

Three.II Homomorphisms

Linear Algebra

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Definition

Homomorphism

1.1 *Definition* A function between vector spaces $h: V \rightarrow W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

Example The function $h: \mathcal{P}_2 \rightarrow \mathbb{R}^2$ given by

$$h(a + bx + cx^2) = \begin{pmatrix} a + c \\ 0 \end{pmatrix}$$

is a homomorphism (it is neither one-to-one nor onto). We will verify that it respects the addition and scalar multiplication operations.

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Addition is routine.

$$\begin{aligned} & h((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) \\ &= h((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) \\ &= \begin{pmatrix} a_1 + a_2 + c_1 + c_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + c_1 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 + c_2 \\ 0 \end{pmatrix} \\ &= h(a_1 + b_1x + c_1x^2) + h(a_2 + b_2x + c_2x^2) \end{aligned}$$

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So is scalar multiplication.

$$r \cdot h(a + bx + cx^2) = r \cdot \begin{pmatrix} a + c \\ 0 \end{pmatrix} = \begin{pmatrix} ra + rc \\ 0 \end{pmatrix} = h(r(a + bx + cx^2))$$

Example Of these two maps $h, g: \mathbb{R}^2 \rightarrow \mathbb{R}$, the first is linear while the second is not.

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{h} 2x - 3y \qquad \begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{g} 2x - 3y + 1$$

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The map h respects addition

$$\begin{aligned} h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= h\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) = 2(x_1 + x_2) - 3(y_1 + y_2) \\ &= (2x_1 - 3y_1) + (2x_2 - 3y_2) = h\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + h\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \end{aligned}$$

and scalar multiplication.

$$r \cdot h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = r \cdot (2x - 3y) = 2rx - 3ry = (2r)x - (3r)y = h\left(r \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

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In contrast, g does not respect addition.

$$g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -17 \quad \text{while} \quad g\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) + g\left(\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = -16$$

We proved the following two in the context of studying isomorphisms.

1.6 *Lemma* A homomorphism sends the zero vector to the zero vector.

1.7 *Lemma* The following are equivalent for any map $f: V \rightarrow W$ between vector spaces.

- (1) f is a homomorphism
- (2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$
- (3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \dots, c_n \in \mathbb{R}$ and $\vec{v}_1, \dots, \vec{v}_n \in V$

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Example Between any two vector spaces the zero map $Z: V \rightarrow W$, defined by $Z(\vec{v}) = \vec{0}_W$ is a homomorphism. The check is:
 $Z(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \vec{0}_W = \vec{0}_W + \vec{0}_W = c_1 Z(\vec{v}_1) + c_2 Z(\vec{v}_2).$

Example The *inclusion map* $\iota: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$\iota\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism. Here is the verification.

$$\begin{aligned} \iota\left(c_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= \iota\left(\begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_1 x_1 \\ c_1 y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} c_2 x_2 \\ c_2 y_2 \\ 0 \end{pmatrix} \\ &= c_1 \cdot \iota\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + c_2 \cdot \iota\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \end{aligned}$$

Example Consider this function $h: \mathcal{P}_1 \rightarrow \mathcal{P}_1$.

$$h(a + bx) = b + bx$$

Here are two examples of the action of this function:

$$h(1 + 2x) = 2 + 2x \text{ and } h(3 - x) = -1 - x.$$

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This function is linear.

$$\begin{aligned} h(c_1 \cdot (a_1 + b_1x) + c_2 \cdot (a_2 + b_2x)) \\ &= h((c_1a_1 + c_2a_2) + (c_1b_1 + c_2b_2)x) \\ &= (c_1b_1 + c_2b_2) + (c_1b_1 + c_2b_2)x \\ &= (c_1b_1 + c_1b_1x) + (c_2b_2 + c_2b_2x) \\ &= c_1 \cdot h(a_1 + b_1x) + c_2 \cdot h(a_2 + b_2x) \end{aligned}$$

Example The derivative map $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$ is given by $d/dx(ax^2 + bx + c) = 2ax + b$. For instance, $d/dx(3x^2 - 2x + 4) = 6x - 2$ and $d/dx(x^2 + 1) = 2x$.

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$$\begin{aligned} & d/dx \left(r_1(a_1x^2 + b_1x + c_1) + r_2(a_2x^2 + b_2x + c_2) \right) \\ &= d/dx \left((r_1a_1 + r_2a_2)x^2 + (r_1b_1 + r_2b_2)x + (r_1c_1 + r_2c_2) \right) \\ &= 2(r_1a_1 + r_2a_2)x + (r_1b_1 + r_2b_2) \\ &= (2r_1a_1x + r_1b_1) + (2r_2a_2x + r_2b_2) \\ &= r_1 \cdot d/dx(a_1x^2 + b_1x + c_1) + r_2 \cdot d/dx(a_2x^2 + b_2x + c_2) \end{aligned}$$

Example The *trace* of a square matrix is the sum down the upper-left to lower-right diagonal. Thus $\text{Tr}: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}$ is this.

$$\text{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + b$$

It is linear.

$$\begin{aligned} \text{Tr}\left(r_1 \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \\ = \text{Tr}\left(\begin{pmatrix} r_1 a_1 + r_2 a_2 & r_1 b_1 + r_2 b_2 \\ r_1 c_1 + r_2 c_2 & r_1 d_1 + r_2 d_2 \end{pmatrix}\right) \\ = (r_1 a_1 + r_2 a_2) + (r_1 d_1 + r_2 d_2) \\ = r_1(a_1 + d_1) + r_2(a_2 + d_2) \\ = r_1 \cdot \text{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right) + r_2 \cdot \text{Tr}\left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \end{aligned}$$

1.9 *Theorem* A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ and W is a vector space with elements $\vec{w}_1, \dots, \vec{w}_n$ (perhaps not distinct elements) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to \vec{w}_i , and that homomorphism is unique.

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Proof For any input $\vec{v} \in V$ let its expression with respect to the basis be $\vec{v} = c_1 \vec{\beta}_1 + \dots + c_n \vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v}) = c_1 \vec{w}_1 + \dots + c_n \vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector \vec{v} is unique.

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This map is a homomorphism because it preserves linear combinations: where $\vec{v}_1 = c_1 \vec{\beta}_1 + \dots + c_n \vec{\beta}_n$ and $\vec{v}_2 = d_1 \vec{\beta}_1 + \dots + d_n \vec{\beta}_n$, here is the calculation.

$$\begin{aligned} h(r_1 \vec{v}_1 + r_2 \vec{v}_2) &= h((r_1 c_1 + r_2 d_1) \vec{\beta}_1 + \dots + (r_1 c_n + r_2 d_n) \vec{\beta}_n) \\ &= (r_1 c_1 + r_2 d_1) \vec{w}_1 + \dots + (r_1 c_n + r_2 d_n) \vec{w}_n \\ &= r_1 h(\vec{v}_1) + r_2 h(\vec{v}_2) \end{aligned}$$

This map is unique because if $\hat{h}: V \rightarrow W$ is another homomorphism satisfying that $h'(\vec{\beta}_i) = \vec{w}_i$ for each i then h and h' agree on all of the vectors in the domain.

$$\begin{aligned} h'(\vec{v}) &= h'(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h'(\vec{\beta}_1) + \cdots + c_nh'(\vec{\beta}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = h(\vec{v}) \end{aligned}$$

They have the same action so they are the same function. QED

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1.10 *Definition* Let V and W be vector spaces and let $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ be a basis for V . A function defined on that basis $f: B \rightarrow W$ is *extended linearly* to a function $\hat{f}: V \rightarrow W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$.

Example One basis of the space of quadratic polynomials \mathcal{P}_2 is $B = \langle x^2, x, 1 \rangle$. We can define a map $\text{eval}_3: \mathcal{P}_2 \rightarrow \mathbb{R}$ by specifying its action on that basis

$$x^2 \xrightarrow{\text{eval}_3} 9 \quad x \xrightarrow{\text{eval}_3} 3 \quad 1 \xrightarrow{\text{eval}_3} 1$$

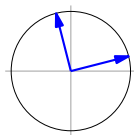
and then extending linearly.

$$\begin{aligned} \text{eval}_3(ax^2 + bx + c) &= a \cdot \text{eval}_3(x^2) + b \cdot \text{eval}_3(x) + c \cdot \text{eval}_3(1) \\ &= 9a + 3b + c \end{aligned}$$

The action of this map on the basis elements is to plug the value 3 in for x . That remains true when we extend linearly, so $\text{eval}_3(p(x)) = p(3)$. For instance, $\text{eval}_3(x^2 + 2x + 3) = 18$.

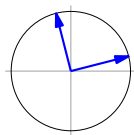
Example Consider the standard basis \mathcal{E}_2 for the vector space \mathbb{R}^2 . We can specify a rotation of the two basis vectors as here.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$$



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Extend linearly to get a homomorphism $t_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

$$\begin{aligned} t_\theta\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= t_\theta\left(x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x \cdot t_\theta\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y \cdot t_\theta\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= x \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + y \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{pmatrix} \end{aligned}$$

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Example In \mathbb{R}^3 the function f_{yz} that reflects vectors over the yz -plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{f_{yz}} \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

is a linear transformation.

$$\begin{aligned} f_{yz}\left(r_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= f_{yz}\left(\begin{pmatrix} r_1 x_1 + r_2 x_2 \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix}\right) = \begin{pmatrix} -(r_1 x_1 + r_2 x_2) \\ r_1 y_1 + r_2 y_2 \\ r_1 z_1 + r_2 z_2 \end{pmatrix} \\ &= r_1 \begin{pmatrix} -x_1 \\ y_1 \\ z_1 \end{pmatrix} + r_2 \begin{pmatrix} -x_2 \\ y_2 \\ z_2 \end{pmatrix} = r_1 f_{yz}\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + r_2 f_{yz}\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) \end{aligned}$$

1.17 *Lemma* For vector spaces V and W , the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W .

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

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Proof This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under the operations. Let $f, g: V \rightarrow W$ be linear. Then the operation of function addition is preserved

$$\begin{aligned}(f + g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1f(\vec{v}_1) + c_2f(\vec{v}_2) + c_1g(\vec{v}_1) + c_2g(\vec{v}_2) \\ &= c_1(f + g)(\vec{v}_1) + c_2(f + g)(\vec{v}_2)\end{aligned}$$

as is the operation of scalar multiplication of a function.

$$\begin{aligned}(r \cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1f(\vec{v}_1) + c_2f(\vec{v}_2)) \\ &= c_1(r \cdot f)(\vec{v}_1) + c_2(r \cdot f)(\vec{v}_2)\end{aligned}$$

Hence $\mathcal{L}(V, W)$ is a subspace.

QED

Example Consider $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$. Fix these bases for the two spaces.

$$B_{\mathbb{R}} = \mathcal{E}_1 = \langle 1 \rangle \quad B_{\mathbb{R}^2} = \mathcal{E}_2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

A linear map is determined by its action on a basis of the domain space. Thus the elements of $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$

$$1 \xrightarrow{t} c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are determined by c_1, c_2 ; we could write each such map as $t = t_{c_1, c_2}$. So $\mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ is a dimension 2 space.

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Example Similarly, for $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$, fix $\mathcal{E}_2 = \langle \vec{e}_1, \vec{e}_2 \rangle$ and $\mathcal{E}_3 = \langle \vec{e}_1, \vec{e}_2, \vec{e}_3 \rangle$ (note that one \vec{e}_1 is two-tall while the other is three-tall). We can characterize the elements in this way.

$$\vec{e}_1 \xrightarrow{t} c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3 \quad \vec{e}_2 \xrightarrow{t} d_1 \vec{e}_1 + d_2 \vec{e}_2 + d_3 \vec{e}_3$$

So, $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is six-dimensional.

Range space and null space

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Example For any angle θ , the function $t_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates vectors counterclockwise by θ is a homomorphism. In the domain \mathbb{R}^2 each line through the origin is a subspace. The image of that line under this map is another line through the origin, a subspace of the codomain \mathbb{R}^2 .

Range space

2.2 *Definition* The *range space* of a homomorphism $h: V \rightarrow W$ is

$$\mathcal{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

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Example Projection $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

is a linear map; the check is routine. The range space is $\mathcal{R}(\pi) = \mathbb{R}^2$ because given a vector $\vec{w} \in \mathbb{R}^2$

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix}$$

we can find a $\vec{v} \in \mathbb{R}^3$ that maps to it, specifically any vector with first component a and second component b . Thus the rank of π is 2.

Example The derivative map $d/dx: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is linear. Its range is $\mathcal{R}(d/dx) = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$. (Verifying that every member of that space is the derivative of a fourth degree polynomial is easy.) The rank of the derivative function is 3.

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Example This map from $\mathcal{M}_{2 \times 2}$ to \mathbb{R}^2 is linear; the check is routine.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

The range space is this line through the origin

$$\left\{ \begin{pmatrix} t \\ 2t \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

(every member of that set is the image

$$\begin{pmatrix} t \\ 2t \end{pmatrix} = h\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right)$$

of a 2×2 matrix). The rank of this map is 1.

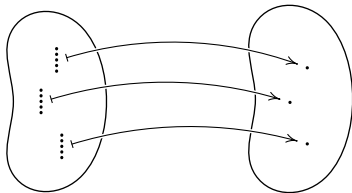
Homomorphisms organize the domain

When we moved from studying isomorphisms to studying homomorphisms we dropped the requirements that the maps be onto and one-to-one. We've seen that dropping the onto condition has no effect in the sense that any homomorphism $h: V \rightarrow W$ is onto some space, namely its range space $\mathcal{R}(h)$.

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Consider the effect of dropping the one-to-one condition. Now, for some vector $\vec{w} \in W$ in the range there may be many vectors $\vec{v} \in V$ mapped to \vec{w} .



Recall that for any function $h: V \rightarrow W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$. Above, the left side shows three inverse image sets.

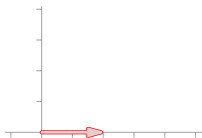
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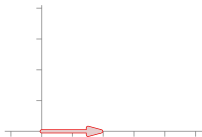
We can identify the codomain \mathbb{R} with the x -axis in \mathbb{R}^2 . Here is a member of the x -axis, drawn in red.



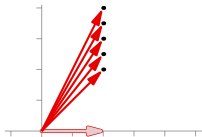
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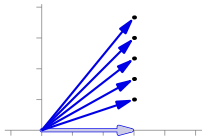


Next are some elements of $\pi^{-1}(2)$, shown both as dots (as in the bean diagram) and as vectors. These are also shown in red because they are associated by π with 2.

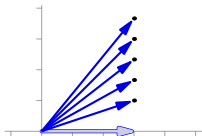


Think of these as “2 vectors.”

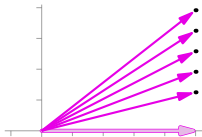
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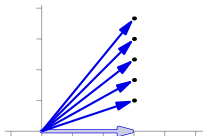


The definition of addition preservation is that $\pi(\vec{u} + \vec{v}) = \pi(\vec{u}) + \pi(\vec{v})$. Therefore where $\pi(\vec{u}) = 2$ and $\pi(\vec{v}) = 3$, the vector sum $\vec{u} + \vec{v}$ will be mapped by π to 5.

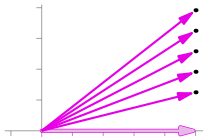


Thus, we can understand the definition of addition preservation as: red plus blue makes purple — a “2 vector” plus a “3 vector” sums to a “5 vector.”

Here are some “3-vectors,” inverse images of 3.



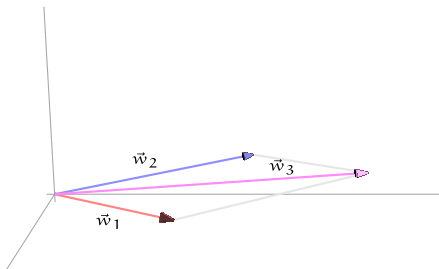
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Thus, we can understand the definition of addition preservation as: red plus blue makes purple—a “2 vector” plus a “3 vector” sums to a “5 vector.” Preservation of scalar multiplication has a similar interpretation.

The same analysis holds for any homomorphism.

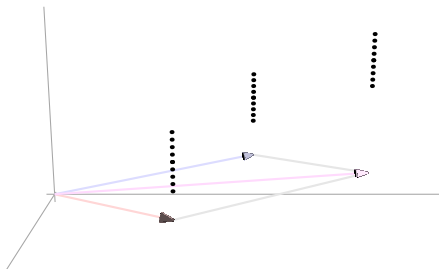
Example The projection $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear map. As above we can identify the codomain with the xy -plane inside of \mathbb{R}^3 .



In the xy -plane, red plus blue makes purple as shown by the parallelogram. We will illustrate that a \vec{w}_1 vector plus a \vec{w}_2 vector is a \vec{w}_3 vector

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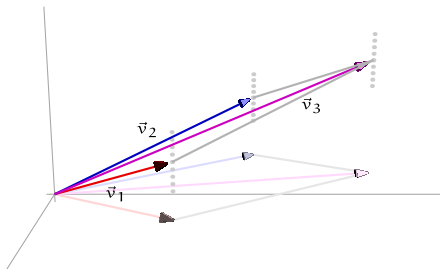


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Consider the inverse image sets; the diagram shows some of the infinitely many points in each $\pi^{-1}(\vec{w}_i)$. If we take a $\vec{v}_1 \in \pi^{-1}(\vec{w}_1)$ and a $\vec{v}_2 \in \pi^{-1}(\vec{w}_2)$ then they sum to a $\vec{v}_3 \in \pi^{-1}(\vec{w}_3)$.

This also holds when the spaces are not ones that we can conveniently draw.

Example Consider $h: \mathcal{P}_2 \rightarrow \mathbb{R}^2$

$$ax^2 + bx + c \mapsto \begin{pmatrix} b \\ c \end{pmatrix}$$

and consider these three members of the range.

$$\vec{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Think of these as “ \vec{w}_1 vectors.” Some examples are

$3x^2 + x + 1$, $3x^2 + x - 4$, and $-2x^2 + x$.

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As above, any $\vec{v}_1 \in h^{-1}(\vec{w}_1)$ plus any $\vec{v}_2 \in h^{-1}(\vec{w}_2)$ equals a $\vec{v}_3 \in h^{-1}(\vec{w}_3)$: a quadratic with an x coefficient of 1 plus a quadratic with an x coefficient of -1 equals a quadratic with an x coefficient of 0. That is, a “ \vec{w}_1 vector” plus a “ \vec{w}_2 vector” is a “ \vec{w}_3 vector.”

In each of those examples, because there is a homomorphism $h: V \rightarrow W$ we can view the domain V as organized into the inverse images $h^{-1}(\vec{w})$ for each $\vec{w} \in \mathcal{R}(h)$.

It is “organized” because these inverse image sets reflect the structure of the range in that a “ \vec{w}_1 vector” plus a “ \vec{w}_2 vector” equals a “ $\vec{w}_1 + \vec{w}_2$ vector.”

Null space

Vector spaces have a distinguished element, $\vec{0}$. So we next consider the inverse image of that element $h^{-1}(\vec{0})$.

2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

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2.10 *Lemma* For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

Proof Let $h: V \rightarrow W$ be a homomorphism and let S be a subspace of the range space of h . Consider the inverse image of S . It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$ and $\vec{0}_W$ is an element of S as S is a subspace. To finish we show that $h^{-1}(S)$ is closed under linear combinations. Let \vec{v}_1 and \vec{v}_2 be two of its elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S . Then $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of the inverse image $h^{-1}(S)$ because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1h(\vec{v}_1) + c_2h(\vec{v}_2)$ is a member of S . QED

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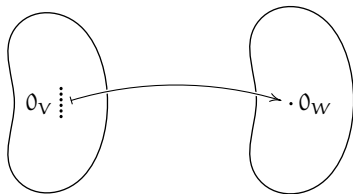
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Note This result complements Lemma 2.1 that for any subspace of the domain, its image is a subspace of the range.

2.11 *Definition* The *null space* or *kernel* of a linear map $h: V \rightarrow W$ is the inverse image of $\vec{0}_W$.

$$\mathcal{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

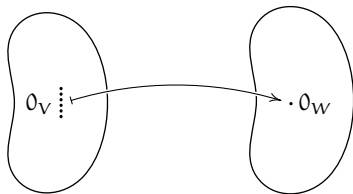
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Note Strictly, the trivial subspace of the codomain is not $\vec{0}_W$, it is $\{\vec{0}_W\}$, and so we may think to write the nullspace as $h^{-1}(\{\vec{0}_W\})$. But we have defined the two sets $h^{-1}(\vec{w})$ and $h^{-1}(\{\vec{w}\})$ to be equal and the first is easier to write.

Example The derivative function $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$ is linear.

$$\mathcal{N}(d/dx) = \{ax^2 + bx + c \mid 2ax + b = 0\}$$

The polynomial $2ax + b$ equals the zero polynomial if and only if they have the same constant coefficient (which implies that $b = 0$), the same coefficient of x (which implies that $a = 0$), and the same coefficient of x^2 (which gives no restriction). Thus the nullspace is this, and the nullity is 1.

$$\mathcal{N}(d/dx) = \{ax^2 + bx + c \mid a = 0, b = 0, c \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

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$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this null space.

$$\mathcal{N}(h) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid 2a + b = 0 \right\} = \left\{ \begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Its nullity is 1.

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Example The dilation function $d_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has a trivial null space $\mathcal{N}(d_3) = \{\vec{0}\}$ and its nullity is 0.

Rank plus nullity

2.14 *Theorem* A linear map's rank plus its nullity equals the dimension of its domain.

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Proof Let $h: V \rightarrow W$ be linear and let $B_N = \langle \vec{\beta}_1, \dots, \vec{\beta}_k \rangle$ be a basis for the null space. Expand that to a basis $B_V = \langle \vec{\beta}_1, \dots, \vec{\beta}_k, \vec{\beta}_{k+1}, \dots, \vec{\beta}_n \rangle$ for the entire domain, using Corollary Two.III.2.13 . We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \dots, h(\vec{\beta}_n) \rangle$ is a basis for the range space. Then counting the size of the bases gives the result.

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To see that B_R is linearly independent, consider $\vec{0}_W = c_{k+1}h(\vec{\beta}_{k+1}) + \dots + c_n h(\vec{\beta}_n)$. We have $\vec{0}_W = h(c_{k+1}\vec{\beta}_{k+1} + \dots + c_n \vec{\beta}_n)$ and so $c_{k+1}\vec{\beta}_{k+1} + \dots + c_n \vec{\beta}_n$ is in the null space of h . As B_N is a basis for the null space there are scalars c_1, \dots, c_k satisfying this relationship.

$$c_1 \vec{\beta}_1 + \dots + c_k \vec{\beta}_k = c_{k+1} \vec{\beta}_{k+1} + \dots + c_n \vec{\beta}_n$$

But this is an equation among members of B_V , which is a basis for V , so each c_i equals 0. Therefore B_R is linearly independent.

To show that B_R spans the range space consider a member of the range space $h(\vec{v})$. Express \vec{v} as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V . This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1h(\vec{\beta}_1) + \cdots + c_kh(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$ and since $\vec{\beta}_1, \dots, \vec{\beta}_k$ are in the null space, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_nh(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R , and so B_R spans the range space. QED

Example Projection $\pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

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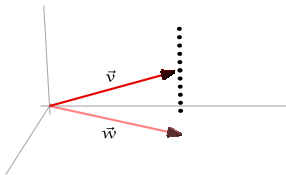
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This example illustrates the proof's steps particularly clearly. Take the basis $B_N = \langle \vec{e}_3 \rangle$ for the null space. Expand that to the basis \mathcal{E}_3 for the entire domain. The action of π is that the third dimension collapses: a linear combinations in the domain $\vec{v} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3$ is sent to the combination $\vec{w} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + \vec{0}$ in the range. Geometrically, the domain is organized into inverse images that are vertical lines, just like the null space.



Example The derivative function $d/dx: \mathcal{P}_2 \rightarrow \mathcal{P}_1$ has this range space

$$\mathcal{R}(d/dx) = \{2ax + b \mid a, b \in \mathbb{R}\} = \mathcal{P}_1$$

(any $cx + d \in \mathcal{P}_1$ is the image of $ax^2 + bx + c$ where $a = c/2$, $b = d$, and c can be any real) and this null space (calculated earlier).

$$\mathcal{N}(d/dx) = \{c \mid c \in \mathbb{R}\}$$

The rank is 2 while the nullity is 1, and they add to the dimension of the domain \mathcal{P}_2 .

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Example The function $h: \mathbb{R}^2 \rightarrow \mathbb{R}^1$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto 2a + b$$

has this range space

$$\mathcal{R}(h) = \{2a + b \mid a, b \in \mathbb{R}\} = \{c \mid c \in \mathbb{R}\}$$

and this null space (calculated earlier).

$$\mathcal{N}(h) = \left\{ \begin{pmatrix} -b/2 \\ b \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Its rank is 1 and its nullity is 1. Its domain \mathbb{R}^2 has dimension 2.

Example The dilation function $d_3: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 3a \\ 3b \end{pmatrix}$$

has range space \mathbb{R}^2 and a trivial nullspace $\mathcal{N}(d_3) = \{\vec{0}\}$. So its rank is 2 and its nullity is 0.

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Example The homomorphism $f: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}^2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{f} \begin{pmatrix} a + b \\ c + d \end{pmatrix}$$

has range space equal to \mathbb{R}^2 (to get a vector with a first component of x and a second component of y we can take $a = x$, $b = 0$, $c = y$, and $d = 0$). Thus f 's rank is 2. We found its null space earlier

$$\mathcal{N}(f) = \left\{ \begin{pmatrix} -b & b \\ -d & d \end{pmatrix} \mid b, d \in \mathbb{R} \right\}$$

and its nullity is 2.

2.18 *Lemma* Under a linear map, the image of a linearly dependent set is linearly dependent.

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Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1h(\vec{v}_1) + \cdots + c_nh(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

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Example The trace function $\text{Tr}: \mathcal{M}_{2 \times 2} \rightarrow \mathbb{R}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

is linear. This set of matrices is dependent.

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

The three matrices map to 1, 0, and 2 respectively. The set $\{1, 0, 2\}$ is linearly dependent in \mathbb{R} .

2.20 *Theorem* In an n -dimensional vector space V , these are equivalent statements about a linear map $h: V \rightarrow W$.

- (1) h is one-to-one
- (2) h has an inverse from its range to its domain that is a linear map
- (3) $\mathcal{N}(h) = \{\vec{0}\}$, that is, $\text{nullity}(h) = 0$
- (4) $\text{rank}(h) = n$
- (5) if $\langle \vec{\beta}_1, \dots, \vec{\beta}_n \rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$ is a basis for $\mathcal{R}(h)$

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Proof We will first show that $(1) \iff (2)$. We will then show that $(1) \implies (3) \implies (4) \implies (5) \implies (2)$.

For $(1) \implies (2)$, suppose that the linear map h is one-to-one, and therefore has an inverse $h^{-1}: \mathcal{R}(h) \rightarrow V$. The domain of that inverse is the range of h and thus a linear combination of two members of it has the form $c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$. On that combination, the inverse h^{-1} gives this.

$$\begin{aligned} h^{-1}(c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)) &= h^{-1}(h(c_1 \vec{v}_1 + c_2 \vec{v}_2)) \\ &= h^{-1} \circ h (c_1 \vec{v}_1 + c_2 \vec{v}_2) \\ &= c_1 \vec{v}_1 + c_2 \vec{v}_2 \\ &= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2)) \end{aligned}$$

Thus if a linear map has an inverse then the inverse must be linear. But this also gives the $(2) \implies (1)$ implication, because the inverse itself must be one-to-one.

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Of the remaining implications, $(1) \implies (3)$ holds because any homomorphism maps $\vec{0}_V$ to $\vec{0}_W$, but a one-to-one map sends at most one member of V to $\vec{0}_W$.

Next, $(3) \implies (4)$ is true since rank plus nullity equals the dimension of the domain.

For (4) \implies (5), to show that $\langle h(\vec{\beta}_1), \dots, h(\vec{\beta}_n) \rangle$ is a basis for the range space we need only show that it is a spanning set, because by assumption the range has dimension n . Consider $h(\vec{v}) \in \mathcal{R}(h)$. Expressing \vec{v} as a linear combination of basis elements produces $h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \dots + c_n\vec{\beta}_n)$, which gives that $h(\vec{v}) = c_1h(\vec{\beta}_1) + \dots + c_nh(\vec{\beta}_n)$, as desired. QED