Three.III Computing Linear Maps

Linear Algebra

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Representing Linear Maps with Matrices

Linear maps are determined by the action on a basis

We've seen that if we fix a domain space V, a codomain space W, and a basis $B_V = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain, and for any linear map $h\colon V \to W$ we fix the action of h on the basis elements $h(\vec{\beta}_1)$, $\ldots$, $h(\vec{\beta}_n)$,

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$$h(\vec{v}) = h(c_1 \cdot \vec{\beta}_1 + \cdots + c_n \cdot \vec{\beta}_n) = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n) \quad (*)$$

We next develop a scheme to easily do those calculations easily.

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We next develop a scheme to easily do those calculations easily.

Example Let the domain be $V = \mathcal{P}_2$ with the basis $B_V = \langle 1, 1+x, 1+x+x^2 \rangle$. Let the codomain be $\mathbb{R}^2$ with this basis.

$$B_W = \langle \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle$$

Let the map $h\colon \mathcal{P}_2 \to \mathbb{R}^2$ have this action on the domain basis.

$$h(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad h(1+x) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad h(1+x+x^2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

First find the representation, with respect to the codomain's basis B_W, of the action of h on B_V.

$$\mathrm{Rep}_{B_W}(h(\vec{\beta}_1)) = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \quad \text{since } (1/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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Summarize that action by writing those three vectors side-by-side, in order, to make a matrix.

$$\begin{pmatrix} 1/2 & 5/2 & -3/2 \\ 1 & 2 & -1 \end{pmatrix}$$

Consider the domain vector $\vec{v} = c_1 \cdot \vec{\beta}_1 + c_2 \cdot \vec{\beta}_2 + c_3 \cdot \vec{\beta}_3$.

$$\mathrm{Rep}_{B_V}(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Apply equation $(*)$.

$$\begin{aligned} c_1 h(\vec{\beta}_1) + c_2 h(\vec{\beta}_2) + c_3 h(\vec{\beta}_3) = c_1(\,(1/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \\ + c_2(\,(5/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \\ + c_3(\,(-3/2) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}\,) \end{aligned}$$

Regroup.

$$= ((1/2)c_1 + (5/2)c_2 - (3/2)c_3) \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (1c_1 + 2c_2 - 1c_3) \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\mathrm{Rep}_{B_W}(h(\vec{v})) = \begin{pmatrix} (1/2)c_1 + (5/2)c_2 - (3/2)c_3 \\ 1c_1 + 2c_2 - 1c_3 \end{pmatrix}$$

So the effect of the linear map summarized by this matrix

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on the domain element represented in this way

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This is the scheme: get the representation of the output by taking the dot product of each row of the matrix with the single column representing the input.

Matrix representation of a linear map

1.2 *Definition* Suppose that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If

$$\mathrm{Rep}_D(h(\vec{\beta}_1)) = \begin{pmatrix} h_{1,1} \\ h_{2,1} \\ \vdots \\ h_{m,1} \end{pmatrix}_D \quad \ldots \quad \mathrm{Rep}_D(h(\vec{\beta}_n)) = \begin{pmatrix} h_{1,n} \\ h_{2,n} \\ \vdots \\ h_{m,n} \end{pmatrix}_D$$

then

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

is the *matrix representation of* h *with respect to* B, D.

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is the *matrix representation of* h *with respect to* B, D.

We often omit the subscript on the matrix.

Example Consider projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis.

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

If we take the input and output bases to be

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle \qquad D = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

then we compute

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

and therefore this is the matrix representing π.

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix}$$

Example Again consider projection onto the x-axis

$$\begin{pmatrix} a \\ b \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

but this time we take the input and output bases to be the standard.

$$B = D = \mathcal{E}_2 = \langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

We have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_1)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{so } \mathrm{Rep}_D(\pi(\vec{\beta}_2)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so this is $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example Consider $h\colon \mathbb{R}^2 \to \mathbb{R}$ given by this.

$$\begin{pmatrix} a \\ b \end{pmatrix} \xmapsto{h} 2a + 3b$$

With the standard bases $\mathcal{E}_2, \mathcal{E}_1$ we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto 2 \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto 3$$

and so this 1×2 matrix represents the map.

$$H = \operatorname{Rep}_{\mathcal{E}_2,\mathcal{E}_1}(h) = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

1.4 *Theorem* Assume that V and W are vector spaces of dimensions n and m with bases B and D, and that $h\colon V \to W$ is a linear map. If h is represented by

$$\mathrm{Rep}_{B,D}(h) = \begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,n} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,n} \\ & \vdots & & \\ h_{m,1} & h_{m,2} & \ldots & h_{m,n} \end{pmatrix}_{B,D}$$

and $\vec{v} \in V$ is represented by

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$

then the representation of the image of $\vec{v}$ is this.

$$\mathrm{Rep}_D(\,h(\vec{v})\,) = \begin{pmatrix} h_{1,1}c_1 + h_{1,2}c_2 + \cdots + h_{1,n}c_n \\ h_{2,1}c_1 + h_{2,2}c_2 + \cdots + h_{2,n}c_n \\ \vdots \\ h_{m,1}c_1 + h_{m,2}c_2 + \cdots + h_{m,n}c_n \end{pmatrix}_D$$

Proof This formalizes the example that began this subsection. See Exercise 29 . QED

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1.5 *Definition* The *matrix-vector product* of a $m \times n$ matrix and a $n \times 1$ vector is this.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \vdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{m,n} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_{1,1}c_1 + \cdots + a_{1,n}c_n \\ a_{2,1}c_1 + \cdots + a_{2,n}c_n \\ \vdots \\ a_{m,1}c_1 + \cdots + a_{m,n}c_n \end{pmatrix}$$

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Example We can perform the operation without any reference to spaces and bases.

$$\begin{pmatrix} 3 & 1 & 2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 4 + 1 \cdot (-1) + 2 \cdot (-3) \\ 0 \cdot 4 - 2 \cdot (-1) + 5 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

Example Recall the two matrices

$$\mathrm{Rep}_{B,D}(\pi) = \begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(\pi) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

representing projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2$ onto the x-axis with respect to

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle, \; D = \langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle$$

and also with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_2$.

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and also with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_2$. This domain vector

$$\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

has these representations with respect to the two domain bases.

$$\mathrm{Rep}_{B}(\vec{v}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

The matrix-vector products

$$\begin{pmatrix} -1 & 1 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

give the two representations $\mathrm{Rep}_{D}(\pi(\vec{v}))$ and $\mathrm{Rep}_{\mathcal{E}_2}(\pi(\vec{v}))$. In both cases

$$1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (1/2) \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad -1 \cdot \vec{e}_1 + 0 \cdot \vec{e}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

they compute this projection.

$$\pi(\begin{pmatrix} -1 \\ 5 \end{pmatrix}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Example Recall also that $h\colon \mathbb{R}^2 \to \mathbb{R}$ with this action

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{h} 2a + 3b$$

is represented with respect to the standard bases $\mathcal{E}_2, \mathcal{E}_1$ by a 1×2 matrix.

$$\mathrm{rep} h \mathcal{E}_2, \mathcal{E}_1 = \begin{pmatrix} 2 & 3 \end{pmatrix}$$

The domain vector

$$\vec{v} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \qquad \mathrm{Rep}_{\mathcal{E}_2}(\vec{v}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

has this image.

$$\mathrm{Rep}_{\mathcal{E}_1}(h(\vec{v})) = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2$$

Any Matrix Represents a Linear Map

Example The prior subsection shows how to start with a linear map and produce its matrix representation. What about the converse? That is, suppose instead that we start with a matrix.

$$H = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Fix a domain and codomain, and bases.

$$\mathcal{E}_2 \subset \mathbb{R}^2 \quad \langle 1-x, 1+x \rangle \subset \mathcal{P}_1$$

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Consider $h\colon \mathbb{R}^2 \to \mathcal{P}_1$ defined by: for any domain vector $\vec{v}$, represent it with respect to the domain basis, multiply that representation by H, and then $h(\vec{v})$ is the codomain vector represented by the result. We will verify that h is a linear function.

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Note first that h is a function. This is because the representation of a vector with respect to a basis can be done in one and only one way, so the outcome is well-defined. That is, for a given input, the output from h exists and is unique.

Now we show that h is linear. Fix domain vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ and represent them with respect to the domain basis. Multiply $c \cdot \mathrm{Rep}_B(\vec{u}) + d \cdot \mathrm{Rep}_D(\vec{w})$ by H.

$$\begin{aligned}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \left(c \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{pmatrix} \\
&= \begin{pmatrix} 1(cu_1 + dv_1) + 2(cu_2 + dv_2) \\ 3(cu_1 + dv_1) + 4(cu_2 + dv_2) \end{pmatrix} \\
&= \begin{pmatrix} 1cu_1 + 2cu_2 \\ 3cu_1 + 4cu_2 \end{pmatrix} + \begin{pmatrix} 1dv_1 + 2dv_2 \\ 3dv_1 + 4dv_2 \end{pmatrix} \\
&= c \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\end{aligned}$$

By the definition of h, the result is $c \cdot \mathrm{Rep}_D(h(\vec{u})) + d \cdot \mathrm{Rep}_D(h(\vec{v}))$.

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Proof We must check that for any matrix H and any domain and codomain bases B, D, the defined map h is linear. If $\vec{v}, \vec{u} \in V$ are such that

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \mathrm{Rep}_B(\vec{u}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

and $c, d \in \mathbb{R}$ then the calculation

$$\begin{aligned} h(c\vec{v} + d\vec{u}) &= \bigl(h_{1,1}(cv_1 + du_1) + \cdots + h_{1,n}(cv_n + du_n)\bigr) \cdot \vec{\delta}_1 + \\ &\qquad \cdots + \bigl(h_{m,1}(cv_1 + du_1) + \cdots + h_{m,n}(cv_n + du_n)\bigr) \cdot \vec{\delta}_m \\ &= c \cdot h(\vec{v}) + d \cdot h(\vec{u}) \end{aligned}$$

supplies that check. QED

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We start with an easy one: we claim that $h \colon V \to W$ is the zero map if and only if it is represented, with respect to any bases, by the zero matrix.

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First assume h is the zero map. Then for any bases B, D we have $h(\vec{\beta}_i) = \vec{0}_W$, which is represented with respect to D by the zero vector. Thus h is represented by the zero matrix.

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Now assume that there are bases B, D such that $\mathrm{Rep}_{B,D}(h)$ is the zero matrix. Then for each $\vec{\beta}_i$ we have that $\mathrm{Rep}_D(h(\vec{\beta}_1))$ is a column vector of zeros, and so $h(\vec{\beta}_i)$ is $\vec{0}_W$. Extending linearly, we have that h maps each $\vec{v} \in V$ to $\vec{0}_W$, and h is the zero map.

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Example The zero map $z\colon \mathbb{R}^2 \to \mathbb{R}^3$ is represented by the 2×3 zero matrix

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to any pair of bases.

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Proof Suppose that the matrix H is $m\times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

$$\begin{aligned}\{h(\vec{v})\mid\vec{v}\in V\}&=\{h(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)\mid c_1,\ldots,c_n\in\mathbb{R}\}\\&=\{c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)\mid c_1,\ldots,c_n\in\mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1),\ldots,h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

2.4 *Theorem* The rank of a matrix equals the rank of any map that it represents.

Proof Suppose that the matrix H is $m\times n$. Fix domain and codomain spaces V and W of dimension n and m with bases $B=\langle\vec{\beta}_1,\ldots,\vec{\beta}_n\rangle$ and D. Then H represents some linear map h between those spaces with respect to these bases whose range space

$$\begin{aligned}\{h(\vec{v})\mid\vec{v}\in V\}&=\{h(c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n)\mid c_1,\ldots,c_n\in\mathbb{R}\}\\&=\{c_1h(\vec{\beta}_1)+\cdots+c_nh(\vec{\beta}_n)\mid c_1,\ldots,c_n\in\mathbb{R}\}\end{aligned}$$

is the span $[\{h(\vec{\beta}_1),\ldots,h(\vec{\beta}_n)\}]$. The rank of the map h is the dimension of this range space.

The rank of the matrix is the dimension of its column space, the span of the set of its columns $[\{\mathrm{Rep}_D(h(\vec{\beta}_1)),\ldots,\mathrm{Rep}_D(h(\vec{\beta}_n))\}]$.

To see that the two spans have the same dimension, recall from the proof of Lemma I.2.5 that if we fix a basis then representation with respect to that basis gives an isomorphism $\mathrm{Rep}_D : W \to \mathbb{R}^m$. Under this isomorphism there is a linear relationship among members of the range space if and only if the same relationship holds in the column space, e.g, $\vec{0} = c_1 \cdot h(\vec{\beta}_1) + \cdots + c_n \cdot h(\vec{\beta}_n)$ if and only if $\vec{0} = c_1 \cdot \mathrm{Rep}_D(h(\vec{\beta}_1)) + \cdots + c_n \cdot \mathrm{Rep}_D(h(\vec{\beta}_n))$. Hence, a subset of the range space is linearly independent if and only if the corresponding subset of the column space is linearly independent. Therefore the size of the largest linearly independent subset of the range space equals the size of the largest linearly independent subset of the column space, and so the two spaces have the same dimension. QED

Example The range of the linear transformation $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 2a-b \\ 2a-b \end{pmatrix}$$

is the line $x = y$. Thus the map t's rank is 1.

Represent t first with respect the standard bases $\mathcal{E}_2, \mathcal{E}_2$ and then with respect to these.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle, \ D = \langle \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \end{pmatrix} \rangle$$

The standard basis case is easy. This is the other calculation.

$$\mathrm{Rep}_D(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix} -3 \\ -3 \end{pmatrix}) = \begin{pmatrix} -6 \\ -9 \end{pmatrix}$$

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We end with these two representations, each matrices of rank 1.

$$\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \qquad \mathrm{Rep}(t) = \begin{pmatrix} 2 & -6 \\ 3 & -9 \end{pmatrix}$$

2.6 *Corollary* Let h be a linear map represented by a matrix H. Then h is onto if and only if the rank of H equals the number of its rows, and h is one-to-one if and only if the rank of H equals the number of its columns.

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Proof For the onto half, the dimension of the range space of h is the rank of h, which equals the rank of H by the theorem. Since the dimension of the codomain of h equals the number of rows in H, if the rank of H equals the number of rows then the dimension of the range space equals the dimension of the codomain. But a subspace with the same dimension as its superspace must equal that superspace (because any basis for the range space is a linearly independent subset of the codomain whose size is equal to the dimension of the codomain, and thus so this basis for the range space must also be a basis for the codomain).

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For the other half, a linear map is one-to-one if and only if it is an isomorphism between its domain and its range, that is, if and only if its domain has the same dimension as its range. The number of columns in h is the dimension of h's domain and by the theorem the rank of H equals the dimension of h's range. QED

2.7 *Definition* A linear map that is one-to-one and onto is ***nonsingular***, otherwise it is ***singular***. That is, a linear map is nonsingular if and only if it is an isomorphism.

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Proof Assume that the map $h: V \to W$ is nonsingular. Corollary 2.6 says that for any matrix H representing that map, because h is onto the number of rows of H equals the rank of H, and because h is one-to-one the number of columns of H is also equal to the rank of H. Hence H is square.

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Next assume that H is square, $n \times n$. The matrix H is nonsingular if and only if its row rank is n, which is true if and only if H's rank is n by Theorem Two.III.3.11 , which is true if and only if h's rank is n by Theorem 2.4 , which is true if and only if h is an isomorphism by Theorem I.2.3 . (This last holds because the domain of h is n-dimensional as it is the number of columns in H.) QED

Example This matrix

$$\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

is nonsingular since by inspection its two rows form a linearly independent set. So any map, with any domain and codomain, and represented by this matrix with respect to any bases, is an isomorphism.

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Example Gauss's method will show that this matrix

$$\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 1 \\ -1 & 0 & 5 \end{pmatrix}$$

is singular so any map that it represents will be a homomorphism that is not an isomorphism.