Three.I Isomorphisms

Linear Algebra
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Definition and examples

Example People often have the intuition that the two vector spaces $\mathbb{R}^2$ and $\mathcal{P}_1$ are "the same," for instance in that

$$\begin{pmatrix}1\\2\end{pmatrix} \quad \text{is just like} \quad 1+2x$$
$$\text{and} \begin{pmatrix}-3\\1/2\end{pmatrix} \quad \text{is just like} \quad -3-(1/2)x$$

etc. What makes the one "just like" the other is that this association holds through the operations of addition

$$\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-3\\1/2\end{pmatrix} = \begin{pmatrix}-2\\5/2\end{pmatrix}$$
$$\text{is just like} \quad (1+2x)+(-3+(1/2)x)=-2+(5/2)x$$

and scalar multiplication.

$$3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} \quad \text{is just like} \quad 3(1+2x)=3+6x$$

More generally, we can express the same-ness of the spaces by associating each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

such that the association holds through the vector space operations of addition

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$
$$\longleftrightarrow \quad (a_1 + b_1 x) + (a_2 + b_2 x) = (a_1 + a_2) + (b_1 + b_2)x$$

and scalar multiplication.

$$r \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \quad \longleftrightarrow \quad r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, this association matches these two.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

Example We can think of $\mathcal{M}_{2\times2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad\longleftrightarrow\quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, this association matches these two.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad\longleftrightarrow\quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

This association holds up under addition.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix}$$

Here is an example of that with particular vectors.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -5 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -5 \end{pmatrix}$$

Here is an example of that with particular vectors.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -5 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ -5 \end{pmatrix}$$

The association also holds under scalar multiplication.

$$r \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} \quad \longleftrightarrow \quad r \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \\ rd \end{pmatrix}$$

This illustrates with particular vectors.

$$2 \cdot \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \quad \longleftrightarrow \quad 2 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -4 \end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1. is a correspondence: f is one-to-one and onto;
2. *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

   and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to the space $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1x + a_2x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

For instance, here is the action of f on two inputs.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check conditions (1) and (2).

The first part of (1) is that f is one-to-one. We usually verify this by assuming that the function yields the same output on two inputs and then show that the two inputs must therefore be equal. So assume that $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. By definition of f we have

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the starting inputs are equal $a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$ and so f is one-to-one.

The first part of (1) is that f is one-to-one. We usually verify this by assuming that the function yields the same output on two inputs and then show that the two inputs must therefore be equal. So assume that $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. By definition of f we have

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the starting inputs are equal $a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$ and so f is one-to-one.

The second part of (1) is that f is onto. We usually verify this by taking an element of the codomain and producing an element of the domain that maps to it. So consider this member of $\mathbb{R}^3$.

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Observe that it is the image under f of the member $u + vx + wx^2$ of the domain. Thus f is onto.

Condition (2) also has two halves. First we must show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$f(\,(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\,) = f((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2)$$

The function maps the linear polynomial on the right to this vector.

$$\begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

The right side is $f(a_0+a_1x+a_2x^2)+f(b_0+b_1x+b_2x^2)$, as required.

Condition (2) also has two halves. First we must show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$f(\,(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\,) = f((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2)$$

The function maps the linear polynomial on the right to this vector.

$$\begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

The right side is $f(a_0+a_1x+a_2x^2)+f(b_0+b_1x+b_2x^2)$, as required.

We finish by checking that f preserves scalar multiplication.

$$r\cdot f(a_0+a_1x+a_2x^2) = r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} = f(\,(ra_0)+(ra_1)x+(ra_2)x^2)$$

QED

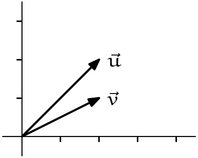
Automorphisms

1.6 *Definition* An *automorphism* is an isomorphism of a space with itself.

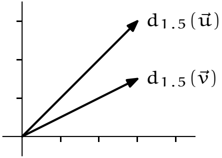
Automorphisms

1.6 *Definition* An *automorphism* is an isomorphism of a space with itself.

1.7 *Example* A *dilation* map $d_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



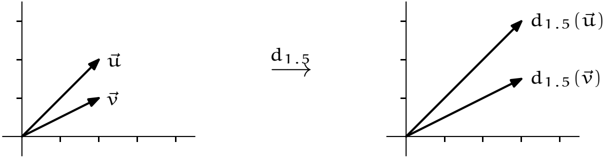
$\xrightarrow{d_{1.5}}$



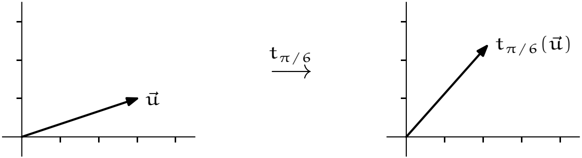
Automorphisms

1.6 *Definition* An ***automorphism*** is an isomorphism of a space with itself.

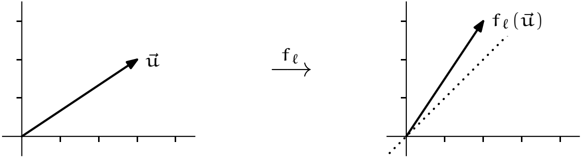
1.7 *Example* A ***dilation*** map $d_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



A ***rotation*** or ***turning map*** $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ is an automorphism.



A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell\colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



Checking that each of these is an isomorphism is an exercise.

1.9 *Lemma* An isomorphism maps a zero vector to a zero vector.

1.9 *Lemma* An isomorphism maps a zero vector to a zero vector.

Proof Where $f\colon V \to W$ is an isomorphism, fix any $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.10 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

1.10 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

Proof Since the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear, we need only show that (1) $\Longrightarrow$ (3). Assume statement (1). We will prove statement (3) by induction on the number of summands n.

1.10 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\,f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

Proof Since the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear, we need only show that (1) $\Longrightarrow$ (3). Assume statement (1). We will prove statement (3) by induction on the number of summands n.

The one-summand base case, that $f(c\vec{v}_1) = c\,f(\vec{v}_1)$, is covered by the assumption of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands, that is, whenever $n = 1$, or $n = 2$, $\ldots$, or $n = k$. Consider the $k+1$-summand case. Use the first half of (1) to break the sum along the final '+'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the k-term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Now the second half of (1) gives

$$= c_1\, f(\vec{v}_1) + \cdots + c_k\, f(\vec{v}_k) + c_{k+1}\, f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED

That result shortens the checking that a function preserves the structure of a vector space.

Example This line through the origin

$$L = \{ t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

is a vector space under the addition and scalar multiplication operations that it inherits from $\mathbb{R}^2$.

$$t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (t_1 + t_2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad r\left(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = (rt) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This space is isomorphic to $\mathbb{R}^1$ under this map.

$$f\left(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = t$$

We first verify that f is one-to-one. Suppose that f maps two members of L to the same output.

$$f(\,t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,) = f(\,t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,)$$

Then by definition of the function f we have that $t_1 = t_2$ and so the two members of L are equal.

We first verify that f is one-to-one. Suppose that f maps two members of L to the same output.

$$f(\, t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) = f(\, t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

Then by definition of the function f we have that $t_1 = t_2$ and so the two members of L are equal.

Next we check that f is an onto map. Consider this member of the codomain: $r \in \mathbb{R}$. There is a member of the domain that maps to it, namely this member of L.

$$f(\, r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

We first verify that f is one-to-one. Suppose that f maps two members of L to the same output.

$$f(\, t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) = f(\, t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

Then by definition of the function f we have that $t_1 = t_2$ and so the two members of L are equal.

Next we check that f is an onto map. Consider this member of the codomain: $r \in \mathbb{R}$. There is a member of the domain that maps to it, namely this member of L.

$$f(\, r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

To finish, we use the lemma to check that f preserves structure.

$$f(\, t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) = f(\, \begin{pmatrix} t_1 + t_2 \\ 2(t_1 + t_2) \end{pmatrix} \,) = t_1 + t_2 = f(\, t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,) + f(\, t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \,)$$

Dimension characterizes isomorphism

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. Because an isomorphism is a correspondence, f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. Because an isomorphism is a correspondence, f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

To finish we will show that because f preserves linear combinations, so also does f^{-1}. Let $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$

$$\begin{aligned}
f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(\, c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\big) \\
&= f^{-1}(\, f\big(c_1\vec{v}_1 + c_2\vec{v}_2)\,\big) \\
&= c_1\vec{v}_1 + c_2\vec{v}_2 \\
&= c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2)
\end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.10 this map preserves structure. QED

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

Proof We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This calculation shows that it also preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

Proof We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This calculation shows that it also preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

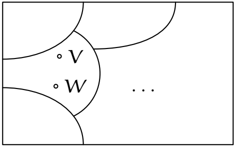
Symmetry, that if V is isomorphic to W then also W is isomorphic to V, holds by Lemma 2.1 since an isomprphism map from V to W is paired with an isomorphism from W to V.

Finally, we must check transitivity, that if V is isomorphic to W and if W is isomorphic to U then also V is isomorphic to U. Let $f\colon V \to W$ and $g\colon W \to U$ be isomorphisms and consider the composition $g \circ f\colon V \to U$. The composition of correspondences is a correspondence so we need only check that the composition preserves linear combinations.

$$\begin{aligned}
g \circ f\,\bigl(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\bigr) &= g\bigl(\,f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2)\,\bigr) \\
&= g\bigl(\,c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\,\bigr) \\
&= c_1 \cdot g\bigl(f(\vec{v}_1)\bigr) + c_2 \cdot g\bigl(f(\vec{v}_2)\bigr) \\
&= c_1 \cdot (g \circ f)\,(\vec{v}_1) + c_2 \cdot (g \circ f)\,(\vec{v}_2)
\end{aligned}$$

Thus the composition is an isomorphism. QED

Thus the collection of all finite-dimensional vector spaces of partitioned into classes. Two spaces are in the same class if they are isomorphic.



The next result characterizes the classes.

2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is the next two lemmas.

2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is the next two lemmas.

2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is the next two lemmas.

2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

Proof We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, we shall show that if $f\colon V \to W$ is an isomorphism and a basis for the domain V is $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, then the image set $D = \langle f(\vec{\beta}_1), \ldots, f(\vec{\beta}_n) \rangle$ is a basis for the codomain W. The other half of the correspondence — that for any basis of W the inverse image is a basis for V — follows from Lemma 2.1 , that if f is an isomorphism then f^{-1} is also an isomorphism, and applying the prior sentence to f^{-1}.

To see that D spans W, fix any $\vec{w} \in W$, note that f is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$, and expand $\vec{v}$ as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of D, if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$$

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have that $\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, implying that all of the c's are zero. QED

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

Proof We will prove that any space of dimension n is isomorphic to $\mathbb{R}^n$. Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

Proof We will prove that any space of dimension n is isomorphic to $\mathbb{R}^n$. Then we will have that all such spaces are isomorphic to each other by transitivity, which was shown in Theorem 2.2 .

Let V be n-dimensional. Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain V. Consider the operation of representing the members of V with respect to B as a function from V to $\mathbb{R}^n$.

$$\vec{v} = v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n \quad \stackrel{\text{Rep}_B}{\longmapsto} \quad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \mathrm{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and so $u_1 = v_1, \ldots, u_n = v_n$, implying that the original arguments $u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$ are equal.

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \mathrm{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and so $u_1 = v_1$, ..., $u_n = v_n$, implying that the original arguments $u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n$ are equal.

This function is onto; any member of $\mathbb{R}^n$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

is the image of some $\vec{v} \in V$, namely $\vec{w} = \mathrm{Rep}_B(w_1\vec{\beta}_1 + \cdots + w_n\vec{\beta}_n)$.

Finally, this function preserves structure.

$$\begin{aligned}
\mathrm{Rep}_B(r\cdot\vec{u}+s\cdot\vec{v}) &= \mathrm{Rep}_B(\,(ru_1+sv_1)\vec{\beta}_1+\cdots+(ru_n+sv_n)\vec{\beta}_n\,) \\
&= \begin{pmatrix} ru_1+sv_1 \\ \vdots \\ ru_n+sv_n \end{pmatrix} \\
&= r\cdot\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}+s\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= r\cdot\mathrm{Rep}_B(\vec{u})+s\cdot\mathrm{Rep}_B(\vec{v})
\end{aligned}$$

Thus, the Rep_B function is an isomorphism and therefore any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

Finally, this function preserves structure.

$$\begin{aligned}
\mathrm{Rep}_B(r\cdot\vec{u}+s\cdot\vec{v}) &= \mathrm{Rep}_B(\,(ru_1+sv_1)\vec{\beta}_1+\cdots+(ru_n+sv_n)\vec{\beta}_n\,) \\
&= \begin{pmatrix} ru_1+sv_1 \\ \vdots \\ ru_n+sv_n \end{pmatrix} \\
&= r\cdot\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}+s\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= r\cdot\mathrm{Rep}_B(\vec{u})+s\cdot\mathrm{Rep}_B(\vec{v})
\end{aligned}$$

Thus, the Rep_B function is an isomorphism and therefore any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

Note The second paragraph's representation map Rep_B is a well-defined function since every vector $\vec{v}$ has a unique representation, with respect to a particular basis.

Example The plane $2x - y + z = 0$ through the origin in $\mathbb{R}^3$ is a vector space. Considering that a one-equation linear system and paramatrizing with the free variables

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} z \;\big|\; y, z \in \mathbb{R} \}$$

gives a basis.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} \rangle$$

This is a dimension 2 space. For instance, it is isomorphic to $\mathbb{R}^2$.

2.7 *Corollary* A finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

2.7 *Corollary* A finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

Thus the real spaces $\mathbb{R}^n$ form a set of canonical representatives of the isomorphism classes.