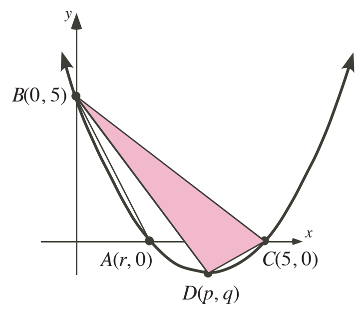
Problem of the Week

Problem E and Solution

Find the Area - No Parabolem!



Problem

In the diagram, $D(p,q)$ is the vertex of the parabola. The parabola cuts the x-axis at $A(r,0)$ and $C(5,0)$. The area of $\triangle ABC$ is 5. Determine the area of $\triangle DBC$.

Solution

Solution 1

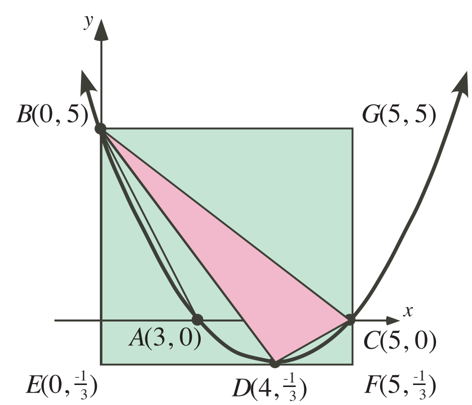
In $\triangle ABC$, the height is the distance from the x-axis to $B(0,5)$. Therefore the height is 5 units. The base is $AC = 5 - r$. Using the formula for the area of a triangle, Area $\triangle ABC = \frac{(5-r)(5)}{2} = 5$. Then $5 - r = 2$ and $r = 3$ follows. The coordinates of A are $(3,0)$.

The axis of symmetry is a vertical line through the midpoint of AC which is $(4,0)$. It follows that the x-coordinate of the vertex is $p = 4$. Therefore, the vertex is $D(4,q)$.

Since the two x-intercepts of the parabola are 3 and 5, the equation of the parabola in factored form can be written as $y = a(x-3)(x-5)$. Since $B(0,5)$ is on the parabola, we can solve for a by substituting $x = 0$ and $y = 5$ into $y = a(x-3)(x-5)$. This leads to $a = \frac{1}{3}$ and $y = \frac{1}{3}(x-3)(x-5)$.

To find q, the y-coordinate of D, substitute $x = 4,\ y = q$ into $y = \frac{1}{3}(x-3)(x-5)$. Then $q = \frac{1}{3}(4-3)(4-5) = -\frac{1}{3}$.

Construct a rectangle with sides parallel to the x and y-axes so that the points B, D, and C are on the rectangle. B is one vertex of the rectangle. We can determine the coordinates of the other three vertices of the rectangle. The results are shown on the diagram to the right. It is straight forward to then determine the lengths required for finding various areas: $BE = 5\frac{1}{3} = \frac{16}{3}$, $ED = 4$, $DF = 1$, $FC = \frac{1}{3}$, $CG = 5$ and $BG = 5$.



To find the area of $\triangle DBC$, subtract the areas of $\triangle BED$, $\triangle DFC$, and $\triangle BCG$ from the area of rectangle $BEFG$.

The area of rectangle $BEFG = BE \times BG = \frac{16}{3} \times 5 = \frac{80}{3}$ units^2.

The area of $\triangle BED = \frac{BE \times ED}{2} = \frac{\frac{16}{3} \times 4}{2} = \frac{32}{3}$ units^2.

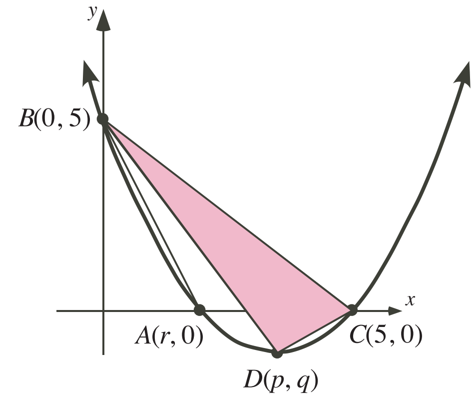
The area of $\triangle DFC = \frac{DF \times FC}{2} = \frac{1 \times \frac{1}{3}}{2} = \frac{1}{6}$ units^2.

The area of $\triangle BCG = \frac{CG \times BG}{2} = \frac{5 \times 5}{2} = \frac{25}{2}$ units^2.

$$\begin{aligned}
\text{Area } \triangle DBC &= \text{Area Rectangle } BEFG - \text{Area } \triangle BED - \text{Area } \triangle DFC - \text{Area } \triangle BCG \\
&= \frac{80}{3} - \frac{32}{3} - \frac{1}{6} - \frac{25}{2} \\
&= \frac{10}{3} \text{ units}^2
\end{aligned}$$

$\therefore$ the area of $\triangle DBC = \frac{10}{3}$ units^2.

Solution 2

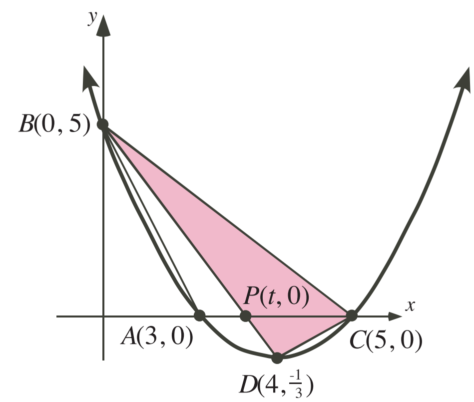


To begin this solution we will pick up a result from the first solution and do not repeat the work here. In solution 1, we found vertex D to be the point $(4, -\frac{1}{3})$.

Let $P(t, 0)$ be the point where the line through B and D crosses the x-axis.

We will determine the equation of the line containing B, P, and D.

The slope of the line is $\frac{5+\frac{1}{3}}{0-4} = \frac{\frac{16}{3}}{-4} = -\frac{4}{3}$.



The y-intercept of the line is 5. Therefore the equation of the line through B, P, and D is $y = -\frac{4}{3}x + 5$.

To find t, the x-coordinate of P we substitute $x = t$ and $y = 0$ into the equation of the line.

$$\begin{aligned} y &= -\frac{4}{3}x + 5 \\ 0 &= -\frac{4}{3}t + 5 \\ 4t &= 15 \\ t &= \frac{15}{4} \end{aligned}$$

To determine the area of $\triangle BDC$, we find the sum of the area of $\triangle BPC$ and the area of $\triangle DPC$. We will use PC as the base in both triangles. Then $PC = 5 - \frac{15}{4} = \frac{5}{4}$.

In $\triangle BPC$, the height is the perpendicular distance from the x-axis to point B. The height is 5. The area of $\triangle BPC = \frac{\frac{5}{4}\times 5}{2} = \frac{25}{8}$ units^2.

In $\triangle DPC$, the height is the perpendicular distance from the x-axis to point D. The height is $\frac{1}{3}$. The area of $\triangle DPC = \frac{\frac{5}{4}\times\frac{1}{3}}{2} = \frac{5}{24}$ units^2.

$$\begin{aligned} \therefore \text{Area } \triangle DBC &= \text{Area } \triangle BPC + \text{Area } \triangle DPC \\ &= \frac{25}{8} + \frac{5}{24} \\ &= \frac{75}{24} + \frac{5}{24} \\ &= \frac{80}{24} \\ &= \frac{10}{3} \text{ units}^2 \end{aligned}$$

$\therefore$ the area of $\triangle DBC = \frac{10}{3}$ units^2.