Two.II Linear Independence

Linear Algebra

Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition and examples

Linear independence

1.4 *Definition* A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others.[1] Otherwise it is *linearly dependent*.

[1] More information on multisets is in the appendix. Most of the time we won't need the set-multiset distinction and we will follow the standard terminology of referring to a linearly independent or dependent 'set'. Remark 1.12 explains why the definition requires a multiset, strictly speaking.

Linear independence

1.4 *Definition* A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others.[1] Otherwise it is *linearly dependent*.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets off $\vec{s}_0$, algebraically there is nothing special about that vector in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite to isolate $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side.

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1.5 *Lemma* A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \ldots, c_n = 0$.

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Proof If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

Example This set of vectors in the plane $\mathbb{R}^2$ is linearly independent.

$$\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Example In the vector space of cubic polynomials $\mathcal{P}_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R}\}$ the set $\{1 - x, 1 + x^2\}$ is linearly independent. The equation $c_0(1 - x) + c_1(1 + x^2) = 0$ leads to this linear system

$$\begin{aligned} c_0 - c_1 &= 0 \\ c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.

Example The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows make a linearly independent set.

Example This subset of $\mathbb{R}^3$ is linearly dependent.

$$\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}\}$$

One way to show that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 \phantom{{}+c_2} + 6c_3 &= 0 \end{aligned}$$

and note that there are more solutions than just the trivial one.

?? *Lemma*

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Proof

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QED

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Example In $\mathcal{P}_2$ consider the set $S = \{1 - x, 1 + x\}$. The span $[S]$ is the subset of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$. (The span is a subset of the linear polynomials because no member of S has a quadratic term. To see that the span is all of the set of linear polynomials, consider a linear polynomial $a + bx$ and use the equation $a + bx = r_1(1 - x) + r_2(1 + x)$ to get a linear system that solves as $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b$.)

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If we add a linear polynomial $S_1 = S \cup \{2 + 2x\}$ then the span is unchanged $[S] = [S_1]$. This is because span of S is all of the linear polynomials and the new member does not add any quadratic terms.

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If we add a quadratic polynomial $S_2 = S \cup \{2 + x^2\}$ then we enlarge the span: the span of S_2 is all of $\mathcal{P}_2$. To see this, consider a quadratic $a + bx + cx^2$ and use $a + bx + cx^2 = r_1(1 - x) + r_2(1 + x) + r_3(2 + x^2)$ to get a linear system that has the solution $r_3 = c$, $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b - c$.

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Proof If $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector $\vec{v}_1$ that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

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If S_1 is linearly independent then we are done. Otherwise iterate: take a vector $\vec{v}_2$ that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. QED

Example Consider this subset of $\mathbb{R}^2$.

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$$

The linear relationship

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (*)$$

gives a system of equations.

$$\begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}- r_4} + r_5 &=& 0 \\ 2r_1 + 3r_2 + 4r_3 - r_4 - r_5 &=& 0 \end{array}$$

$$\xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}- r_4} + r_5 &=& 0 \\ +3r_3 - r_4 - 2r_5 &=& 0 \end{array}$$

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}\}$$

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Set $r_5 = 1$ and set the other two parameters to 0 to get $r_1 = -5/6$ and $r_3 = 2/3$. This instance of $(*)$

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

shows that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$.

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shows that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows $\vec{s}_4$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows $\vec{s}_2$ is in the span of the same set.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

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The set $\{\vec{s}_1, \vec{s}_3\}$ is linearly independent and so we cannot omit any members without shrinking the span. (In $(*)$ note that $\vec{s}_2$ is linearly dependent on $\vec{s}_1$ and r_2 did not end as a leading variable.)

1.18 *Corollary* A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s_i}$ is a linear combination of the vectors $\vec{s}_1$, ..., $\vec{s}_{i-1}$ listed before it.

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Proof Consider $S_0 = \{\}$, $S_1 = \{\vec{s_1}\}$, $S_2 = \{\vec{s}_1, \vec{s}_2\}$, etc. Some index $i \geqslant 1$ is the first one with $S_{i-1} \cup \{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i \in [S_{i-1}]$. QED

Linear independence and subset

1.19 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

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This table summarizes the cases.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S *independent*	$\hat{S}$ must be independent	$\hat{S}$ may be either
S *dependent*	$\hat{S}$ may be either	$\hat{S}$ must be dependent

An example of the lower left is that the set S of all vectors in the space $\mathbb{R}^2$ is linearly dependent but the subset S_1 consisting of only the unit vector on the x-axis is independent. By interchanging S_1 with S that's also an example of the upper right.