Online class: MAT 350.4/350.5 23/11/2025

System of ODEs

Solve:
$$\dot{x} = \underbrace{\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}}_{A} x$$

Soln:
$$|(A - \lambda I)| = 0$$

$$\left| \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} (1-\lambda) & -2 & 2 \\ -2 & (1-\lambda) & -2 \\ 2 & -2 & (1-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)(1-\lambda) - 4\right] + 2\left[-2(1-\lambda) + 4\right] + 2\left[4 - 2(1-\lambda)\right] = 0$$

$$\Rightarrow \quad (1-\lambda)\quad (1-2\lambda+\lambda^2-4) + 2(-2+2\lambda+4)$$
$$+2(4-2+2\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda^2-2\lambda-3) + 4(\lambda+1) + 4(\lambda+1) = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda-3)(\lambda+1) + 8(\lambda+1) = 0$$

$$(\lambda+1)\left[(1-\lambda)(\lambda-3)+8\right] = 0$$

$$(\lambda+1)(\lambda-\lambda^2-3+3\lambda+8)$$

$$(\lambda+1)(-\lambda^2+4\lambda+5)$$

$$-(\lambda+1)(\lambda^2-4\lambda-5) = 0$$

$$-(\lambda+1)(\lambda^2-5\lambda+\lambda-5) = 0$$

$$-(\lambda+1)(\lambda-5)(\lambda+1) = 0$$

$$\boxed{\lambda = -1, -1, 5}$$

For $\lambda = -1$, we find the eigenvector: K

$$(A-\lambda I)K = 0$$

$$\begin{bmatrix} 1-(-1) & -2 & 2 \\ -2 & 1-(-1) & -2 \\ 2 & -2 & 1-(-1) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3^{new} = R_3 + R_2$

$$\approx \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2^{new} = R_2 + R_1$

$$\approx \begin{pmatrix} 2 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we have Two free variables.

$\longrightarrow$ let k_2 & k_3 free variables.

$\longrightarrow$ $k_2 = 1$, & $k_3 = 0$ $\Rightarrow$ $2k_1 - 2k_2 + 2k_3 = 0$

$2k_1 - 2 + 0 = 0$

$k_1 = 1$

$$\therefore K = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \hookleftarrow$$

Also, let $k_3 = 1$ & $k_2 = 0$ $\Rightarrow$ $2k_1 - 2k_2 + 2k_3 = 0$

$2k_1 - 2\cdot 0 + 2\cdot 1 = 0$

$k_1 = -1$

$$K = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \hookleftarrow$$

Also, for $\lambda = 5$, we find the eigenvector K as:

$$\begin{pmatrix} 1-5 & -2 & 2 \\ -2 & 1-5 & -2 \\ 2 & -2 & 1-5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 2 & -2 & -4 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3^{new} = R_3 + R_2$

$$\approx \begin{pmatrix} -4 & -2 & 2 \\ -2 & -4 & -2 \\ 0 & -6 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2^{new} = 2R_2 - R_1$

$$\approx \begin{pmatrix} -4 & -2 & 2 \\ 0 & -6 & -6 \\ 0 & -6 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\approx \begin{pmatrix} -4 & -2 & 2 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let k_3 be free variable. & $k_3 = 2$

$\therefore \quad -6k_2 - 6k_3 = 0$

$-6k_2 - 6 * 2 = 0 \qquad\qquad k_2 = -2$

&

$$-4k_1 - 2k_2 + 2k_3 = 0$$

$$-4k_1 - 2(-2) + 2(2) = 0$$

$$-4k_1 + 4 + 4 = 0$$

$$-4k_1 = -8,$$

$$k_1 = 2$$

$$K = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

General solution:

$$X = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} e^{5t}$$

Ans

Chp# 8.3

Non-Homogeneous ODE

$$\dot{X} = AX + F(t), \qquad F(t) \neq 0$$

In this case, The general solution X

$$X = X_c + X_p$$

X_c: comes by solving the Homogeneous part.

X_p: various process

(i) Superposition

(ii) Variation of parameters.

Ex: Solve: $\dot{X} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X + \underbrace{\begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}}_{F(t)}$

Sol General solution:

$$X = X_c + X_p$$

For X_c: we solve: $\dot{X} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} X$

$\lambda = 2, 7$

$\lambda = 2$: $K = \begin{pmatrix} -1/4 \\ 1 \end{pmatrix}$ $\qquad$ $\lambda = 7$: $K = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$$X_c = c_1 \begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{7t}$$

For X_p: we follow the superposition method:

$$F(t) = \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 6 \\ -10 \end{pmatrix}}_{A} \underset{t}{t} + \underset{B}{\begin{pmatrix} 0 \\ 4 \end{pmatrix}}$$

$$\therefore X_p = At + B$$

$$X_p = \underline{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} t + \underline{\underline{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}}$$

So, $X_p' = \underline{A}\,\underline{X_p} + F(t)$

$$\underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\dot{X}_p} = \underbrace{\begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}}_{A} \underbrace{\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right]}_{X(p)} + \underbrace{\begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}}_{F(t)}$$

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$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} \boxed{a_1 t + b_1} \\ \boxed{a_2 t + b_2} \end{pmatrix} + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 6a_1 t + 6b_1 + a_2 t + b_2 + 6t \\ 4a_1 t + 4b_1 + 3a_2 t + 3b_2 - 10t + 4 \end{pmatrix}$$

$$\begin{pmatrix} \underline{a_1} \\ \underline{a_2} \end{pmatrix} = \begin{bmatrix} (\underline{6a_1 + a_2 + 6})t + (\underline{6b_1 + b_2}) \\ (\underline{4a_1 + 3a_2 - 10})t + (\underline{4b_1 + 3b_2 + 4}) \end{bmatrix}$$

Equating coefficient from both sides:

$$\left.\begin{aligned} 6a_1 + a_2 + 6 &= 0 \\ 4a_1 + 3a_2 - 10 &= 0 \end{aligned}\right]$$

$$\left.\begin{aligned} 6b_1 + b_2 &= \cancel{a_1}\ -2 \\ 4b_1 + 3b_2 + 4 &= \cancel{a_2}\ 6 \end{aligned}\right]$$

$\downarrow$

solve:

$a_1 = -2$

$a_2 \quad 6$

solve:

$b_1 = -4/7$

$b_2 = 10/7$

$$x_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -4/7 \\ 10/7 \end{pmatrix}$$

$$X = X_c + X_p$$

$$= c_1 \begin{pmatrix} -1/4 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{7t}$$

$$+ \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -4/7 \\ 10/7 \end{pmatrix} \quad \underline{\underline{\text{Ans}}}$$