Three.I Isomorphisms

Linear Algebra

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Definition and examples

Example People often have the intuition that the two vector spaces $\mathbb{R}^2$ and $\mathcal{P}_1$ are "the same," for instance in that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{is just like} \quad 1 + 2x$$
$$\text{and} \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} \quad \text{is just like} \quad -3 - (1/2)x$$

etc. What makes the one "just like" the other is that this association holds through the operations of addition

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5/2 \end{pmatrix}$$
$$\text{is just like} \quad (1 + 2x) + (-3 + (1/2)x) = -2 + (5/2)x$$

and scalar multiplication.

$$3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \quad \text{is just like} \quad 3(1 + 2x) = 3 + 6x$$

More generally, we can express the same-ness of the spaces by associating each two-tall vector with a linear polynomial.

$$\begin{pmatrix} a \\ b \end{pmatrix} \quad \longleftrightarrow \quad a + bx$$

such that the association holds through the vector space operations of addition

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$
$$\longleftrightarrow \quad (a_1 + b_1 x) + (a_2 + b_2 x) = (a_1 + a_2) + (b_1 + b_2)x$$

and scalar multiplication.

$$r\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ra \\ rb \end{pmatrix} \quad \longleftrightarrow \quad r(a + bx) = (ra) + (rb)x$$

We say that the association *preserves the structure* of the spaces.

Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, this association matches these two.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

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This association holds up under addition.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$\longleftrightarrow\quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix}$$

Here is an example of that with particular vectors.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & -5 \end{pmatrix}$$

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$$\begin{pmatrix}1&-1\\2&-2\end{pmatrix}+\begin{pmatrix}0&4\\3&-3\end{pmatrix}=\begin{pmatrix}1&3\\5&-5\end{pmatrix}$$
$$\longleftrightarrow\quad\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}+\begin{pmatrix}0\\4\\3\\-3\end{pmatrix}=\begin{pmatrix}1\\3\\5\\-5\end{pmatrix}$$

The association also holds under scalar multiplication.

$$r\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}ra&rb\\rc&rd\end{pmatrix}\quad\longleftrightarrow\quad r\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}ra\\rb\\rc\\rd\end{pmatrix}$$

This illustrates with particular vectors.

$$2\cdot\begin{pmatrix}1&-1\\2&-2\end{pmatrix}=\begin{pmatrix}2&-2\\4&-4\end{pmatrix}\quad\longleftrightarrow\quad 2\cdot\begin{pmatrix}1\\-1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\4\\-4\end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1) is a correspondence: f is one-to-one and onto;
2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to the space $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

For instance, here is the action of f on two inputs.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check conditions (1) and (2).

The first part of (1) is that f is one-to-one. We usually verify this by assuming that the function yields the same output on two inputs and then show that the two inputs must therefore be equal. So assume that $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$. By definition of f we have

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the starting inputs are equal $a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$ and so f is one-to-one.

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The second part of (1) is that f is onto. We usually verify this by taking an element of the codomain and producing an element of the domain that maps to it. So consider this member of $\mathbb{R}^3$.

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Observe that it is the image under f of the member $u + vx + wx^2$ of the domain. Thus f is onto.

Condition (2) also has two halves. First we must show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$f(\,(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2)\,) = f((a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2)$$

The function maps the linear polynomial on the right to this vector.

$$\begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_2+b_2 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

The right side is $f(a_0+a_1x+a_2x^2)+f(b_0+b_1x+b_2x^2)$, as required.

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The right side is $f(a_0+a_1x+a_2x^2)+f(b_0+b_1x+b_2x^2)$, as required.

We finish by checking that f preserves scalar multiplication.

$$r\cdot f(a_0+a_1x+a_2x^2) = r\cdot\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} = f(\,(ra_0)+(ra_1)x+(ra_2)x^2)$$

QED

Automorphisms

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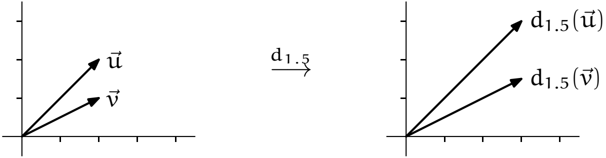
1.8 *Example* A *dilation* map $d_s \colon \mathbb{R}^2 \to \mathbb{R}^2$ that multiplies all vectors by a nonzero scalar s is an automorphism of $\mathbb{R}^2$.



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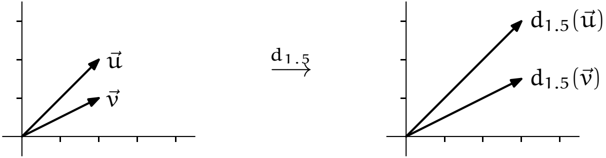
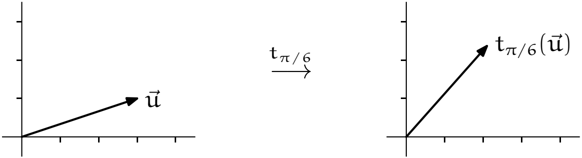
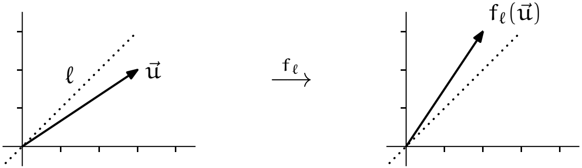


Another automorphism is a *rotation* or *turning map*, $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through an angle θ.



A third type of automorphism of $\mathbb{R}^2$ is a map $f_\ell \colon \mathbb{R}^2 \to \mathbb{R}^2$ that *flips* or *reflects* all vectors over a line ℓ through the origin.



Checking that each of these is an isomorphism is an exercise.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

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Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.11 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\, f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

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Proof Since the implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are clear, we need only show that (1) $\Longrightarrow$ (3). Assume statement (1). We will prove statement (3) by induction on the number of summands n.

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The one-summand base case, that $f(c\vec{v}_1) = c\, f(\vec{v}_1)$, is covered by the second clause of statement (1).

For the inductive step assume that statement (3) holds whenever there are k or fewer summands. Consider the $k+1$-summand case. Use the first half of (1) to break the sum along the final '+'.

$$f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}) = f(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Use the inductive hypothesis to break up the k-term sum on the left.

$$= f(c_1\vec{v}_1) + \cdots + f(c_k\vec{v}_k) + f(c_{k+1}\vec{v}_{k+1})$$

Now the second half of (1) gives

$$= c_1\,f(\vec{v}_1) + \cdots + c_k\,f(\vec{v}_k) + c_{k+1}\,f(\vec{v}_{k+1})$$

when applied $k+1$ times. QED

That result shortens the checking that a function preserves the structure of a vector space.

Example This line through the origin

$$L = \{ t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

is a vector space under the addition and scalar multiplication operations that it inherits from $\mathbb{R}^2$.

$$t_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (t_1 + t_2) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad r\bigl(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\bigr) = (rt) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

This space is isomorphic to $\mathbb{R}^1$ under this map.

$$f(t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}) = t$$

We first verify that f is one-to-one. Suppose that f maps two members of L to the same output.

$$f(\,t_1 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) = f(\,t_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,)$$

Then by definition of the function f we have that $t_1 = t_2$ and so the two members of L are equal.

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Next we check that f is an onto map. Consider this member of the codomain: $r \in \mathbb{R}$. There is a member of the domain that maps to it, namely this member of L.

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$$f(\,r \cdot \begin{pmatrix}1\\2\end{pmatrix}\,)$$

To finish, we use the lemma to check that f preserves structure.

$$f(\,t_1 \cdot \begin{pmatrix}1\\2\end{pmatrix} + t_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) = f(\,\begin{pmatrix}t_1 + t_2\\2(t_1 + t_2)\end{pmatrix}\,) = t_1 + t_2 = f(\,t_1 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,) + f(\,t_2 \cdot \begin{pmatrix}1\\2\end{pmatrix}\,)$$

Dimension characterizes isomorphism

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

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Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

2.1 *Lemma* The inverse of an isomorphism is also an isomorphism.

Proof Suppose that V is isomorphic to W via $f\colon V \to W$. An isomorphism is a correspondence so f has an inverse function $f^{-1}\colon W \to V$ that is also a correspondence.

We will show that because f preserves linear combinations, so also does f^{-1}. Suppose that $\vec{w}_1, \vec{w}_2 \in W$. Because it is an isomorphism, f is onto and there are $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{w}_1 = f(\vec{v}_1)$ and $\vec{w}_2 = f(\vec{v}_2)$. Then

$$\begin{aligned} f^{-1}(c_1 \cdot \vec{w}_1 + c_2 \cdot \vec{w}_2) &= f^{-1}\big(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)\big) \\ &= f^{-1}\big(f(c_1\vec{v}_1 + c_2\vec{v}_2)\big) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot f^{-1}(\vec{w}_1) + c_2 \cdot f^{-1}(\vec{w}_2) \end{aligned}$$

since $f^{-1}(\vec{w}_1) = \vec{v}_1$ and $f^{-1}(\vec{w}_2) = \vec{v}_2$. With that, by Lemma 1.11 's second item, this map preserves structure. QED

2.2 *Theorem* Isomorphism is an equivalence relation between vector spaces.

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Proof We must prove that the relation is symmetric, reflexive, and transitive.

To check reflexivity, that any space is isomorphic to itself, consider the identity map. It is clearly one-to-one and onto. This calculation shows that it also preserves linear combinations.

$$\mathrm{id}(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1 \cdot \mathrm{id}(\vec{v}_1) + c_2 \cdot \mathrm{id}(\vec{v}_2)$$

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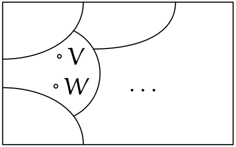
Symmetry, that if V is isomorphic to W then also W is isomorphic to V, holds by Lemma 2.1 since each isomorphism map from V to W is paired with an isomorphism from W to V.

To finish we must check transitivity, that if V is isomorphic to W and W is isomorphic to U then V is isomorphic to U. Let $f\colon V \to W$ and $g\colon W \to U$ be isomorphisms. Consider their composition $g \circ f\colon V \to U$. The composition of correspondences is a correspondence so we need only check that the composition preserves linear combinations.

$$\begin{aligned}
g \circ f \left(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\right) &= g\big(f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) \big) \\
&= g\big(c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2) \big) \\
&= c_1 \cdot g\big(f(\vec{v}_1)\big) + c_2 \cdot g\big(f(\vec{v}_2)\big) \\
&= c_1 \cdot (g \circ f)(\vec{v}_1) + c_2 \cdot (g \circ f)(\vec{v}_2)
\end{aligned}$$

Thus the composition is an isomorphism. QED

Thus the collection of all finite-dimensional vector spaces of partitioned into classes. Two spaces are in the same class if they are isomorphic.



The next result characterizes the classes.

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2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

Proof We shall show that an isomorphism of two spaces gives a correspondence between their bases. That is, we shall show that if $f\colon V \to W$ is an isomorphism and a basis for the domain V is $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ then its image $D = \langle f(\vec{\beta}_1), \ldots, f(\vec{\beta}_n) \rangle$ is a basis for the codomain W. (The other half of the correspondence, that for any basis of W the inverse image is a basis for V, follows from the fact that f^{-1} is also an isomorphism and so we can apply the prior sentence to f^{-1}.)

To see that D spans W, fix any $\vec{w} \in W$. Because f is an isomorphism it is onto and so there is a $\vec{v} \in V$ with $\vec{w} = f(\vec{v})$. Expand $\vec{v}$ as a combination of basis vectors.

$$\vec{w} = f(\vec{v}) = f(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n) = v_1 \cdot f(\vec{\beta}_1) + \cdots + v_n \cdot f(\vec{\beta}_n)$$

For linear independence of D, if

$$\vec{0}_W = c_1 f(\vec{\beta}_1) + \cdots + c_n f(\vec{\beta}_n) = f(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n)$$

then, since f is one-to-one and so the only vector sent to $\vec{0}_W$ is $\vec{0}_V$, we have that $\vec{0}_V = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, which implies that all of the c's are zero. QED

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Let V be n-dimensional. Fix a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ for the domain V. Consider the operation of representing the members of V with respect to B as a function from V to $\mathbb{R}^n$.

$$\vec{v} = v_1 \vec{\beta}_1 + \cdots + v_n \vec{\beta}_n \stackrel{\text{Rep}_B}{\longmapsto} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1+\cdots+u_n\vec{\beta}_n)=\mathrm{Rep}_B(v_1\vec{\beta}_1+\cdots+v_n\vec{\beta}_n)$$

then

$$\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$$

and so $u_1=v_1$, ..., $u_n=v_n$, implying that the original arguments $u_1\vec{\beta}_1+\cdots+u_n\vec{\beta}_n$ and $v_1\vec{\beta}_1+\cdots+v_n\vec{\beta}_n$ are equal.

This function is one-to-one because if

$$\mathrm{Rep}_B(u_1\vec{\beta}_1 + \cdots + u_n\vec{\beta}_n) = \mathrm{Rep}_B(v_1\vec{\beta}_1 + \cdots + v_n\vec{\beta}_n)$$

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This function is onto; any member of $\mathbb{R}^n$

$$\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

is the image of some $\vec{v} \in V$, namely $\vec{w} = \mathrm{Rep}_B(w_1\vec{\beta}_1 + \cdots + w_n\vec{\beta}_n)$.

Finally, this function preserves structure.

$$\begin{aligned}
\mathrm{Rep}_B(r\cdot\vec{u}+s\cdot\vec{v}) &= \mathrm{Rep}_B(\,(ru_1+sv_1)\vec{\beta}_1+\cdots+(ru_n+sv_n)\vec{\beta}_n\,) \\
&= \begin{pmatrix} ru_1+sv_1 \\ \vdots \\ ru_n+sv_n \end{pmatrix} \\
&= r\cdot\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + s\cdot\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\
&= r\cdot\mathrm{Rep}_B(\vec{u})+s\cdot\mathrm{Rep}_B(\vec{v})
\end{aligned}$$

Therefore Rep_B is an isomorphism. Consequently any n-dimensional space is isomorphic to $\mathbb{R}^n$. QED

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Note The second paragraph's representation map Rep_B is a well-defined function since every vector $\vec{v}$ has a unique representation, with respect to a particular basis.

Example The plane $2x - y + z = 0$ through the origin in $\mathbb{R}^3$ is a vector space. Considering that a one-equation linear system and paramatrizing with the free variables

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

gives a basis.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} \rangle$$

This is a dimension 2 space. For instance, it is isomorphic to $\mathbb{R}^2$.

2.7 *Corollary* A finite-dimensional vector space is isomorphic to one and only one of the $\mathbb{R}^n$.

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Thus the real spaces $\mathbb{R}^n$ form a set of canonical representatives of the isomorphism classes.