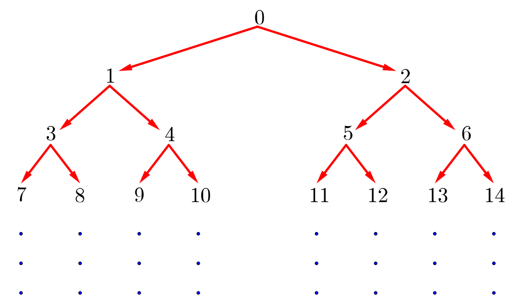
Problem of the Week
Problem E and Solution
How Did I Get Here?

Problem

Your friend writes down all of the integers starting from 0 as shown in the diagram to the right. Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R). What is the sequence of left (L) and right (R) movements to get to the number 1000 in the tree?

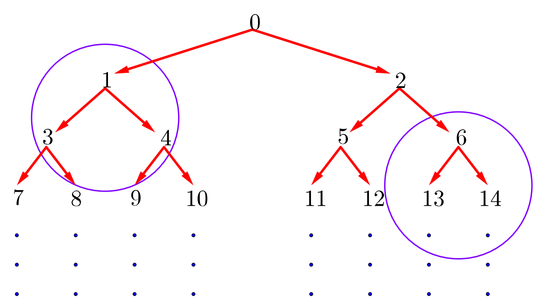


Solution

To begin, we will make an observation concerning the tree. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. So the final move to get to the number 1000 was a move to the right (R) since 1000 is an even number. Is there a general formula which can be used when asked to move right (R)? Is there a general formula which can be used when asked to move left (L)?

The diagram below has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing the same with 6 would not get us to 13 and 14.

As we go down the tree, each new row has twice as many numbers as the row above. Let's try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1 we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.



So it would appear that if we make a move left (L) from any number a in the tree, the resulting number is one more than twice the value of a. That is, a move left (L) from a takes us to the number $2a + 1$ in the tree.

It would appear that if we make a move right (R) from any number a in the tree, the resulting number is two more than twice the value of a. That is, a move right (R) from a takes us to the number $2a + 2$ in the tree.

The results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

To go back up the tree, we could determine an inverse move to undo a move to the left (L) or to the right (R).

A move to the left (L) takes a to an odd number n such that $n = 2a + 1$. Solving this equation for a we get $a = \frac{n-1}{2}$. A move to the right (R) takes a to an even number n such that $n = 2a + 2$. Solving this equation for a we get $a = \frac{n-2}{2}$.
We can now move from 1000 up the tree to zero using the appropriate undoing formula each time.

Initial Number	Odd or Even	Calculation	Previous Number
1000	even	$\frac{1000-2}{2}$	499
499	odd	$\frac{499-1}{2}$	249
249	odd	$\frac{249-1}{2}$	124
124	even	$\frac{124-2}{2}$	61
61	odd	$\frac{61-1}{2}$	30
30	even	$\frac{30-2}{2}$	14
14	even	$\frac{14-2}{2}$	6
6	even	$\frac{6-2}{2}$	2
2	even	$\frac{2-2}{2}$	0

The path to 1000 goes through the following numbers:

$$0 \to 2 \to 6 \to 14 \to 30 \to 61 \to 124 \to 249 \to 499 \to 1000$$

By looking at each successive number in the sequence in terms of its parity (odd or even) we can determine the required moves.

$$0 \xrightarrow{\text{R}} 2 \xrightarrow{\text{R}} 6 \xrightarrow{\text{R}} 14 \xrightarrow{\text{R}} 30 \xrightarrow{\text{L}} 61 \xrightarrow{\text{R}} 124 \xrightarrow{\text{L}} 249 \xrightarrow{\text{L}} 499 \xrightarrow{\text{R}} 1000$$