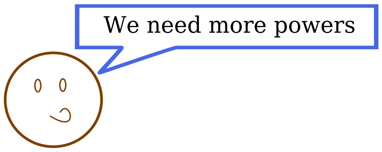

Problem of the Week

Problem E and Solution

Powerful Factors

Problem

The product of the integers 1 to 64 can be written in an abbreviated form as 64! and we say 64 *factorial*. So $64! = 64 \times 63 \times 62 \times \cdots \times 3 \times 2 \times 1$. In general, the product of the positive integers 1 to m is

$$m! = m \times (m-1) \times (m-2) \times \cdots \times 3 \times 2 \times 1.$$

Determine the largest positive integer value of n so that 64! is divisible by 2016^n.

Solution

Let $P = 64!$.

The prime factorization of 2016 is $2^5 \times 3^2 \times 7$. We must determine how many times the factors $2^5 \times 3^2 \times 7$ are repeated in the factorization of P.

First we will count the number of factors of 2 in P by looking at the 32 even numbers. Each of the numbers $\{2, 4, 6, \cdots, 60, 62, 64\}$ contains a factor of 2. That is a total of 32 factors of 2.

Dividing the even numbers in $\{2, 4, 6, \cdots, 60, 62, 64\}$ by 2, we obtain the numbers $\{1, 2, 3, \cdots, 30, 31, 32\}$. This list contains 16 even numbers so we gain another 16 factors of 2 bringing the total to $32 + 16 = 48$.

Dividing the even numbers in $\{1, 2, 3, \cdots, 30, 31, 32\}$ by 2, we obtain the numbers $\{1, 2, 3, \cdots, 14, 15, 16\}$. This list contains 8 even numbers so we gain another 8 factors of 2 bringing the total to $48 + 8 = 56$.

Dividing the even numbers in $\{1, 2, 3, \cdots, 14, 15, 16\}$ by 2, we obtain the numbers $\{1, 2, 3, \cdots, 6, 7, 8\}$. This list contains 4 even numbers so we gain another 4 factors of 2 bringing the total to $56 + 4 = 60$.

Dividing the even numbers in $\{1, 2, 3, \cdots, 6, 7, 8\}$ by 2, we obtain the numbers $\{1, 2, 3, 4\}$. This list contains 2 even numbers so we gain another 2 factors of 2 bringing the total to $60 + 2 = 62$.

Finally, dividing the even numbers in $\{1, 2, 3, 4\}$ by 2, we obtain the numbers $\{1, 2\}$. This list contains 1 even number so we gain another factor of 2 bringing the total to $62 + 1 = 63$. So when P is factored there are 63 factors of 2. In fact, the largest power of 2 that P is divisible by is 2^{63}.

Next we will count the number of factors of 3 in P by looking at the 21 multiples of 3, namely $\{3, 6, 9, \cdots, 57, 60, 63\}$. Each of these 21 numbers contains a factor of 3.

Dividing the multiples of 3 in $\{3, 6, 9, \cdots, 57, 60, 63\}$ by 3, we obtain the numbers $\{1, 2, 3, \cdots, 19, 20, 21\}$. This list contains 7 multiples of 3 so we gain another 7 factors of 3 bringing the total to $21 + 7 = 28$.

Dividing the multiples of 3 in $\{1, 2, 3, \cdots, 19, 20, 21\}$ by 3, we obtain the numbers $\{1, 2, 3, 4, 5, 6, 7\}$. This list contains 2 multiples of 3 so we gain another 2 factors of 3 bringing the total to $28 + 2 = 30$.

Dividing the multiples of 3 in $\{1, 2, 3, 4, 5, 6, 7\}$ by 3, we obtain the numbers $\{1, 2\}$. This list contains no additional multiples of 3 so when P is factored there is a total of 30 factors of 3. In fact, the largest power of 3 that P is divisible by is 3^{30}.

Finally we will count the number of factors of 7 in P by looking at the 9 multiples of 7, namely $\{7, 14, 21, \cdots, 49, 56, 63\}$. Each of these 9 numbers contains a factor of 7.

Dividing the multiples of 7 in $\{7, 14, 21, \cdots, 49, 56, 63\}$ by 7, we obtain the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. This list contains 1 multiple of 7 so we gain another factor of 7 bringing the total to $9 + 1 = 10$.

Dividing the multiples of 7 in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by 7, we obtain the number $\{1\}$. The number $\{1\}$ is not a multiple of 7 so we gain no additional factors of 7. When P is factored there is a total of 10 factors of 7. In fact, the largest power of 7 that P is divisible by is 7^{10}.

From the 63 factors of 2, 30 factors of 3 and 10 factors of 7, we want to create as many powers of $2^5 \times 3^2 \times 7$ as possible.

$$\begin{aligned}
\text{So, } P \text{ is divisible by } 2^{63} \times 3^{30} \times 7^{10} &= 2^{13} \times 2^{50} \times 3^{10} \times 3^{20} \times 7^{10} \\
&= 2^{13} \times 3^{10} \times (2^5)^{10} \times (3^2)^{10} \times 7^{10} \\
&= 2^{13} \times 3^{10} \times (2^5 \times 3^2 \times 7)^{10} \\
&= 2^{13} \times 3^{10} \times (2016)^{10}
\end{aligned}$$

$\therefore$ P is divisble by 2016^{10} and the largest value of n is 10. (Since all of the powers of 7 have been used, none remain to combine with any of the remaining 2's and 3's to form any additional factors of 2016.)