Four.III Laplace's Expansion

Linear Algebra

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Laplace's formula for the determinant

1.1 *Example* In this permutation expansion

$$\begin{aligned}\begin{vmatrix} t_{1,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \\ t_{3,1} & t_{3,2} & t_{3,3} \end{vmatrix} &= t_{1,1}t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,1}t_{2,3}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \\ &\quad + t_{1,2}t_{2,1}t_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{1,2}t_{2,3}t_{3,1} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ &\quad + t_{1,3}t_{2,1}t_{3,2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{1,3}t_{2,2}t_{3,1} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}\end{aligned}$$

we can factor out the entries from the first row $t_{1,1}$, $t_{1,2}$, $t_{1,3}$

$$
\begin{aligned}
&= t_{1,1} \cdot \left[t_{2,2}t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,2} \cdot \left[t_{2,1}t_{3,3} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3}t_{3,1} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,3} \cdot \left[t_{2,1}t_{3,2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + t_{2,2}t_{3,1} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \right]
\end{aligned}
$$

and in the permutation matrices swap to get the first rows into place.

$$
\begin{aligned}
&= t_{1,1} \cdot \left[t_{2,2} t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3} t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad - t_{1,2} \cdot \left[t_{2,1} t_{3,3} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,3} t_{3,1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right] \\
&\quad + t_{1,3} \cdot \left[t_{2,1} t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,2} t_{3,1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right]
\end{aligned}
$$

$$
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&\quad + t_{1,3} \cdot \left[t_{2,1}t_{3,2} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + t_{2,2}t_{3,1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \right]
\end{aligned}
$$

The point of the swapping (one swap to each of the permutation matrices on the second line and two swaps to each on the third line) is that the three lines simplify to three terms.

$$
= t_{1,1} \cdot \begin{vmatrix} t_{2,2} & t_{2,3} \\ t_{3,2} & t_{3,3} \end{vmatrix} - t_{1,2} \cdot \begin{vmatrix} t_{2,1} & t_{2,3} \\ t_{3,1} & t_{3,3} \end{vmatrix} + t_{1,3} \cdot \begin{vmatrix} t_{2,1} & t_{2,2} \\ t_{3,1} & t_{3,2} \end{vmatrix}
$$

Minor

1.2 *Definition* For any $n \times n$ matrix T, the $(n-1) \times (n-1)$ matrix formed by deleting row i and column j of T is the i, j *minor* of T. The i, j *cofactor* $T_{i,j}$ of T is $(-1)^{i+j}$ times the determinant of the i, j minor of T.

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Example For this matrix

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

the $2, 3$ minor is

$$\begin{pmatrix} 3 & 1 \\ 7 & 0 \end{pmatrix}$$

so the associated cofactor is $S_{2,3} = (-1)^5 \cdot (-7) = 7$.

1.5 *Theorem* Where T is an $n \times n$ matrix, we can find the determinant by expanding by cofactors on any row i or column j.

$$
\begin{aligned}
|T| &= t_{i,1} \cdot T_{i,1} + t_{i,2} \cdot T_{i,2} + \cdots + t_{i,n} \cdot T_{i,n} \\
&= t_{1,j} \cdot T_{1,j} + t_{2,j} \cdot T_{2,j} + \cdots + t_{n,j} \cdot T_{n,j}
\end{aligned}
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Proof Exercise 27 . QED

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We can find this determinant

$$\begin{vmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{vmatrix}$$

by expanding along the second row. Besides $S_{2,3} = 7$, the other two cofactors are here.

$$S_{2,1} = (-1)^3 \cdot \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 3 \quad S_{2,2} = (-1)^4 \cdot \begin{vmatrix} 3 & 2 \\ 7 & -3 \end{vmatrix} = -23$$

The Laplace expansion gives $5 \cdot 3 + 4 \cdot (-23) - 1 \cdot 7 = -84$.

Adjoint

1.8 *Definition* The matrix *adjoint* to the square matrix T is

$$\operatorname{adj}(T) = \begin{pmatrix} T_{1,1} & T_{2,1} & \dots & T_{n,1} \\ T_{1,2} & T_{2,2} & \dots & T_{n,2} \\ & \vdots & & \\ T_{1,n} & T_{2,n} & \dots & T_{n,n} \end{pmatrix}$$

where $T_{j,i}$ is the j, i cofactor.

Note that the order of the subscripts in this matrix is opposite to the order that you might expect.

Example The matrix adjoint to this

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

is this (some of these cofactors we have calculated above).

$$\begin{pmatrix} S_{1,1} & S_{2,1} & S_{3,1} \\ S_{1,2} & S_{2,2} & S_{3,2} \\ S_{1,3} & S_{2,3} & S_{3,3} \end{pmatrix} = \begin{pmatrix} -12 & 3 & -9 \\ 8 & -23 & 13 \\ -28 & 7 & 7 \end{pmatrix}$$

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This summarizes.

$$\begin{pmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} \\ t_{2,1} & t_{2,2} & \ldots & t_{2,n} \\ & \vdots & & \\ t_{n,1} & t_{n,2} & \ldots & t_{n,n} \end{pmatrix} \begin{pmatrix} T_{1,1} & T_{2,1} & \ldots & T_{n,1} \\ T_{1,2} & T_{2,2} & \ldots & T_{n,2} \\ & \vdots & & \\ T_{1,n} & T_{2,n} & \ldots & T_{n,n} \end{pmatrix} = \begin{pmatrix} |T| & 0 & \ldots & 0 \\ 0 & |T| & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & |T| \end{pmatrix}$$

1.9 *Proof* Theorem 1.5 says we can calculate the determinant of an $n \times n$ matrix T by taking linear combinations of entries from a row and their associated cofactors.

$$t_{i,1} \cdot T_{i,1} + t_{i,2} \cdot T_{i,2} + \cdots + t_{i,n} \cdot T_{i,n} = |T|$$

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For the off-diagonal entries, recall that a matrix with two identical rows has a determinant of 0. Thus, for any matrix T, weighing the cofactors by entries from row k with $k \neq i$ gives 0

$$t_{i,1} \cdot T_{k,1} + t_{i,2} \cdot T_{k,2} + \cdots + t_{i,n} \cdot T_{k,n} = 0$$

because it represents the expansion along the row k of a matrix with row i equal to row k. QED

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Example The inverse of this matrix

$$S = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 4 & -1 \\ 7 & 0 & -3 \end{pmatrix}$$

is this.

$$\frac{1}{|S|} \cdot \mathrm{adj}(S) = \frac{1}{84} \cdot \begin{pmatrix} -12 & 3 & -9 \\ 8 & -23 & 13 \\ -28 & 7 & 7 \end{pmatrix}$$

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Note The formulas from this section are not the best choice for computations with arbitrary matrices because they typically require more arithmetic than the Gauss-Jordan method.