

# Problem of the Week

## Problem E and Solution

### Analytic Geometry Tools Required

#### Problem

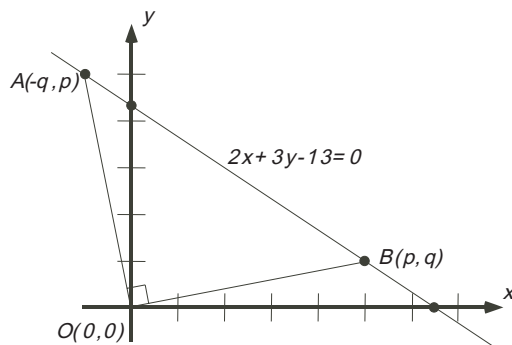
$OAB$  is an isosceles right triangle with vertex  $O$  at the origin  $(0,0)$ , vertices  $A$  and  $B$  on the line  $2x + 3y - 13 = 0$ , and  $\angle AOB = 90^\circ$ . Determine the area of  $\triangle OAB$ .

#### Solution

##### Solution 1

In analytic geometry problems, a representative diagram is important and often provides clues for the solution of the problem. The following diagram has the given information plus a couple of pieces of information that will be justified now.

Let  $B$  have coordinates  $(p, q)$ . The slope of  $OB = \frac{q}{p}$ . Since  $\angle AOB = 90^\circ$ ,  $OB \perp OA$  and the slope of  $OA$  is the negative reciprocal of  $OB$ . Therefore the slope of  $OA = \frac{p}{-q}$ . Since the triangle is isosceles,  $OA = OB$  and it follows that the coordinates of  $A$  are  $(-q, p)$ . (We can verify this by finding the length of  $OA$  and the length of  $OB$  and showing that both are equal to  $\sqrt{p^2 + q^2}$ .)



Since  $B(p, q)$  is on the line  $2x + 3y - 13 = 0$ , it satisfies the equation of the line.

$$\therefore 2p + 3q - 13 = 0 \quad (1)$$

Since  $A(-q, p)$  is on the line  $2x + 3y - 13 = 0$ , it satisfies the equation of the line.

$$\therefore -2q + 3p - 13 = 0 \text{ or } 3p - 2q - 13 = 0 \quad (2)$$

Since we have two equations and two unknowns, we can use elimination to solve for  $p$  and  $q$ .

$$(1) \times 2 : \quad 4p + 6q - 26 = 0$$

$$(2) \times 3 : \quad 9p - 6q - 39 = 0$$

$$\text{Adding, we obtain :} \quad 13p - 65 = 0$$

$$\therefore p = 5$$

$$\text{Substituting in (1) :} \quad 10 + 3q - 13 = 0$$

$$3q = 3$$

$$\therefore q = 1$$

The point  $B$  is  $(5, 1)$  and the length of  $OB = \sqrt{5^2 + 1^2} = \sqrt{26}$ . Since  $OA = OB$ ,  $OA = \sqrt{26}$ .  $\triangle AOB$  is a right triangle so we can use  $OB$  as the base and  $OA$  as the height in the formula for the area of a triangle. Then the area of  $\triangle AOB = \frac{OA \times OB}{2} = \frac{\sqrt{26} \sqrt{26}}{2} = 13$ .

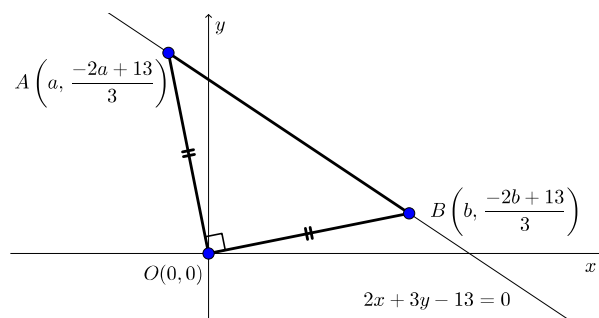
$\therefore$  the area of  $\triangle AOB$  is 13 units<sup>2</sup>.





## Solution 2

By rearranging the given equation for the line, we obtain  $y = \frac{-2x+13}{3}$ . Since the points  $A$  and  $B$  are on the line, their coordinates satisfy the equation of the line. If  $A$  has  $x$ -coordinate  $a$ , then  $A$  has coordinates  $(a, \frac{-2a+13}{3})$ . If  $B$  has  $x$ -coordinate  $b$ , then  $B$  has coordinates  $(b, \frac{-2b+13}{3})$ . Since  $\triangle OAB$  is isosceles, we know that  $OA = OB$ . Then



$$\begin{aligned}
 OA^2 &= OB^2 \\
 a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
 a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
 \text{Multiplying by 9 : } 9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
 \text{Simplifying : } 13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
 \text{Rearranging : } 13a^2 - 13b^2 - 52a + 52b &= 0 \\
 \text{Dividing by 13 : } a^2 - b^2 - 4a + 4b &= 0 \\
 \text{Factoring pairs : } (a+b)(a-b) - 4(a-b) &= 0 \\
 \text{Common factoring : } (a-b)(a+b-4) &= 0
 \end{aligned}$$

Solving,  $a = b$  or  $a = 4 - b$ . Since  $A$  and  $B$  are distinct points,  $a \neq b$ . Therefore,  $a = 4 - b$ . We can rewrite  $A(a, \frac{-2a+13}{3})$  as  $A(4-b, \frac{-2(4-b)+13}{3})$  which simplifies to  $A(4-b, \frac{2b+5}{3})$ .

Since  $\triangle OAB$  is a right triangle, we can use the Pythagorean theorem and  $AB^2 = OA^2 + OB^2$  follows. But  $OA = OB$  so this can be written  $AB^2 = 2OB^2$ .

$$\begin{aligned}
 AB^2 &= 2OB^2 \\
 (b - (4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2 \left[ b^2 + \left(\frac{-2b+13}{3}\right)^2 \right] \\
 (2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2 \left[ b^2 + \frac{4b^2 - 52b + 169}{9} \right] \\
 4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9} \\
 \text{Multiplying by 9 : } 36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
 \text{Simplifying : } 52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
 \text{Rearranging : } 26b^2 - 104b - 130 &= 0 \\
 \text{Dividing by 26 : } b^2 - 4b - 5 &= 0 \\
 \text{Factoring : } (b-5)(b+1) &= 0
 \end{aligned}$$

It follows that  $b = 5$  or  $b = -1$ . When  $b = 5$ , the point  $A$  is  $(-1, 5)$  and the point  $B$  is  $(5, 1)$ . When  $b = -1$ , the point  $A$  is  $(5, 1)$  and the point  $B$  is  $(-1, 5)$ . There are only two points. The area work shown in Solution 1 follows from here. Therefore, the area of  $\triangle OAB$  is 13 units<sup>2</sup>.

