Course: Digital Signal and Data Processing

Windowed-Sinc Filters

Lecture 3

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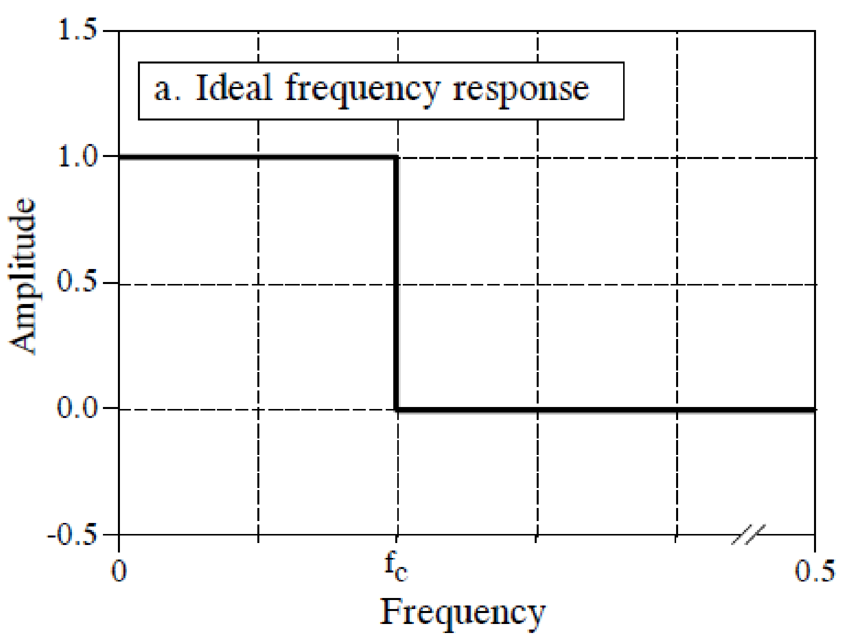
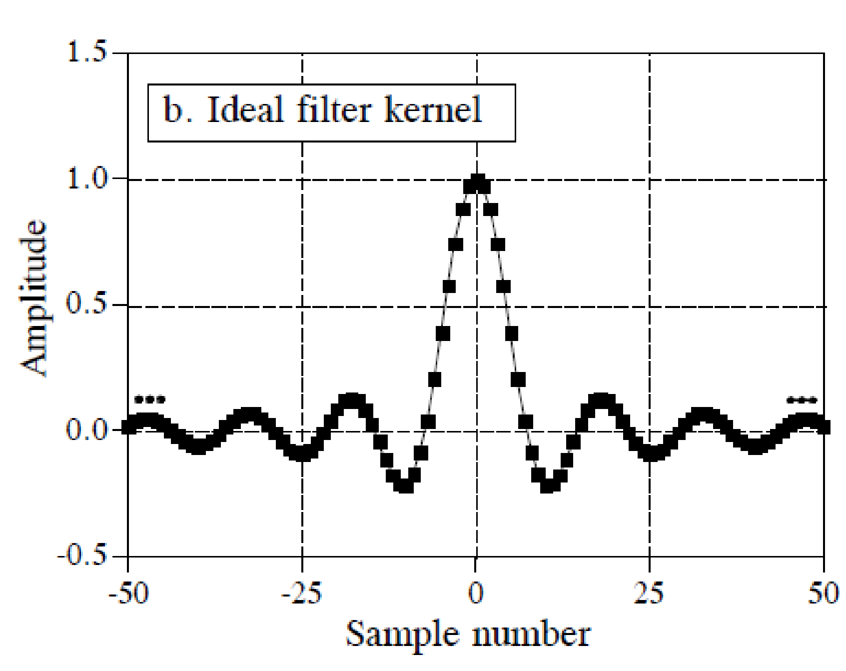
Ideal Low-pass Filter

The frequency response of the *ideal* low-pass filter is shown in (a). All frequencies below the cutoff frequency, f_c , are passed with unity amplitude, while all higher frequencies are blocked. The passband is perfectly **flat**, the attenuation in the stopband is **infinite**, and the transition between the two is infinitesimally **small**. Taking the Inverse Fourier Transform of this ideal frequency response produces the ideal filter kernel (impulse response) shown in (b). This curve is $\sin(x)/x$, called the **sinc function**, given by:

$$h[i] = \frac{\sin(2\pi f_C i)}{i\pi}$$





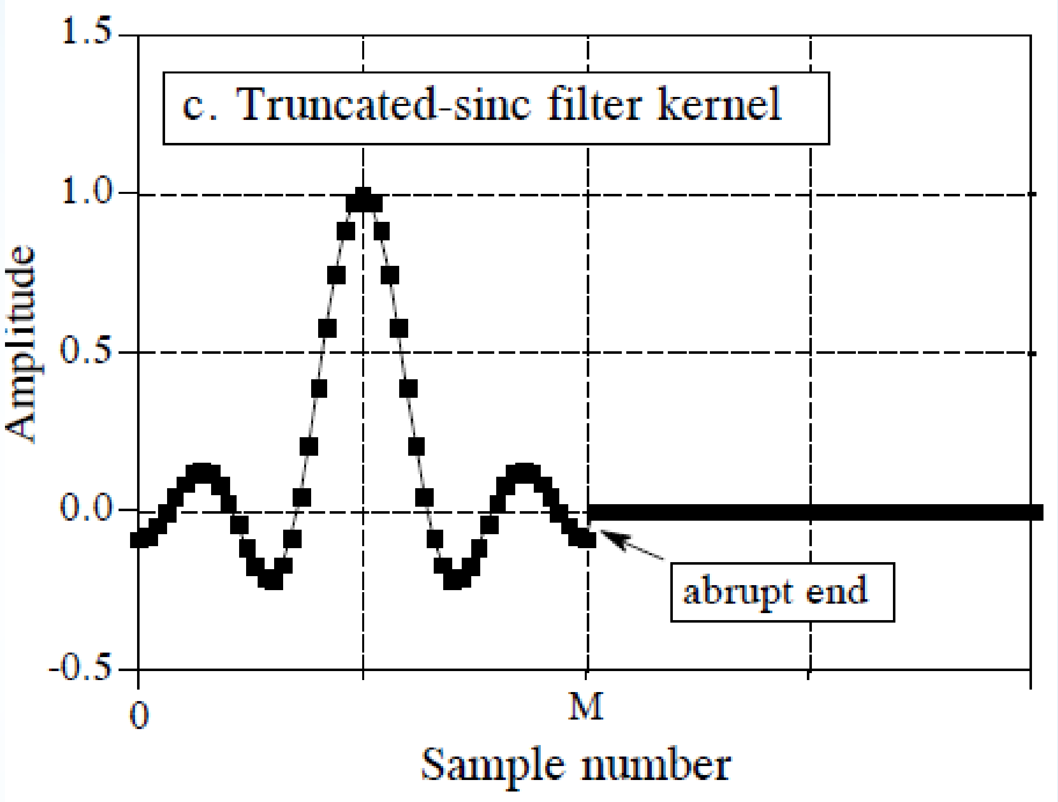
Ideal Low-pass Filter

- Convolving an input signal with this filter kernel provides a *perfect* low-pass filter.
- **The problem is, the sinc function continues to both negative and positive infinity without dropping to zero amplitude.** While this infinite length is not a problem for *mathematics*, it is a problem for ***computers***.

Truncated Sinc-filter

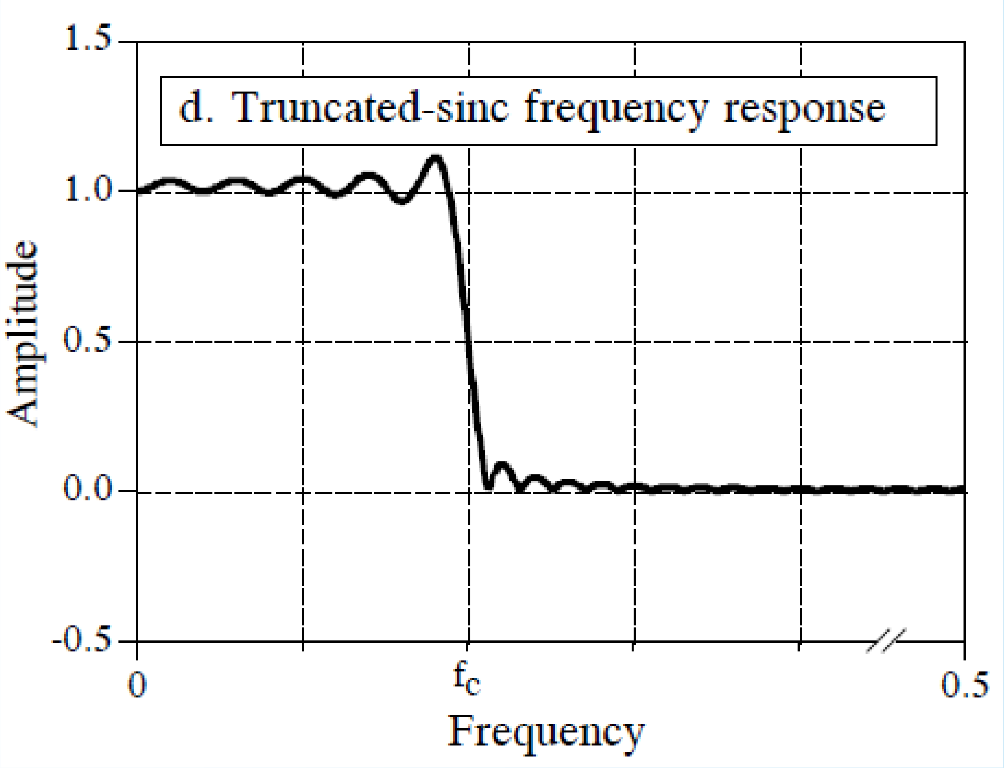
Two modifications to the ***sinc*** function in (b), resulting in the waveform shown in (c).

1. It is **truncated** to ***M*+1** points, symmetrically chosen around the main lobe, where ***M*** is an **even number**. All samples outside these M+1 points are set to zero, or simply ignored.
2. The entire sequence is **shifted** to the right so that it runs from 0 to M. This allows the filter kernel to be represented using ***only positive* indexes**.



Truncated Sinc-filter

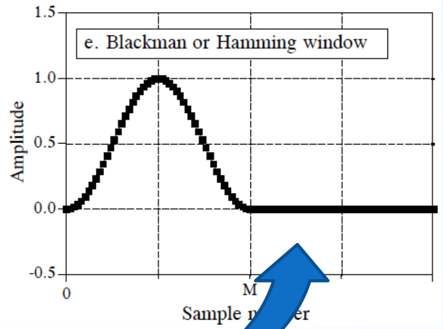
- Since the modified filter kernel is only an approximation to the ideal filter kernel, it will not have an ideal frequency response
- To find the frequency response that is obtained, the Fourier transform can be taken of the signal in (c), resulting in the curve in (d).
- There is **extreme ripple in the passband** and **poor attenuation in the stopband** (Gibbs effect). These problems result from the sudden discontinuity at the ends of the truncated sinc function.

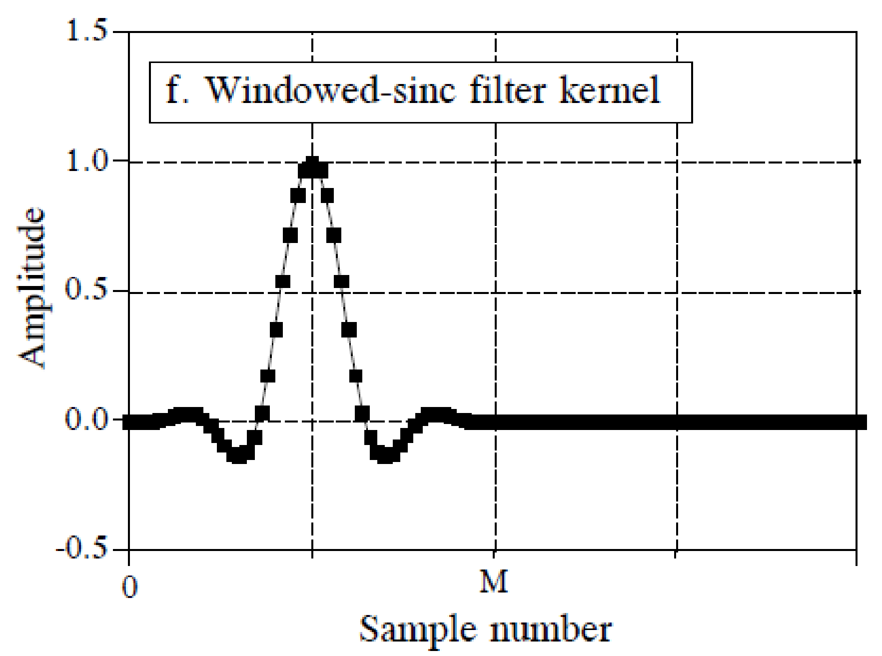


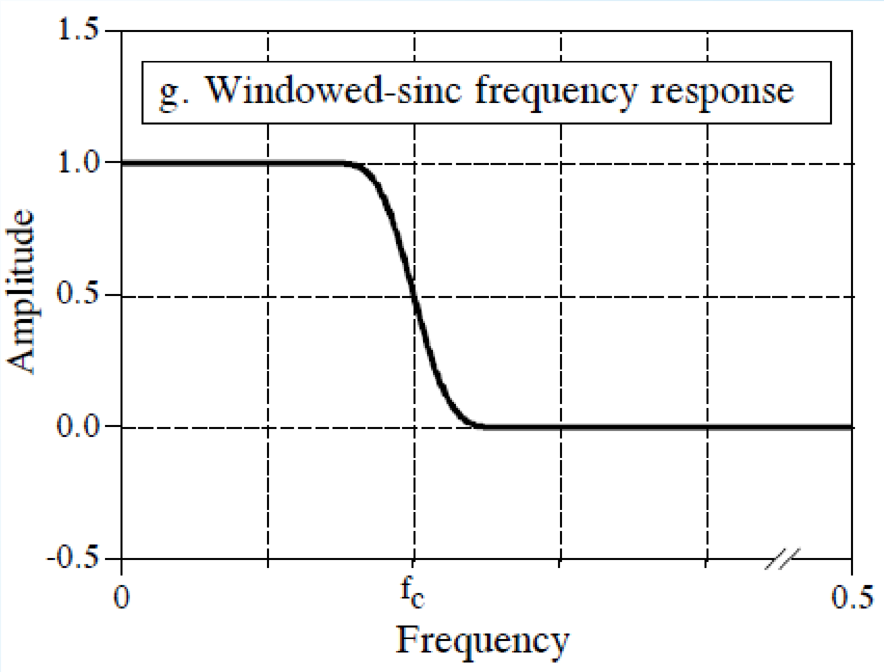
Increasing the length of the filter kernel does not reduce these problems; the discontinuity is significant no matter how long *M* is made.

Windowed Sinc-filter

- Multiplying the truncated-sinc, (c), by the **Blackman window**, results in the **windowed sinc** filter kernel shown in (f).
- Windowed Sinc-filter **reduces** the uncertainty in truncated ends and thereby **improve** the frequency response.
- Figure (g) shows this improvement. The **passband is now flat**, and the **stopband** attenuation **is so good** it cannot be seen in this graph.







Window filters

Several different windows are available, most of them named after their original developers in the 1950s. Only two are widely used, the **Hamming window** and the **Blackman window.**

- The **Hamming window**. These windows run from *i* =0 to *M*, for a total of *M* +1 points.

$$w[i] = 0.54 - 0.46\cos(2\pi i/M)$$

- The **Blackman window.**

$$w[i] = 0.42 - 0.5\cos(2\pi i/M) + 0.08\cos(4\pi i/M)$$

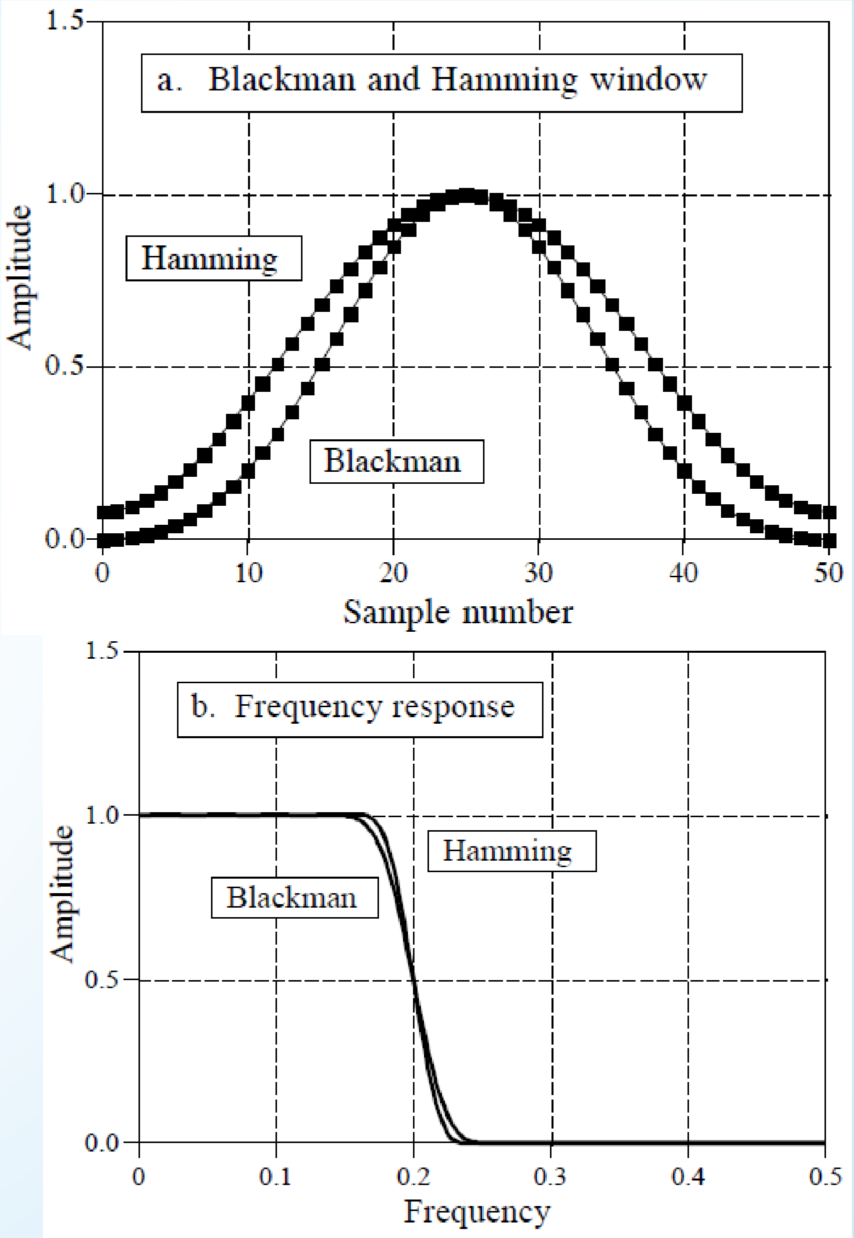
Blackman and Hamming windows

Figure (a) shows the shape of two window filters for M=50.

Which of these two windows should you use?

It's a trade-off between parameters.

As shown in Fig. (b), the Hamming window has about a **20% faster *roll-off*** than the Blackman.



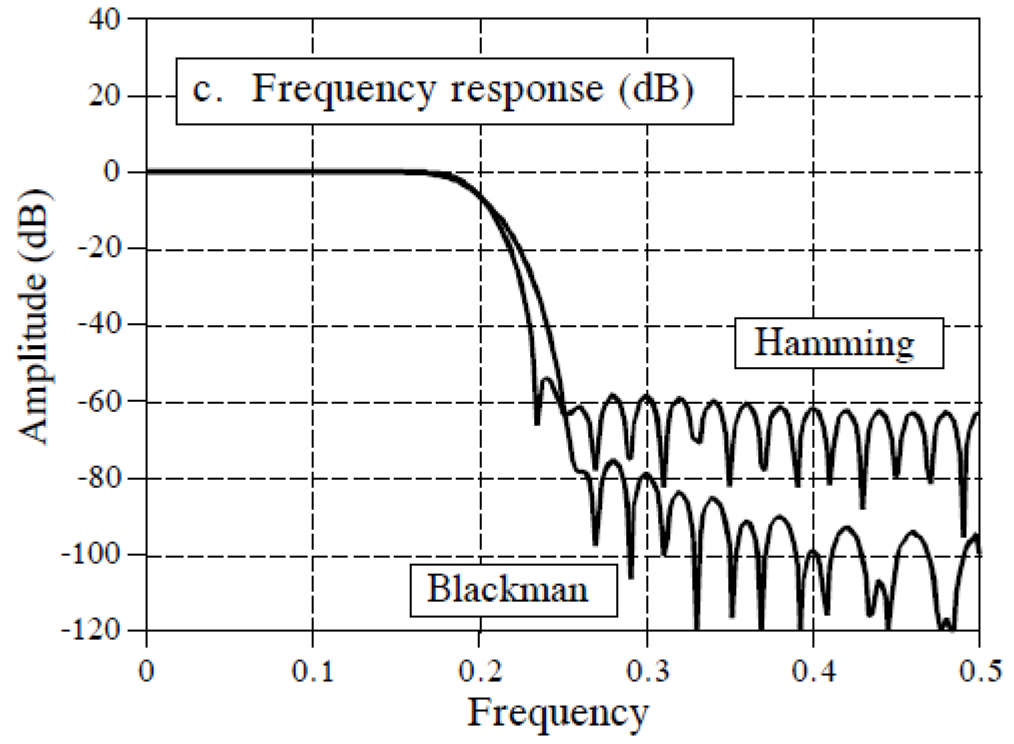
Blackman and Hamming windows

Figure(c) shows that the Blackman has a ***better stopband attenuation***.

The stopband attenuation for the Blackman is -74dB (-0.02%), while the Hamming is only -53dB (-0.2%).

Although it cannot be seen in these graphs, the Blackman has a ***passband ripple*** of only about 0.02%, while the Hamming is typically 0.2%.

The Blackman should be your first choice; a slow roll-off is easier to handle than poor stopband attenuation.



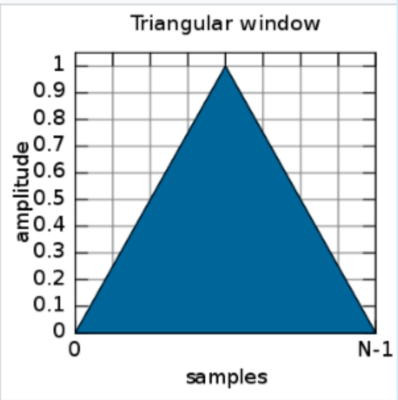
Other Window Filters

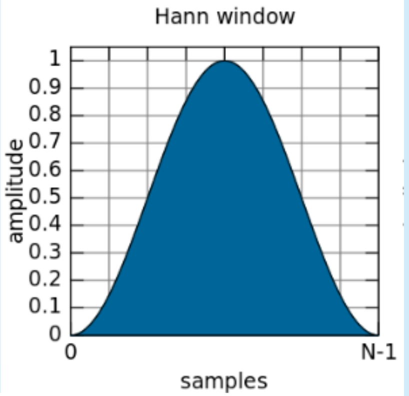
- The **Bartlett window** is a triangle, using straight lines for the taper.
- The **Hanning window**, also called the **raised cosine window**, is given by:

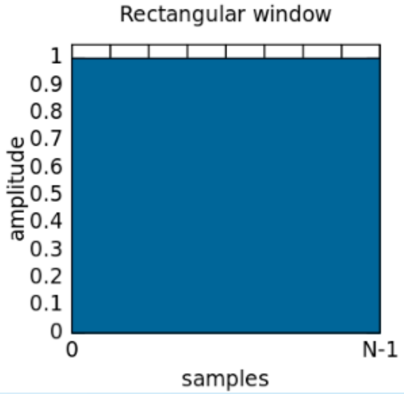
$$w[i] = 0.5 - 0.5\cos(2\pi i/M)\ .$$

These two windows have about the same roll-off speed as the Hamming, but worse stopband attenuation (Bartlett: -25dB or 5.6%, Hanning -44dB or 0.63%).

- **Rectangular window**. This is the same as *no* window, just a truncation of the tails. While the roll-off is -2.5 times faster than the Blackman, the stopband attenuation is only -21dB (8.9%).







Designing the Filter

To design a windowed-sinc, two parameters must be selected: the **cutoff frequency, f_c** , and the **length of the filter kernel, *M***.

The cutoff frequency is expressed as a fraction of the sampling rate, and therefore must be between 0 and 0.5. The value for *M* sets the *roll-off* according to the approximation:

$$M \approx \frac{4}{BW}$$

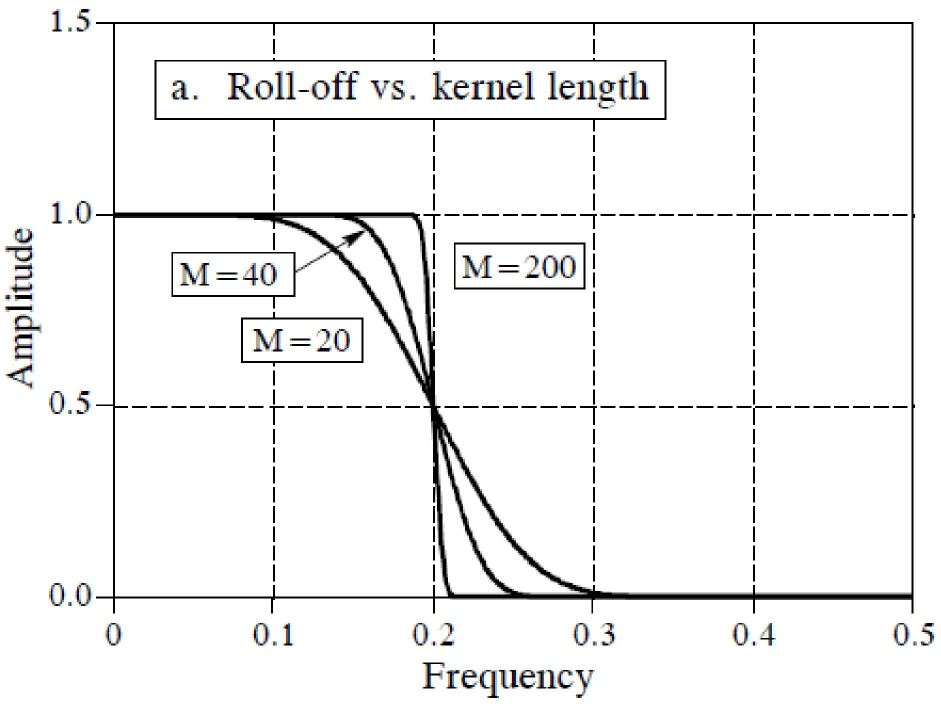
where ***BW*** is the width of the transition band, measured from where the curve just barely leaves one, to where it almost reaches zero (say, 99% to 1% of the curve). The transition bandwidth is also expressed as a fraction of the sampling frequency, and must between 0 and 0.5.

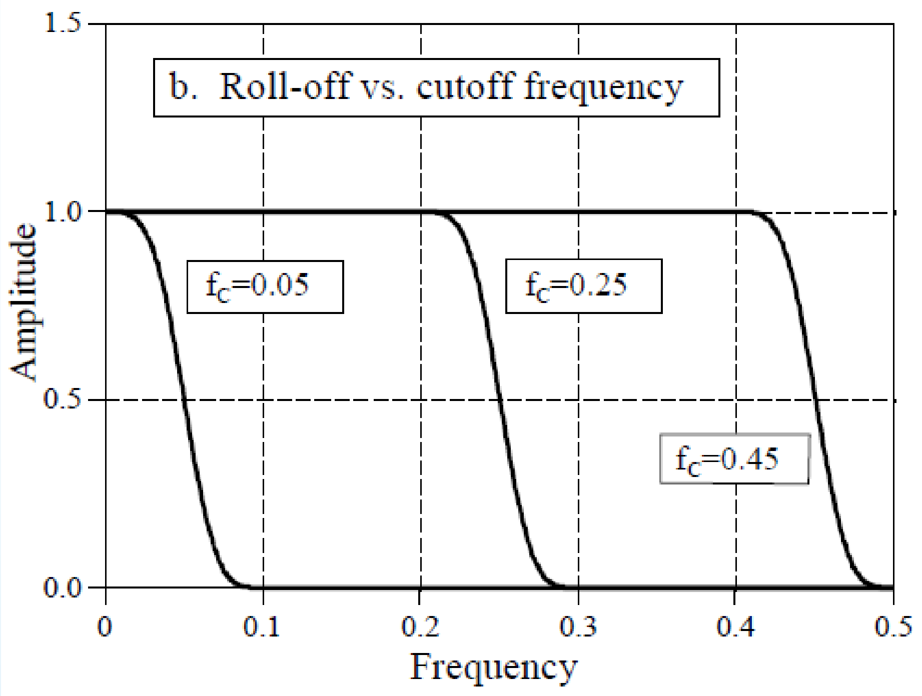
Designing the Filter

Figure (a) shows an example of how this approximation is used. The three curves shown are generated from filter kernels with: *M*=20, 40, and 200.

From above Equation, the transition bandwidths are: *BW*=0.2, 0.1, and 0.02 , respectively.

Figure (b) shows that **the shape of the frequency response does not depend on the cutoff frequency selected.**





Designing the Filter

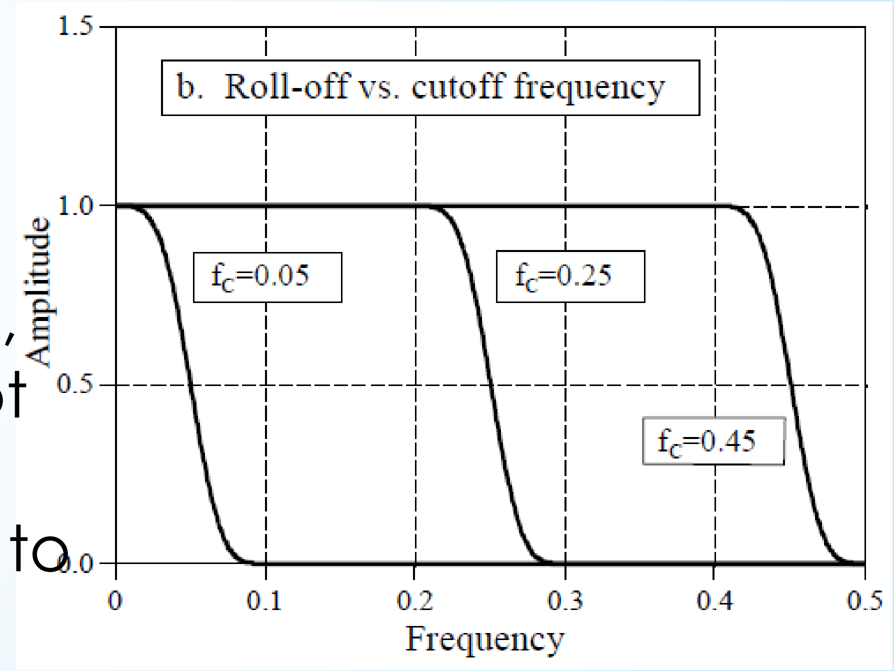
$$M \approx \frac{4}{BW}$$

- Since the time required for a convolution is proportional to the length of the signals, Equation shown above expresses a trade-off between ***computation time*** (depends on the value of *M*) and ***filter sharpness*** (the value of *BW*). For instance, the 20% slower roll-off of the Blackman window (as compared with the Hamming) can be compensated for by using a filter kernel 20% longer.
- It could be said that the **Blackman window is 20% slower to execute that an equivalent roll-off Hamming window**. This is important because the execution speed of windowed-sinc filters is already terribly slow.

Designing the Filter

The **cutoff frequency** of the windowed-sinc filter is measured at the **one-half amplitude** point.

Why use 0.5 instead of the standard 0.707 (-3dB)? This is because the windowed sinc's frequency response is ***symmetrical*** between the passband and the stopband. For instance, the Hamming window results in a passband ripple of 0.2%, and an *identical* stopband attenuation (i.e., ripple in the stopband) of 0.2%. Other filters do not show this symmetry, and therefore have no advantage in using the one-half amplitude point to mark the cutoff frequency. This symmetry makes the windowed-sinc ideal for ***spectral inversion***.



Designing the Filter

- After f_c and ***M*** have been selected, the filter kernel is calculated from the relation:

$$h[i] = K \frac{\sin(2\pi f_c (i - M/2))}{i - M/2} \left[0.42 - 0.5 \cos\left(\frac{2\pi i}{M} \right) + 0.08 \cos\left(\frac{4\pi i}{M} \right) \right]$$

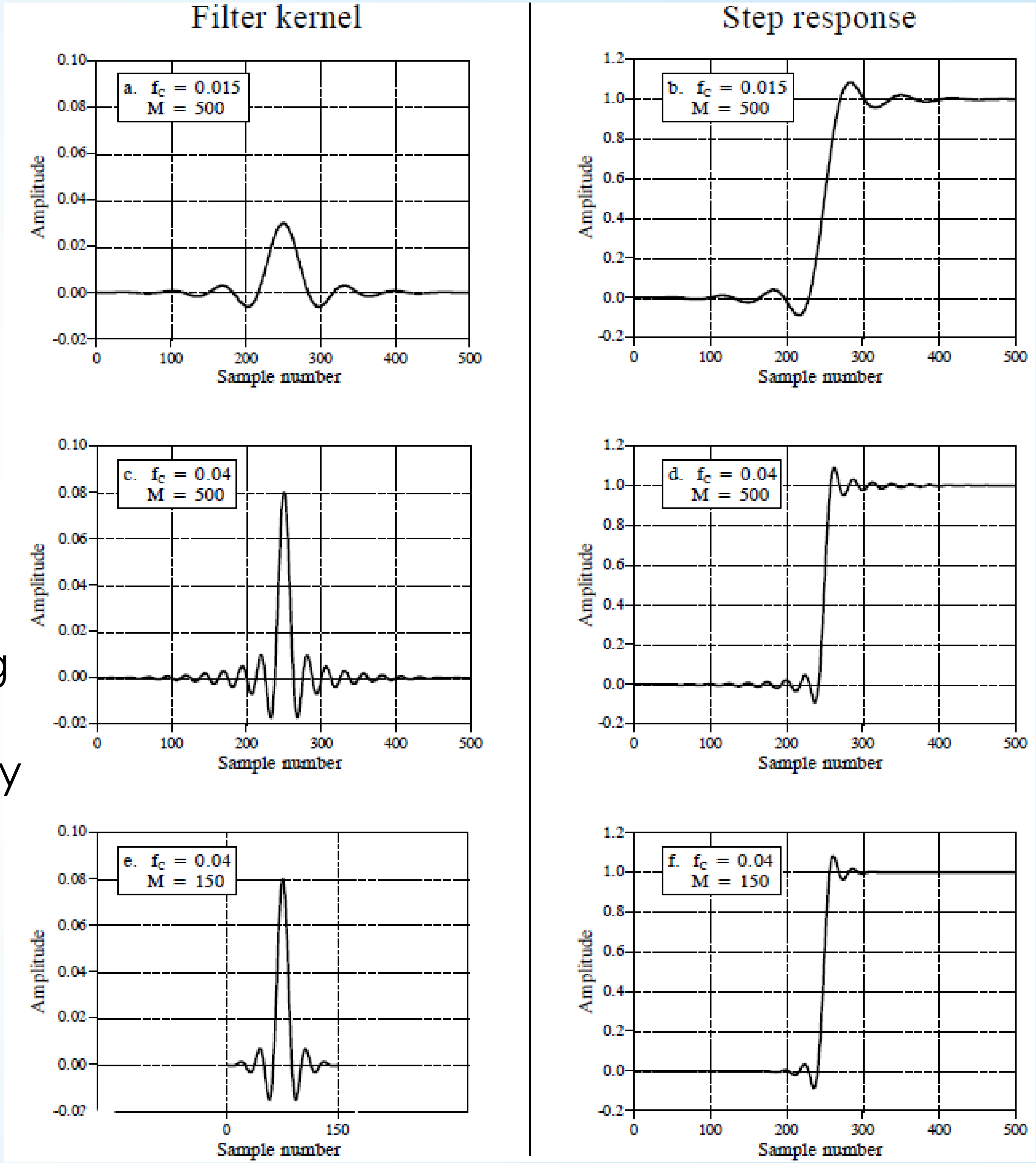
- You should be able to identify three components: the ***sinc function***, the ***M/2 shift***, and the ***Blackman window***. For the filter to have **unity gain at DC**, the constant *K* must be chosen such that the **sum of all the samples is equal to one**.
- In practice, **ignore *K*** during the calculation of the filter kernel, and then ***normalize*** all of the samples as needed.

Designing the Filter

Figure shows examples of windowed-sinc **filter kernels**, and their corresponding **step responses**. The samples at the beginning and end of the filter kernels are so small that they can't even be seen in the graphs. Don't make the mistake of thinking they are unimportant! These samples may be small in value; however, **they collectively have a large effect on the performance of the filter**. This is also why floating point representation is typically used to implement windowed-sinc filters. Integers usually *don't have enough dynamic range* to capture the large variation of values contained in the filter kernel.

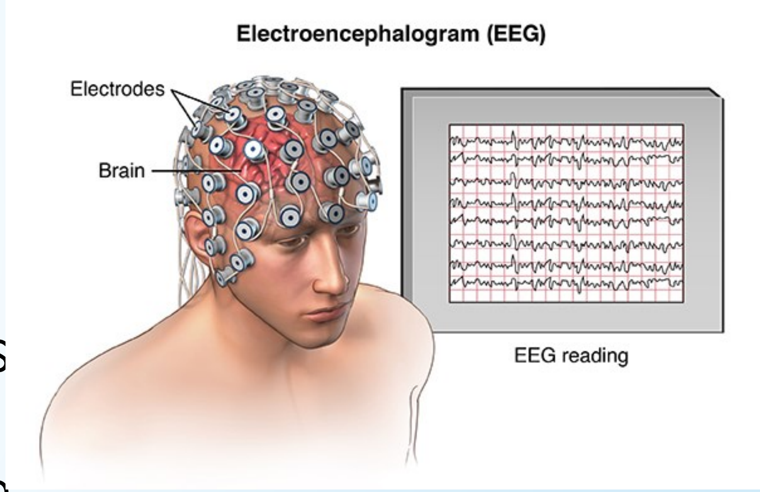
The step response has overshoot and ringing.

This is *not* a filter for signals with information encoded in the time domain!



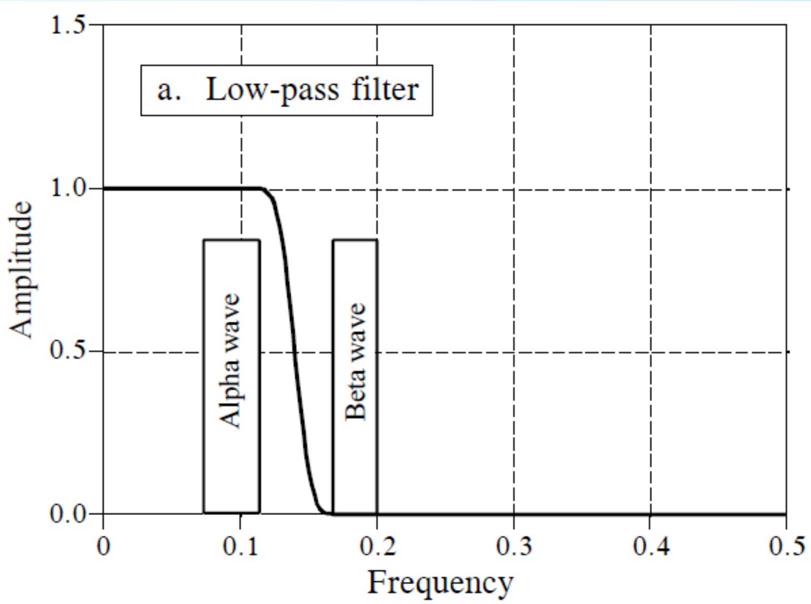
Examples of Windowed-Sinc Filters

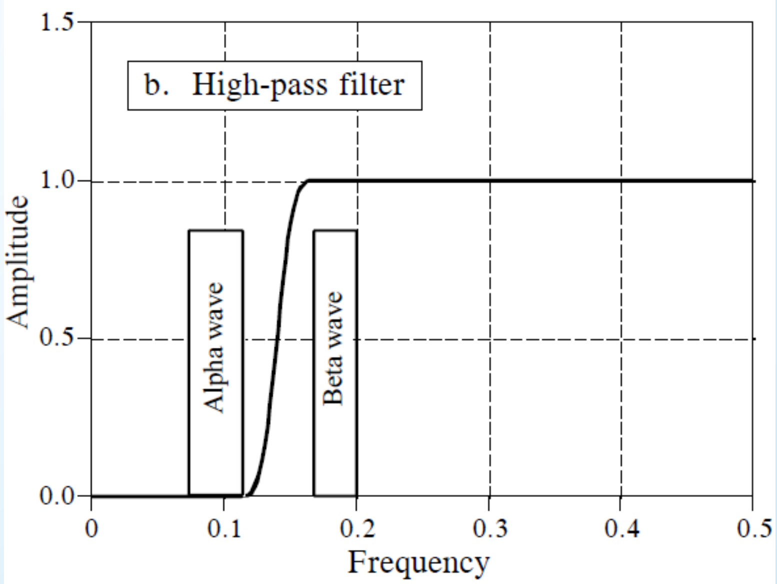
An **electroencephalogram**, or EEG, is a measurement of the electrical activity of the brain. It can be detected as millivolt level signals appearing on electrodes attached to the surface of the head. Each nerve cell in the brain generates small electrical pulses. The EEG is the combined result of an enormous number of these electrical pulses being generated in a coordinated manner. Although the relationship between thought and this electrical coordination is very poorly understood, different frequencies in the EEG can be identified with specific mental states. If you close your eyes and relax, the predominant EEG pattern will be a slow oscillation between about 7 and 12 hertz. This waveform is called the **alpha rhythm**, and is associated with contentment and a decreased level of attention. Opening your eyes and looking around causes the EEG to change to the **beta rhythm**, occurring between about 17 and 20 hertz.



Example 1: Windowed-Sinc Filters

We will assume that the EEG signal has been amplified by analog electronics, and then digitized at a sampling rate of 100 samples per second. Acquiring data for 50 seconds produces a signal of 5,000 points. **Our goal is to separate the alpha from the beta rhythms!** To do this, we will design a digital low-pass filter with a cutoff frequency of 14 hertz, or 0.14 of the sampling rate. The transition bandwidth will be set at 4 hertz, or 0.04 of the sampling rate. From Equation, the filter kernel needs to be about 101 points long, and we will arbitrarily choose to use a Hamming window. The frequency response of the filter, obtained by taking the Fourier Transform of the filter kernel, is shown in Figure.

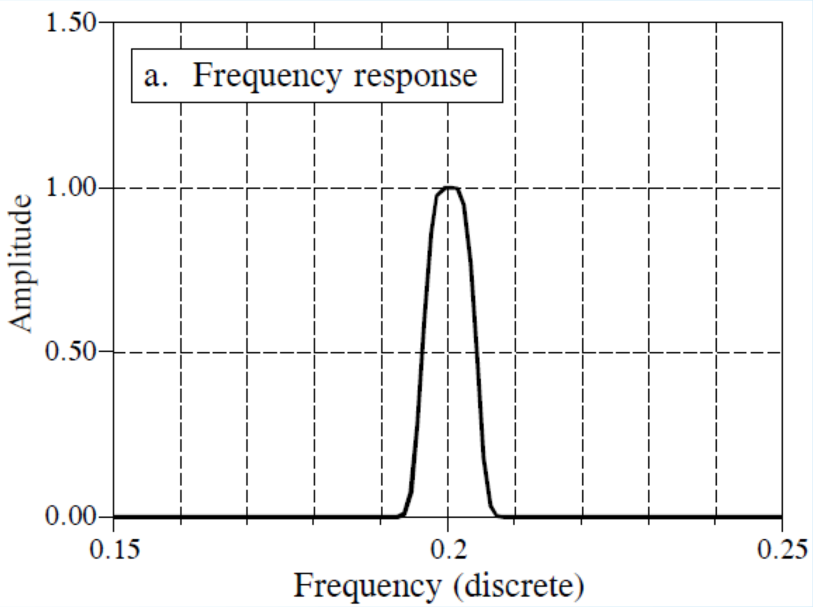


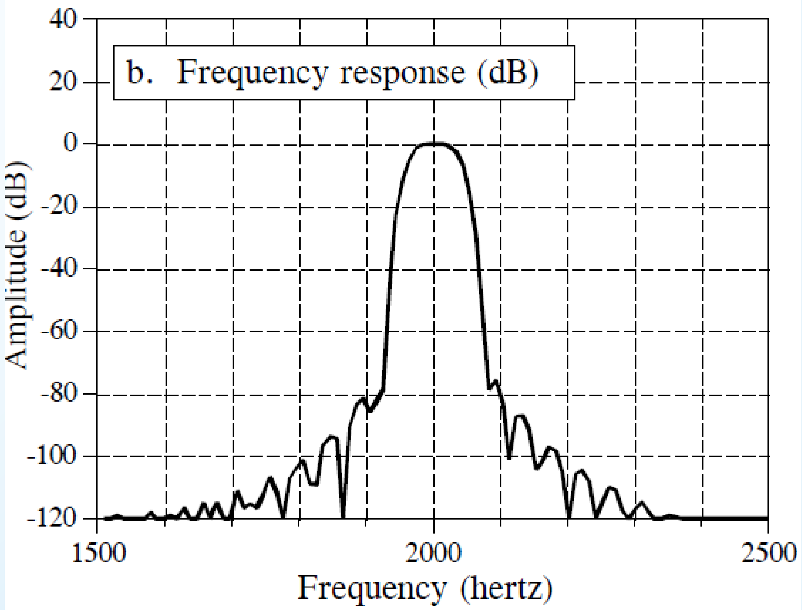


Example 2: Windowed-Sinc Filters

We will design a **band-pass filter** to isolate a *signaling tone* in an audio signal, such as when a button on a telephone is pressed. We will assume that the signal has been digitized at 10 kHz, and the goal is to isolate an 80 hertz band of frequencies centered on 2 kHz. In terms of the sampling rate, we want to block all frequencies below 0.196 and above 0.204 (corresponding to 1960 hertz and 2040 hertz, respectively). To achieve a transition bandwidth of 50 hertz (0.005 of the sampling rate), we will make the filter kernel 801 points long, and use a Blackman window. Figure shows the frequency response. The design involves several steps.

1. ***Two low-pass*** filters are designed, one with a cutoff at 0.196, and the other with a cutoff at 0.204.
2. This second filter is then ***spectrally inverted***, making it a high-pass filter.
3. The **two filter kernels are added**, resulting in a band-reject filter.
4. Another ***spectral inversion*** makes this into the desired band-pass filter.





High performance windowed-sinc filters

Suppose, you need to **isolate a 1 *millivolt*** signal riding on a 120 *volt* power line. The low-pass filter will need a stopband attenuation of at least -120dB. The Blackman window only provides -74dB.

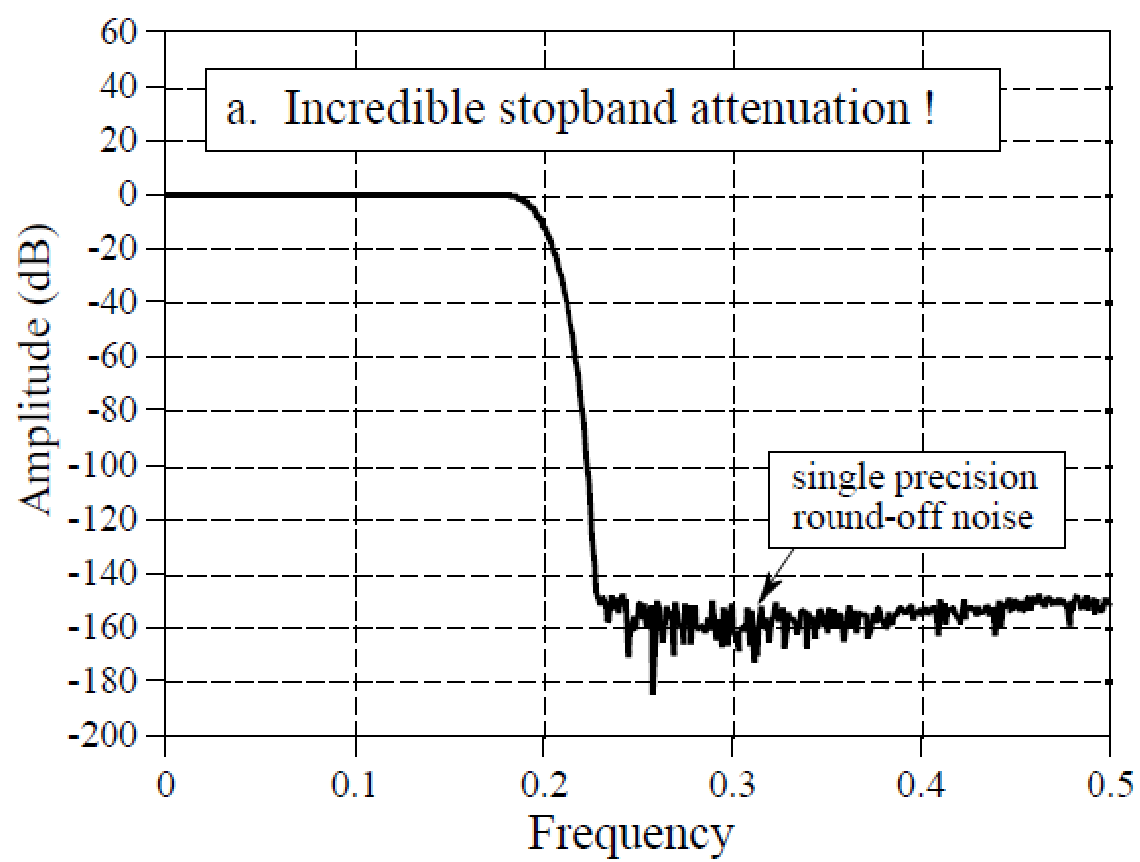
Fortunately, greater stopband attenuation is easy to obtain!

- ✓ The input signal can be filtered using a conventional windowed-sinc filter kernel, providing an intermediate signal.
- ✓ The intermediate signal can then be passed through the filter a second time, further increasing the stopband attenuation to -148dB.

It is also possible to combine the two stages into a single filter. The kernel of the combined filter is equal to the ***convolution*** of the filter kernels of the two stages. This also means that **convolving any filter kernel *with itself* results in a filter kernel with a much improved stopband attenuation**. The price you pay is a **longer filter kernel** and a **slower roll-off**.

High performance windowed-sinc filters

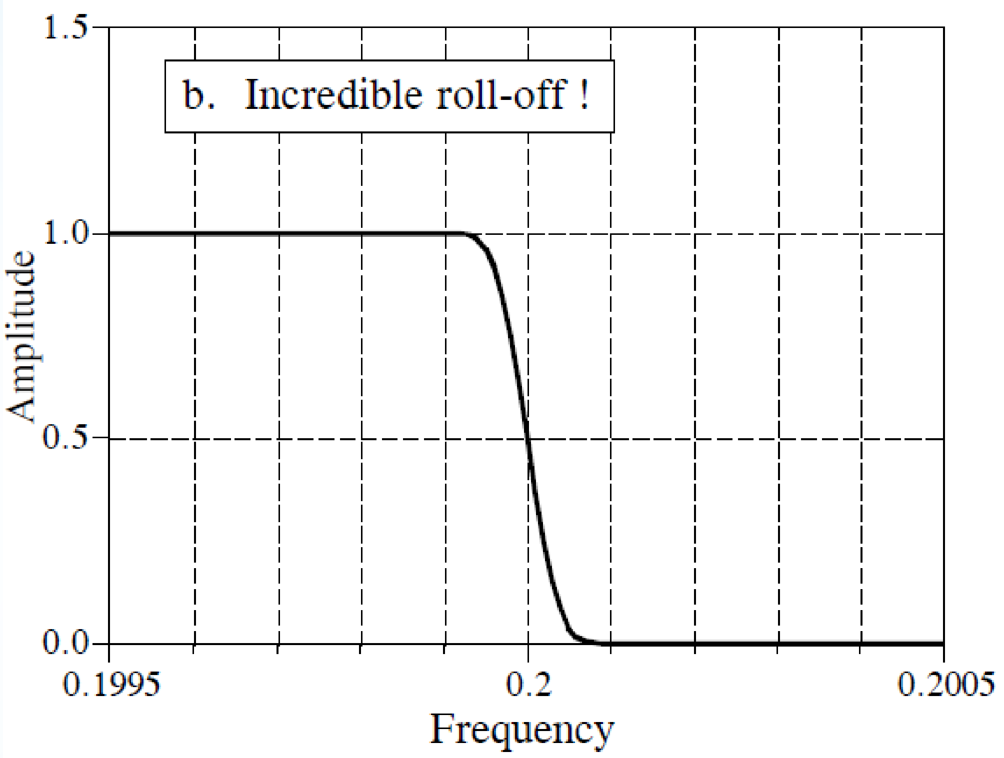
- Figure (a) shows the frequency response of a 201 point lowpass filter, formed by convolving a 101 point Blackman windowed-sinc with itself.
- Amazing performance! (If you really need more than -100dB of stopband attenuation, you should use **double precision**. Single precision round-off noise on signals in the *passband* can appear in the *stopband* with amplitudes in the -100dB to -120dB range).



High performance windowed-sinc filters

Figure shows another example of the high performance windowed-sinc's: a low-pass filter with 32,001 points in the kernel. The frequency response appears as expected, with a roll-off of 0.000125 of the sampling rate. How good is this filter? Try building an analog electronic filter that passes signals from DC to 1000 hertz with less than a 0.02% variation, and blocks all frequencies above 1001 hertz with less than 0.02% residue. Now that's a filter!

Both the filters in Figure use ***single precision***. **Using *double precision* allows these performance levels to be extended by a *million* times**.



High performance windowed-sinc filters

The strongest limitation of the windowed-sinc filter is the ***execution time***; it can be **very long** if there are many points in the filter kernel and standard convolution is used. **A high-speed algorithm for this filter is FFT convolution**.

!! Recursive filters also provide good frequency separation and are a reasonable alternative to the windowed-sinc filter.

Is the windowed-sinc the optimal filter kernel for separating frequencies? **No**, filter kernels resulting from more sophisticated techniques can be better. The windowed-sinc will provide ***any*** level of performance that you could possibly need.

Another advanced filter design methods may provide is a **slightly shorter filter kernel** for a given level of performance, and a slightly faster execution speed.