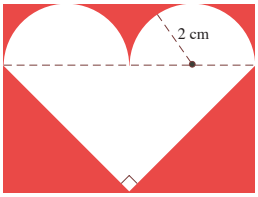


Problem of the Week
Problem E and Solution
Love is Blind Valentine

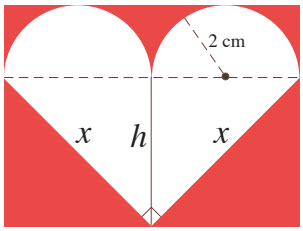
Problem

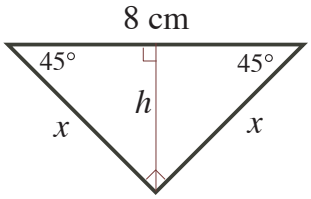
A heart is constructed by attaching two white semi-circles, each with radius 2 cm, onto the hypotenuse of an isosceles right triangle. The heart is then pasted onto a rectangular sheet of red construction paper as illustrated below. (The dashed lines, the side measurement and the right angle symbol will not actually be on the finished card.) Cupid shoots an arrow that randomly lands somewhere on the card. Determine the probability that Cupid's arrow lands on the heart.

Solution

The semi-circles are placed along the hypotenuse of the triangle. The hypotenuse is equal in length to four radii and is therefore $4 \times 2 = 8$ cm long. Let x represent the length of the sides of equal length in the triangle and let h represent the height of the triangle drawn from the vertex between the two equal sides to the opposite side.

Place the given information on the diagrams below.





Since the triangle is a right triangle we can use Pythagoras' Theorem to find the value of x.

$$\begin{aligned} x^2 + x^2 &= 8^2 \\ 2x^2 &= 64 \\ x^2 &= 32 \\ x &= 4\sqrt{2}, \quad \text{since } x > 0 \end{aligned}$$

The altitude of an isosceles, right triangle bisects the hypotenuse so we can determine the height of the triangle, h. This statement is justified at the end of the solution.

$$\begin{aligned} h^2 + 4^2 &= x^2 \\ h^2 + 16 &= 32 \\ h^2 &= 16 \\ h &= 4, \quad \text{since } h > 0 \end{aligned}$$

We now have enough information to find the area of the heart and the entire shape.

The total height of the rectangle is equal to the height of the triangle plus the radius of the semi-circle. The height of the rectangle is $4 + 2 = 6$ cm. The width of the rectangle is the same as the length of the hypotenuse and is therefore equal to 8 cm. The area of the rectangle is $8 \times 6 = 48$ cm^2.

The area of the heart is equal to the area of the triangle plus the area of two congruent semi-circles of radius 2 cm.

Since the triangle is isosceles right we can use one of the equal sides as the base and the other as the height. The area of the triangle is $\frac{bh}{2} = \frac{(4\sqrt{2})\times(4\sqrt{2})}{2} = \frac{32}{2} = 16$ cm^2.

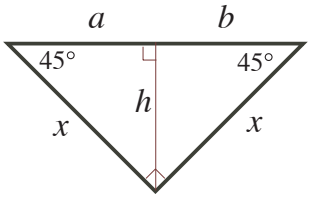
The area of the two semi-circles is the same as the area of one circle with radius 2 cm. The area of the two semi-circles is $\pi r^2 = \pi(2)^2 = 4\pi$ cm^2.

The total area of the heart is $(4\pi + 16)$ cm^2.

The probability that Cupid's arrow lands on the heart is equal to the area of the heart divided by the area of the rectangle. The probability equals $\frac{4\pi+16}{48} \doteq 0.595$. In other words, Cupid has about a 60% chance of landing his arrow on the heart. Happy Valentine's Day.

Justification of the statement: "The altitude of an isosceles right triangle bisects the hypotenuse."

There are many ways to justify this. Since the triangle is an isosceles right triangle, the two angles opposite the sides of equal length are equal. In a triangle, the sum of the angles is 180° so the two equal angles must add to the remaining 90° and each angle is 45°. Construct the altitude, h, as shown in the following diagram, splitting the opposite sides into two parts, labelled a and b.



The altitude hits the opposite side at 90° so each of the two smaller triangles contains a right angle and a 45° angle. The missing angle in each of the two smaller triangles is then 45°. Each of the smaller triangles is then isosceles and it follows that $h = a$ and $h = b$. $\therefore a = b$ and the altitude bisects the hypotenuse.