Solving Linear Systems

Linear Algebra

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Gauss’s Method

Linear systems

1.1 *Definition* A *linear combination* of $x_1, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the combination's *coefficients*.

Example This is a linear combination of x, y, and z.

$$(1/4)x + y - z$$

1.1 *Definition* A *linear equation* in the variables $x_1, \ldots, x_n$ has the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$ where $d \in \mathbb{R}$ is the *constant*.

An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a *solution* of, or *satisfies*, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A *system of linear equations*

$$\begin{aligned}
a_{1,1}x_1 + \; a_{1,2}x_2 + \cdots + \; a_{1,n}x_n &= d_1 \\
a_{2,1}x_1 + \; a_{2,2}x_2 + \cdots + \; a_{2,n}x_n &= d_2 \\
&\;\;\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m
\end{aligned}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations.

Example There are three linear equations in this linear system.

$$\begin{aligned}
(1/4)x + \;\; y - \;\; z &= \;\; 0 \\
x + 4y + 2z &= 12 \\
2x - 3y - \;\; z &= \;\; 3
\end{aligned}$$

Solving a linear system

Example To find the solution of this system

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Next use the first row to act on the rows below, eliminating their x terms.

$$\xrightarrow[-2\rho_1+\rho_3]{-1\rho_1+\rho_2} \quad \begin{aligned} x + 4y - 4z &= 0 \\ +6z &= 12 \\ -11y + 7z &= 3 \end{aligned}$$

Then swap to bring a y term to the second row.

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{aligned} x + \quad 4y - 4z &= 0 \\ -11y + 7z &= 3 \\ 6z &= 12 \end{aligned}$$

Then swap to bring a y term to the second row.

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{aligned} x + \quad 4y - 4z &= 0 \\ -11y + 7z &= 3 \\ 6z &= 12 \end{aligned}$$

Now solve the bottom row: $z = 2$.

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Now solve the bottom row: $z = 2$. With that, the shape of the transformed system lets us solve for y by substituting into the second row: $-11y + 7(2) = 3$ shows $y = 1$.

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1.10 *Definition* In each row of a system, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it (except for the leading variable in the first row).

Example

$$\begin{array}{rcl} \begin{aligned} 2x - 3y - z + 2w &= -2 \\ x \phantom{{}-3y} + 3z + 1w &= 6 \\ 2x - 3y - z + 3w &= -3 \\ y + z - 2w &= 4 \end{aligned} & \xrightarrow[-\rho_1+\rho_3]{(-1/2)\rho_1+\rho_2} & \begin{aligned} 2x - 3y - z + 2w &= -2 \\ (3/2)y + (7/2)z &= 7 \\ w &= -1 \\ y + z - 2w &= 4 \end{aligned} \end{array}$$

Example

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\begin{array}{rcl}
2x - 3y - z + 2w &=& -2 \\
x \phantom{{}-3y} + 3z + 1w &=& 6 \\
2x - 3y - z + 3w &=& -3 \\
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$$

$$
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$$\begin{array}{rcl}
\begin{aligned}2x-3y-z+2w&=-2\\ x+3z+1w&=6\\ 2x-3y-z+3w&=-3\\ y+z-2w&=4\end{aligned}
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$$\xrightarrow{\rho_3\leftrightarrow\rho_4}\quad
\begin{aligned}2x-3y-z+2w&=-2\\ (3/2)y+(7/2)z&=7\\ -(4/3)z-2w&=-2/3\\ w&=-1\end{aligned}$$

The fourth equation says $w=-1$. Substituting back into the third equation gives $z=2$. Then back substitution into the second and first rows gives $y=0$ and $x=1$. The unique solution is $(1,0,2,-1)$.

Gauss's Method

1.5 *Theorem* If a linear system is changed to another by one of these operations

1) an equation is swapped with another
2) an equation has both sides multiplied by a nonzero constant
3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

1.6 *Definition* The three operations from Theorem 1.5 are the *elementary reduction operations*, or *row operations*, or *Gaussian operations*. They are *swapping*, *multiplying by a scalar* (or *rescaling*), and *row combination*.

1.5 *Proof* We verify the result for operation (1). The other two are similar.

Consider a linear system.

$$\begin{aligned}
a_{1,1}x_1 + \; a_{1,2}x_2 + \cdots + \; a_{1,n}x_n &= \; d_1 \\
&\vdots \\
a_{i,1}x_1 + \; a_{i,2}x_2 + \cdots + \; a_{i,n}x_n &= \; d_i \\
&\vdots \\
a_{j,1}x_1 + \; a_{j,2}x_2 + \cdots + \; a_{j,n}x_n &= \; d_j \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m
\end{aligned}$$

The tuple $(s_1, \ldots, s_n)$ satisfies this system if and only if substituting the values for the variables, the s's for the x's, gives a conjunction of true statements: $a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and $\ldots$ $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and $\ldots$ $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and $\ldots$ $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$.

In a list of statements joined with ‘and’ we can rearrange the order of the statements. Thus this requirement is met if and only if $a_{1,1}s_1 + a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and $\ldots$ $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and $\ldots$ $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and $\ldots$ $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$. This is exactly the requirement that $(s_1, \ldots, s_n)$ solves the system after the row swap. QED

Systems without a unique solution

Example This system has no solution.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + z &= 8 \\ 2x + 3y + 2z &= 13 \end{aligned}$$

On the left side of the equals sign the sum of the first two rows equals the third row, while on the right that is not so. So there is no triple of reals that makes all three equations true.

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Gauss' Method makes the inconsistency clear.

$$\xrightarrow[-2\rho_1+\rho_3]{-\rho_1+\rho_2} \begin{aligned} x + y + z &= 6 \\ y &= 2 \\ y &= 1 \end{aligned} \xrightarrow{-\rho_2+\rho_3} \begin{aligned} x + y + z &= 6 \\ y &= 2 \\ 0 &= -1 \end{aligned}$$

Example This system has infinitely many solutions.

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 4 \\ x & + & y & - & 2z & = & -1 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcrcrcr} x & - & y & + & z & = & 4 \\ & & 2y & - & 3z & = & -5 \end{array}$$

Taking $z = 0$ gives the solution $(3/2, -5/2, 0)$. Taking $z = -1$ gives $(1, -4, -1)$.

Example This system has infinitely many solutions.

$$\begin{array}{rcl} x-y+z &=& 4 \\ x+y-2z &=& -1 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} x-y+z &=& 4 \\ 2y-3z &=& -5 \end{array}$$

Taking $z=0$ gives the solution $(3/2,-5/2,0)$. Taking $z=-1$ gives $(1,-4,-1)$.

Example Another system with infinitely many solutions.

$$\begin{array}{rcl} -x-y+3z &=& 3 \\ x+z &=& 3 \\ 3x-y+7z &=& 15 \end{array} \xrightarrow[3\rho_1+\rho_3]{-\rho_1+\rho_2} \begin{array}{rcl} -x-y+3z &=& 3 \\ -y+4z &=& 6 \\ -4y+16z &=& 24 \end{array}$$

$$\xrightarrow{-4\rho_2+\rho_3} \begin{array}{rcl} -x-y+3z &=& 3 \\ -y+4z &=& 6 \\ 0 &=& 0 \end{array}$$

Taking $z=0$ gives $(3,-6,0)$ while taking $z=1$ gives $(2,-2,1)$.

Describing the solution set

Parametrizing

We've seen that this system has infinitely many solutions.

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We want to describe the solution set.

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We want to describe the solution set.

Use the second row to express y in terms of z as $y = -6 + 4z$. Now substitute into the first row $-x - (-6 + 4z) + 3z = 3$ to express x also in terms of z with $x = 3 - z$.

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We want to describe the solution set.

Use the second row to express y in terms of z as $y=-6+4z$. Now substitute into the first row $-x-(-6+4z)+3z=3$ to express x also in terms of z with $x=3-z$.

2.2 *Definition* In an echelon form linear system the variables that are not leading are *free*.

A variable that we use to describe a family of solutions is a *parameter*.

We shall routinely parametrize linear systems using the free variables.

Example

$$\begin{array}{rcl}
x - y + 2z + 3w &=& 14 \\
2x - 2y - z + 2w &=& 6 \\
-3z + 2w &=& 0
\end{array}
\xrightarrow{-2\rho_1+\rho_2}
\begin{array}{rcl}
x - y + 2z + 3w &=& 14 \\
-5z - 4w &=& -22 \\
-3z + 2w &=& 0
\end{array}$$

$$\xrightarrow{-(3/5)\rho_2+\rho_3}
\begin{array}{rcl}
x - y + 2z + 3w &=& 14 \\
-5z - 4w &=& -22 \\
(22/5)w &=& 66/5
\end{array}$$

The leading variables are x, z, and w. We will parametrize with the free variable y.

Example

$$\begin{array}{rcl} x - y + 2z + 3w = 14 \\ 2x - 2y - z + 2w = 6 \\ -3z + 2w = 0 \end{array} \xrightarrow{-2\rho_1+\rho_2} \begin{array}{r} x - y + 2z + 3w = 14 \\ -5z - 4w = -22 \\ -3z + 2w = 0 \end{array}$$

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The leading variables are x, z, and w. We will parametrize with the free variable y.

The bottom row gives $w = 3$ and substituting that into the next row up gives $z = 2$. The top equation is $x - y + 2 \cdot 2 + 3 \cdot 3 = 14$ so we have $x = 1 - y$.

Example This system is already in echelon form.

$$\begin{aligned} -2x + y - z + w &= 3/2 \\ 2z - w &= 1/2 \end{aligned}$$

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The second row gives $z = 1/4 + (1/2)w$. Substituting back into the first row gives $-2x + y - ((1/4) + (1/2)w) + w = 3/2$, and solving for x leaves $x = -(7/8) + (1/2)y + (1/4)w$.

Matrices and vectors

2.6 *Definition* An $m \times n$ *matrix* is a rectangular array of numbers with m *rows* and n *columns*. Each number in the matrix is an *entry*.

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Example This is a 2×3 matrix

$$B = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & 6 \end{pmatrix}$$

because it has 2 rows and 3 columns. The entry in row 2 and column 1 is $b_{2,1} = 4$.

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2.8 *Definition* A *vector* (or *column vector*) is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*. A column or row vector whose components are all zeros is a *zero vector*.

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We denote a vector with an over-arrow (many authors use boldface).

Example This column vector has three components.

$$\vec{v} = \begin{pmatrix} -1 \\ -0.5 \\ 0 \end{pmatrix}$$

Example This row vector has three components

$$\vec{w} = \begin{pmatrix} -1 & -0.5 & 0 \end{pmatrix}$$

Example This is the two-component zero vector.

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Vector operations

2.10 *Definition* The *vector sum* of $\vec{u}$ and $\vec{v}$ is the vector of the sums.

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

2.11 *Definition* The *scalar multiplication* of the real number r and the vector $\vec{v}$ is the vector of the multiples.

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

Example

$$3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Matrix notation for linear systems

Example We can simplify the clerical load in reducing this system

$$\begin{aligned} -3x \qquad\quad + 2z &= -1 \\ x - 2y + 2z &= -5/3 \\ -x - 4y + 6z &= -13/3 \end{aligned}$$

by writing it as an *augmented matrix*.

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 1 & -2 & 2 & -5/3 \\ -1 & -4 & 6 & -13/3 \end{array}\right) \xrightarrow[-(1/3)\rho_1+\rho_3]{(1/3)\rho_1+\rho_2} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & -4 & 16/3 & -4 \end{array}\right)$$

$$\xrightarrow{-2\rho_2+\rho_3} \left(\begin{array}{ccc|c} -3 & 0 & 2 & -1 \\ 0 & -2 & 8/3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.

Parametrizing $-3x + 2z = -1$ and $-2y + (8/3)z = -2$ gives $y = 1 + (4/3)z$ and $x = (1/3) + (2/3)z$.

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We can write the solution set in vector notation.

$$\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 4/3 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \}$$

Example Reducing this system

$$\begin{array}{rcl} x + 2y - z \phantom{{}+w} &=& 2 \\ 2x - y - 2z + w &=& 5 \end{array}$$

using the augmented matrix notation

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 5 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & -5 & 0 & 1 & 1 \end{array}\right)$$

gives this vector description of the solution set.

$$\{ \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \}$$

$$\text{General} = \text{Particular} + \text{Homogeneous}$$

Form of solution sets

Example This system

$$\begin{aligned} x + 2y - z \phantom{{}+w} &= 2 \\ 2x - y - 2z + w &= 5 \end{aligned}$$

has solutions of this form.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 12/5 \\ -1/5 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \qquad z, w \in \mathbb{R}$$

Taking $z = w = 0$ shows that the first vector is a particular solution of the system.

3.2 *Definition* A linear equation is *homogeneous* if it has a constant of zero, so that it can be written as $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

Example Consider the system of homogeneous equations derived from the above system by changing the constants to 0's.

$$\begin{aligned} x + 2y - z \phantom{{}+w} &= 0 \\ 2x - y - 2z + w &= 0 \end{aligned}$$

The same Gauss's Method steps reduce it to echelon form.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 2 & -1 & -2 & 1 & 0 \end{array}\right) \xrightarrow{-2\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \end{array}\right)$$

The vector description of the solution set

$$\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -2/5 \\ 1/5 \\ 0 \\ 1 \end{pmatrix} w \mid z, w \in \mathbb{R} \}$$

is the same as earlier but with a particular solution that is the zero vector.

3.6 *Lemma* For any homogeneous linear system there exist vectors $\vec{\beta}_1$, $\ldots$, $\vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where k is the number of free variables in an echelon form version of the system.

Example Consider this system of homogeneous equations.

$$\begin{aligned} x + y + z + w &= 0 \\ y - z + w &= 0 \end{aligned}$$

With the bottom equation express the leading variable y in terms of the free variables $y = z - w$. Move up to the equation above, substitute $x + (z - w) + z + w = 0$, and solve for the leading variable $x = -2z$.

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$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} w \qquad z, w \in \mathbb{R}$$

and recognize $\vec{\beta}_1$ and $\vec{\beta}_2$ as the vectors associated with z and w.

3.6 *Proof* Apply Gauss's Method to get to echelon form. There may be some $0 = 0$ equations; we ignore these (if the system consists only of $0 = 0$ equations then the lemma is trivially true because there are no leading variables). But because the system is homogeneous there are no contradictory equations.

We will use induction to verify that each leading variable can be expressed in terms of free variables. That will finish the proof because we can use the free variables as parameters and the $\vec{\beta}$'s are the vectors of coefficients of those free variables.

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For the base step consider the bottom-most equation

$$a_{m,\ell_m}x_{\ell_m} + a_{m,\ell_m+1}x_{\ell_m+1} + \cdots + a_{m,n}x_n = 0 \qquad (*)$$

where $a_{m,\ell_m} \neq 0$.

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where $a_{m,\ell_m} \neq 0$. This is the bottom row so any variables after the leading one must be free. Move these to the right hand side and divide by a_{m,ℓ_m}

$$x_{\ell_m} = (-a_{m,\ell_m+1}/a_{m,\ell_m})x_{\ell_m+1} + \cdots + (-a_{m,n}/a_{m,\ell_m})x_n$$

to express the leading variable in terms of free variables.

For the inductive step assume that the statement holds for the bottom-most t rows, with $0 \leqslant t < m-1$. That is, assume that for the m-th equation, and the $(m-1)$-th equation, etc., up to and including the $(m-t)$-th equation, we can express the leading variable in terms of free ones. We must verify that this then also holds for the next equation up, the $(m-(t+1))$-th equation. For that, take each variable that leads in a lower equation $x_{\ell_m}, \ldots, x_{\ell_{m-t}}$ and substitute its expression in terms of free variables. We only need expressions for leading variables from lower equations because the system is in echelon form, so leading variables in higher equation do not appear in this equation. The result has a leading term of $a_{m-(t+1),\ell_{m-(t+1)}} x_{\ell_{m-(t+1)}}$ with $a_{m-(t+1),\ell_{m-(t+1)}} \neq 0$, and the rest of the left hand side is a linear combination of free variables. Move the free variables to the right side and divide by $a_{m-(t+1),\ell_{m-(t+1)}}$ to end with this equation's leading variable $x_{\ell_{m-(t+1)}}$ in terms of free variables.

We have done both the base step and the inductive step so by the principle of mathematical induction the proposition is true. QED

3.7 *Lemma* For a linear system, where $\vec{p}$ is any particular solution, the solution set equals this set.

$$\{\vec{p}+\vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$$

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3.7 *Proof* For set inclusion the first way, that if a vector solves the system then it is in the set described above, assume that $\vec{s}$ solves the system. Then $\vec{s}-\vec{p}$ solves the associated homogeneous system since for each equation index i,

$$\begin{aligned}
&a_{i,1}(s_1-p_1)+\cdots+a_{i,n}(s_n-p_n)\\
&=(a_{i,1}s_1+\cdots+a_{i,n}s_n)-(a_{i,1}p_1+\cdots+a_{i,n}p_n)=d_i-d_i=0
\end{aligned}$$

where p_j and s_j are the j-th components of $\vec{p}$ and $\vec{s}$. Express $\vec{s}$ in the required $\vec{p}+\vec{h}$ form by writing $\vec{s}-\vec{p}$ as $\vec{h}$.

For set inclusion the other way, take a vector of the form $\vec{p}+\vec{h}$, where $\vec{p}$ solves the system and $\vec{h}$ solves the associated homogeneous system and note that $\vec{p}+\vec{h}$ solves the given system since for any equation index i,

$$\begin{aligned}
&a_{i,1}(p_1+h_1)+\cdots+a_{i,n}(p_n+h_n)\\
&=(a_{i,1}p_1+\cdots+a_{i,n}p_n)+(a_{i,1}h_1+\cdots+a_{i,n}h_n)=d_i+0=d_i
\end{aligned}$$

where as earlier p_j and h_j are the j-th components of $\vec{p}$ and $\vec{h}$. QED

3.1 *Theorem* Any linear system's solution set has the form

$$\{\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where $\vec{p}$ is any particular solution and where the number of vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ equals the number of free variables that the system has after a Gaussian reduction.

Proof This restates the prior two lemmas. QED

3.10 *Corollary* Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

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3.10 *Proof* We've seen examples of all three happening so we need only prove that there are no other possibilities.

First observe a homogeneous system with at least one non-$\vec{0}$ solution $\vec{v}$ has infinitely many solutions. This is because any scalar multiple of $\vec{v}$ also solves the homogeneous system and there are infinitely many vectors in the set of scalar multiples of $\vec{v}$: if $s, t \in \mathbb{R}$ are unequal then $s\vec{v} \neq t\vec{v}$, since $s\vec{v} - t\vec{v} = (s - t)\vec{v}$ is non-$\vec{0}$ as any non-0 component of $\vec{v}$, when rescaled by the non-0 factor $s - t$, will give a non-0 value.

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Now apply Lemma 3.7 to conclude that a solution set

$$\{\vec{p} + \vec{h} \mid \vec{h} \text{ solves the associated homogeneous system}\}$$

is either empty (if there is no particular solution $\vec{p}$), or has one element (if there is a $\vec{p}$ and the homogeneous system has the unique solution $\vec{0}$), or is infinite (if there is a $\vec{p}$ and the homogeneous system has a non-$\vec{0}$ solution, and thus by the prior paragraph has infinitely many solutions). QED

Summary: Kinds of Solution Sets

		number of solutions of the homogeneous system	
		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
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An important special case is when there are the same number of equations as unknowns.

3.11 *Definition* A square matrix is *nonsingular* if it is the matrix of coefficients of a homogeneous system with a unique solution. It is *singular* otherwise, that is, if it is the matrix of coefficients of a homogeneous system with infinitely many solutions.