Supplementary document for galIMF.py

Based on C. Schulz, J. Pflamm-Altenburg, and P. Kroupa 2015 and Yan, Z., Jerabkova, T., Kroupa, P. 2017

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This document developed the mathematical forms to optimal sampling (Schulz et al 2015) the object mass from a 3-slop-power-law stellar initial mass function (IMF) and a 1-slop-power-law embedded cluster mass function (ECMF).

This document serves especially as a supplementary documentation for galIMF module written in Python 3 programming language. The galIMF calculate galaxy-wide stellar initial mass function based on IGIMF theory (Yan et al 2017) and optimal sampling. All equations used in the galIMF module are derived in this file. For example, in the case of Part I. IMF, equation 1 to 10 are assumptions coming from Yan et al (2017) and Schulz et al (2015) while others are derivations for different case that is useful for the computer code.

The symbols used here are similar as in the code (i.e. M_L in the code is written M_L here) but different from publications. We summarize the meaning of the symbols before the derivation.

Equations in the galIMF.py are labeled with the same equation number here, allowing the user to go through the code and get oriented in exactly what each function computes.

The assumptions or the starting point of this derivation are given in Yan et al (2017) and Schulz et al (2015). We strongly recommend go through this two papers before the use of galIMF module. A resulting galaxy-wide IMF grid using galIMF can be found in Jerabkova et al. (2017).

Part I. IMF

1. Symbols

M_L and M_U are the physical possible star mass limits in theoretical sense.
M_{max} is maximum mass star's integration upper limit.

M_1 are existing maximum star masses in observational sense.

M_L, M_{max} and M_U possess an index number name that will be used from equation 33
$M_L = m_{N_{tot}}$
$M_{max} = m_1$
$M_U = m_0$
Lower case letter m are all integration limits, not real star mass.
Existing star mass are indicated by M_1, M_2... in uppercase letter with a index start from most massive to less massive stars.

In summary $M_U = m_0 > M_{max} = m_1 > M_1 > m_2 > M_2$...

The default values in galIMF.py are:
$M_L = 0.08$ solar mass (hydrogen burning mass limit),
$M_{turn} = 0.5$ and $M_{turn,2} = 1$ is where the IMF slope changes,
$M_U = 150$ (Massey assertion).

2. Derivation

$$\xi_{star} = \begin{cases} k_1 M^{-\alpha_1}, & \text{if } M_L < M < M_{turn} \\ k_2 M^{-\alpha_2}, & \text{if } M_{turn} < M < M_{turn,2} \\ k_3 M^{-\alpha_3}, & \text{if } M_{turn,2} < M < M_U \end{cases} \tag{1}$$

$$\begin{aligned} k_1 &= k_2 M_{turn}^{\alpha_1-\alpha_2} \\ k_2 &= k_3 M_{turn,2}^{\alpha_2-\alpha_3} \\ &= k_3, \text{ if } M_{turn,2} = 1 \end{aligned} \tag{2}$$

where $m_{turn} = 0.5$, $m_{turn,2} = 1$, $\alpha_1 = 1.3 \pm 0.3$, $\alpha_2 = 2.3 \pm 0.36$,

$$\alpha_3 = \begin{cases} 2.3, & x < -0.87 \\ -0.41x + 1.94, & x \geq -0.87 \end{cases} \tag{3}$$

with $x = -0.14[\mathrm{Fe/H}] + 0.99 \log_{10}(\rho_{cl}/(10^6 M_{\odot}\mathrm{pc}^{-3}))$.

$$N_{tot} = \int_{M_L}^{M_{max}} \xi(M)\,\mathrm{d}M, \tag{4}$$

$$M_{ecl} = M_{tot} = \int_{M_L}^{M_{max}} M\xi(M)\,\mathrm{d}M. \tag{5}$$

Divide both integrals 29 and 30 into N_{tot} separate integrals, each integral representing one individual object:

$$\begin{aligned} N_{tot} = &\int_{M_L}^{m_{N_{tot}-1}} \xi(M)\,\mathrm{d}M + \int_{m_{N_{tot}-1}}^{m_{N_{tot}-2}} \xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} \xi(M)\,\mathrm{d}M, \end{aligned} \tag{6}$$

$$\begin{aligned} M_{tot} = &\int_{M_L}^{m_{N_{tot}-1}} M\xi(M)\,\mathrm{d}M + \int_{m_{N_{tot}-1}}^{m_{N_{tot}-2}} M\xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{m_1} M\xi(M)\,\mathrm{d}M. \end{aligned} \tag{7}$$

where $m_1 = M_{max}$.

Each integral must give one object:

$$1 = \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M, \tag{8}$$

$$M_i = \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M, \tag{9}$$

where the first integral from M_L to $m_{N_{tot}-1}$ can be smallar than 1 and became a redundant mass that doesn't form a star.

Optimally sampling condition is:

$$I = \int_{m_1}^{M_U} \xi(M)\,\mathrm{d}M = \int_{M_{max}}^{M_U} \xi(M)\,\mathrm{d}M. \tag{10}$$

This equation define M_{max} for given M_U. $I = 1$ is the canonical optimally sampling condition. It is easy to change I to match the observed M_1-M_{ecl} relation, where M_1 is the most massive star mass in the cluster.

2.1. For $M_{turn,2} < M_{max}$

Inserting 28 to 35 for $\alpha_3 \neq 1$

$$I = k_3 \int_{M_{max}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M = \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{max}^{1-\alpha_3}). \tag{11}$$

then

$$k_3 = \frac{I(1-\alpha_3)}{M_U^{1-\alpha_3} - M_{max}^{1-\alpha_3}}. \tag{12}$$

We consider only the case $\alpha_3 \approx \alpha_2 > \alpha_1 > 1$ as indicated by observations.

For simplicity, consider only $\alpha_2 \neq 2$ and $\alpha_3 \neq 2$. 2 can be proximate by $2+\Delta$ if needed.

Inserting 28 to 30

$$
\begin{aligned}
&M_{tot} \\
&= \int_{M_L}^{M_{turn}} k_1 M^{1-\alpha_1}\,\mathrm{d}M + \int_{M_{turn}}^{M_{turn,2}} k_2 M^{1-\alpha_2}\,\mathrm{d}M + \int_{M_{turn,2}}^{M_{max}} k_3 M^{1-\alpha_3}\,\mathrm{d}M \\
&= \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_2}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) + \frac{k_3}{2-\alpha_3}(M_{max}^{2-\alpha_3} - M_{turn,2}^{2-\alpha_3}) \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) \\
&+ \frac{k_3}{2-\alpha_3}(M_{max}^{2-\alpha_3} - M_{turn,2}^{2-\alpha_3})
\end{aligned}
\tag{13}
$$

Inserting 37 to 39

$$
\begin{aligned}
&\frac{M_{tot} M_U^{1-\alpha_3}}{I(1-\alpha_3)} - \frac{M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) \\
&- \frac{M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) + \frac{M_{turn,2}^{2-\alpha_3}}{2-\alpha_3} \\
&= \frac{M_{max}^{2-\alpha_3}}{2-\alpha_3} + \frac{M_{tot} M_{max}^{1-\alpha_3}}{I(1-\alpha_3)}
\end{aligned}
\tag{14}
$$

This give us $M_{max} = M_{max}(M_{tot}, I, M_L, M_{turn}, M_{turn,2}, M_U, \alpha_1, \alpha_2, \alpha_3)$.

M_{tot} is given by the cluster mass M_{ecl}.

Inserting 37 to 28

$$
\xi_{star} = \xi(M; M_{ecl}, I, M_L, M_{turn}, M_{turn,2}, M_U, \alpha_1, \alpha_2, \alpha_3). \tag{15}
$$

Use 28 and 33 can calculate every m_i:

$$m_{i+1}=\begin{cases}\left(m_i^{1-\alpha_3}-\frac{1-\alpha_3}{k_3}\right)^{\frac{1}{1-\alpha_3}}\\ \text{, if } M_{turn,2}<m_{i+1}<m_i<M_U\\ \left(M_{turn,2}^{1-\alpha_2}+\frac{k_3}{k_2}\frac{1-\alpha_2}{1-\alpha_3}(m_i^{1-\alpha_3}-M_{turn,2}^{1-\alpha_3})-\frac{1-\alpha_2}{k_2}\right)^{\frac{1}{1-\alpha_2}}\\ \text{, if } M_{turn}<m_{i+1}<M_{turn,2}<m_i<M_U\\ \left(m_i^{1-\alpha_2}-\frac{1-\alpha_2}{k_2}\right)^{\frac{1}{1-\alpha_2}}\\ \text{, if } M_{turn}<m_{i+1}<m_i<M_{turn,2}\\ \left(M_{turn}^{1-\alpha_1}+\frac{k_2}{k_1}\frac{1-\alpha_1}{1-\alpha_2}(m_i^{1-\alpha_2}-M_{turn}^{1-\alpha_2})-\frac{1-\alpha_1}{k_1}\right)^{\frac{1}{1-\alpha_1}}\\ \text{, if } M_L<m_{i+1}<M_{turn}<m_i<M_{turn,2}\\ \left(m_i^{1-\alpha_1}-\frac{1-\alpha_1}{k_1}\right)^{\frac{1}{1-\alpha_1}}\\ \text{, if } M_L<m_{i+1}<m_i<M_{turn}\end{cases}\tag{16}$$

$$m_{i+n}=\left(m_i^{1-\alpha}-n\cdot\frac{1-\alpha}{k}\right)^{\frac{1}{1-\alpha}},\tag{17}$$

where $\alpha=\alpha_1$ when $m_{i+1}<m_i<M_{turn}$, $\alpha=\alpha_2$ when $M_{turn}<m_{i+1}<m_i<M_{turn,2}$ and $\alpha=\alpha_3$ when $M_{turn,2}<m_{i+1}<m_i<M_U$.

Then use 34 to calculate every M_i.

Remember we consider only $\alpha\neq 2$:

$$n\cdot\bar{M}=M_i+M_{i+1}+...+M_{i+n-1}=$$
$$\begin{cases}\frac{k_1}{2-\alpha_1}(m_i^{2-\alpha_1}-m_{i+n}^{2-\alpha_1}), & \text{if } M_L<m_{i+n}<m_i<M_{turn}\\ \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1}-m_{i+n}^{2-\alpha_1})+\frac{k_2}{2-\alpha_2}(M_i^{2-\alpha_2}-M_{turn}^{2-\alpha_2}), & \text{if } M_L<m_{i+n}<M_{turn}<m_i<M_{turn,2}\\ \frac{k_2}{2-\alpha_2}(m_i^{2-\alpha_2}-m_{i+n}^{2-\alpha_2}), & \text{if } M_{turn}<m_{i+n}<m_i<M_{turn,2}\\ \frac{k_2}{2-\alpha_2}(M_{turn,2}^{2-\alpha_2}-m_{i+n}^{2-\alpha_2})+\frac{k_3}{2-\alpha_3}(M_i^{2-\alpha_3}-M_{turn,2}^{2-\alpha_3}), & \text{if } M_{turn}<m_{i+n}<M_{turn,2}<m_i<M_U\\ \frac{k_3}{2-\alpha_3}(m_i^{2-\alpha_3}-m_{i+n}^{2-\alpha_3}), & \text{if } M_{turn,2}<m_{i+n}<m_i<M_U\end{cases}\tag{18}$$

and
$$n'\cdot\bar{M}=\frac{k_1}{2-\alpha_1}(m_i^{2-\alpha_1}-M_L^{2-\alpha_1}),\ \text{if } m_{i+n}<M_L<m_i<M_{turn}\tag{19}$$

2.2. For $M_{turn}<M_{max}<M_{turn,2}$

If the cluster mass is small enough, $M_{max}<M_{turn,2}$ is possible. This happens when $M_{ecl,L}<2.72$.

Similar to 36, inserting 28 to 35 for $\alpha_2 \neq 1$

$$\begin{aligned}
I &= k_2 \int_{M_{max}}^{M_{turn,2}} M^{-\alpha_2}\,\mathrm{d}M + k_3 \int_{M_{turn,2}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M \\
&= \frac{k_2}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}). \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}).
\end{aligned} \tag{20}$$

from this we get

$$k_3 = I \cdot \left[\frac{M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{max}^{1-\alpha_2}) + \frac{1}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}) \right]^{-1}. \tag{21}$$

for $\alpha_2 \neq 2$:

$$\begin{aligned}
M_{tot} &= \int_{M_L}^{M_{turn}} k_1 M^{1-\alpha_1}\,\mathrm{d}M + \int_{M_{turn}}^{M_{max}} k_2 M^{1-\alpha_2}\,\mathrm{d}M \\
&= \frac{k_1}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_2}{2-\alpha_2}(M_{max}^{2-\alpha_2} - M_{turn}^{2-\alpha_2}) \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{2-\alpha_2}(M_{max}^{2-\alpha_2} - M_{turn}^{2-\alpha_2})
\end{aligned} \tag{22}$$

Insert 21 gives M_{max}:

$$\begin{aligned}
&\frac{M_{max}^{2-\alpha_2}}{2-\alpha_2} + \frac{M_{tot}}{I(1-\alpha_2)} M_{max}^{1-\alpha_2} \\
=&\frac{M_{tot} M_{turn,2}^{1-\alpha_2}}{I(1-\alpha_2)} + \frac{M_{tot} M_{turn,2}^{\alpha_3-\alpha_2}}{I(1-\alpha_3)}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}) \\
&- \frac{M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{turn}^{2-\alpha_1} - M_L^{2-\alpha_1}) + \frac{M_{turn}^{2-\alpha_2}}{2-\alpha_2}
\end{aligned} \tag{23}$$

m_{i+1} is the same as equation 16 but use the new k_3 in 21.

Finally, the mass of stars are the same as in equation 18

2.3. For $M_{max} < M_{turn}$

This happens when $M_{ecl,L} <$.

Inserting 28 to 35 for $\alpha_2 \neq 1$

$$\begin{aligned}
I &= k_1 \int_{M_{max}}^{M_{turn}} M^{-\alpha_1}\,\mathrm{d}M + k_2 \int_{M_{turn}}^{M_{turn,2}} M^{-\alpha_2}\,\mathrm{d}M + k_3 \int_{M_{turn,2}}^{M_U} M^{-\alpha_3}\,\mathrm{d}M \\
&= \frac{k_1}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) + \frac{k_2}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\
&+ \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}). \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) + \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\
&+ \frac{k_3}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}).
\end{aligned} \tag{24}$$

from this we get k_3

$$\begin{aligned}
I \cdot k_3^{-1} =& \frac{M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{1-\alpha_1}(M_{turn}^{1-\alpha_1} - M_{max}^{1-\alpha_1}) \\
&+ \frac{M_{turn,2}^{\alpha_2-\alpha_3}}{1-\alpha_2}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) + \frac{1}{1-\alpha_3}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}).
\end{aligned} \tag{25}$$

$$\begin{aligned}
M_{tot} &= \int_{M_L}^{M_{max}} k_1 M^{1-\alpha_1}\,\mathrm{d}M = \frac{k_1}{2-\alpha_1}(M_{max}^{2-\alpha_1} - M_L^{2-\alpha_1}) \\
&= \frac{k_3 M_{turn,2}^{\alpha_2-\alpha_3} M_{turn}^{\alpha_1-\alpha_2}}{2-\alpha_1}(M_{max}^{2-\alpha_1} - M_L^{2-\alpha_1})
\end{aligned} \tag{26}$$

Insert new k_3 in 25 gives M_{max}:

$$\begin{aligned}
&\frac{M_{max}^{2-\alpha_1}}{2-\alpha_1} + \frac{M_{tot}}{I(1-\alpha_1)} M_{max}^{1-\alpha_1} \\
=& \frac{M_{tot} M_{turn}^{1-\alpha_1}}{I(1-\alpha_1)} + \frac{M_{tot} M_{turn}^{\alpha_2-\alpha_1}}{I(1-\alpha_2)}(M_{turn,2}^{1-\alpha_2} - M_{turn}^{1-\alpha_2}) \\
&+ \frac{M_{tot} M_{turn,2}^{\alpha_3-\alpha_2} M_{turn}^{\alpha_2-\alpha_1}}{I(1-\alpha_3)}(M_U^{1-\alpha_3} - M_{turn,2}^{1-\alpha_3}) + \frac{M_L^{2-\alpha_1}}{2-\alpha_1}
\end{aligned} \tag{27}$$

m_{i+1} is the same as equation 16 but use the new k_3 in 25.

Finally, the mass of stars are the same as in equation 18

Part II.
ECMF

3. Symbols

M_L and M_U are the physical possible cluster mass limits in theoretical sense.
M_{max} is maximum mass cluster's integration upper limit.

M_L, M_{max} and M_U possess an index number name that will be used from equation 33
$M_L = m_{N_{tot}}$
$M_{max} = m_1$
$M_U = m_0$
Lower case letter m are all integration limits, not real cluster mass.
Existing cluster mass are indicated by M_1, M_2... in uppercase letter with a index start from most massive to less massive clusters.

In summary $M_U = m_0 > M_{max} = m_1 > M_1 > m_2 > M_2$...

$M_L = 5$ solar mass
M_U adopt 10^9

4. Derivation

$$\xi = kM^{-\beta}, \tag{28}$$

$$N_{tot} = \int_{M_L}^{M_{max}} \xi(M)\,\mathrm{d}M, \tag{29}$$

$$SFR * \delta t = M_{tot} = \int_{M_L}^{M_{max}} M\xi(M)\,\mathrm{d}M. \tag{30}$$

Divide both integrals 29 and 30 into N_{tot} separate integrals, each integral representing one individual object:

$$\begin{aligned} N_{tot} = &\int_{M_L}^{m_{N_{tot}-1}} \xi(M)\,\mathrm{d}M + \int_{m_{N_{tot}-1}}^{m_{N_{tot}-2}} \xi(M)\,\mathrm{d}M + ... \\ &+ \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{M_{max}} \xi(M)\,\mathrm{d}M, \end{aligned} \tag{31}$$

$$M_{tot} = \int_{M_L}^{m_{N_{tot}-1}} M\xi(M)\,\mathrm{d}M + \int_{m_{N_{tot}-1}}^{m_{N_{tot}-2}} M\xi(M)\,\mathrm{d}M + ... + \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M + ... + \int_{m_2}^{M_{max}} M\xi(M)\,\mathrm{d}M. \quad (32)$$

Each integral must give one object:

$$1 = \int_{m_{i+1}}^{m_i} \xi(M)\,\mathrm{d}M, \quad (33)$$

$$M_i = \int_{m_{i+1}}^{m_i} M\xi(M)\,\mathrm{d}M, \quad (34)$$

here $m_1 = M_{max}$ and $m_{N_{tot}} = M_L$.

optimally sampling condition:

$$I = \int_{m_1}^{m_0} \xi(M)\,\mathrm{d}M = \int_{M_{max}}^{M_U} \xi(M)\,\mathrm{d}M. \quad (35)$$

Inserting 28 to 35 for $\beta \neq 1$

$$I = k\int_{M_{max}}^{M_U} M^{-\beta}\,\mathrm{d}M = \frac{k}{1-\beta}(M_U^{1-\beta} - M_{max}^{1-\beta}). \quad (36)$$

$$k = \frac{I(1-\beta)}{M_U^{1-\beta} - M_{max}^{1-\beta}}. \quad (37)$$

We consider only the case $\beta > 1$ as indicated by observations.

If $M_U = \infty$ then

$$k = \frac{I(\beta-1)}{M_{max}^{1-\beta}} \quad (38)$$

For $\beta \neq 2$:

Inserting 28 to 30

$$M_{tot} = k\int_{M_L}^{M_{max}} M^{1-\beta}\,\mathrm{d}M = \frac{k}{2-\beta}(M_{max}^{2-\beta} - M_L^{2-\beta}). \quad (39)$$

Inserting 37 to 39

$$M_{tot} = I \cdot \frac{1-\beta}{2-\beta} \cdot \frac{M_{max}^{2-\beta} - M_L^{2-\beta}}{M_U^{1-\beta} - M_{max}^{1-\beta}} \quad (40)$$

For $\beta = 2$:

Equation 37 and 38 reduced to

$$k = \frac{I}{M_{max}^{-1} - M_U^{-1}}. \tag{41}$$

$$k = IM_{max} \tag{42}$$

Again, inserting 28 to 30

$$M_{tot} = k \int_{M_L}^{M_{max}} M^{-1}\,\mathrm{d}M = k(\ln M_{max} - \ln M_L). \tag{43}$$

Inserting 41 to 43

$$M_{tot} = I \cdot \frac{\ln M_{max} - \ln M_L}{M_{max}^{-1} - M_U^{-1}}, \tag{44}$$

Equation 40 and 44 give us

$$M_{max} = M_{max}(M_{tot}, I, M_U, M_L, \beta) \tag{45}$$

M_{tot} can be given from $SFR * \delta t$, with a fixed $\delta t \approx 10$ Myr and an assumed SFH. Thus M_{max} is determine by:

$$M_{max} = M_{max}(SFR, \delta t, I, M_U, M_L, \beta). \tag{46}$$

Inserting 37 and 46 in 28

$$\xi = \xi(M; SFR, \delta t, I, M_U, M_L, \beta). \tag{47}$$

Then the mass of each cluster generated can be calculated from equation 33 and 34.

Use 28, 33 and 37 can calculate every m_i:

$$m_{i+n} = \left(m_i^{1-\beta} - n \cdot \frac{1-\beta}{k} \right)^{\frac{1}{1-\beta}} \tag{48}$$

for $i > 0$.

Then use 28, 34 and 37 to calculate every M_i. For $\beta \neq 2$:

$$n \cdot \bar{M} = M_i + M_{i+1} + ... + M_{i+n-1} = \frac{k}{2-\beta} \cdot (m_i^{2-\beta} - m_{i+n}^{2-\beta}) \tag{49}$$

for $\beta = 2$:

$$n \cdot \bar{M} = M_i + M_{i+1} + ... + M_{i+n-1} = k(\ln m_i - \ln m_{i+n}). \tag{50}$$