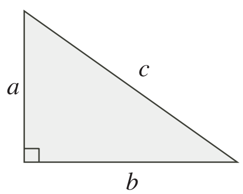


# Problem of the Week

## Problem E and Solution

## This Triangle is All Right

### Problem

A right-angled triangle has sides whose lengths are two-digit integers. The digits of the length of the hypotenuse are the reverse of the digits of the length of one of the other sides. Determine the smallest possible length of the hypotenuse.

## Solution

Let $x$ represent the tens digit of the hypotenuse such that $x$ is an integer from 1 to 9. Let $y$ represent the units digit of the hypotenuse such that $y$ is an integer from 1 to 9. Then the length of the hypotenuse is $10x + y$.

Since one of the sides has the same digits as the hypotenuse in reverse order, the length of this side is $10y + x$.

Let the third side be $z$ such that $z$ is a two digit integer.

$$\begin{aligned}
\text{Using the Pythagorean Theorem:} \quad (10y+x)^2 + z^2 &= (10x+y)^2 \\
\text{Expanding:} \quad 100y^2 + 20xy + x^2 + z^2 &= 100x^2 + 20xy + y^2 \\
\text{Rearranging:} \quad z^2 &= 99x^2 - 99y^2 \\
\text{Factoring:} \quad z^2 &= 99(x-y)(x+y)
\end{aligned}$$

Since $z^2$ is a perfect square, $99(x+y)(x-y)$ must also be a perfect square. But $99(x+y)(x-y) = 9(11)(x+y)(x-y)$. To be a perfect square $(x+y)(x-y)$ must be a multiple of 11 and contain a factor which is a perfect square. Since $x$ and $y$ are each integers from 1 to 9, $x - y$ cannot be 11, and so we must have $x + y = 11$. Again, since $x$ and $y$ are integers from 1 to 9, there are three possibilities for $x - y$ that give a perfect square; $x - y = 1$, $x - y = 4$ or $x - y = 9$. These are the three possibilities: $(x+y)(x-y) = 11(1)$ or $(x+y)(x-y) = 11(4)$ or $(x+y)(x-y) = 11(9)$.

If $x + y = 11$ and $x - y = 1$, we solve the system of equations obtaining $x = 6$ and $y = 5$. This gives a hypotenuse of $10x + y = 65$ and second side $10y + x = 56$. Then solving for $z$, $z^2 = 99(x+y)(x-y) = 99(11)(1) = 1089$ and $z = 33$. This solution is easily confirmed but is it the only solution?

If $x + y = 11$ and $x - y = 4$, we solve the system of equations obtaining $x = 7.5$ and $y = 3.5$. But $x$ and $y$ are both integers so this solution is inadmissible.

If $x + y = 11$ and $x - y = 9$, we solve the system of equations obtaining $x = 10$ and $y = 1$. But $x$ must be an integer from 1 to 9 so this solution is inadmissible.

Therefore the only solution is a right triangle with hypotenuse of length 65.