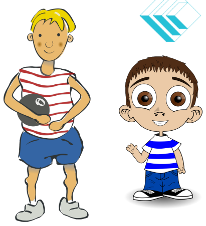
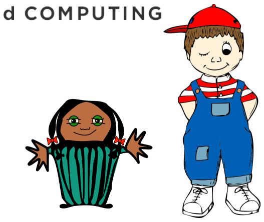


Problem of the Week
Problem E and Solution
A Product for the Ages



Problem

A mother has four children, each with a different age. The product of their ages is 17 280. The sum of the ages of the three oldest children is 40 and the sum of the ages of the three youngest children is 32. Determine the ages of the four children.

Solution

Let the ages of the children from youngest to oldest be $a,\ b,\ c,\ d$.

Since the ages of the three oldest children sum to 40, $b + c + d = 40$. (1)
Since the ages of the three youngest children sum to 32, $a + b + c = 32$. (2)

Subtracting (2) from (1), we obtain $d - a = 8$. This means that the difference between the age of the oldest child and the age of the youngest child is 8.

Now the prime factorization of 17 280 is $2^7 \times 3^3 \times 5 = 12 \times 12 \times 12 \times 10$.

Is it possible that the oldest child is 12? If the oldest child is 12, then the youngest child would be $12 - 8 = 4$. The largest product that could be generated using two more different ages between 4 and 12 would be $12 \times 11 \times 10 \times 4 = 5280 < 17\,280$. It follows that the oldest child must be at least 13 and that the youngest child is at least 5.

Is it possible that the youngest child is 12? If the youngest child is 12, then the oldest child would be $12 + 8 = 20$. The smallest product that could be generated using two more different ages between 12 and 20 would be $12 \times 13 \times 14 \times 20 = 43\,680 > 17\,280$. It follows that the youngest child must be 11 or younger and that the oldest child is at most 19.

It follows that there are a limited number of possibilities such that the difference between the oldest and youngest is 8 which also satisfy the condition that the youngest is 11 or younger and the oldest is 13 or older. The possibilities for youngest and oldest are (5,13), (6,14), (7,15), (8,16), (9,17), (10,18) and (11,19). No other combination would be possible since the oldest child must be 13 or older and the youngest child must be 11 or younger.

The numbers 7,11, 13, and 17 are primes and are not factors of 17 280. Therefore we can eliminate the possibilities where an age is one of 7,11, 13, or 17, leaving (6,14), (8,16) and (10,18). But 14 contains a prime factor of 7 which is not a factor of 17 280 so we can eliminate (6,14). Now there are only two possibilities left to consider, namely (8,16) and (10,18).

Consider the age 10 for the youngest and 18 for the oldest. The product of these ages is 180. When 17 280 is divided by 180, the quotient is 96. The minimum product of any two different numbers between 10 and 18 is $11 \times 12 = 132 > 96$. It is not possible for the youngest to be 10 and the oldest to be 18.

We must then conclude that the youngest is 8 and the oldest is 16. Now, $17\,280 = 8 \times 16 \times 3^3 \times 5$. Using the remaining factors 3^3 and 5, we need to create two numbers between 8 and 16. The only possibilities are $3^2 = 9$ and $3 \times 5 = 15$.

Therefore the ages of the children are 8, 9, 15, and 16. It is easy to verify that this is the correct solution.