

## Three.VI Projection

*Linear Algebra*

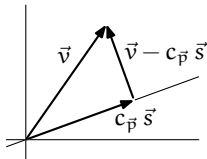
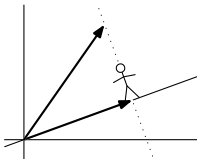
Jim Hefferon

<http://joshua.smcvt.edu/linearalgebra>

## Orthogonal Projection Into a Line

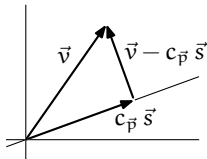
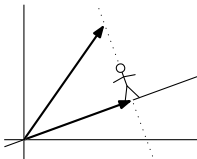
## Project a vector into a line

This shows a figure walking out on the line until they are at a point  $\vec{p}$  such that the tip of  $\vec{v}$  is directly above them, where “above” does not mean parallel to the y-axis but instead means orthogonal to the line.



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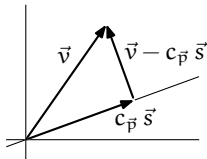
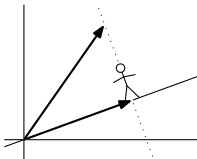
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Since the line is the span of some vector  $\ell = \{c \cdot \vec{s} \mid c \in \mathbb{R}\}$ , we have a coefficient  $c_{\vec{p}}$  with the property that  $\vec{v} - c_{\vec{p}}\vec{s}$  is orthogonal to  $c_{\vec{p}}\vec{s}$ .

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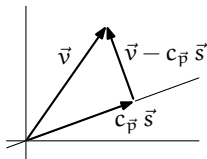
To solve for this coefficient, observe that because  $\vec{v} - c_{\vec{p}}\vec{s}$  is orthogonal to a scalar multiple of  $\vec{s}$ , it must be orthogonal to  $\vec{s}$  itself. Then  $(\vec{v} - c_{\vec{p}}\vec{s}) \cdot \vec{s} = 0$  gives that  $c_{\vec{p}} = \vec{v} \cdot \vec{s} / \vec{s} \cdot \vec{s}$ .

Note two things about orthogonal projection.

- 1) We have an idea of 'angle' in  $\mathbb{R}^n$  but we have not given such a definition in some other spaces. This section's results will stick to spaces in which we have covered 'orthogonal,' namely the real spaces.

Note two things about orthogonal projection.

- 1) We have an idea of ‘angle’ in  $\mathbb{R}^n$  but we have not given such a definition in some other spaces. This section’s results will stick to spaces in which we have covered ‘orthogonal,’ namely the real spaces.
- 2) We have decomposed  $\vec{v}$  into two parts.



Intuitively, some of  $\vec{v}$  lies with the line and that gives the first part  $c_{\vec{p}} \vec{s}$ . The part of  $\vec{v}$  that lies with a line orthogonal to the given line is  $\vec{v} - c_{\vec{p}} \vec{s}$ . What’s compelling about pairing these two parts is that they don’t interact in that the projection of one into the line spanned by the other is the zero vector.

1.1 *Definition* The *orthogonal projection of  $\vec{v}$  into the line spanned by a nonzero  $\vec{s}$*  is this vector.

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

*Example* The projection of this  $\mathbb{R}^3$  vector into the line

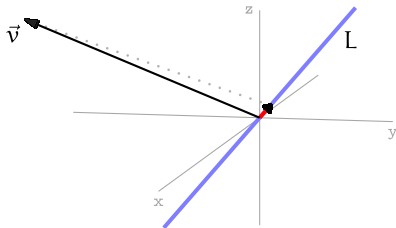
$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad L = \{c \cdot \vec{s} = c \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R}\}$$

is this vector.

$$\frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/6 \end{pmatrix}$$



This picture of that projection



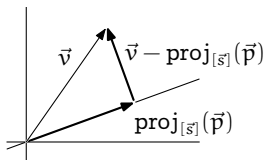
brings out that the projection vector is quite short:

$\|\vec{v}\| = \sqrt{6} \approx 2.45$  while  $\|\text{proj}_{[\vec{s}]}(\vec{v})\| = \sqrt{1/6} \approx 0.41$ . Only a small part of the vector  $\vec{v}$  lies in the direction of the line L.

# Gram-Schmidt Orthogonalization

## Mutually orthogonal vectors

The prior subsection suggests that projecting a vector  $\vec{v}$  into the line spanned by  $\vec{s}$  decomposes  $\vec{v}$  into two parts

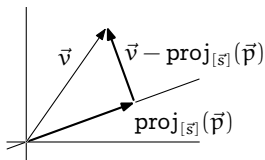


$$\vec{v} = \text{proj}_{[\vec{s}]}(\vec{v}) + (\vec{v} - \text{proj}_{[\vec{s}]}(\vec{v}))$$

that are orthogonal and so are not-interacting. We will now develop that suggestion.

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2.1 *Definition* Vectors  $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$  are *mutually orthogonal* when any two are orthogonal: if  $i \neq j$  then the dot product  $\vec{v}_i \cdot \vec{v}_j$  is zero.

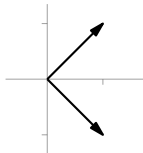
*Example* The vectors of the standard basis  $\mathcal{E}_3 \subset \mathbb{R}^3$  are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This remains true if we rotate this basis.

*Example* These two vectors in  $\mathbb{R}^2$  are mutually orthogonal.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



2.2 *Theorem* If the vectors in a set  $\{\vec{v}_1, \dots, \vec{v}_k\} \subset \mathbb{R}^n$  are mutually orthogonal and nonzero then that set is linearly independent.

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*Proof* Consider  $\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$ . For  $i \in \{1, \dots, k\}$ , taking the dot product of  $\vec{v}_i$  with both sides of the equation  $\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k) = \vec{v}_i \cdot \vec{0}$ , which gives  $c_i \cdot (\vec{v}_i \cdot \vec{v}_i) = 0$ , shows that  $c_i = 0$  since  $\vec{v}_i \neq \vec{0}$ . QED

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2.5 *Definition* An *orthogonal basis* for a vector space is a basis of mutually orthogonal vectors.

2.7 *Theorem* If  $\langle \vec{\beta}_1, \dots, \vec{\beta}_k \rangle$  is a basis for a subspace of  $\mathbb{R}^n$  then the vectors

$$\vec{\kappa}_1 = \vec{\beta}_1$$

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$$

$$\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$$

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$$\vec{\kappa}_k = \vec{\beta}_k - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_k) - \dots - \text{proj}_{[\vec{\kappa}_{k-1}]}(\vec{\beta}_k)$$

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*Proof* We will use induction to check that each  $\vec{\kappa}_i$  is nonzero, is in the span of  $\langle \vec{\beta}_1, \dots, \vec{\beta}_i \rangle$ , and is orthogonal to all preceding vectors  $\vec{\kappa}_1 \cdot \vec{\kappa}_i = \dots = \vec{\kappa}_{i-1} \cdot \vec{\kappa}_i = 0$ . Then Corollary 2.3 gives that  $\langle \vec{\kappa}_1, \dots, \vec{\kappa}_k \rangle$  is a basis for the same space as is the starting basis.

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We shall only cover the cases up to  $i = 3$ , to give the sense of the argument. The full argument is Exercise 25 .

The  $i = 1$  case is trivial; taking  $\vec{\kappa}_1$  to be  $\vec{\beta}_1$  makes it a nonzero vector since  $\vec{\beta}_1$  is a member of a basis, it is obviously in the span of  $\langle \vec{\beta}_1 \rangle$ , and the ‘orthogonal to all preceding vectors’ condition is satisfied vacuously.

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In the  $i = 2$  case the expansion

$$\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2) = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 = \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1$$

shows that  $\vec{\kappa}_2 \neq \vec{0}$  or else this would be a non-trivial linear dependence among the  $\vec{\beta}$ ’s (it is nontrivial because the coefficient of  $\vec{\beta}_2$  is 1). It also shows that  $\vec{\kappa}_2$  is in the span of  $\langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ . And,  $\vec{\kappa}_2$  is orthogonal to the only preceding vector

$$\vec{\kappa}_1 \cdot \vec{\kappa}_2 = \vec{\kappa}_1 \cdot (\vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)) = 0$$

because this projection is orthogonal.

The  $i = 3$  case is the same as the  $i = 2$  case except for one detail. As in the  $i = 2$  case, expand the definition.

$$\begin{aligned}\vec{\kappa}_3 &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\kappa}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \vec{\kappa}_2 \\ &= \vec{\beta}_3 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1 - \frac{\vec{\beta}_3 \cdot \vec{\kappa}_2}{\vec{\kappa}_2 \cdot \vec{\kappa}_2} \cdot \left( \vec{\beta}_2 - \frac{\vec{\beta}_2 \cdot \vec{\kappa}_1}{\vec{\kappa}_1 \cdot \vec{\kappa}_1} \cdot \vec{\beta}_1 \right)\end{aligned}$$

By the first line  $\vec{\kappa}_3 \neq \vec{0}$ , since  $\vec{\beta}_3$  isn't in the span  $[\vec{\beta}_1, \vec{\beta}_2]$  and therefore by the inductive hypothesis it isn't in the span  $[\vec{\kappa}_1, \vec{\kappa}_2]$ . By the second line  $\vec{\kappa}_3$  is in the span of the first three  $\vec{\beta}$ 's. Finally, the calculation below shows that  $\vec{\kappa}_3$  is orthogonal to  $\vec{\kappa}_1$ .



$$\begin{aligned}
\vec{\kappa}_1 \cdot \vec{\kappa}_3 &= \vec{\kappa}_1 \cdot ( \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) ) \\
&= \vec{\kappa}_1 \cdot ( \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) ) - \vec{\kappa}_1 \cdot \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3) \\
&= 0
\end{aligned}$$

(Here is the difference with the  $i = 2$  case: as happened for  $i = 2$  the first term is 0 because this projection is orthogonal, but here the second term in the second line is 0 because  $\vec{\kappa}_1$  is orthogonal to  $\vec{\kappa}_2$  and so is orthogonal to any vector in the line spanned by  $\vec{\kappa}_2$ .) A similar check shows that  $\vec{\kappa}_3$  is also orthogonal to  $\vec{\kappa}_2$ . QED

*Example* This is a basis for  $\mathbb{R}^3$ .

$$B = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\rangle$$

We produce the new basis by starting with  $\vec{\beta}_1$ .

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The next step is  $\vec{\kappa}_2 = \vec{\beta}_2 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_2)$ .

$$\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

The third step is  $\vec{\kappa}_3 = \vec{\beta}_3 - \text{proj}_{[\vec{\kappa}_1]}(\vec{\beta}_3) - \text{proj}_{[\vec{\kappa}_2]}(\vec{\beta}_3)$ .

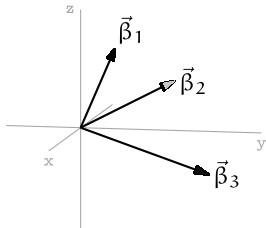
$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

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The members of this basis are mutually orthogonal.

$$K = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix} \right\rangle$$

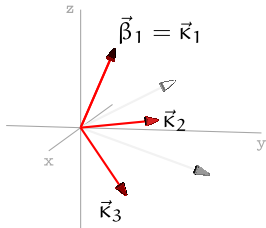


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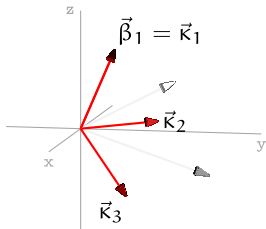


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$$\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix}$$

The members of this basis are mutually orthogonal.

$$K = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 4/3 \\ -4/3 \end{pmatrix} \right\rangle$$



We could go on to make this basis even more like  $\mathcal{E}_3$  by normalizing all of its members to have length 1, making an *orthonormal* basis.