# Problem of the Week
Problem D and Solution
Left, Right, Left, Right, …

Problem

Your friend writes down all of the integers starting from 0 as shown in the diagram to the right. Specifically, below every number there are two numbers: one on the left and one on the right. For example, below 3, the number 7 is on the left, and the number 8 is on the right. The numbers can be read in increasing order from top row to bottom row and from left-to-right within a row. Notice that we can get from 0 to 12 by going right (R), left (L) then right (R). What number do you end at if you take the following path from 0:
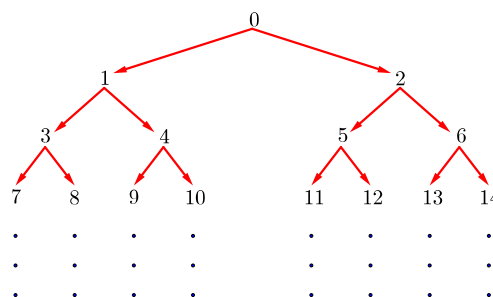
$$L \to L \to R \to L \to L \to R \to L \to R \to L \to R \to L$$

Solution

At first it may not be obvious how to proceed. We can see from the given diagram that $L \to L \to R$ takes us from 0 to 1 to 3 to 8. But from there where do we go? We could write out more rows of the chart until we are able to make the required number of moves and then read off the final answer. This approach would work for a relatively small number of moves but would not be practical in general for "longer" strings of moves.

We will proceed by making an observation. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. Is there a general formula which can be used when asked to move left (L)? Is there a general formula which can be used when asked to move right (R)?

The diagram to the right has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing the same with 6 would not get us to 13 and 14. As we go down the chart, each new row has twice as many numbers as the row above. Let's try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1 we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.
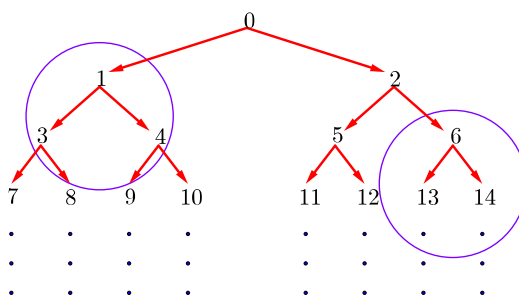
So it would appear that if we make a move left (L) from any number a in the tree, the resulting number is one more than twice the value of a. That is, a move left (L) from a takes us to the number $2a + 1$ in the tree.

It would appear that if we make a move right (R) from any number a in the tree, the resulting number is two more than twice the value of a. That is, a move right (R) from a takes us to the number $2a + 2$ in the tree.

These results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

The following table shows the result of performing the given sequence of moves

$$L \to L \to R \to L \to L \to R \to L \to R \to L \to R \to L.$$

Initial Number	Move	Calculation	Next Number
0	L	$2(0) + 1$	1
1	L	$2(1) + 1$	3
3	R	$2(3) + 2$	8
8	L	$2(8) + 1$	17
17	L	$2(17) + 1$	35
35	R	$2(35) + 2$	72
72	L	$2(72) + 1$	145
145	R	$2(145) + 2$	292
292	L	$2(292) + 1$	585
585	R	$2(585) + 2$	1172
1172	L	$2(1172) + 1$	2345

Starting at 0 and making the moves $L \to L \to R \to L \to L \to R \to L \to R \to L \to R \to L$, we end at the number 2345.

Once we determined the operations required to make a move left (L) and a move right (R), the problem was quite straightforward to solve. It would be possible to write a computer program which would accept any length sequence of Ls and Rs, and get the computer to determine the final position in this specific tree.