Review of linear models

Linear and logistic regression

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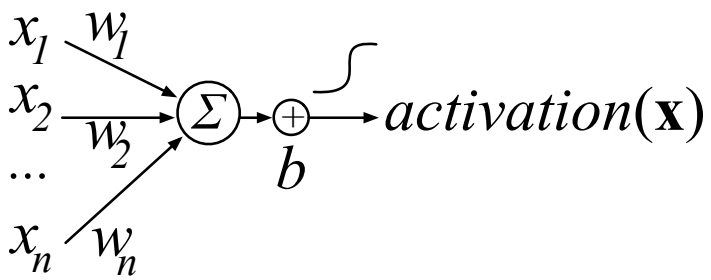
Why do we study linear models?

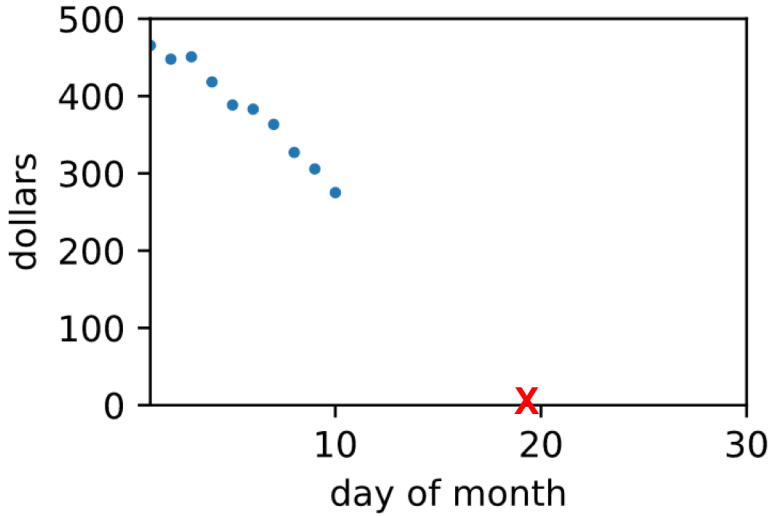
- Simple, interpretable, super fast, can't be beat for linear relations
- Usually a lower bound on power but they often form the basis of other more powerful techniques, such as LOESS and…
- Combining multiple models into a lattice, yields a neural network and those are insanely useful and powerful
- Logistic regression model is a 1-neuron neural network



Linear regression

What problem are we solving?

- In college, I was given a fixed $500 for food every month
- Wanted to know, at current rate of pizza consumption, how fast I'd run out of money so I plotted it and "eyeballed zero x point"



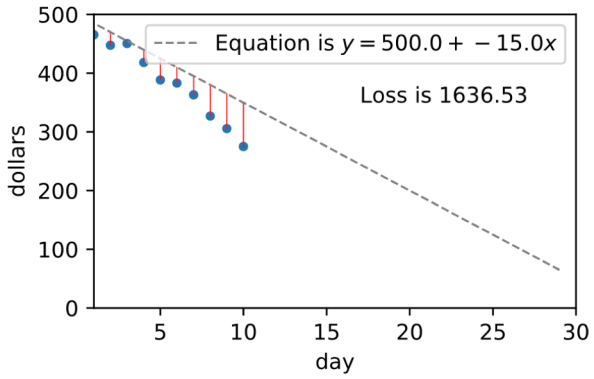
[Car computers that show number of miles remaining are solving the same problem]

See https://github.com/parrt/msds621/blob/master/notebooks/linear-models/sklearn-linear-models.ipynb

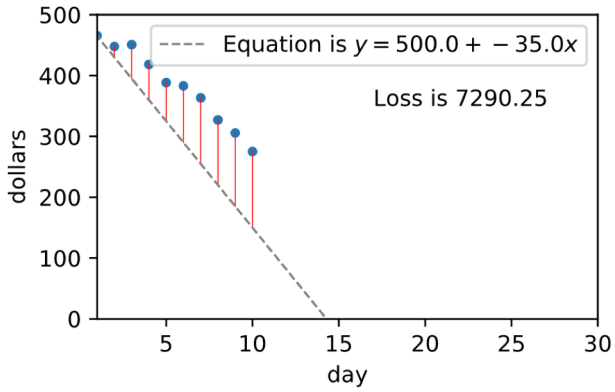
Draw line, manually finding coefficients

- I knew to draw line to project into future, but how can we figure out slope of line? (y-intercept is clearly starting amount)
- Measure error (loss) by computing average squared residual error then just wiggle slope until we find min loss

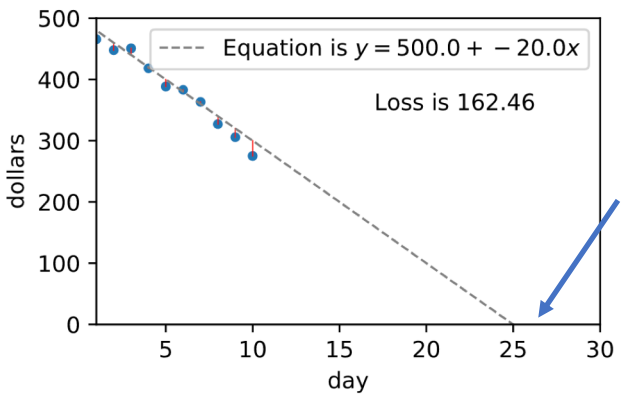
bad fit!



bad fit!



good fit!



sell blood for 5 days

See https://github.com/parrt/msds621/blob/master/notebooks/linear-models/sklearn-linear-models.ipynb

Review of linear regression notation

- Given (X, y) where X is *n* x *p* explanatory matrix and y is target or response vector, we seek coefficients that describe best hyper plane through (X, y) data
- Each $x^{(i)}$ in X maps to $y^{(i)}$and $x^{(i)}$ = $[x_1, x_2, \ldots, x_p]$

$$\hat{y} = \beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p = \beta_0 + \sum_{i=1}^{p} \beta_i x_i$$

- In vector notation, β is column vector $[\beta_1, \beta_2, \ldots, \beta_p]$

$$\hat{y} = \beta_0 + \mathbf{x} \cdot \vec{\beta} = \beta_0 + \mathbf{x}\vec{\beta}$$

Augment with “1” trick

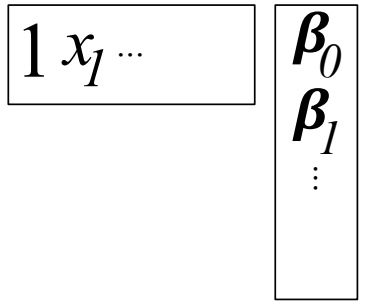
- Adding β_0 is messy so augment x with 1:

$$x' = [1, x_1, x_2, \ldots, x_p]$$

then β is column vector

$$\beta = [\beta_0, \beta_1, \beta_2, \ldots, \beta_p]$$

and we get the much simpler equation: $\hat{y} = \mathbf{x}'\vec{\beta}$



Training/fitting means finding coefficients

- Finding vector β amounts to finding those β that minimize the mean-squared error, which is our loss function:

$$MSE(\beta) = \frac{1}{n}\sum_{i=1}^{n}(y^{(i)} - \hat{y}^{(i)})^2$$

- Ignoring 1/n and substituting $\hat{y} = \mathbf{x}'\vec{\beta}$, we get:

rows augmented

$$\mathscr{L}(\beta) = \sum_{i=1}^{n}(y^{(i)} - (\mathbf{x}'^{(i)} \cdot \beta))^2 = (\mathbf{y} - \mathbf{X}'\beta) \cdot (\mathbf{y} - \mathbf{X}'\beta)$$

Solutions for finding regression β

- Loss function is a (convex) quadratic with exact, symbolic solution and you've learned how to solve for coefficients directly
- Many regularized loss functions and logistic regression have no direct solutions, though
- You'll use an iterative solution (gradient descent) for all regression problems in your project

Training/testing of linear models in Python

- Retype Boston dataset example from into a notebook:
  https://github.com/parrt/msds621/blob/master/notebooks/linear-models/sklearn-linear-models.ipynb

```
boston = load_boston()
X, y = boston.data, boston.target

X_train, X_test, y_train, y_test = \
     train_test_split(X, y, test_size=0.2)

lm = LinearRegression()      # OLS
lm.fit(X_train, y_train)
s = lm.score(X_test, y_test) # R^2 = 0.66
```

Logistic regression

Review of logistic regression

- For classification, response y is discrete int value like {0,1}
- Could use linear regression, but line would exceed [0,1] range
- Sigmoid is a much better transition from class 0 to class 1 and gives a probability of class 1
- Just run output of linear model into sigmoid
- Threshold at p(x)=0.5 to get classifier

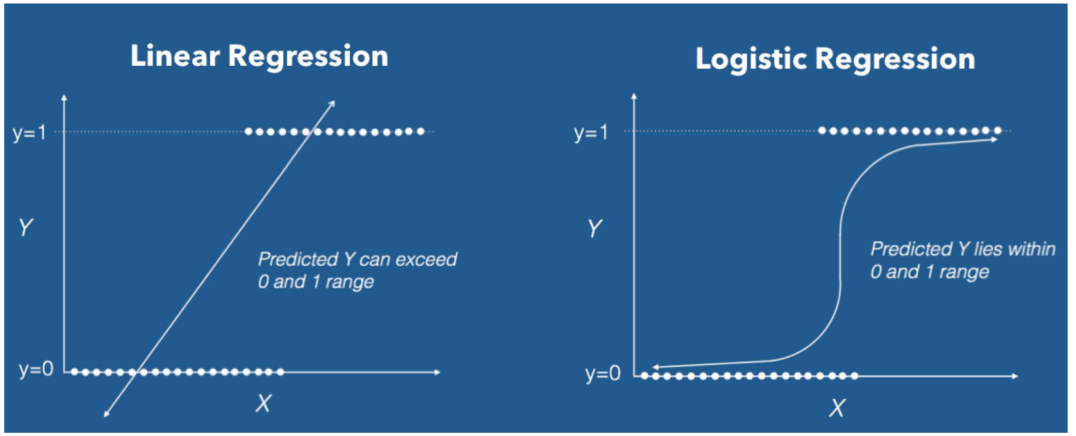


Image credit: https://www.machinelearningplus.com/machine-learning/logistic-regression-tutorial-examples-r

Logistic regression notation

- Sigmoid function

$$\sigma(z) = \frac{1}{1+e^{-z}} = \frac{e^z}{1+e^z}$$

- Substituting vectorized linear eqn into sigmoid:

$$p(\mathbf{x}') = \sigma(\mathbf{x}'\beta) = \frac{1}{1+e^{-\mathbf{x}'\beta}}$$

- Using odds = p/(1-p), subst in p(x'), simplify, take log; we get:

$$log(odds) = \mathbf{x}'\beta$$

Solving for logistic β

- Same idea: define loss function as mean-squared-error and solve for β that gives min loss function value
- Loss function has nonlinearity in it now though:

$$\mathscr{L}(\beta) = \sum_{i=1}^{n} \left\{ y^{(i)} \mathbf{x}'^{(i)} \beta - log(1 + e^{\mathbf{x}'\beta}) \right\}$$

- Logistic regression requires an iterative solution because there is no direct, symbolic solution, unlike linear regression

Training/testing of logistic regression models in Python

- Retype Wine dataset example from into a notebook: https://github.com/parrt/msds621/blob/master/notebooks/linear-models/classifier-regularization.ipynb

```
wine = load_wine()
df_wine = pd.DataFrame(data=wine.data,
columns=wine.feature_names)
df_wine['y'] = wine.target
df_wine = df_wine[df_wine['y']<2] # do 2-class problem {0,1}
X, y= df_wine.drop('y', axis=1), df_wine['y']
normalize(X)
lm = LogisticRegression(solver='lbfgs', max_iter=1000)
lm.fit(X.values, y)
lm.score(X.values, y)
```