Online Appendix A

1. Proof of Lemma 1

Clearly $\mu_0 = E[X^0] = 1$. For a location-scale random variable X, we have $X = \theta_1 + \theta_2 Z$ where Z is the standardized random variable corresponding to X, when $\theta_1 = 0$ and $\theta_2 = 1$. Denoting $E[Z^j] = \kappa_j$, we obtain claim (a) from

$$E[X^j] = E[(\theta_1 + \theta_2 Z)^j] = \sum_{i=0}^{j} \binom{j}{i} \theta_1^i \theta_2^{j-i} \kappa_{j-i} \tag{16}$$

where the last equality follows from the binomial expansion of the term $(\theta_1 + \theta_2 Z)^j$.
To obtain (b) we substitute $X = \theta_1 + \theta_2 Z$ and $\mu_1 = \theta_1 + \theta_2 \kappa_1$ from (16) into $E[(X - \mu_1)^j]$ to get

$$E[(X - \mu_1)^j] = E[(\theta_1 + Z\theta_2 - \theta_1 - \kappa_1 \theta_2)^j] = \theta_2^j E[(Z - \kappa_1)^j] \tag{17}$$

From the binomial expansion of the last term, and using $E[Z^{j-i}] = \kappa_{j-i}$, we obtain $\theta_2^j E[(Z - \kappa_1)^j] = \theta_2^j \sum_{i=0}^{j} \binom{j}{i} (-\kappa_1)^i \kappa_{j-i}$. □

2. Proof of Theorem 1

The Lagrangian of the problem is

$$L = \mathbf{w}_k{}^{\mathrm{T}} \Omega \mathbf{w}_k - [\mathbf{Z}^{\mathrm{T}} \mathbf{w}_k - \mathbf{a}_k][\boldsymbol{\lambda}] \tag{18}$$

where $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]^{\mathrm{T}}$ is the vector of Lagrange multipliers for vector $\boldsymbol{\theta}^{\mathrm{T}} = (\theta_1, \theta_2) \in R \times R_{++}$. Then, the first order optimality conditions are given by the following $m + 2$ equations:

$$\nabla_{w_k} L = 2\Omega \mathbf{w}_k - \mathbf{Z}\boldsymbol{\lambda} = 0, \tag{19}$$

$$\nabla_{\lambda} L = -\mathbf{Z}^{\mathrm{T}} \mathbf{w}_k + \mathbf{a}_k = 0. \tag{20}$$

This is a set of $m + 2$ linear equations with the same number of unknowns. It can also be seen from the formulation that this system of equations has a full rank so a unique solution exists. We first reduce (19) by solving for $\boldsymbol{\lambda}$; to this end we pre-multiply (19) by $\mathbf{Z}^{\mathrm{T}} \Omega^{-1}$, and obtain

$$2\mathbf{Z}^{\mathrm{T}} \mathbf{w}_k = \mathbf{Z}^{\mathrm{T}} \Omega^{-1} \mathbf{Z} \boldsymbol{\lambda} \tag{21}$$

The inverse Ω^{-1} exists because Ω is positive definite. Now, substituting $\mathbf{Z}^{\mathrm{T}} \mathbf{w}_k = \mathbf{a}_k$ from (20), we obtain from (21)

$$\boldsymbol{\lambda} = 2(\mathbf{Z}^{\mathrm{T}} \Omega^{-1} \mathbf{Z})^{-1} \mathbf{a}_k \tag{22}$$

The inverse $(\mathbf{Z}^{\mathrm{T}} \Omega^{-1} \mathbf{Z})^{-1}$ exists because $\mathbf{Z}$ has a full rank and Ω^{-1} is positive definite. Substituting $\boldsymbol{\lambda}$ from (22) into (19) reduces (19) to a system of m equations with m unknowns, and we solve this system as follows

$$2\Omega \mathbf{w}_k - \mathbf{Z}\boldsymbol{\lambda} = 0 \Rightarrow 2\Omega \mathbf{w}_k = 2\mathbf{Z}(\mathbf{Z}^{\mathrm{T}} \Omega^{-1} \mathbf{Z})^{-1} \mathbf{a}_k \Rightarrow \mathbf{w}_k = \Omega^{-1} \mathbf{Z} (\mathbf{Z}^{\mathrm{T}} \Omega^{-1} \mathbf{Z})^{-1} \mathbf{a}_k \tag{23}$$

□

3. Proof of Proposition 2

We showed above that

$$\mathbf{w}_k{}^{\mathrm{T}}\mathbf{Z} = \mathbf{a}_k{}^{\mathrm{T}}. \tag{24}$$

For the estimation of mean μ_1, we have $\mathbf{a}_1 = [1, \kappa_1]$, which then implies that $\sum w^*_{1_i} = 1$. Similarly, for the estimation of standard deviation μ_2, $\mathbf{a}_2 = \left[0, \sqrt{\kappa_2 - \kappa_1^2}\right]$, which implies that $\sum w^*_{2i} = 0$. □

4. Proof of the Variance of $\hat{\mu}_k$ for Section 5.1

We are interested in the variance of $\mathbf{w}_k{}^{\mathrm{T}}\hat{\mathbf{q}}$. Since $\mathbf{w}_k{}^{\mathrm{T}}\hat{\mathbf{q}} = \mathbf{w}_k{}^{\mathrm{T}}(\mathbf{Z}\boldsymbol{\theta} + \boldsymbol{\epsilon}) = \mathbf{w}_k{}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\theta} + \mathbf{w}_k{}^{\mathrm{T}}\boldsymbol{\epsilon}$ and the term $\mathbf{w}_k{}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\theta}$ is a constant, it follows that the variance of $\mathbf{w}_k{}^{\mathrm{T}}\hat{\mathbf{q}}$ is equal to the variance of $\mathbf{w}_k{}^{\mathrm{T}}\boldsymbol{\epsilon}$ which is given by $\mathbf{w}_k{}^{\mathrm{T}}\Omega\mathbf{w}_k$ for $k = 1, 2$. Replacing the expressions for $\mathbf{w}_k$ obtained in Theorem 1 we obtain the variance as

$$\mathbf{w}_k{}^{\mathrm{T}}\Omega\mathbf{w}_k = \mathbf{a}_k{}^{\mathrm{T}}(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\Omega\Omega^{-1}\mathbf{Z}(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{a}_k = \mathbf{a}_k{}^{\mathrm{T}}(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{a}_k. \tag{25}$$

Note that $\mathbf{a}_1 = [1, \kappa_1]^{\mathrm{T}}$ for estimating μ_1, $\mathbf{a}_2 = \left[0, \sqrt{\kappa_2 - \kappa_1^2}\right]^{\mathrm{T}}$ for estimating μ_2.

□

5. Proof of Proposition 3

(a) For the mean, we note that variance of the distribution of a sample mean of size N_1 is equal to μ_2^2/N_1 where μ_2^2 is the population variation. Equating this variance with the variance $[1, \kappa_1](\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}[1, \kappa_1]^{\mathrm{T}}$ obtained above in (25), we obtain $N_1 = \frac{{\mu_2}^2}{[1,\kappa_1](\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}[1,\kappa_1]^{\mathrm{T}}}$.

(b) Stuart and Ord (1994) show on p 352 that the variance of sample standard deviation can be approximated as

$$Var(S) \approx \frac{E[(x-\mu_1)^4] - (E[(x-\mu_1)^j])^2}{4N_2\mu_2^2} \tag{26}$$

We know from Lemma 1 that

$$E[(X-\mu_1)^i] = \theta_2^i \sum_{j=0}^{i} (-\kappa_1)^j \kappa_{i-j} \tag{27}$$

From Lemma 1, we know that $\theta_2 = \mu_2/\sqrt{\kappa_2 - \kappa_1^2}$. Substituting this in (27), we obtain $E[(X-\mu_1)^i] = \mu_2^i \sum_{j=0}^{i}(-\kappa_1)^j\kappa_{i-j} / \left(\sqrt{\kappa_2 - \kappa_1^2}\right)^i$ Substituting this in (26), we obtain

$$Var(S) \approx \frac{{\mu_2}^2}{4N_2}\left(\frac{\sum_{j=0}^{4}(-\kappa_1)^j\kappa_{4-j}}{(\kappa_2-\kappa_1^2)^2} - \frac{\left(\sum_{j=0}^{2}(-\kappa_1)^j\kappa_{2-j}\right)^2}{(\kappa_2-\kappa_1^2)^2}\right) \tag{28}$$

Equating this with the variance in (25) $[0, \sqrt{\kappa_2 - \kappa_1^2}](\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}[0, \sqrt{\kappa_2 - \kappa_1^2}]^{\mathrm{T}}$, we obtain

$$N_2 \approx \frac{{\mu_2}^2\left(\frac{\sum_{j=1}^{4}(-\kappa_1)^j\kappa_{4-j}}{(\kappa_2-\kappa_1^2)^2} - \frac{\left(\sum_{j=1}^{2}(-\kappa_1)^j\kappa_{2-j}\right)^2}{(\kappa_2-\kappa_1^2)^2}\right)}{4[0,\sqrt{\kappa_2-\kappa_1^2}](\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}[0,\sqrt{\kappa_2-\kappa_1^2}]^{\mathrm{T}}}.$$ □

6. Proof of Proposition 4

(a) We start by noting the general result in Theorem 1: $\mathbf{w}_k^{*\mathrm{T}} = \mathbf{a}_k{}^{\mathrm{T}}(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}\Omega^{-1}$. Next, we define each of these components when j=*1,2,...,n* experts provide quantile judgments. The matrix $\mathbf{Z}^{\mathrm{T}} = [\mathbf{Z_0}^{\mathrm{T}}, \mathbf{Z_0}^{\mathrm{T}}, ..., \mathbf{Z_0}^{\mathrm{T}}]$ where $\mathbf{Z_0}^{\mathrm{T}}$ appears n times, once for each expert. The subscript 0 simply suggests that this is a constant matrix since all experts provide judgments for the same set of quantiles.. The matrix Ω is a $mn \times mn$ block diagonal matrix with diagonal blocks $r_j\Omega_0$ where Ω_0 is of size $m \times m$. Then,

1. $(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z}) = \mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}\mathbf{Z}_0[\sum_{j=1}^{N}(1/r_j)] = \mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}\mathbf{Z}_0 R$ where $R = [\sum_{j=1}^{N}(1/r_j)]$
2. $\mathbf{Z}^{\mathrm{T}}\Omega^{-1} = [\mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}(1/r_1), \mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}(1/r_2), ..., \mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}(1/r_N)]$
3. It follows from points 1 and 2 above that $\mathbf{w}_k^{*\mathrm{T}} = \mathbf{a}_k{}^{\mathrm{T}}(\mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}\mathbf{Z}_0)^{-1}\mathbf{Z}_0^{\mathrm{T}}\Omega_0^{-1}[(1/r_1)/R, (1/r_2)/R, ..., (1/r_N)/R]$.

Now we can write this expression as $\mathbf{w}_k^{*\mathrm{T}} = \mathbf{w}_k^{c\mathrm{T}}[(1/r_1)/R, (1/r_2)/R, ..., (1/r_N)/R]$ where $\mathbf{w}_k^{c\mathrm{T}} = \mathbf{a}_k{}^{\mathrm{T}}(\mathbf{Z_0}^{\mathrm{T}}\Omega_0^{-1}\mathbf{Z}_0)^{-1}\mathbf{Z}_0^{\mathrm{T}}\Omega_0^{-1}$. Further, the vector $\mathbf{w}_k^{*\mathrm{T}}$ is composed of the m weights for each expert j: $\mathbf{w}_k^{*\mathrm{T}} = [\mathbf{w}_k^{1\mathrm{T}}, \mathbf{w}_k^{2\mathrm{T}}, ..., \mathbf{w}_k^{n\mathrm{T}}]$. It follows from these two relations that $\mathbf{w}_k^j = \alpha_j \mathbf{w}_k^c$ where the expert j's marginal weight is equal to $\alpha_j = (1/r_j)/R$.

(b) Consider the case when expert j is the only expert available with matrix Ω_0 for eliciting quantiles $\mathbf{Z}_0$. Then $R = (1/r_j)$. Substituting this expression in point 3 above, it follows that the weight for this expert is equal to $(1/r_j)/R = 1$, and the weights for his quantile judgments are equal to his independent weights.

□

7. Proof for Proposition 5: 1) The proof follows from the text above the proposition.

2) We will first obtain explicit expressions for the weights $\mathbf{w}_k^* = \Omega^{-1}\mathbf{Z}(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{a}_k$ with $\Omega = K\Omega'$ with $\Omega' = I$, and then show that these weights are identical to the weights obtained using the formulation $\min_{\mu_1,\mu_2}\left\{\sum_{i=1}^{m}\left(\Phi^{-1}(p_i;\mu_1,\mu_2) - \hat{q}_i\right)^2\right\}$. For rigor purposes, we provide the result for a more general case when the diagonals elements of Ω' are equal to 1, and the off-diagonal elements are equal to the correlation value ρ, and for brevity we will provide the proof for μ_1. The analysis for μ_2 is analogous.

To obtain the explicit expressions of the weights we need three intermediate results.

(a) $\Omega^{-1} = \frac{1}{(1-\rho)K^2}\left(I - \frac{\rho M_1}{1+(m-1)\rho}\right)$. To establish this claim define M_1 as an $(m \times m)$ matrix of ones and verify that

$$\begin{aligned}\Omega\Omega^{-1} &= \frac{1}{1-\rho}\Big((1-\rho)I + \rho M_1\Big)\Big(I - \frac{\rho}{1+(m-1)\rho}M_1\Big),\\ &= \frac{1}{1-\rho}\left[(1-\rho)I + \rho M_1 - \frac{\rho(1-\rho)}{1+(m-1)\rho}M_1 - \frac{\rho^2 m}{1+(m-1)\rho}M_1\right] = I.\end{aligned}$$

(b)

$$\mathbf{Z}^{\mathrm{T}}\Omega^{-1} = \frac{1}{K^2(1+(m-1)\rho)}\begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \frac{(1+(m-1)\rho)z_1 - S_1\rho}{1-\rho} & \cdots & \frac{(1+(m-1)\rho)z_i - S_1\rho}{1-\rho} & \cdots & \frac{(1+(m-1)\rho)z_m - S_1\rho}{1-\rho}\end{pmatrix}.$$

To establish this claim we use (a) to obtain

$$\begin{aligned}\mathbf{Z}^{\mathrm{T}}\Omega^{-1} &= \frac{1}{(1-\rho)K^2}\begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ z_1 & \cdots & z_i & \cdots & z_m\end{pmatrix}\Big(I - \frac{\rho}{1+(m-1)\rho}M_1\Big)\\ &= \frac{1}{K^2(1+(m-1)\rho)}\begin{pmatrix} 1 & \cdots & 1 \\ \frac{(1+(m-1)\rho)z_1 - S_1\rho}{1-\rho} & \cdots & \frac{(1+(m-1)\rho)z_m - S_1\rho}{1-\rho}\end{pmatrix}.\end{aligned}$$

(c) Denote $S_1 \equiv \sum_{i=1}^{m} z_i$ and $S_2 \equiv \sum_{i=1}^{m} z_i^2$, then the inverse of (2×2) matrix $\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z}$ is obtained as

$$(\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1} = \frac{K^2}{mS_2 - S_1^2}\begin{pmatrix} S_2(1+(m-1)\rho) - S_1^2\rho & -S_1(1-\rho) \\ -S_1(1-\rho) & m(1-\rho)\end{pmatrix}.$$

Therefore, the weights for the mean are obtained as $\mathbf{w}^*_{\mu_1} = [1, \kappa_1](\mathbf{Z}^{\mathrm{T}}\Omega^{-1}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}\Omega^{-1}$, which on simplification reduce to

$$w_i = \big(S_2 - z_i S_1 + n z_i \kappa_1 - S_1\kappa_1 - 2\kappa_1 S_1 + 2m\kappa_1^2\big) m S_2 - S_1^2 (29)$$

2) We now show that these weights coincide with the weights obtained for the minimization of the least squares $\min_{\mu_1,\mu_2} F = \left\{ \sum_{i=1}^m \left(\mu_2 - \frac{\kappa_1}{\sqrt{\kappa_2-\kappa_1^2}}\mu_2 + z_i \frac{\mu_2}{\sqrt{\kappa_2-\kappa_1^2}} - \hat{q}_i\right)^2 \right\}$. We will drop the indices over the summation in the rest of the proof. Taking the first order derivatives, we obtain:

$$\frac{\partial F}{\partial \mu_1} = 2\sum \left(\mu_1 + \mu_2 \frac{z_i - \kappa_1}{\sqrt{\kappa_2 - \kappa_1^2}} - \hat{q}_i\right) = 0 \tag{30}$$

$$\frac{\partial F}{\partial \mu_2} = 2\sum \left(\mu_1 + \mu_2 \frac{z_i - \kappa_1}{\sqrt{\kappa_2 - \kappa_1^2}} - \hat{q}_i\right) \frac{z_i - \kappa_1}{\sqrt{\kappa_2 - \kappa_1^2}} = 0 \tag{31}$$

Now, we can write (30) as $m\mu_1 + \mu_2 \frac{\sum z_i}{\sqrt{\kappa_2-\kappa_1^2}} - \mu_2 \frac{m\kappa_1}{\sqrt{\kappa_2-\kappa_1^2}} - \sum \hat{q}_i = 0$, or, using $\sum z_i = S_1$ equivalently,

$$\mu_1 = \frac{\mu_2(m\kappa_1 - S_1)}{m\sqrt{\kappa_2 - \kappa_1^2}} + \frac{\sum \hat{q}_i}{m} \tag{32}$$

Next, we can simplify (31) using $\sum z_i^2 = S_2$ and obtain,

$$\mu_2 = \frac{\sqrt{\kappa_2 - \kappa_1^2}\left(\sum q_i z_i - \sum q_i \kappa_1 + \mu_1(S_1 - m\kappa_1)\right)}{S_2 - 2\kappa_1 S_1 + n\kappa_1^2} \tag{33}$$

Now, substituting (33) into (32), and simplification, we obtain

$$\mu_1 = \frac{\sum q_i \big(S_2 - z_i S_1 + n z_i \kappa_1 - S_1\kappa_1 - 2\kappa_1 S_1 + 2m\kappa_1^2\big)}{m S_2 - S_1^2} \tag{34}$$

which implies the weights of

$$w_{\mu_i} = \frac{\big(S_2 - z_i S_1 + n z_i \kappa_1 - S_1\kappa_1 - 2\kappa_1 S_1 + 2m\kappa_1^2\big)}{m S_2 - S_1^2} \tag{35}$$

These weights are identical to the weights obtained in the least squares formulation in (29). The weights for μ_2 can be shown to be equal similarly. □

References:

Stuart, Alan, J Keith Ord. 1994. Kendalls advanced theory of statistics. vol. i. distribution theory. *Arnold, London* .