Five.I Complex Vector Spaces

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Chapter Five. Similarity

This chapter requires that we factor polynomials, but many polynomials do not factor over the real numbers. For instance, $x^2 + 1$ does not factor into a product of two linear polynomials with real coefficients, instead it requires complex numbers $x^2 + 1 = (x - i)(x + i)$.

Therefore, in this chapter we shall use complex numbers for our scalars, including entries in vectors and matrices. That is, we are shifting from studying vector spaces over the real numbers to vector spaces over the complex numbers.

Any real number is a complex number and in this chapter most of the examples use only real numbers. Nonetheless, the critical theorems require that the scalars be complex so the first section is a quick review of complex numbers.

Review of Factoring and Complex Numbers

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So, $3x^3 + 2x^2 - x + 4 = (x^2 + x) \cdot (3x) + 4$.

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$$c(x) = m(x) \cdot q(x) + r(x)$$

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If a divisor $m(x)$ goes into a dividend $c(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $m(x)$ is a *factor* of $c(x)$. Any *root* of the factor (any $\lambda \in \mathbb{R}$ such that $m(\lambda) = 0$) is a root of $c(x)$ since $c(\lambda) = m(\lambda) \cdot q(\lambda) = 0$.

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Proof By the above corollary $c(x) = (x - \lambda) \cdot q(x) + c(\lambda)$. Since λ is a root, $c(\lambda) = 0$ so $x - \lambda$ is a factor. QED

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1.7 *Corollary* Any polynomial with real coefficients can be factored into linear and irreducible quadratic polynomials. That factorization is unique; any two factorizations have the same powers of the same factors.

Complex numbers

While $x^2 + 1$ has no real roots and so doesn't factor over the real numbers, if we imagine a root — traditionally denoted i so that $i^2 + 1 = 0$ — then $x^2 + 1$ factors into a product of linears $(x - i)(x + i)$.

So we adjoin this root i to the reals and close the new system with respect to addition, multiplication, etc. (i.e., we also add $3 + i$, and $2i$, and $3 + 2i$, etc., putting in all linear combinations of 1 and i). We then get a new structure, the *complex numbers* $\mathbb{C}$.

In $\mathbb{C}$ we can factor (obviously, at least some) quadratics that would be irreducible if we were to stick to the real numbers. Surprisingly, in $\mathbb{C}$ we can not only factor $x^2 + 1$ and its close relatives, we can factor any quadratic.

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1.11 *Theorem* [Fundamental Theorem of Algebra] Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

Complex Representations

Recall the definitions of the complex number addition

$$(a+bi)\ +\ (c+di)=(a+c)+(b+d)i$$

and multiplication.

$$\begin{aligned}(a+bi)(c+di)&=ac+adi+bci+bd(-1)\\&=(ac-bd)+(ad+bc)i\end{aligned}$$

With those rules for scalars, all of the operations that we've covered for real vector spaces carry over unchanged.

Example

$$\begin{pmatrix}2-i & 1+i\\ i & 4\end{pmatrix}\begin{pmatrix}0 & 3+3i\\ 1-i & 2\end{pmatrix}=\begin{pmatrix}2 & 9+3i\\ 4-4i & 5+3i\end{pmatrix}$$

We shall carry over unchanged from the previous work everything else that we can. For instance, we shall call this

$$\langle \begin{pmatrix} 1+0i \\ 0+0i \\ \vdots \\ 0+0i \end{pmatrix}, \ldots, \begin{pmatrix} 0+0i \\ 0+0i \\ \vdots \\ 1+0i \end{pmatrix} \rangle$$

the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and again denote it $\mathcal{E}_n$.