Two.I Definition of Vector Space

Linear Algebra

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Definition and examples

Vector space

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Where $\vec{v}, \vec{w} \in V$, (1) their *vector sum* $\vec{v} + \vec{w}$ is an element of V. If $\vec{u}, \vec{v}, \vec{w} \in V$ then (2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ and (3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$. (4) There is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$. (5) Each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$.

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If r, s are *scalars*, members of $\mathbb{R}$, and $\vec{v}, \vec{w} \in V$ then (6) each *scalar multiple* $r \cdot \vec{v}$ is in V. If $r, s \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$ then (7) $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$, and (8) $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$, and (9) $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$, and (10) $1 \cdot \vec{v} = \vec{v}$.

Example Consider this reduced echelon form matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

In any matrix row-equivalent to A each row must be a multiple of the vector $(1 \;\; 2)$. We will verify that this set of row vectors

$$V = \{ (a \;\; 2a) \mid a \in \mathbb{R} \}$$

is a vector space under the natural operations of addition

$$(a_1 \;\; 2a_1) + (a_2 \;\; 2a_2) = (a_1 + a_2 \;\; 2a_1 + a_2)$$

and scalar multiplication.

$$r(a_1 \;\; a_2) = (ra_1 \;\; ra_2)$$

For that we will check each of the conditions. (This first time through, we verify these at length.)

We first check *closure under addition* (1), that the sum of two members of V is also a member of V. Take $\vec{v}$ and $\vec{w}$ to be members of V so that

$$\vec{v} = \begin{pmatrix} v_1 & 2v_1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 & 2w_1 \end{pmatrix}$$

and note that their sum

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 & 2v_1 + 2w_1 \end{pmatrix}$$

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Condition (2), commutativity of addition, is easy. The first sum is

$$\vec{v} + \vec{w} = (v_1 + w_1 \quad 2(v_1 + w_1))$$

and the second sum is

$$\vec{w} + \vec{v} = (w_1 + v_1 \quad 2(w_1 + v_1))$$

and the two are equal because the sum of real numbers $v_1 + w_1$ equals the sum of real numbers $w_1 + v_1$.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v}+\vec{w})+\vec{u} = ((v_1+w_1)+u_1 \quad (2v_1+2w_1)+2u_1)$$

while the right side is this.

$$\vec{v}+(\vec{w}+\vec{u}) = (v_1+(w_1+u_1) \quad 2v_1+(2w_1+2u_1))$$

The two are equal because real number addition is associative $(v_1+w_1)+u_1 = v_1+(w_1+u_1)$.

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The two are equal because real number addition is associative $(v_1+w_1)+u_1=v_1+(w_1+u_1)$.

For condition (4) we just produce the member of V with the desired property. So consider the vector of 0's. Note that it is a member of V since its second component is twice its first, and note that it is the *identity element* with respect to addition.

$$\begin{aligned}\vec{v}+\vec{0}&=(v_1 \quad 2v_1)+(0 \quad 0)\\&=(v_1 \quad 2v_1)\\&=\vec{v}\end{aligned}$$

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \;\; 2v_1)$ of V, consider $\vec{w} = (w_1 \;\; 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels $\vec{v}$.

$$\vec{w} + \vec{v} = (-v_1 \;\; -2v_1) + (v_1 \;\; 2v_1) = \vec{0}$$

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Next we verify the conditions for scalar multiplication. First, condition (6) is *closure under scalar multiplication*. We consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} = (v_1 \;\; 2v_2) \in V$. The scalar multiple $r\vec{v} = (rv_1 \;\; r2v_1)$ is also a member of V because the second component is twice the first.

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Condition (7) is that *real number addition distributes over scalar multiplication*. Let the scalars be $r, s \in \mathbb{R}$ and let the vector be $\vec{v} = (v_1 \;\; 2v_1) \in V$. Then $(r+s)\vec{v} = ((r+s)v_1 \;\; (r+2)2v_1)$, which equals $(rv_1 \;\; 2rv_1) + (sv_1 \;\; 2sv_1) = r\vec{v} + s\vec{v}$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v} + \vec{w}) = (rv_1 \quad 2rv_1) + (rw_1 \quad 2rw_1)$, which equals $(rv_1 + rw_1 \quad 2rv_1 + 2rw_1)$, which equals $r(v_1 \quad 2v_1) + r(w_1 \quad 2w_1) = r\vec{v} + r\vec{w}$.

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For condition (9) take $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \;\; 2v_1) \in V$. The left side is $(rs)(v_1 \;\; 2v_1) = ((rs)v_1 \;\; (rs)2v_1)$, while the right side is $r(s(v_1 \;\; 2v_1)) = r(sv_1 \;\; s2v_1) = (r(sv_1) \;\; r(s2v_1))$. The two are equal because, as they are real number multiplications, $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$.

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The final condition is straightforward: for any $\vec{v} \in V$ we have $1\vec{v} = 1(v_1 \;\; 2v_1) = (1 \cdot v_1 \;\; 1 \cdot 2v_1) = \vec{v}$.

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Thus the set $V = \{(a \;\; 2a) \mid a \in \mathbb{R}\}$ is a vector space under the natural addition and scalar multiplication operations.

Example The set $\mathbb{R}^3$ is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that we will check the conditions (more briefly than for the prior example).

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Condition (1) is closure under addition. This is clear because the only condition for membership in the set $\mathbb{R}^3$ is to be a three-tall vector of reals, and the sum of two three-tall vectors of reals is also a three-tall vector of reals.

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Condition (2) is routine

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{w} + \vec{v}$$

Condition (3) is also a direct consequence of the related real number property.

$$
\begin{aligned}
(\vec{v}+\vec{w})+\vec{u} &= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1+w_1+u_1 \\ v_2+w_2+u_2 \\ v_3+w_3+u_3 \end{pmatrix} \\
&= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1+u_1 \\ w_2+u_2 \\ w_3+u_3 \end{pmatrix} = \vec{v}+(\vec{w}+\vec{u})
\end{aligned}
$$

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$$=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}+\begin{pmatrix}w_1+u_1\\w_2+u_2\\w_3+u_3\end{pmatrix}=\vec{v}+(\vec{w}+\vec{u})$$

For condition (4) take the vector of 0's.

$$\begin{pmatrix}0\\0\\0\end{pmatrix}+\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$$

For condition (5), given $\vec{v}\in\mathbb{R}^3$, use $\vec{w}=-1\vec{v}$ as the additive inverse.

$$\begin{pmatrix}-v_1\\-v_2\\-v_3\end{pmatrix}+\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Condition (6) is closure under scalar multiplication. Let the scalar be $r \in \mathbb{R}$ and the vector be $\vec{v} \in \mathbb{R}^3$. Then $r\vec{v}$ is a three-tall vector of reals, so $r\vec{v} \in \mathbb{R}^3$.

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Conditions (7)

$$(r+s)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \\ (r+s)v_3 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \\ rv_3 + sv_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix} + \begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}+s\vec{v}$$

and (8)

$$r(\vec{v}+\vec{w}) = r(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}) = r\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \\ rv_3 + rw_3 \end{pmatrix} = r\vec{v}+r\vec{w}$$

are straightforward.

Condition (9) is similar.

$$(rs)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (rs)v_1 \\ (rs)v_2 \\ (rs)v_3 \end{pmatrix} = r\begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v}(s\vec{v})$$

And (10) is also easy.

$$1\vec{v} = 1\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ 1 \cdot v_2 \\ 1 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$

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So the set $\mathbb{R}^3$ is a vector space under the usual operations of vector addition and scalar-vector multiplication.

Example The set $\mathcal{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0+a_1x+a_2x^2)+(b_0+b_1x+b_2x^2) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1 x + a_2 x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

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Example The set of 3×3 matrices

$$\mathcal{M}_{3\times3} = \{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \mid a_{i,j} \in \mathbb{R}\}$$

is a vector space under the usual matrix addition and scalar multiplication.

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Example The set consisting only of the two-tall vector of 0's

$$V = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \}$$

is a vector space (under the usual vector addition and scalar multiplication operations).

1.7 *Definition* A one-element vector space is a *trivial* space.

1.16 *Lemma* In any vector space V, for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have (1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

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Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

$$\begin{aligned} \vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\ \vec{0} &= \vec{0} + 0 \cdot \vec{v} \\ \vec{0} &= 0 \cdot \vec{v} \end{aligned}$$

Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1+1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$. For (3), $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do.

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Proof For (1) note that $\vec{v} = (1+0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of $\vec{v}$, the vector $\vec{w}$ such that $\vec{w} + \vec{v} = \vec{0}$.

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\vec{0} &= \vec{0} + 0 \cdot \vec{v} \\
\vec{0} &= 0 \cdot \vec{v}
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\vec{w} + \vec{v} &= \vec{w} + \vec{v} + 0 \cdot \vec{v} \\
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QED

Subspaces and spanning sets

Subspace

2.1 *Definition* For any vector space, a ***subspace*** is a subset that is itself a vector space, under the inherited operations.

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Example In the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{\begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R}\}$$

is a subspace. The operations, as required by the definition, are the ones from $\mathbb{R}^2$. We can check all the conditions to show it is a vector space, but the next result gives an easier way.

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Example This subset of $\mathcal{M}_{2\times 2}$ is a subspace.

$$S = \{\begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R}\} = \{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} b \mid a, b \in \mathbb{R}\}$$

As above, addition and scalar multiplication are the same as in $\mathcal{M}_{2\times 2}$.

Example This is not a subspace of $\mathbb{R}^3$.

$$T = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$$

It is a subset of $\mathbb{R}^3$ but it is not a vector space. One condition that it violates is that it is not closed under vector addition: here are two elements of T that sum to a vector that is not an element of T.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(Another reason that it is not a vector space is that it does not contain the zero vector.)

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations the following are equivalent statements.

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is an element of S.

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'The following are equivalent' means that each pair of statements are equivalent.

$$(1) \iff (2) \qquad (2) \iff (3) \qquad (3) \iff (1)$$

We will prove the equivalence by establishing that $(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by the observation that the implications $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue that $(2) \implies (1)$.

2.9 *Proof* Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The vector space definition has five conditions on addition. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$ is a linear combination of a pair of vectors and we are assuming that S is closed under those. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V, the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V, and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S. The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that (where $\vec{s}$ is any member of the nonempty set S) $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is in S; checking that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations of pairs of vectors shows that $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is an element of S; checking that it is the additive inverse of $\vec{s}$ under the inherited operations is routine.

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Example The vector space of quadratic polynomials $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ has a subspace comprised of the linear polynomials $L = \{b_0 + b_1x \mid b_0, b_1 \in \mathbb{R}\}$. To verify that, take scalars $r, s \in \mathbb{R}$ and consider a linear combination.

$$r(b_0 + b_1x) + s(c_0 + c_1x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

The right side is a linear polynomial with real coefficients, and so is a member of L. Thus L is closed under linear combinations.

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Example Another subspace of $\mathcal{P}_2$ is the set of quadratic polynomials with all three coefficients equal.

$$M = \{a + ax + ax^2 \mid a \in \mathbb{R}\} = \{(1 + x + x^2)a \mid a \in \mathbb{R}\}$$

Verify that it is a subspace by taking two scalars $r, s \in \mathbb{R}$ and considering a linear combination of polynomials with all three coefficients the same.

$$r(a+ax+ax^2)+s(b+bx+bx^2) = (ra+sb)+(ra+sb)x+(ra+sb)x^2$$

The result is a quadratic polynomial with all three coefficents the same, and so M is closed under linear combinations.

The above examples of subspace paramatrize the description.

Example This set is a plane inside of $\mathbb{R}^3$.

$$P = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0 \}$$

We could verify that it is a subspace by checking that it is closed under linear combination as above.

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That's easier if we first paramatrize the one-equation linear system $2x - y + z = 0$ using the free variables y and z.

$$P = \{ \begin{pmatrix} (1/2)y - (1/2)z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \} = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

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Now members of P are described as a linear combination of those two vectors. Verifying that P is closed then involves taking a linear combination of linear combinations, which makes a linear combination.

Span

2.13 *Definition* The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S.

$$[S] = \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \mid c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

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The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are ‘span(S)’ and ‘sp(S)’.

Example Inside the vector space of all two-wide row vectors, the span of this one-element set

$$S = \{\begin{pmatrix}1 & 2\end{pmatrix}\}$$

is this.

$$[S] = \{\begin{pmatrix}a & 2a\end{pmatrix} \mid a \in \mathbb{R}\} = \{\begin{pmatrix}1 & 2\end{pmatrix}a \mid a \in \mathbb{R}\}$$

Example This is a subset of $\mathbb{R}^3$.

$$S = \{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}$$

Any vector in the xy-plane is a member of the span $[S]$; for instance, this system has a solution.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

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$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

But vectors not in the xy-plane are not in the span; for instance, this system does not have a solution.

$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

(just consider the third components).

2.15 *Lemma* In a vector space, the span of any subset is a subspace.

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Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations of pairs of elements. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p\cdot(c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r\cdot(c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

is a linear combination of elements of S and so is an element of $[S]$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$ but that does not matter). QED