Problem of the Week

Problem E and Solution

Pick a Card - Any Card

Problem

Luke has a unique deck of cards. Each card in the deck has a positive three digit number on it. There is exactly one card in the deck for every three digit positive integer. The deck is shuffled. Luke selects a card at random from the deck and calculates the sum of the digits. If the sum of the digits is 15, the card is a winner. Determine the probability that Luke selects a winning card.

Solution

To begin, we need to determine the number of cards in the deck. Since there is a card for each three digit positive integer there are 900 cards in the deck. We must be careful calculating this number. There are 999 positive integers less than 1000. Of this set, 90 are two digit numbers and 9 are single digit numbers. Therefore there are $999 - 90 - 9 = 900$ three digit positive integers.

Next we need to determine the digit combinations on a card that have a sum of 15. We will determine the possibilities using cases. Then we will look at the specific groups of numbers that sum to 15 to count the number of cards produced from each group.

1. One of the digits on the card is a zero. The other two digits on the card must add to 15. This leads to two groups of numbers: (0,6,9) and (0,7,8).
2. One of the digits on the card is a one but the number does not contain a zero. The other two digits on the card must add to 14. This leads to three groups of numbers: (1,5,9), (1,6,8) and (1,7,7).
3. One of the digits on the card is a two but the number does not contain a zero or one. The other two digits on the card must add to 13. This leads to three groups of numbers: (2,4,9), (2,5,8) and (2,6,7).
4. One of the digits on the card is a three but the number does not contain a zero, one or two. The other two digits on the card must add to 12. This leads to four groups of numbers: (3,3,9), (3,4,8), (3,5,7) and (3,6,6).
5. One of the digits on the card is a four but the number does not contain a zero, one, two or three. The other two digits on the card must add to 11. This leads to two groups of numbers: (4,4,7) and (4,5,6).
6. One of the digits on the card is a five but the number does not contain a zero, one, two, three or four. The other two digits on the card must add to 10. This leads to only one group of numbers: (5,5,5).

Now that we have the groups of numbers, we can determine the number of cards that can be created from each group of three numbers. We will do this again with cases: groups containing a zero, groups containing three distinct numbers but not zero, groups containing exactly two numbers the same but not zero, and groups containing three numbers the same but not zero.

1. One of the numbers on the card is zero. Earlier we found that there were two such groups: (0,6,9) and (0,7,8). This is a special case since zero cannot appear in the number as the hundreds digit for the number to be a three digit number. For each of the two groups of numbers, the zero can be placed in two ways, in the tens digit or the units digit. Once the zero is placed, the other two numbers can be placed in the remaining two spots in two ways. Each group can form $2 \times 2 = 4$ three digit numbers. Since there are two groups, there are $2 \times 4 = 8$ cards in the deck that contain a zero and add to 15.

2. All three digits on the card are different but the number does not contain a zero. From the earlier cases there are eight groups in which all three numbers are different: (1,5,9), (1,6,8), (2,4,9), (2,5,8), (2,6,7), (3,4,8), (3,5,7), and (4,5,6). For each of these groups, the hundreds digit can be filled three ways. For each of these three choices for hundreds digit, the tens digit can be filled two ways. Once the hundreds digit and tens digit are selected the units digit must get the third number. So each group can form $3 \times 2 = 6$ different numbers. Since there are eight groups, there are $8 \times 6 = 48$ cards in the deck that contain three different digits other than zero and add to 15.

3. Two of the digits on the card are the same and the number does not contain a zero. From the earlier cases, there are four groups of numbers in which exactly two of the numbers in the group are the same: (1,7,7), (3,3,9), (3,6,6), and (4,4,7). For each of these groups, the unique number can be placed in one of three spots. Once the unique number is placed the other two numbers must go in the remaining two spots. So each group can form 3 different numbers. Since there are four groups, there are $4 \times 3 = 12$ cards in the deck that do not contain a zero but contain two digits the same and add to 15.

4. The three digits on the card are the same. From the earlier cases we discovered only one such group: (5,5,5). Only one card can be produced using the numbers from this group.

Combining the counts from the above four cases, there are $8 + 48 + 12 + 1 = 69$ cards in the deck with a digit sum of 15. Therefore the probability that Luke selects a winning card is $\frac{69}{900} = \frac{23}{300}$. This translates to approximately a 7.7% chance of winning.

A game is considered *fair* if there is close to a 50% chance of winning. This game is definitely not fair. If you changed the game to "if the card chosen has a sum that is divisible by 5", there is a 20% chance of winning. This is better but still not fair. If you changed the game to "if the card chosen has a sum less than 14", there is a 46% chance of winning. This game is much fairer but not really fun.

Can you create a game using this specific deck of cards that is reasonably fair and "fun" to play?