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Power System State Estimation Theory and Implementation

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Chapter 2

Weighted Least Squares State Estimation

2.1 Introduction

Static state estimation refers to the procedure of obtaining the voltage phasors at all of the system buses at a given point in time. This can be achieved by direct means which involve very accurate synchronized phasor measurements of all bus voltages in the system. However, such an approach would be very vulnerable to measurement errors or telemetery failures. Instead, state estimation procedure makes use of a set of redundant measurements in order to filter out such errors and find an optimal estimate. The measurements may include not only the conventional power and voltage measurements, but also those others such as the current magnitude or synchronized voltage phasor measurements as well. Simultaneous measurement of quantities at different parts of the system is practically impossible, hence a certain amount of time skew between measurements is commonly tolerated. This tolerance is justified due to the slowly varying operating conditions of the power systems under normal operating conditions.

The definition of the system state usually includes the steady state bus voltage phasors only. This implies that the network topology and parameters are perfectly known. However, errors in the network parameters or topology do exist occasionally, due to various reasons such as unreported outages, transmission line sags on hot days, etc. Detection and correction of such errors will be separately discussed later on in chapters 7 and 8.

2.2 Component Modeling and Assumptions

Power system is assumed to operate in the steady state under balanced conditions. This implies that all bus loads and branch power flows will be three phase and balanced, all transmission lines are fully transposed, and all other series or shunt devices are symmetrical in the three phases. These assumptions allow the use of single phase positive sequence equivalent circuit for modeling the entire power system. The solution that will be obtained by using such a network model, will also be the positive sequence component of the system state during balanced steady state operation. As in the case of the power flow, all network data as well as the network variables, are expressed in the per unit system. The following component models will thus be used in representing the entire network.

2.2.1 Transmission Lines

Transmission lines are represented by a two-port π-model whose parameters correspond to the positive sequence equivalent circuit of transmission lines. A transmission line with a positive sequence series impedance of $R+jX$ and total line charging susceptance of $j2B$, will be modelled by the equivalent circuit shown in Figure 2.1.

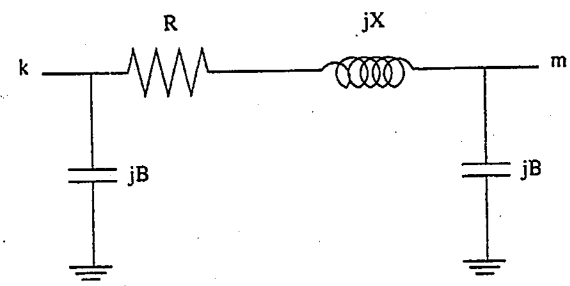

Figure 2.1. Equivalent circuit for a transmission line

2.2.2 Shunt Capacitors or Reactors

Shunt capacitors or reactors which may be used for voltage and/or reactive power control, are represented by their per phase susceptance at the corresponding bus. The sign of the susceptance value will determine the type of the shunt element. It will be positive or negative corresponding to a shunt capacitor or reactor respectively.

2.2.3 Tap Changing and Phase Shifting Transformers

Transformers with off-nominal but in-phase taps, can be modeled as series impedances in series with ideal transformers as shown in Figure 2.2. The two transformer terminal buses m and k are commonly designated as the impedance side and the tap side bus respectively.

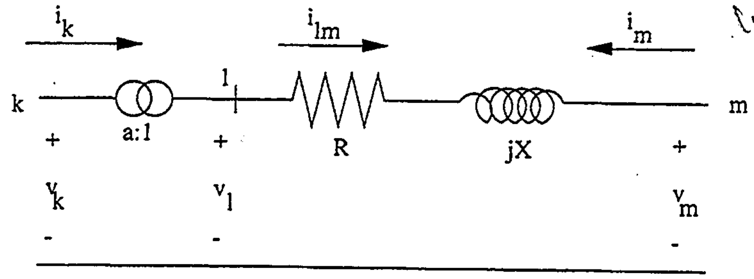

Figure 2.2. Equivalent circuit for an off-nominal tap transformer

The nodal equations of the two port circuit of Figure 2.2 can be derived by first expressing the current flows $i_{\ell m}$ and i_m at each end of the series branch $R + jX$. Denoting the admittance of this branch $\ell - m$ by y, the terminal current injections will be given by:

$$\begin{bmatrix} i_{\ell m} \\ i_m \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \begin{bmatrix} v_\ell \\ v_m \end{bmatrix} \tag{2.1}$$

Substituting for $i_{\ell m}$ and v_ℓ:

$$\begin{aligned} i_{\ell m} &= a \cdot i_k \\ v_\ell &= v_k / a \end{aligned}$$

the final form will be obtained as follows:

$$\begin{bmatrix} i_k \\ i_m \end{bmatrix} = \begin{bmatrix} y/a^2 & -y/a \\ -y/a & y \end{bmatrix} \begin{bmatrix} v_k \\ v_m \end{bmatrix} \tag{2.2}$$

where a is the in phase tap ratio. Figure 2.3 shows the corresponding two port equivalent circuit for the above set of nodal equations.

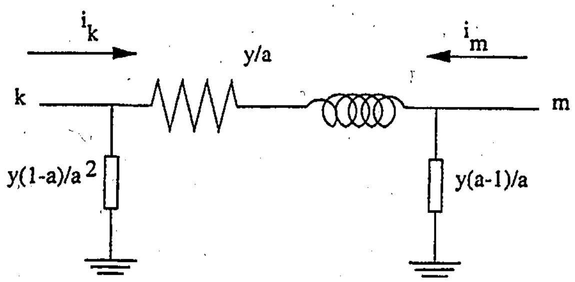

Figure 2.3. Equivalent circuit of an in-phase tap changer

For a phase shifting transformer where the off-nominal tap value a, is complex, the equations will slightly change as:

$$\begin{aligned} a^* i_k &= i_{\ell m} \\ a v_\ell &= v_k \end{aligned}$$

yielding the following set of nodal equations:

$$\begin{bmatrix} i_k \\ i_m \end{bmatrix} = \begin{bmatrix} y/\mid a \mid^2 & -y/a^* \\ -y/a & y \end{bmatrix} \begin{bmatrix} v_k \\ v_m \end{bmatrix} \tag{2.3}$$

Note the loss of reciprocity as the admittance matrix is no longer symmetrical. Therefore, a passive equivalent circuit such as the one shown in Figure 2.3 for the in-phase tap changer, can no longer be realized for the phase shifting transformer. However, the circuit equations can still be solved as before by only modifying the admittance matrix which is no longer symmetrical.

2.2.4 Loads and Generators

Loads and generators are modeled as equivalent complex power injections and therefore have no effect on the network model. Exceptions are constant impedance type loads which are included as shunt admittances at the corresponding buses.

2.3 Building the Network Model

The above-described component models can be used to build the network model for the entire power system. This is accomplished by writing a set of nodal equations which are derived by applying Kirchhoff's current law at each bus. Denoting the vector of net current injections by I, and the vector of bus voltage phasors by V, these equations will take the following form:

$$I = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = Y \cdot V \tag{2.4}$$

where
i_k is the net current injection phasor at bus k.
v_k is the voltage phasor at bus k.
Y_{km} is the (k,m)th element of Y.

Note that, as a convention, currents (or power) entering a bus will be assumed to be positive injections throughout the rest of the book. Matrix Y is referred to as the bus admittance matrix, and has the following properties:

1. It is in general complex, and can be written as $G + jB$.
2. It is structurally symmetric. It may also be numerically symmetric depending upon the absence of certain network components such as phase shifters, with non-symmetrical nodal equations.

3. It is very sparse.

4. It is non-singular provided that each island in the network has at least one shunt connection to ground.

Equation (2.4) is valid for any N-port passive circuit with external current injections defined by the vector I. This representation of the network by nodal equations, facilitates the modification of equations in case of topology changes. Adding or removing a k-port sub-circuit can be easily done by adding or subtracting the corresponding entries of the admittance matrix.

As an example, consider a two-port model of a transformer connected between bus k and m, having a series admittance of y_t and a tap ratio of a, represented by the following nodal equations:

$$\begin{bmatrix} i_k \\ i_m \end{bmatrix} = \begin{bmatrix} y_t / \mid a \mid^2 & -y_t/a^* \\ -y_t/a & y_t \end{bmatrix} \begin{bmatrix} v_k \\ v_m \end{bmatrix} \tag{2.5}$$

Given the bus admittance matrix Y for the entire system, the transformer model can be introduced by modifying the following 4 entries in Y:

$$\begin{aligned} Y_{kk}^{new} &= Y_{kk} + y_t / \mid a \mid^2 \\ Y_{km}^{new} &= Y_{km} - y_t/a^* \\ Y_{mk}^{new} &= Y_{mk} - y_t/a \\ Y_{mm}^{new} &= Y_{mm} + y_t \end{aligned}$$

Hence, the bus admittance matrix Y of a large power system can be built from scratch by introducing one subsystem at a time and modifying the corresponding entries of Y until all branches are processed. One of the simplest subsystems is a two-port network such as the model of a transformer or a transmission line as shown in Figures 2.1 and 2.3.

Example 2.1:

Consider the 4-bus power system whose one-line diagram is given in Figure 2.4. Network data and the steady state bus voltages are listed below. The susceptance of the shunt capacitor at bus 3 is given as 0.5 per unit.

From Bus	To Bus	R pu	X pu	Total Line Charging Susceptance	Tap a	Tap Side Bus
1	2	0.02	0.06	0.20	–	–
1	3	0.02	0.06	0.25	–	–
2	3	0.05	0.10	0.00	–	–
2	4	0.00	0.08	0.00	0.98	2

Bus No.	Voltage Mag. pu	Phase Angle degrees
1	1.0000	0.00
2	0.9629	-2.76
3	0.9597	-3.58
4	0.9742	-3.96

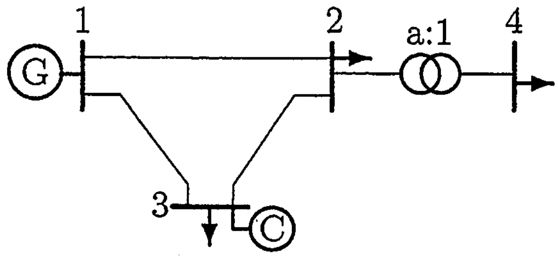

Figure 2.4. One-line diagram of a 4-bus power system

- Write the nodal equations for the 2-port π-model of the transformer connected between bus 2 and 4.
- Form the bus admittance matrix, Y for the entire system.
- Calculate the net complex power injections at each bus.

Solution:

The nodal equations for the transformer branch will be obtained by substituting for y and a in Equation (2.2):

$$\begin{bmatrix} i_2 \\ i_4 \end{bmatrix} = \begin{bmatrix} -j13.02 & j12.75 \\ j12.75 & -j12.50 \end{bmatrix} \begin{bmatrix} v_2 \\ v_4 \end{bmatrix}$$

Bus admittance matrix for the entire system can be obtained by including one branch at a time and expanding the above admittance matrix to a 4x4 matrix:

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix} = \begin{bmatrix} 10.00 - j29.77 & -5.00 + j15.00 & -5.00 + j15.00 & 0 \\ -5.00j15.00 & 9.00 - j35.91 & -4.00 + j8.00 & j12.75 \\ -5.00j15.00 & -4.00 + j8.00 & 9.00 - j22.37 & 0 \\ 0 & j12.75 & 0 & -j12.50 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Complex power injection at bus k will be given by:

$$S_k = v_k \cdot i_k^*$$

Substituting for i_k from the above nodal equation:

$$S_k = v_k \cdot \sum_{j=1}^{4} Y_{kj}^* v_j^*$$

Evaluating them for $k = 1, \ldots, 4$ yields:

$$\begin{aligned} S_1 &= 2.00 + j0.45 \\ S_2 &= -0.50 - j0.30 \\ S_3 &= -1.20 - j0.80 \\ S_4 &= -0.25 - j0.10 \end{aligned}$$

2.4 Maximum Likelihood Estimation

The objective of state estimation is to determine the most likely state of the system based on the quantities that are measured. One way to accomplish this is by maximum likelihood estimation (MLE), a method widely used in statistics. The measurement errors are assumed to have a known probability distribution with unknown parameters. The joint probability density function for all the measurements can then be written in terms of these unknown parameters. This function is referred to as the likelihood function and will attain its peak value when the unknown parameters are chosen to be closest to their actual values. Hence, an optimization problem can be set up in order to maximize the likelihood function as a function of these unknown parameters. The solution will give the maximum likelihood estimates for the parameters of interest.

The measurement errors are commonly assumed to have a Gaussian (Normal) distribution and the parameters for such a distribution are its mean, μ and its variance, σ^2. The problem of maximum likelihood estimation is then solved for these two parameters. The Gaussian probability density function (p.d.f.) and the corresponding probability distribution function (d.f.) will be reviewed below briefly before describing the maximum likelihood estimation method.

2.4.1 Gaussian (Normal) probability density function

The Normal probability density function for a random variable z is defined as:

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\{\frac{z-\mu}{\sigma}\}^2}$$

where z : random variable
μ : mean (or expected value) of $z = E(z)$
σ : standard deviation of z

The function $f(z)$ will change its shape depending on the parameters μ and σ. However, its shape can be standardized by using the following change of variables:

$$u = \frac{z - \mu}{\sigma}$$

which yields:

$$\begin{aligned} E(u) &= \frac{1}{\sigma}(E(z) - \mu) = 0 \\ Var(u) &= \frac{1}{\sigma^2}Var(z - \mu) = \frac{\sigma^2}{\sigma^2} = 1.0 \end{aligned}$$

Hence, the new function becomes:

$$\Phi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$$

A plot of $\Phi(u)$, which is referred to as the Standard Normal (Gaussian) Probability Density Function, is shown in Figure 2.5.

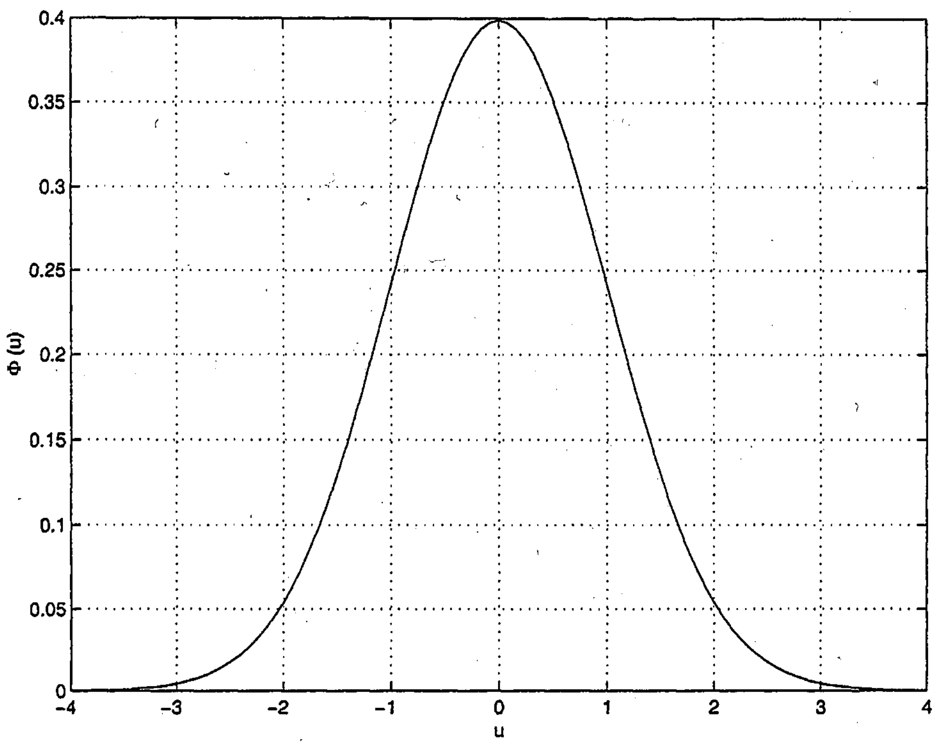

Figure 2.5. Standard Gaussian (Normal) Probability Density Function, $\Phi(u)$

2.4.2 The likelihood function

Consider the joint probability density function which represents the probability of measuring m independent measurements, each having the same Gaussian p.d.f. The joint p.d.f can simply be expressed as the product of individual p.d.f's if each measurement is assumed to be independent of the rest:

$$f_m(z) = f(z_1)f(z_2)\cdots f(z_m)$$

$$\text{where} \quad \begin{aligned} z_i &: i\text{th measurement} \\ z^T &: [z_1, z_2, \cdots, z_m] \end{aligned}$$

The function $f_m(z)$ is called the likelihood function for z. Essentially it is a measure of the probability of observing the particular set of measurements in the vector z.

The objective of maximum likelihood estimation is to maximize this likelihood function by varying the assumed parameters of the density function, namely its mean μ and its standard deviation σ. In determining the optimum parameter values, the function is commonly replaced by its logarithm, in order to simplify the optimization procedure. The modified function is called the Log-Likelihood Function, $\mathcal{L}$ and is given by:

$$\begin{aligned} \mathcal{L} = \log f_m(z) &= \sum_{i=1}^{m} \log f(z_i) \\ &= -\frac{1}{2}\sum_{i=1}^{m}\left(\frac{z_i - \mu_i}{\sigma_i}\right)^2 - \frac{m}{2}\log 2\pi - \sum_{i=1}^{m}\log \sigma_i \end{aligned}$$

MLE will maximize the likelihood (or log-likelihood) function for a given set of observations $z_1, z_2, \ldots, z_m$. Hence, it can be obtained by solving the following problem:

$$\begin{array}{ll} \text{maximize} & \log f_m(z) \\ \text{OR} & \\ \text{minimize} & \displaystyle\sum_{i=1}^{m}\left(\frac{z_i - \mu_i}{\sigma_i}\right)^2 \end{array} \tag{2.6}$$

This minimization problem can be re-written in terms of the *residual* r_i of measurement i, which is defined as:

$$r_i = z_i - \mu_i = z_i - E(z_i)$$

where the mean μ_i, or the expected value $E(z_i)$ of the measurement z_i can be expressed as $h_i(x)$, a nonlinear function relating the system state vector x to the ith measurement. Square of each residual r_i^2 is weighted by W_{ii}

$= \sigma_i^{-2}$, which is inversely related to the assumed error variance for that measurement. Hence, the minimization problem of Equation (2.6) will be equivalent to minimizing the weighted sum of squares of the residuals or solving the following optimization problem for the state vector x:

$$\text{minimize} \quad \sum_{i=1}^{m} W_{ii} r_i^2 \tag{2.7}$$

$$\text{subject to} \quad z_i = h_i(x) + r_i, \quad i = 1, \ldots, m. \tag{2.8}$$

The solution of the above optimization problem is called the *weighted least squares* (WLS) estimator for x. A review of the measurement model and the associated assumptions will be given next, before discussing the numerical solution methods.

2.5 Measurement Model and Assumptions

Consider the set of measurements given by the vector z:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} = \begin{bmatrix} h_1(x_1, x_2, \ldots, x_n) \\ h_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ h_m(x_1, x_2, \ldots, x_n) \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} = h(x) + e \tag{2.9}$$

where:
$h^T = [h_1(x),\ h_2(x),\ \ldots,\ h_m(x)]$
$h_i(x)$ is the nonlinear function relating measurement i to the state vector x
$x^T = [x_1,\ x_2,\ \ldots,\ x_n]$ is the system state vector
$e^T = [e_1,\ e_2,\ \ldots,\ e_m]$ is the vector of measurement errors.

The following assumptions are commonly made, regarding the statistical properties of the measurement errors:

- $\mathrm{E}(e_i) = 0, \quad i = 1, \ldots, m.$
- Measurement errors are independent, i.e. $E[e_i e_j] = 0$.
  Hence, $Cov(e) = E[e \cdot e^T] = R = \text{diag}\ \{\ \sigma_1^2,\ \sigma_2^2,\ \cdots,\ \sigma_m^2\ \}$.

The standard deviation σ_i of each measurement i is calculated to reflect the expected accuracy of the corresponding meter used.

The WLS estimator will minimize the following objective function:

$$\begin{aligned} J(x) &= \sum_{i=1}^{m} (z_i - h_i(x))^2 / R_{ii} \\ &= [z - h(x)]^T R^{-1} [z - h(x)] \end{aligned} \tag{2.10}$$

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At the minimum, the first-order optimality conditions will have to be satisfied. These can be expressed in compact form as follows:

$$g(x) = \frac{\partial J(x)}{\partial x} = -H^T(x)R^{-1}[z - h(x)] = 0 \tag{2.11}$$

$$\text{where} \quad H(x) = \left[\frac{\partial h(x)}{\partial x}\right]$$

Expanding the non-linear function $g(x)$ into its Taylor series around the state vector x^k yields:

$$g(x) = g(x^k) + G(x^k)(x - x^k) + \cdots = 0$$

Neglecting the higher order terms leads to an iterative solution scheme known as the Gauss-Newton method as shown below:

$$x^{k+1} = x^k - [G(x^k)]^{-1} \cdot g(x^k)$$

where k is the iteration index,

x^k is the solution vector at iteration k,

$$G(x^k) = \frac{\partial g(x^k)}{\partial x} = H^T(x^k) \cdot R^{-1} \cdot H(x^k)$$

$$g(x^k) = -H^T(x^k) \cdot R^{-1} \cdot (z - h(x^k)).$$

$G(x)$ is called the *gain matrix*. It is sparse, positive definite and symmetric provided that the system is fully observable. The issue of observability will be discussed in detail in Chapter 4. The matrix $G(x)$ is typically not inverted (the inverse will in general be a full matrix, whereas $G(x)$ itself is quite sparse), but instead it is decomposed into its triangular factors and the following sparse linear set of equations are solved using forward/back substitutions at each iteration k:

$$[G(x^k)]\Delta x^{k+1} = H^T(x^k)R^{-1}[z - h(x^k)] \tag{2.12}$$

where $\Delta x^{k+1} = x^{k+1} - x^k$. The set of equations given by Equation (2.12) is also referred to as the Normal Equations.

Example 2.2:

Consider the 3-bus power system shown in Figure 2.6. The network data are presented in the table below:

Line		Resistance	Reactance	Total Susceptance
From Bus	To Bus	R (pu)	X (pu)	$2b_s$ (pu)
1	2	0.01	0.03	0.0
1	3	0.02	0.05	0.0
2	3	0.03	0.08	0.0

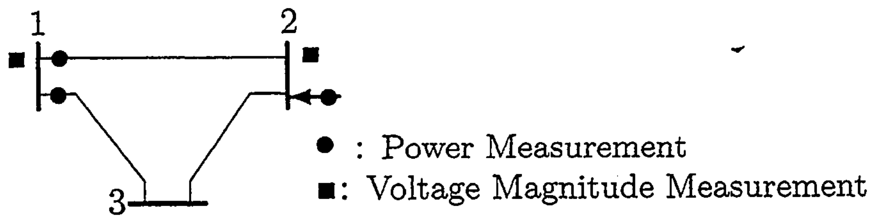

Figure 2.6. One-line diagram and measurement configuration of a 3-bus power system

The system is monitored by 8 measurements, hence $m = 8$ in Equation (2.9). Measurement values and their associated error standard deviations $\sqrt{R_{ii}} = \sigma_i$, are given as:

Measurement, i	Type	Value (pu)	$\sqrt{R_{ii}}$ (pu)
1	p_{12}	0.888	0.008
2	p_{13}	1.173	0.008
3	p_2	-0.501	0.010
4	q_{12}	0.568	0.008
5	q_{13}	0.663	0.008
6	q_2	-0.286	0.010
7	V_1	1.006	0.004
8	V_2	0.968	0.004

The state vector x will have 5 elements in this case ($n = 5$),

$$x^T = [\theta_2, \theta_3, V_1, V_2, V_3]$$

$\theta_1 = 0$ is chosen as the arbitrary reference angle.

2.6 WLS State Estimation Algorithm

WLS State Estimation involves the iterative solution of the Normal equations given by Equation (2.12). An initial guess has to be made for the state vector x^0. As in the case of the power flow solution, this guess typically corresponds to the flat voltage profile, where all bus voltages are assumed to be 1.0 per unit and in phase with each other.

The iterative solution algorithm for WLS state estimation problem can be outlined as follows:

1. Start iterations, set the iteration index $k = 0$.

2. Initialize the state vector x^k, typically as a flat start.

3. Calculate the gain matrix, $G(x^k)$.
4. Calculate the right hand side $t^k = H(x^k)^T R^{-1}(z - h(x^k))$.
5. Decompose $G(x^k)$ and solve for Δx^k.
6. Test for convergence, $\max \mid \Delta x^k \mid \leq \epsilon$?
7. If no, update $x^{k+1} = x^k + \Delta x^k$, $k = k + 1$, and go to step 3. Else, stop.

The above algorithm essentially involves the following computations in each iteration, k:

1. Calculation of the right hand side of Equation (2.12).
   (a) Calculating the measurement function, $h(x^k)$.
   (b) Building the measurement Jacobian, $H(x^k)$.
2. Calculation of $G(x^k)$ and solution of Equation (2.12).
   (a) Building the gain matrix, $G(x^k)$.
   (b) Decomposing $G(x^k)$ into its Cholesky factors.
   (c) Performing the forward/back substitutions to solve for Δx^{k+1}.

2.6.1 The Measurement Function, $h(x^k)$

Measurements can be of a variety of types. Most commonly used measurements are the line power flows, bus power injections, bus voltage magnitudes and line current flow magnitudes. These measurements can be expressed in terms of the state variables either using the rectangular or the polar coordinates. When using the polar coordinates for a system containing N buses, the state vector will have $(2N - 1)$ elements, N bus voltage magnitudes and $(N - 1)$ phase angles, where the phase angle of one reference bus is set equal to an arbitrary value, such as 0. The state vector x will have the following form assuming bus 1 is chosen as the reference:

$$x^T = [\theta_2 \theta_3 \dots \theta_N V_1 V_2 \dots V_N]$$

The expressions for each of the above types of measurements are given below, assuming the general two-port π-model for the network branches, shown in Figure 2.7:

- Real and reactive power injection at bus i:

$$\begin{aligned} P_i &= V_i \sum_{j \in \aleph_i} V_j (G_{ij} \cos \theta_{ij} + B_{ij} \sin \theta_{ij}) \\ Q_i &= V_i \sum_{j \in \aleph_i} V_j (G_{ij} \sin \theta_{ij} - B_{ij} \cos \theta_{ij}) \end{aligned}$$

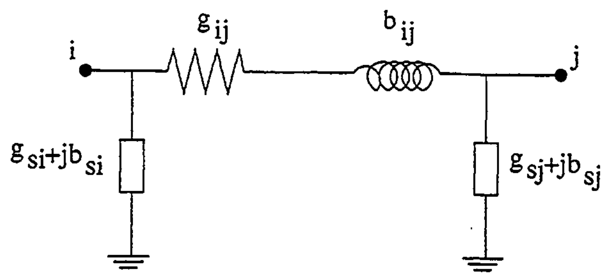

Figure 2.7. Two-port π-model of a network branch

- Real and reactive power flow from bus i to bus j:

$$\begin{aligned} P_{ij} &= V_i^2(g_{si}+g_{ij}) - V_iV_j(g_{ij}\cos\theta_{ij} + b_{ij}\sin\theta_{ij}) \\ Q_{ij} &= -V_i^2(b_{si}+b_{ij}) - V_iV_j(g_{ij}\sin\theta_{ij} - b_{ij}\cos\theta_{ij}) \end{aligned}$$

- Line current flow magnitude from bus i to bus j:

$$I_{ij} = \frac{\sqrt{P_{ij}^2 + Q_{ij}^2}}{V_i}$$

or ignoring the shunt admittance $(g_{si} + jb_{si})$:

$$I_{ij} = \sqrt{(g_{ij}^2 + b_{ij}^2)(V_i^2 + V_j^2 - 2V_iV_j\cos\theta_{ij})}$$

where
V_i, θ_i is the voltage magnitude and phase angle at bus i.
$\theta_{ij} = \theta_i - \theta_j$.
$G_{ij} + jB_{ij}$ is the ijth element of the complex bus admittance matrix.
$g_{ij} + jb_{ij}$ is the admittance of the series branch connecting buses i and j.
$g_{si} + jb_{si}$ is the admittance of the shunt branch connected at bus i as shown in Figure 2.7.
$\aleph_i$ is the set of bus numbers that are directly connected to bus i.

2.6.2 The Measurement Jacobian, H

The structure of the measurement Jacobian H will be as follows:

$$H = \begin{bmatrix} \frac{\partial P_{inj}}{\partial \theta} & \frac{\partial P_{inj}}{\partial V} \\ \frac{\partial P_{flow}}{\partial \theta} & \frac{\partial P_{flow}}{\partial V} \\ \frac{\partial Q_{inj}}{\partial \theta} & \frac{\partial Q_{inj}}{\partial V} \\ \frac{\partial Q_{flow}}{\partial \theta} & \frac{\partial Q_{flow}}{\partial V} \\ \frac{\partial I_{mag}}{\partial \theta} & \frac{\partial I_{mag}}{\partial V} \\ 0 & \frac{\partial V_{mag}}{\partial V} \end{bmatrix}$$

The expressions for each partition are given below:

- Elements corresponding to real power injection measurements:

$$\begin{aligned} \frac{\partial P_i}{\partial \theta_i} &= \sum_{j=1}^{N} V_i V_j (-G_{ij} \sin\theta_{ij} + B_{ij} \cos\theta_{ij}) - V_i^2 B_{ii} \\ \frac{\partial P_i}{\partial \theta_j} &= V_i V_j (G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij}) \\ \frac{\partial P_i}{\partial V_i} &= \sum_{j=1}^{N} V_j (G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij}) + V_i G_{ii} \\ \frac{\partial P_i}{\partial V_j} &= V_i (G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij}) \end{aligned}$$

- Elements corresponding to reactive power injection measurements:

$$\begin{aligned} \frac{\partial Q_i}{\partial \theta_i} &= \sum_{j=1}^{N} V_i V_j (G_{ij} \cos\theta_{ij} + B_{ij} \sin\theta_{ij}) - V_i^2 G_{ii} \\ \frac{\partial Q_i}{\partial \theta_j} &= V_i V_j (-G_{ij} \cos\theta_{ij} - B_{ij} \sin\theta_{ij}) \\ \frac{\partial Q_i}{\partial V_i} &= \sum_{j=1}^{N} V_j (G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij}) - V_i B_{ii} \\ \frac{\partial Q_i}{\partial V_j} &= V_i (G_{ij} \sin\theta_{ij} - B_{ij} \cos\theta_{ij}) \end{aligned}$$

- Elements corresponding to real power flow measurements:

$$\begin{aligned}
\frac{\partial P_{ij}}{\partial \theta_i} &= V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \\
\frac{\partial P_{ij}}{\partial \theta_j} &= -V_i V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) \\
\frac{\partial P_{ij}}{\partial V_i} &= -V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) + 2\,(g_{ij} + g_{si}) V_i \\
\frac{\partial P_{ij}}{\partial V_j} &= -V_i (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij})
\end{aligned}$$

- Elements corresponding to reactive power flow measurements:

$$\begin{aligned}
\frac{\partial Q_{ij}}{\partial \theta_i} &= -V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \\
\frac{\partial Q_{ij}}{\partial \theta_j} &= V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij}) \\
\frac{\partial Q_{ij}}{\partial V_i} &= -V_j (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij}) - 2\,V_i (b_{ij} + b_{si}) \\
\frac{\partial Q_{ij}}{\partial V_j} &= -V_i (g_{ij} \sin \theta_{ij} - b_{ij} \cos \theta_{ij})
\end{aligned}$$

- Elements corresponding to voltage magnitude measurements:

$$\frac{\partial V_i}{\partial V_i} = 1, \frac{\partial V_i}{\partial V_j} = 0, \frac{\partial V_i}{\partial \theta_i} = 0, \frac{\partial V_i}{\partial \theta_j} = 0$$

- Elements corresponding to current magnitude measurements (ignoring the shunt admittance of the branch):

$$\begin{aligned}
\frac{\partial I_{ij}}{\partial \theta_i} &= \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin \theta_{ij} \\
\frac{\partial I_{ij}}{\partial \theta_j} &= -\frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} V_i V_j \sin \theta_{ij} \\
\frac{\partial I_{ij}}{\partial V_i} &= \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_i - V_j \cos \theta_{ij}) \\
\frac{\partial I_{ij}}{\partial V_j} &= \frac{g_{ij}^2 + b_{ij}^2}{I_{ij}} (V_j - V_i \cos \theta_{ij})
\end{aligned}$$

Example 2.3:

Consider the same system and measurement configuration shown in example 2.2. Assume flat start conditions, where the state vector is equal to:

$$x^0 = \begin{array}{c} \theta_2 \\ \theta_3 \\ V_1 \\ V_2 \\ V_3 \end{array} \begin{bmatrix} 0 \\ 0 \\ 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

Then, the measurement Jacobian can be evaluated as follows, using the expressions given above:

$$H(x^0) = \begin{array}{cc} & \begin{array}{ccccc} \partial\theta_2 & \partial\theta_3 & \partial V_1 & \partial V_2 & \partial V_3 \end{array} \\ \begin{array}{c} \partial p_{12} \\ \partial p_{13} \\ \partial p_2 \\ \partial q_{12} \\ \partial q_{13} \\ \partial q_2 \\ \partial V_1 \\ \partial V_2 \end{array} & \left[\begin{array}{cc|ccc} -30.0 & & 10.0 & -10.0 & \\ & -17.2 & 6.9 & & -6.9 \\ 40.9 & -10.9 & -10.0 & 14.1 & -4.1 \\ \hline 10.0 & & 30.0 & -30.0 & \\ & 6.9 & 17.2 & & -17.2 \\ -14.1 & 4.1 & -30.0 & 40.9 & -10.9 \\ & & 1.0 & & \\ & & & 1.0 & \end{array}\right] \end{array}$$

Note that the dimension of H is $m \times n = 8 \times 5$, and it is a sparse matrix. Its sparsity becomes more pronounced for large scale systems, where the number of nonzeros per row stays fairly constant, irrespective of the system size.

2.6.3 The Gain Matrix, G

Gain matrix is formed using the measurement Jacobian H and the measurement error covariance matrix, R. The covariance matrix is assumed to be diagonal having measurement variances as its diagonal entries. Since G is formed as:

$$G(x^k) = H^T R^{-1} H$$

it has the following properties:

1. It is structurally and numerically symmetric.
2. It is sparse, yet less sparse compared to H.
3. In general it is a non-negative definite matrix, i.e. all of its eigenvalues are non-negative. It is positive definite for fully observable networks.

G is built and stored as a sparse matrix for computational efficiency and memory considerations. It is built by processing one measurement at a time. Consider the measurement jacobian H and the covariance matrix for a set of m measurements, each one corresponding to one row, as shown below:

$$H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_m \end{bmatrix}, R = \begin{bmatrix} R_{11} & 0 & \cdots & 0 \\ 0 & R_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & R_{mm} \end{bmatrix}$$

Then, the gain matrix can be re-written as follows:

$$G = \sum_{i=1}^{m} H_i^T R_{ii}^{-1} H_i$$

Since H_i arrays are very sparse row vectors, their product will also yield a sparse matrix. Nonzero terms in G can thus be calculated and stored in sparse form.

Example 2.4:

Using the measurement jacobian $H(x^0)$ evaluated in example 2.3, the gain matrix $G(x^0)$ will be obtained as follows:

$$G(x^0) = 10^7 \left[\begin{array}{rr|rrr} 3.4392 & -0.5068 & 0.0137 & & -0.0137 \\ -0.5068 & 0.6758 & -0.0137 & 0.0137 & 0.0000 \\ \hline 0.0137 & -0.0137 & 3.1075 & -2.9324 & -0.1689 \\ & 0.0137 & -2.9324 & 3.4455 & -0.5068 \\ -0.0137 & 0.0000 & -0.1689 & -0.5068 & 0.6758 \end{array} \right]$$

Gain matrix is 5×5, symmetric and less sparse than the corresponding measurement jacobian $H(x^0)$. Its eigenvalues can be computed as:

$$Eigen(G) = 10^7 \begin{bmatrix} 3.5293 \\ 6.2254 \\ 0.5857 \\ 0.9992 \\ 0.0042 \end{bmatrix}$$

confirming that it is positive definite.