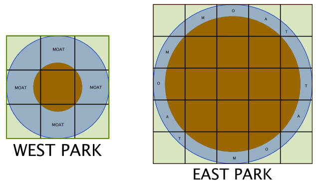

Problem of the Week

Problem D and Solution

Moats for Boats

Problem

A municipality has the opportunity to upgrade one of two square parks by building a circular moat for paddle boats in one of them. The surface of the moat is the area outside of a smaller circle and inside a second larger concentric circle. (From above, the moat looks somewhat like a donut.) West Park is 300 m by 300 m and East Park is 500 m by 500 m. Both parks are divided by horizontal and vertical grid lines spaced 100 m apart, as shown, creating nine and twenty five equal squares, respectively. Both moat designs shown are based on municipal park rules:

- the outer edge of the moat must touch the midpoint of each of the four outer sides of the park; and
- the inner edge of the moat must pass through the four corners of the largest square totally inside the park created by the grid lines.

The city is interested in conserving water and will choose the plan which uses less water. Assuming that both moats will have an equal and constant depth, which Park will be chosen for the moat?

Solution

To find the volume of each moat we need to find the area of each donut-like ring and multiply by the depth of the water. Since the depth of the water in each moat is the same and is constant, we need only compare the areas to determine which one is larger.

For the West Park moat, let the diameter of the inner circle be d_1, the diameter of the larger circle be D_1, the radius of the inner circle be r_1 and the radius of the larger circle be R_1.

For the East Park moat, let the diameter of the inner circle be d_2, the diameter of the larger circle be D_2, the radius of the inner circle be r_2 and the radius of the larger circle be R_2.

The calculations for West Park will be shown on the left and the calculations for East Park will be shown on the right.

West Park

The diameter of the inner circle is the length of the diagonal of the contained 100 m by 100 m square. Using the Pythagorean Theorem,

$$\begin{aligned} d_1 &= \sqrt{100^2 + 100^2} \\ &= \sqrt{20\,000} \\ &= 100\sqrt{2} \text{ m} \\ r_1 &= \frac{1}{2} d_1 \\ &= 50\sqrt{2} \text{ m} \end{aligned}$$

East Park

The diameter of the inner circle is the length of the diagonal of the contained 300 m by 300 m square. Using the Pythagorean Theorem,

$$\begin{aligned} d_2 &= \sqrt{300^2 + 300^2} \\ &= \sqrt{180\,000} \\ &= 300\sqrt{2} \text{ m} \\ r_2 &= \frac{1}{2} d_2 \\ &= 150\sqrt{2} \text{ m} \end{aligned}$$

West Park

The diameter of the outer circle is the width of the 300 m by 300 m square. It follows that

$$\begin{aligned} D_1 &= 300 \text{ m} \\ R_1 &= \frac{1}{2}D_1 \\ &= 150 \text{ m} \end{aligned}$$

East Park

The diameter of the outer circle is the width of the 500 m by 500 m square. It follows that

$$\begin{aligned} D_2 &= 500 \text{ m} \\ R_2 &= \frac{1}{2}D_2 \\ &= 250 \text{ m} \end{aligned}$$

The surface area of each moat can be determined by subtracting the area of the inner circle from the area of the outer circle in each case.

Let A_1 be the surface area of the West Park moat and A_2 be the surface area of the East Park moat.

West Park

$$\begin{aligned} A_1 &= \pi(R_1)^2 - \pi(r_1)^2 \\ &= \pi(150)^2 - \pi(50\sqrt{2})^2 \\ &= 22500\pi - 5000\pi \\ &= 17500\pi \text{ m}^2 \end{aligned}$$

East Park

$$\begin{aligned} A_1 &= \pi(R_2)^2 - \pi(r_2)^2 \\ &= \pi(250)^2 - \pi(150\sqrt{2})^2 \\ &= 62500\pi - 45000\pi \\ &= 17500\pi \text{ m}^2 \end{aligned}$$

This may be a surprising result. Both moats have equal surface areas. Since the depths of the moats are equal and uniform, the volume of water in each moat will be the same.

The municipality can choose either moat and base their decision on other things.

For Further Thought:

North Park is a 700 m by 700 m park. If a moat were constructed in a similar manner to the moat in either East Park or West Park, how would the volume of the North Park moat compare? Can you explain what is happening here?