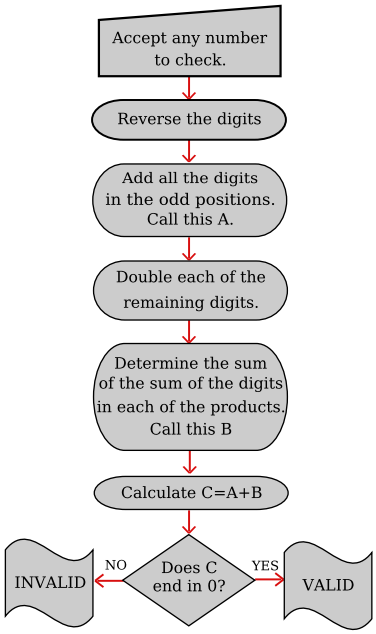
Problem of the Week
Problem D and Solution
Check Please

Problem

Debit and credit cards contain account numbers which consist of many digits. Often, when purchasing items online you are asked to type in your account number. Because there are so many digits it is easy to type the number incorrectly. Most companies use a "check digit" to quickly verify the validity of the number. One commonly used algorithm for verifying that numbers have been entered correctly is called the Luhn Algorithm. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid. The steps performed in the Luhn Algorithm are illustrated in the flowchart to the right.



The number 8664 $R8R4$ $R6R9$ 0359 is a valid number when verified by the Luhn Algorithm. R is an integer from 0 to 9 occurring four times in the number. (It may also be one of the existing known digits.) Determine all possible values of R.

Solution

Solution 1

When the digits of the number are reversed the resulting number is 9530 $9R6R$ $4R8R$ 4668. The sum of the digits in the odd positions is

$$A = 9 + 3 + 9 + 6 + 4 + 8 + 4 + 6 = 49$$

When the digits in the remaining positions are doubled, the following products are obtained:
$2 \times 5 = 10;\ 2 \times 0 = 0;\ 2 \times R = 2R;\ 2 \times R = 2R;\ 2 \times R = 2R;\ 2 \times R = 2R;\ 2 \times 6 = 12;$ and $2 \times 8 = 16$.

Let x represent the sum of the digits of $2R$.

When the digit sums from each of the products are added, the sum is:

$$B = (1 + 0) + 0 + x + x + x + x + (1 + 2) + (1 + 6) = 1 + 0 + 4x + 3 + 7 = 4x + 11$$

C is the sum of A and B, so $C = 49 + 4x + 11 = 60 + 4x$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what are the possible sums which can be obtained?

Original Digit $\rightarrow R$	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit $\rightarrow 2R$	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2R$	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of twice the original digit can only be an integer from 0 to 9 inclusive. It follows that the only values for x are the integers from 0 to 9.

To satisfy the Luhn Algorithm, the units digit of C must be zero. We want $60 + 4x$ to be an integer greater than or equal to 60 such that the units digit is 0.

Can $60 + 4x = 60$? When $4x = 0$, then $x = 0$, $60 + 4x = 0$ and $R = 0$. That is, when $R = 0$, $2R = 0$ and the sum of the digits of $2R$ is $x = 0$. This value of R produces a valid number.

Can $60 + 4x = 70$? When $4x = 10$, then $x = 2.5$ and $60 + 4x = 70$. But x in an integer value so this is not possible.

Can $60 + 4x = 80$? When $4x = 20$, then $x = 5$, $60 + 4x = 80$ and $R = 7$. That is, when $R = 7$, $2R = 14$ and the sum of the digits of $2R$ is $x = 5$. This value of R produces a valid number.

Can $60 + 4x = 90$? When $4x = 30$, then $x = 7.5$ and $60 + 4x = 90$. But x in an integer value so this is not possible.

Can $60 + 4x = 100$? When $4x = 40$, then $x = 10$ and $60 + 4x = 100$. But x is an integer from 0 to 9 inclusive, so this is not possible.

Every integer ending in zero that is larger than 100 would produce a value for x greater than 10. There are no more possible values for x or R.

The two valid possibilities for R are 0 and 7.

When $R = 0$, the valid number is 8664 0804 0609 0359.

When $R = 7$, the valid number is 8664 7874 7679 0359.

Solution 2

The second solution looks at each of the possible values of R and applies the Luhn Algorithm to the resulting number. A computer program or spreadsheet could be developed to solve this problem efficiently.

Remember that A is the sum of the digits in the odd positions of the reversal. Each of the digits in the even positions of the reversal are doubled and B is the sum of the sum of the digits of each of these products. C is the sum $A + B$.

R	Number	Reversal	A	Double Even Digits	B	C	Valid / Invalid
0	8664 0804 0609 0359	9530 9060 4080 4668	49	10, 0, 0, 0, 0, 0, 12, 16	11	60	Valid
1	8664 1814 1619 0359	9530 9161 4181 4668	49	10, 0, 2, 2, 2, 2, 12, 16	19	68	Invalid
2	8664 2824 2629 0359	9530 9262 4282 4668	49	10, 0, 4, 4, 4, 4, 12, 16	27	76	Invalid
3	8664 3834 3639 0359	9530 9363 4383 4668	49	10, 0, 6, 6, 6, 6, 12, 16	35	84	Invalid
4	8664 4844 4649 0359	9530 9464 4484 4668	49	10, 0, 8, 8, 8, 8, 12, 16	43	92	Invalid
5	8664 5854 5659 0359	9530 9565 4585 4668	49	10, 0, 10, 10, 10, 10, 12, 16	15	64	Invalid
6	8664 6864 6669 0359	9530 9666 4686 4668	49	10, 0, 12, 12, 12, 12, 12, 16	23	72	Invalid
7	8664 7874 7679 0359	9530 9767 4787 4668	49	10, 0, 14, 14, 14, 14, 12, 16	31	80	Valid
8	8664 8884 8689 0359	9530 9868 4888 4668	49	10, 0, 16, 16, 16, 16, 12, 16	39	88	Invalid
9	8664 9894 9699 0359	9530 9969 4989 4668	49	10, 0, 18, 18, 18, 18, 12, 16	47	96	Invalid

The two valid possibilities for R are 0 and 7.