Problem of the Week
Problem D and Solution
Pick a Box, Any Box

Problem

A room contains 150 boxes. The boxes are identical except for colour. Each box contains a card which has one of three messages printed on it. In 78 of the boxes the card says "No Prize". In 66 of the boxes the card says "Winner \$20". And in 6 of the boxes the card says "Winner \$100". Contestants have been assigned a number corresponding to when they will make their selection. On a turn, each contestant gets to randomly choose 2 boxes. Once a box is chosen it is removed from the room. You are the second contestant. What is the probability that you will select at least one of the boxes containing a \$100 prize?

Solution

To start, we will count the total number of ways to select 4 boxes. There are 150 ways to select the first box. For each of these 150 possible selections, there are 149 ways to select the second box. There are then $150 \times 149 = 22\,350$ ways to select the first two boxes. For each of these 22 350 ways to select the first two boxes, there are 148 ways to select the third box. There are then $22\,350 \times 148 = 3\,307\,800$ ways to select the first three boxes. And for each of these selections there are 147 ways to select the fourth box. There are then $3\,307\,800 \times 147 = 486\,246\,600$ ways to select the first four boxes.

Six of the boxes contain a \$100 card and $150 - 6 = 144$ do not contain a \$100 card. Once a \$100 card is selected the number of available \$100 cards decreases by 1. Once any card other than a \$100 card is selected the number of such cards decreases by 1.

We can now approach the problem from one of two different directions. Either we determine the total number of ways that you, the second person, can select at least one \$100 card and then calculate the associated probability. Or we could determine the number of ways that you, the second person, do not select any \$100 card, calculate the related probability and then subtract that result from 1. The first approach is a direct approach. The second approach is an indirect approach.

We will present both approaches but start with the indirect approach first.

Indirect Approach

In how many ways can you not select a box containing a \$100 card? What are the possibilities for Contestant 1?

Case 1 Contestant 1 selects two boxes, neither of which contain a \$100 card.
Then you, Contestant 2, select two boxes, neither of which contain a \$100 card. The number of possible selections is $144 \times 143 \times 142 \times 141 = 412\,293\,024$.

Case 2 Contestant 1 selects two boxes, the first contains a \$100 card and the second one does not.
Then you, Contestant 2, select two boxes, neither of which contain a \$100 card. The number of possible selections is $6 \times 144 \times 143 \times 142 = 17\,544\,384$.

Case 3 Contestant 1 selects two boxes, the first does not contain a \$100 card and the second box does.
Then you, Contestant 2, select two boxes, neither of which contain a \$100 card. The number of possible selections is $144 \times 6 \times 143 \times 142 = 17\,544\,384$.

Case 4 Contestant 1 selects two boxes, both of which contain a \$100 card.
Then you, Contestant 2, select two boxes, neither of which contain a \$100 card. The number of possible selections is $6 \times 5 \times 144 \times 143 = 617\,760$.

The total number of ways in which you, the second contestant, do not select a box containing a \$100 card is

$$412\,293\,024 + 17\,544\,384 + 17\,544\,384 + 617\,760 = 447\,999\,552.$$

The probability you do not select a \$100 card is the total number of ways for you to not select a \$100 card divided by the total number of ways to select four boxes.

$$P(\text{do not select a \$100 card as second contestant}) = \frac{447\,999\,552}{486\,246\,600} = \frac{3432}{3725}$$

The probability of you selecting at least one \$100 card as the second contestant is 1 subtract the probability of you not selecting any \$100 cards as the second contestant

$$P(\text{selecting at least one \$100 card as second contestant}) = 1 - \frac{3432}{3725} = \frac{293}{3725} \approx 0.079$$

As the second contestant, the probability of selecting at least one \$100 card is $\frac{293}{3725}$. You would select at least one \$100 card approximately 8% of the time.

Direct Approach

There are more possibilities to consider using this approach. We will present a simplified solution in a chart to determine the total number of ways for contestant 2 to select at least one \$100 card. If for any selection a \$100 card is selected, we will represent the selection with a G. If for any selection a \$100 card is not selected, we will represent the selection with an N. So $GGNG$ represents the possibility that the first contestant picked two \$100 cards and the second contestant picked a non-\$100 card then a \$100 card. The possible cases are shown in the chart.

Contestant 1 Selection	Contestant 1 Selection	Calculation	Number of Possibilities
GG	GN	$6 \times 5 \times 4 \times 144$	17 280
GG	NG	$6 \times 5 \times 144 \times 4$	17 280
GG	GG	$6 \times 5 \times 4 \times 3$	360
GN	GN	$6 \times 144 \times 5 \times 143$	617 760
GN	NG	$6 \times 144 \times 143 \times 5$	617 760
GN	GG	$6 \times 144 \times 5 \times 4$	17 280
NG	GN	$144 \times 6 \times 5 \times 143$	617 760
NG	NG	$144 \times 6 \times 143 \times 5$	617 760
NG	GG	$144 \times 6 \times 5 \times 4$	17 280
NN	GN	$144 \times 143 \times 6 \times 142$	17 544 384
NN	NG	$144 \times 143 \times 142 \times 6$	17 544 384
NN	GG	$144 \times 143 \times 6 \times 5$	617 760
		Total Possibilities	38 247 048

$$P(\text{selecting at least one \$100 card as second contestant}) = \frac{38\,247\,048}{486\,246\,600} = \frac{293}{3725} \approx 0.079$$

As the second contestant, the probability of selecting at least one \$100 card is $\frac{293}{3725}$. You would select at least one \$100 card approximately 8% of the time.