第四节、换元积分法

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一、第一类换元积分法

问题 $\int \cos 2x dx \overset{?}{=} \sin 2x + C,$

解决方法 利用复合函数，设置中间变量.

过程 令 $t = 2x \Rightarrow dx = \frac{1}{2}dt,$

$$\int \cos 2x dx = \boxed{\frac{1}{2}\int \cos t dt} = \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C.$$

分析：

在一般情况下：

设 $F'(u)=f(u)$，则 $\int f(u)du=F(u)+C.$

设 $F'(u)=f(u)$，$\underline{u=\varphi(x)}$ 可导，则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore \quad \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C=F(u)+C\Big|_{u=\varphi(x)}$$

$$=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\mathrm{d}u$$

定理1. 设 $f(u)$ 有原函数，$u=\varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\underline{\varphi'(x)\mathrm{d}x} = \int f(u)\mathrm{d}u \Big|_{u=\varphi(x)}$$

即 $\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$

(也称配元法, 凑微分法)

注 使用此公式的关键在于将

$\int g(x)dx$ 化为 $\int f[\varphi(x)]\varphi'(x)dx.$

观察重点不同，所得结论不同.

例1. 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1)$.

解: 令 $u = ax+b$,则 $\mathrm{d}u = a\mathrm{d}x$, 故

$$\text{原式} = \int u^m \ \frac{1}{a}\mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$$

$$= \frac{1}{a(m+1)}(ax+b)^{m+1} + C$$

注: 当 $m = -1$ 时

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

例2 求 $\int \sin 2x dx.$

解法1 $\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

解法2 $\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C;$

解法3 $\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C.$

一般常用的第一类换元积分法有如下几种形式:

$$\boxed{(1)\ \int f(ax+b)\mathrm{d}x =} \frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$=\frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$$

例3 求 $\int \frac{1}{3+2x}dx.$

解 $$\int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x}\cdot(3+2x)'dx$$

$$=\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|3+2x|+C.$$

例4. 求 $\int \frac{\mathrm{d}x}{x^2-a^2}$.

解:

$$\because \frac{1}{x^2-a^2} = \frac{1}{2a}\frac{(x+a)-(x-a)}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\therefore \text{原式} = \frac{1}{2a}\left[\int\frac{\mathrm{d}x}{x-a}-\int\frac{\mathrm{d}x}{x+a}\right]$$

$$= \frac{1}{2a}\left[\int\frac{\mathrm{d}(x-a)}{x-a}-\int\frac{\mathrm{d}(x+a)}{x+a}\right]$$

$$= \frac{1}{2a}\left[\ln|x-a|-\ln|x+a|\right]+C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

例5 求 $\int \frac{x}{(1+x)^3}dx.$

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]d(1+x)$

$= -\frac{1}{1+x} + \frac{1}{2(1+x)^2} + C.$

一般常用的第一类换元积分法有如下几种形式:

$$(1)\ \boxed{\int f(ax+b)\mathrm{d}x =} \frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$(2)\ \boxed{\int f(x^n)x^{n-1}\,\mathrm{d}x =} \frac{1}{n}\int f(x^n)\ \mathrm{d}x^n$$

例6 求 $\displaystyle\int \frac{x}{4+x^2}\mathrm{d}x$

解 $$\int \frac{x}{4+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2)}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(4+x^2)}{4+x^2}$$

$$= \frac{1}{2}\ln(4+x^2)+C$$

例7. 求 $\displaystyle\int \frac{x^3}{(x^2+a^2)^{3/2}}\,\mathrm{d}x$.

解: 原式 $\displaystyle= \frac{1}{2}\int \frac{x^2\,\mathrm{d}x^2}{(x^2+a^2)^{3/2}} = \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)$$

$$-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C$$

一般常用的第一类换元积分法有如下几种形式:

$$(1)\ \int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\ \mathrm{d}(ax+b)$$

$$(2)\ \int f(x^n)x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\ \mathrm{d}x^n$$

$$(3)\ \int f(\sqrt{x})\frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\int f(\sqrt{x})\ \mathrm{d}\sqrt{x}$$

例8. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$.

解: 原式 $= 2\int e^{3\sqrt{x}}\mathrm{d}\sqrt{x} = \frac{2}{3}\int e^{3\sqrt{x}}\mathrm{d}(3\sqrt{x})$

$$= \frac{2}{3}e^{3\sqrt{x}} + C$$

一般常用的第一类换元积分法有如下几种形式:

$$\boxed{(4)\ \int f(\sin x)\cos x\,\mathrm{d}x =} \int f(\sin x)\,\mathrm{d}\sin x$$

$$\boxed{(5)\ \int f(\cos x)\sin x\,\mathrm{d}x =} -\int f(\cos x)\,\mathrm{d}\cos x$$

$$\boxed{(6)\ \int f(\tan x)\sec^2 x\,\mathrm{d}x =} \int f(\tan x)\,\mathrm{d}\tan x$$

例9. 求$\int \tan x\,\mathrm{d}x$.

解: $\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$$= -\ln\left|\cos x\right| + C$$

类似

$$\int \cot x\mathrm{d}x = \int \frac{\cos x\,\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$$

$$= \ln\left|\sin x\right| + C$$

例10. 求$\int \sec^6 x\mathrm{d}x$.

解: 原式 $= \int \sec^4 x\cdot \sec^2 x dx$

$$= \int (\tan^2 x + 1)^2 d\tan x$$

$$= \int (\tan^4 x + 2\tan^2 x + 1)\mathrm{d}\tan x$$

$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$$

例11. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \int \frac{1}{2}\left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right) \mathrm{d}\sin x$$

$$= \frac{1}{2}\left[\ln\left|1+\sin x\right| - \ln\left|1-\sin x\right|\right] + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

解法 2 $\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \mathrm{d}x$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left| \sec x + \tan x \right| + C$$

同样可证

$$\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$$

或 $\int \csc x \mathrm{d}x = \ln\left|\tan \frac{x}{2}\right| + C$

例12. 求 $\int \cos^4 x \,\mathrm{d}x$.

解: $\because \quad \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$$= \tfrac{1}{4}(1+2\cos 2x+\cos^2 2x)$$

$$= \tfrac{1}{4}(1+2\cos 2x+\tfrac{1+\cos 4x}{2})$$

$$= \tfrac{1}{4}(\tfrac{3}{2}+2\cos 2x+\tfrac{1}{2}\cos 4x)$$

$$\therefore \quad \int \cos^4 x \,\mathrm{d}x = \tfrac{1}{4}\int (\tfrac{3}{2}+2\cos 2x+\tfrac{1}{2}\cos 4x)\,\mathrm{d}x$$

$$= \tfrac{1}{4}\left[\tfrac{3}{2}\int \mathrm{d}x + \int \cos 2x \,\mathrm{d}(2x) + \tfrac{1}{8}\int \cos 4x \,\mathrm{d}(4x)\right]$$

$$= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

例13. 求 $\int \sin^2 x \cos^2 3x \,\mathrm{d}x$.

解: $\because \quad \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1 - \cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1 - \cos 4x)$$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x \,\mathrm{d}(8x)$

$$- \frac{1}{2}\int \sin^2 2x \,\mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x \,\mathrm{d}(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

(7) $\int f(\arctan\frac{x}{a})\frac{1}{a^2+x^2}\mathrm{d}x=\frac{1}{a}\int f(\arctan\frac{x}{a})\,\mathrm{d}\arctan\frac{x}{a}$

例14. 求 $\int\frac{\mathrm{d}x}{a^2+x^2}$.

想到公式

$$\int\frac{\mathrm{d}u}{1+u^2}$$

$$=\arctan u+C$$

解: $\int\frac{\mathrm{d}x}{a^2+x^2}=\frac{1}{a^2}\int\frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u=\frac{1}{a}\mathrm{d}x$

$$=\frac{1}{a}\int\frac{\mathrm{d}u}{1+u^2}=\frac{1}{a}\arctan u+C$$

$$=\frac{1}{a}\arctan(\frac{x}{a})+C$$

例15 求 $\int \frac{1}{x^2-8x+25}dx.$

解 $\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

$$(8)\ \int f(\arcsin x)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \int f(\arcsin x)\,\mathrm{d}\arcsin x$$

例16. 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}\ (a>0)$.

解:
$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$$

$$= \arcsin\frac{x}{a} + C$$

想到公式

$$\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C$$

$$(8)\ \int f(\arcsin x)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\int f(\arcsin x)\,\mathrm{d}\arcsin x$$

例17. 求 $\displaystyle\int\frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}}\ (a>0).$

解: 原式$\displaystyle=\int\frac{\mathrm{d}\arcsin x}{(\arcsin x)^2}$

$$=-\frac{1}{\arcsin x}+C$$

$$\text{(9)}\ \boxed{\int f(e^x)e^x\,\mathrm{d}x =} \int f(e^x)\,\mathrm{d}e^x$$

例18 求 $\int \frac{\mathrm{d}x}{1+e^x}$.

解

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+e^x)}{1+e^x}$$

$$= x - \ln(1+e^x) + C$$

例19 求 $\int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx.$

解 $\because \left(x+\frac{1}{x}\right)' = 1-\frac{1}{x^2},$

$$\therefore \int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}d(x+\frac{1}{x}) = e^{x+\frac{1}{x}}+C.$$

$$(10)\ \int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$$

例20. 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例21. 求

$$\int \frac{x+1}{x(1+xe^x)}\mathrm{d}x.$$

解: 原式= $\displaystyle\int \frac{(x+1)e^x}{xe^x(1+xe^x)}\mathrm{d}x = \int\left(\frac{1}{xe^x}-\frac{1}{1+xe^x}\right)\mathrm{d}(xe^x)$

$$=\ln\left|xe^x\right|-\ln\left|1+xe^x\right|+C$$

$$=x+\ln\left|x\right|-\ln\left|1+xe^x\right|+C$$

分析: $\displaystyle\frac{1}{xe^x(1+xe^x)}=\frac{1+xe^x-xe^x}{xe^x(1+xe^x)}=\frac{1}{xe^x}-\frac{1}{1+xe^x}$

$$(x+1)e^x\,\mathrm{d}x = xe^x\,\mathrm{d}x+e^x\,\mathrm{d}x=\mathrm{d}(xe^x)$$

二、第二类换元积分法

第一类换元法解决的问题

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(u)\mathrm{d}u \Big|_{u=\varphi(x)}$$

难求 易求

若所求积分 $\int f(u)\mathrm{d}u$ 难求，

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求，

则得第二类换元积分法 .

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\mathrm{d}u \Big|_{u=\varphi(x)}$$

定理2. 设 $x=\psi(t)$ 是单调可导函数，且 $\psi'(t)\neq 0$，$f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $x=\psi(t)$ 的反函数

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数 $\Phi(t)$，令

$$F(x)=\Phi[\psi^{-1}(x)]$$

$$\boxed{\Phi'(t)=f[\psi(t)]\psi'(t)}$$

则 $$F'(x)=\frac{\mathrm{d}\Phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[\psi(t)]\cancel{\psi'(t)}\cdot\frac{1}{\cancel{\psi'(t)}}=f(x)$$

$$\therefore\ \int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C$$

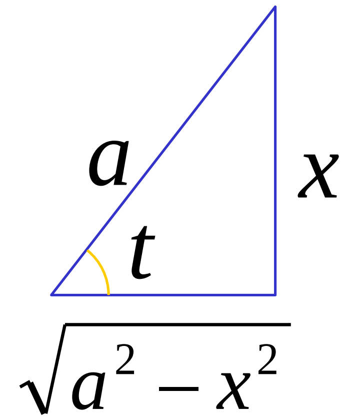
$$=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

例22. 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x \ \ (a>0)$.

解: 令 $x=a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$

$$\mathrm{d}x=a\cos t\,\mathrm{d}t$$



$$\therefore\ 原式=\int a\cos t\cdot a\cos t\,\mathrm{d}t=a^2\int\cos^2 t\,\mathrm{d}t$$

$$=a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$$\sin 2t=2\sin t\cos t=2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

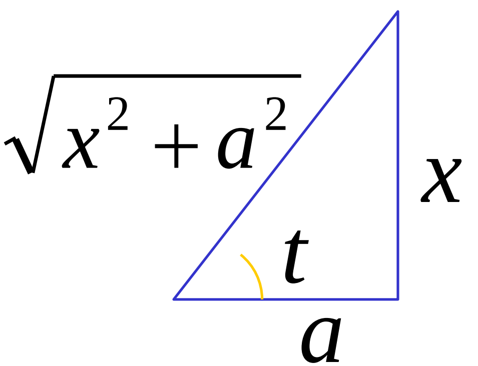
例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0)$.

解: 令 $x = a\tan t,\ t \in (-\frac{\pi}{2}, \frac{\pi}{2})$，则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$



$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\right] + C \qquad (C = C_1 - \ln a)$$

例24. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} \quad (a>0).$

解: 当 $x>a$ 时，令 $x=a\sec t,\ t\in(0,\frac{\pi}{2})$，则

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2 t-a^2}=a\tan t$$

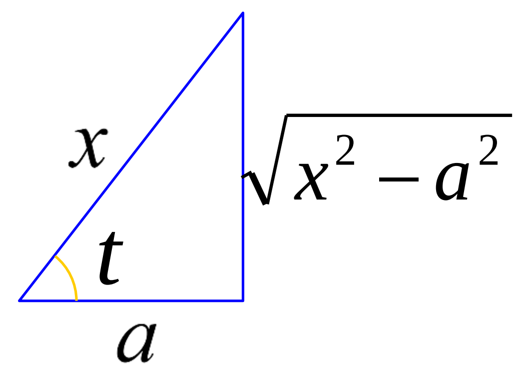
$$\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t$$

$$\therefore \text{原式}=\int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$

$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C \qquad (C=C_1-\ln a)$$



当 $x<-a$ 时, 令 $x=-u$, 则 $u>a$, 于是

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=-\int\frac{\mathrm{d}u}{\sqrt{u^2-a^2}}=-\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$=-\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$=-\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\quad(C=C_1-2\ln a)$$

$x>a$ 时, $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|+C$

注

被积函数含有 $\sqrt{x^2+a^2}$ 或 $\sqrt{x^2-a^2}$ 时, 除采用

三角代换外, 还可利用公式

$$\mathrm{ch}^2 t-\mathrm{sh}^2 t=1$$

采用双曲代换

$$x=a\,\mathrm{sh}\,t \quad \text{或} \quad x=a\,\mathrm{ch}\,t$$

消去根式 , 所得结果一致 .

小结:

1. 第二类换元法常见类型:

(1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$, 令 $t = \sqrt[n]{ax+b}$

(2) $\int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x$, 令 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

(3) $\int f(x, \sqrt{a^2 - x^2})\,\mathrm{d}x$, 令 $x = a\sin t$ 或 $x = a\cos t$

(4) $\int f(x, \sqrt{a^2 + x^2})\,\mathrm{d}x$, 令 $x = a\tan t$ 或 $x = a\,\mathrm{sh}\,t$

(5) $\int f(x, \sqrt{x^2 - a^2})\,\mathrm{d}x$, 令 $x = a\sec t$ 或 $x = a\,\mathrm{ch}\,t$

(6) $\int f(a^x)\,\mathrm{d}x$，令 $t=a^x$

2. 常用基本积分公式的补充

(16) $\int \tan x \,\mathrm{d}x = -\ln\left|\cos x\right| + C$

(17) $\int \cot x \mathrm{d}x = \ln\left|\sin x\right| + C$

(18) $\int \sec x \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$

(19) $\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$

(20) $$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$$

(21) $$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(22) $$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$$

(23) $$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x+\sqrt{x^2+a^2}) + C$$

(24) $$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

例26. 求 $\int \frac{\mathrm{d}x}{x^2+2x+3}$.

解: 原式 $=\int \frac{1}{(x+1)^2+(\sqrt{2})^2}\mathrm{d}(x+1)$

$$=\frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

例27. 求 $I=\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I=\frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}}=\frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

例28. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}} = \arcsin\frac{2x-1}{\sqrt{5}}+C$

例29. 求 $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

解: 原式 $= -\int \frac{\mathrm{d}e^{-x}}{\sqrt{1-e^{-2x}}} = -\arcsin e^{-x}+C$

三、倒代换

例30. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$，得

$$\text{原式}=-\int \frac{t}{\sqrt{a^2t^2+1}}\mathrm{d}t=-\frac{1}{2a^2}\int \frac{\mathrm{d}(a^2t^2+1)}{\sqrt{a^2t^2+1}}$$

$$=-\frac{1}{a^2}\sqrt{a^2t^2+1}+C=-\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}t$

$$\text{原式}=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}t=-\int(a^2t^2-1)^{\frac{1}{2}}\left|t\right|\mathrm{d}t$$

当 $x>0$时,

$$\text{原式}=-\frac{1}{2a^2}\int(a^2t^2-1)^{\frac{1}{2}}\ \mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C\ =-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得同样结果 .

例31. 求 $\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $=\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$

$$\boxed{\text{令 } x+1=\frac{1}{t}}$$

$$=\int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t=-\int \frac{t^2}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\mathrm{d}t=\boxed{\int \sqrt{1-t^2}\,\mathrm{d}t}-\int \frac{1}{\sqrt{1-t^2}}\mathrm{d}t$$

$$=\tfrac{1}{2}t\sqrt{1-t^2}+\tfrac{1}{2}\arcsin t-\arcsin t+C$$

$$=\frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2}-\frac{1}{2}\arcsin\frac{1}{x+1}+C$$

四、小结、思考题

思考与练习

1. 下列积分应如何换元才使积分简便？

(1) $\displaystyle\int \frac{x^5}{\sqrt{1+x^2}}\mathrm{d}x$

令 $t=\sqrt{1+x^2}$

(2) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+e^x}}$

令 $t=\sqrt{1+e^x}$

(3) $\displaystyle\int \frac{\mathrm{d}x}{x(x^7+2)}$　　令 $t=\dfrac{1}{x}$

2. 已知 $\int x^5 f(x)\,\mathrm{d}x = \sqrt{x^2-1}+C$, 求 $\int f(x)\,\mathrm{d}x$.

解: 两边求导, 得 $x^5 f(x) = \dfrac{x}{\sqrt{x^2-1}}$, 则

$$\int f(x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^4\sqrt{x^2-1}} \quad (令\ t=\frac{1}{x})$$

$$= \int \frac{-t^3\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{2}\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t^2$$

$$= \frac{-1}{2}\int (1-t^2)^{\frac{1}{2}}\,\mathrm{d}(1-t^2) + \frac{1}{2}\int (1-t^2)^{-\frac{1}{2}}\,\mathrm{d}(1-t^2)$$

$$= \frac{-1}{3}(1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C = \cdots$$

$\boxed{\text{(代回原变量)}}$

例题 **1.** 求下列积分:

1) $$\int x^2 \frac{1}{\sqrt{x^3+1}}\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\mathrm{d}(x^3+1)$$

$$= \frac{2}{3}\sqrt{x^3+1}+C$$

2) $$\int \frac{2x+3}{\sqrt{1+2x-x^2}}\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2}+5\arcsin\frac{x-1}{\sqrt{2}}+C$$

2. 求不定积分 $\int \frac{2\sin x\cos x\sqrt{1+\sin^2 x}}{2+\sin^2 x}\,\mathrm{d}x$.

解: 利用凑微分法，得

$$\text{原式} = \int \frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x}\,\mathrm{d}(1+\sin^2 x)$$

令 $t = \sqrt{1+\sin^2 x}$

$$= \int \frac{2t^2}{1+t^2}\,\mathrm{d}t = 2\int \left(1 - \frac{1}{1+t^2}\right)\mathrm{d}t$$

$$= 2t - 2\arctan t + C$$

$$= 2\left[\sqrt{1+\sin^2 x} - \arctan\sqrt{1+\sin^2 x}\right] + C$$

3. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\,\mathrm{d}x.$

解: 令 $x=\sin t$, $1+x^2=1+\sin^2 t$, $\mathrm{d}x=\cos t\,\mathrm{d}t$

原式 $=\int\frac{1}{1+\sin^2 t}\,\mathrm{d}t$

分子分母同除以 $\cos^2 t$

$$=\int\frac{\sec^2 t}{\sec^2 t+\tan^2 t}\,\mathrm{d}t=\int\frac{1}{1+2\tan^2 t}\,\mathrm{d}\tan t$$

$$=\frac{1}{\sqrt{2}}\int\frac{1}{1+(\sqrt{2}\tan t)^2}\,\mathrm{d}\sqrt{2}\tan t$$

$$=\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan t)+C=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}x}{\sqrt{1-x^2}}+C$$

五、作业

习题

4-4：7（1-20双数题，21-26）