REVIEW OF MODERN ANALYSIS I

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The goal of these notes is to consolidate our understanding of the material presented in the Fall 2019 offering of the undergraduate course "Introduction to Modern Analysis I" at Columbia University. The course follows closely Chapters 1–7 of the book authored by Rudin ([R+64]), and this is reflected in the organization of these notes. The idea will be to provide an outline of the central concepts and results as opposed to a transcription of the lectures; as such, many proofs have been omitted at the discretion of the authors. For the details omitted, we refer the reader to [R+64].

1. From the Rational Numbers to Real Numbers

Proposition 1.1. *The set $A = \{p \in \mathbb{Q} \mid p > 0, p^2 < 2\}$ has no largest element.*

Proof. Choose any $p \in A$. Then

$$q = \frac{2p+2}{p+2} = \frac{p^2+2p+2-p^2}{p+2} = p + \frac{2-p^2}{p+2}$$

is an element in A larger than p. We first argue q is in A. It is clear from the leftmost expression for q that $q > 0$ (because $p > 0$). The manipulations below show that $q^2 < 2$.

$$q^2 = \frac{(2p+2)^2}{(p+2)^2} = \frac{4p^2+8p+4}{(p+2)^2} = 2 - \frac{4-2p^2}{(p+2)^2} < 2.$$

Hence, $q \in A$. Since $2 - p^2 > 0$, the rightmost expression for q makes clear that $q > p$. $\square$

Similarly, it can be shown that $B = \{p \in \mathbb{Q} \mid p > 0, p^2 > 2\}$ has no smallest element. Together, these two statements reveal that even though $\sqrt{2}$ is not a rational number, there do exist arbitrarily close rational approximations from both above and below. Thus, the rational numbers have gaps "of zero length." In order to close these gaps, we introduce the notion of *least upper bound*.

Definition 1.2. *Let S be an ordered set, and let E be a subset of S. If there exists β in S such that $\beta \geq x$ for all x in S, then we say that β is an **upper bound** on E, and that E is **bounded above**.*

Definition 1.3. *Let S be an ordered set, and let E be a subset of S bounded above. Suppose there exists an element α in S such that*

(1) α is an upper bound of E, and
(2) an element $\gamma \in S$ is not an upper bound if $\gamma < \alpha$.

*That is, there is no smaller upper bound on E than α. Then α is the **least upper bound** on E.*

Definition 1.4. *An ordered set is said to satisfy **the least upper bound property** if every nonempty subset bounded above has a least upper bound.*

The least upper bound property will ensure for us that we do not have the gaps exhibited by $\mathbb{Q}$.

Definition 1.5. *An **ordered field** is an ordered set that is also a field, and such that*

Date: September 2020.

(1) if $x > y$, then $x + z > y + z$, and
(2) if $x > 0$ and $y > 0$, then $xy > 0$,

for any three elements x, y, z.

The two conditions in the definition of an ordered field ensure that the two structures (that of an ordered set, and that of a field) are compatible.

Theorem 1.6. *There exists an ordered field extension of $\mathbb{Q}$ which satisfies the least upper bound property. Moreover, this field is unique up to isomorphism.*

In the sequel, we denote this field by $\mathbb{R}$. The following result lists some of its key properties.

Theorem 1.7. *Let x, y be real numbers.*
(1) If $x > 0$, then there exists $n > 0$ such that $nx > y$.
(2) If $x < y$, then there exists a rational number q such that $x < q < y$.

The first property is known as the Archimedean Property and in essence asserts that you can surpass any finite quantity in finitely many finite-sized steps. The second property asserts that $\mathbb{Q}$ is dense in $\mathbb{R}$.

2. Cardinality of Sets

Definition 2.1. *For any positive integer n, let J_n denote the set $\{1, \ldots, n\}$. For any set A, we say*
*(1) A is **finite** if A is empty or if A is in bijection with J_n for some positive integer n,*
*(2) A is **infinite** if A is not finite,*
*(3) A is **countable** if A is in bijection with $\mathbb{N}$,*
*(4) A is **uncountable** if A is neither countable nor finite,*
*(5) A is **at most countable** if A is countable or finite.*

Theorem 2.2. *Every infinite subset of a countable set is countable.*

Definition 2.3. *A sequence is a function from $\mathbb{N}$ to a set.*

Theorem 2.4. *Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets, all countable. Then $\bigcup_{n=1}^{\infty} E_n$ is countable.*

3. Topology of Euclidean Space

Definition 3.1. *A **metric space** is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a function satisfying the following three properties.*
(1) d is positive definite, i.e., $d(p, q) = 0$ if and only if $p = q$.
(2) d is symmetric, i.e., $d(p, q) = d(q, p)$.
(3) d satisfies the triangle inequality, i.e., $d(p, q) \leq d(p, r) + d(r, q)$.

*Any function with these three properties is called a **metric** on X.*

Let (X, d) be a metric space. The metric d induces a topology on X as follows. For any point $x \in X$ and any positive $r > 0$, we define the open ball of radius r centered at x to be the set

$$N_r(x) = \{y \in X \mid d(x, y) < r\}.$$

We now define the topology on X induced by d to be the topology generated by all open balls. As Euclidean spaces are metric spaces, we obtain a topology on Euclidean space via this procedure.

Definition 3.2. *An **open cover** of a set E is a collection $\{G_\alpha\}_{\alpha \in A}$ of open sets such that $E \subseteq \bigcup_{\alpha \in A} G_\alpha$.*

Definition 3.3. *A set K is said to be **compact** every open cover $\{G_\alpha\}$ of K admits a finite subcover.*

Theorem 3.4 (Heine-Borel Theorem)**.** *The following are equivalent for all subsets E of $\mathbb{R}^k$.*

1. *E is closed and bounded.*
2. *E is compact.*
3. *Every infinite subset of E has a limit point in E.*

In general topological spaces, one still has the equivalence of (2) and (3), but the equivalence of (1) and (2) is particular to Euclidean spaces. It is this latter equivalence that is the assertion of the Heine-Borel Theorem.

Definition 3.5. *A set E is said to be **connected** if for any pair of open subsets A, B partitioning E, one of A or B is empty.*

Why do we care about compact and connected sets? The notion of compactness will lead us to the Extreme Value Theorem, and the notion of connectedness will lead us to the Intermediate Value Theorem. These are two central results in the theory of real-valued functions on $\mathbb{R}$.

4. Sequences and Series of Numbers

In this section, we turn our attention to sequences of real and complex numbers, but we before doing so, we begin in the setting of arbitrary metric spaces.

Definition 4.1 (Cauchy)**.** *Let $\{p_n\}$ be a sequence in a metric space (X, d), i.e., fix a function $f : \mathbb{N} \to X$ and put $p_n = f(n)$. We say $\{p_n\}$ is **Cauchy** if for every $\epsilon > 0$, we have $d(p_m, p_n) < \epsilon$ for large enough m, n; to be more precise, we have N in $\mathbb{N}$ such that $d(p_m, p_n) < \epsilon$ for all $m, n > N$.*

In other words, the points of $\{p_n\}$ tend to crowd together as n increases. The following notion of *diameter* is one measure of such crowding, and as it turns out, diameter provides an alternative characterization of Cauchy sequences.

Definition 4.2 (Diameter)**.** *Let (X, d) be a metric space and let E be a nonempty subset of X. Then the **diameter** $\operatorname{diam} E$ of E is defined to be the supremum of the set $\{d(p, q) \mid p, q \in E\}$.*

Note that the diameter of a nonempty set E is always well-defined, for if p is a point in E, then $0 = d(p, p) \in \{d(p, q) \mid p, q \in E\}$. Note also that diameter is an extended real number, because $\{d(p, q) \mid p, q \in E\}$ is in general not bounded above.

Theorem 4.3. *Let $\{p_n\}$ be a sequence in a metric space (X, d), and put $S_m = \{p_n \mid n \geq m\}$. Then $\{p_n\}$ is Cauchy if and only if*

$$\inf_{m \in \mathbb{N}} \operatorname{diam} S_m = 0.$$

Definition 4.4 (Convergence)**.** *Let $\{p_n\}$ be a sequence in a metric space (X, d). Then $\{p_n\}$ is said to **converge** to a point p in X if for every $\epsilon > 0$, we have $d(p_n, p) < \epsilon$ for large enough n; to be more precise, we have $N \in \mathbb{N}$ such that $d(p_n, p) < \epsilon$ for all $n > N$. The point p is then said to be the **limit point** of $\{p_n\}$, and $\{p_n\}$ is said to be **convergent**.*

Theorem 4.5. *Convergent sequences are Cauchy.*

In general, the converse does not hold. However, if it does hold for a metric space (X, d), i.e., every Cauchy sequence in X converges, then (X, d) is said to be *complete*.

Theorem 4.6. $(\mathbb{R}, |\cdot|)$ *is a complete metric space.*

5. Continuous Functions

Continuous functions arise naturally in the study of general topological spaces for the reason that they preserve (intrinsic) topological properties such as compactness and connectedness. Moreover, if one considers the category **Top** of topological spaces where morphisms are continuous functions, then isomorphisms correspond to *homeomorphisms*, and homeomorphisms provide an effective notion of equivalence for topological spaces. As a result, when investigating maps of topological spaces, continuous functions are a go-to candidate.

Of course, continuous functions find utility outside of topology as well; the typical math student is first acquainted with continuous functions in a first calculus course for a reason. Indeed, in the context of analysis, we see two of the most fundamental results of calculus—namely, the Extreme Value Theorem and the Intermediate Value Theorem—follow immediately from basic properties of continuous functions. We begin this section with a brief introduction to continuous functions on metric spaces and their properties. (There is not much to say.) We then demonstrate that the Extreme Value and Intermediate Value theorems are trivial consequences of continuity.

There is a characterization of real continuous functions on the real line using sequences, with which the reader is likely familiar with, but we begin instead from the more general definition of continuity below.

Definition 5.1 (Continuous)**.** *A mapping f of a metric space X into a metric space Y is* ***continuous*** *if $f^{-1}(V)$ is open in X for every open set V in Y.*

Equivalently, a function $f : X \to Y$ is continuous if preimages of closed sets are closed. From this definition and the characterization of closed sets as those subsets which contain their limit points, one recovers the usual analytic notion of continuity.

Theorem 5.2 (Sequential Continuity)**.** *Suppose X and Y are metric spaces, $E \subseteq X$ is any subset, and $p \in E$ is a limit point of E. Then a function $f : E \to Y$ is continuous at p if and only if*

$$\lim_{x \to p} f(x) = f(p).$$

Because the notions of continuous functions on metric spaces and of continuous functions on topological spaces coincide (here by definition), we immediately deduce from the principles of point-set topology the basic properties of continuous functions. In particular, we have the following.

Theorem 5.3 (Compositions of continuous functions are continuous)**.** *Suppose X, Y, Z are metric spaces, $E \subseteq X$ is any subset, f maps E into Y, g maps $f(E)$ into Z, and h is a mapping of E into Z defined by*

$$h(x) = g(f(x)) \qquad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at $f(p)$, then h is continuous at p.

Theorem 5.4. *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

From this and the Heine-Borel theorem, we obtain the Extreme Value Theorem.

Corollary 5.5 (Extreme Value)**.** *Suppose f is a continuous real function on a compact metric space X, and*

$$M = \sup_{p \in X} f(p), \qquad m = \inf_{q \in X} f(q).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Theorem 5.6. *Suppose f is a continuous mapping of a metric space X into a metric space Y. If E is a connected subset of X, then $f(E)$ is connected.*

From this last result, we immediately deduce the Intermediate Value Theorem.

Corollary 5.7 (Intermediate Value)**.** *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

The following is a special case of the more general statement that a continuous injection from a compact space into a Hausdorff space is a topological embedding.

Theorem 5.8. *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y is a continuous mapping of Y onto X.*

Here is a useful result particular to functions on $\mathbb{R}$.

Theorem 5.9. *Let f be monotonic on (a, b). Then the set of points (a, b) at which f is discontinuous is at most countable.*

6. Differentiable Functions

Definition 6.1 (derivative)**.** *Let f be defined on $[a, b]$. For any $x \in [a, b]$ form the quotient*

$$\phi(t) = \frac{f(t) - f(x)}{t - x}$$

and define

$$f'(x) = \lim_{t \to 0} \phi(t),$$

provided that this limit exists. We thus associate with the function f a function f' whose domain is the set of points x at which the limit exists; f' is called the ***derivative*** *of f.*

The following theorem tells us that differentiable functions are at least as nice as continuous functions.

Theorem 6.2 (differentiable $\implies$ continuous)**.** *Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

We thereby obtain a nested hierarchy of functions on any given interval $[a, b]$, starting from the continuous functions, the once-differentiable functions, the twice-differentiable functions, and so on. The space of all n-times differentiable functions with continuous nth derivative is denoted by $C^n([a, b])$, and the space of infinitely differentiable functions by $C^\infty([a, b])$.

$$C^0([a, b]) \supseteq C^1([a, b]) \supseteq C^2([a, b]) \supseteq \cdots$$

All of these spaces are $\mathbb{R}$-vector spaces.

7. Sequences and Series of Functions

In this section, we investigate sequences of functions and their calculus. Naturally, we expect a sequence of functions to converge to a function, but what function? And before that, how do we even define convergence? After all, the space of functions on a set does not have a canonical metric or topology.

Here is a natural definition. Let $\{f_n\}$ be a sequence of real functions enumerated by the positive integers, all defined on a common set E. For any given point $x \in E$, we can then define a new function $f : E \to \mathbb{R}$ at x by

$$f(x) = \lim_{n\to\infty} f_n(x),$$

provided that the limit exists for all $x \in E$, of course. If the limits exist, we say the f_n converge *pointwise*, and that f is the *limit function* of the sequence.

This notion of pointwise convergence we just defined is perfectly well-defined. The problem is that it is also perfectly useless. For instance, we would like for the limit of a sequence of continuous functions to also be continuous, for this allows us to construct continuous functions by limiting processes. However, this is simply not true for pointwise convergent sequences.

References

[R+64] Walter Rudin et al., *Principles of mathematical analysis*, vol. 3, McGraw-hill New York, 1964.