A Summations

When an algorithm contains an iterative control construct such as a **while** or **for** loop, its running time can be expressed as the sum of the times spent on each execution of the body of the loop. For example, we found in Section 2.2 that the jth iteration of insertion sort took time proportional to j in the worst case. By adding up the time spent on each iteration, we obtained the summation (or series)

$$\sum_{j=2}^{n} j .$$

Evaluating this summation yielded a bound of $\Theta(n^2)$ on the worst-case running time of the algorithm. This example indicates the general importance of understanding how to manipulate and bound summations.

Section A.1 lists several basic formulas involving summations. Section A.2 offers useful techniques for bounding summations. The formulas in Section A.1 are given without proof, though proofs for some of them are presented in Section A.2 to illustrate the methods of that section. Most of the other proofs can be found in any calculus text.

A.1 Summation formulas and properties

Given a sequence $a_1, a_2, \ldots$ of numbers, the finite sum $a_1 + a_2 + \cdots + a_n$, where n is an nonnegative integer, can be written

$$\sum_{k=1}^{n} a_k .$$

If $n = 0$, the value of the summation is defined to be 0. The value of a finite series is always well defined, and its terms can be added in any order.

Given a sequence $a_1, a_2, \ldots$ of numbers, the infinite sum $a_1 + a_2 + \cdots$ can be written

$$\sum_{k=1}^{\infty} a_k ,$$

which is interpreted to mean

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k .$$

If the limit does not exist, the series ***diverges***; otherwise, it ***converges***. The terms of a convergent series cannot always be added in any order. We can, however, rearrange the terms of an ***absolutely convergent series***, that is, a series $\sum_{k=1}^{\infty} a_k$ for which the series $\sum_{k=1}^{\infty} |a_k|$ also converges.

Linearity

For any real number c and any finite sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k .$$

The linearity property is also obeyed by infinite convergent series.

The linearity property can be exploited to manipulate summations incorporating asymptotic notation. For example,

$$\sum_{k=1}^{n} \Theta(f(k)) = \Theta\left(\sum_{k=1}^{n} f(k)\right) .$$

In this equation, the Θ-notation on the left-hand side applies to the variable k, but on the right-hand side, it applies to n. Such manipulations can also be applied to infinite convergent series.

Arithmetic series

The summation

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n ,$$

is an ***arithmetic series*** and has the value

$$\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) \qquad \text{(A.1)}$$

$$= \Theta(n^2) . \qquad \text{(A.2)}$$

Sums of squares and cubes

We have the following summations of squares and cubes:

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} , \tag{A.3}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} . \tag{A.4}$$

Geometric series

For real $x \neq 1$, the summation

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$$

is a ***geometric*** or ***exponential series*** and has the value

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} . \tag{A.5}$$

When the summation is infinite and $|x| < 1$, we have the infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} . \tag{A.6}$$

Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \\ &= \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln n + O(1) . \end{aligned} \tag{A.7}$$

(We shall prove this bound in Section A.2.)

Integrating and differentiating series

Additional formulas can be obtained by integrating or differentiating the formulas above. For example, by differentiating both sides of the infinite geometric series (A.6) and multiplying by x, we get

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \tag{A.8}$$

for $|x| < 1$.

Telescoping series

For any sequence $a_0, a_1, \ldots, a_n$,

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 , \tag{A.9}$$

since each of the terms $a_1, a_2, \ldots, a_{n-1}$ is added in exactly once and subtracted out exactly once. We say that the sum ***telescopes***. Similarly,

$$\sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n .$$

As an example of a telescoping sum, consider the series

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} .$$

Since we can rewrite each term as

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} ,$$

we get

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} &= \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{n} . \end{aligned}$$

Products

The finite product $a_1 a_2 \cdots a_n$ can be written

$$\prod_{k=1}^{n} a_k .$$

If $n = 0$, the value of the product is defined to be 1. We can convert a formula with a product to a formula with a summation by using the identity

$$\lg \left(\prod_{k=1}^{n} a_k \right) = \sum_{k=1}^{n} \lg a_k .$$

Exercises

A.1-1
Find a simple formula for $\sum_{k=1}^{n}(2k-1)$.

A.1-2 ⋆
Show that $\sum_{k=1}^{n} 1/(2k-1) = \ln(\sqrt{n}) + O(1)$ by manipulating the harmonic series.

A.1-3
Show that $\sum_{k=0}^{\infty} k^2 x^k = x(1+x)/(1-x)^3$ for $0 < |x| < 1$.

A.1-4 ⋆
Show that $\sum_{k=0}^{\infty}(k-1)/2^k = 0$.

A.1-5 ⋆
Evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

A.1-6
Prove that $\sum_{k=1}^{n} O(f_k(n)) = O\left(\sum_{k=1}^{n} f_k(n)\right)$ by using the linearity property of summations.

A.1-7
Evaluate the product $\prod_{k=1}^{n} 2 \cdot 4^k$.

A.1-8 ⋆
Evaluate the product $\prod_{k=2}^{n}(1-1/k^2)$.

A.2 Bounding summations

There are many techniques available for bounding the summations that describe the running times of algorithms. Here are some of the most frequently used methods.

Mathematical induction

The most basic way to evaluate a series is to use mathematical induction. As an example, let us prove that the arithmetic series $\sum_{k=1}^{n} k$ evaluates to $\frac{1}{2}n(n+1)$. We can easily verify this for $n = 1$, so we make the inductive assumption that it holds for n and prove that it holds for $n+1$. We have

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$$