only if $\det_{n \times n}(T) \neq 0$.

We will define the determinant function by listing some conditions. These are convenient for computing the value of the determinant on an input square matrix. Then we will show that there is only one function satisfying those conditions.

Definition of determinant

2.1 *Definition* A $n \times n$ ***determinant*** is a function $\det\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ such that

1) $\det(\vec{\rho}_1, \ldots, k\cdot\vec{\rho}_i + \vec{\rho}_j, \ldots, \vec{\rho}_n) = \det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
2) $\det(\vec{\rho}_1, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n) = -\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_j, \ldots, \vec{\rho}_n)$ for $i \neq j$
3) $\det(\vec{\rho}_1, \ldots, k\vec{\rho}_i, \ldots, \vec{\rho}_n) = k\cdot\det(\vec{\rho}_1, \ldots, \vec{\rho}_i, \ldots, \vec{\rho}_n)$ for any scalar k
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(the $\vec{\rho}$'s are the rows of the matrix). We often write $|T|$ for $\det(T)$.

2.2 *Remark* Condition (2) is redundant since

$T \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_j+\rho_i} \xrightarrow{\rho_i+\rho_j} \xrightarrow{-\rho_i} \hat{T}$ swaps rows i and j. We have only listed it for convenience.

Consequences of the definition

2.4 *Lemma* A matrix with two identical rows has a determinant of zero. A matrix with a zero row has a determinant of zero. A matrix is nonsingular if and only if its determinant is nonzero. The determinant of an echelon form matrix is the product down its diagonal.

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Proof To verify the first sentence swap the two equal rows. The sign of the determinant changes but the matrix is the same and so its determinant is the same. Thus the determinant is zero.

For the second sentence multiply the zero row by two. That doubles the determinant but it also leaves the row unchanged, and hence leaves the determinant unchanged. Thus the determinant must be zero.

Do Gauss-Jordan reduction for the third sentence, $T \rightarrow \cdots \rightarrow \hat{T}$. By the first three properties the determinant of T is zero if and only if the determinant of $\hat{T}$ is zero (although the two could differ in sign or magnitude). A nonsingular matrix T Gauss-Jordan reduces to an identity matrix and so has a nonzero determinant. A singular T reduces to a $\hat{T}$ with a zero row; by the second sentence of this lemma its determinant is zero.

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The fourth sentence has two cases. If the echelon form matrix is singular then it has a zero row. Thus it has a zero on its diagonal and the product down its diagonal is zero. By the third sentence of this result the determinant is zero and therefore this matrix's determinant equals the product down its diagonal.

If the echelon form matrix is nonsingular then none of its diagonal entries is zero so we can use condition (3) to get 1's on the diagonal.

$$\begin{vmatrix} t_{1,1} & t_{1,2} & & t_{1,n} \\ 0 & t_{2,2} & & t_{2,n} \\ & & \ddots & \\ 0 & & & t_{n,n} \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & t_{1,2}/t_{1,1} & & t_{1,n}/t_{1,1} \\ 0 & 1 & & t_{2,n}/t_{2,2} \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix}$$

(We need that diagonal entries are nonzero to write, e.g., $t_{1,2}/t_{1,1}$.) Then the Jordan half of Gauss-Jordan elimination leaves the identity matrix.

$$= t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot \begin{vmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{vmatrix} = t_{1,1} \cdot t_{2,2} \cdots t_{n,n} \cdot 1$$

So in this case also, the determinant is the product down the diagonal. QED

We can compute the determinant of a matrix using Gauss's Method (presuming that a function satisfying the definition exists; we will prove later that it does).

Example On this matrix the Gauss's Method reduces the first column with $-2\rho_1 + \rho_2$ and $-3\rho_1 + \rho_3$. Condition (1) says that these row operations leave the determinant unchanged.

$$\begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -6 & -8 \\ 0 & -10 & -11 \end{vmatrix}$$

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Reduce the second column with $-(5/3)\rho_2+\rho_3$. Again, by condition (1) the determinant stays the same.

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By the prior lemma we can now find the determinant by taking the product down the diagonal.

$$= 1 \cdot (-6) \cdot (-7/3) = 14$$

Example To reduce this matrix we do a row swap, which changes the sign of the determinant.

$$\begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 0 \\ 1 & 5 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 5 & 2 \end{vmatrix}$$

Performing $-\rho_1 + \rho_3$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 2 \end{vmatrix}$$

and $-\rho_2 + \rho_3$

$$= -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

and then multiplying down the diagonal gives that the determinant of the original matrix is -3.

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So if there is a function mapping $\mathcal{M}_{n \times n}$ to $\mathbb{R}$ with the four conditions of the definition then there is only one such function. The next two subsections show that for each n a determinant function exists.

The Permutation Expansion

The prior subsection defines a function to be a determinant if it satisfies four conditions and shows that there is at most one $n \times n$ determinant function for each n. What is left is to show that for each n such a function exists.

But, we easily compute determinants: we use Gauss's Method, keeping track of the sign changes from row swaps, and end by multiplying down the diagonal. How could they not exist?

The difficulty is to show that the computation gives a well-defined — that is, unique — result. Consider these two Gauss's Method reductions of the same matrix, the first without any row swap

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

and the second with one.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \xrightarrow{-(1/3)\rho_1+\rho_2} \begin{pmatrix} 3 & 4 \\ 0 & 2/3 \end{pmatrix}$$

Both yield the determinant -2 since in the second one we note that the row swap changes the sign of the result we get by multiplying down the diagonal.

That the above computation gives a consistent result for these two ways to do a reduction on one matrix does not ensure that determinants always give a well-defined value. Our algorithm for computing determinant values does not plainly eliminate the possibility that there might be, say, two reductions of some 7×7 matrix that lead to different determinant outputs. In that case there would exist no determinant function, since functions must have the property that for each input there is exactly one output.

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To show that determinants are well-defined we will give an alternative way to compute the value of a determinant. This new way is less useful in practice since it makes the computations awkward and slow, which is why we didn't start with it. But it is useful for theory since it makes easier the proof that we need now.

The determinant function is not linear

Example The second matrix here is twice the first but the determinant does not double.

$$\begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} = -72 \qquad \begin{vmatrix} 6 & -6 & 18 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} = -576$$

Instead, by condition (3) of Definition 2.1 the determinant scales one row at a time.

$$\begin{aligned} \begin{vmatrix} 6 & -6 & 18 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} &= 2 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 2 & -2 & 14 \\ 4 & 8 & 0 \end{vmatrix} \\ &= 4 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 4 & 8 & 0 \end{vmatrix} \\ &= 8 \cdot \begin{vmatrix} 3 & -3 & 9 \\ 1 & -1 & 7 \\ 2 & 4 & 0 \end{vmatrix} \end{aligned}$$

Multilinear

3.2 *Definition* Let V be a vector space. A map $f\colon V^n \to \mathbb{R}$ is *multilinear* if

1) $f(\vec{\rho}_1, \ldots, \vec{v}+\vec{w}, \ldots, \vec{\rho}_n) =$
   $f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n) + f(\vec{\rho}_1, \ldots, \vec{w}, \ldots, \vec{\rho}_n)$
2) $f(\vec{\rho}_1, \ldots, k\vec{v}, \ldots, \vec{\rho}_n) = k \cdot f(\vec{\rho}_1, \ldots, \vec{v}, \ldots, \vec{\rho}_n)$

for $\vec{v}, \vec{w} \in V$ and $k \in \mathbb{R}$.

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There are two cases. If the set of other rows $\{\vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n\}$ is linearly dependent then all three matrices are singular and so all three determinants are zero and the equality is trivial.

Therefore assume that the set of other rows is linearly independent. We can make a basis by adding one more vector $\langle \vec{\rho}_1, \ldots, \vec{\rho}_{i-1}, \vec{\beta}, \vec{\rho}_{i+1}, \ldots, \vec{\rho}_n \rangle$. Express $\vec{v}$ and $\vec{w}$ with respect to this basis

$$\vec{v} = v_1\vec{\rho}_1 + \cdots + v_{i-1}\vec{\rho}_{i-1} + v_i\vec{\beta} + v_{i+1}\vec{\rho}_{i+1} + \cdots + v_n\vec{\rho}_n$$
$$\vec{w} = w_1\vec{\rho}_1 + \cdots + w_{i-1}\vec{\rho}_{i-1} + w_i\vec{\beta} + w_{i+1}\vec{\rho}_{i+1} + \cdots + w_n\vec{\rho}_n$$

and add.

$$\vec{v} + \vec{w} = (v_1 + w_1)\vec{\rho}_1 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

Consider the left side of (1) and expand $\vec{v} + \vec{w}$.

$$\det(\vec{\rho}_1, \ldots, (v_1 + w_1)\vec{\rho}_1 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n, \ldots, \vec{\rho}_n) \tag{*}$$

By the definition of determinant's condition (1), the value of $(*)$ is unchanged by the operation of adding $-(v_1 + w_1)\vec{\rho}_1$ to the i-th row $\vec{v} + \vec{w}$. The i-th row becomes this.

$$\vec{v} + \vec{w} - (v_1 + w_1)\vec{\rho}_1 = (v_2 + w_2)\vec{\rho}_2 + \cdots + (v_i + w_i)\vec{\beta} + \cdots + (v_n + w_n)\vec{\rho}_n$$

Next add $-(v_2 + w_2)\vec{\rho}_2$, etc., to eliminate all of the terms from the other rows. Apply condition (3) from the definition of determinant.

$$\begin{aligned}
\det(\vec{\rho}_1, \ldots, \vec{v} + \vec{w}, \ldots, \vec{\rho}_n)& \\
&= \det(\vec{\rho}_1, \ldots, (v_i + w_i) \cdot \vec{\beta}, \ldots, \vec{\rho}_n) \\
&= (v_i + w_i) \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) \\
&= v_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n) + w_i \cdot \det(\vec{\rho}_1, \ldots, \vec{\beta}, \ldots, \vec{\rho}_n)
\end{aligned}$$

Now this is a sum of two determinants. To finish, bring v_i and w_i back inside in front of the $\vec{\beta}$'s and use row combinations again, this time to reconstruct the expressions of $\vec{v}$ and $\vec{w}$ in terms of the basis. That is, start with the operations of adding $v_1\vec{\rho}_1$ to $v_i\vec{\beta}$ and $w_1\vec{\rho}_1$ to $w_i\vec{\rho}_1$, etc., to get the expansions of $\vec{v}$ and $\vec{w}$. QED

Use multilinearity to break a determinant into a sum of simple determinants.

Example We can expand this determinant

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

along the first row

$$= \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 4 \end{vmatrix}$$

and the second row.

$$= \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 4 \end{vmatrix}$$

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The first and last of the four determinants are 0 because the matrices are nonsingular (since one row is a multiple of the other). We are left with the two matrices, each having one entry from each row and column from the starting matrix.

Example Similarly we can start to evaluate this determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

by breaking it into a sum of determinants of matrices having one entry in each row from the starting matrix.

$$= \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \cdots + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 9 \end{vmatrix}$$

This gives a number of matrices, each all 0's except that each row has a single entry from the original matrix.

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For any of these determinants, if two rows have their original matrix entry in the same column then the determinant is 0, since if either entry is 0 then the matrix has a zero row while if neither is 0 then each row is a multiple of the other.

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For any of these determinants, if two rows have their original matrix entry in the same column then the determinant is 0, since if either entry is 0 then the matrix has a zero row while if neither is 0 then each row is a multiple of the other. Therefore, the above reduces to a sum of determinants, each all 0's but for a single entry in each row and column from the original matrix.

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 8 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 6 \\ 7 & 0 & 0 \end{vmatrix} \\
&\quad + \begin{vmatrix} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{vmatrix} \\
&= 45 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 48 \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\
&\quad + 72 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 84 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\
&\quad + 96 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + 105 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}
\end{aligned}
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Permutation matrices

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3.7 *Definition* An *n-permutation* is a function on the first n positive integers $\phi\colon\{1,\ldots,n\}\to\{1,\ldots,n\}$ that is one-to-one and onto.

So, in a permutation each number 1, ..., n appears as the output for one and only one input. We sometimes denote a permutation as the sequence $\phi=\langle\phi(1),\phi(2),\ldots,\phi(n)\rangle$.

?? *Example* The 2-permutations are the functions $\phi_1\colon\{1,2\}\to\{1,2\}$ given by $\phi_1(1)=1$, $\phi_1(2)=2$, and $\phi_2\colon\{1,2\}\to\{1,2\}$ given by $\phi_2(1)=2$, $\phi_2(2)=1$. The sequence notation is shorter: $\phi_1=\langle 1,2\rangle$ and $\phi_2=\langle 2,1\rangle$.

3.9 *Example* In the sequence notation the 3-permutations are $\phi_1=\langle 1,2,3\rangle$, $\phi_2=\langle 1,3,2\rangle$, $\phi_3=\langle 2,1,3\rangle$, $\phi_4=\langle 2,3,1\rangle$, $\phi_5=\langle 3,1,2\rangle$, and $\phi_6=\langle 3,2,1\rangle$.

Let ι_j be the row vector that is all 0's except for a 1 in entry j, so that the four-wide ι_2 is $\begin{pmatrix}0&1&0&0\end{pmatrix}$. Then our notation for permutation matrices is: with any $\phi = \langle \phi(1), \ldots, \phi(n) \rangle$ associate the matrix whose rows are $\iota_{\phi(1)}, \ldots, \iota_{\phi(n)}$.

Example Associated with the 4-permutation $\psi = \langle 2, 4, 3, 1 \rangle$ is the matrix whose rows are the matching ι's.

$$P_\psi = \begin{pmatrix} \iota_2 \\ \iota_4 \\ \iota_3 \\ \iota_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Permutation expansion

3.12 *Definition* The *permutation expansion* for determinants is

$$\begin{vmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{vmatrix} = \begin{aligned} & t_{1,\phi_1(1)} t_{2,\phi_1(2)} \cdots t_{n,\phi_1(n)} |P_{\phi_1}| \\ & \quad + t_{1,\phi_2(1)} t_{2,\phi_2(2)} \cdots t_{n,\phi_2(n)} |P_{\phi_2}| \\ & \quad \vdots \\ & \quad + t_{1,\phi_k(1)} t_{2,\phi_k(2)} \cdots t_{n,\phi_k(n)} |P_{\phi_k}| \end{aligned}$$

where $\phi_1, \ldots, \phi_k$ are all of the n-permutations.

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where $\phi_1, \ldots, \phi_k$ are all of the n-permutations.

We can restate the formula in *summation notation*

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \, |P_\phi|$$

read aloud as, "the sum, over all permutations ϕ, of terms having the form $t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} |P_\phi|$."

Example Recall that there are two 2-permutations $\phi_1 = \langle 1, 2 \rangle$ and $\phi_2 = \langle 2, 1 \rangle$. So for the 2×2 case, the sum over all permutations has two terms.

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These are the associated permutation matrices

$$P_{\phi_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{\phi_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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$$P_{\phi_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P_{\phi_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

giving this expansion.

$$\begin{aligned} \begin{vmatrix} t_{1,1} & t_{1,2} \\ t_{2,1} & t_{2,2} \end{vmatrix} &= t_{1,1}t_{2,2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + t_{1,2}t_{2,1} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\ &= t_{1,1}t_{2,2} \cdot 1 + t_{1,2}t_{2,1} \cdot (-1) \end{aligned}$$

The determinant $|P_{\phi_2}|$ equals -1 because we can bring that to the identity matrix with one row swap.

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The determinant $|P_{\phi_2}|$ equals -1 because we can bring that to the identity matrix with one row swap. Renaming the matrix entries gives the familiar 2×2 formula.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Proofs for these two theorems are in the next subsection.

3.14 *Theorem* For each n there is an $n \times n$ determinant function.

3.15 *Theorem* The determinant of a matrix equals the determinant of its transpose.

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3.15 *Theorem* The determinant of a matrix equals the determinant of its transpose.

3.16 *Corollary* A matrix with two equal columns is singular. Column swaps change the sign of a determinant. Determinants are multilinear in their columns.

Proof For the first statement, transposing the matrix results in a matrix with the same determinant, and with two equal rows, and hence a determinant of zero. Prove the other two in the same way.

QED

Determinants Exist

Inversion

4.1 *Definition* In a permutation $\phi = \langle \ldots, k, \ldots, j, \ldots \rangle$ elements such that $k > j$ are in an *inversion* of their natural order. Similarly, in a permutation matrix two rows

$$P_\phi = \begin{pmatrix} \vdots \\ \iota_k \\ \vdots \\ \iota_j \\ \vdots \end{pmatrix}$$

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Example The permutation $\phi = \langle 3, 2, 1 \rangle$ has three inversions: 3 is before 2, 3 is before 1, and 2 is before 1.

Example Here there are two inversions:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

row one is inverted with respect to row two and row three is inverted with respect to row four.

4.4 *Lemma* A row-swap in a permutation matrix changes the number of inversions from even to odd, or from odd to even.

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Proof Consider a swap of rows j and k, where $k > j$.

If the two rows are adjacent

$$P_\phi = \begin{pmatrix} \vdots \\ \iota_{\phi(j)} \\ \iota_{\phi(k)} \\ \vdots \end{pmatrix} \xrightarrow{\rho_k \leftrightarrow \rho_j} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j)} \\ \vdots \end{pmatrix}$$

then since inversions involving rows not in this pair are not affected, the swap changes the total number of inversions by one, either removing or producing one inversion depending on whether $\phi(j) > \phi(k)$ or not. Consequently, the total number of inversions changes from odd to even or from even to odd.

If the rows are not adjacent then we can swap them via a sequence of adjacent swaps, first bringing row k up

$$\begin{pmatrix} \vdots \\ \iota_{\phi(j)} \\ \iota_{\phi(j+1)} \\ \iota_{\phi(j+2)} \\ \vdots \\ \iota_{\phi(k)} \\ \vdots \end{pmatrix} \xrightarrow{\rho_k \leftrightarrow \rho_{k-1}} \xrightarrow{\rho_{k-1} \leftrightarrow \rho_{k-2}} \cdots \xrightarrow{\rho_{j+1} \leftrightarrow \rho_j} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j)} \\ \iota_{\phi(j+1)} \\ \vdots \\ \iota_{\phi(k-1)} \\ \vdots \end{pmatrix}$$

and then bringing row j down.

$$\xrightarrow{\rho_{j+1} \leftrightarrow \rho_{j+2}} \xrightarrow{\rho_{j+2} \leftrightarrow \rho_{j+3}} \cdots \xrightarrow{\rho_{k-1} \leftrightarrow \rho_{k}} \begin{pmatrix} \vdots \\ \iota_{\phi(k)} \\ \iota_{\phi(j+1)} \\ \iota_{\phi(j+2)} \\ \vdots \\ \iota_{\phi(j)} \\ \vdots \end{pmatrix}$$

Each of these adjacent swaps changes the number of inversions from odd to even or from even to odd. The total number of swaps $(k-j)+(k-j-1)$ is odd. Thus, in aggregate, the number of inversions changes from even to odd, or from odd to even. QED

Signum

4.7 *Definition* The ***signum*** of a permutation $\mathrm{sgn}(\phi)$ is -1 if the number of inversions in ϕ is odd and is $+1$ if the number of inversions is even.

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$$P_\phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

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Example The permutation $\psi = \langle 3, 2, 4, 1 \rangle$ has four inversions: 3 is before 2 and 1, 2 is before 1, and 4 is before 1. So $\operatorname{sgn}(\psi) = +1$.

4.5 *Corollary* If a permutation matrix has an odd number of inversions then swapping it to the identity takes an odd number of swaps. If it has an even number of inversions then swapping to the identity takes an even number.

4.5 *Corollary* If a permutation matrix has an odd number of inversions then swapping it to the identity takes an odd number of swaps. If it has an even number of inversions then swapping to the identity takes an even number.

Proof The identity matrix has zero inversions. To change an odd number to zero requires an odd number of swaps, and to change an even number to zero requires an even number of swaps. QED

Determinants exist

We still have not shown that the determinant function is well-defined because we have not considered row operations on permutation matrices other than row swaps. We will finesse this issue. We will define a function $d\colon \mathcal{M}_{n\times n} \to \mathbb{R}$ by altering the permutation expansion formula, replacing $|P_\phi|$ with $\operatorname{sgn}(\phi)$.

$$d(T) = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

The advantage of this formula is that the number of inversions is clearly well-defined — just count them. Therefore, we will be finished showing that an $n\times n$ determinant function exists when we show that this d satisfies the conditions required of a determinant.

4.9 *Lemma* The function d above is a determinant. Hence determinants exist for every n.

Proof We must check that it has the four conditions from the definition of determinant, Definition 2.1 .

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Condition (4) is easy: where I is the $n \times n$ identity, in

$$d(I) = \sum_{\text{perm } \phi} \iota_{1,\phi(1)} \iota_{2,\phi(2)} \cdots \iota_{n,\phi(n)} \operatorname{sgn}(\phi)$$

all of the terms in the summation are zero except for the one where the permutation ϕ is the identity, which gives the product down the diagonal, which is one.

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all of the terms in the summation are zero except for the one where the permutation ϕ is the identity, which gives the product down the diagonal, which is one.

For condition (3) suppose that $T \xrightarrow{k\rho_i} \hat{T}$ and consider $d(\hat{T})$.

$$\sum_{\text{perm } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi)$$

$$= \sum_{\phi} t_{1,\phi(1)} \cdots k t_{i,\phi(i)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

Factor out k to get the desired equality.

$$= k \cdot \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) = k \cdot d(T)$$

For (2) suppose that $T \xrightarrow{\rho_i \leftrightarrow \rho_j} \hat{T}$. We must show that $d(\hat{T})$ is the negative of $d(T)$.

$$d(\hat{T}) = \sum_{\text{perm } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi) \qquad (*)$$

We will show that each term in $(*)$ is associated with a term in $d(T)$, and that the two terms are negatives of each other. Consider the matrix from the multilinear expansion of $d(\hat{T})$ giving the term $\hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi)$.

$$\begin{pmatrix} & \vdots & \\ \hat{t}_{i,\phi(i)} & & \\ & \vdots & \\ & & \hat{t}_{j,\phi(j)} \\ & \vdots & \end{pmatrix}$$

It is the result of the $\rho_i \leftrightarrow \rho_j$ operation performed on this matrix.

$$\begin{pmatrix} & \vdots & \\ & & t_{i,\phi(j)} \\ & \vdots & \\ t_{j,\phi(i)} & & \\ & \vdots & \end{pmatrix}$$

That is, the term with hatted t's is associated with this term from the $d(T)$ expansion: $t_{1,\sigma(1)} \cdots t_{j,\sigma(j)} \cdots t_{i,\sigma(i)} \cdots t_{n,\sigma(n)} \operatorname{sgn}(\sigma)$, where the permutation σ equals ϕ but with the i-th and j-th numbers interchanged, $\sigma(i) = \phi(j)$ and $\sigma(j) = \phi(i)$. The two terms have the same multiplicands $\hat{t}_{1,\phi(1)} = t_{1,\sigma(1)}$, ..., including the entries from the swapped rows $\hat{t}_{i,\phi(i)} = t_{j,\phi(i)} = t_{j,\sigma(j)}$ and $\hat{t}_{j,\phi(j)} = t_{i,\phi(j)} = t_{i,\sigma(i)}$. But the two terms are negatives of each other since $\operatorname{sgn}(\phi) = -\operatorname{sgn}(\sigma)$ by Lemma 4.4 .

Now, any permutation ϕ can be derived from some other permutation σ by such a swap, in one and only one way. Therefore the summation in $(*)$ is in fact a sum over all permutations, taken once and only once.

$$\begin{aligned} d(\hat{T}) &= \sum_{\text{perm } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi) \\ &= \sum_{\text{perm } \sigma} t_{1,\sigma(1)} \cdots t_{j,\sigma(j)} \cdots t_{i,\sigma(i)} \cdots t_{n,\sigma(n)} \cdot \big(-\operatorname{sgn}(\sigma)\big) \end{aligned}$$

Thus $d(\hat{T}) = -d(T)$.

Finally, for condition (1) suppose that $T \xrightarrow{k\rho_i+\rho_j} \hat{T}$.

$$
\begin{aligned}
d(\hat{T}) &= \sum_{\text{perm } \phi} \hat{t}_{1,\phi(1)} \cdots \hat{t}_{i,\phi(i)} \cdots \hat{t}_{j,\phi(j)} \cdots \hat{t}_{n,\phi(n)} \operatorname{sgn}(\phi) \\
&= \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots (k t_{i,\phi(j)} + t_{j,\phi(j)}) \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)
\end{aligned}
$$

Distribute over the addition in $k t_{i,\phi(j)} + t_{j,\phi(j)}$.

$$
\begin{aligned}
= \sum_{\phi} \bigl[& t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots k t_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
& + t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{j,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \bigr]
\end{aligned}
$$

Break it into two summations.

$$
\begin{aligned}
= \sum_{\phi} & t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots k t_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
& + \sum_{\phi} t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{j,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)
\end{aligned}
$$

Recognize the second one.

$$
\begin{aligned}
= k \cdot \sum_{\phi} & t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi) \\
& + d(T)
\end{aligned}
$$

Consider the terms $t_{1,\phi(1)} \cdots t_{i,\phi(i)} \cdots t_{i,\phi(j)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$. Notice the subscripts; the entry is $t_{i,\phi(j)}$, not $t_{j,\phi(j)}$. The sum of these terms is the determinant of a matrix S that is equal to T except that row j of S is a copy of row i of T, that is, S has two equal rows. In the same way that we proved Lemma 2.4 we can see that $d(S) = 0$: a swap of S's equal rows will change the sign of $d(S)$ but since the matrix is unchanged by that swap the value of $d(S)$ must also be unchanged, and so that value must be zero. QED

The determinant of the transpose

4.10 *Theorem* The determinant of a matrix equals the determinant of its transpose.

Proof The proof is best understood by doing the general 3×3 case. That the argument applies to the $n\times n$ case will be clear.

Compare the permutation expansion of the matrix T

$$\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} = t_{1,1}t_{2,2}t_{3,3}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{2,3}t_{3,2}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$
$$+ t_{1,2}t_{2,1}t_{3,3}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,2}t_{2,3}t_{3,1}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$+ t_{1,3}t_{2,1}t_{3,2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{1,3}t_{2,2}t_{3,1}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

with the permutation expansion of its transpose.

$$\begin{vmatrix} t_{1,1} & t_{2,1} & t_{3,1} \\ t_{1,2} & t_{2,2} & t_{3,2} \\ t_{1,3} & t_{2,3} & t_{3,3} \end{vmatrix} = t_{1,1}t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{3,2}t_{2,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$
$$+ t_{2,1}t_{1,2}t_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,1}t_{3,2}t_{1,3} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$
$$+ t_{3,1}t_{1,2}t_{2,3} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{3,1}t_{2,2}t_{1,3} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

Compare first the six products of t's. The ones in the expansion of T are the same as the ones in the expansion of the transpose; for instance, $t_{1,2}t_{2,3}t_{3,1}$ is in the top and $t_{3,1}t_{1,2}t_{2,3}$ is in the bottom. That's perfectly sensible — the six in the top arise from all of the ways of picking one entry of T from each row and column while the six in the bottom are all of the ways of picking one entry of T from each column and row, so of course they are the same set.

Next observe that in the two expansions, each t-product expression is not necessarily associated with the same permutation matrix. For instance, on the top $t_{1,2}t_{2,3}t_{3,1}$ is associated with the matrix for the map $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. On the bottom $t_{3,1}t_{1,2}t_{2,3}$ is associated with the matrix for the map $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$. The second map is inverse to the first. This is also perfectly sensible — both the matrix transpose and the map inverse flip the $1, 2$ to $2, 1$, flip the $2, 3$ to $3, 2$, and flip $3, 1$ to $1, 3$.

We finish by noting that the determinant of P_ϕ equals the determinant of $P_{\phi^{-1}}$, as shown in Exercise 16 . QED

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We finish by noting that the determinant of P_ϕ equals the determinant of $P_{\phi^{-1}}$, as shown in Exercise 16 . QED

Example We know the formula for 2×2 matrices.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$$