Problem of the Week
Problem E and Solution
Looking for Possibilities

Problem

Determine all possible ordered pairs of positive integers (a, b) such that $\frac{1}{a} + \frac{2}{b} = \frac{8}{2a+b}$ and $1963 \leq 4a + 7b \leq 2016$.

Solution

Since a and b are positive integers satisfying $1963 \leq 4a + 7b \leq 2016$, we could write out all ordered pairs that satisfy this inequality and then determine which ones also satisfy the first equation. There will be a large number of possibilities to check so we need to find a way to reduce the number of possibilities. We will work with the first equation.

$$\begin{array}{rrcl} & \frac{b+2a}{ab} & = & \frac{8}{2a+b} \\ \text{Multiply both sides by } ab(2a+b)\text{:} & (b+2a)(2a+b) & = & 8ab \\ \text{Expand and simplify:} & 4a^2 + 4ab + b^2 & = & 8ab \\ \text{Rearrange:} & 4a^2 - 4ab + b^2 & = & 0 \\ \text{Factor:} & (2a-b)^2 & = & 0 \end{array}$$

It follows that $2a - b = 0$ and $b = 2a$.

Each of the ordered pairs (a, b) will look like $(a, 2a)$. We substitute $2a$ for b in the inequality obtaining $1963 \leq 4a + 7(2a) \leq 2016$ or $1963 \leq 18a \leq 2016$. We could work with the parts of the inequality separately. However, we can also divide each term in the inequality by 18.

$$\begin{array}{ccccc} 1963 & \leq & 18a & \leq & 2016 \\ \frac{1963}{18} & \leq & \frac{18a}{18} & \leq & \frac{2016}{18} \\ 109\frac{1}{18} & \leq & a & \leq & 112 \end{array}$$

Since a is a positive integer, a can only take on integer values $110,\ 111,\ 112$. Since $b = 2a$, the corresponding values of b are $220,\ 222,\ 224$.

The ordered pairs of positive integers (a, b) that satisfy $\frac{1}{a} + \frac{2}{b} = \frac{8}{2a+b}$ and $1963 \leq 4a + 7b \leq 2016$ are $(110, 220), (111, 222)$ and $(112, 224)$.