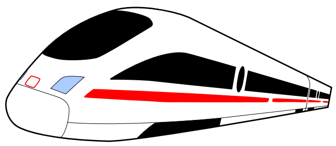


Problem of the Week

Problem E and Solution

Advanced Training

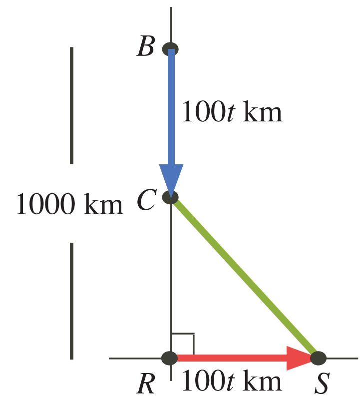
Problem

A blue train is 1000 km north of a red train. The blue train is travelling south, and the red train is travelling east. Both trains travel at the same speed of 100 km/h. How much time elapses until the **total** distance travelled by the two trains is equal to the distance between the trains.

Solution

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.



The diagram to the right represents the positions of the two trains after t hours. The blue train starts at B and moves to C. The red train starts at R and moves to S. Then $BC = RS = 100t$ and $CR = 1000 - 100t$. We want the time t when $CS = BC + RS = 100t + 100t = 200t$.

$$\begin{array}{rrcl}
\text{But } \triangle CRS \text{ is right angled so} & CS^2 & = & CR^2 + RS^2 \\
& (200t)^2 & = & (1000 - 100t)^2 + (100t)^2 \\
& 40000t^2 & = & 1000000 - 200000t + 10000t^2 + 10000t^2 \\
& 20000t^2 + 200000t - 1000000 & = & 0 \\
& t^2 + 10t - 50 & = & 0 \\
\text{Using the Quadratic Formula,} & t & = & \dfrac{-10 \pm \sqrt{100 - 4(-50)}}{2} \\
& t & = & \dfrac{-10 \pm 10\sqrt{3}}{2} \\
& t & = & -5 \pm 5\sqrt{3}
\end{array}$$

Since $t > 0$, $-5 - 5\sqrt{3}$ is inadmissible. $\therefore t = -5 + 5\sqrt{3} \doteq 3.66$ hours.

The distance between the two trains will be equal to their total distance travelled in $(-5 + 5\sqrt{3})$ hours which is approximately 3 hours and 40 minutes after they leave their initial positions.

Could the diagram be drawn any other way? On the next page the other two possible diagrams are briefly discussed.

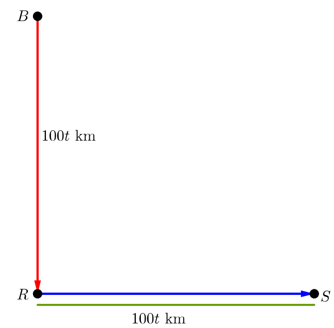
We will repeat some of the initial work and then apply it to a second diagram.

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.

Maybe the train travelling south gets to a point in line with the west to east direction of the second train.

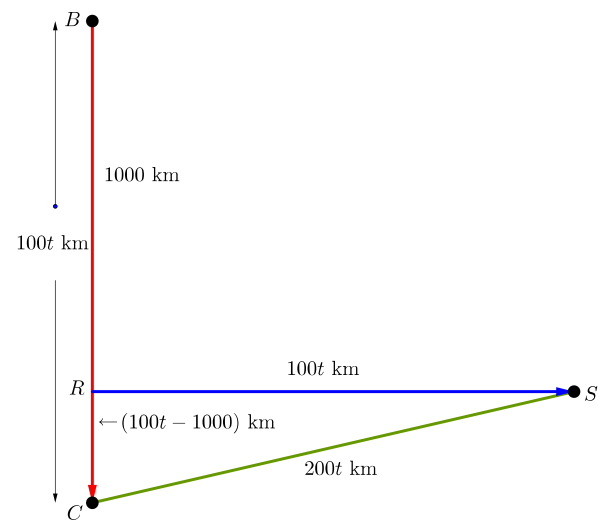
The diagram to the right represents the positions of the two trains after t hours. $BR = 1000$ km and it follows that $t = 10$ hours. The second train travels from R to S, a total of 1000 km, in the same time. But RS is supposed to be the total distance travelled by the two trains. The total distance travelled is 2000 km but $RS = 1000$ km. This diagram is not possible.



Is it possible that the train travelling south goes lower than the west to east line along which the second train travels?

The diagram to the right represents the positions of the two trains after t hours. The distance travelled by the first train is represented by $BC = 100t$ km and it follows that $RC = (100t - 1000)$ km. The second train travels from R to S, a total of $100t$ km in the same time. The total distance is $CS = 200t$ km.

In any triangle, the sum of the lengths of any two sides of the triangle must be greater than the length of the third side. This is known as the triangle inequality law.



In $\triangle RSC$, $RC + RS = (100t - 1000) + 100t = 200t - 1000 < 200t = CS$.

The triangle inequality law is not satisfied. This "triangle" cannot represent the situation presented in this problem. It can be shown that the triangle shown on the previous page satisfies the triangle inequality law. It is the only possibility.