ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

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The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173–198. check it out

have valuable applications. In particular, it is shown (§ 11) that the Hilbertian Entscheidungsproblem can have no solution.

In a recent paper Alonzo Church† has introduced an idea of "effective calculability", which is equivalent to my "computability", but is very differently defined. Church also reaches similar conclusions about the Entscheidungsproblem‡. The proof of equivalence between "computability" and "effective calculability" is outlined in an appendix to the present paper.

1. *Computing machines.*

We have said that the computable numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach § 9. For the present I shall only say that the justification lies in the fact that the human memory is necessarily limited.

We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions $q_1, q_2, \ldots, q_R$ which will be called "m-configurations". The machine is supplied with a "tape" (the analogue of paper) running through it, and divided into sections (called "squares") each capable of bearing a "symbol". At any moment there is just one square, say the r-th, bearing the symbol $\mathfrak{S}(r)$ which is "in the machine". We may call this square the "scanned square". The symbol on the scanned square may be called the "scanned symbol". The "scanned symbol" is the only one of which the machine is, so to speak, "directly aware". However, by altering its m-configuration the machine can effectively remember some of the symbols which it has "seen" (scanned) previously. The possible behaviour of the machine at any moment is determined by the m-configuration q_n and the scanned symbol $\mathfrak{S}(r)$. This pair $q_n, \mathfrak{S}(r)$ will be called the "configuration": thus the configuration determines the possible behaviour of the machine. In some of the configurations in which the scanned square is blank (*i.e.* bears no symbol) the machine writes down a new symbol on the scanned square: in other configurations it erases the scanned symbol. The machine may also change the square which is being scanned, but only by shifting it one place to right or left. In addition to any of these operations the m-configuration may be changed. Some of the symbols written down

† Alonzo Church, "An unsolvable problem of elementary number theory", *American J. of Math.*, 58 (1936), 345–363.

‡ Alonzo Church, "A note on the Entscheidungsproblem", *J. of Symbolic Logic*, 1 (1936), 40–41.

Check it out as well

From (vi) we deduce that all real algebraic numbers are computable.

From (vi) and (x) we deduce that the real zeros of the Bessel functions are computable.

Proof of (ii).

Let $H(x, y)$ mean "$\eta(x)=y$", and let $K(x, y, z)$ mean "$\phi(x, y)=z$". $\mathfrak{A}_\phi$ is the axiom for $\phi(x, y)$. We take $\mathfrak{A}_\eta$ to be

$$\mathfrak{A}_\phi \;\&\; P \;\&\; \big(F(x, y) \rightarrow G(x, y)\big) \;\&\; \big(G(x, y) \;\&\; G(y, z) \rightarrow G(x, z)\big)$$

$$\& \; \big(F^{(r)} \rightarrow H(u, u^{(r)})\big) \;\&\; \big(F(v, w) \;\&\; H(v, x) \;\&\; K(w, x, z) \rightarrow H(w, z)\big)$$

$$\& \; \Big[H(w, z) \;\&\; G(z, t) \vee G(t, z) \rightarrow \big(-H(w, t)\big)\Big].$$

I shall not give the proof of consistency of $\mathfrak{A}_\eta$. Such a proof may be constructed by the methods used in Hilbert and Bernays, *Grundlagen der Mathematik* (Berlin, 1934), p. 209 *et seq*. The consistency is also clear from the meaning.

Suppose that, for some n, N, we have shown

$$\mathfrak{A}_\eta \;\&\; F^{(N)} \rightarrow H(u^{(n-1)}, u^{(\eta(n-1))}),$$

then, for some M,

$$\mathfrak{A}_\phi \;\&\; F^{(M)} \rightarrow K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

$$\mathfrak{A}_\eta \;\&\; F^{(M)} \rightarrow F(u^{(n-1)}, u^{(n)}) \;\&\; H(u^{(n-1)}, u^{(\eta(n-1))})$$

$$\& \; K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

and

$$\mathfrak{A}_\eta \;\&\; F^{(M)} \rightarrow [F(u^{(n-1)}, u^{(n)}) \;\&\; H(u^{(n-1)}, u^{(\eta(n-1))})$$

$$\& \; K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}) \rightarrow H(u^{(n)}, u^{(\eta(n))})].$$

Hence
$$\mathfrak{A}_\eta \;\&\; F^{(M)} \rightarrow H(u^{(n)}, u^{(\eta(n))}).$$

Also
$$\mathfrak{A}_\eta \;\&\; F^{(r)} \rightarrow H(u, u^{(\eta(0))}).$$

Hence for each n some formula of the form

$$\mathfrak{A}_\eta \;\&\; F^{(M)} \rightarrow H(u^{(n)}, u^{(\eta(n))})$$

is provable. Also, if $M' \geqslant M$ and $M' \geqslant m$ and $m \neq \eta(u)$, then

$$\mathfrak{A}_\eta \;\&\; F^{(M')} \rightarrow G(u^{\eta((n))}, u^{(m)}) \,\nu\, G(u^{(m)}, u^{(\eta(n))})$$