(1) Д-ть проектор P явл ортогон-м $\Leftrightarrow$ $P = P^T$

Д-во:

1) Пусть $P = P^T$.

Тогда

$\forall x, y$ $\quad (Px, (I-P)y) = x^T P^T (I-P) y =$

$= x^T (P^T - P^T P) y = x^T (P - P^2) y = 0$

$\Rightarrow$ range $(P) \perp$ range $(I-P)$ $\Rightarrow$

$\Rightarrow$ P – ортогон проекция

2) Пусть P – орт проекция.

Тогда range $(P) \perp$ range $(I-P)$

$Px \in$ range (P)

$(I-P)y \in$ range $(I-P)$ $\quad \Rightarrow$

$\Rightarrow$ $(Px, (I-P)y) = 0$

$x^T P^T (I-P) y = 0,$

$x^T (P^T - P^T P) y = 0$

т.к. x и y – произв-е, то

$P^T - P^T P = 0,$

$P^T = P^T P$

$P^T = (PP)^T \Rightarrow P^T P^T = P^T P \Rightarrow P = P^T$ ■

Пусть P - орт-й проектор,

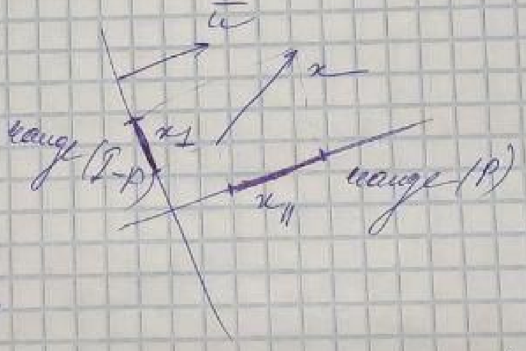
тогда $(I - 2P)(I - 2P)^T =$

$= (I - 2P)(I - 2P) = I^2 - 2P - 2P + 4P^2 =$

$= I \Rightarrow (I - 2P)$ - ~~унитарна~~ ортогон-на.

$$x = \underbrace{x_{\parallel}}_{\in \text{range}(P)} + \underbrace{x_{\perp}}_{\in \text{range}(I-P)}$$

$(I - 2P)x = x - 2Px = x - 2x_{\parallel}$



$x_{\parallel} = (u, x)\, u,$

$(I - 2P)x = x - 2u(u, x) = (x - 2uu^T x) =$

$= (I - 2uu^T)x \Rightarrow (I - 2P)$ - зеркальное отобр-е отн-но range$(I-P)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\quad\ \ a_1 \quad a_2$$

$$z_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$z_2 = \frac{a_2 - (a_2, z_1) z_1}{\| a_2 - (a_2, z_1) z_1 \|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{y} = (y, z_1) z_1 + (y, z_2) z_2 = Py$$

$$\hat{e}_1 = (e_1, z_1) z_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{e}_2 = (e_2, z_2) z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{e}_3 = (e_3, z_1) z_1 = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$P = \begin{pmatrix} Pe_1 & Pe_2 & Pe_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^2 = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & 0 & \frac{1}{4}+\frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{4}+\frac{1}{4} & 0 & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = P$$

$P = P^T \Rightarrow P$ ортогон. проекция

$$B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ b_1 & b_2 \end{pmatrix} \qquad f_1 = \frac{b_1}{\|b_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$f_2 = \frac{b_2 - (b_2, f_1) f_1}{\| b_2 - (b_2, f_1) f_1 \|} = \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \cdot \frac{1}{\| \ldots \|} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{\sqrt{3}}$$

$$Px = (x, f_1) f_1 + (x, f_2) f_2$$

$$Pe_1 = (e_1, f_1) f_1 + (e_1, f_2) f_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

$$Pe_2 = (e_2, f_2) f_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$Pe_3 = (e_3, f_1) f_1 + (e_3, f_2) f_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{5}{6} \end{pmatrix} \qquad P = \begin{pmatrix} Pe_1 & Pe_2 & Pe_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

$$P^2 = \begin{pmatrix} \frac{25}{36} + \frac{1}{9} + \frac{1}{36} & \frac{5}{18} + \frac{1}{9} - \frac{1}{18} & \frac{5}{36} - \frac{1}{9} + \frac{5}{36} \\ \frac{5}{18} + \frac{1}{9} - \frac{1}{18} & \frac{1}{9} + \frac{1}{9} + \frac{1}{9} & \frac{1}{18} - \frac{1}{9} - \frac{5}{18} \\ \frac{5}{36} - \frac{1}{9} + \frac{5}{36} & \frac{1}{18} - \frac{1}{9} - \frac{5}{18} & \frac{1}{36} + \frac{1}{9} + \frac{25}{36} \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix} = P = P^T$$

① $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$$A = QR \Rightarrow R = Q^T A =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

② $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\qquad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$

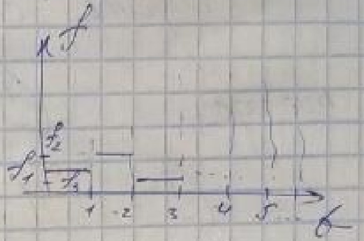
$$B = QR \Rightarrow R = Q^T B =$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

$m = 1, \; t_0 = 0, \; x = 0, \; v = 0$

$$f = \begin{cases} f_1, & 0 < t \le 1 \\ f_2, & 1 < t \le 2 \\ \vdots \\ f_{10}, & 9 < t \le 10 \end{cases}$$



$$a = \begin{pmatrix} x(t=10) \\ v(t=10) \end{pmatrix}$$

$$a = Af$$

$$f = \int_{t_0}^{t} v \, d\tau$$

$$v = \begin{cases} \int_0^t f_1 \, d\tau, & 0 < t \le 1 \\ f_1 + \int_1^t f_2 \, d\tau, & 1 < t \le 2 \\ f_1 + f_2 + \int_2^t f_3 \, d\tau, & 2 < t \le 3 \\ \vdots \\ \sum_{k=1}^{9} f_k + \int_9^t f_{10} \, d\tau, & 9 < t \le 10 \end{cases}$$

$$v = \begin{cases} f_1 t, & 0 < t \le 1 \\ f_1 + f_2 (t-1), & 1 < t \le 2 \\ \vdots \\ \sum_{k=1}^{9} f_k + f_{10} (t-9), & 9 < t \le 10 \end{cases}$$

$\frac{t^2}{2} - 9t \Big|_9^{10}$

$50 - 90 - \frac{81}{2} + 81$

$$x = \int_{t_0}^{t} v(\tau) \, d\tau ,$$

$$x(t=10) = \left(\int_0^1 + \int_1^2 + \ldots + \int_9^{10} \right) v \, dt =$$

$$= f_1 \frac{t^2}{2} \Big|_0^1 + f_1 t \Big|_1^2 + f_2 \left(\frac{t^2}{2} - t \right) \Big|_1^2 + \ldots =$$

$$= \sum_{k=1}^{9} f_k (10-k) + \sum_{k=1}^{10} f_k \cdot \frac{1}{2}$$

$$a = \begin{pmatrix} \sum_{k=1}^{9} f_k (10-k) + \sum_{k=1}^{10} f_k \frac{1}{2} \\ \sum_{k=1}^{10} f_k \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{10} f_k \left((10-k) + \frac{1}{2} \right) \\ \sum_{k=1}^{10} f_k \end{pmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{10} \end{pmatrix}$$

$$a = Af \Rightarrow A = \begin{pmatrix} \frac{19}{2} & \frac{17}{2} & \frac{15}{2} & \dots & \frac{1}{2} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

$A = U \Sigma V^T \Rightarrow$

$A' = V \Sigma^{-1} U^T$ – псевдообр-я,

$A'a = f$