# Problem of the Week
## Problem E and Solution
## Looking for Possibilities

### Problem

Determine all possible ordered pairs of positive integers $(a, b)$ such that $\dfrac{1}{a} + \dfrac{2}{b} = \dfrac{8}{2a+b}$ and $1963 \leq 4a + 7b \leq 2016$.

## Solution

Since $a$ and $b$ are positive integers satisfying $1963 \leq 4a + 7b \leq 2016$, we could write out all ordered pairs that satisfy this inequality and then determine which ones also satisfy the first equation. There will be a large number of possibilities to check so we need to find a way to reduce the number of possibilities. We will work with the first equation.

$$\begin{array}{rrcl} & \dfrac{b+2a}{ab} & = & \dfrac{8}{2a+b} \\ \text{Multiply both sides by } ab(2a+b)\text{:} & (b+2a)(2a+b) & = & 8ab \\ \text{Expand and simplify:} & 4a^2 + 4ab + b^2 & = & 8ab \\ \text{Rearrange:} & 4a^2 - 4ab + b^2 & = & 0 \\ \text{Factor:} & (2a-b)^2 & = & 0 \end{array}$$

It follows that $2a - b = 0$ and $b = 2a$.

Each of the ordered pairs $(a, b)$ will look like $(a, 2a)$. We substitute $2a$ for $b$ in the inequality obtaining $1963 \leq 4a + 7(2a) \leq 2016$ or $1963 \leq 18a \leq 2016$. We could work with the parts of the inequality separately. However, we can also divide each term in the inequality by 18.

$$\begin{array}{ccccc} 1963 & \leq & 18a & \leq & 2016 \\ \dfrac{1963}{18} & \leq & \dfrac{18a}{18} & \leq & \dfrac{2016}{18} \\ 109\dfrac{1}{18} & \leq & a & \leq & 112 \end{array}$$

Since $a$ is a positive integer, $a$ can only take on integer values $110,\ 111,\ 112$. Since $b = 2a$, the corresponding values of $b$ are $220,\ 222,\ 224$.

The ordered pairs of positive integers $(a, b)$ that satisfy $\dfrac{1}{a} + \dfrac{2}{b} = \dfrac{8}{2a+b}$ and $1963 \leq 4a + 7b \leq 2016$ are $(110, 220), (111, 222)$ and $(112, 224)$.