

Two.II Linear Independence

Linear Algebra

Jim Hefferon

<http://joshua.smcvt.edu/linearalgebra>

Definition and examples

Linear independence

- 1.4 *Definition* A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.

Linear independence

1.4 *Definition* A multiset subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \cdots + c_n \vec{s}_n$$

visually sets off \vec{s}_0 , algebraically there is nothing special about that vector in that equation. For any \vec{s}_i with a coefficient c_i that is non-0 we can rewrite to isolate \vec{s}_i .

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \dots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side.

1.5 *Lemma* A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \dots, c_n = 0$.

1.5 *Lemma* A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \dots, c_n = 0$.

Proof If S is linearly independent then no vector \vec{s}_i is a linear combination of other vectors from S so there is no linear relationship where some of the \vec{s} 's have nonzero coefficients.

If S is not linearly independent then some \vec{s}_i is a linear combination $\vec{s}_i = c_1 \vec{s}_1 + \cdots + c_{i-1} \vec{s}_{i-1} + c_{i+1} \vec{s}_{i+1} + \cdots + c_n \vec{s}_n$ of other vectors from S . Subtracting \vec{s}_i from both sides gives a relationship involving a nonzero coefficient, the -1 in front of \vec{s}_i . QED

Example This set of vectors in the plane \mathbb{R}^2 is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the trivial solution $c_1 = 0, c_2 = 0$.

Example This set of vectors in the plane \mathbb{R}^2 is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the trivial solution $c_1 = 0, c_2 = 0$.

Example In the vector space of cubic polynomials $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$ the set $\{1 - x, 1 + x^2\}$ is linearly independent. The equation $c_0(1 - x) + c_1(1 + x^2) = 0$ leads to this linear system

$$\begin{aligned} c_0 - c_1 &= 0 \\ c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.

Example The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows form a linearly independent set.

Example The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows form a linearly independent set.

Example This subset of \mathbb{R}^3 is linearly dependent.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$$

One way to check that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 + 6c_3 &= 0 \end{aligned}$$

and note that there are more solutions than just the trivial one.

1.2 *Lemma* Where V is a vector space, S is a subset of that space, and \vec{v} is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

1.2 *Lemma* Where V is a vector space, S is a subset of that space, and \vec{v} is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

Proof Half of the if and only if is immediate: if $\vec{v} \notin [S]$ then the sets are not equal because $\vec{v} \in [S \cup \{\vec{v}\}]$.

For the other half assume that $\vec{v} \in [S]$ so that $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$ for some scalars c_i and vectors $\vec{s}_i \in S$. We will use mutual containment to show that the sets $[S \cup \{\vec{v}\}]$ and $[S]$ are equal. The containment $[S \cup \{\vec{v}\}] \supseteq [S]$ is clear.

1.2 *Lemma* Where V is a vector space, S is a subset of that space, and \vec{v} is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

Proof Half of the if and only if is immediate: if $\vec{v} \notin [S]$ then the sets are not equal because $\vec{v} \in [S \cup \{\vec{v}\}]$.

For the other half assume that $\vec{v} \in [S]$ so that $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$ for some scalars c_i and vectors $\vec{s}_i \in S$. We will use mutual containment to show that the sets $[S \cup \{\vec{v}\}]$ and $[S]$ are equal. The containment $[S \cup \{\vec{v}\}] \supseteq [S]$ is clear.

To show containment in the other direction let \vec{w} be an element of $[S \cup \{\vec{v}\}]$. Then \vec{w} is a linear combination of elements of $S \cup \{\vec{v}\}$, which we can write as $\vec{w} = c_{n+1} \vec{s}_{n+1} + \cdots + c_{n+k} \vec{s}_{n+k} + c_{n+k+1} \vec{v}$. (Possibly some of the \vec{s}_i 's from \vec{w} 's equation are the same as some of those from \vec{v} 's equation but that does not matter.) Expand \vec{v} .

$$\vec{w} = c_{n+1} \vec{s}_{n+1} + \cdots + c_{n+k} \vec{s}_{n+k} + c_{n+k+1} \cdot (c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n)$$

Recognize the right hand side as a linear combination of linear combinations of vectors from S . Thus $\vec{w} \in [S]$.

QED

1.3 *Corollary* For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

Example These two subsets of \mathbb{R}^3 have the same span

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

because in the first set $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$.

1.3 *Corollary* For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

Example These two subsets of \mathbb{R}^3 have the same span

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$$

because in the first set $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$.

1.13 *Corollary* A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span $[S - \{\vec{v}\}] \subset [S]$.

Proof This follows from Corollary 1.3. If S is linearly independent then none of its vectors is dependent on the other elements, so removal of any vector will shrink the span. If S is not linearly independent then it contains a vector that is dependent on other elements of the set, and removal of that vector will not shrink the span. QED

1.14 *Lemma* Suppose that S is linearly independent and that $\vec{v} \notin S$.
Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

1.14 *Lemma* Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

Proof We will show that $S \cup \{\vec{v}\}$ is not linearly independent if and only if $\vec{v} \in [S]$.

Suppose first that $v \in [S]$. Express \vec{v} as a combination $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$. Rewrite that $\vec{0} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n - 1 \cdot \vec{v}$. Since $v \notin S$, it does not equal any of the \vec{s}_i so this is a nontrivial linear dependence among the elements of $S \cup \{\vec{v}\}$. Thus that set is not linearly independent.

1.14 *Lemma* Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

Proof We will show that $S \cup \{\vec{v}\}$ is not linearly independent if and only if $\vec{v} \in [S]$.

Suppose first that $v \in [S]$. Express \vec{v} as a combination $\vec{v} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n$. Rewrite that $\vec{0} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n - 1 \cdot \vec{v}$. Since $v \notin S$, it does not equal any of the \vec{s}_i so this is a nontrivial linear dependence among the elements of $S \cup \{\vec{v}\}$. Thus that set is not linearly independent.

Now suppose that $S \cup \{\vec{v}\}$ is not linearly independent and consider a nontrivial dependence among its members $\vec{0} = c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n + c_{n+1} \cdot \vec{v}$. If $c_{n+1} = 0$ then that is a dependence among the elements of S , but we are assuming that S is independent, so $c_{n+1} \neq 0$. Rewrite the equation as $\vec{v} = (c_1/c_{n+1})\vec{s}_1 + \cdots + (c_n/c_{n+1})\vec{s}_n$ to get $\vec{v} \in [S]$

QED

Example In \mathcal{P}_2 consider the set $S = \{1 - x, 1 + x\}$. The span $[S]$ is the subset of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$. (The span is a subset of the linear polynomials because no member of S has a quadratic term. To see that the span is all of the set of linear polynomials, consider a linear polynomial $a + bx$ and use the equation $a + bx = r_1(1 - x) + r_2(1 + x)$ to get a linear system that solves as $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b$.)

Example In \mathcal{P}_2 consider the set $S = \{1 - x, 1 + x\}$. The span $[S]$ is the subset of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$. (The span is a subset of the linear polynomials because no member of S has a quadratic term. To see that the span is all of the set of linear polynomials, consider a linear polynomial $a + bx$ and use the equation $a + bx = r_1(1 - x) + r_2(1 + x)$ to get a linear system that solves as $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b$.)

If we add a linear polynomial $S_1 = S \cup \{2 + 2x\}$ then the span is unchanged $[S] = [S_1]$. This is because span of S is all of the linear polynomials and the new member does not add any quadratic terms.

Example In \mathcal{P}_2 consider the set $S = \{1 - x, 1 + x\}$. The span $[S]$ is the subset of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$. (The span is a subset of the linear polynomials because no member of S has a quadratic term. To see that the span is all of the set of linear polynomials, consider a linear polynomial $a + bx$ and use the equation $a + bx = r_1(1 - x) + r_2(1 + x)$ to get a linear system that solves as $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b$.)

If we add a linear polynomial $S_1 = S \cup \{2 + 2x\}$ then the span is unchanged $[S] = [S_1]$. This is because span of S is all of the linear polynomials and the new member does not add any quadratic terms.

If we add a quadratic polynomial $S_2 = S \cup \{2 + x^2\}$ then we enlarge the span: the span of S_2 is all of \mathcal{P}_2 . To see this, consider a quadratic $a + bx + cx^2$ and use $a + bx + cx^2 = r_1(1 - x) + r_2(1 + x) + r_3(2 + x^2)$ to get a linear system that has the solution $r_3 = c$, $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b - c$.

1.16 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

1.16 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

Proof If $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

1.16 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

Proof If $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector \vec{v}_1 that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

1.16 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

Proof If $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector \vec{v}_1 that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

If S_1 is linearly independent then we are done. Otherwise iterate: take a vector \vec{v}_2 that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. QED

Example Consider this subset of \mathbb{R}^2 .

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The linear relationship

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives a system of equations.

$$2r_1 + 3r_2 + r_3 + r_5 = 0$$

$$2r_1 + 3r_2 + 4r_3 - r_4 - r_5 = 0$$

$$\begin{array}{rcl} & & -\rho_1 + \rho_2 \rightarrow \\ & 2r_1 + 3r_2 + r_3 + r_5 = 0 \\ & + 3r_3 - r_4 - 2r_5 = 0 \end{array}$$

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R} \right\}$$

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R} \right\}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that \vec{s}_5 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix} = \begin{Bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} r_2 + \begin{Bmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{Bmatrix} r_4 + \begin{Bmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{Bmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that \vec{s}_5 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows \vec{s}_4 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows \vec{s}_2 is in the span of the same set.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\left\{ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R} \right\}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that \vec{s}_5 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows \vec{s}_4 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows \vec{s}_2 is in the span of the same set. So we can omit the vectors $\vec{s}_2, \vec{s}_4, \vec{s}_5$ associated with the free variables without shrinking the span.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{Bmatrix} = \begin{Bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} r_2 + \begin{Bmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{Bmatrix} r_4 + \begin{Bmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{Bmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that \vec{s}_5 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows \vec{s}_4 is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows \vec{s}_2 is in the span of the same set. So we can omit the vectors \vec{s}_2 , \vec{s}_4 , \vec{s}_5 associated with the free variables without shrinking the span. The set $\{\vec{s}_1, \vec{s}_3\}$ is linearly independent and so we cannot omit any members without shrinking the span.

1.18 *Corollary* A subset $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some \vec{s}_i is a linear combination of the vectors $\vec{s}_1, \dots, \vec{s}_{i-1}$ listed before it.

1.18 *Corollary* A subset $S = \{\vec{s}_1, \dots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some \vec{s}_i is a linear combination of the vectors $\vec{s}_1, \dots, \vec{s}_{i-1}$ listed before it.

Proof Consider $S_0 = \{\}$, $S_1 = \{\vec{s}_1\}$, $S_2 = \{\vec{s}_1, \vec{s}_2\}$, etc. Some index $i \geq 1$ is the first one with $S_{i-1} \cup \{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i \in [S_{i-1}]$. QED

Linear independence and subset

1.19 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof Both are clear.

QED

Linear independence and subset

1.19 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

Proof Both are clear.

QED

This table summarizes the cases.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S independent	\hat{S} must be independent	\hat{S} may be either
S dependent	\hat{S} may be either	\hat{S} must be dependent

An example of the lower left is that the set S of all vectors in the space \mathbb{R}^2 is linearly dependent but the subset \hat{S} consisting of only the unit vector on the x -axis is independent. By interchanging \hat{S} with S that's also an example of the upper right.