Mathematics Toolkit

Problem of the Week
Problem E and Solution
Analytic Geometry Tools Required

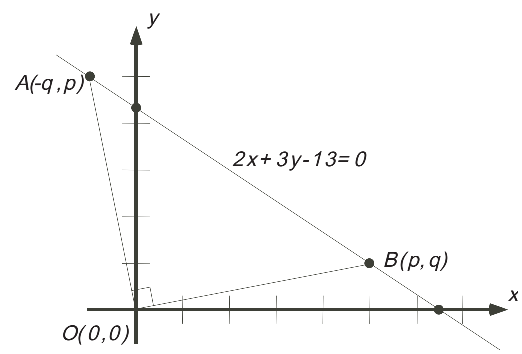
Problem

OAB is an isosceles right triangle with vertex O at the origin $(0, 0)$, vertices A and B on the line $2x + 3y - 13 = 0$, and $\angle AOB = 90°$. Determine the area of $\triangle OAB$.

Solution

Solution 1

In analytic geometry problems, a representative diagram is important and often provides clues for the solution of the problem. The following diagram has the given information plus a couple of pieces of information that will be justified now.



Let B have coordinates (p, q). The slope of $OB = \frac{q}{p}$. Since $\angle AOB = 90°$, $OB \perp OA$ and the slope of OA is the negative reciprocal of OB. Therefore the slope of $OA = \frac{p}{-q}$. Since the triangle is isosceles, $OA = OB$ and it follows that the coordinates of A are $(-q, p)$. (We can verify this by finding the length of OA and the length of OB and showing that both are equal to $\sqrt{p^2 + q^2}$.)

Since $B(p, q)$ is on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.

$$\therefore 2p + 3q - 13 = 0 \quad (1)$$

Since $A(-q, p)$ is on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.

$$\therefore -2q + 3p - 13 = 0 \text{ or } 3p - 2q - 13 = 0 \quad (2)$$

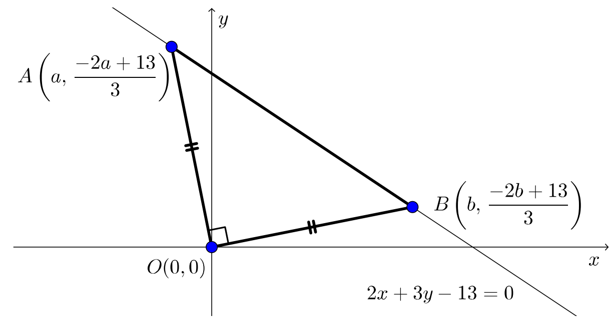
Since we have two equations and two unknowns, we can use elimination to solve for p and q.

$$\begin{array}{rrcl}
(1) \times 2: & 4p + 6q - 26 & = & 0 \\
(2) \times 3: & 9p - 6q - 39 & = & 0 \\
\text{Adding, we obtain}: & 13p - 65 & = & 0 \\
& \therefore p & = & 5 \\
\text{Substituting in (1)}: & 10 + 3q - 13 & = & 0 \\
& 3q & = & 3 \\
& \therefore q & = & 1
\end{array}$$

The point B is $(5, 1)$ and the length of $OB = \sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$. $\triangle AOB$ is a right triangle so we can use OB as the base and OA as the height in the formula for the area of a triangle. Then the area of $\triangle AOB = \frac{OA \times OB}{2} = \frac{\sqrt{26}\sqrt{26}}{2} = 13$.

$\therefore$ the area of $\triangle AOB$ is 13 units^2.

Solution 2



By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$. Since the points A and B are on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$. Since $\triangle OAB$ is isosceles, we know that $OA = OB$. Then

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
\text{Multiplying by 9 :} \quad 9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
\text{Simplifying :} \quad 13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
\text{Rearranging :} \quad 13a^2 - 13b^2 - 52a + 52b &= 0 \\
\text{Dividing by 13 :} \quad a^2 - b^2 - 4a + 4b &= 0 \\
\text{Factoring pairs :} \quad (a+b)(a-b) - 4(a-b) &= 0 \\
\text{Common factoring :} \quad (a-b)(a+b-4) &= 0
\end{aligned}$$

Solving, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$. We can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is a right triangle, we can use the Pythagorean theorem and $AB^2 = OA^2 + OB^2$ follows. But $OA = OB$ so this can be written $AB^2 = 2OB^2$.

$$\begin{aligned}
AB^2 &= 2OB^2 \\
(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left[b^2 + \left(\frac{-2b+13}{3}\right)^2\right] \\
(2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left[b^2 + \frac{4b^2 - 52b + 169}{9}\right] \\
4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9} \\
\text{Multiplying by 9 :} \quad 36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
\text{Simplifying :} \quad 52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
\text{Rearranging :} \quad 26b^2 - 104b - 130 &= 0 \\
\text{Dividing by 26 :} \quad b^2 - 4b - 5 &= 0 \\
\text{Factoring :} \quad (b-5)(b+1) &= 0
\end{aligned}$$

It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$. When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$. There are only two points. The area work shown in Solution 1 follows from here. Therefore, the area of $\triangle OAB$ is 13 units^2.