Problem of the Week
Problem D and Solution
ABACUS Counting

Problem

We are going to count with the abacus. To be more precise, we are going to count certain six letter arrangements of the letters of the word $ABACUS$ in which every letter is used exactly once. The six letters, A, B, A, C, U, and S are arranged to form six letter "words". When examining the "words", how many of the words have the vowels A, A and U appearing in alphabetical order and the consonants B, C and S not appearing in alphabetical order. The vowels may or may not be adjacent to each other and the consonants may or may not be adjacent to each other.

Solution

Solution 1

In this solution we will count all of the valid six letter arrangements of $A,\ A,\ B,\ C,\ S,$ and U directly.

First consider the number of ways to place the vowels alphabetically in the arrangement.

- If the first A is in the first position, the U and the second A can be placed as follows:
  1. $A\ A\ \_\ \_\ \_\ \_$ and the U can be placed in 4 ways after the second A.
  2. $A\ \_\ A\ \_\ \_\ \_$ and the U can be placed in 3 ways after the second A.
  3. $A\ \_\ \_\ A\ \_\ \_$ and the U can be placed in 2 ways after the second A.
  4. $A\ \_\ \_\ \_\ A\ \_$ and the U can be placed in 1 way after the second A.

  There is a total of $4+3+2+1=10$ ways to place the vowels in alphabetical order so that the first A is in the first position.
- If the first A is in the second position, the U and the second A can be placed as follows:
  1. $\_\ A\ A\ \_\ \_\ \_$ and the U can be placed in 3 ways after the second A.
  2. $\_\ A\ \_\ A\ \_\ \_$ and the U can be placed in 2 ways after the second A.
  3. $\_\ A\ \_\ \_\ A\ \_$ and the U can be placed in 1 way after the second A.

  There is a total of $3+2+1=6$ ways to place the vowels in alphabetical order so the first A is in the second position.
- If the first A is in the third position, the U and the second A can be placed as follows:
  1. $\_\ \_\ A\ A\ \_\ \_$ and the U can be placed in 2 ways after the second A.
  2. $\_\ \_\ A\ \_\ A\ \_$ and the U can be placed in 1 way after the second A.

  There is a total of $2+1=3$ ways to place the vowels in alphabetical order so the first A is in the third position.
- If the first A is in the fourth position, the U and the second A can be placed as follows:
  1. $\_\ \_\ \_\ A\ A\ \_$ and the U can be placed in 1 way after the second A.

  There is a total of 1 way to place the vowels in alphabetical order so the first A is in the fourth position.

Adding each of the above results, there is a total of $10+6+3+1=20$ ways to place the vowels so that they are in alphabetical order.

For each of the 20 ways to place the vowels in alphabetical order, we find the number of valid ways to fill the remaining three positions with the consonants so that they are not in alphabetical order.

It turns out that the consonants can be placed in the three remaining positions in six possible orders: BCS, BSC, CBS, CSB, SBC, and SCB. One of the arrangements of the consonants is in alphabetical order and five of the arrangements are not in alphabetical order.

So for each of the 20 arrangements in which the vowels are in alphabetical order there are 5 arrangements of the consonants so they are not in alphabetical order.

Therefore, there are $20 \times 5 = 100$ arrangements of the letters of the word $ABACUS$ in which the vowels appear alphabetically and the consonants do not appear alphabetically.

Solution 2

In this solution we will first count the total number of arrangements of the letters in the word $ABACUS$.

Notice that there are two As. We need to be careful not to count arrangements twice. So we will place the distinct letters B, C, S, and U first. There are 6 places for the B. For each of these placements of B, there are 5 ways to place the C. This gives a total of $6 \times 5 = 30$ ways to place the B and C. For each of these placements of B and C, there are 4 ways to place the S. This gives a total of $30 \times 4 = 120$ ways to place the B, C and S. For each of these placements of B, C and S, there are 3 ways to place the U. This gives a total of $120 \times 3 = 360$ ways to place the B, C, S, and U. The two As must go in the remaining two spots and this can be done in only 1 way. Therefore, there is a total of 360 ways to arrange the six letters of the word $ABACUS$.

In examining the 360 arrangements, you would see the vowels appearing in one of three orders: AAU, AUA and UAA, only one of which is in alphabetical order. So in one-third of the 360 arrangements or 120 arrangements, the vowels will appear in alphabetical order.

In examining the 120 arrangements in which the vowels appear alphabetically, the consonants will appear in one of six different orders: BCS, BSC, CBS, CSB, SBC, and SCB. Five-sixths of these orderings have the consonants out of alphabetical order. So five-sixths of the 120 arrangements or 100 arrangements will be such that the vowels appear alphabetically and the consonants do not appear alphabetically.

Solution 3

This solution will be outlined only and is left as an exercise to be completed by the solver.

If we were to count the total number of possibilities as we did in the second solution, we could then subtract the arrangements which are invalid. The invalid arrangements include the number of arrangements in which the vowels and consonants are both not in alphabetical order, the number of arrangements in which the vowels and consonants are both in alphabetical order and the number of arrangements in which the vowels are not in alphabetical order but the consonants are in alphabetical order. Be careful not to count arrangements twice.