Course work solution

1. Solve the ODE $4y'' + y' - 3y = 2e^{-5t}$
i.c. $y'(0) = 0$ and $y(0) = 1$

take the Laplace transform of each term:

$$4\,L\{y''\} = 4s^2 Y(s) - 4s\,y(0) - 4y'(0)$$
$$L\{y'\} = sY(s) - y(0)$$
$$-3\,L\{y\} = -3Y(s)$$
$$2\,L\{e^{-5t}\} = 2/s+5$$

Sub in the i.c.

$$4s^2Y(s) - 4s + sY(s) - 1 - 3Y(s) = 2/s+5$$

$$Y(s)\left[4s^2 + s - 3\right] = \frac{2}{s+5} + 1 + 4s = \frac{4s^2+21s+7}{s+5}$$

$$Y(s) = \frac{4s^2+21s+7}{(s+5)(4s^2+s-3)} = \frac{4s^2+21s+7}{(s+5)(4s-3)(s+1)}$$

$$\frac{4s^2+21s+7}{(s+5)(4s-3)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1} + \frac{C}{4s-3}$$

$$4s^2+21s+7 = A(s+1)(4s-3) + B(s+5)(4s-3) + C(s+5)(s+1)$$

using the cover-up rule:

when $s = -1$

$$-10 = B(4)(-7) = -28B \Rightarrow B = 10/28 = 5/14$$

when $s = -5$

$$2 = A(-4)(-23) = 92A \Rightarrow A = 2/92 = 1/46$$

when $s = 3/4$

$$25 = C(23/4)(7/4) = \frac{161}{16}C \Rightarrow C = \frac{25\times16}{161} = 400/161$$

$$Y(s) = \frac{1}{46(s+5)} + \frac{5}{14(s+1)} + \frac{400}{161(4s-3)}$$

invert the Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{46(s+5)}\right\} = \frac{1}{46}e^{-5t}$$

$$\mathcal{L}^{-1}\left\{\frac{5}{14(s+1)}\right\} = \frac{5e^{-t}}{14}$$

$$\mathcal{L}^{-1}\left\{\frac{400}{161(4s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{100}{161(s-3/4)}\right\} = \frac{100}{161}e^{3/4t}$$

$$y(t) = \frac{e^{-5t}}{46} + \frac{5e^{-t}}{14} + \frac{100e^{3/4t}}{161}$$

2. Find the Laplace transform of $x(t) = \begin{cases} 4-t & 0 < t < 3 \\ t^3 & 3 < t < 5 \\ t-2 & 5 < t \end{cases}$

using the Heaviside function

$$x(t) = (4-t)H(t) - (4-\overset{*}{t})H(t-3) + \overset{*}{t}^3 H(t-3) - \overset{*}{t}^3 H(t-5) + (\overset{*}{t}-2)H(t-5)$$

$t = t-3+3$ and $t^3 = (t-3)^3 + 9(t-3)^2 + 27(t-3) + 27$

$t = t-5+5$ and $t^3 = (t-5)^3 + 15(t-5)^2 + 75(t-5) + 125$

$$x(t) = (4-t)H(t) \underbrace{-\left(4-[(t-3)+3]\right)}_{1} H(t-3) + (t-3)^3 H(t-3)$$
$$+9(t-3)^2 H(t-3) + 27(t-3)H(t-3) + 27H(t-3) - (t-5)^3 H(t-5)$$
$$-15(t-5)^2 H(t-5) - 75(t-5)H(t-5) - 125H(t-5)$$
$$+\underbrace{\left((t-5)+5-2\right)}_{2} H(t-5)$$

$$1 \Rightarrow -(4-(t-3)-3) = (t-3)-1$$
$$2 \Rightarrow (t-5)+3$$

collect the terms:

$$x(t) = (4-t)H(t) + 26H(t-3) + 28(t-3)H(t-3) + 9(t-3)^2 H(t-3)$$
$$+ (t-3)^3 H(t-3) - (t-5)^3 H(t-5) - 15(t-5)^2 H(t-5)$$
$$- 74(t-5)H(t-5) - 122H(t-5)$$

take the Laplace transform using the time-shift property:

$$X(s) = \frac{4}{s} - \frac{1}{s^2} + \frac{26e^{-3s}}{s} + \frac{28e^{-3s}}{s^2} + \frac{18e^{-3s}}{s^3} + \frac{6e^{-3s}}{s^4}$$
$$-\frac{6e^{-5s}}{s^4} - \frac{30e^{-5s}}{s^3} - \frac{74e^{-5s}}{s^2} - \frac{122e^{-5s}}{s}$$

3. Use the convolution theorem to find the inverse of

$$X(s) = \frac{5}{s^2(s+2)}$$

Let $A(s) = \frac{5}{s^2}$ and $B(s) = \frac{1}{s+2}$

$$\Rightarrow a(t) = 5t$$
$$b(t) = e^{-2t}$$

$$a(t) * b(t) = \int_{\tau=0}^{t} 5e^{(-2(t-\tau))} . \tau \, d\tau$$

$$= 5\int_0^t \tau e^{-2t} e^{2\tau} \, d\tau$$

$$= 5e^{-2t} \int_0^t \tau e^{2\tau} \, d\tau \qquad u = \tau \quad v' = e^{2\tau}$$

$$u' = 1 \quad v = \frac{e^{2\tau}}{2}$$

$$= 5e^{-2t} \left[\frac{\tau e^{2\tau}}{2} - \int \frac{e^{2\tau}}{2} \, d\tau \right]_0^t$$

$$= 5e^{-2t} \left[\frac{\tau e^{2\tau}}{2} - \frac{e^{2\tau}}{4} \right]_0^t$$

$$= 5e^{-2t} \left[\frac{te^{2t}}{2} - \frac{e^{2t}}{4} - \left(0 - \frac{1}{4}\right) \right]$$

$$= \frac{5t}{2} - \frac{5}{4} + \frac{5e^{-2t}}{4}$$

4. Signal $x(t) = \begin{cases} t^3 & -2 < t < 0 \\ 0 & 0 < t < 2 \\ t+2 & 2 < t < 4. \end{cases}$

$$F[x(t)] = \underbrace{\int_{-2}^{0} t^3 e^{-2j\pi f t} \, dt}_{1} + \underbrace{\int_0^2 0 \cdot e^{-2j\pi f t} \, dt}_{=0} + \underbrace{\int_2^4 (t+2) e^{-2j\pi f t} \, dt}_{2}$$

(1) $\displaystyle\int_{-2}^{0} t^3 e^{-j\pi f t} \, dt \qquad u = t^3 \quad v' = e^{-2j\pi f t}$

$$u' = 3t^2 \quad v = \frac{-e^{-2j\pi f t}}{2j\pi f}$$

$$= \left[\frac{t^3 e^{-2j\pi f t}}{2j\pi f} - \int \frac{-3t^2 e^{-2j\pi f t}}{2j\pi f} \, dt \right]_{-2}^{0}$$

$$= \left[\frac{-t^3 e^{-2j\pi f t}}{2j\pi f} + \frac{3}{2j\pi f} \int t^2 e^{-2j\pi f t}\, dt \right]_{-2}^{0} \qquad \begin{array}{ll} u = t^2 & u' = 2t \\ v' = e^{-2j\pi f t} & v = \dfrac{-e^{-2j\pi f t}}{2j\pi f} \end{array}$$

$$= \left[\frac{-t^3 e^{-2j\pi f t}}{2j\pi f} + \frac{3}{2j\pi f} \left[\frac{-t^2 e^{-2j\pi f t}}{2j\pi f} - \int \frac{-2t e^{-2j\pi f t}}{2j\pi f}\, dt \right] \right]_{-2}^{0}$$

$$= \left[\frac{-t^3 e^{-2j\pi f t}}{2j\pi f} - \frac{6t^2 e^{-2j\pi f t}}{(2j\pi f)^2} + \frac{6}{(2j\pi f)^2} \int t e^{-2j\pi f t}\, dt \right]_{-2}^{0} \qquad \begin{array}{ll} u = t & u' = 1 \\ v' = e^{-2j\pi f t} & v = \dfrac{-e^{-2j\pi f t}}{2j\pi f} \end{array}$$

$$= \left[\frac{-t^3 e^{-2j\pi f t}}{2j\pi f} - \frac{6t^2 e^{-2j\pi f t}}{(2j\pi f)^2} + \frac{6}{(2j\pi f)^2} \left[\frac{-t e^{-2j\pi f t}}{2j\pi f} - \int \frac{-e^{-2j\pi f t}}{2j\pi f}\, dt \right] \right]_{-2}^{0}$$

$$= \left[\frac{-t^3 e^{-2j\pi f t}}{2j\pi f} - \frac{6t^2 e^{-2j\pi f t}}{(2j\pi f)^2} - \frac{6t e^{-2j\pi f t}}{(2j\pi f)^3} - \frac{6 e^{-2j\pi f t}}{(2j\pi f)^4} \right]_{-2}^{0}$$

putting in the limits:

$$\frac{-6}{(2j\pi f)^4} - \left[\frac{8e^{4j\pi f}}{2j\pi f} - \frac{24e^{4j\pi f}}{(2j\pi f)^2} + \frac{12e^{4j\pi f}}{(2j\pi f)^3} - \frac{6e^{4j\pi f}}{(2j\pi f)^4} \right]$$

multiply out bracket, remove j from denominator and cancel powers of 2:

$$-\frac{3}{8(\pi f)^4} + \frac{4je^{4j\pi f}}{\pi f} - \frac{6e^{4j\pi f}}{(\pi f)^2} - \frac{3je^{4j\pi f}}{2(\pi f)^3} + \frac{3e^{4j\pi f}}{8(\pi f)^4}$$

(2) $$\int_2^4 (t+2) e^{-2j\pi f t}\, dt = \int_2^4 t e^{-2j\pi f t}\, dt + 2\int_2^4 e^{-2j\pi f t}\, dt$$

$u = t \quad u' = 1$

$v' = e^{-2j\pi f t} \quad v = \dfrac{-e^{-2j\pi f t}}{2j\pi f}$

$$= \left[\frac{-te^{-2j\pi\gamma t}}{2j\pi\gamma} - \int \frac{-e^{-2j\pi\gamma t}}{2j\pi\gamma}\,dt - \frac{2e^{-2j\pi\gamma t}}{2j\pi\gamma} \right]_2^4$$

$$= \left[\frac{-te^{-2j\pi\gamma t}}{2j\pi\gamma} - \frac{e^{-2j\pi\gamma t}}{(2j\pi\gamma)^2} - \frac{2e^{-2j\pi\gamma t}}{2j\pi\gamma} \right]_2^4$$

putting in the limits:

$$\frac{-4e^{-8j\pi\gamma}}{2j\pi\gamma} \quad \frac{-e^{-8j\pi\gamma}}{(2j\pi\gamma)^2} \quad -\frac{2e^{-8j\pi\gamma}}{2j\pi\gamma} - \left[\frac{-2e^{-4j\pi\gamma}}{2j\pi\gamma} \quad \frac{-e^{-4j\pi\gamma}}{(2j\pi\gamma)^2} \quad \frac{-2e^{-4j\pi\gamma}}{2j\pi\gamma} \right]$$

remove j from the denominator, collect terms and cancel powers of 2

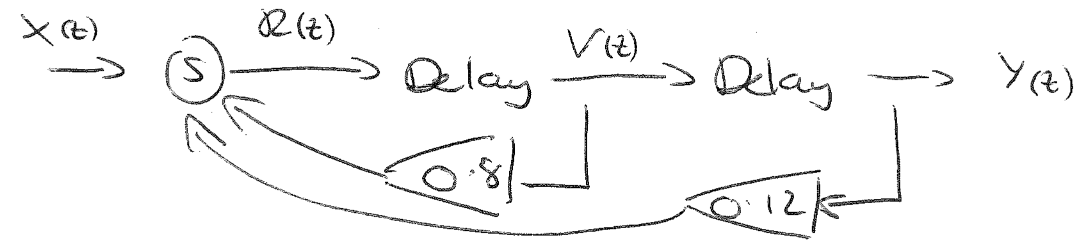
$$\frac{3je^{-8j\pi\gamma}}{\pi\gamma} + \frac{e^{-8j\pi\gamma}}{(2\pi\gamma)^2} - \frac{e^{-4j\pi\gamma}}{(2\pi\gamma)^2} - \frac{2je^{-4j\pi\gamma}}{\pi\gamma}$$

combine the two integrals together:

$$X(\gamma) = \frac{-3}{8(\pi\gamma)^4} + \frac{4je^{4j\pi\gamma}}{\pi\gamma} - \frac{6e^{4j\pi\gamma}}{(\pi\gamma)^2} - \frac{3je^{4j\pi\gamma}}{2(\pi\gamma)^3} + \frac{3e^{4j\pi\gamma}}{8(\pi\gamma)^4}$$

$$+ \frac{3je^{-8j\pi\gamma}}{\pi\gamma} + \frac{e^{-8j\pi\gamma}}{(2\pi\gamma)^2} - \frac{e^{-4j\pi\gamma}}{(2\pi\gamma)^2} - \frac{2je^{-4j\pi\gamma}}{\pi\gamma}$$

5.



a) $Y(z) = \frac{1}{z} V(z)$, $V(z) = \frac{1}{z} R(z)$

$R(z) = X(z) - 0.8\, V(z) - 0.12\, Y(z)$

$V(z) = z\, Y(z) \qquad R(z) = z^2\, Y(z)$

$X(z) = Y(z) \left[z^2 + 0.8 z + 0.12 \right]$

Transfer function : $G(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^2 + 0.8 z + 0.12}$

Stability is given by the poles, must lie in the unit circle

$z^2 + 0.8 z + 0.12 = 0$

$(z + 0.2)(z + 0.6) = 0 \implies z = -0.2$

$z = -0.6$

both lie in the unit circle, so system is stable.

b. impulse response is the inversion of the transfer function.

$G(z) = \frac{1}{(z+0.2)(z+0.6)} \qquad \mathcal{Z}^{-1}\left[G(z)\right]$

$\frac{G(z)}{z} = \frac{1}{z(z+0.2)(z+0.6)}$

Split into partial fractions:

$$\frac{1}{z(z+0.2)(z+0.6)} = \frac{A}{z} + \frac{B}{z+0.2} + \frac{C}{z+0.6}$$

$$1 = A(z+0.2)(z+0.6) + B(z)(z+0.6) + Cz(z+0.6)$$

when $z=0$

$$1 = A(0.2)(0.6) = \frac{3}{25}A \qquad A = 25/3$$

when $z=-0.2$

$$1 = B(-0.2)(0.4) = -2/25\,B \qquad B = -25/2$$

when $z=-0.6$

$$1 = C(-0.6)(-0.4) = 6/25\,C \qquad C = 25/6$$

$$G(z) = \frac{25}{3} + \frac{25}{6}\frac{z}{z+0.6} - \frac{25}{2}\frac{z}{(z+0.2)}$$

impulse responce: $h(n) = \frac{25}{3}\delta(n) + \frac{25}{6}(-0.6)^n - \frac{25}{2}(-0.2)^n$

c. $x(n) = 3^n \qquad X(z) = \frac{z}{z-3} \qquad Y(z) = G(z)\cdot X(z)$

$$Y(z) = \frac{1}{(z+0.2)(z+0.6)} \cdot \frac{z}{z-3}$$

responce: $y(nT) = Z^{-1}[Y(z)]$

$$\frac{Y(z)}{z} = \frac{1}{(z+0.2)(z+0.3)(z-3)}$$

$$\frac{1}{(z+0.2)(z+0.3)(z-3)} = \frac{A}{z+0.2} + \frac{B}{z+0.6} + \frac{C}{z-3}$$

$$1 = A(z+0.6)(z-3) + B(z+0.2)(z-3) + C(z+0.2)(z+0.6)$$

when $z = -0.2$

$$1 = A(0.4)(-3.2) = \frac{-32}{25}A \implies A = -25/32$$

when $z = -0.6$

$$1 = B(-0.4)(-3.6) = 36/25\, B \implies B = 25/36$$

when $z = 3$

$$1 = C(3.2)(3.6) = 288/25\, C \implies C = 25/288$$

$$Y(z) = \frac{-25}{32}\frac{z}{z+0.2} + \frac{25}{36}\frac{z}{z+0.6} + \frac{25}{288}\frac{z}{z-3}$$

$$y(n) = \frac{-25}{32}(-0.2)^n + \frac{25}{36}(-0.6)^n + \frac{25}{288}3^n$$

First 5 samples:

$x(0) = 3^0 = 1$ $\quad y(0) = \frac{-25}{32} + \frac{25}{36} + \frac{25}{288} = 0$

$x(1) = 3^1 = 3$ $\quad y(1) = 5/32 - 5/12 + 25/96 = 0$

$x(2) = 3^2 = 9$ $\quad y(2) = -1/32 + 1/4 + 25/32 = 1$

$x(3) = 3^3 = 27$ $\quad y(3) = 1/160 - 3/20 + 75/32 = 11/5$

$x(4) = 3^4 = 81$ $\quad y(4) = -1/800 + 9/100 + 225/32 = 178/25$