Problem of the Week
Problem D and Solution
Several Areas of Interest

Problem

The area of $\triangle ACD$ is twice the area of square $BCDE$. Square $BCDE$ has sides of length 12 cm. AD intersects BE at F. Determine the area of quadrilateral $BCDF$.

Solution

The area of square $BCDE = 12 \times 12 = 144$ cm^2. The area of $\triangle ACD$ equals twice the area of square $BCDE$. Therefore area $\triangle ACD = 288$ cm^2.

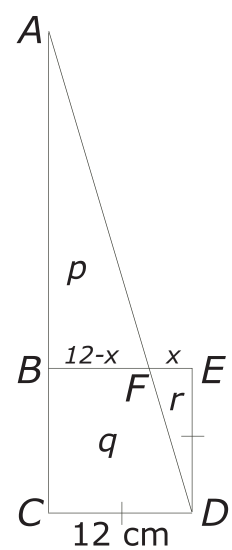
The area of a triangle is calculated using the formula $base \times height \div 2$. It follows that:

$$\begin{aligned} \text{Area } \triangle ACD &= (CD) \times (AC) \div 2 \\ 288 &= 12\ AC \div 2 \\ 288 &= 6\ AC \\ 48 \text{ cm} &= AC \end{aligned}$$

But $AC = AB + BC$ so $48 = AB + 12$ and it follows that $AB = 36$ cm.

From this point, we will present three different approaches to obtaining the required area.

Method 1



Let the area of $\triangle ABF$ be p, quadrilateral $BCDF$ be q and $\triangle DEF$ be r. Let $FE = x$. Since $BE = 12$, $BF = 12 - x$.

Area $\triangle ACD = 288 = p + q$ (1) and area square $BCDE = 144 = r + q$ (2).
Subtracting (2) from (1), $p - r = 144$ and $p = r + 144$ follows.

Area $\triangle ABF = p = r + 144 = (AB)(BF) \div 2 = 36(12 - x) \div 2 = 18(12 - x)$

$$\begin{aligned} \therefore r + 144 &= 18(12 - x) \\ r &= 216 - 18x - 144 \\ r &= 72 - 18x \qquad (3) \end{aligned}$$

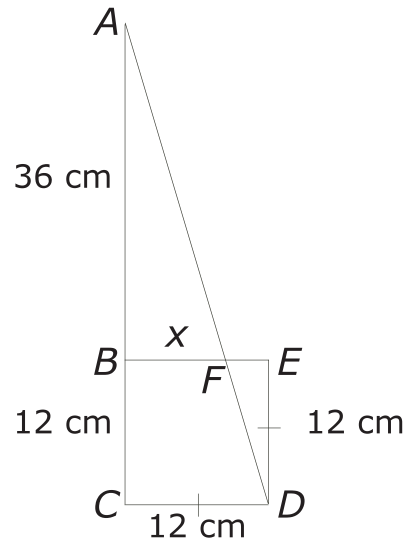
Area $\triangle DEF = r = (DE)(EF) \div 2 = 12x \div 2 = 6x$. $\therefore r = 6x \quad (4)$
Using (3) and (4), since $r = r$, $6x = 72 - 18x$. Solving, $x = 3$ and $r = 18$ follow.

$$\begin{aligned} \text{The area of quadrilateral } BCDF &= \text{area of square } BCDE - \text{ area } \triangle DEF \\ &= 144 - r \\ &= 144 - 18 \\ &= 126 \text{ cm}^2 \end{aligned}$$

Therefore the area of quadrilateral $BCDF$ is 126 cm^2.

See alternative methods on the next page.

Method 2



Let x represent the length of BF.

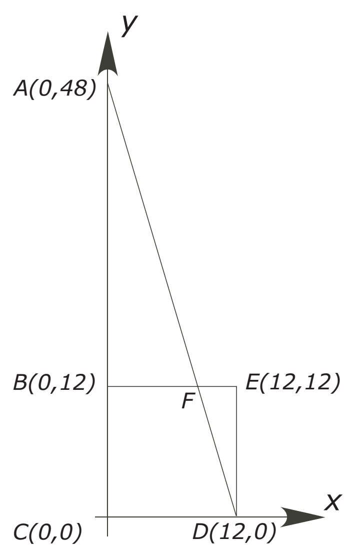
In $\triangle ABF$ and $\triangle ACD$, $\angle A$ is common and $\angle ABF = \angle ACD = 90°$. It follows that $\triangle ABF$ is similar to $\triangle ACD$ and $\frac{AB}{AC} = \frac{BF}{CD}$. Therefore $\frac{36}{48} = \frac{x}{12}$ and $x = 9$.

Since $BCDE$ is a square, $BE \parallel CD$. Then quadrilateral $BCDF$ is a trapezoid.

$$\begin{aligned}\text{Area Trapezoid } BCDF &= (BC)(BF + CD) \div 2\\ &= 12(9 + 12) \div 2\\ &= 6(21)\\ &= 126 \text{ cm}^2\end{aligned}$$

Therefore the area of quadrilateral $BCDF$ is 126 cm^2.

Method 3



Position the diagram so that C is at the origin, A and B are on the positive y-axis and D is on the positive x-axis. C has coordinates (0,0), B has coordinates (0,12),A has coordinates (0,48), and D has coordinates (12, 0).

Find the equation of the line containing AD. The slope is $\frac{-48}{12} = -4$ and the line crosses the y-axis at A so the y-intercept is 48. Therefore the equation is $y = -4x + 48$.

The line containing BE is horizontal crossing the y-axis at B. The equation of the line containing BE is $y = 12$.

F is the intersection of $y = -4x + 48$ and $y = 12$. Since $y = y$ at the point of intersection, $-4x + 48 = 12$ and $x = 9$ follows. Therefore F has coordinates (9,12) and $BF = 9$ cm.

Since $BCDE$ is a square, $BE \parallel CD$. Then quadrilateral $BCDF$ is a trapezoid.

$$\begin{aligned}\text{Area Trapezoid } BCDF &= (BC)(BF + CD) \div 2\\ &= 12(9 + 12) \div 2\\ &= 6(21)\\ &= 126 \text{ cm}^2\end{aligned}$$

Therefore the area of quadrilateral $BCDF$ is 126 cm^2.