

## 第三节、函数的求导法则



一、四则运算求导法则



二、反函数的求导法则



三、复合函数求导法则



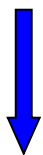
四、初等函数的求导问题



五、微分运算法则

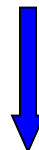
思路:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{构造性定义})$$

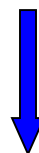


$$\left\{ \begin{array}{l} (C)' = 0 \\ (\cos x)' = -\sin x \\ (\ln x)' = \frac{1}{x} \end{array} \right.$$

本节内容



求导法则



其它基本初等  
函数求导公式

初等函数求导问题

## 一、四则运算求导法则

**定理1** 函数  $u = u(x)$  及  $v = v(x)$  都在点  $x$  具有导数

→  $u=u(x)$  及  $v=v(x)$  的和、差、积、商 (除分母为0的点外) 都在点  $x$  可导, 且

$$(1) \quad (u(x) \pm v(x))' = u'(x) \pm v'(x)$$

$$(2) \quad (u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

$$(3) \quad \left( \frac{u(x)}{v(x)} \right)' = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{(v(x))^2} \quad (v(x) \neq 0)$$

下面分三部分加以证明, 同时给出相应的推论和例题.

$$(1) \quad (u \pm v)' = u' \pm v'$$

证： 设  $f(x) = u(x) \pm v(x)$ , 则

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(u(x+h) \pm v(x+h)) - (u(x) \pm v(x))}{h} \\ &= \lim_{h \rightarrow 0} \frac{(u(x+h) - u(x))}{h} \pm \lim_{h \rightarrow 0} \frac{(v(x+h) - v(x))}{h} \\ &= u'(x) \pm v'(x). \end{aligned} \quad \text{故结论成立.}$$

此法则可推广到任意有限项的情形. 例如,

$$(u + v - w)' = u' + v' - w'$$

$$(2) \quad (uv)' = u'v + uv'$$

证 设  $f(x) = u(x)v(x)$ , 则有

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{u(x+h) - u(x)}{h} v(x+h) + u(x) \frac{v(x+h) - v(x)}{h} \right] \\ &= u'(x)v(x) + u(x)v'(x). \quad \text{故结论成立.} \end{aligned}$$

推论 1)  $(Cu)' = Cu'$  ( $C$ 为常数)

$$2) \quad (uvw)' = u'vw + uv'w + uvw'$$

$$3) \quad (\log_a x)' = \left( \frac{\ln x}{\ln a} \right)' = \frac{1}{x \ln a}$$

例1.  $y = \sqrt{x} (x^3 - 4\cos x - \sin 1)$ , 求  $y'$  及  $y'|_{x=1}$ .

解: 
$$\begin{aligned} y' &= (\sqrt{x})' (x^3 - 4\cos x - \sin 1) \\ &\quad + \sqrt{x} (x^3 - 4\cos x - \sin 1)' \\ &= \frac{1}{2\sqrt{x}} (x^3 - 4\cos x - \sin 1) + \sqrt{x} (3x^2 + 4\sin x) \\ y'|_{x=1} &= \frac{1}{2} (1 - 4\cos 1 - \sin 1) + (3 + 4\sin 1) \\ &= \frac{7}{2} + \frac{7}{2}\sin 1 - 2\cos 1. \end{aligned}$$

$$(3) \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2}, \quad v \neq 0.$$

证： 设  $f(x) = \frac{u(x)}{v(x)}$ ， 则有

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{\frac{u(x+h) - u(x)}{h} v(x) - u(x) \frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)} \right] \end{aligned}$$

$$= \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

故结论成立.

推论：  $\left( \frac{C}{v} \right)' = \frac{-C v'}{v^2} \quad (C \text{ 为常数})$

例2 求证  $(\tan x)' = \sec^2 x$  ,  $(\csc x)' = -\csc x \cot x$  .


证 
$$(\tan x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x .$$

$$(\csc x)' = \left( \frac{1}{\sin x} \right)' = \frac{-(\sin x)'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$
$$= -\csc x \cot x .$$

类似可证:  $(\cot x)' = -\csc^2 x$  ,  $(\sec x)' = \sec x \tan x$  .



## 二、反函数的求导法则

**定理2.** 设  $y = f(x)$  为  $x = f^{-1}(y)$  的反函数,  $f^{-1}(y)$  在  $y$  的某邻域内单调可导, 且  $[f^{-1}(y)]' \neq 0$  

$$f'(x) = \frac{1}{[f^{-1}(y)]'} \quad \text{或} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

**证** 在  $x$  处给增量  $\Delta x \neq 0$ , 由反函数的单调性知

$$\Delta y = f(x + \Delta x) - f(x) \neq 0, \quad \therefore \frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$$

且由反函数的连续性知  $\Delta x \rightarrow 0$  时必有  $\Delta y \rightarrow 0$ , 因此

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \rightarrow 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{[f^{-1}(y)]'}$$

例3 求反三角函数  $y = \arcsin x$  的导数。

解 设  $y = \arcsin x$ , 则  $x = \sin y$ ,  $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ,

$\therefore \cos y > 0$ , 则

$$(\arcsin x)' = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

利用

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

类似可求得

$$(\arctan x)' = \frac{1}{1 + x^2}, \quad (\operatorname{arccot} x)' = -\frac{1}{1 + x^2}.$$

## 小结

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

### 三、复合函数求导法则

定理3.  $u = g(x)$  在点  $x$  可导,  $y = f(u)$  在点  $u = g(x)$  可导  $\longrightarrow$  复合函数  $y = f(g(x))$  在点  $x$  可导, 且

$$\frac{dy}{dx} = f'(u)g'(x)$$

证  $\because y = f(u)$  在点  $u$  可导, 故  $\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = f'(u),$

$\therefore \Delta y = f'(u)\Delta u + \alpha\Delta u,$  当  $\Delta u \rightarrow 0$  时,  $\alpha \rightarrow 0,$

故有  $\frac{\Delta y}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \alpha\frac{\Delta u}{\Delta x} \quad (\Delta x \neq 0)$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left[ f'(u)\frac{\Delta u}{\Delta x} + \alpha\frac{\Delta u}{\Delta x} \right] = f'(u)g'(x)$$

补充定义

$$\alpha|_{\Delta u=0} = 0$$

推广：此法则可推广到多个中间变量的情形.

例如  $y = f(u), u = \varphi(v), v = \psi(x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\ &= f'(u) \cdot \varphi'(v) \cdot \psi'(x).\end{aligned}$$

关键：搞清复合函数结构，由外向内逐层求导.

$y$   
 $\downarrow$   
 $u$   
 $\downarrow$   
 $v$   
 $\downarrow$   
 $x$

$$\begin{array}{l} y=f(u) \\ u=g(x) \end{array} \quad \frac{dy}{dx} = f'(u)g'(x)$$

例4. 求下列导数: (1)  $(x^\mu)'$ ; (2)  $(x^x)'$ ; (3)  $(\operatorname{sh} x)'$ .

解: (1)  $(x^\mu)' = (e^{\mu \ln x})' = e^{\mu \ln x} \cdot (\mu \ln x)' = x^\mu \cdot \frac{\mu}{x} = \mu x^{\mu-1}$

(2)  $(x^x)' = (e^{x \ln x})' = e^{x \ln x} \cdot (x \ln x)' = x^x (\ln x + 1)$

(3)  $(\operatorname{sh} x)' = \left( \frac{e^x - e^{-x}}{2} \right)' = \frac{e^x + e^{-x}}{2} = \operatorname{ch} x.$

说明: 类似可得

$$(\operatorname{ch} x)' = \operatorname{sh} x; \quad (\operatorname{th} x)' = \frac{1}{\operatorname{ch}^2 x}; \quad (a^x)' = a^x \ln a.$$

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x}$$

$$a^x = e^{x \ln a}$$

例5. 设  $y = e^{\cos(e^x)}$ , 求  $\frac{dy}{dx}$ .

解 
$$\begin{aligned}\frac{dy}{dx} &= e^{\cos(e^x)} (\cos(e^x))' \\ &= e^{\cos(e^x)} \cdot (-\sin(e^x)) \cdot e^x = -e^{x+\cos(e^x)} \cdot \sin(e^x)\end{aligned}$$

思考: 若  $f'(u)$  存在, 如何求  $f(\ln \cos(e^x))$  的导数?

$$\frac{df}{dx} = f'(\ln \cos(e^x)) \cdot (\ln \cos(e^x))' = \dots$$

这两个记号含义不同

$$f'(u) \Big|_{u=\ln \cos(e^x)}$$


练习: 设  $y = f(f(f(x)))$ , 其中  $f(x)$  可导, 求  $y'$ .

例6. 设  $y = \ln(x + \sqrt{x^2 + 1})$ , 求  $y'$ .

解  $y' = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left( 1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \right) = \frac{1}{\sqrt{x^2 + 1}}$

记  $\operatorname{arsh} x = \ln(x + \sqrt{x^2 + 1})$ , 则  
(反双曲正弦)

$$(\operatorname{arsh} x)' = \frac{1}{\sqrt{x^2 + 1}}$$



$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2}$$

的反函数



## 四、初等函数的求导问题

### 1. 常数和基本初等函数的导数

$$(C)' = 0$$

$$(\sin x)' = \cos x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(a^x)' = a^x \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(x^\mu)' = \mu x^{\mu-1}$$

$$(\cos x)' = -\sin x$$

$$(\cot x)' = -\csc^2 x$$

$$(\csc x)' = -\csc x \cot x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

## 2. 有限次四则运算的求导法则

$$(u \pm v)' = u' \pm v'$$

$$(Cu)' = Cu' \quad (C \text{ 为常数})$$

$$(uv)' = u'v + uv'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$$

## 3. 复合函数求导法则

$$y = f(u), u = \varphi(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot \varphi'(x)$$

说明：最基本的公式

$$(C)' = 0$$

$$(\sin x)' = \cos x$$

$$(\ln x)' = \frac{1}{x}$$

由定义证，其它公式  
用求导法则推出。

例7.  $y = \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}$ , 求  $y'$ .

解:  $\because y = \frac{2x - 2\sqrt{x^2 - 1}}{2} = x - \sqrt{x^2 - 1}$

$$\therefore y' = 1 - \frac{1}{2\sqrt{x^2 - 1}} \cdot (2x) = 1 - \frac{x}{\sqrt{x^2 - 1}}$$

例8. 设  $y = x^{a^a} + a^{x^a} + a^{a^x} (a > 0)$ , 求  $y'$ .

解  $y' = a^a x^{a^a - 1} + a^{x^a} \ln a \cdot a x^{a-1} + a^{a^x} \ln a \cdot a^x \ln a$

**例9**  $y = e^{\sin x^2} \arctan \sqrt{x^2 - 1}$ , 求  $y'$ .

**解**

$$\begin{aligned} y' &= (e^{\sin x^2} \cdot \cos x^2 \cdot 2x) \arctan \sqrt{x^2 - 1} \\ &\quad + e^{\sin x^2} \left( \frac{1}{x^2} \cdot \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x \right) \\ &= 2x \cos x^2 e^{\sin x^2} \arctan \sqrt{x^2 - 1} + \frac{1}{x\sqrt{x^2 - 1}} e^{\sin x^2} \end{aligned}$$

**关键：** 搞清复合函数结构  
由外向内逐层求导

**例10** 设  $y = \frac{1}{2} \arctan \sqrt{1+x^2} + \frac{1}{4} \ln \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}$ , 求  $y'$ .

解

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{x}{\sqrt{1+x^2}} \\ &+ \frac{1}{4} \left( \frac{1}{\sqrt{1+x^2}+1} \cdot \frac{x}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}-1} \cdot \frac{x}{\sqrt{1+x^2}} \right) \\ &= \frac{1}{2} \frac{x}{\sqrt{1+x^2}} \left( \frac{1}{2+x^2} - \frac{1}{x^2} \right) \\ &= \frac{-1}{(2x+x^3)\sqrt{1+x^2}} \end{aligned}$$