Problem of the Week
Problem E and Solution
Five Prime Mates

Problem

The product of five different **odd** prime numbers is a five-digit number of the form $strst$, where $r = 0$. Determine all possible numbers.

Solution

Solution 1

Be sure to look at solution 2 after solution 1.

A number is divisible by 11 if the difference between the sum of the even position numbers and sum of the odd position numbers is a multiple of 11. (This problem can still be done without knowing this divisibility fact but the task is made simpler with it.) The sum of the digits in the even positions is $strst$ is $t + s$. The sum of the digits in the odd positions is $s + r + t$ but $r = 0$ so the sum is $s + t$. The difference of the two sums is $(t + s) - (s + t) = 0$ which is a multiple of 11. Therefore, $st0st$ is divisible by 11.

Only odd factors are used so the product will be odd. This means that the product looks like _10_1, _30_3, _50_5, _70_7, or _90_9. So we begin to systematically look at the possibilities.

First, we will examine numbers that have 3 (and 11) as a factor. To be divisible by three, the sum of the digits will be divisible by three. To be divisible by nine, the sum of the digits will be divisible by nine. But if the number is divisible by nine, it is divisible by 3^2 and would have a repeated prime factor which is not allowed. So we want numbers divisible by 3 but not 9. The possibilities are as follows: 21021, 51051, 33033, 93093, 15015, 75075, 57057, 87087, 39039, and 69069. The sum of the digits of these numbers is divisible by three so the numbers are divisible by three. The numbers 81081, 63063, 45045, 27027 and 99099 are divisible by 9 and have therefore been eliminated.

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$21021 = 3 \times 11 \times 637 = 3 \times 11 \times 7 \times 91 = 3 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$51051 = 3 \times 11 \times 1547 = 3 \times 11 \times 7 \times 221 = 3 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 51051 is a valid number.

$$33033 = 3 \times 11 \times 1001 = 3 \times 11 \times 7 \times 143 = 3 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$93093 = 3 \times 11 \times 2821 = 3 \times 11 \times 7 \times 403 = 3 \times 11 \times 7 \times 13 \times 31$$

Since there are 5 different odd prime factors, 93093 is a valid number.

$$15015 = 3 \times 11 \times 455 = 3 \times 11 \times 5 \times 91 = 3 \times 11 \times 5 \times 7 \times 13$$

Since there are 5 different odd prime factors, 15015 is a valid number.

$$75075 = 3 \times 11 \times 2275 = 3 \times 11 \times 5 \times 455 = 3 \times 11 \times 5 \times 5 \times 91 = 3 \times 11 \times 5 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated and there are six prime factors, this is not a valid number.

$$57057 = 3 \times 11 \times 1729 = 3 \times 11 \times 7 \times 247 = 3 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 57057 is a valid number.

$$87087 = 3 \times 11 \times 2639 = 3 \times 11 \times 7 \times 377 = 3 \times 11 \times 7 \times 13 \times 29$$

Since there are 5 different odd prime factors, 87087 is a valid number.

$$39039 = 3 \times 11 \times 1183 = 3 \times 11 \times 7 \times 169 = 3 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$69069 = 3 \times 11 \times 2093 = 3 \times 11 \times 7 \times 299 = 3 \times 11 \times 7 \times 13 \times 23$$

Since there are 5 different odd prime factors, 69069 is a valid number.

Second, we will examine numbers that are divisible by 5 but not 3, since divisibility by three has been examined. If a number is divisible by 5 it ends in 5 or 0. Since the number is odd, we can exclude any number ending in 0 leaving 25025, 35035, 55055, 65065, 85085 and 95095 as possible numbers. (15015, 45045, 75075 were examined above and have been excluded.)

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$25025 = 5 \times 11 \times 455 = 5 \times 11 \times 5 \times 91 = 5 \times 11 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated, this is not a valid number.

$$35035 = 5 \times 11 \times 637 = 5 \times 11 \times 7 \times 91 = 5 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$55055 = 5 \times 11 \times 1001 = 5 \times 11 \times 7 \times 143 = 5 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$65065 = 5 \times 11 \times 1183 = 5 \times 11 \times 7 \times 169 = 5 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$85085 = 5 \times 11 \times 1547 = 5 \times 11 \times 7 \times 221 = 5 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 85085 is a valid number.

$$95095 = 5 \times 11 \times 1729 = 5 \times 11 \times 7 \times 247 = 5 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 95095 is a valid number.

Thirdly, we will look at numbers that are divisible by 7 but not 3 or 5. If we multiply 7 by the next four odd prime numbers we get $7 \times 11 \times 13 \times 17 \times 19 = 323323$, a six digit number so we are beyond all possible solutions.

Therefore there are 8 numbers of the form $st0st$ which are the product of five different odd prime numbers, namely 51051, 93093, 15015, 57057, 87087, 69069, 85085 and 95095.

Look at Solution 2 for a much more insightful approach to the problem.

Solution 2

Since $r = 0$, the number is of the form $st0st$. Then

$$st0st = st(1000) + st = st(1000 + 1) = st(1001)$$

This means that the number $st0st$ is divisible by 1001 which is the product of the three odd prime factors 7, 11, and 13. So st is a two digit number which is the product of two different odd prime factors none of which can be 7, 11 or 13. It is now a straight forward matter of generating all possible two digit products, $a \times b$ say, using odd prime factors other than 7, 11 and 13.

Prime Factor a	Prime Factor b	st $= a \times b$	Five Different Odd Primes	Product $st0st$
3	5	15	$3, 5, 7, 11, 13$	15015
3	17	51	$3, 7, 11, 13, 17$	51051
3	19	57	$3, 7, 11, 13, 19$	57057
3	23	69	$3, 7, 11, 13, 23$	69069
3	29	87	$3, 7, 11, 13, 29$	87087
3	31	93	$3, 7, 11, 13, 31$	93093
5	17	85	$5, 7, 11, 13, 17$	85085
5	19	95	$5, 7, 11, 13, 19$	95095

No other two digit product of two different odd prime factors other than 7, 11 and 13 exists.

Therefore there are 8 numbers of the form $st0st$ which are the product of five different odd prime numbers, namely 15015, 51051, 57057, 69069, 87087, 93093, 85085 and 95095.