Two.III Basis and Dimension

Linear Algebra Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Basis

Definition of basis

1.1 *Definition* A ***basis*** for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Definition of basis

1.1 *Definition* A ***basis*** for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Because a basis is a sequence,meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle\vec{\beta}_1, \vec{\beta}_2, \ldots\rangle$.

Definition of basis

1.1 *Definition* A *basis* for a vector space is a sequence of vectors that is linearly independent and that spans the space.

Because a basis is a sequence,meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle\vec{\beta}_1, \vec{\beta}_2, \ldots\rangle$.

Example This is a basis for $\mathbb{R}^2$.

$$\langle\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\rangle$$

It is linearly independent.

$$c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \implies \begin{array}{r} c_1 + c_2 = 0\\ -c_1 + c_2 = 0\end{array} \implies c_1 = 0,\ c_2 = 0$$

And it spans $\mathbb{R}^2$ since

$$c_1\begin{pmatrix}1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \implies \begin{array}{r} c_1 + c_2 = x\\ -c_1 + c_2 = y\end{array}$$

has the solution $c_1 = (1/2)x - (1/2)y$ and $c_2 = (1/2)x + (1/2)y$.

Example This is a basis for $\mathbb{R}^3$.

$$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Calculus books sometimes call those $im\vec{a}th$, $jm\vec{a}th$, and $\vec{k}$.

Example This is a basis for $\mathbb{R}^3$.

$$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Calculus books sometimes call those $im\vec{a}th$, $jm\vec{a}th$, and $\vec{k}$.

1.5 *Definition* For any $\mathbb{R}^n$

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \dots, \vec{e}_n$.

Checking that $\mathcal{E}_n$ is a basis for $\mathbb{R}^n$ is routine.

1.12 *Theorem* In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

1.12 *Theorem* In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

Proof A sequence is a basis if and only if its vectors form a set that spans and that is linearly independent. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Consider two expressions of a vector as a linear combination of the members of the subset. Rearrange the two sums, and if necessary add some $0 \cdot \vec{\beta}_i$ terms, so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ and $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$$

holds if and only if

$$(c_1 - d_1)\vec{\beta}_1 + \cdots + (c_n - d_n)\vec{\beta}_n = \vec{0}$$

holds. So, asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i, that is, that every vector is expressible as a linear combination of the $\vec{\beta}$'s in a unique way. QED

1.13 *Definition* In a vector space with basis B the *representation of $\vec{v}$ with respect to B* is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to B*.

Example In the vector space of linear polynomials $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$.

Check that is a basis by verifying that it is linearly independent

$$0 = c_1(1+x)+c_2(1-x) \implies 0 = c_1+c_2,\ 0 = c_1-c_2 \implies c_1 = c_2 = 0$$

and that it spans the space.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

Example In the vector space of linear polynomials $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$.

Check that is a basis by verifying that it is linearly independent

$$0 = c_1(1+x)+c_2(1-x) \implies 0 = c_1+c_2,\ 0 = c_1-c_2 \implies c_1 = c_2 = 0$$

and that it spans the space.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

Note that the solutions to those systems are unique. For instance, the polynomial $3 + 4x$ has a unique expression as a combination of the two basis vectors.

$$3 + 4x = (7/2) \cdot (1 + x) + (-1/2) \cdot (1 - x)$$

We write this.

$$\mathrm{Rep}_B(3 + 4x) = \begin{pmatrix} 7/2 \\ -1/2 \end{pmatrix}$$

Example With respect to $\mathbb{R}^3$'s standard basis $\mathcal{E}_3$ the vector

$$\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 1/2 \end{pmatrix}$$

has this representation.

$$\operatorname{Rep}_{\mathcal{E}_3}(\vec{v}) = \begin{pmatrix} 2 \\ -3 \\ 1/2 \end{pmatrix}$$

In general, any $\vec{w} \in \mathbb{R}^n$ has $\operatorname{Rep}_{\mathcal{E}_n}(\vec{w}) = \vec{w}$.

Dimension

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2 \rangle$, so it is finite-dimensional.

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2 \rangle$, so it is finite-dimensional.

Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is finite-dimensional. Here is one basis with finitely many members.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

Definition of dimension

2.1 *Definition* A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2\rangle$, so it is finite-dimensional.

Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is finite-dimensional. Here is one basis with finitely many members.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

Note From this point on, we will restrict our attention to vector spaces that are finite-dimensional. All the later examples, definitions, and theorems carry this presumption of the spaces even if they don't explicitly state that.

Exchange Lemma

2.4 *Lemma* Assume that $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Exchange Lemma

2.4 *Lemma* Assume that $B = \langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Proof Call the outcome of the exchange $\hat{B} = \langle\vec{\beta}_1, \ldots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \ldots, \vec{\beta}_n\rangle$.

We first show that $\hat{B}$ is linearly independent. Any relationship $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n = \vec{0}$ among the members of $\hat{B}$, after substitution for $\vec{v}$,

$$d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_i\vec{\beta}_i + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n = \vec{0} \quad (*)$$

gives a linear relationship among the members of B. The basis B is linearly independent so the coefficient $d_i c_i$ of $\vec{\beta}_i$ is zero. Because we assumed that c_i is nonzero, $d_i = 0$. Using this in equation $(*)$ gives that all of the other d's are also zero. Therefore $\hat{B}$ is linearly independent.

We finish by showing that $\hat{B}$ has the same span as B. Half of this argument, that $[\hat{B}] \subseteq [B]$, is easy; we can write any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n$ of $[\hat{B}]$ as $d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n$, which is a linear combination of linear combinations of members of B, and hence is in $[B]$. For the $[B] \subseteq [\hat{B}]$ half of the argument, recall that if $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ with $c_i \neq 0$ then we can rearrange the equation to $\vec{\beta}_i = (-c_1/c_i)\vec{\beta}_1 + \cdots + (1/c_i)\vec{v} + \cdots + (-c_n/c_i)\vec{\beta}_n$. Now, consider any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{\beta}_i + \cdots + d_n\vec{\beta}_n$ of $[B]$, substitute for $\vec{\beta}_i$ its expression as a linear combination of the members of $\hat{B}$, and recognize, as in the first half of this argument, that the result is a linear combination of linear combinations of members of $\hat{B}$, and hence is in $[\hat{B}]$. QED

All of a space's bases are the same size

2.5 *Theorem* In any finite-dimensional vector space, all bases have the same number of elements.

All of a space's bases are the same size

2.5 *Theorem* In any finite-dimensional vector space, all bases have the same number of elements.

Proof Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

All of a space's bases are the same size

2.5 *Theorem* In any finite-dimensional vector space, all bases have the same number of elements.

Proof Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

The basis B spans the space and $\vec{\delta}_1$ is in the space, so $\vec{\delta}_1$ is a nontrivial linear combination of elements of B. By the Exchange Lemma, we can swap $\vec{\delta}_1$ for a vector from B, resulting in a basis B_1, where one element is $\vec{\delta}_1$ and all of the $n-1$ other elements are $\vec{\beta}$'s.

The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leqslant k < n$) containing k members of D and $n-k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leqslant k < n$) containing k members of D and $n-k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

Repeat that until no $\vec{\beta}$'s remain, so that B_n contains $\vec{\delta}_1, \ldots, \vec{\delta}_n$. Now, D cannot have more than these n vectors because any $\vec{\delta}_{n+1}$ that remains would be in the span of B_n (since it is a basis) and hence would be a linear combination of the other $\vec{\delta}$'s, contradicting that D is linearly independent. QED

Example Each of these is a basis for $\mathcal{P}_2$.

$$B_0 = \langle 1, 1+x, 1+x+x^2 \rangle$$
$$B_1 = \langle 1+x+x^2, 1+x, 1 \rangle$$
$$B_2 = \langle x^2, 1+x, 1-x \rangle$$
$$B_3 = \langle 1, x, x^2 \rangle$$

Each has two elements.

Example Each of these is a basis for $\mathcal{P}_2$.

$$
\begin{aligned}
B_0 &= \langle 1, 1+x, 1+x+x^2 \rangle \\
B_1 &= \langle 1+x+x^2, 1+x, 1 \rangle \\
B_2 &= \langle x^2, 1+x, 1-x \rangle \\
B_3 &= \langle 1, x, x^2 \rangle
\end{aligned}
$$

Each has two elements.

Example Here are two different bases for $\mathcal{M}_{2\times 2}$.

$$
\begin{aligned}
B_0 &= \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle \\
B_1 &= \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle
\end{aligned}
$$

Each has four elements.

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Example The vector space $\mathcal{P}_2$ has dimension 3 because one of its bases is $\langle 1, x, x^2 \rangle$.

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Example The vector space $\mathcal{P}_2$ has dimension 3 because one of its bases is $\langle 1, x, x^2 \rangle$. More generally, $\mathcal{P}_n$ has dimension $n+1$.

Definition of dimension

2.6 *Definition* The *dimension* of a vector space is the number of vectors in any of its bases.

Example The vector space $\mathbb{R}^n$ has dimension n because that is how many members are in $\mathcal{E}_n$.

Example The vector space $\mathcal{P}_2$ has dimension 3 because one of its bases is $\langle 1, x, x^2 \rangle$. More generally, $\mathcal{P}_n$ has dimension $n+1$.

Example The vector space $\mathcal{M}_{n\times m}$ has dimension nm. A natural basis consists of matrices with a single 1 and the other entries 0's.

Example The solution set of this system

$$\begin{aligned} x - y + z &= 0 \\ -x + 2y - z + 2w &= 0 \\ -x + 3y - z + 4w &= 0 \end{aligned}$$

is a vector space (this is easy to check for any homogeneous system). Solving the system

$$\left(\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 2 & 0 \\ 1 & 3 & -1 & 4 & 0 \end{array}\right) \xrightarrow[\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and parametrizing

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\}$$

gives a basis for the space, a sequence of those two vectors.

2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

2.13 *Corollary* Any linearly independent set can be expanded to make a basis.

2.11 *Corollary* No linearly independent set can have a size greater than the dimension of the enclosing space.

Proof The proof of Theorem 2.5 never uses that D spans the space, only that it is linearly independent. QED

Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

2.13 *Corollary* Any linearly independent set can be expanded to make a basis.

Proof If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence. Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps. QED

2.14 *Corollary* Any spanning set can be shrunk to a basis.

2.14 *Corollary* Any spanning set can be shrunk to a basis.

Proof Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s}_2 \rangle$; if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps. QED

2.15 *Corollary* In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

2.15 *Corollary* In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

Proof First we will show that a subset with n vectors is linearly independent if and only if it is a basis. The 'if' is trivially true — bases are linearly independent. 'Only if' holds because a linearly independent set can be expanded to a basis, but a basis has n elements, so this expansion is actually the set that we began with.

To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with.

QED

Vector Spaces and Linear Systems

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 *Lemma* If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 *Lemma* If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

Proof Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then $\text{Rowspace}(B) \subseteq \text{Rowspace}(A)$ follows because a member of the set $\text{Rowspace}(B)$ is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of $\text{Rowspace}(A)$.

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$ follows as in the previous paragraph.

QED

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$ follows as in the previous paragraph.

QED

3.3 *Lemma* The nonzero rows of an echelon form matrix make up a linearly independent set.

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$ follows as in the previous paragraph.

QED

3.3 *Lemma* The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result just restates that in this chapter's terminology. QED

Example The matrix before Gauss's Method and the matrix after have equal row spaces.

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ -1 & -2 & 2 & 2 & 0 \\ 2 & 4 & 5 & 2 & 9 \end{pmatrix} \underset{-2\rho_1+\rho_3}{\xrightarrow{\rho_1+\rho_2}} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of the latter matrix form a basis for Rowspace(M).

$$B = \langle \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 & 2 & 3 \end{pmatrix} \rangle$$

The row rank is 2.

So Gauss's Method produces a basis for the row space of a matrix. It has found the "repeat" information, that M's third row is three times the first plus the second, and eliminated that extra row.

Column space

3.6 *Definition* The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Column space

3.6 *Definition* The *column space* of a matrix is the span of the set of its columns. The *column rank* is the dimension of the column space, the number of linearly independent columns.

Example This system

$$\begin{aligned} 2x + \quad 3y &= d_1 \\ -x + (1/2)y &= d_2 \end{aligned}$$

has a solution for those $d_1, d_2 \in \mathbb{R}$ that we can find to satisfy this vector equation.

$$x \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad x, y \in \mathbb{R}$$

That is, the system has a solution if and only if the vector on the right is in the column space of this matrix.

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

Transpose

3.8 *Definition* The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.

Transpose

3.8 *Definition* The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.

Example To find a basis for the column space of a matrix

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

we can temporarily transpose

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix}$$

linearly reduce

$$\begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix} \xrightarrow{(-3/2)\rho_1+\rho_2} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

and transpose back.

$$\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$$

This basis

$$B = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

shows that the column space is the entire vector space $\mathbb{R}^2$.

3.10 *Lemma* Row operations do not change the column rank.

3.10 *Lemma* Row operations do not change the column rank.

Proof Restated, if A reduces to B then the column rank of B equals the column rank of A.

This proof will be finished if we show that row operations do not affect linear relationships among columns, because the column rank is the size of the largest set of unrelated columns. That is, we will show that a relationship exists among columns (such as that the fifth column is twice the second plus the fourth) if and only if that relationship exists after the row operation. But this is exactly the first theorem of this book, Theorem One.I.1.5 : in a relationship among columns,

$$c_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \cdot \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

row operations leave unchanged the set of solutions $(c_1, \ldots, c_n)$.

QED

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

Proof Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

Proof Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

Proof Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal. QED

3.12 *Definition* The *rank* of a matrix is its row rank or column rank.

Example The column rank of this matrix

$$\begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

is 3. Its largest set of linearly independent columns is size 3 because that's the size of its largest set of linearly independent rows.

$$\xrightarrow[\substack{-2\rho_1+\rho_3 \\ -(1/2)\rho_1+\rho_4}]{-(3/2)\rho_1+\rho_2} \xrightarrow{-(1/3)\rho_2+\rho_4} \xrightarrow{\rho_3\leftrightarrow\rho_4} \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 0 & 3/2 & -7/2 & -1/2 & 4 & -5/2 \\ 0 & 0 & 8/3 & -1/3 & -4/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.13 *Theorem* For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

3.13 *Theorem* For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r

(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

are equivalent.

Proof The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables. QED

3.14 *Corollary* Where the matrix A is $n \times n$, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

3.14 *Corollary* Where the matrix A is $n \times n$, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

Proof Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis. QED