

Two.I Definition of Vector Space

Linear Algebra

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Definition and examples

Vector space

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Where $\vec{v}, \vec{w} \in V$, (1) their *vector sum* $\vec{v} + \vec{w}$ is an element of V . If $\vec{u}, \vec{v}, \vec{w} \in V$ then (2) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ and (3) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$. (4) There is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$. (5) Each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$.

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If r, s are *scalars*, members of \mathbb{R} , and $\vec{v}, \vec{w} \in V$ then (6) each *scalar multiple* $r \cdot \vec{v}$ is in V . If $r, s \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$ then (7) $(r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$, and (8) $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$, and (9) $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$, and (10) $1 \cdot \vec{v} = \vec{v}$.

Example Consider this reduced echelon form matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

In any matrix row-equivalent to A each row must be a multiple of the vector $(1 \ 2)$. We will verify that this set of row vectors

$$V = \{(a \ 2a) \mid a \in \mathbb{R}\}$$

is a vector space under the natural operations of addition

$$(a_1 \ 2a_1) + (a_2 \ 2a_2) = (a_1 + a_2 \ 2a_1 + a_2)$$

and scalar multiplication.

$$r(a_1 \ a_2) = (ra_1 \ ra_2)$$

For that we will check each of the conditions. (This first time through, we verify these at length.)

We first check *closure under addition* (1), that the sum of two members of V is also a member of V . Take \vec{v} and \vec{w} to be members of V so that

$$\vec{v} = (v_1 \quad 2v_1) \quad \vec{w} = (w_1 \quad 2w_1)$$

and note that their sum

$$\vec{v} + \vec{w} = (v_1 + w_1 \quad 2v_1 + 2w_1)$$

is also a member of V , because its second entry is twice its first.

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Condition (2), commutativity of addition, is easy. The first sum is

$$\vec{v} + \vec{w} = (v_1 + w_1 \quad 2(v_1 + w_1))$$

and the second sum is

$$\vec{w} + \vec{v} = (w_1 + v_1 \quad 2(w_1 + v_1))$$

and the two are equal because the sum of real numbers $v_1 + w_1$ equals the sum of real numbers $w_1 + v_1$.

Condition (3), associativity of addition, is like the prior one. The left side is

$$(\vec{v} + \vec{w}) + \vec{u} = ((v_1 + w_1) + u_1 \quad (2v_1 + 2w_1) + 2u_1)$$

while the right side is this.

$$\vec{v} + (\vec{w} + \vec{u}) = (v_1 + (w_1 + u_1) \quad 2v_1 + (2w_1 + 2u_1))$$

The two are equal because real number addition is associative $(v_1 + w_1) + u_1 = v_1 + (w_1 + u_1)$.

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For condition (4) we just produce the member of V with the desired property. So consider the vector of 0's. Note that it is a member of V since its second component is twice its first, and note that it is the *identity element* with respect to addition.

$$\begin{aligned}\vec{v} + \vec{0} &= (v_1 \quad 2v_1) + (0 \quad 0) \\ &= (v_1 \quad 2v_1) \\ &= \vec{v}\end{aligned}$$

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \quad 2v_1)$ of V , consider $\vec{w} = (w_1 \quad 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels \vec{v} .

$$\vec{w} + \vec{v} = (-v_1 \quad -2v_1) + (v_1 \quad 2v_1) = \vec{0}$$

Condition (5), *existence of an additive inverse*, is also a matter of producing the desired element. Given a member $\vec{v} = (v_1 \ 2v_1)$ of V , consider $\vec{w} = (w_1 \ 2w_1)$ where $w_1 = -v_1$. Note that $\vec{w} \in V$ and note also that it additively cancels \vec{v} .

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Next we verify the conditions for scalar multiplication. First, condition (6) is *closure under scalar multiplication*. We consider a scalar $r \in \mathbb{R}$ and a vector $\vec{v} = (v_1 \ 2v_1) \in V$. The scalar multiple $r\vec{v} = (rv_1 \ r2v_1)$ is also a member of V because the second component is twice the first.

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Condition (7) is that *real number addition distributes over scalar multiplication*. Let the scalars be $r, s \in \mathbb{R}$ and let the vector be $\vec{v} = (v_1 \ 2v_1) \in V$. Then $(r + s)\vec{v} = ((r + s)v_1 \ (r + 2)2v_1)$, which equals $(rv_1 \ 2rv_1) + (sv_1 \ 2sv_1) = r\vec{v} + s\vec{v}$.

For *distributivity of vector addition over scalar multiplication* (8), let the scalar be $r \in \mathbb{R}$ and let the vectors be $\vec{v}, \vec{w} \in V$. Then $r(\vec{v} + \vec{w}) = (rv_1 \ 2rv_1) + (rw_1 \ 2rw_1)$, which equals $(rv_1 + rw_1 \ 2rv_1 + 2rw_1)$, which equals $r(v_1 \ 2v_1) + r(w_1 \ 2w_1) = r\vec{v} + r\vec{w}$.

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For condition (9) take $r, s \in \mathbb{R}$ and $\vec{v} = (v_1 \quad 2v_1) \in V$. The left side is $(rs)(v_1 \quad 2v_1) = ((rs)v_1 \quad (rs)2v_1)$, while the right side is $r(s(v_1 \quad 2v_1)) = r(sv_1 \quad s2v_1) = (r(sv_1) \quad r(s2v_1))$. The two are equal because, as they are real number multiplications, $(rs)v_1 = r(sv_1)$ and $(rs)2v_1 = r(s2v_1)$.

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The final condition is straightforward: for any $\vec{v} \in V$ we have $1\vec{v} = 1(v_1 \quad 2v_1) = (1 \cdot v_1 \quad 1 \cdot 2v_1) = \vec{v}$.

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Thus the set $V = \{(a \quad 2a) \mid a \in \mathbb{R}\}$ is a vector space under the natural addition and scalar multiplication operations.

Example The set \mathbb{R}^3 is a vector space under the usual vector addition and scalar multiplication operations.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} \quad \text{and} \quad r \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix}$$

To verify that we will check the conditions (more briefly than for the prior example).

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Condition (1) is closure under addition. This is clear because the only condition for membership in the set \mathbb{R}^3 is to be a three-tall vector of reals, and the sum of two three-tall vectors of reals is also a three-tall vector of reals.

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Condition (2) is routine

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{w} + \vec{v}$$

Condition (3) is also a direct consequence of the related real number property.

$$\begin{aligned}(\vec{v} + \vec{w}) + \vec{u} &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 + u_1 \\ v_2 + w_2 + u_2 \\ v_3 + w_3 + u_3 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 + u_1 \\ w_2 + u_2 \\ w_3 + u_3 \end{pmatrix} = \vec{v} + (\vec{w} + \vec{u})\end{aligned}$$

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For condition (4) take the vector of 0's.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

For condition (5), given $\vec{v} \in \mathbb{R}^3$, use $\vec{w} = -1\vec{v}$ as the additive inverse.

$$\begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Condition (6) is closure under scalar multiplication. Let the scalar be $r \in \mathbb{R}$ and the vector be $\vec{v} \in \mathbb{R}^3$. Then $r\vec{v}$ is a three-tall vector of reals, so $r\vec{v} \in \mathbb{R}^3$.

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Conditions (7)

$$(r+s) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (r+s)v_1 \\ (r+s)v_2 \\ (r+s)v_3 \end{pmatrix} = \begin{pmatrix} rv_1 + sv_1 \\ rv_2 + sv_2 \\ rv_3 + sv_3 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \\ rv_3 \end{pmatrix} + \begin{pmatrix} sv_1 \\ sv_2 \\ sv_3 \end{pmatrix} = r\vec{v} + s\vec{v}$$

and (8)

$$r(\vec{v} + \vec{w}) = r \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right) = r \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} rv_1 + rw_1 \\ rv_2 + rw_2 \\ rv_3 + rw_3 \end{pmatrix} = r\vec{v} + r\vec{w}$$

are straightforward.

Condition (9) is similar.

$$(\mathbf{rs}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{rs})v_1 \\ (\mathbf{rs})v_2 \\ (\mathbf{rs})v_3 \end{pmatrix} = \mathbf{r} \begin{pmatrix} \mathbf{s}v_1 \\ \mathbf{s}v_2 \\ \mathbf{s}v_3 \end{pmatrix} = \mathbf{r}\vec{v}(\mathbf{s}\vec{v})$$

And (10) is also easy.

$$1\vec{v} = 1 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 \\ 1 \cdot v_2 \\ 1 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{v}$$

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So the set \mathbb{R}^3 is a vector space under the usual operations of vector addition and scalar-vector multiplication.

Example The set $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

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Example The set of 3×3 matrices

$$\mathcal{M}_{3 \times 3} = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \mid a_{i,j} \in \mathbb{R} \right\}$$

is a vector space under the usual matrix addition and scalar multiplication.

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Example The set consisting only of the two-tall vector of 0's

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

is a vector space (under the usual vector addition and scalar multiplication operations).

1.7 *Definition* A one-element vector space is a *trivial* space.

1.16 *Lemma* In any vector space V , for any $\vec{v} \in V$ and $r \in \mathbb{R}$, we have
(1) $0 \cdot \vec{v} = \vec{0}$, and (2) $(-1 \cdot \vec{v}) + \vec{v} = \vec{0}$, and (3) $r \cdot \vec{0} = \vec{0}$.

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Proof For (1), note that $\vec{v} = (1 + 0) \cdot \vec{v} = \vec{v} + (0 \cdot \vec{v})$. Add to both sides the additive inverse of \vec{v} , the vector \vec{w} such that $\vec{w} + \vec{v} = \vec{0}$.

$$\vec{w} + \vec{v} = \vec{w} + \vec{v} + 0 \cdot \vec{v}$$

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Item (2) is easy: $(-1 \cdot \vec{v}) + \vec{v} = (-1 + 1) \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0}$ shows that we can write ‘ $-\vec{v}$ ’ for the additive inverse of \vec{v} without worrying about possible confusion with $(-1) \cdot \vec{v}$.

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For (3) $r \cdot \vec{0} = r \cdot (0 \cdot \vec{0}) = (r \cdot 0) \cdot \vec{0} = \vec{0}$ will do.

QED

Subspaces and spanning sets

Subspace

- 2.1 *Definition* For any vector space, a *subspace* is a subset that is itself a vector space, under the inherited operations.

Subspace

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Example In the vector space \mathbb{R}^2 , the line $y = 2x$

$$S = \left\{ \begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R} \right\}$$

is a subspace. The operations, as required by the definition, are the ones from \mathbb{R}^2 . We can check all the conditions to show it is a vector space, but the next result gives an easier way.

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Example This subset of $\mathcal{M}_{2 \times 2}$ is a subspace.

$$S = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} b \mid a, b \in \mathbb{R} \right\}$$

As above, addition and scalar multiplication are the same as in $\mathcal{M}_{2 \times 2}$.

Example This is not a subspace of \mathbb{R}^3 .

$$T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \right\}$$

It is a subset of \mathbb{R}^3 but it is not a vector space. One condition that it violates is that it is not closed under vector addition: here are two elements of T that sum to a vector that is not an element of T .

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(Another reason that it is not a vector space is that it does not contain the zero vector.)

2.9 *Lemma* For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements.

- (1) S is a subspace of that vector space
- (2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S
- (3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \dots, \vec{s}_n \in S$ and scalars r_1, \dots, r_n the vector $r_1\vec{s}_1 + \dots + r_n\vec{s}_n$ is in S .

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‘The following are equivalent’ means that each pair of statements are equivalent.

$$(1) \iff (2) \quad (2) \iff (3) \quad (3) \iff (1)$$

We will prove the equivalence by establishing that

$(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by the observation that $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue the single implication $(2) \implies (1)$.

2.9 *Proof* Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

The first item in the vector space definition has five conditions. First, for closure under addition, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$. Second, for any $\vec{s}_1, \vec{s}_2 \in S$, because addition is inherited from V , the sum $\vec{s}_1 + \vec{s}_2$ in S equals the sum $\vec{s}_1 + \vec{s}_2$ in V , and that equals the sum $\vec{s}_2 + \vec{s}_1$ in V (because V is a vector space, its addition is commutative), and that in turn equals the sum $\vec{s}_2 + \vec{s}_1$ in S . The argument for the third condition is similar to that for the second. For the fourth, consider the zero vector of V and note that closure of S under linear combinations of pairs of vectors gives that (where \vec{s} is any member of the nonempty set S) $0 \cdot \vec{s} + 0 \cdot \vec{s} = \vec{0}$ is in S ; showing that $\vec{0}$ acts under the inherited operations as the additive identity of S is easy. The fifth condition is satisfied because for any $\vec{s} \in S$, closure under linear combinations shows that the vector $0 \cdot \vec{0} + (-1) \cdot \vec{s}$ is in S ; showing that it is the additive inverse of \vec{s} under the inherited operations is routine.

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The conditions for scalar multiplication are similar.

QED

Example The vector space of quadratic polynomials $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ has a subspace comprised of the linear polynomials $L = \{b_0 + b_1x \mid b_0, b_1 \in \mathbb{R}\}$. To verify that, take scalars $r, s \in \mathbb{R}$ and consider a linear combination.

$$r(b_0 + b_1x) + s(c_0 + c_1x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

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Example Another subspace of \mathcal{P}_2 is the set of quadratic polynomials with all three coefficients equal.

$$M = \{a + ax + ax^2 \mid a \in \mathbb{R}\} = \{(1 + x + x^2)a \mid a \in \mathbb{R}\}$$

Verify that it is a subspace by taking two scalars $r, s \in \mathbb{R}$ and considering a linear combination of polynomials with all three coefficients the same.

$$r(a + ax + ax^2) + s(b + bx + bx^2) = (ra + sb) + (ra + sb)x + (ra + sb)x^2$$

The result is a quadratic polynomial with all three coefficients the same, and so M is closed under linear combinations.

The above examples of subspace parametrize the description.

Example This set is a plane inside of \mathbb{R}^3 .

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid 2x - y + z = 0 \right\}$$

We could verify that it is a subspace by checking that it is closed under linear combination as above.

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That's easier if we first parametrize the one-equation linear system $2x - y + z = 0$ using the free variables y and z .

$$P = \left\{ \begin{pmatrix} (1/2)y - (1/2)z \\ y \\ z \end{pmatrix} \mid y, z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \right\}$$

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Now members of P are described as a linear combination of those two vectors. Verifying that P is closed then involves taking a linear combination of linear combinations, which makes a linear combination.

Span

2.13 *Definition* The *span* (or *linear closure*) of a nonempty subset S of a vector space is the set of all linear combinations of vectors from S .

$$[S] = \{c_1 \vec{s}_1 + \cdots + c_n \vec{s}_n \mid c_1, \dots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \dots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is the trivial subspace.

No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

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No notation for the span is completely standard. The square brackets used here are common but so are 'span(S)' and 'sp(S)'.

Example Inside the vector space of all two-wide row vectors, the span of this one-element set

$$S = \{(1 \quad 2)\}$$

is this.

$$[S] = \{(a \quad 2a) \mid a \in \mathbb{R}\} = \{(1 \quad 2)a \mid a \in \mathbb{R}\}$$

Example This is a subset of \mathbb{R}^3 .

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Any vector in the xy -plane is a member of the span $[S]$; for instance, this system has a solution.

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

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$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

But vectors not in the xy -plane are not in the span; for instance, this system does not have a solution.

$$\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} c_2$$

(just consider the third components).

2.15 *Lemma* In a vector space, the span of any subset is a subspace.

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Proof If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then by Lemma 2.9 we need only check that the span $[S]$ is closed under linear combinations. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

(p, r scalars) is a linear combination of elements of S and so is in $[S]$ (possibly some of the \vec{s}_i 's from \vec{v} equal some of the \vec{s}_j 's from \vec{w} , but it does not matter). QED