Homework 5

Antonio Zea Jr

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Problems

Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. We describe the functions $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ in the following tables. Answer each part and give a reason for each negative answer.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

n	$g(n)$
1	10
2	9
3	8
4	7
5	6

[A]**a.** Is f one-to-one?
b. Is f onto?
c. Is f a correspondence?
[A]**d.** Is g one-to-one?
e. Is g onto?
f. Is g a correspondence?

0.1

A) Is f one-to-one?
No, f is not one-to-one because $f(2) = f(4)$.

A) Is f onto?
No, $\forall y \in Y \nexists \quad x \in X : f(x) = y$. For example, nothing is mapped on to 8 in Y.

A) Is f a correspondence?
No, f is a correspondence iff. it is both one-to-one and onto.

A) Is g one-to-one?
Yes, $\forall x_1, x_2 \in X : g(x_1) = g(x_2) \implies x_1 = x_2$

A) Is g onto?
Yes, $\forall y \in Y \quad \exists x \in X : f(x) = y$.

A) Is g a correspondence?
Yes, g is a correspondence iff. it is both one-to-one and onto.

0.2 Prove that the set of all infinite sequences over {0,1} is uncountable.

Suppose that a corrspondence f existed between $\mathbb{N}$ and the set of all infinite sequences over $\{0, 1\}$ referred to as A.

n	$f(n)$
1	000...
2	010...
3	100...
4	001...
$\vdots$	$\vdots$

Lets now go about constructing x, a sequence in A but not represented by the mapping described by f. Construct x such that
$x \neq f(1)$ by choosing the first digit to be 1.
$x \neq f(2)$ by choosing the second digit to be 0.
$x \neq f(3)$ by choosing the third digit to be 1.
$x \neq f(4)$ by choosing the fourth digit to be the opposite of $f(4)$'s fourth digit and so on.
$x = 101...$

By continuing this process, x will be a sequence that is not represented by f. This is a contradiction therefore the set of all infinite sequences over {0,1} is uncountable.

0.3 Use the result from the previous proof (item 2) to prove that the power set of any infinite countable set is uncountable.

Suppose that the power set of natural numbers, $2^{\mathbb{N}}$ is countable. Then $\exists f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$.

i	$f(i)$	$1 \in f(i)$	$2 \in f(i)$	$3 \in f(i)$	$4 \in f(i)$	$\cdots$
1	$\mathbb{N}$	(yes)	yes	yes	yes	$\cdots$
2	$\emptyset$	no	(no)	no	no	$\cdots$
3	the set of even numbers	no	yes	(no)	yes	$\cdots$
4	the set of odd numbers	yes	no	yes	(no)	$\cdots$
5	{1}	yes	no	no	no	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Lets now go about constructing x, a set in $2^{\mathbb{N}}$ but not represented by the mapping described by f. Construct x by changing each element on the diagonal.

	$1 \in f(i)$	$2 \in f(i)$	$3 \in f(i)$	$4 \in f(i)$	$\cdots$
x	no	yes	yes	yes	$\cdots$

By continuing this process, x will be a set that is not represented by f. This is a contradiction therefore the power set of $\mathbb{N}$ is uncountable. Lastly, by the definition of countable there is a direct mapping from any infinitely countable set to $\mathbb{N}$. Therefore the power set of any infinite countable set is uncountable.

0.4 $B = \{\langle M \rangle | M$ is a TM and M accepts the two's complement of binary number N whenever it accepts $N\}$. Prove that language B is undecidable

We assume that B is decidable. Suppose that H is a decider for B. ON input $\langle M, N \rangle$, where M is a TM and N is a string, H halts and accepts if M accepts N. Furthermore, H halts and rejects if M fails to accept N.

$$H(\langle M, N \rangle) = \begin{cases} \text{accept} & \text{if } M \text{accepts } N \\ \text{reject} & \text{if } M \text{does not accept N} \end{cases}$$

Now we construct a new Turing machine D with H as a subroutine. This new TM calls H to determine what M does when the input to M is its own description $\langle M \rangle$. Once D has determined this information, it does the opposite.

$D =$ "On input $\langle M \rangle$, where M is a TM:

1. Run Hon input $\langle M, \langle M \rangle \rangle$.
2. Output the opposite of what H outputs. That is, if H accepts, *reject*; and if H rejects, *accept*."

So we have the following:

$$D(\langle M \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \text{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

What happens when we run D with its own description $\langle D \rangle$ as input? In that case, we get

$$D(\langle D \rangle) = \begin{cases} \text{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \text{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

No matter what D does, it is forced to do the opposite, which is a contradiction. Therefore, TM $D$and TM Hcan not exits. Language B is therefore undecidable.