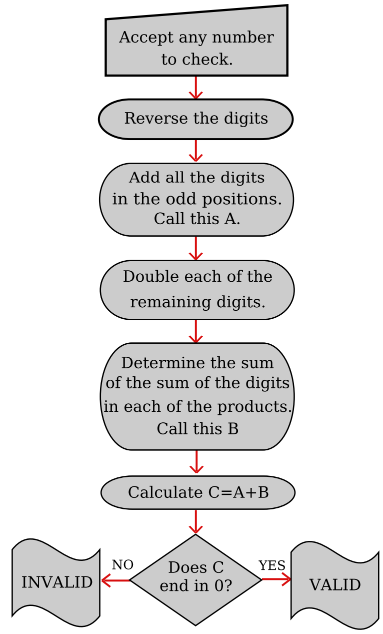
Problem of the Week

Problem E and Solution

Double Check

Problem

Debit and credit cards contain account numbers which consist of many digits. Often, when purchasing items online you are asked to type in your account number. Because there are so many digits it is easy to type the number incorrectly. Most companies use a "check digit" to quickly verify the validity of the number. One commonly used algorithm for verifying that numbers have been entered correctly is called the Luhn Algorithm. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid. The steps performed in the Luhn Algorithm are illustrated in the flowchart to the right.



Suppose the number 2853 $0QPP$ 362 is a valid number when verified by the Luhn Algorithm. P and Q are each integers from 0 to 9 such that $P \leq Q$. Determine all possible values of P and Q.

Solution

When the digits of the number are reversed the resulting number is 263 $PPQ0$ 3582. The sum of the digits in the odd positions is $A = 2 + 3 + P + 0 + 5 + 2 = 12 + P$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$2 \times 6 = 12;\ \ 2 \times P = 2P;\ \ 2 \times Q = 2Q;\ \ 2 \times 3 = 6;\ \text{ and } 2 \times 8 = 16$$

Let x represent the sum of the digits of $2P$ and y represent the sum of the digits of $2Q$. When the digit sums from each of the products are added, the sum is:

$$B = (1 + 2) + x + y + 6 + (1 + 6) = 3 + x + y + 6 + 7 = x + y + 16$$

C is the sum of A and B, so $C = 12 + P + x + y + 16 = 28 + P + x + y$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what are the possible sums which can be obtained?

Original Digit (P or Q)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2P$ or $2Q$)	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2P$ or $2Q$ (x or y)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of twice the original digit can only be an integer from 0 to 9 inclusive. It follows that the only values for x or y are the integers from 0 to 9.

The first row and the last row of this chart will be reprinted at the top of the next page as the results are used in the solution. We will use it to determine x from P and to determine Q from y.

Original Digit (P or Q)	0	1	2	3	4	5	6	7	8	9
The Sum of the Digits of $2P$ or $2Q$ (x or y)	0	2	4	6	8	1	3	5	7	9

Since the number satisfies the Luhn algorithm, $C = 28 + P + x + y$ must end in zero.

What are the possible values to consider for C? Since the maximum value for each of P, x and y is 9, the maximum value for $P + x + y$ is 27 and the maximum value for $C = 28 + P + x + y$ is 55. It follows that the only valid possibilities for C that end in zero are 30, 40 and 50. We will consider each of the three possibilities.

1. $\underline{C = 30 \text{ and } P + x + y = 2}$

   If $P = 0$, then $x = 0$, $y = 2$ and $Q = 1$.
   There are no other valid possibilities for P so that $P + x + y = 2$.

   Therefore, this case produces one valid possibility for (P, Q): $(0, 1)$.

2. $\underline{C = 40 \text{ and } P + x + y = 12}$

P	x	$y = 12 - P - x$	Q	Valid or Invalid	(P, Q)
0	0	12		not valid, $y > 9$	
1	2	9	9	valid, $P \le Q$	$(1, 9)$
2	4	6	3	valid, $P \le Q$	$(2, 3)$
3	6	3	6	valid, $P \le Q$	$(3, 6)$
4	8	0	0	invalid, $P > Q$	
5	1	6	3	invalid, $P > Q$	
6	3	3	6	valid, $P \le Q$	$(6, 6)$
7	5	0	0	invalid, $P > Q$	
8	7	-3		invalid, $y < 0$	
9	9	-6		invalid, $y < 0$	

   Therefore, this case produces four valid possibilities for (P, Q): $(1, 9)$, $(2, 3)$, $(3, 6)$ and $(6, 6)$.

3. $\underline{C = 50 \text{ and } P + x + y = 22}$

P	x	$y = 22 - P - x$	Q	Valid or Invalid	(P, Q)
0	0	22		not valid, $y > 9$	
1	2	19		not valid, $y > 9$	
2	4	16		not valid, $y > 9$	
3	6	13		not valid, $y > 9$	
4	8	10		not valid, $y > 9$	
5	1	16		not valid, $y > 9$	
6	3	13		not valid, $y > 9$	
7	5	10		not valid, $y > 9$	
8	7	7	8	valid, $P \le Q$	$(8, 8)$
9	9	4	2	not valid, $P > Q$	

   Therefore, this case produces one valid possibility for (P, Q): $(8, 8)$.

We have therefore examined all possible values for C. Therefore, there are six valid possibilities for (P, Q): $(0, 1)$, $(1, 9)$, $(2, 3)$, $(3, 6)$, $(6, 6)$ and $(8, 8)$.