Four.II Geometry of Determinants

Linear Algebra

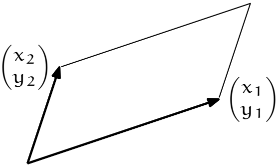
Jim Hefferon

http://joshua.smcvt.edu/linearalgebra

Determinants as size functions

Box

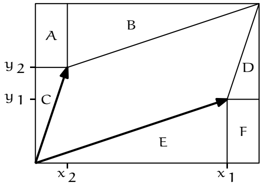
This parallelogram is defined by the two vectors.



1.1 *Definition* In $\mathbb{R}^n$ the *box* (or *parallelepiped*) formed by $\langle \vec{v}_1, \ldots, \vec{v}_n \rangle$ is the set $\{t_1\vec{v}_1 + \cdots + t_n\vec{v}_n \mid t_1, \ldots, t_n \in [0 \ldots 1]\}$.

The box is a subset the span, with the scalars limited to the unit interval.

Area of a box



$$\begin{aligned}\text{box area} &= \text{rectangle area} - \text{area of } A - \cdots - \text{area of F} \\ &= (x_1 + x_2)(y_1 + y_2) - x_2 y_1 - x_1 y_1/2 \\ &\quad - x_2 y_2/2 - x_2 y_2/2 - x_1 y_1/2 - x_2 y_1 \\ &= x_1 y_2 - x_2 y_1\end{aligned}$$

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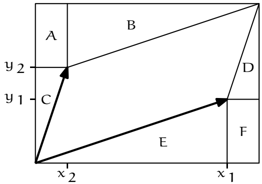

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&= x_1 y_2 - x_2 y_1
\end{aligned}$$

We recognize that as the value of the determinant.

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1$$

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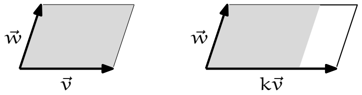
Because the determinant of a matrix equals the determinant of its transpose, we can change the determinant conditions from statements about rows to statements about columns. For instance, the first condition is that a determinant is unaffected by a row operation $k\rho_i + \rho_j$ with $i \neq j$. By transposing we have that a determinant does not change under combinations of columns.

Definition of determinant reinterpreted

Condition (3) from the definition of determinant is that rescaling a column rescales the entire determinant $\det(\ldots, k\vec{v}_i, \ldots) = k \cdot \det(\ldots, \vec{v}_i, \ldots)$. This fits with the idea that the determinant gives the size of the box formed by the columns of the matrix: if we scale a column by a factor k then the size of the box scales by that factor.



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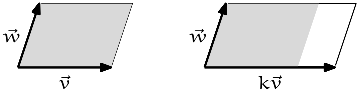
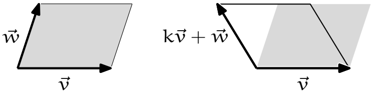


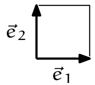
Condition (1) is that the determinant is unaffected by combining columns. The picture



shows that the box formed by $\vec{v}$ and $k\vec{v} + \vec{w}$ is slanted at a different angle than the original box but the two have the same base and height, and hence the same area.

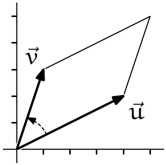
As we noted after the definition, condition (2) is a consequence of the others so we leave it aside for the moment.

Condition (4) is that the determinant of the identity matrix is 1.

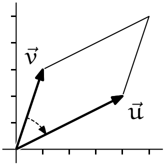


Orientation in two space

1.2 *Remark* Although condition (2) is redundant, it says something notable. Consider these two. Swapping changes the sign.



$$\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$$



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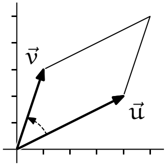

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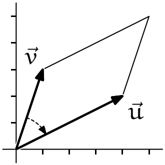

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In the picture on the left, $\vec{u}$ is first in the matrix and then we follow the counterclockwise arc to $\vec{v}$, and get a positive size. On the right following the clockwise arc gives a negative size. The sign returned by the size function, the determinant, reflects the *orientation* or *sense* of the box.

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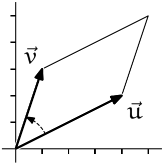

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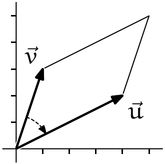

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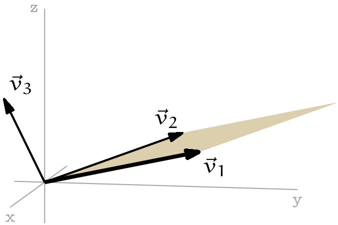
We see the same thing if we use Condition (3) with a negative scalar.

More on orientation: three space

Starting with these two vectors we want to form a three-space box with positive size.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$



Those two vectors span a plane, which divides three-space in two. The $\vec{v}_3$ shown is on the side of the plane containing vectors with this property:

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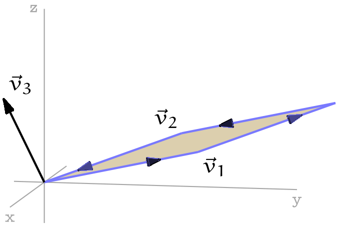

Those two vectors span a plane, which divides three-space in two. The $\vec{v}_3$ shown is on the side of the plane containing vectors with this property: if a person at the tip of $\vec{v}_3$ looks down at the box and traces out the parallelogram from $\vec{0}$, to $\vec{v}_1$, to $\vec{v}_1 + \vec{v}_2$, to $\vec{v}_2$, and back to $\vec{0}$ then their trace looks to them to be counterclockwise. Any such vector will make a positive-sized box.

The vector shown is this.

$$\vec{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \qquad \begin{vmatrix} 1 & -2 & 0 \\ 4 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 25$$

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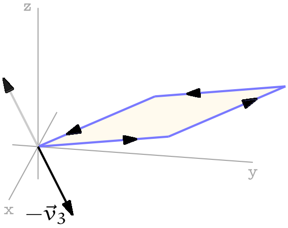

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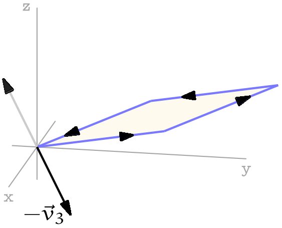

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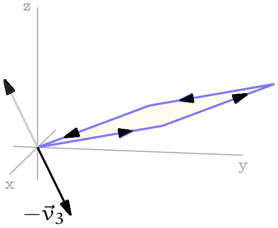

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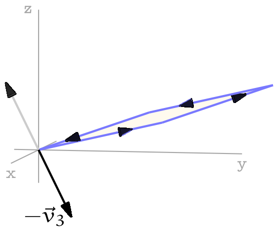

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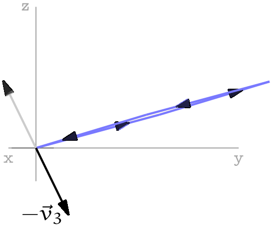

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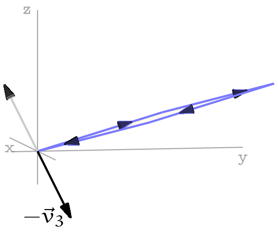

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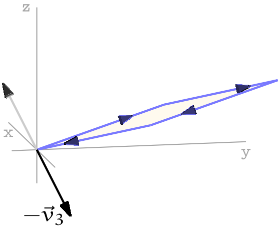

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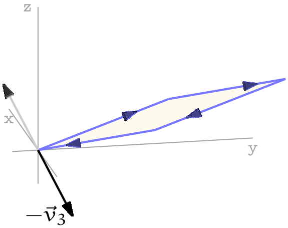

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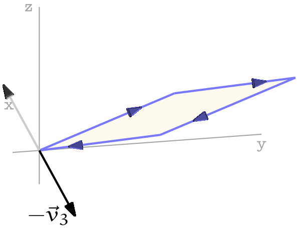

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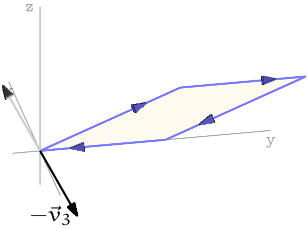

Determinants are multiplicative

1.5 *Theorem* A transformation $t\colon \mathbb{R}^n \to \mathbb{R}^n$ changes the size of all boxes by the same factor, namely, the size of the image of a box $|t(S)|$ is $|T|$ times the size of the box $|S|$, where T is the matrix representing t with respect to the standard basis.

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Proof First consider the $|T| = 0$ case, the case that T is singular and does not have an inverse. Observe that if TS is invertible then there is an M such that $(TS)M = I$, so $T(SM) = I$, and so T is invertible. The contrapositive of that observation is that if T is not invertible then neither is TS — if $|T| = 0$ then $|TS| = 0$.

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Now consider the case that T is nonsingular. Any nonsingular matrix factors into a product of elementary matrices $T = E_1 E_2 \cdots E_r$. To finish this argument we will verify that $|ES| = |E| \cdot |S|$ for all matrices S and elementary matrices E. The result will then follow because $|TS| = |E_1 \cdots E_r S| = |E_1| \cdots |E_r| \cdot |S| = |E_1 \cdots E_r| \cdot |S| = |T| \cdot |S|$.

There are three types of elementary matrix. We will cover the $M_i(k)$ case; the $P_{i,j}$ and $C_{i,j}(k)$ checks are similar. The matrix $M_i(k)S$ equals S except that row i is multiplied by k. The third condition of determinant functions then gives that $|M_i(k)S| = k \cdot |S|$. But $|M_i(k)| = k$, again by the third condition because $M_i(k)$ is derived from the identity by multiplication of row i by k. Thus $|ES| = |E| \cdot |S|$ holds for $E = M_i(k)$. QED

Example The transformation $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through a counterclockwise angle θ is represented by this matrix.

$$T_\theta = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$

Observe that t_θ doesn't change the size of any boxes, it just rotates the entire box as a rigid whole. Note that $|T_\theta| = 1$.

Example The transformation $t_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ that rotates all vectors through a counterclockwise angle θ is represented by this matrix.

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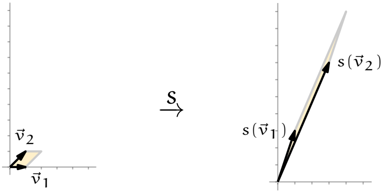
Example The linear transformation $s\colon \mathbb{R}^2 \to \mathbb{R}^2$ represented with respect to the standard basis by this matrix

$$S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

will, by the theorem, change the size of a box by a factor of $|S| = -2$. Here is s acting on a typical box.

The box defined by the two vectors $\vec{v}_1 = \binom{1}{0}$ and $\vec{v}_2 = \binom{1}{1}$ is transformed by s to the box defined by the two vectors $s(\vec{v}_1) = \binom{1}{3}$ and $s(\vec{v}_2) = \binom{3}{7}$.



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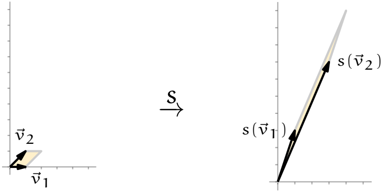

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The two sizes are easy.

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} = -2$$

Determinant of the inverse

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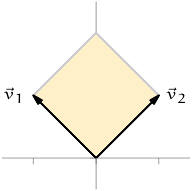
Proof $1 = |I| = |TT^{-1}| = |T| \cdot |T^{-1}|$ QED

Volume

1.3 *Definition* The *volume* of a box is the absolute value of the determinant of a matrix with those vectors as columns.

Example The box formed by the vectors

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$



gives this determinant

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

so its volume is 2.

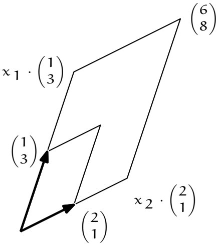
Cramer's Rule

Geometric interpretation of linear systems

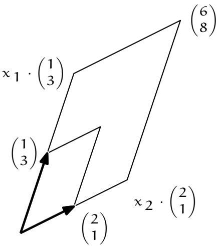
A linear system is equivalent to a linear relationship among vectors.

$$\begin{aligned} x_1 + 2x_2 &= 6 \\ 3x_1 + x_2 &= 8 \end{aligned} \qquad \Longleftrightarrow \qquad x_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

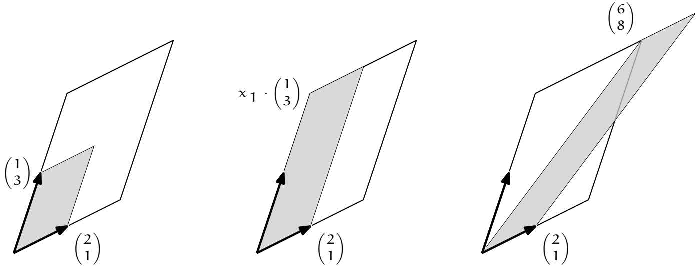
In the picture below the small parallelogram is formed from sides that are the vectors $\binom{1}{3}$ and $\binom{2}{1}$. It is nested inside a parallelogram with sides $x_1\binom{1}{3}$ and $x_2\binom{2}{1}$. By the vector equation, the far corner of the larger parallelogram is $\binom{6}{8}$.



This drawing restates the algebraic question of finding the solution of a linear system into geometric terms: by what factors x_1 and x_2 must we dilate the sides of the starting parallelogram so that it will fill the other one?



Consider expanding only one side of the parallelogram. Compare the sizes of these shaded boxes.



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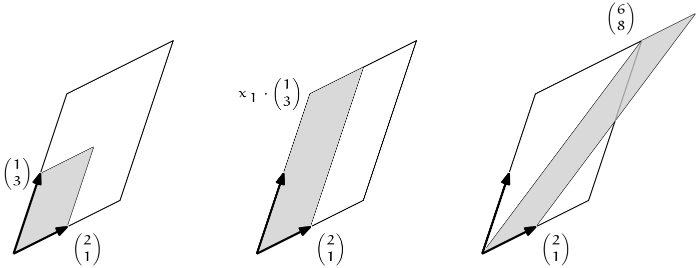

The second is defined by the vectors $x_1\binom{1}{3}$ and $\binom{2}{1}$ and one of the properties of the size function — the determinant — is that therefore the size of the second box is x_1 times the size of the first. The third box is derived from the second by shearing, adding $x_2\binom{2}{1}$ to $x_1\binom{1}{3}$ to get $x_1\binom{1}{3} + x_2\binom{2}{1} = \binom{6}{8}$, along with $\binom{2}{1}$. The determinant is not affected by shearing so the size of the third box equals that of the second.

Taken together, we have this.

$$\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix} = \begin{vmatrix} x_1 \cdot 1 & 2 \\ x_1 \cdot 3 & 1 \end{vmatrix} = x_1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

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Solving gives the value of one of the variables.

$$x_1 = \frac{\begin{vmatrix} 6 & 2 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

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The symmetric argument for the other side gives this.

$$x_2 = \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}} = 2$$

Cramer's Rule

Let A be an $n \times n$ matrix, let $\vec{b}$ be an n-tall column vector, and consider the linear system $A\vec{x} = \vec{b}$. For any $i \in [1, \ldots, n]$ let B_i be the matrix obtained by substituting $\vec{b}$ for column i of A. Then the value of the i-th unknown is $x_i = |B_i|/|A|$.

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Exercise 3 gives the proof.

Example Given this system

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 + 3x_2 \phantom{{}-5x_3} &= 2 \\ x_2 - 5x_3 &= 0 \end{aligned}$$

we can rewrite it as

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

and

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ 0 & 1 & -5 \end{vmatrix} = -26 \qquad |B_2| = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -5 \end{vmatrix} = 0$$

so $x_2 = 0/-26 = 0$.

A caution

Cramer's Rule is an interesting application of the geometry that we have developed. And it allows us to mentally solve systems with two or three variables that use simple numbers. But don't use it for systems having many variables. Taking a determinant of a general large matrix is very slow.