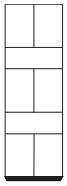
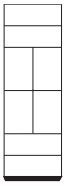

Problem of the Week
Problem D and Solution
A Pane - Full Experience

Problem

A contractor is building new houses. Each house has a large window with width 2 feet and height 8 feet to be made up of eight identical 1 foot by 2 feet glass panes. The contractor wants each house to look slightly different. The window size is fixed, but he plans on arranging the glass panes in the windows so that no two windows have the same configuration. Two possible arrangements of the glass panes are given below. Assuming that the glass panes cannot be cut, how many different arrangements can be made?

Solution

Let's consider the ways that the glass panes can be arranged. First, notice that there must always be an even number of window panes that have a vertical orientation (standing on end).

- All glass panes are horizontal, none are vertical

  This can only be done one way.

- Six glass panes are horizontal, two are vertical

  There could be 0, 1, 2, 3, 4, 5, or 6 horizontal panes below the two vertical panes. So there are 7 ways that six glass panes are horizontal and two are vertical.

- Four glass panes are horizontal, four are vertical

  We need to consider sub cases:

  - Case 1: There are no horizontal panes between the vertical panes.
    There could be 0, 1, 2, 3, or 4 horizontal panes below the four vertical panes. So there are 5 ways that four glass panes are horizontal, four are vertical and there are no horizontal panes between the vertical panes.
  - Case 2: There is one horizontal pane between the vertical panes.
    There could be 0, 1, 2, or 3 horizontal panes below the bottom two vertical panes. So there are 4 ways that four glass panes are horizontal, four are vertical and there is one horizontal pane between the vertical panes.
  - Case 3: There are two horizontal panes between the vertical panes.
    There could be 0, 1 or 2 horizontal panes below the bottom two vertical panes. So there are 3 ways that four glass panes are horizontal, four are vertical and there are two horizontal panes between the vertical panes.

- Case 4: There are three horizontal panes between the vertical panes.
  There could be 0 or 1 horizontal pane below the bottom two vertical panes. So there are 2 ways that four glass panes are horizontal, four are vertical and there are three horizontal panes between the vertical panes.
- Case 5: There are four horizontal panes between the vertical panes.
  Since there are no more glass panes left, there can only can be 0 horizontal panes below the bottom two vertical panes. So there is 1 way that four glass panes are horizontal, four are vertical and there are four horizontal panes between the vertical panes.

- Two glass panes are horizontal, six are vertical
  - Case 1: There are no vertical panes between the horizontal panes.
    There could be 0, 2, 4 or 6 vertical panes below the bottom horizontal pane. So there are 4 ways that two glass panes are horizontal, six are vertical and there are no vertical panes between the horizontal panes.
  - Case 2: There are two vertical panes between the horizontal panes.
    There could be 0, 2 or 4 vertical panes below the bottom horizontal pane. So there are 3 ways that two glass panes are horizontal, six are vertical and there are two vertical panes between the horizontal panes.
  - Case 3: There are four vertical panes between the horizontal panes.
    There could be 0 or 2 vertical panes below the bottom horizontal pane. So there are 2 ways that two glass panes are horizontal, six are vertical and there are four vertical panes between the horizontal panes.
  - Case 4: There are six vertical panes between the horizontal panes.
    Since there are no more glass panes left, there can only can be 0 vertical panes below the bottom horizontal pane. So there is 1 way that two glass panes are horizontal, six are vertical and there are six vertical panes between the horizontal panes.

- All glass panes are vertical, none are horizontal

  This can only be done one way.

Therefore, the total number of different configurations of the glass panes is

$$1 + 7 + (5 + 4 + 3 + 2 + 1) + (4 + 3 + 2 + 1) + 1 = 34.$$

So the contractor can build 34 houses before he has to start duplicating window patterns.