$$\dot u = f(t,u)$$

① $$\frac{u_n - u_{n-1}}{\tau} = \frac{3}{2} f_{n-1} - \frac{1}{2} f_{n-2}$$

$$u_{n-1} = u(t_n - \tau) = u_n - \tau \dot u_n + \frac{\tau^2}{2}\ddot u_n + O(\tau^2)$$

$$u_{n-2} = u(t_n - 2\tau) = u_n - 2\tau \dot u_n + \frac{\cancel{4}^{2}\tau^2}{\cancel{2}_{1}}\ddot u_n + O(\tau^2)$$

$$f_{n-1} = f(t_{n-1}, u_{n-1}) = f(t_n - \tau,\ u_n - \tau\dot u_n + \frac{\tau^2}{2}\ddot u_n + O(\tau^2)) =$$
$$= f_n + \frac{\partial f}{\partial t}\Big|_{t_n,u_n}(-\tau) + \frac{\partial f}{\partial u}\Big|_{(t_n,u_n)}(-\tau\dot u_n + \frac{\tau^2}{2}\ddot u_n + O(\tau^2)) + O(\tau^2 + \dots)$$

$$f_{n-2} = f(t_{n-2}, u_{n-2}) = f(t_n - 2\tau,\ u_n - 2\tau\dot u_n + 2\tau^2\ddot u_n + O(\tau^2)) = f_n + (-2\tau)\frac{\partial f}{\partial t}\Big|_{(t_n,u_n)} + \frac{\partial f}{\partial u}\Big|_{(t_n,u_n)}(-2\tau\dot u_n + 2\tau^2\ddot u_n + O(\tau^2)) + O(\tau^2 + \dots)$$

$$\frac{\cancel{u_n} - \cancel{u_n} + \tau\dot u_n - \frac{\tau^2}{2}\ddot u_n + O(\tau^2)}{\tau} = \frac{3}{2}\left(f_n + \frac{\partial f}{\partial t}_n(-\tau) + (-\tau\dot u_n + \frac{\tau^2}{2}\ddot u_n + O(\tau^2))\frac{\partial f}{\partial u}_n\right) -$$
$$- \frac{1}{2}\left(f_n - 2\tau\frac{\partial f}{\partial t}_n + \frac{\partial f}{\partial u}_n(-2\tau\dot u_n + 2\tau^2\ddot u_n + O(\tau^2))\right) + O(\tau^2 + \dots)$$

$$\dot u_n - \cancel{\frac{\tau}{2}\ddot u_n} + O(\tau) = f_n - \cancel{\frac{1}{2}\tau\frac{\partial f}{\partial t}_n} - \cancel{\frac{1}{2}\tau\dot u_n\frac{\partial f}{\partial u}_n} - \frac{1}{\cancel{8}\,4}\tau^2\ddot u_n\frac{\partial f}{\partial u}_n + O(\tau^2)$$

Т.к.

$$\ddot u = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t},\ \text{то}$$

$$\dot u_n = f_n - \frac{\tau^2}{4}\ddot u_n\frac{\partial f}{\partial u}_n + O(\tau^2)$$

$\Rightarrow$ Порядок аппроксимации квадратичный.

(2) $\frac{3}{2}u_n - 2u_{n-1} + \frac{1}{2}u_{n-2} = \tau f_n$

$u_{n-1} = u_n - \tau \dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + O(\tau^2)$

$u_{n-2} = u_n - 2\tau \dot{u}_n + 2\tau^2 \ddot{u}_n + O(\tau^2)$

$\cancel{\frac{3}{2}u_n} - 2(\cancel{u_n} - \tau\dot{u}_n + \cancel{\frac{\tau^2}{2}\ddot{u}_n} + O(\tau^2)) + \frac{1}{2}(\cancel{u_n} - 2\tau\dot{u}_n + \cancel{2\tau^2\ddot{u}_n} + O(\tau^2)) = \tau f_n$

$\tau\dot{u}_n + O(\tau^2) = \tau f_n \quad | : \tau$

$\dot{u}_n + O(\tau) = f_n$

=> Порядок линейный

(3) $$\frac{u_n + 4u_{n-1} - 5u_{n-2}}{6\tau} = \frac{2}{3}f_{n-1} + \frac{1}{3}f_{n-2}$$

$u_{n-1} = u_n - \tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + O(\tau^2)$

$u_{n-2} = u_n - 2\tau\dot{u}_n + 2\tau^2\ddot{u}_n + O(\tau^2)$

$f_{n-1} = f_n - \tau\frac{\partial f}{\partial t}_n + (-\tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + O(\tau^2))\frac{\partial f}{\partial u}_n + O(\tau^2 + \ldots)$

$f_{n-2} = f_n - 2\tau\frac{\partial f}{\partial t}_n + (-2\tau\dot{u}_n + 2\tau^2\ddot{u}_n + O(\tau^2))\frac{\partial f}{\partial u}_n + O(\tau^2 + \ldots)$

$$\frac{\cancel{u_n} + 4(\cancel{u_n} - \tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + O(\tau^2)) - 5(\cancel{u_n} - 2\tau\dot{u}_n + 2\tau^2\ddot{u}_n + O(\tau^2))}{6\tau} =$$

$$= \frac{2}{3}(f_n - \tau\frac{\partial f}{\partial t}_n + (-\tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + O(\tau^2))\frac{\partial f}{\partial u}_n + \frac{1}{3}(f_n - 2\tau\frac{\partial f}{\partial t}_n + (-2\tau\dot{u}_n + 2\tau^2\ddot{u}_n + O(\tau^2))\frac{\partial f}{\partial u}_n)$$

$\dot{u}_n - \cancel{\frac{4}{3}\tau\ddot{u}_n} + O(\tau) = f_n - \cancel{\frac{4}{3}\tau\frac{\partial f}{\partial t}_n} - \cancel{\frac{4}{3}\tau\dot{u}_n\frac{\partial f}{\partial u}_n} + \tau^2\ddot{u}_n\frac{\partial f}{\partial u}_n + O(\tau^2)$

$\dot{u}_n - \tau^2\ddot{u}_n\frac{\partial f}{\partial u}_n + O(\tau^2) = f_n$,

=> Порядок аппроксимации квадрат.

№2.

1

$$\frac{u_n - u_{n-1}}{\tau} = 0 \quad u_n = q^n$$

$q^2 - q = 0$, $\left[\begin{array}{l} q = 1 \\ q = 0 \end{array}\right.$ $\Rightarrow$ схема явл. нуль устойчивой.

2

$\frac{3}{2} u_n - 2u_{n-1} + \frac{1}{2} u_{n-2} = 0$, $u_n = q^n$

$$\frac{3}{2} q^n - 2q^{n-1} + \frac{1}{2} q^{n-2} = 0$$

$$\frac{3}{2} q^2 - 2q + \frac{1}{2} = 0.$$

$$3q^2 - 4q + 1 = 0$$

$$3q(q-1) - (q-1) = 0$$

$(q-1)(q-\frac{1}{3}) = 0 \Rightarrow \left[\begin{array}{l} q = 1 \\ q = \frac{1}{3} \end{array}\right.$ $|q| \le 1 \Rightarrow$ схема явл. нуль устойчивой

3

$\frac{u_n + 4u_{n-1} - 5u_{n-2}}{6\tau} = 0$, $u_n = q^n$

$$q^2 + 4q - 5 = 0$$

$\left[\begin{array}{l} q_1 = 1 \\ q_2 = -5 \end{array}\right.$ – $|q_2| > 1 \Rightarrow$ схема не явл-ся нуль устойчивой.

a) $u_n - u_{n-1} = \frac{3}{2}\tau\lambda u_{n-1} - \frac{1}{2}\lambda u_{n-2}\tau$

$$q^2 - q = \frac{3}{2}\lambda\tau q - \frac{1}{2}\lambda\tau$$

$$q^2 - q\left(1+\frac{3}{2}\lambda\tau\right) + \frac{1}{2}\lambda\tau = 0$$

Пусть $\cancel{\lambda\tau = z}$ $z = \lambda\tau$

$$q^2 - q(1+z) + \frac{1}{2}z = 0$$

$$D = (1+z)^2 - 2z$$

$$\sqrt{D} = \sqrt{(1+z)^2 - 2z}$$

$$q_1 = \frac{1+z-\sqrt{(1+z)^2-2z}}{2} \qquad |q_1| < 1$$

$$q_2 = \frac{1+z+\sqrt{(1+z)^2-2z}}{2} \qquad |q_2| < 1$$

$$\begin{cases} \left|1+z-\sqrt{(1+z)^2-2z}\right| < 2 \\ \left|1+z+\sqrt{(1+z)^2-2z}\right| < 2 \end{cases}$$

если взять $z = -3 + i\cdot 0$

$$|-1-3-\sqrt{10}| > 2$$

$\Rightarrow$ есть z: $\operatorname{Re} z < 0$ и не удовл. нер-ву

$\Rightarrow$ нет асимпт. уст-ти.

8) $\frac{3}{2}u_n - 2u_{n-1} + \frac{1}{2}u_{n-2} = \lambda\tau u_n$

$$\frac{3}{2}q^2 - 2q + \frac{1}{2} = \lambda\tau q^2$$

$$q^2\left(\frac{3}{2} - \lambda\tau\right) - 2q + \frac{1}{2} = 0$$

Пусть $z = \lambda\tau$

$$q^2\left(\frac{3}{2} - z\right) - 2q + \frac{1}{2} = 0$$

$$D = 4 - 2\left(\frac{3}{2} - z\right)$$

$$\sqrt{D} = \sqrt{4 - 3 + 2z} = \sqrt{1 + 2z}$$

$$q_{1,2} = \frac{2 \pm \sqrt{1+2z}}{2\left(\frac{3}{2} - z\right)}$$

$$\begin{cases} \left|\dfrac{2 + \sqrt{1+2z}}{3 - 2z}\right| < 1 \\ \left|\dfrac{2 - \sqrt{1+2z}}{3 - 2z}\right| < 1 \end{cases}$$

Пусть $z \in \mathbb{R}$,

~~$0 < z < 0,5$~~

$0 > z > -0,5$

$$\frac{2 + \sqrt{1+2z}}{3 - 2z} < 1$$

$z < -0,5$

$$\frac{|2 + i\sqrt{z}|}{3 - 2z} = \frac{\sqrt{2 + z^2}}{3 - 2z} < 1$$

Пусть $z \in \mathbb{C}$

$z = x + iy$

$$\left|\frac{2 + \sqrt{1 + 2x + 2yi}}{3 - 2x - 2iy}\right| = \frac{|(x' + y'i)(3 - 2x + 2yi)|}{(3-2x)^2 + 4y^2} = \frac{|x'(3-2x) - 2yy' + (2yx' + y'(3-2x))i|}{(3-2x)^2 + 4y^2} =$$

$$= \sqrt{\frac{(x'(3-2x) - 2yy')^2 + (2yx' + y'(3-2x))^2}{(3-2x)^2 + 4y^2}} < 1 \Rightarrow A\text{-устойчиво}$$

б) $q^2+4q-5 = 4\lambda q + 2\lambda z$

$q^2 + (4-4\lambda z)q - 5 + 2\lambda z = 0$

$D = 16(1-z)^2 + 20 - 8z = 16 - 32z + z^2 + 20 - 8z =$

$= z^2 - 40z + 36$

$\sqrt{D} = \sqrt{z^2-40z+36}$

$$q_{1,2} = \frac{4(1-z) \pm \sqrt{z^2-40z+36}}{2}$$

$$\begin{cases} \left|4 - z + \sqrt{z^2-40z+36}\right| < 2 \\ \left|4 - z - \sqrt{z^2-40z+36}\right| < 2 \end{cases}$$

При $z = -15$

Получим $19 - 18,4i$,

$|19 - 18,4i| > 2$

$\Rightarrow$ нет А-уст-ти

④ $\frac{du}{dt} = f(u)$, $f(u) = \frac{1}{u}$

$$u_{i+1} = u_i + \tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)$$

$$\frac{u_{i+1} - u_i}{\tau} = f_{i+1},$$

$$f_{i+1} = f(u_{i+1}) = f\left(u_i + \tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)\right) =$$

$$= f_i + \frac{df}{du}\left(\tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)\right) =$$

$$= f_i + \left(-\frac{1}{u_i^2}\right)\left(\tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)\right)$$

$$\frac{-\cancel{u_i} + \cancel{u_i} + \tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)}{\tau} = f_i - \frac{1}{u_i^2}\left(\tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)\right)$$

$$\dot{u}_i + \frac{\tau}{2}\ddot{u}_i + O(\tau) + \frac{1}{u_i^2}\left(\tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i + O(\tau^2)\right) = f_i$$

$$\dot{u}_i\left(1 + \frac{\tau}{u_i^2}\right) + \ddot{u}_i\frac{\tau}{2} + O(\tau) = f_i$$

$\Rightarrow$ порядок апроксимации линейной