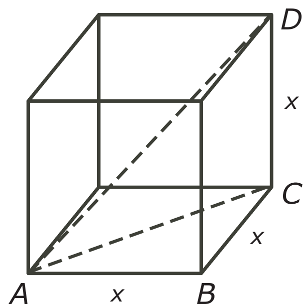
Problem of the Week

Problem E and Solution

Sphere Pressure

Problem

A cube rests inside a sphere so that each vertex touches the sphere. The radius of the sphere is 6 cm. Determine the volume of the cube.



Solution

Label four vertices of the cube A, B, C, D as shown in the diagram. Let x represent the side length of the cube. Then $AB = BC = CD = x$.

The diagonals of a cube intersect in a point such that the distance from the intersection point to each vertex is equal. Since each vertex of the cube touches the sphere, the diagonal of the cube, AD, is equal in length to the diameter of the sphere. Therefore $AD = 2(6) = 12$ cm.

Each face of a cube is a square so $\angle ABC = 90°$. Using Pythagoras' Theorem,

$$AC^2 = AB^2 + BC^2 = x^2 + x^2 = 2x^2.$$

In a cube the sides are perpendicular to the base. In particular, DC is perpendicular to the base and it follows that $DC \perp AC$. Therefore $\triangle DCA$ is a right angled triangle. Using Pythagoras' Theorem,

$$AD^2 = AC^2 + CD^2 = 2x^2 + x^2 = 3x^2.$$

But $AD = 12$ so $AD^2 = 144$. Then $3x^2 = 144$, $x^2 = 48$ and $x = 4\sqrt{3}$ since $x > 0$. The volume of the cube is $x^3 = (4\sqrt{3})^3 = 192\sqrt{3}$ cm^3.

$\therefore$ the volume of the cube is $192\sqrt{3}$ cm^3.