Problem of the Week

Problem E and Solution

The New and Improved Lane

Problem

Lane had a problem with being late for his first class of the day over his first three years of high school. However, this year he has made a conscious effort to improve.

On the first day of the week, usually Monday, Lane's probability of being on time for his first class of the day is $\frac{2}{3}$. If he is on time on any particular day, the probability that he will be on time the next day is also $\frac{2}{3}$. However, if he is late on any particular day, the probability of being late on the next day is one-half the probability of being late on the previous day.

Lane attends school each of the five days from Monday to Friday one week. What is the probability that he will be late for his first class at least three times in that week?

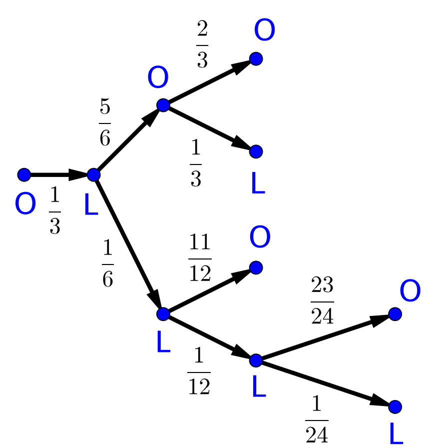
Solution

On any given day of the week, Lane could be late or on time. If we use L to represent a late and O to represent on time, there are $2 \times 2 \times 2 \times 2 \times 2 = 32$ five-letter arrangements using only L's and O's.

We are only interested in arrangements that contain three or more L's. We will consider possibilities for the third late.

1. The third late occurs on the Friday.
   On the first four days of the week we need to assign two O's and two L's. This can be done in six ways: $OOLL$, $OLOL$, $OLLO$, $LOOL$, $LOLO$, and $LLOO$. There are six ways for Lane to be late three times with the third late occurring on the Friday: $OOLLL$, $OLOLL$, $OLLOL$, $LOOLL$, $LOLOL$, and $LLOOL$.
2. The third late occurs on the Thursday.
   On the first three days of the week we need to assign one O and two L's. This can be done in three ways: OLL, LOL, LLO. There are three ways for Lane to be late at least three times with the third late occurring on the Thursday: $OLLL$, $LOLL$, $LLOL$.
3. The third late occurs on the Wednesday.
   This can happen only one way: LLL.

This may be a perfect time to discuss how the probabilities change as a result of Lane being late. If Lane is on time, the probability of being on time the next day remains $\frac{2}{3}$ and the probability of being late is $1 - \frac{2}{3} = \frac{1}{3}$. However, if Lane is late on Monday or on any day following an on time day, the probability of him being late a second day in a row reduces to $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ and his probability of being on time increases to $1 - \frac{1}{6} = \frac{5}{6}$. The tree diagram shows some of the probabilities that will be of interest to us in solving the problem. The points are labelled L or O. The arrows connecting adjacent points show the probability of an event happening given that an event took place.



The following chart illustrates the possible cases and shows the calculation of the probability of that particular case happening.

MON	TUES	WED	THUR	FRI	Probability
O	O	L	L	L	$P(OOLLL) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} = \frac{1}{486} = \frac{4}{1944}$
O	L	O	L	L	$P(OLOLL) = \frac{2}{3} \times \frac{1}{3} \times \frac{5}{6} \times \frac{1}{3} \times \frac{1}{6} = \frac{5}{486} = \frac{20}{1944}$
O	L	L	O	L	$P(OLLOL) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{11}{12} \times \frac{1}{3} = \frac{11}{972} = \frac{22}{1944}$
L	O	O	L	L	$P(LOOLL) = \frac{1}{3} \times \frac{5}{6} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} = \frac{5}{486} = \frac{20}{1944}$
L	O	L	O	L	$P(LOLOL) = \frac{1}{3} \times \frac{5}{6} \times \frac{1}{3} \times \frac{5}{6} \times \frac{1}{3} = \frac{25}{972} = \frac{50}{1944}$
L	L	O	O	L	$P(LLOOL) = \frac{1}{3} \times \frac{1}{6} \times \frac{11}{12} \times \frac{2}{3} \times \frac{1}{3} = \frac{11}{972} = \frac{22}{1944}$
O	L	L	L		$P(OLLL) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} = \frac{1}{324} = \frac{6}{1944}$
L	O	L	L		$P(LOLL) = \frac{1}{3} \times \frac{5}{6} \times \frac{1}{3} \times \frac{1}{6} = \frac{5}{324} = \frac{30}{1944}$
L	L	O	L		$P(LLOL) = \frac{1}{3} \times \frac{1}{6} \times \frac{11}{12} \times \frac{1}{3} = \frac{11}{648} = \frac{33}{1944}$
L	L	L			$P(LLL) = \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} = \frac{1}{216} = \frac{9}{1944}$

Adding the probabilities, we determine that the probability of Lane being late at least three times in the week is $\frac{216}{1944} = \frac{1}{9}$. As a percentage, there is approximately a 11% chance that Lane will be late three or more times in a week.

In the cases listed below the double line in the chart, we stop once we hit a third late. By doing so, are we getting the correct probability in each case?

We will elaborate on the $OLLL$ case and let you expand on the others if you wish. The probability of being on time on Monday and then late the next three days is $\frac{1}{324}$. Our claim is that this probability is equal to the probability of being on time on Monday, late the next three days, on time on Friday plus the probability of being on time on Monday, and late the next four days. In other words, $P(OLLL) = P(OLLLO) + P(OLLLL)$. We will show that this is true on the next page.

We know that $P(OLLL) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} = \frac{1}{324}$.

$$\begin{aligned}
P(OLLLO) + P(OLLLL) &= \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} \times \frac{23}{24}\right) + \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} \times \frac{1}{24}\right) \\
&= \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12}\right)\left(\frac{23}{24} + \frac{1}{24}\right) && \text{Common Factor} \\
&= \left(\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12}\right)(1) && \text{Simplify} \\
&= \frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} \times \frac{1}{12} \\
&= P(OLLL)
\end{aligned}$$

We could also show this for each of the remaining cases below the double horizontal line in the previous table.