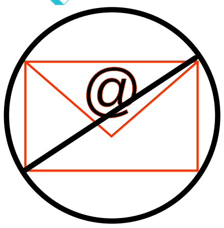

Problem of the Week

Problem E and Solution

No More Emails!

Problem

Jesse decided to stop answering emails indefinitely. However, he would look each day and count the number of emails but leave all of the emails unread. At the end of the first day there were 2 unread emails. At the end of the second day there were 5 unread emails, an increase of 3 emails from the previous day. At the end of the third day there were 12 unread emails, an increase of 7 emails from the previous day. At the end of the fourth day there were 23 unread emails, an increase of 11 emails from the previous day.

Jesse noticed that the increases from day to day so far formed an arithmetic sequence. (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3,5,7,9 is an arithmetic sequence with four terms and constant difference 2.)

Jesse's email provider allows him to have a maximum of 6330 unread emails in his inbox. Once this limit is reached his account is will no longer be able to receive emails. Supposing that the pattern of the daily increases continues, for how many days can Jesse receive emails before new emails are no longer allowed?

Solution

Solution 1

Let the number of unread emails on day i be a_i. That is, on day 1, $a_1 = 2$; on day 2, $a_2 = 5$; on day 3, $a_3 = 12$; and on day 4, $a_4 = 23$. We want to determine the day number, n, so that $a_n \leq 6330$.

Let the arithmetic sequence of daily differences be $t_1,\ t_2,\ t_3,\ \cdots\ ,\ t_n,\ \cdots$.

Then $t_1 = a_2 - a_1 = 5 - 2 = 3$, $t_2 = a_3 - a_2 = 12 - 5 = 7$, $t_3 = a_4 - a_3 = 23 - 12 = 11$ and $t_4 = a_5 - a_4 = 38 - 23 = 15$. Since the new sequence is arithmetic, the difference between consecutive terms is constant. Therefore the constant difference is $d = t_2 - t_1 = 7 - 3 = 4$. To generate t_5 we add the constant difference 4 to t_4. So $t_5 = t_4 + 4 = 15 + 4 = 19$. Then $a_6 = a_5 + t_5 = 38 + 19 = 57$. This means that the number of unread emails on day 6 is 57. We could continue generating daily differences and new daily totals until the maximum is reached.

Let's take a closer look at the number of unread emails.

$a_1 = 2$
$a_2 = a_1 + t_1 = 2 + 3 = 5$
$a_3 = a_2 + t_2 = 2 + 3 + 7 = 12$
$a_4 = a_3 + t_3 = 2 + 3 + 7 + 11 = 23$
$a_5 = a_4 + t_4 = 2 + 3 + 7 + 11 + 15 = 38$
$a_6 = a_5 + t_5 = 2 + 3 + 7 + 11 + 15 + 19 = 57$

We can present the information in table form.

Day Number n	Number of emails a_n	Daily Differences t_n	Difference of the Daily Differences d
1	2	3	4
2	5	7	4
3	12	11	4
4	23	15	4
5	38	19	
6	57		

Since the second difference is constant we can represent the general term of the first sequence with a quadratic function. Let $a_n = pn^2 + qn + r$.

For $n = 1$, $a_1 = 2 = p(1)^2 + q(1) + r$. $\therefore\ p + q + r = 2$. (1)
For $n = 2$, $a_2 = 5 = p(2)^2 + q(2) + r$. $\therefore 4p + 2q + r = 5$. (2)
For $n = 3$, $a_3 = 12 = p(3)^2 + q(3) + r$. $\therefore 9p + 3q + r = 12$. (3)

Subtracting (1) from (2), $3p + q = 3$. (4)
Subtracting (2) from (3), $5p + q = 7$. (5)

Subtracting (4) from (5), $2p = 4$ and $p = 2$ follows.
Substituting $p = 2$ into (4), $3(2) + q = 3$ and $q = -3$ follows.

Substituting $p = 2,\ q = -3$ into (1), $2 - 3 + r = 2$ and $r = 3$ follows.

$\therefore a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.

We want to find the value of n so that $2n^2 - 3n + 3 \leq 6330$. If we solve $2n^2 - 3n + 3 = 6330$, then we can find the largest value of n. Rearranging the equation, $2n^2 - 3n - 6327 = 0$. Factoring, we obtain $(2n + 111)(n - 57) = 0$. Solving for n, $n = -55.5$ or $n = 57$. Since n is the day number, $n > 0$ and $n = -55.5$ is inadmissible. It follows that on day 57 there would be exactly 6330 unread emails. On day 58 no new emails would be allowed.

(Instead of factoring we could have used the quadratic formula to get $n = 57$ or $n = -55.5$.)

Solution 2

This solution approaches the problem in a more general sense.

As in Solution 1, let the number of unread emails on day i be a_i and let the sequence of daily differences be $t_1,\ t_2,\ t_3,\ \cdots\ ,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the sequence of daily differences is arithmetic, the constant difference is $d = 7 - 3 = 4$. We can generate more terms: $t_3 = 11,\ t_4 = 15,\ t_5 = 19,\ \cdots$.

Each term in the new sequence is the difference between consecutive terms of the original sequence.

$$\begin{array}{rcccl}
\cancel{a_2} & - & a_1 & = & t_1 \\
\cancel{a_3} & - & \cancel{a_2} & = & t_2 \\
\cancel{a_4} & - & \cancel{a_3} & = & t_3 \\
\vdots & & \vdots & & \vdots \\
\cancel{a_{n-2}} & - & \cancel{a_{n-3}} & = & t_{n-3} \\
\cancel{a_{n-1}} & - & \cancel{a_{n-2}} & = & t_{n-2} \\
a_n & - & \cancel{a_{n-1}} & = & t_{n-1} \\
\hline
\text{Adding,}\quad a_n & - & a_1 & = & t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1} = S_{n-1}. \quad (1)
\end{array}$$

To find the sum $S_{n-1} = t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1}$ we can use the formula $S_n = \dfrac{n}{2}[2a + (n-1)d]$ with $a = 3$ and $d = 4$ for $(n-1)$ terms. So

$$\begin{aligned}
S_{n-1} &= t_1 + t_2 + t_3 + \ \cdots\ + t_{n-2} + t_{n-1} \\
&= \frac{n-1}{2}[2a + ((n-1)-1)d] \\
&= \frac{n-1}{2}[2(3) + ((n-1)-1)(4)] \\
&= \frac{n-1}{2}[6 + (n-2)(4)] \\
&= \frac{n-1}{2}[6 + 4n - 8] \\
&= \frac{n-1}{2}[4n - 2] \\
&= (n-1)(2n-1)
\end{aligned}$$

From (1) above, $a_n - a_1 = S_{n-1}$ so $a_n = S_{n-1} + a_1 = (n-1)(2n-1) + 2 = 2n^2 - 3n + 3$.

$\therefore a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.

Then, as in solution 1, we would find the value of n so that $2n^2 - 3n + 3 \leq 6330$. Without repeating the work here, on the 57^{th} day there would be 6330 unread emails. No more emails would be received after day 57.

Solution 3

This solution approaches the problem by examining what the n^{th} term, a_n, looks like.

As in Solution 1, let the number of unread emails on day i be a_i and let the sequence of daily differences be $t_1,\ t_2,\ t_3,\ \cdots\ ,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the new sequence is arithmetic, the constant difference is $d = 7 - 3 = 4$. Using the formula for the general term of an arithmetic sequence, $t_n = a + (n-1)d$, the general term of the sequence of differences is $t_n = (3) + (n-1)(4) = 3 + 4n - 4 = 4n - 1$.

To generate the original sequence we start with the first term and add more terms from the arithmetic sequence of differences. For example,

$$a_1 = 3,$$
$$a_2 = 2 + t_1 = 2 + [4(1) - 1] = 2 + 3 = 5, \text{and}$$
$$a_3 = 2 + t_1 + t_2 = 2 + [4(1) - 1] + [4(2) - 1] = 2 + 3 + 7 = 12$$

$$\begin{aligned}
\text{So,}\ \ a_n &= 2 + t_1 + t_2 + t_3 + \ \cdots\ + t_{n-1} \\
&= 2 + [4(1) - 1] + [4(2) - 1] + [4(3) - 1] + \ \cdots\ + [4(n-1) - 1] \\
&= 2 + [4(1) + 4(2) + 4(3) + \ \cdots\ + 4(n-1)] + (n-1)(-1) \\
&= 2 + 4[1 + 2 + 3 + \ \cdots\ + (n-1)] - n + 1 \\
&= 3 - n + 4\left[\frac{(n-1)(n)}{2}\right] \qquad\qquad (1) \\
&= 3 - n + 2(n^2 - n) \\
&= 3 - n + 2n^2 - 2n \\
&= 2n^2 - 3n + 3
\end{aligned}$$

An explanation is provided here to show how (1) above is obtained. $1 + 2 + 3 + \ \cdots\ + (n-1)$ is an arithmetic sequence with $b_1 = 1$, $b_{n-1} = (n-1)$ and the number of terms $(n-1)$ terms. So to get to (1) above we use the formula for the sum of the terms of an arithmetic sequence, $S_n = n\left(\dfrac{b_1 + b_n}{2}\right)$, to obtain

$$S_{n-1} = (n-1)\left(\frac{1 + (n-1)}{2}\right) = (n-1)\left(\frac{n}{2}\right) = \frac{(n-1)(n)}{2}$$

$\therefore a_n = 2n^2 - 3n + 3$ is the general term of the given sequence in terms of n.

Then, as in solution 1, we would find the value of n so that $2n^2 - 3n + 3 \leq 6330$. Without repeating the work here, on the 57^{th} day there would be 6330 unread emails. No more emails would be received after day 57.