Markov Chain

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2.1 Difinition

Definition 2.1.1 (Markov Chain)**.** A discrete time stochastic process $\{X_n, n = 1, 2, 3, \ldots\}$ is defined to be *Discrete Time Markov Chain* or simply *Markov Chain* if it takes value the state space $\mathbf{S}$, and for every $n \geq 0$ it satisfy the property

$$\mathbf{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \mathbf{P}(X_{n+1} = j | X_n = i) \tag{2.1}$$

Unless otherwise mentioned we take the state space $\mathbf{S}$ to be $\{0, 1, 2, 3, \ldots\}$. If $X_n = i$ we say that the process is in ith state at time n. In the definition eq. (2.1) may be interpreted as for Markov Chain, the conditional distribution of any future state X_{n+1}, given the past states $X_0, X_1, \ldots, X_{n-1}$ and the present state X_n, is independent of the past and only depend on the present state. This property is called *Markovian Property*. In other word for markov chain predicting the future we only need information about the present state.

Note: Notice that Assumption we are making for markov chain to forget the past as long as present in known is very strong assumption.

2.2 Homogeneous Markov Chain

Definition 2.2.1 (Homogeneous Markov Chain)**.** We say a markov chain $\{X_n, n \geq 0\}$ is homogeneous if $\mathbf{P}(X_{n+1} = j | X_n = i) = \mathbf{P}(X_2 = j | X_1 = i)\ \forall n > 0$.

The quantity $\mathbf{P}(X_{n+1} = j | X_n = i)$ is called the *transition probability* from state i to state j. For homogeneous Markov Chain we can specify the transition probabilities $\mathbf{P}(X_{n+1} = j | X_n = i)$ by a sequence of value $p_{xy} = \mathbf{P}(X_{n+1} = y | X_n = x)$.

For the case of finite state Markov chain, say the state space is $\{1, 2, 3, \ldots\}$. Then the transition probabilities are p_{ij}, $1 \leq i, j \leq N$ for transition from state i to state j. The $N \times$

N matrix $P = (p_{ij})_{N\times N}$ is called The *Transition Matrix* of chain. Since probabilities are nonnegative and since the process must make a transition into some state, we have

$$P_{ij} \geq 0, \quad i,j \geq 0, \text{ and } \sum_{j=0}^{N} P_{ij} = 1, \quad \forall\, i = 0,1,2,\ldots.$$

For, infinite state markov chain the probability transition matrix will be infinite order. Then,

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{11} & P_{12} & p_{12} & \cdots \\ \vdots & \vdots & \vdots & \ldots \\ p_{i0} & \cdots & p_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

2.3 Chapman-Kolmogorov Equations

We have already defined the one step transition probabilities p_{ij}. We now define the n-step transition probabilities p_{ij}^n to be the probability that a process in state i will be in state j after n additional transitions. i.e.

$$p_{ij}^n = \mathbf{P}(X_{n+m} = j | X_m = i), \; n \geq 0, \; i,j \geq 0.$$

By definition of markov chain we get,

$$\begin{aligned} p_{ij}^{n+m} &= \mathbf{P}(X_{n+m} = j | X_0 = i) \\ &= \sum_{k=0}^{\infty} \mathbf{P}(X_{n+m} = j, X_n = k | X_0 = i) \text{ (By theorem of total probability)} \\ &= \sum_{k=0}^{\infty} \mathbf{P}(X_{n+m} = j | X_n = k, X_0 = i)\mathbf{P}(X_n = k | X_0 = i) \\ &= \sum_{k=0}^{\infty} \mathbf{P}(X_{n+m} = j | X_n = k)\mathbf{P}(X_n = k | X_0 = i) \\ &= \sum_{k=0}^{\infty} p_{kj}^m p_{ik}^n. \end{aligned} \tag{2.2}$$

If we take $n = m = 1$. Then

$$p_{ij}^2 = \sum_{k=0}^{\infty} p_{kj} p_{ik} \tag{2.3}$$

the above expression is (i,j) element of P^2 matrix then we see eq. (2.3) in matrix form,

$$P^2 = P * P$$

Hence, eq. (2.2) can also written in matrix form,

$$P^{n+m} = P^n * P^m$$

Where $P^n \& P^m$