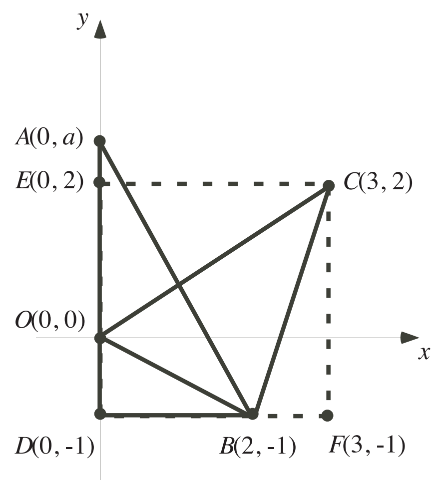
Problem of the Week
Problem D and Solution
Tangled Triangles

Problem

In the diagram, $A(0, a)$ lies on the y-axis above the origin. If $\triangle ABD$ and $\triangle COB$ have the same area, determine the value of a.



Solution

Solution 1

Draw rectangle $EDFC$ with sides parallel to the x and y-axes so that $O(0, 0)$ is on ED and $B(2, -1)$ is on DF. Since EC is parallel to the x-axis and E is on the y-axis, E has coordinates $(0, 2)$. Since CF is parallel to the y-axis, F has the same x-coordinate as C. Since FD is parallel to the x-axis, F has the same y-coordinate as D and B. Therefore the coordinates of F are $(3, -1)$.

To find the area of $\triangle COB$, subtract the areas of $\triangle CEO$, $\triangle ODB$, and $\triangle BFC$ from the area of rectangle $EDFC$.

In rectangle $EDFC$, $EC = 3 - 0 = 3$ and $ED = 2 - (-1) = 3$. The area of rectangle $EDFC = EC \times ED = 3 \times 3 = 9$ units^2.

In $\triangle CEO$, $EC = 3$ and $EO = 2 - 0 = 2$. The area of $\triangle ECO = \frac{EC \times EO}{2} = \frac{3 \times 2}{2} = 3$ units^2.

In $\triangle ODB$, $OD = 0 - (-1) = 1$ and $DB = 2 - 0 = 2$. The area of $\triangle ODB = \frac{OD \times DB}{2} = \frac{1 \times 2}{2} = 1$ unit^2.

In $\triangle BFC$, $BF = 3 - 2 = 1$ and $CF = 2 - (-1) = 3$. The area of $\triangle BFC = \frac{BF \times CF}{2} = \frac{1 \times 3}{2} = 1.5$ units^2.

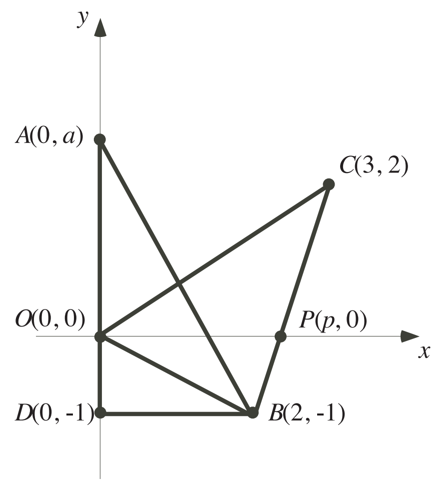
$$\begin{aligned}
\text{Area } \triangle COB &= \text{Area Rectangle } EDFC - \triangle CEO - \triangle ODB - \triangle BFC \\
&= 9 - 3 - 1 - 1.5 \\
&= 3.5 \text{ units}^2
\end{aligned}$$

But the area $\triangle ABD = \triangle COB$ so the area of $\triangle ABD = 3.5$ units^2.

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$ so

$$\begin{aligned}
\text{Area } \triangle ABD &= \frac{AD \times DB}{2} \\
3.5 &= \frac{(a+1) \times 2}{2} \\
3.5 &= a + 1 \\
2.5 &= a
\end{aligned}$$

$\therefore$ the value of a is 2.5.



Solution 2

Determine the equation of the line containing $C(3,2)$ and $B(2,-1)$.

The slope of the line is $\frac{2-(-1)}{3-2} = 3$. The equation of the line is of the form $y = 3x + b$. Substitute $x = 3,\ y = 2$ to determine the value of b. $2 = 3(3) + b$ and $b = -7$ follows. Therefore the equation of the line containing C and B is $y = 3x - 7$.

Let $P(p,0)$ be the x-intercept of the line. Substituting into $y = 3x - 7$ we obtain $0 = 3p - 7$ and $p = \frac{7}{3}$ follows.

To determine the area of $\triangle COB$ determine the sum of the areas of $\triangle COP$ and $\triangle BOP$.

In $\triangle COP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the x-axis to $C(3,2)$, which is 2 units. The area of $\triangle COP = \frac{\frac{7}{3}\times 2}{2} = \frac{7}{3}$ units^2.

In $\triangle BOP$, $OP = \frac{7}{3}$ and the height is the perpendicular distance from the x-axis to $B(2,-1)$, which is 1 unit. The area of $\triangle BOP = \frac{\frac{7}{3}\times 1}{2} = \frac{7}{6}$ units^2.

$$\begin{aligned}
\text{Area } \triangle COB &= \text{Area } \triangle COP + \text{Area } \triangle BOP \\
&= \tfrac{7}{3} + \tfrac{7}{6} \\
&= \tfrac{14}{6} + \tfrac{7}{6} \\
&= \tfrac{21}{6} \\
&= \tfrac{7}{2} \text{ units}^2
\end{aligned}$$

But the area $\triangle ABD = \triangle COB$ so the area of $\triangle ABD = \frac{7}{2}$ units^2.

In $\triangle ABD$, $AD = a - (-1) = a + 1$ and $DB = 2 - 0 = 2$ so

$$\begin{aligned}
\text{Area } \triangle ABD &= \frac{AD \times DB}{2} \\
\frac{7}{2} &= \frac{(a+1)\times 2}{2} \\
7 &= 2a + 2 \\
5 &= 2a
\end{aligned}$$

$$\therefore\ a = \frac{5}{2} = 2.5.$$