①

Д-ть

$\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$, в x длины m

$\|A\|_\infty \le \sqrt{n}\,\|A\|_2$, где A - $m \times n$

Д-во

1) $\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}$, $\|x\|_\infty = \max_i |x_i|$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2} \le \sqrt{m\cdot(\max_i |x_i|)^2} = \sqrt{m}\,\max_i |x_i| = \|x\|_\infty$$

Пример насыщения: $k \le \sqrt{3}\cdot l$, $k = \sqrt{3}$, $l = 1$

$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $m = 3$,

$\|x\|_2 = 1$, $\|x\|_\infty = 1 \Rightarrow \sqrt{3} = \sqrt{3}\cdot 1$

2) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sup_{\|x\|_2 = 1} \|Ax\|_2$$

$$\|A\|_\infty = \sup_{\|x\|\ne 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \sup_{\|x\|\ne 0} \frac{\sqrt{n}\,\|Ax\|_\infty}{\|x\|_2} \le$$

$$\le \sup_{\|x\|\ne 0} \frac{\sqrt{n}\,\|Ax\|_2}{\|x\|_2} = \sqrt{2}\,\|A\|_2$$

②

a) $X = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$

$X^T X = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

$\lambda_1 = 9$

$\lambda_2 = 4$

$\Sigma^2 = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$

$X^T X = V \Sigma^2 V^T$

$\lambda_1 : \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \end{pmatrix} \Rightarrow \begin{cases} 9x = 9x \\ 4y = 9y \end{cases} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$X v_1 = \sigma_1 u_1 \Rightarrow u_1 = X v_1 \frac{1}{\sigma_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_2 = X v_2 \frac{1}{\sigma_2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

в)

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^T A = V \Sigma^2 V^T$$

$$A^T A = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

$\lambda_1 = 4$

$\lambda_2 = 0$

$$\Sigma'^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^T A v_1 = \lambda_1 v_1 \Rightarrow$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix} \Rightarrow 4y = 4y$$

$y = 1,$

$x = 0$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A A^T = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\lambda_1 = 4$

$\lambda_2 = 0$

$\lambda_3 = 0$

$$\Sigma^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A A^T u_1 = \lambda_1 u_1$$

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix} \Rightarrow$$

$x = 1$

$y = 0$

$z = 0$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c) $$B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$B = U \Sigma V^T$$

$$B^T B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \Rightarrow \lambda_1 = 4, \quad \lambda_2 = 0$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix} \Rightarrow x = y, \quad v_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$B v_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} B v_1 = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} \frac{1}{2} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \Rightarrow u_2 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$B = U \Sigma V^T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

(4) $f(I) = X(X^T X)^{-1} =$

$= U\Sigma V^T (U\Sigma^T U^T U \Sigma V^T)^{-1} =$

$= U\Sigma V^T (V\Sigma^T \Sigma V^T)^{-1} =$

$= U\Sigma V^T V \Sigma^{-1} (\Sigma^T)^{-1} V^T = U(\Sigma^T)^{-1} V^T$

$f(XX^T + I) = (XX^T + I)^{-1} X (X^T (XX^T + I)^{-1} X)^{-1} =$

$= (U\Sigma^2 U^T + I)^{-1} U\Sigma V^T (V\Sigma^T U^T (U\Sigma^2 U^T + I)^{-1} \cdot$

$\cdot\ U\Sigma V^T)^{-1} = (U\Sigma^2 U^T + I)^{-1} U\Sigma V^T V \Sigma^{-1} U^T \cdot$

$\cdot\ (U\Sigma^2 U^T + I)\, U(\Sigma^T)^{-1} V^T = U(\Sigma^T)^{-1} V^T$

$f(I) = f(XX^T + I)$ ■