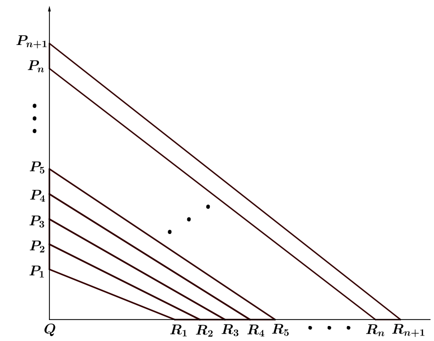
Problem of the Week
Problem E and Solution
A Skinny Quadrilateral

Problem

In the diagram, $\triangle QP_1R_1$ is right-angled with $QP_1 = 2$ and $QR_1 = 5$. Lines QP_1 and QR_1 are extended and many more points are labelled at intervals of 1 unit, so that $P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = \cdots = 1$, and $R_1R_2 = R_2R_3 = R_3R_4 = R_4R_5 = \cdots = 1$. In fact, $P_1P_j = j-1$ and $R_1R_k = k-1$ for any positive integers j and k. For example, $P_1P_5 = 5-1 = 4$ and $R_1R_4 = 4-1 = 3$. Determine the value of n so that the area of quadrilateral $P_nP_{n+1}R_{n+1}R_n$ is 2016.



Solution

Solution 1

In order to solve the problem, looking at the calculation of a specific area may prove helpful.

So let's determine the area of quadrilateral $P_4P_5R_5R_4$.

$$\begin{aligned}
\text{Area of quadrilateral } P_4P_5R_5R_4 &= \text{Area } \triangle P_5QR_5 - \text{Area } \triangle P_4QR_4 \\
&= \frac{1}{2}(QP_5)(QR_5) - \frac{1}{2}(QP_4)(QR_4) \\
&= \frac{1}{2}(2+(5-1))(5+(5-1)) - \frac{1}{2}(2+(4-1))(5+(4-1)) \\
&= \frac{1}{2}(6)(9) - \frac{1}{2}(5)(8) \\
&= 27 - 20 \\
&= 7 \text{ units}^2
\end{aligned}$$

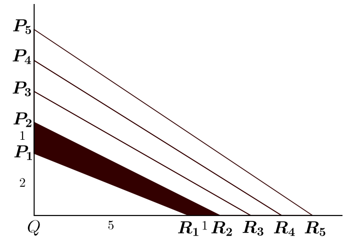
We will pattern the more general solution off the above example.

$$\begin{aligned}
\text{Area of quad. } P_nP_{n+1}R_{n+1}R_n &= \text{Area } \triangle P_{n+1}QR_{n+1} - \text{Area } \triangle P_nQR_n \\
2016 &= \frac{1}{2}(QP_{n+1})(QR_{n+1}) - \frac{1}{2}(QP_n)(QR_n) \\
2016 &= \frac{1}{2}(2+((n+1)-1))(5+((n+1)-1)) - \frac{1}{2}(2+(n-1))(5+(n-1)) \\
2016 &= \frac{1}{2}(2+n)(5+n) - \frac{1}{2}(1+n)(4+n) \\
4032 &= (2+n)(5+n) - (1+n)(4+n) \qquad \text{multiplying by 2} \\
4032 &= n^2 + 7n + 10 - (n^2 + 5n + 4) \\
4032 &= n^2 + 7n + 10 - n^2 - 5n - 4 \\
4032 &= 2n + 6 \\
4026 &= 2n \\
2013 &= n
\end{aligned}$$

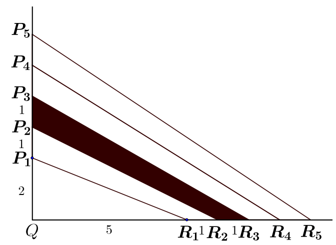
Therefore, the value of n is 2013. Using the method of the specific example, this result is easily confirmed.

Solution 2

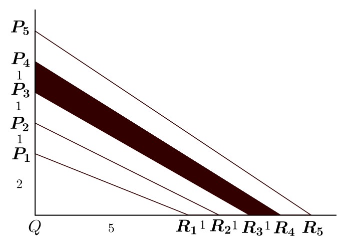
In this solution we look for a pattern in the area calculations.



$$\begin{aligned}
\text{Area of quadrilateral } P_1P_2R_2R_1 &= \text{Area } \triangle P_2QR_2 - \text{Area } \triangle P_1QR_1\\
&= \frac{1}{2}(QP_2)(QR_2) - \frac{1}{2}(QP_1)(QR_1)\\
&= \frac{1}{2}(3)(6) - \frac{1}{2}(2)(5)\\
&= 9 - 5\\
\text{Area of first quadrilateral} &= 4 \text{ units}^2
\end{aligned}$$



$$\begin{aligned}
\text{Area of quadrilateral } P_2P_3R_3R_2 &= \text{Area } \triangle P_3QR_3 - \text{Area } \triangle P_2QR_2\\
&= \frac{1}{2}(QP_3)(QR_3) - \frac{1}{2}(QP_2)(QR_2)\\
&= \frac{1}{2}(4)(7) - \frac{1}{2}(3)(6)\\
&= 14 - 9\\
\text{Area of second quadrilateral} &= 5 \text{ units}^2
\end{aligned}$$



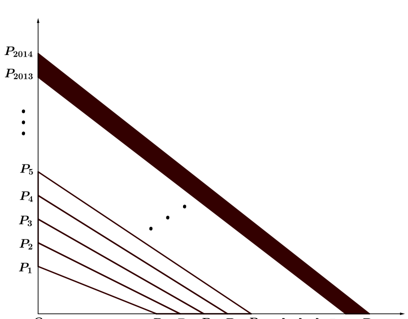
$$\begin{aligned}
\text{Area of quadrilateral } P_3P_4R_4R_3 &= \text{Area } \triangle P_4QR_4 - \text{Area } \triangle P_3QR_3\\
&= \frac{1}{2}(QP_4)(QR_4) - \frac{1}{2}(QP_3)(QR_3)\\
&= \frac{1}{2}(5)(8) - \frac{1}{2}(4)(7)\\
&= 20 - 14\\
\text{Area of third quadrilateral} &= 6 \text{ units}^2
\end{aligned}$$

A possible pattern has emerged. The area of the quadrilateral appears to be three more than the position of the quadrilateral on the stack of consecutive quadrilaterals. If we want the area to be 2016 then it should be the 2013^{th} quadrilateral. That is, it should be the quadrilateral with vertices $P_{2013}P_{2014}R_{2014}R_{2013}$. Therefore, the value of n is 2013.

We can verify this value of n using the area calculation. (Recall from the problem statement, $P_1P_j = j - 1$ and $R_1R_k = k - 1$ for any positive integers j and k.)

So, $P_1P_{2013} = 2013 - 1 = 2012$. Then $QP_{2013} = QP_1 + P_1P_{2013} = 2 + 2012 = 2014$.
Since $QP_{2014} = QP_{2013} + 1$, it follows that $QP_{2014} = 2015$.

Also, $R_1R_{2013} = 2013 - 1 = 2012$. Then $QR_{2013} = QR_1 + R_1R_{2013} = 5 + 2012 = 2017$.
Since $QR_{2014} = QR_{2013} + 1$, it follows that $QR_{2014} = 2018$.



$$\begin{aligned}
&\text{Area of quadrilateral } P_{2013}P_{2014}R_{2014}R_{2013}\\
= \;&\text{Area } \triangle P_{2014}QR_{2014} - \text{Area } \triangle P_{2013}QR_{2013}\\
= \;&\frac{1}{2}(QP_{2014})(QR_{2014}) - \frac{1}{2}(QP_{2013})(QR_{2013})\\
= \;&\frac{1}{2}(2015)(2018) - \frac{1}{2}(2014)(2017)\\
= \;&2\,033\,135 - 2\,031\,119\\
= \;&2016 \text{ units}^2
\end{aligned}$$