Five.II Similarity

Linear Algebra

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We've defined two matrices H and $\hat{H}$ to be matrix equivalent if there are nonsingular P and Q such that $\hat{H} = PHQ$. We were motivated by this diagram showing H and $\hat{H}$ both representing a map h, but with respect to different pairs of bases, B, D and $\hat{B}, \hat{D}$.

$$\begin{tikzcd}
V_{\text{wrt } B} \arrow[r, "h"', "H"] \arrow[d, "\text{id}"'] & W_{\text{wrt } D} \arrow[d, "\text{id}"'] \\
V_{\text{wrt } \hat{B}} \arrow[r, "h", "\hat{H}"'] & W_{\text{wrt } \hat{D}}
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$$\begin{array}{ccc} V_{\textit{wrt}\ B} & \xrightarrow[H]{h} & W_{\textit{wrt}\ D} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\textit{wrt}\ \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\textit{wrt}\ \hat{D}} \end{array}$$

We now consider the special case of transformations, where the codomain equals the domain, and we add the requirement that the codomain's basis equals the domain's basis. So, we are considering representations with respect to B, B and D, D.

$$\begin{array}{ccc} V_{\textit{wrt}\ B} & \xrightarrow[T]{t} & V_{\textit{wrt}\ B} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\textit{wrt}\ D} & \xrightarrow[\hat{T}]{t} & V_{\textit{wrt}\ D} \end{array}$$

In matrix terms, $\mathrm{Rep}_{D,D}(t) = \mathrm{Rep}_{B,D}(\mathrm{id})\ \mathrm{Rep}_{B,B}(t)\ \big(\mathrm{Rep}_{B,D}(\mathrm{id})\big)^{-1}$.

Definition and Examples

Similar matrices

1.1 *Definition* The matrices T and $\hat{T}$ are *similar* if there is a nonsingular P such that $\hat{T} = PTP^{-1}$.

Example Consider the derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$. Fix the basis $B = \langle 1, x, x^2 \rangle$ and the basis $D = \langle 1, 1+x, 1+x+x^2 \rangle$. In this arrow diagram we will first get T, and then calculate $\hat{T}$ from it.

$$\begin{tikzcd}
V_{\mathit{wrt}\ B} \arrow[r, "t", "T"'] \arrow[d, "\mathrm{id}"'] & V_{\mathit{wrt}\ B} \arrow[d, "\mathrm{id}"'] \\
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$$\begin{array}{ccc} V_{wrt\ B} & \xrightarrow[T]{t} & V_{wrt\ B} \\ {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow \\ V_{wrt\ D} & \xrightarrow[\hat{T}]{t} & V_{wrt\ D} \end{array}$$

The action of d/dx on the elements of the basis B is $1 \mapsto 0$, $x \mapsto 1$, and $x^2 \mapsto 2x$.

$$\mathrm{Rep}_B(\frac{d}{dx}(1)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(\frac{d}{dx}(x)) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(\frac{d}{dx}(x^2)) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

So we have this matrix representation of the map.

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The matrix changing bases from B to D is $\mathrm{Rep}_{B,D}(\mathrm{id})$. We find these by eye

$$\mathrm{Rep}_D(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{Rep}_D(x) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \mathrm{Rep}_D(x^2) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

to get this.

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, by following the arrow diagram we have $\hat{T} = PTP^{-1}$.

$$\hat{T} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

To check that, and to underline what the arrow diagram says,

$$\begin{tikzcd}
V_{\mathit{wrt}\ B} \arrow[r, "t", "T"'] \arrow[d, "\mathrm{id}"'] & V_{\mathit{wrt}\ B} \arrow[d, "\mathrm{id}"'] \\
V_{\mathit{wrt}\ D} \arrow[r, "t", "\hat{T}"'] & V_{\mathit{wrt}\ D}
\end{tikzcd}$$

we calculating T directly. The effect of the map on the basis elements is $d/dx(1) = 0$, $d/dx(1+x) = 1$, and $d/dx(1+x+x^2) = 1+2x$. Representing of those with respect to D

$$\mathrm{Rep}_D(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_D(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_D(1+2x) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

gives the same matrix $\hat{T} = \mathrm{Rep}_{D,D}(d/dx)$ as we found above.

The definition doesn't require that we consider the underlying maps. We can just multiply matrices.

Example Where

$$T = \begin{pmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 4 & 5 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(note that P is nonsingular) we can compute this $\hat{T} = PTP^{-1}$.

$$\hat{T} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 4/3 \\ 27/2 & 3/2 & 2 \end{pmatrix}$$

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$$\hat{T} = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 4/3 \\ 27/2 & 3/2 & 2 \end{pmatrix}$$

1.3 *Example* The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The identity matrix has the same property: $PIP^{-1} = PP^{-1} = I$.

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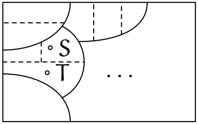
So some matrix equivalence classes split into two or more similarity classes — similarity gives a finer partition than does equivalence. This pictures some matrix equivalence classes subdivided into similarity classes.



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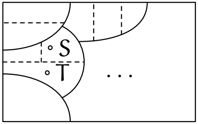

We naturally want a canonical form to represent the similarity classes. Some classes, but not all, are represented by a diagonal form.

Diagonalizability

2.1 *Definition* A transformation is *diagonalizable* if it has a diagonal representation with respect to the same basis for the codomain as for the domain. A *diagonalizable matrix* is one that is similar to a diagonal matrix: T is diagonalizable if there is a nonsingular P such that PTP^{-1} is diagonal.

Example This matrix

$$\begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix}$$

is diagonalizable by using this

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \\ -1/2 & -3/4 & 1/4 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

to get this $D = PSP^{-1}$.

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Example We will show that this matrix is not diagonalizable.

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The fact that N is not the zero matrix means that it cannot be similar to the zero matrix, because the zero matrix is similar only to itself. Thus if N were to be similar to a diagonal matrix then that matrix would have have at least one nonzero entry on its diagonal.

The square of N is the zero matrix. This imples that for any map n represented by N (with respect to some B, B) the composition $n \circ n$ is the zero map. This in turn implies that for any matrix representing n (with respect to some $\hat{B}, \hat{B}$), its square is the zero matrix. But the square of a nonzero diagonal matrix cannot be the zero matrix, because the square of a diagonal matrix is the diagonal matrix whose entries are the squares of the entries from the starting matrix. Thus there is no $\hat{B}, \hat{B}$ such that n is represented by a diagonal matrix — the matrix N is not diagonalizable.

2.4 *Lemma* A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

2.4 *Lemma* A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

Proof Consider a diagonal representation matrix.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_B(t(\vec{\beta}_1)) & \cdots & \mathrm{Rep}_B(t(\vec{\beta}_n)) \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}$$

Consider the representation of a member of this basis with respect to the basis $\mathrm{Rep}_B(\vec{\beta}_i)$. The product of the diagonal matrix and the representation vector

$$\mathrm{Rep}_B(t(\vec{\beta}_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix}$$

has the stated action. QED

Example This matrix is not diagonal because of the nonzero entry in the upper right.

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Suppose $T = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ for $t\colon \mathbb{R}^2 \to \mathbb{R}^2$. We will find a basis $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ giving a diagonal representation.

$$D = \mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Here is the arrow diagram.

$$\begin{array}{ccc} V_{\mathit{wrt}\ \mathcal{E}_2} & \xrightarrow[T]{t} & V_{\mathit{wrt}\ \mathcal{E}_2} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\mathit{wrt}\ B} & \xrightarrow[D]{t} & V_{\mathit{wrt}\ B} \end{array}$$

We want λ_1 and λ_2 making these true.

$$\begin{pmatrix} 4 & 1 \\ 0 & -1 \end{pmatrix} \vec{\beta}_1 = \lambda_1 \cdot \vec{\beta}_1 \qquad \begin{pmatrix} 4 & 1 \\ 0 & -1 \end{pmatrix} \vec{\beta}_2 = \lambda_2 \cdot \vec{\beta}_2$$

More precisely, we want all scalars $x \in \mathbb{C}$ such that this system

$$\begin{pmatrix} 4 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

has solutions $b_1, b_2 \in \mathbb{C}$ that are not both zero (the zero vector is not an element of any basis).

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Rewrite that as a linear system.

$$\begin{aligned} (4 - x) \cdot b_1 + \qquad\qquad b_2 &= 0 \\ (-1 - x) \cdot b_2 &= 0 \end{aligned}$$

One solution is $\lambda_1 = -1$, associated with those (b_1, b_2) such that $b_1 = (-1/5)b_2$. The other solution is $\lambda_2 = 4$, associated with the (b_1, b_2) such that $b_2 = 0$.

Thus the original matrix

$$T = \begin{pmatrix} 4 & 1 \\ 0 & -1 \end{pmatrix}$$

is diagonalizable to

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

where this is a basis.

$$B = \langle \begin{pmatrix} -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle$$

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

3.1 *Definition* A transformation $t\colon V \to V$ has a scalar *eigenvalue* λ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

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3.6 *Definition* A square matrix T has a scalar *eigenvalue* λ associated with the nonzero *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

Example The matrix

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

has one eigenvalue $\lambda_1 = 4$ and a second eigenvalue $\lambda_2 = 2$. The first is true because an associated eigenvector is $\vec{e_1}$

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and similarly for the second an associated eigenvector is e_2.

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thinking of the matrix as representing a transformation of the plane, the transformation acts on those vectors in a particularly simple way, by rescaling.

Not every vector is simply rescaled.

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq x \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Matrices that are similar have the same eigenvalues, but needn't have the same eigenvectors.

Example These two are similar

$$T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix} \qquad S = \begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix}$$

since $S = PTP^{-1}$ for this P.

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \\ -1/2 & -3/4 & 1/4 \end{pmatrix}$$

Suppose that $t\colon \mathbb{C}^3 \to \mathbb{C}^3$ is represented by T with respect to the standard basis. Then this is the action of t.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 4x \\ 8y \\ 12z \end{pmatrix}$$

Suppose that $t\colon \mathbb{C}^3 \to \mathbb{C}^3$ is represented by T with respect to the standard basis. Then this is the action of t.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 4x \\ 8y \\ 12z \end{pmatrix}$$

By eye we see that the eigenvalues of t are $\lambda_1 = 4$, $\lambda_2 = 8$, $\lambda_3 = 12$; for instance this holds.

$$T \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Contrast that with S. Of course, $S = PTP^{-1}$ represents the same function, but with respect to a different basis.

$$\begin{tikzcd}
V_{\mathit{wrt}\ \mathcal{E}_3} \arrow[r, "t"', "T"] \arrow[d, "\mathrm{id}"'] & V_{\mathit{wrt}\ \mathcal{E}_3} \arrow[d, "\mathrm{id}"'] \\
V_{\mathit{wrt}\ B} \arrow[r, "t", "S"'] & V_{\mathit{wrt}\ B}
\end{tikzcd}$$

We can easily find the basis B. Since $P^{-1} = \mathrm{Rep}_{B,\mathcal{E}_3}(\mathrm{id})$, its first column is $\mathrm{Rep}_{\mathcal{E}_3}(\mathrm{id}(\vec{\beta}_1)) = \mathrm{Rep}_{\mathcal{E}_3}(\vec{\beta}_1)$. With respect to the standard basis any vector is represented by itself so the first basis element $\vec{\beta}_1$ is the first column of P^{-1}. The same goes for the other two columns.

$$B = \langle \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \\ -3/4 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \rangle$$

Now, since each represents the transformation t, the matrices T and S reflect the same action $\vec{e}_1 \mapsto 4\vec{e}_1$.

$$\begin{aligned}
\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(t) \cdot \mathrm{Rep}_{\mathcal{E}_3}(\vec{e}_1) &= T \cdot \mathrm{Rep}_{\mathcal{E}_3}(\vec{e}_1) = 4 \cdot \mathrm{Rep}_{\mathcal{E}_3}(\vec{e}_1) \\
\mathrm{Rep}_{B,B}(t) \cdot \mathrm{Rep}_{B}(\vec{e}_1) &= S \cdot \mathrm{Rep}_{B}(\vec{e}_1) = 4 \cdot \mathrm{Rep}_{B}(\vec{e}_1)
\end{aligned}$$

But, while in the two equations the 4's are the same, the vectors representations are not.

$$\begin{aligned}
T \cdot \mathrm{Rep}_{\mathcal{E}_3}(\vec{e}_1) = T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
S \cdot \mathrm{Rep}_{B}(\vec{e}_1) = S \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} &= 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}
\end{aligned}$$

So the two matrices have the same eigenvalues but different eigenvectors.

Computing eigenvalues and eigenvectors

Example We will find the eigenvalues and associated eigenvectors of this matrix.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

We want to find scalars x such that $T\vec{\zeta} = x\vec{\zeta}$ for some nonzero $\vec{\zeta}$. Bring the terms to the left side.

$$\begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - x \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and factor.

$$\begin{pmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

This homogeneous system has nonzero solutions if and only if the matrix is nonsingular, that is, has a determinant of zero.

Some computation gives the determinant and its factors.

$$
\begin{aligned}
0 &= \begin{vmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{vmatrix} \\
&= x^3 - 11x^2 + 38x - 40 = (x-5)(x-4)(x-2)
\end{aligned}
$$

So the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, and $\lambda_3 = 2$.

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\end{aligned}$$

So the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, and $\lambda_3 = 2$.

To find the eigenvectors associated with the eigenvalue of 5 specialize equation $(*)$ for $x = 5$.

$$\begin{pmatrix} -5 & 5 & 7 \\ -2 & 2 & 7 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this solution set; its nonzero elements are the eigenvectors.

$$V_5 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

Similarly, to find the eigenvectors associated with the eigenvalue of 4 specialize equation $(*)$ for $x = 4$.

$$\begin{pmatrix} -4 & 5 & 7 \\ -2 & 3 & 7 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this.

$$V_4 = \{ \begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

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Specializing $(*)$ for $x = 2$

$$\begin{pmatrix} -2 & 5 & 7 \\ -2 & 5 & 7 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives this.

$$V_2 = \{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

Example To find the eigenvalues and associated eigenvectors for the matrix

$$T = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

start with this equation.

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 3-x & 1 \\ 1 & 3-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (*)$$

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That system has a nontrivial solution if this determinant is nonzero.

$$\begin{vmatrix} 3-x & 1 \\ 1 & 3-x \end{vmatrix} = x^2 - 6x + 8 = (x-2)(x-4)$$

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$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = x \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} 3-x & 1 \\ 1 & 3-x \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (*)$$

That system has a nontrivial solution if this determinant is nonzero.

$$\begin{vmatrix} 3-x & 1 \\ 1 & 3-x \end{vmatrix} = x^2 - 6x + 8 = (x-2)(x-4)$$

First take the $x = 2$ version of $(*)$.

$$\begin{aligned} 1 \cdot b_1 + \phantom{1 \cdot{}} b_2 &= 0 \\ b_1 + 1 \cdot b_2 &= 0 \end{aligned} \quad \Longrightarrow \quad V_2 = \{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mid b_1 = -b_2 \text{ where } b_2 \in \mathbb{C} \}$$

Solving the second system is just as easy.

$$\begin{aligned} -1 \cdot b_1 + \phantom{1 \cdot{}} b_2 &= 0 \\ b_1 - 1 \cdot b_2 &= 0 \end{aligned} \quad \Longrightarrow \quad V_4 = \{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \mid b_1 = b_2 \text{ where } b_2 \in \mathbb{C} \}$$

Example If the matrix is upper diagonal or lower diagonal

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

then the polynomial is easy to factor.

$$0 = \begin{vmatrix} 2-x & 1 & 0 \\ 0 & 3-x & 1 \\ 0 & 0 & 2-x \end{vmatrix} = (3-x)(2-x)^2$$

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These are the solutions for $\lambda_1 = 3$.

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad V_3 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

These are for $\lambda_2 = 2$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad V_2 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z_1 \mid z_1 \in \mathbb{C} \}$$

Characteristic polynomial

3.11 *Definition* The *characteristic polynomial of a square matrix* T is the determinant $|T - xI|$ where x is a variable. The *characteristic equation* is $|T - xI| = 0$. The *characteristic polynomial of a transformation* t is the characteristic polynomial of any matrix representation $\text{Rep}_{B,B}(t)$.

Note Exercise 32 checks that the characteristic polynomial of a transformation is well-defined, that is, that the characteristic polynomial is the same no matter which basis we use for the representation.

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Remark This result is why we switched in this chapter from working with real number scalars to complex number scalars.

Eigenspace

3.14 *Definition* The *eigenspace of a transformation t associated with the eigenvalue λ* is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}) = \lambda\vec{\zeta}\}$. The eigenspace of a matrix is analogous.

Example Recall that this matrix has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Earlier, we found that these are the eigenspaces.

$$V_5 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c \mid c \in \mathbb{C}\} \quad V_4 = \{ \begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\} \quad V_2 = \{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\}$$

3.15 *Lemma* An eigenspace is a subspace.

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Proof Fix an eigenvalue λ. Notice first that V_λ contains the zero vector since $t(\vec{0})=\vec{0}$, which equals $\lambda\vec{0}$. So the eigenspace is a nonempty subset of the space. What remains is to check closure of this set under linear combinations. Take $\vec{\zeta}_1,\ldots,\vec{\zeta}_n\in V_\lambda$ and then verify

$$\begin{aligned}
t(c_1\vec{\zeta}_1+c_2\vec{\zeta}_2+\cdots+c_n\vec{\zeta}_n) &= c_1t(\vec{\zeta}_1)+\cdots+c_nt(\vec{\zeta}_n)\\
&= c_1\lambda\vec{\zeta}_1+\cdots+c_n\lambda\vec{\zeta}_n\\
&= \lambda(c_1\vec{\zeta}_1+\cdots+c_n\vec{\zeta}_n)
\end{aligned}$$

that the combination is also an element of V_λ. QED

3.19 *Theorem* For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

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Proof We will use induction on the number of eigenvalues. The base step is that there are zero eigenvalues. Then the set of associated vectors is empty and so is linearly independent.

For the inductive step assume that the statement is true for any set of $k \geqslant 0$ distinct eigenvalues. Consider distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ and let $\vec{v}_1, \ldots, \vec{v}_{k+1}$ be associated eigenvectors. Suppose that $\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}$. Derive two equations from that, the first by multiplying by λ_{k+1} on both sides $\vec{0} = c_1\lambda_{k+1}\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ and the second by applying the map to both sides $\vec{0} = c_1 t(\vec{v}_1) + \cdots + c_{k+1} t(\vec{v}_{k+1}) = c_1\lambda_1\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ (applying the matrix gives the same result). Subtract the second from the first.

$$\vec{0} = c_1(\lambda_{k+1} - \lambda_1)\vec{v}_1 + \cdots + c_k(\lambda_{k+1} - \lambda_k)\vec{v}_k + c_{k+1}(\lambda_{k+1} - \lambda_{k+1})\vec{v}_{k+1}$$

The $\vec{v}_{k+1}$ term vanishes. Then the induction hypothesis gives that $c_1(\lambda_{k+1} - \lambda_1) = 0, \ldots, c_k(\lambda_{k+1} - \lambda_k) = 0$. The eigenvalues are distinct so the coefficients $c_1, \ldots, c_k$ are all 0. With that we are left with the equation $\vec{0} = c_{k+1}\vec{v}_{k+1}$ so c_{k+1} is also 0. QED

Example This matrix has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Picking a nonzero vector from each eigenspace we get this linearly independent set (which is a basis because it has three elements).

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -14 \\ -14 \\ 2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \}$$

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Example This upper-triangluar matrix has the eigenvalues 2 and 3

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Picking a vector from each of V_3 and V_2 gives this linearly independent set.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \}$$

3.21 *Corollary* An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof Form a basis of eigenvectors. Apply Lemma 2.4 . QED