0.3 Let A be the set $\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$ and B be the set $\{\mathtt{x}, \mathtt{y}\}$.

- **a.** Is A a subset of B?
- **b.** Is B a subset of A?
- **c.** What is $A \cup B$?
- **d.** What is $A \cap B$?
- **e.** What is $A \times B$?
- **f.** What is the power set of B?

a. No

b. Yes

c.

$$A \cup B = \{x, y, z\}$$

d.

$$A \cap B = \{x, y\}$$

e.

$$A \times B = \{(x,x), (x,y), (y,x), (y,y), (z,x), (z,y)\}$$

f. $\mathcal{P}(B) = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$

0.4 If A has a elements and B has b elements, how many elements are in $A \times B$? Explain your answer.

$$|A \times B| = a \cdot b$$
$$A \times B = \{(x,y) | x \in A \text{ and } B\}$$

0.5 If C is a set with c elements, how many elements are in the power set of C? Explain your answer.

$$|\mathcal{P}(C)| = 2^c$$

0.6 Let X be the set $\{1,2,3,4,5\}$ and Y be the set $\{6,7,8,9,10\}$. The unary function $f\colon X \longrightarrow Y$ and the binary function $g\colon X \times Y \longrightarrow Y$ are described in the following tables.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

a. What is the value of $f(2)$?

b. What are the range and domain of f?

c. What is the value of $g(2,10)$?

d. What are the range and domain of g?

e. What is the value of $g(4, f(4))$?

a. $f(2) = 7$

b. $D: \{1,2,3,4,5\} \quad R: \{6,7\}$

c. $g(2,10) = 6$

d. $D: \{1,2,3,4,5\} \quad R: \{6,7,8,9,10\}$

e. $g(4, f(4)) = 8$

0.7 For each part, give a relation that satisfies the condition.

a. Reflexive and symmetric but not transitive

b. Reflexive and transitive but not symmetric

c. Symmetric and transitive but not reflexive

$A = \{1,2,3\}$

a. $R_1 = \{(1,1),(2,2),(3,3),(2,1),(1,2),(2,3),(3,2)\}$
is reflexive, symmetric, but not transitive on A
"x is a synonym of y" on the set of all words

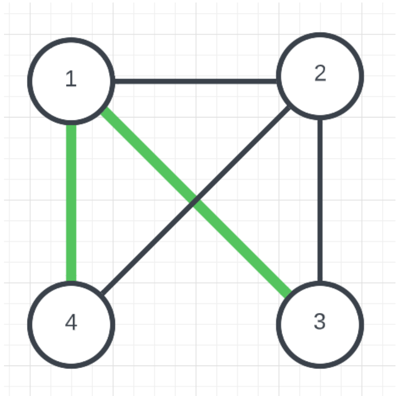
b.

$R_2 = \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$
is reflexive, transitive, but not symmetric on A
$x, y, z \in \mathbb{R}$, "less than or equal to"
$x, y \in \mathbb{R}$ "divisible by"

c.

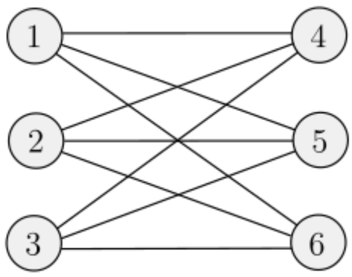
$R_2 = \{(1,1),(2,2),(1,2),(2,1)\}$
is symmetric, transitive, but not reflexive on A
"x goes to the same school as y" on the set of all students

0.8 Consider the undirected graph $G = (V, E)$ where V, the set of nodes, is $\{1, 2, 3, 4\}$ and E, the set of edges, is $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$. Draw the graph G. What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of G.



$$
\begin{aligned}
deg(1) &= 3\\
deg(2) &= 3\\
deg(3) &= 2\\
deg(4) &= 2
\end{aligned}
$$

0.9 Write a formal description of the following graph.



$$(\{1, 2, 3, 4, 5, 6\}, \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\})$$

0.10 Find the error in the following proof that $2 = 1$.
Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$ to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

If $a = 1$ then $(a - b) = 0$. When we divided both sides by $(a - b)$, we were dividing by zero which is undefined.

0.11 Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n, to arrive at the curious conclusion that $C(n) = S^2(n)$ for every n.

a. $S(n) = \frac{1}{2}n(n+1)$.

b. $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

$n = 1:$

$S(n) = \dfrac{n(n+1)}{2}$

$S(1) \stackrel{?}{=} \dfrac{1(1+1)}{2}$

$1 = 1$

$n = k:$

$S(k) = \dfrac{k(k+1)}{2}$

Induction Hypothesis

$n = k+1:$

$S(k+1) \stackrel{?}{=} \dfrac{(k+1)((k+1)+1)}{2}$

$$1 + 2 + ... + k + (k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$

$$\frac{k(k+1)}{2} + (k+1) \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$ by the Induction Hypothesis

$$\frac{k(k+1) + 2(k+1)}{2} \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$ equivalent fractions, multiply numerator and denominator by 2

$$\frac{(k+1)(k+2)}{2} \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$ factoring, factor out (k+1)

$$\frac{(k+1)((k+1)+1)}{2} \stackrel{?}{=} \frac{(k+1)((k+1)+1)}{2}$$ associative property of addition

QED

0.11 b.

Let $n \in \mathbb{N}$ and $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove $C(n) = \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n+1)^2}{4}$ by induction on n.

Proof: We will prove by induction that,

$$C(n) = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n+1)^2}{4} \tag{1}$$

Base case: When $n = 1$, the left side of (1) is $C(1) = 1^3 = 1$, and the right side is $\frac{1^2(1+1)^2}{4} = \frac{1 \cdot 2^2}{4} = \frac{4}{4} = 1$, so both sides are equal and (1) is true for $n = 1$.

Induction step: Let $k \in \mathbb{N}$ be given and suppose (1) is true for $n = k$. Then

$$\begin{aligned}
C(k+1) &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
C(k) + (k+1)^3 &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
\frac{k^2(k+1)^2}{4} + (k+1)^3 &= \frac{(k+1)^2((k+1)+1)^2}{4} \quad \text{(by induction hypothesis)} \\
\frac{k^2(k+1)^2 + 4(k+1)^3}{4} &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
\frac{(k+1)^2(k^2 + 4(k+1))}{4} &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
\frac{(k+1)^2(k^2 + 4k + 4)}{4} &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
\frac{(k+1)^2(k+2)^2}{4} &= \frac{(k+1)^2((k+1)+1)^2}{4} \\
\frac{(k+1)^2((k+1)+1)^2}{4} &= \frac{(k+1)^2((k+1)+1)^2}{4}
\end{aligned} \tag{2}$$

Therefore (1) holds for $n = k + 1$, and the proof of the induction step is complete.

Conclusion: By the principle of induction, (1) is true for all $n \in \mathbb{N}$.

Corollary: $C(n) = S^2(n)$

Given by 0.11 b $C(n) = \frac{n^2(n+1)^2}{4}$ and by 0.11 a $S(n) = \frac{n(n+1)}{2}$. Then

$$\begin{aligned}
C(n) &= \frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2} \\
C(n) &= S(n) \cdot S(n) \\
C(n) &= S^2(n)
\end{aligned}$$

QED

Prove that for all integers *n*, if $n^3 + 5$ is odd then *n* is even. Use proof by contradiction.

Let's suppose that

1) $n^3 + 5$ is odd $\implies$ n is odd

2) $n^3 + 5$ is odd $\implies$ n^3 is even, because 5 is odd,

n^3 must be even for $n^3 + 5$ to be odd.

3) n^3 is even $\implies$ n is even,

because at least one factor has to be even for a product to be even.

4) This contradicts step 1

$\therefore n^3 + 5$ is odd $\implies$ n is even

QED

Prove that in a set of 51 integers, randomly chosen from the interval [1,100], there are at least two integers that divide each other without a remainder.

Start by writing every number randomly choosen as $a = 2^k m$, where m is an odd number between 1 and 99. Note that there 50 odd numbers in the interval [1,100]. Since we choosen 51 numbers from [1,100], each represented as a product of an odd number and power of 2, there must be two numbers in our random selection with the same odd part. If two numbers, have the same odd part then $2^i m$ and $2^j m$ so one must divide the other.