Three.IV Matrix Operations

Linear Algebra
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Sums and Scalar Products

Representing operations on linear functions

Recall that the collection of linear functions $\mathcal{L}(V, W)$ between two spaces V, W is itself a vector space. That is, we can add two functions $f, g \colon V \to W$,

$$\vec{v} \stackrel{f+g}{\longmapsto} f(\vec{v}) + g(\vec{v})$$

and we can take the scalar multiple $r \cdot f$ of a function

$$\vec{v} \stackrel{r \cdot f}{\longmapsto} r \cdot (f(\vec{v}))$$

where $r \in \mathbb{R}$. We now see how the matrix representations $\text{Rep}_{B,D}(f)$ and $\text{Rep}_{B,D}(g)$ combine to give $\text{Rep}_{B,D}(f+g)$ and $\text{Rep}_{B,D}(r \cdot f)$.

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Example Suppose that $f\colon V\to W$ is the linear map represented with respect to some bases B,D by this matrix.

$$F=\mathrm{Rep}_{B,D}(f)=\begin{pmatrix}2&1\\3&4\end{pmatrix}$$

We will first find the matrix representing the function $6f$ and compare it with this matrix, to see the effect of the scalar 6.

Suppose that $f(\vec{v}) = \vec{w}$ and that these are the representations.

$$\mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \mathrm{Rep}_{D}(\vec{w}) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

The action of the function $6f\colon V \to W$ is $6f(\vec{v}) = 6\vec{w}$. Since $6\vec{w} = 6 \cdot (w_1\vec{\delta}_1 + w_2\vec{\delta}_2) = (6w_1)\vec{\delta}_1 + (6w_2)\vec{\delta}_2$, application of the function $6f\colon V \to W$ yields this representation of the codomain vector.

$$\mathrm{Rep}_{D}(6\vec{w}) = \begin{pmatrix} 6w_1 \\ 6w_2 \end{pmatrix}$$

This shows how the representation of $6f$ compares with the representation of f. The matrix for $6f$ gives vector outputs where each entry is 6 times as big as the entry given by the matrix for f.

Here is the representation of the action of f.

$$\mathrm{Rep}_{B,D}(f) \cdot \mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 \\ 3v_1 + 4v_2 \end{pmatrix}$$

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The prior slide shows that the matrix representing $6f$ does this.

$$\mathrm{Rep}_{B,D}(6f) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6(2v_1 + v_2) \\ 6(3v_1 + 4v_2) \end{pmatrix} = \begin{pmatrix} 12v_1 + 6v_2 \\ 18v_1 + 24v_2 \end{pmatrix}$$

Therefore

$$\mathrm{Rep}_{B,D}(6f) = \begin{pmatrix} 12 & 6 \\ 18 & 24 \end{pmatrix}$$

and so going from the function f to the function $6f$ has the effect on the representation matrix of multiplying all the entries by 6.

Example Suppose that $f, g \colon V \to W$ are linear maps represented with respect to some bases by these matrices.

$$F = \mathrm{Rep}_{B,D}(f) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \qquad G = \mathrm{Rep}_{B,D}(g) = \begin{pmatrix} 5 & 8 \\ 7 & 6 \end{pmatrix}$$

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We will find the matrix representing the function $f+g$.

If $f(\vec{v}) = \vec{u}$ and $g(\vec{v}) = \vec{w}$ then $f+g$ does this.

$$\begin{aligned} \vec{v} \overset{f+g}{\longmapsto} \vec{u} + \vec{v} &= (u_1\vec{\delta}_1 + u_2\vec{\delta}_2) + (w_1\vec{\delta}_1 + w_2\vec{\delta}_2) \\ &= (u_1 + w_1)\vec{\delta}_1 + (u_2 + w_2)\vec{\delta}_2 \end{aligned}$$

Thus for any $\vec{v}$, the effect on the representatives

$$\mathrm{Rep}_D(f(\vec{v})) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \mathrm{Rep}_D(g(\vec{v})) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

of adding the functions is to add the column vectors.

$$\mathrm{Rep}_D((f+g)(\vec{v})) = \begin{pmatrix} u_1 + w_1 \\ u_2 + w_2 \end{pmatrix}$$

The particular functions in this example have this effect on the representations.

$$\mathrm{Rep}_D\left(f(\vec{v})\right) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 \\ 3v_1 + 4v_2 \end{pmatrix}$$

$$\mathrm{Rep}_D\left(g(\vec{v})\right) = \begin{pmatrix} 5 & 8 \\ 7 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5v_1 + 8v_2 \\ 7v_1 + 6v_2 \end{pmatrix}$$

So we want the matrix with this effect.

$$\mathrm{Rep}_D\left((f+g)\,(\vec{v})\right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 7v_1 + 9v_2 \\ 10v_1 + 10v_2 \end{pmatrix}$$

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We want this.

$$\mathrm{Rep}_{B,D}(f+g) = \begin{pmatrix} 7 & 9 \\ 10 & 10 \end{pmatrix}$$

So, the representation of the sum is the entry-by-entry sum of the representations.

Definition of matrix sum and scalar multiple

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Example Where

$$A=\begin{pmatrix}1&-1\\2&3\end{pmatrix}\quad B=\begin{pmatrix}0&0&2\\9&-1/2&5\end{pmatrix}\quad C=\begin{pmatrix}1&0\\8&-1\end{pmatrix}$$

Then

$$A+C=\begin{pmatrix}2&-1\\10&2\end{pmatrix}\qquad 5B=\begin{pmatrix}0&0&10\\45&-5/2&25\end{pmatrix}$$

Because the sizes don't match, none of these is defined: $A+B$, $B+A$, $B+C$, $C+B$.

1.4 *Theorem* Let $h, g\colon V \to W$ be linear maps represented with respect to bases B, D by the matrices H and G and let r be a scalar. Then with respect to B, D the map $r \cdot h\colon V \to W$ is represented by rH and the map $h + g\colon V \to W$ is represented by $H + G$.

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Example The zero matrix is the identity element for matrix addition.

$$\begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix}$$

A zero function $Z\colon V \to W$ is the identity element for function addition, and the matrix fact accords with the map fact.

Matrix Multiplication

Representing composition

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Proof Let $h\colon V \to W$ and $g\colon W \to U$ be linear. The calculation

$$\begin{aligned} g\circ h\,\big(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2\big) &= g\big(\,h(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2)\,\big) = g\big(\,c_1\cdot h(\vec{v}_1)+c_2\cdot h(\vec{v}_2)\,\big) \\ &= c_1\cdot g\big(h(\vec{v}_1)\big)+c_2\cdot g\big(h(\vec{v}_2)\big) = c_1\cdot(g\circ h)(\vec{v}_1)+c_2\cdot(g\circ h)(\vec{v}_2) \end{aligned}$$

shows that $g\circ h\colon V \to U$ preserves linear combinations, and so is linear. QED

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We will see how the matrix representations of the two functions combine to make the matrix representation of their composition.

Example Consider two linear functions $h\colon V \to W$ and $g\colon W \to X$ represented as here.

$$\mathrm{Rep}_{B,C}(h) = \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} \qquad \mathrm{Rep}_{C,D}(g) = \begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix}$$

The sizes of the matrices show that V has dimension 2, W has dimension 3, and X has dimension 2.

We want to see how these two matrices combine to represent the map $g \circ h\colon V \to X$.

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We will represent the action of $g \circ h$ by first representing the action of h on a vector $\vec{v} \in V$.

$$\begin{aligned} \mathrm{Rep}_C(h(\vec{v})) &= \mathrm{Rep}_{B,C}(h) \cdot \mathrm{Rep}_B(\vec{v}) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 + v_2 \\ 2v_1 + 5v_2 \\ 4v_1 + 6v_2 \end{pmatrix} \end{aligned}$$

Apply g to the vector $\mathrm{Rep}_C(h(\vec{v}))$.

$$\mathrm{Rep}_{C,D}(g) \cdot \mathrm{Rep}_C(h(\vec{v})) = \begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix} \begin{pmatrix} 3v_1 + v_2 \\ 2v_1 + 5v_2 \\ 4v_1 + 6v_2 \end{pmatrix}$$
$$= \begin{pmatrix} 8(3v_1 + v_2) + 7(2v_1 + 5v_2) + 11(4v_1 + 6v_2) \\ 9(3v_1 + v_2) + 10(2v_1 + 5v_2) + 12(4v_1 + 6v_2) \end{pmatrix}$$

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$$\mathrm{Rep}_{C,D}(g)\cdot\mathrm{Rep}_C(h(\vec{v}))=\begin{pmatrix}8&7&11\\9&10&12\end{pmatrix}\begin{pmatrix}3v_1+v_2\\2v_1+5v_2\\4v_1+6v_2\end{pmatrix}$$
$$=\begin{pmatrix}8(3v_1+v_2)+7(2v_1+5v_2)+11(4v_1+6v_2)\\9(3v_1+v_2)+10(2v_1+5v_2)+12(4v_1+6v_2)\end{pmatrix}$$

Gather terms.

$$=\begin{pmatrix}(8\cdot3+7\cdot2+11\cdot4)v_1+(8\cdot1+7\cdot5+11\cdot6)v_2\\(9\cdot3+10\cdot2+12\cdot4)v_1+(9\cdot1+10\cdot5+12\cdot6)v_2\end{pmatrix}$$

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Gather terms.

$$= \begin{pmatrix} (8 \cdot 3 + 7 \cdot 2 + 11 \cdot 4)v_1 + (8 \cdot 1 + 7 \cdot 5 + 11 \cdot 6)v_2 \\ (9 \cdot 3 + 10 \cdot 2 + 12 \cdot 4)v_1 + (9 \cdot 1 + 10 \cdot 5 + 12 \cdot 6)v_2 \end{pmatrix}$$

Rewrite that as a matrix-vector multiplication.

$$\begin{pmatrix} 8 \cdot 3 + 7 \cdot 2 + 11 \cdot 4 & 8 \cdot 1 + 7 \cdot 5 + 11 \cdot 6 \\ 9 \cdot 3 + 10 \cdot 2 + 12 \cdot 4 & 9 \cdot 1 + 10 \cdot 5 + 12 \cdot 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Apply g to the vector $\mathrm{Rep}_C(h(\vec{v}))$.

$$\mathrm{Rep}_{C,D}(g)\cdot\mathrm{Rep}_C(h(\vec{v}))=\begin{pmatrix}8&7&11\\9&10&12\end{pmatrix}\begin{pmatrix}3v_1+v_2\\2v_1+5v_2\\4v_1+6v_2\end{pmatrix}$$
$$=\begin{pmatrix}8(3v_1+v_2)+7(2v_1+5v_2)+11(4v_1+6v_2)\\9(3v_1+v_2)+10(2v_1+5v_2)+12(4v_1+6v_2)\end{pmatrix}$$

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$$=\begin{pmatrix}(8\cdot3+7\cdot2+11\cdot4)v_1+(8\cdot1+7\cdot5+11\cdot6)v_2\\(9\cdot3+10\cdot2+12\cdot4)v_1+(9\cdot1+10\cdot5+12\cdot6)v_2\end{pmatrix}$$

Rewrite that as a matrix-vector multiplication.

$$\begin{pmatrix}8\cdot3+7\cdot2+11\cdot4&8\cdot1+7\cdot5+11\cdot6\\9\cdot3+10\cdot2+12\cdot4&9\cdot1+10\cdot5+12\cdot6\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}$$

So here is how the parts combine.

$$\begin{pmatrix}8&7&11\\9&10&12\end{pmatrix}\begin{pmatrix}3&1\\2&5\\4&6\end{pmatrix}=\begin{pmatrix}8\cdot3+7\cdot2+11\cdot4&8\cdot1+7\cdot5+11\cdot6\\9\cdot3+10\cdot2+12\cdot4&9\cdot1+10\cdot5+12\cdot6\end{pmatrix}$$

Definition of matrix multiplication

2.3 *Definition* The *matrix-multiplicative product* of the $m \times r$ matrix G and the $r \times n$ matrix H is the $m \times n$ matrix P, where

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

so that the i, j-th entry of the product is the dot product of the i-th row of the first matrix with the j-th column of the second.

$$GH = \begin{pmatrix} & \vdots & & \\ g_{i,1} & g_{i,2} & \cdots & g_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & h_{1,j} & \\ \cdots & h_{2,j} & \cdots \\ & \vdots & \\ & h_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & p_{i,j} & \cdots \\ & \vdots & \end{pmatrix}$$

Example

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 1 & -3 & 5 \\ 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 31 & 9 & 59 \\ 45 & 3 & 96 \end{pmatrix}$$

Example This product is not defined.

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 7 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

The number of columns on the left must equal the number of rows on the right.

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Example Square matrices of the same size have a defined product.

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 7 & 1 \\ 2 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 14 & -1 \\ 0 & 0 & 0 \\ 10 & 14 & 2 \end{pmatrix}$$

This reflects the fact that we can compose two functions from a space to itself $f, g\colon V \to V$.

Matrix multiplication represents composition

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Proof Let $h\colon V \to W$ and $g\colon W \to X$ be represented by H and G with respect to bases $B \subset V$, $C \subset W$, and $D \subset X$, of sizes n, r, and m. For any $\vec{v} \in V$ the k-th component of $\mathrm{Rep}_C(\,h(\vec{v})\,)$ is

$$h_{k,1}v_1 + \cdots + h_{k,n}v_n$$

and so the i-th component of $\mathrm{Rep}_D(\,g \circ h\,(\vec{v})\,)$ is this.

$$\begin{aligned} g_{i,1} \cdot (h_{1,1}v_1 + \cdots + h_{1,n}v_n) + g_{i,2} \cdot (h_{2,1}v_1 + \cdots + h_{2,n}v_n) \\ + \cdots + g_{i,r} \cdot (h_{r,1}v_1 + \cdots + h_{r,n}v_n) \end{aligned}$$

Distribute and regroup on the v's.

$$\begin{aligned} = (g_{i,1}h_{1,1} + g_{i,2}h_{2,1} + \cdots + g_{i,r}h_{r,1}) \cdot v_1 \\ + \cdots + (g_{i,1}h_{1,n} + g_{i,2}h_{2,n} + \cdots + g_{i,r}h_{r,n}) \cdot v_n \end{aligned}$$

Finish by recognizing that the coefficient of each v_j

$$g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

matches the definition of the i, j entry of the product GH. QED

Arrow diagrams

This pictures the relationship between maps and matrices.

$$\begin{tikzcd}
 & W_{\mathit{wrt}\ C} \arrow[dr, "g", "G"'] & \\
V_{\mathit{wrt}\ B} \arrow[ur, "h", "H"'] \arrow[rr, "g \circ h", "GH"'] & & X_{\mathit{wrt}\ D}
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\end{tikzcd}
$$

Above the arrows, the maps show that the two ways of going from V to X, straight over via the composition or else in two steps by way of W, have the same effect

$$\vec{v} \xmapsto{g \circ h} g(h(\vec{v})) \qquad \vec{v} \xmapsto{h} h(\vec{v}) \xmapsto{g} g(h(\vec{v}))$$

(this is just the definition of composition). Below the arrows, the matrices indicate that multiplying GH into the column vector $\mathrm{Rep}_B(\vec{v})$ has the same effect as multiplying the column vector first by H and then multiplying the result by G.

$$\mathrm{Rep}_{B,D}(g \circ h) = GH \qquad \mathrm{Rep}_{C,D}(g)\ \mathrm{Rep}_{B,C}(h) = GH$$

Order, dimensions, and sizes

Consider the composition of two function.

First consider the order in which we write the composition. In $g \circ h$ the function written first, g, is the function applied second.

$$\vec{v} \xmapsto{h} h(\vec{v}) \xmapsto{g} g(h(\vec{v}))$$

We write g first to match the definition $g \circ h(\vec{v}) = g(\, h(\vec{v})\,)$. That order carries over to matrices: $g \circ h$ is represented by GH.

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Now consider the dimensions of the domain and codomain spaces.

$$\text{dimension } n \text{ space} \xrightarrow{h} \text{dimension } r \text{ space} \xrightarrow{g} \text{dimension } m \text{ space}$$

The representing $m \times n$ matrix GH is the product of an $m \times r$ matrix G and a $r \times n$ matrix H. Briefly, $m \times r$ times $r \times n$ equals $m \times n$.

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Function composition is not a commutative operation — $\cos(x^2)$ is different than $\cos^2(x)$. This holds even in the special case of composition of linear functions.

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Example Changing the order in which we multiply these matrices

$$\begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 33 \\ 24 & 20 \end{pmatrix}$$

changes the result.

$$\begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -6 & 18 \\ 18 & 38 \end{pmatrix}$$

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Example The product of these matrices

$$\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 8 & 12 & 0 \\ -4 & 0 & 1/2 \end{pmatrix}$$

is defined in one order and not defined in the other.

Although the matrix operation of multiplication does not have the property of being commutative, it does have some nice algebraic properties.

2.12 *Theorem* If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G+H) = FG + FH$ and $(G+H)F = GF + HF$.

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Proof Associativity holds because matrix multiplication represents function composition, which is associative: the maps $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as both send $\vec{v}$ to $f(g(h(\vec{v})))$.

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Distributivity is similar. For instance, the first one goes $f \circ (g + h)\,(\vec{v}) = f\big(\,(g + h)(\vec{v})\,\big) = f\big(\,g(\vec{v}) + h(\vec{v})\,\big) = f(g(\vec{v})) + f(h(\vec{v})) = f \circ g(\vec{v}) + f \circ h(\vec{v})$ (the third equality uses the linearity of f). Right-distributivity goes the same way. QED

Mechanics of Matrix Multiplication

Combinatorics of multiplication

The striking thing about matrix multiplication is the way rows and columns combine. Here a second row and a third column combine to make a $2, 3$ entry.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 13 & 17 \\ 5 & 7 & 9 \\ 4 & 6 & 8 \end{pmatrix}$$

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The i,j entry of the matrix product GH is the dot product of row i of the left matrix G with column j of the right one H.

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

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$$p_{i,j} = g_{i,1} h_{1,j} + g_{i,2} h_{2,j} + \cdots + g_{i,r} h_{r,j}$$

We can view this as the left matrix acting by multiplying its rows into the columns of the right matrix. Or we could see it as the right matrix using its columns to act on the left matrix's rows.

3.2 *Definition* A matrix with all 0's except for a 1 in the i, j entry is an i, j *unit* matrix.

Example The $2, 1$ unit 2×3 matrix multiplies from the left

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 5 & 6 & 4 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

to copy row 1 of the multiplicand into row 2 of the result.

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Example From the right the $2,1$ unit 2×3 matrix

$$\begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

copies column 2 of the first matrix into column 1 of the result.

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copies column 2 of the first matrix into column 1 of the result.

Example Rescaling the unit matrix rescales the result.

$$\begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 15 & 0 & 0 \end{pmatrix}$$

3.7 *Lemma* In a product of two matrices G and H, the columns of GH are formed by taking G times the columns of H

$$G \cdot \begin{pmatrix} \vdots & & \vdots \\ \vec{h}_1 & \cdots & \vec{h}_n \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & & \vdots \\ G \cdot \vec{h}_1 & \cdots & G \cdot \vec{h}_n \\ \vdots & & \vdots \end{pmatrix}$$

and the rows of GH are formed by taking the rows of G times H

$$\begin{pmatrix} \cdots & \vec{g}_1 & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r & \cdots \end{pmatrix} \cdot H = \begin{pmatrix} \cdots & \vec{g}_1 \cdot H & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r \cdot H & \cdots \end{pmatrix}$$

(ignoring the extra parentheses).

Proof We will check that in a product of 2×2 matrices, the rows of the product equal the product of the rows of G with the entire matrix H.

$$\begin{aligned}
\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix}\begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} &= \begin{pmatrix} (g_{1,1}\ \ g_{1,2})H \\ (g_{2,1}\ \ g_{2,2})H \end{pmatrix} \\
&= \begin{pmatrix} (g_{1,1}h_{1,1}+g_{1,2}h_{2,1}\ \ \ g_{1,1}h_{1,2}+g_{1,2}h_{2,2}) \\ (g_{2,1}h_{1,1}+g_{2,2}h_{2,1}\ \ \ g_{2,1}h_{1,2}+g_{2,2}h_{2,2}) \end{pmatrix}
\end{aligned}$$

We leave the more general check as an exercise. QED

3.8 *Definition* The *main diagonal* (or *principle diagonal* or *diagonal*) of a square matrix goes from the upper left to the lower right.

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3.9 *Definition* An *identity matrix* is square and every entry is 0 except for 1's in the main diagonal.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

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Taking the product with an identity matrix returns the multiplicand.

Example Multiplication by an identity from the left

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

or from the right leaves the matrix unchanged.

$$\begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

3.12 *Definition* A ***diagonal matrix*** is square and has 0's off the main diagonal.

$$\begin{pmatrix} a_{1,1} & 0 & \ldots & 0 \\ 0 & a_{2,2} & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & a_{n,n} \end{pmatrix}$$

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Example Multiplication from the left by a diagonal matrix rescales the rows.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -3 & 15 \end{pmatrix}$$

From the right it rescales the columns.

$$\begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -2 & 15 \end{pmatrix}$$

3.14 *Definition* A *permutation matrix* is square and is all 0's except for a single 1 in each row and column.

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Example Multiplication by a permutation matrix from the left will swap rows.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

From the right it swaps columns.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{pmatrix}$$

3.19 *Definition* The *elementary reduction matrices* result from applying a one Gaussian operation to an identity matrix.

1) $I \xrightarrow{k\rho_i} M_i(k)$ for $k \neq 0$

2) $I \xrightarrow{\rho_i \leftrightarrow \rho_j} P_{i,j}$ for $i \neq j$

3) $I \xrightarrow{k\rho_i + \rho_j} C_{i,j}(k)$ for $i \neq j$

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Example Here are some 2×2 examples.

$$M_2(3) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad P_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C_{1,2}(-3) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

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Example Some 3×3 examples.

$$P_{2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad C_{2,3}(-4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{pmatrix}$$

Example Multiplying on the left by the 3×3 matrix $M_2(1/2)$ has the effect of the row operation $(1/2)\rho_2$.

$$\begin{pmatrix}1&0&0\\0&1/2&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&3&0\\0&2&5\\0&0&0\end{pmatrix}=\begin{pmatrix}1&3&0\\0&1&5/2\\0&0&0\end{pmatrix}$$

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Example Left multiplication by $C_{1,3}(-2)$ performs the row operation $-2\rho_1+\rho_3$.

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 0\\ 0 & 2 & 5\\ 2 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 3 & 0\\ 0 & 2 & 5\\ 0 & -5 & 1\end{pmatrix}$$

3.20 *Lemma* Matrix multiplication can do Gaussian reduction.

1) If $H \xrightarrow{k\rho_i} G$ then $M_i(k)H = G$.

2) If $H \xrightarrow{\rho_i \leftrightarrow \rho_j} G$ then $P_{i,j}H = G$.

3) If $H \xrightarrow{k\rho_i + \rho_j} G$ then $C_{i,j}(k)H = G$.

Proof Clear. QED

3.23 *Corollary* For any matrix H there are elementary reduction matrices $R_1, \ldots, R_r$ such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

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3.23 *Corollary* For any matrix H there are elementary reduction matrices $R_1, \ldots, R_r$ such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

Example We can bring this augmented matrix to echelon form with matrix multiplication.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right)$$

We first perform $-2\rho_1 + \rho_2$ via left multiplication by $C_{1,2}(-2)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & -5 & -2 \\ 0 & 3 & 1 & 5 \end{array}\right)$$

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Now we swap rows 2 and 3 with $P_{2,3}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & -5 & -2 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -5 & -2 \end{array}\right)$$

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When writing them out in full, remember that the matrix used first appears to the right of the matrix used second.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -5 & -2 \end{array}\right)$$

Inverses

Function inverses

We finish this section by considering how to represent the inverse of a linear map.

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We first recall some things about inverses. Where $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection map and $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$ is the embedding

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

then the composition $\pi \circ \iota$ is the identity map $\pi \circ \iota = \text{id}$ on $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix}$$

We say that ι is a *right inverse* of π or, what is the same thing, that π is a *left inverse* of ι.

However, composition in the other order $\iota \circ \pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\iota \circ \pi$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

However, composition in the other order $\iota \circ \pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\iota \circ \pi$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In fact, π has no left inverse at all. For, if f were to be a left inverse of π then we would have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all of the infinitely many z's. But a function f cannot send a single argument $\binom{x}{y}$ to more than one value.

So a function can have a right inverse but no left inverse, or a left inverse but no right inverse. A function can also fail to have an inverse on either side; one example is the zero transformation on $\mathbb{R}^2$.

Some functions have a *two-sided inverse*, another function that is the inverse both from the left and from the right. For instance, the transformation given by $\vec{v} \mapsto 2 \cdot \vec{v}$ has the two-sided inverse $\vec{v} \mapsto (1/2) \cdot \vec{v}$. The appendix shows that a function has a two-sided inverse if and only if it is both one-to-one and onto. The appendix also shows that if a function f has a two-sided inverse then it is unique, so we call it 'the' inverse and write f^{-1}.

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In addition, recall that we have shown in Theorem II.2.20 that if a linear map has a two-sided inverse then that inverse is also linear.

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In addition, recall that we have shown in Theorem II.2.20 that if a linear map has a two-sided inverse then that inverse is also linear.

Our goal is, where a linear h has an inverse, to find the relationship between the matrices $\mathrm{Rep}_{B,D}(h)$ and $\mathrm{Rep}_{D,B}(h^{-1})$.

Definition of matrix inverse

4.1 *Definition* A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix*. That two-sided inverse is denoted H^{-1}.

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Example This matrix

$$H = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

has a two-sided inverse.

$$H^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

To check that, we can multiply them in both orders. Here is one; the other order is just as easy.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here is an arrow diagram for matrix inverses.

$$\begin{tikzcd}
& W_{\mathit{wrt}\ \mathrm{C}} \arrow[dr, "h^{-1}", "H^{-1}"'] & \\
V_{\mathit{wrt}\ \mathrm{B}} \arrow[ur, "h", "H"'] \arrow[rr, "\mathrm{id}", "I"'] & & V_{\mathit{wrt}\ \mathrm{B}}
\end{tikzcd}$$

4.2 *Lemma* If a matrix has both a left inverse and a right inverse then the two are equal.

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Proof *(For both results.)* Given a matrix H, fix spaces of appropriate dimension for the domain and codomain and fix bases for these spaces. With respect to these bases, H represents a map h. The statements are true about the map and therefore they are true about the matrix. QED

4.4 *Lemma* A product of invertible matrices is invertible: if G and H are invertible and GH is defined then GH is invertible and $(GH)^{-1} = H^{-1}G^{-1}$.

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Proof Because the two matrices are invertible they are square, and because their product is defined they must both be $n \times n$. Fix spaces and bases — say, $\mathbb{R}^n$ with the standard bases — to get maps $g, h \colon \mathbb{R}^n \to \mathbb{R}^n$ that are associated with the matrices, $G = \text{Rep}_{\mathcal{E}_n,\mathcal{E}_n}(g)$ and $H = \text{Rep}_{\mathcal{E}_n,\mathcal{E}_n}(h)$.

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Consider $h^{-1}g^{-1}$. By the prior paragraph this composition is defined. This map is a two-sided inverse of gh since $(h^{-1}g^{-1})(gh) = h^{-1}(\mathrm{id})h = h^{-1}h = \mathrm{id}$ and $(gh)(h^{-1}g^{-1}) = g(\mathrm{id})g^{-1} = gg^{-1} = \mathrm{id}$. The matrices representing the maps reflect this equality. QED

4.7 *Lemma* A matrix H is invertible if and only if it can be written as the product of elementary reduction matrices. We can compute the inverse by applying to the identity matrix the same row steps, in the same order, that Gauss-Jordan reduce H.

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Proof The matrix H is invertible if and only if it is nonsingular and thus Gauss-Jordan reduces to the identity. By Corollary 3.23 we can do this reduction with elementary matrices.

$$R_r \cdot R_{r-1} \ldots R_1 \cdot H = I \qquad (*)$$

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For the first sentence of the result, note that elementary matrices are invertible because elementary row operations are reversible, and that their inverses are also elementary. Apply R_r^{-1} from the left to both sides of $(*)$. Then apply R_{r-1}^{-1}, etc. The result gives H as the product of elementary matrices $H = R_1^{-1} \cdots R_r^{-1} \cdot I$. (The I there covers the case $r = 0$.)

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For the second sentence, group $(*)$ as $(R_r \cdot R_{r-1} \ldots R_1) \cdot H = I$ and recognize what's in the parentheses as the inverse $H^{-1} = R_r \cdot R_{r-1} \ldots R_1 \cdot I$. Restated: applying R_1 to the identity, followed by R_2, etc., yields the inverse of H. QED

Example This matrix is nonsingular and so is invertible.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

To ease the inverse calculation described in the prior proof, we write the matrix A next to the 3×3 identity and as we Gauss-Jordan reduce the matrix on the left, we apply those operations also on the right.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-1/6\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1/2 & -2/3 & -1/6 & 1 \end{array}\right)$$

The right-hand side is in echelon form. We continue with the second half of Gauss-Jordan reduction on the next slide.

$$\xrightarrow[-2\rho_3]{-1/6\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/3 & -1/6 & 0 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow[-(1/2)\rho_3+\rho_2]{-\rho_3+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1/3 & -1/3 & 2 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 2/3 & -1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

This is the inverse.

$$A^{-1} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix}$$

Finding the inverse of a matrix A is a lot of work but once we have it then solving linear systems $A\vec{x} = \vec{b}$ is easy.

Example The linear system

$$\begin{aligned} x + 3y + z &= 2 \\ 2x \qquad - z &= 12 \\ x + 2y \qquad &= 4 \end{aligned}$$

is this matrix equation.

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix}$$

Solve it by multiplying both sides from the left by the inverse that we found earlier.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 16/3 \\ -2/3 \\ -4/3 \end{pmatrix}$$

The inverse of a 2×2 matrix

4.11 *Corollary* The inverse for a 2×2 matrix exists and equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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$$\begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/6 & -2/3 \\ 1/6 & 1/3 \end{pmatrix}$$

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The 3×3 formula is much more complicated. We will cover it in the next chapter.