Five.II Similarity

Linear Algebra

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We've defined two matrices H and $\hat{H}$ to be matrix equivalent if there are nonsingular P and Q such that $\hat{H} = PHQ$. We were motivated by this diagram showing H and $\hat{H}$ both representing a map h, but with respect to different pairs of bases, B, D and $\hat{B}, \hat{D}$.

$$\begin{array}{ccc}
V_{wrt\ B} & \xrightarrow[H]{h} & W_{wrt\ D} \\
{\scriptstyle id}\Big\downarrow & & {\scriptstyle id}\Big\downarrow \\
V_{wrt\ \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{wrt\ \hat{D}}
\end{array}$$

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$$\begin{tikzcd}
V_{\mathit{wrt}\ B} \arrow[r, "h", "H"'] \arrow[d, "\mathrm{id}"'] & W_{\mathit{wrt}\ D} \arrow[d, "\mathrm{id}"'] \\
V_{\mathit{wrt}\ \hat{B}} \arrow[r, "h", "\hat{H}"'] & W_{\mathit{wrt}\ \hat{D}}
\end{tikzcd}$$

We now consider the special case of transformations, where the codomain equals the domain, and we add the requirement that the codomain's basis equals the domain's basis. So, we are considering representations with respect to B, B and D, D.

$$\begin{tikzcd}
V_{\mathit{wrt}\ B} \arrow[r, "t", "S"'] \arrow[d, "\mathrm{id}"'] & V_{\mathit{wrt}\ B} \arrow[d, "\mathrm{id}"'] \\
V_{\mathit{wrt}\ D} \arrow[r, "t", "T"'] & V_{\mathit{wrt}\ D}
\end{tikzcd}$$

In matrix terms, $\mathrm{Rep}_{D,D}(t) = \mathrm{Rep}_{B,D}(\mathrm{id})\ \mathrm{Rep}_{B,B}(t)\ \big(\mathrm{Rep}_{B,D}(\mathrm{id})\big)^{-1}$.

Definition and Examples

Similar matrices

1.1 *Definition* The matrices T and S are *similar* if there is a nonsingular P such that $T = PSP^{-1}$.

Example Consider the derivative map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$. Fix the basis $B = \langle 1, x, x^2 \rangle$ and the basis $D = \langle 1, 1+x, 1+x+x^2 \rangle$. In this arrow diagram we will first get S, and then calculate T from it.

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V_{\mathit{wrt}\ D} \arrow[r, "t", "T"'] & V_{\mathit{wrt}\ D}
\end{tikzcd}$$

The action of d/dx on the elements of the basis B is $1 \mapsto 0$, $x \mapsto 1$, and $x^2 \mapsto 2x$.

$$\mathrm{Rep}_B(d/dx(1)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(d/dx(x)) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_B(d/dx(x^2)) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

So we have this matrix representation of the map.

$$S = \mathrm{Rep}_{B,B}(d/dx) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

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Recall that the matrix changing bases from B to D is $\text{Rep}_{B,D}(\text{id})$. We find these by eye

$$\text{Rep}_D(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{Rep}_D(x) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{Rep}_D(x^2) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

to get this.

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, by following the arrow diagram we have $T = PSP^{-1}$.

$$T = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We underline what the arrow diagram says

$$\begin{tikzcd}
V_{wrt\ B} \arrow[r, "t"', "S"] \arrow[d, "\mathrm{id}"'] & V_{wrt\ B} \arrow[d, "\mathrm{id}"'] \\
V_{wrt\ D} \arrow[r, "t", "T"'] & V_{wrt\ D}
\end{tikzcd}$$

by calculating T directly. The effect of the map on the basis elements is $d/dx(1) = 0$, $d/dx(1 + x) = 1$, and $d/dx(1 + x + x^2) = 1 + 2x$. Representing of those with respect to D

$$\mathrm{Rep}_D(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_D(1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{Rep}_D(1 + 2x) = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

gives the same matrix $\mathrm{Rep}_{D,D}(d/dx)$ as we found above.

We don't need to consider the underlying maps. We can just multiply matrices.

Example Where

$$S = \begin{pmatrix} 0 & -1 & -2 \\ 2 & 3 & 2 \\ 4 & 5 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(note that P is nonsingular) we can compute this $T = PSP^{-1}$.

$$T = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 4/3 \\ 27/2 & 3/2 & 2 \end{pmatrix}$$

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$$T = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 4/3 \\ 27/2 & 3/2 & 2 \end{pmatrix}$$

1.3 *Example* The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The identity matrix has the same property: $PIP^{-1} = PP^{-1} = I$.

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Since matrix similarity is a special case of matrix equivalence, if two matrices are similar then they are matrix equivalent. What about the converse: must any two matrix equivalent square matrices be similar? No; the matrix equivalence class of an identity consists of all nonsingular matrices of that size.

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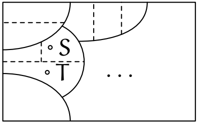
So some matrix equivalence classes split into two or more similarity classes — similarity gives a finer partition than does equivalence. This pictures some matrix equivalence classes subdivided into similarity classes.



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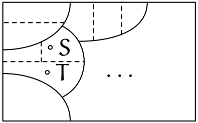

We naturally want a canonical form to represent the similarity classes. Some classes, but not all, are represented by a diagonal form.

Diagonalizability

2.1 *Definition* A transformation is *diagonalizable* if it has a diagonal representation with respect to the same basis for the codomain as for the domain. A *diagonalizable matrix* is one that is similar to a diagonal matrix: T is diagonalizable if there is a nonsingular P such that PTP^{-1} is diagonal.

Example This matrix

$$\begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix}$$

is diagonalizable by using this

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \\ -1/2 & -3/4 & 1/4 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

to get this $T = PSP^{-1}$.

$$T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

Example This matrix is not diagonalizable

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

because it is not the zero matrix but its square is the zero matrix. The fact that N is not the zero matrix means that it cannot be similar to the zero matrix, by Example 1.3 . So if N is similar to a diagonal matrix D then D has at least one nonzero entry on its diagonal. The fact that the square of N is the zero matrix means that for any map n that N represents, the composition $n \circ n$ is the zero map. The only matrix representing the zero map is the zero matrix and thus D^2 would have to be the zero matrix. But the square of a nonzero diagonal matrix cannot be the zero matrix because the square of a diagonal matrix is the diagonal matrix whose entries are the squares of the entries from the starting matrix, and D is not the zero matrix.

2.4 *Lemma* A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

2.4 *Lemma* A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

Proof Consider a diagonal representation matrix.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_B(t(\vec{\beta}_1)) & \cdots & \mathrm{Rep}_B(t(\vec{\beta}_n)) \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}$$

Consider the representation of a member of this basis with respect to the basis $\mathrm{Rep}_B(\vec{\beta}_i)$. The product of the diagonal matrix and the representation vector

$$\mathrm{Rep}_B(t(\vec{\beta}_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix}$$

has the stated action. QED

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

3.1 *Definition* A transformation $t\colon V \to V$ has a scalar *eigenvalue* λ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

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3.6 *Definition* A square matrix T has a scalar *eigenvalue* λ associated with the nonzero *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

Note Similar matrices have the same eigenvalues, because eigenvalues of a map are also the eigenvalues of matrices representing that map. But similar matrices can have different eigenvectors.

Example Recall the example from the prior subsection that these two are similar.

$$S = \begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix} \quad T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

since $T = PSP^{-1}$ for this P.

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \\ -1/2 & -3/4 & 1/4 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

Fix the vector space $\mathbb{C}^3$ and suppose that $t\colon \mathbb{C}^3 \to \mathbb{C}^3$ is the transformation represented by T with respect to the standard basis $T = \operatorname{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(t)$. Then this is the action of t.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 4x \\ 8y \\ 12z \end{pmatrix}$$

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$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 4x \\ 8y \\ 12z \end{pmatrix}$$

By eye we see that the eigenvalues of t are $\lambda_1 = 4$, $\lambda_2 = 8$, $\lambda_3 = 12$ and where

$$V_1 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} a \mid a \in \mathbb{R} \}, \quad V_2 = \{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} b \mid b \in \mathbb{R} \}, \quad V_3 = \{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c \mid c \in \mathbb{R} \}$$

any nonzero member of V_1 is an eigenvector associated with the eigenvalue $\lambda_1 = 4$, etc.

$$\vec{v}_1 = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} 400 \\ 0 \\ 0 \end{pmatrix} = t(\vec{v}_1)$$

Recall that $T = PSP^{-1}$, which we can picture as

$$\begin{array}{ccc} V_{wrt\ B} & \xrightarrow[S]{t} & V_{wrt\ B} \\ {\scriptstyle id}\big\downarrow & & {\scriptstyle id}\big\downarrow \\ V_{wrt\ D} & \xrightarrow[T]{t} & V_{wrt\ D} \end{array}$$

where P is the matrix that changes basis from B to D. In this example we have fixed $D = \mathcal{E}_3$ so we next find B.
We know that $P = \text{Rep}_{B,D}(\text{id})$ so its first column is $\text{Rep}_D(\text{id}(\vec{\beta}_1)) = \text{Rep}_{\mathcal{E}_3}(\vec{\beta}_1) = \vec{\beta}_1$, and the same holds for its other columns.

$$B = \langle \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \\ -3/4 \end{pmatrix}, \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} \rangle$$

Now, we know that the transformation t has eigenvalues of 4, 8, and 12. We know for instance that there is $\vec{v}_1$ such that $t(\vec{v}_1) = 4\vec{v}_1$. Since it represents the transformation, the matrix S also has that behavior.

$$S \cdot \mathrm{Rep}_B(\vec{v}_1) = \begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \\ 200 \end{pmatrix} = \begin{pmatrix} 400 \\ 0 \\ 800 \end{pmatrix} = 4\mathrm{Rep}_B(\vec{v}_1)$$

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Note in particular that

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 400 \\ 0 \\ 0 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 6 & -1 & -1 \\ 2 & 11 & -1 \\ -6 & -5 & 7 \end{pmatrix} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 400 \\ 0 \\ 0 \end{pmatrix}$$

so this is a case where similar matrices, which must share eigenvalues, do not share the eigenvectors associated with those eigenvalues.

Computing eigenvalues and eigenvectors

Example We will find the eigenvalues and associated eigenvectors of this matrix.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

We want to find scalars x such that $T\vec{\zeta} = x\vec{\zeta}$ for some nonzero $\vec{\zeta}$. Bring the terms to the left side.

$$\begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - x \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and factor.

$$\begin{pmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

This homogeneous system has nonzero solutions if and only if the matrix is nonsingular, that is, has a determinant of zero.

Some computation gives the determinant and its factors.

$$0 = \begin{vmatrix} 0-x & 5 & 7 \\ -2 & 7-x & 7 \\ -1 & 1 & 4-x \end{vmatrix} = x^3 - 11x^2 + 38x - 40 = (x-5)(x-4)(x-2)$$

So the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, and $\lambda_3 = 2$.

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So the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 4$, and $\lambda_3 = 2$.

To find the eigenvectors associated with the eigenvalue of 5 specialize equation $(*)$ for $x = 5$.

$$\begin{pmatrix} -5 & 5 & 7 \\ -2 & 2 & 7 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this solution set.

$$V_5 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

Similarly, to find the eigenvectors associated with the eigenvalue of 4 specialize equation $(*)$ for $x = 4$.

$$\begin{pmatrix} -4 & 5 & 7 \\ -2 & 3 & 7 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Gauss's Method gives this.

$$V_4 = \{ \begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

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Specializing $(*)$ for $x = 2$

$$\begin{pmatrix} -2 & 5 & 7 \\ -2 & 5 & 7 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

gives this.

$$V_2 = \{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} z_3 \mid z_3 \in \mathbb{C} \}$$

Example If the matrix is either upper diagonal or lower diagonal then the polynomial is easy to factor.

$$T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

The value of the determinant is easy.

$$0 = \begin{vmatrix} 2-x & 1 & 0 \\ 0 & 3-x & 1 \\ 0 & 0 & 2-x \end{vmatrix} = (3-x)(2-x)^2 \qquad (*)$$

The vectors associated with $\lambda_1 = 3$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

are here.

$$V_3 = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} z_2 \mid z_2 \in \mathbb{C} \}$$

The vectors associated with $\lambda_2 = 2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

are here.

$$V_2 = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} z_1 \mid z_1 \in \mathbb{C} \}$$

Characteristic polynomial

3.11 *Definition* The *characteristic polynomial of a square matrix* T is the determinant $|T - xI|$ where x is a variable. The *characteristic equation* is $|T - xI| = 0$. The *characteristic polynomial of a transformation* t is the characteristic polynomial of any matrix representation $\mathrm{Rep}_{B,B}(t)$.

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Note This result is why we switched from working with real number scalars to scalars that are complex.

Eigenspace

3.14 *Definition* The ***eigenspace of a transformation*** t ***associated with the eigenvalue*** λ is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}) = \lambda\vec{\zeta}\}$. The eigenspace of a matrix is analogous.

Example Recall that this matrix has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Earlier, we found that the associated eigenspaces are these.

$$V_5 = \{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c \mid c \in \mathbb{C}\}$$

$$V_4 = \{\begin{pmatrix} -7 \\ -7 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\}$$

$$V_2 = \{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} c \mid c \in \mathbb{C}\}$$

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Proof Fix an eigenvalue λ. Notice first that V_λ contains the zero vector since $t(\vec{0}) = \vec{0}$, which equals $\lambda\vec{0}$. So the eigenspace is a nonempty subset of the space. What remains is to check closure of this set under linear combinations. Take $\vec{\zeta}_1, \dots, \vec{\zeta}_n \in V_\lambda$ and then verify

$$\begin{aligned}
t(c_1\vec{\zeta}_1 + c_2\vec{\zeta}_2 + \cdots + c_n\vec{\zeta}_n) &= c_1t(\vec{\zeta}_1) + \cdots + c_nt(\vec{\zeta}_n) \\
&= c_1\lambda\vec{\zeta}_1 + \cdots + c_n\lambda\vec{\zeta}_n \\
&= \lambda(c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n)
\end{aligned}$$

that the combination is also an element of V_λ. QED

3.19 *Theorem* For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

3.19 *Theorem* For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

Proof We will use induction on the number of eigenvalues. The base step is that there are zero eigenvalues. Then the set of associated vectors is empty and so is linearly independent.

For the inductive step assume that the statement is true for any set of $k \geqslant 0$ distinct eigenvalues. Consider distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ and let $\vec{v}_1, \ldots, \vec{v}_{k+1}$ be associated eigenvectors. Suppose that $\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}$. Derive two equations from that, the first by multiplying by λ_{k+1} on both sides $\vec{0} = c_1\lambda_{k+1}\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ and the second by applying the map to both sides $\vec{0} = c_1 t(\vec{v}_1) + \cdots + c_{k+1} t(\vec{v}_{k+1}) = c_1\lambda_1\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ (applying the matrix gives the same result). Subtract the second from the first.

$$\vec{0} = c_1(\lambda_{k+1} - \lambda_1)\vec{v}_1 + \cdots + c_k(\lambda_{k+1} - \lambda_k)\vec{v}_k + c_{k+1}(\lambda_{k+1} - \lambda_{k+1})\vec{v}_{k+1}$$

The $\vec{v}_{k+1}$ term vanishes. Then the induction hypothesis gives that $c_1(\lambda_{k+1} - \lambda_1) = 0$, ..., $c_k(\lambda_{k+1} - \lambda_k) = 0$. The eigenvalues are distinct so the coefficients c_1, ..., c_k are all 0. With that we are left with the equation $\vec{0} = c_{k+1}\vec{v}_{k+1}$ so c_{k+1} is also 0. QED

Example This matrix has three eigenvalues, 5, 4, and 2.

$$T = \begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

Picking a nonzero vector from each eigenspace we get this linearly independent set (which is a basis because it has three elements).

$$\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -14 \\ -14 \\ 2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}\}$$

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Example We've also computed that this matrix has the eigenvalues 3 and 2

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Picking a vector from each of V_3 and V_2 gives this linearly independent set.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \}$$