

# Three.I Isomorphisms

*Linear Algebra*

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## Definition

*Example* We have the intuition that the vector spaces  $\mathbb{R}^2$  and  $\mathcal{P}_1$  are “the same,” for instance in that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ is just like } 1 + 2x$$
$$\text{and } \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} \text{ is just like } -3 - (1/2)x$$

etc. What makes the spaces “just like” each other is that this association holds through the operations of addition

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5/2 \end{pmatrix}$$
$$\text{is just like } (1 + 2x) + (-3 + (1/2)x) = -2 + (5/2)x$$

and scalar multiplication.

$$3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ is just like } 3(1 + 2x) = 3 + 6x$$

More formally, we can associate each two-tall vector with a linear polynomial.

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Note that this association holds through the vector space operations of addition

$$\begin{aligned} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} \\ \longleftrightarrow (a_1 + b_1x) + (a_2 + b_2x) &= (a_1 + a_2) + (b_1 + b_2)x \end{aligned}$$

and scalar multiplication.

$$r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \longleftrightarrow r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

*Example* We can think of  $\mathcal{M}_{2 \times 2}$  as “the same” as  $\mathbb{R}^4$  if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

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With the association defined, note that it holds up under addition.

$$\begin{aligned} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \\ &\longleftrightarrow \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix} \end{aligned}$$

Here is an example of that with particular elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -5 \end{pmatrix}$$



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$$\longleftrightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -5 \end{pmatrix}$$

The association also holds under scalar multiplication.

$$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \longleftrightarrow r \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \\ rd \end{pmatrix}$$

This illustrates.

$$2 \cdot \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \longleftrightarrow 2 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}$$

# Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces  $V$  and  $W$  is a map  $f: V \rightarrow W$  that

- 1) is a correspondence:  $f$  is one-to-one and onto;
- 2) *preserves structure*: if  $\vec{v}_1, \vec{v}_2 \in V$  then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if  $\vec{v} \in V$  and  $r \in \mathbb{R}$  then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write  $V \cong W$ , read “ $V$  is isomorphic to  $W$ ”, when such a map exists).

*Example* The space of quadratic polynomials  $\mathcal{P}_2$  is isomorphic to  $\mathbb{R}^3$  under this map.

$$f(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Here are two examples of the action of  $f$ .

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that  $f$  is an isomorphism we must check condition (1), that  $f$  is a correspondence, and condition (2), that  $f$  preserves structure.

The first part of (1) is that  $f$  is one-to-one. We usually verify one-to-oneness by assuming that the function yields the same output on two inputs, and then show that the two inputs must therefore be equal. So assume that  $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$ . By definition of  $f$  we have

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal  $a_0 = b_0$ ,  $a_1 = b_1$ , and  $a_2 = b_2$ . Thus the starting inputs are equal  $a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$  and so  $f$  is one-to-one.

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The second part of (1) is that  $f$  is onto. We usually verify ontoeness by considering an element of the codomain and producing an element of the domain that maps to it. So consider this member of  $\mathbb{R}^3$ .

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Observe that it is the image under  $f$  of the member  $u + vx + wx^2$  of the domain. Thus  $f$  is onto.

Condition (2) also has two halves. First we must show that  $f$  preserves addition. Consider  $f$  acting on the sum of two elements of the domain.

$$\begin{aligned} f((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)) \\ = f((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2) \end{aligned}$$

By definition of  $f$  we have this.

$$= \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

Of course,

$$= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

which gives

$$= f(a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2)$$

as required.

We finish by checking that  $f$  preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned} r \cdot f(a_0 + a_1x + a_2x^2) &= r \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\ &= f(ra_0 + (ra_1)x + (ra_2)x^2) \end{aligned}$$

QED

## Special case: Automorphisms

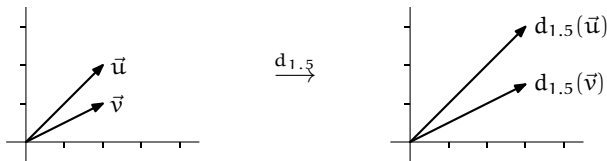
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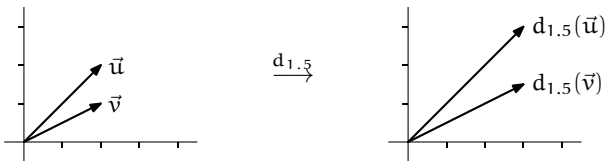
1.8 *Example* A *dilation* map  $d_s: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that multiplies all vectors by a nonzero scalar  $s$  is an automorphism of  $\mathbb{R}^2$ .



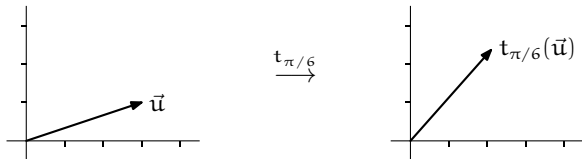
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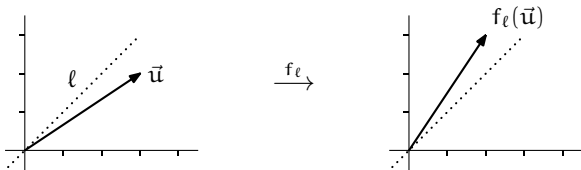
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Another automorphism is a *rotation* or *turning map*,  $t_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that rotates all vectors through an angle  $\theta$ .



A third type of automorphism of  $\mathbb{R}^2$  is a map  $f_\ell: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that *flips* or *reflects* all vectors over a line  $\ell$  through the origin.



Checking that each of these is an isomorphism is an exercise.

1.10 *Lemma*    An isomorphism maps a zero vector to a zero vector.

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*Proof*    Where  $f: V \rightarrow W$  is an isomorphism, fix some  $\vec{v} \in V$ . Then  
 $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$ . QED

1.11 *Lemma* For any map  $f: V \rightarrow W$  between vector spaces these statements are equivalent.

(1)  $f$  preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c f(\vec{v})$$

(2)  $f$  preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3)  $f$  preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

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*Proof* Since the implications  $(3) \implies (2)$  and  $(2) \implies (1)$  are clear, we need only show that  $(1) \implies (3)$ . So assume statement (1). We will prove (3) by induction on the number of summands  $n$ .

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The one-summand base case, that  $f(c\vec{v}_1) = c f(\vec{v}_1)$ , is covered by the second clause of statement (1).



For the inductive step assume that statement (3) holds whenever there are  $k$  or fewer summands. Consider the  $k + 1$ -summand case. Use the first half of (1) to break the sum along the final '+'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the  $k$ -term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Now the second half of (1) gives

$$= c_1 f(\vec{v}_1) + \cdots + c_k f(\vec{v}_k) + c_{k+1} f(\vec{v}_{k+1})$$

when applied  $k + 1$  times.

QED

This result eases checking that a function preserves the structure of a vector space, since we can do it in one step with statement (2).

*Example* This line through the origin

$$L = \{t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R}\}$$

is a vector space under the addition and scalar multiplication operations that it inherits from  $\mathbb{R}^2$ .

$$\begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix} + \begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ 2(t_1 + t_2) \end{pmatrix} \quad r \cdot \begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} rt \\ 2rt \end{pmatrix}$$

We will verify that the map below is an isomorphism between  $L$  and  $\mathbb{R}^1$ .

$$f\left(\begin{pmatrix} t \\ 2t \end{pmatrix}\right) = f\left(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = t$$

We first verify that  $f$  is one-to-one. Suppose that  $f$  maps two members of  $L$  to the same output.

$$f\left(t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = f\left(t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

By the definition of  $f$  we have that  $t_1 = t_2$  and so the two members of  $L$  are equal.

Next we check that  $f$  is onto. Consider this member of the codomain:  $r \in \mathbb{R}$ . There is a member of the domain that maps to it, namely this member of  $L$ .

$$f\left(r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

To finish we check that  $f$  preserves structure with the lemma's (2).

$$f\left(t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = f\left((t_1 + t_2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = t_1 + t_2 = f\left(t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + f\left(t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

Dimension characterizes isomorphism

2.1 *Lemma*    The inverse of an isomorphism is also an isomorphism.

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*Proof*    Suppose that  $V$  is isomorphic to  $W$  via  $f: V \rightarrow W$ . An isomorphism is a correspondence between the sets so  $f$  has an inverse function  $f^{-1}: W \rightarrow V$  that is also a correspondence.

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

*Proof* Suppose that  $V$  is isomorphic to  $W$  via  $f: V \rightarrow W$ . An isomorphism is a correspondence between the sets so  $f$  has an inverse function  $f^{-1}: W \rightarrow V$  that is also a correspondence.

We will show that because  $f$  preserves linear combinations, so also does  $f^{-1}$ . Suppose that  $\vec{w}_1, \vec{w}_2 \in W$ . Because it is an isomorphism,  $f$  is onto and there are  $\vec{v}_1, \vec{v}_2 \in V$  such that  $\vec{w}_1 = f(\vec{v}_1)$  and  $\vec{w}_2 = f(\vec{v}_2)$ . Then

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)) \\ &= f^{-1}(f(c_1 \vec{v}_1 + c_2 \vec{v}_2)) = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

since  $f^{-1}(\vec{w}_1) = \vec{v}_1$  and  $f^{-1}(\vec{w}_2) = \vec{v}_2$ . With that, by Lemma 1.11's second statement, this map preserves structure. QED

*Example* We saw earlier that this planar line through the origin

$$L = \{t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R}\}$$

(under the natural operations) is isomorphic to  $\mathbb{R}^1$  via this function.

$$f\left(\begin{pmatrix} t \\ 2t \end{pmatrix}\right) = f\left(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = t$$

The inverse  $f^{-1}: \mathbb{R} \rightarrow L$  given by

$$f^{-1}(x) = x \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$$

is also an isomorphism.



2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

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*Proof* We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This shows that it preserves linear combinations.

$$\text{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \cdot \text{id}(\vec{v}_1) + c_2 \cdot \text{id}(\vec{v}_2)$$

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Symmetry, that if  $V$  is isomorphic to  $W$  then also  $W$  is isomorphic to  $V$ , holds by Lemma 2.1 since each isomorphism map from  $V$  to  $W$  is paired with an isomorphism from  $W$  to  $V$ .

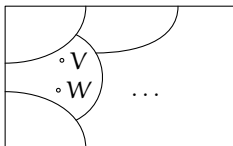
To finish we must check transitivity, that if  $V$  is isomorphic to  $W$  and  $W$  is isomorphic to  $U$  then  $V$  is isomorphic to  $U$ . Let  $f: V \rightarrow W$  and  $g: W \rightarrow U$  be isomorphisms. Consider their composition  $g \circ f: V \rightarrow U$ . Because the composition of correspondences is a correspondence, we need only check that the composition preserves linear combinations.

$$\begin{aligned} g \circ f (c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) &= g( f( c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 ) ) \\ &= g( c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2) ) \\ &= c_1 \cdot g(f(\vec{v}_1)) + c_2 \cdot g(f(\vec{v}_2)) \\ &= c_1 \cdot (g \circ f) (\vec{v}_1) + c_2 \cdot (g \circ f) (\vec{v}_2) \end{aligned}$$

Thus the composition is an isomorphism.

QED

The prior result tells us that the collection of all finite-dimensional vector spaces is partitioned into classes. Two spaces are in the same class if they are isomorphic.



The next result characterizes these classes.

2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is the next two lemmas.

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2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

*Proof* We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, we shall show that if  $f: V \rightarrow W$  is an isomorphism and a basis for the domain  $V$  is  $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$  then its image  $D = \langle f(\vec{\beta}_1), \dots, f(\vec{\beta}_n) \rangle$  is a basis for the codomain  $W$ . (The other half of the correspondence, that for any basis of  $W$  the inverse image is a basis for  $V$ , follows from the fact that  $f^{-1}$  is also an isomorphism and so we can apply the prior sentence to  $f^{-1}$ .)



To see that  $D$  spans  $W$ , fix any  $\vec{w} \in W$ . Because  $f$  is an isomorphism it is onto and so there is a  $\vec{v} \in V$  with  $\vec{w} = f(\vec{v})$ . Expand  $\vec{v}$  as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1 \vec{\beta}_1 + \cdots + v_n \vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of  $D$ , if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1 \vec{\beta}_1 + \cdots + c_n \vec{\beta}_n)$$

then, since  $f$  is one-to-one and so the only vector sent to  $\vec{0}_W$  is  $\vec{0}_V$ , we have that  $\vec{0}_V = c_1 \vec{\beta}_1 + \cdots + c_n \vec{\beta}_n$ , which implies that all of the  $c$ 's are zero. QED

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*Proof* We will prove that any space of dimension  $n$  is isomorphic to  $\mathbb{R}^n$ . Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

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*Proof* We will prove that any space of dimension  $n$  is isomorphic to  $\mathbb{R}^n$ . Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

Let  $V$  be  $n$ -dimensional. Fix a basis  $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$  for the domain  $V$ . Consider the operation of representing the members of  $V$  with respect to  $B$  as a function from  $V$  to  $\mathbb{R}^n$ .

$$\vec{v} = v_1 \vec{\beta}_1 + \dots + v_n \vec{\beta}_n \xrightarrow{\text{Rep}_B} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This function is one-to-one because if

$$\text{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \text{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and so  $u_1 = v_1, \dots, u_n = v_n$ , implying that the original arguments  $u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n$  and  $v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$  are equal.

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This function is onto; any member of  $\mathbb{R}^n$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

is the image of some  $\vec{v} \in V$ , namely  $\vec{w} = \text{Rep}_B(w_1 \vec{\beta}_1 + \cdots + w_n \vec{\beta}_n)$ .

Finally, this function preserves structure.

$$\begin{aligned}\text{Rep}_B(\mathbf{r} \cdot \vec{\mathbf{u}} + s \cdot \vec{\mathbf{v}}) &= \text{Rep}_B((\mathbf{r}u_1 + sv_1)\vec{\beta}_1 + \cdots + (\mathbf{r}u_n + sv_n)\vec{\beta}_n) \\&= \begin{pmatrix} \mathbf{r}u_1 + sv_1 \\ \vdots \\ \mathbf{r}u_n + sv_n \end{pmatrix} \\&= \mathbf{r} \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\&= \mathbf{r} \cdot \text{Rep}_B(\vec{\mathbf{u}}) + s \cdot \text{Rep}_B(\vec{\mathbf{v}})\end{aligned}$$

Therefore  $\text{Rep}_B$  is an isomorphism. Consequently any  $n$ -dimensional space is isomorphic to  $\mathbb{R}^n$ .

QED

Finally, this function preserves structure.

$$\begin{aligned}\text{Rep}_B(r \cdot \vec{u} + s \cdot \vec{v}) &= \text{Rep}_B((ru_1 + sv_1)\vec{\beta}_1 + \cdots + (ru_n + sv_n)\vec{\beta}_n) \\ &= \begin{pmatrix} ru_1 + sv_1 \\ \vdots \\ ru_n + sv_n \end{pmatrix} \\ &= r \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + s \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= r \cdot \text{Rep}_B(\vec{u}) + s \cdot \text{Rep}_B(\vec{v})\end{aligned}$$

Therefore  $\text{Rep}_B$  is an isomorphism. Consequently any  $n$ -dimensional space is isomorphic to  $\mathbb{R}^n$ .

QED

*Note* The second paragraph's representation map  $\text{Rep}_B$  is a well-defined function since for each basis, every vector  $\vec{v}$  has a unique representation with respect to that basis.



*Example* The plane  $2x - y + z = 0$  through the origin in  $\mathbb{R}^3$  is a vector space. Considering that a one-equation linear system and parametrizing with the free variables

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} z \mid y, z \in \mathbb{R} \right\}$$

gives a basis.

$$B = \left\langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

This is a dimension 2 space. For instance, it is isomorphic to  $\mathbb{R}^2$ .

2.7 *Corollary*    A finite-dimensional vector space is isomorphic to one and only one of the  $\mathbb{R}^n$ .

2.7 *Corollary* A finite-dimensional vector space is isomorphic to one and only one of the  $\mathbb{R}^n$ .

Thus the real spaces  $\mathbb{R}^n$  form a set of canonical representatives of the isomorphism classes — every isomorphism class contains one and only one  $\mathbb{R}^n$ .

