Chapter - 06

$$5x_1 + 2x_2 + 3x_3 = 7$$
$$2x_1 + 7x_2 + 8x_3 = 5$$
$$3x_1 + 9x_2 + 2x_3 = 6$$

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 8 \\ 3 & 9 & 2 \end{bmatrix}$$

3×3 matrix = $n \times n$ matrix

Here, the no. of variables = no. of equations

We can solve this using Gaussian Elimination, LU Decomposition, Inverse Matrix.

However, if we added another equation:

$$4x_1 + 2x_2 + 5x_3 = 10.$$

The matrix would look like this,

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 8 \\ 3 & 9 & 2 \\ 4 & 2 & 5 \end{bmatrix} \rightarrow [4 \times 3] \text{ matrix}$$

Not a square matrix

$[m \times n]$ $m > n$.

We cannot directly solve this using Gaussian Elimination LU Decomposition, Inverse Matrix since this is not a square matrix.

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This system is known as overdetermined syst
where the number of equations is greater th
number of variables. The matrix is not a squ
matrix.

4 equations
3 variables

$$\left\{\begin{aligned} 5x_1 + 2x_2 + 3x_3 &= 7 \\ 2x_1 + 7x_2 + 8x_3 &= 5 \\ 3x_1 + 9x_2 + 2x_3 &= 6 \\ 4x_1 + 2x_2 + 5x_3 &= 10 \end{aligned}\right.$$

$$\begin{bmatrix} 5 & 2 & 3 \\ 2 & 7 & 8 \\ 3 & 9 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

$m \times n$

$m > n$.

~~Orthogo~~

Orthonormality : orthogonal + Normal

We need to check if it has the following properties : orthogonality and normality .

$X = [x_0 \; x_1 \; x_2 \; x_3]$

$[1 \times 4]$ matrix

Row Matrix

$y = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$[4 \times 1]$

Column Matrix

$X = [x_0 \; x_1 \; x_2 \; x_3]$ $[1 \times 4]$ matrix

$X^T = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$ → Transpose matrix where row becomes column and vice versa .

$[4 \times 1]$ matrix

$y = \begin{bmatrix} 5 & 4 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ $[2 \times 3]$ matrix

$y^T = \begin{bmatrix} 5 & 1 \\ 4 & 0 \\ 3 & 2 \end{bmatrix}$ $[3 \times 2]$ matrix

$Q_{(m \times n)} = \underset{(n \times m)}{Q^T} \; \underset{(m \times n)}{Q} = I_{(n \times n)}$

$Q^T_{(n \times m)}$

Square matrix

Orthogonality :

** $\boxed{x^T y = 0}$

$\theta = 90$:
$x^T y = |x| |y| \cos 90$
$x^T y = 0$

not important for exam.

Normality :

** $\boxed{\begin{matrix} x^T x = 1 \\ y^T y = 1 \end{matrix}}$

Example : Check if the following set is orthonormal.

$$S = \left\{ \frac{1}{\sqrt{5}} (2, 1)^T, \frac{1}{\sqrt{5}} (1, -2)^T \right\}$$

u matrix v matrix

Normality : ① $u^T u = 1$

$$\frac{1}{\sqrt{5}} [2 \ 1] \times \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5} [2 \ 1] \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5} (4 + 1) = 1.$$

$u^T u = 1 \rightarrow$ True

(ii) $v^T v = 1$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \end{bmatrix} \times \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{5} (1+4) = 1.$$

$v^T v = 1 \rightarrow$ True

Orthogonality :

$$u^T v = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \times \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \frac{1}{5} (2-2)$$

$u^T v = 0 \rightarrow$ True.

Since both normality and orthogonality has been proved so the matrix has orthonormal properties.

Kronecker Delta :

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$u^T u$ { no

$u^T v$ { orth

$\hookrightarrow$ orthonormal

Least Square

$$Ax = b$$

$[m \times n]$

$$A^T A x = A^T b$$

A^T: $[n \times m]$, A: $\hookrightarrow [m \times n]$ $=$ $[n \times n]$

Square matrix.

By multiplying with Transpose matrix on both sides we can solve the problem of ~~square matr~~ overdetermined system and obtain a square matrix.

Example :

$f(-3) = 0$
$f(0) = 0$
$f(6) = 2$

$P_2(x) = a_0 + a_1 x + a_2 x^2$
↳ degree is 2

In order to make this an overdetermined system we can make it :

$P_1(x) = a_0 + a_1 x$
$P_1(-3) = a_0 - 3a_1 = 0$
$P_1(0) = a_0 = 0$
$P_1(6) = a_0 + 6a_1 = 2$

3 equations however 2 variables hence we took $P_1(x)$ instead of $P_2(x)$

Coefficient matrix For $P_1(x)$:

$$\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

The linear system in matrix form :

$$\underset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \underset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

(3×2) matrix

$A^T A x = A^T b$

$$A^T A = \underset{[2\times 3]}{\begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix}} \underset{[3\times 2]}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}}$$

$$= \underset{[2\times 2]}{\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 45 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 12 \end{bmatrix}$$

$$a_0 = 3/7$$

$$a_1 = 5/21$$

$$p_1(x) = \frac{3}{7} + \frac{5}{21} x.$$ [We have solved this overdetermined system using inverse matrix].

overdetermined system

$f(2) = 3$
$f(3) = 5$
$f(5) = 12$
$f(6) = 15$

$P_2(x) = a_0 + a_1 x + a_2 x^2$
$P_2(2) = a_0 + 2a_1 + 4a_2 = 3$
$P_2(3) = a_0 + 3a_1 + 9a_2 = 5$
$P_2(5) = a_0 + 5a_1 + 25a_2 = 12$
$P_2(6) = a_0 + 6a_1 + 36a_2 = 15$

4 equations, 3 variables.

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix}}_{b}$$

$$A^T A x = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}$$

[3×4] matrix [4×3] matrix

$$[3\times3] \text{ matrix} = \begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix} [3\times3] \text{ matrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 6 \\ 4 & 9 & 25 & 36 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 12 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 35 \\ 171 \\ 879 \end{bmatrix}$$

Augmented Matrix :

$$\left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 16 & 74 & 376 & 171 \\ 74 & 376 & 2018 & 879 \end{array}\right]$$

Applying Gaussian Elimination :

$$R_2 = R_2 - \left(\frac{16}{4}\right) R_1$$

$$R_3 = R_3 - \left(\frac{74}{4}\right) R_1$$

$$\left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 0 & 10 & 80 & 31 \\ 0 & 80 & 649 & 231.5 \end{array}\right]$$

$$R_3 = R_3 - \left(\frac{80}{10}\right) R_2$$

$$\left[\begin{array}{ccc|c} 4 & 16 & 74 & 35 \\ 0 & 10 & 80 & 31 \\ 0 & 0 & 9 & -16.5 \end{array}\right]$$

$$9a_2 = -16.5$$

$$10a_1 + 80a_2 = 31$$

$$4a_0 + 16a_1 + 74a_2 = 35.$$

QR Decomposition

$$\underset{m\times n}{A} = QR \qquad Q = \{q_1, q_2, \ldots\}$$

$\hookrightarrow$ orthonormal property.

Q is an orthonormal set of vectors which we can obtain by $\boxed{\text{Gram - Schmidt process}}$.

Suppose we have a coefficient matrix given,

$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \\ 0 & 2 \end{bmatrix}$$

$\quad\; v_1 \quad v_2$

Using Gram - schmidt process we will convert these v into q. $[Q = \{q_1, q_2\}]$

<u>Step 1</u> : $p_k = v_k - \sum_{i=1}^{k-1} (v_k^T q_i) q_i$

$\hookrightarrow$ The result obtained will be a scalar dot product.

<u>Step 2</u> : $q_k = \dfrac{p_k}{|p_k|}$.

$\hookrightarrow$ Gram - schmidt process.

Steps to solve the problem.

Step-1 : $k = 1$.

$P_1 = U_1$

Reason ? We always start from $i=1$

$$P_1 = U_1 - \sum_{i=1}^{0}$$ since $k = 1 (k-1 = 0)$

This is not possible because we can't go from 1 to 0.

$$P_1 = U_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

$$q_1 = \frac{P_1}{|P_1|}$$

$$|P_1| = \sqrt{(3)^2 + (6)^2 + (0)^2} = \sqrt{45}$$

$$q_1 = \frac{1}{\sqrt{45}} \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}$$

Step 2 : $k = 2$

$$P_2 = U_2 - \sum_{i=1}^{1} (U_2^T q_i) q_i$$

$$P_2 = U_2 - (U_2^T q_1) q_1$$

$$P_2 = \begin{pmatrix}1\\2\\2\end{pmatrix} - \left(\begin{pmatrix}1 & 2 & 2\end{pmatrix} \begin{pmatrix}3\\6\\0\end{pmatrix} \frac{1}{\sqrt{45}} \right) q_1$$

$$P_2 = \begin{pmatrix}1\\2\\2\end{pmatrix} - \left(15 \cdot \frac{1}{\sqrt{45}}\right) \left(\frac{1}{\sqrt{45}} \begin{pmatrix}3\\6\\0\end{pmatrix} \right)$$

$$P_2 = \begin{pmatrix}1\\2\\2\end{pmatrix} - \left(\frac{1}{3}\right) \begin{pmatrix}3\\6\\0\end{pmatrix}$$

$$P_2 = \begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$P_2 = \begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$q_2 = \frac{P_2}{|P_2|}$$

$$|P_2| = \sqrt{(0)^2 + (0)^2 + (2)^2} = 2$$

$$q_2 = \frac{1}{2} \begin{pmatrix}0\\0\\2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$$

$$= \begin{array}{cc} \begin{pmatrix} \frac{3}{\sqrt{45}} & 0 \\ 6/\sqrt{45} & 0 \\ 0 & 1 \end{pmatrix} \\ q_1 \quad\quad q_2 \end{array}$$

∴ We have used Gram-schmidt process to convert u into orthonormal set of vectors q_1 and q_2.

$Ax = b$
$\hookrightarrow (m \times n)$

Previously we have used Transpose matrix to convert A into a square matrix.

$$A^T A x = A^T b$$

$$A = QR$$

$$(QR)^T (QR) x = (QR)^T b$$

$$(Q^T R^T)(QR) x = (QR)^T b$$

$$R^T \boxed{Q^T Q} R x = Q^T R^T b$$

$\hookrightarrow$ According to property of normality, $Q^T Q = I$.

$$\cancel{R^T} \cdot I \cdot Rx = Q^T \cancel{R^T} b$$

$$\boxed{Rx = Q^T b} \rightarrow$$ We will use this equation to solve the system by QR Decomposition.

Example : $f(-3) = 0$

$f(0) = 0$

$f(6) = 2$

In order to make this an overdetermined system we will consider P_1.

$$P_1 = a_0 + a_1(x) = f(x)$$

$$P_1(-3) = a_0 - 3a_1 = 0.$$

$$P_1(0) = a_0 = 0.$$

$$P_1(6) = a_0 + 6a_1 = 2.$$

$$\overset{A}{\begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}} \overset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}.$$

↳ (3×2) matrix

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$$

$$\quad u_1 \quad u_2 .$$

We will now convert u_1 and u_2 into their corresponding orthonormal set of vectors q_1 and q_2 using Gram-Schmidt process.

$$P_k = U_k - \sum_{i=1}^{k-1} (U_k^T q_i) q_i$$

Step 1 : $k = 1$

$$P_1 = U_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$q_1 = \frac{P_1}{|P_1|}$$

$$|P_1| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Step 2 : $k = 2$

$$P_2 = U_2 - (U_2^T q_1) q_1$$

$$P_2 = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \left(\begin{pmatrix} -3 & 0 & 6 \end{pmatrix} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) q_1$$

$$P_2 = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \left(\sqrt{3} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right)$$

$$P_2 = \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

$$q_2 = \frac{P_2}{|P_2|}$$

$$|P_2| = \sqrt{(-4)^2 + (-1)^2 + (5)^2} = \sqrt{42}$$

$$q_2 = \frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$$

$$Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt{3} & -4/\sqrt{42} \\ 1/\sqrt{3} & -1/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{pmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ q_1 & q_2 . \end{array}$$

Now we need to find R before solving the system,

$$R = \begin{bmatrix} U_1^T q_1 & U_2^T q_2 \\ 0 & U_2^T q_2 \end{bmatrix} \rightarrow$$ This will work if u_1 & u_2 is given.

$$R = \begin{bmatrix} U_1^T q_1 & U_2^T q_1 & U_3^T q_1 \\ 0 & U_2^T q_2 & U_3^T q_2 \\ 0 & 0 & U_3^T q_3 \end{bmatrix}$$ If u_1, U_2 & u_3 is given.

$$U_1^T a_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \sqrt{3}$$

$U_2^T a_1 = \sqrt{3} \rightarrow$ We got this value from Step.

$$U_2^T a_2 = \begin{pmatrix} -3 & 0 & 6 \end{pmatrix} \begin{pmatrix} -4/\sqrt{42} \\ -1/\sqrt{42} \\ 5/\sqrt{42} \end{pmatrix}$$

$$= \sqrt{42}.$$

$$R = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}.$$

$$Rx = Q^T b$$

$$\underset{R}{\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{42} \end{bmatrix}} \underset{x}{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -4/\sqrt{42} & -1/\sqrt{42} & 5/\sqrt{42} \end{bmatrix}} \underset{b}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}.$$

$a_0 = 3/7$

$a_1 = 5/21.$

<u>Example</u> : $f(1) = 2$

$f(2) = 3$

$f(3) = 6$

$f(4) = 4$.

$\hookrightarrow P_2(x) = a_0 + a_1 x + a_2 x^2$.

$P_2(1) = a_0 + a_1 + a_2 = 2$

$P_2(2) = a_0 + 2a_1 + 4a_2 = 3$.

$P_2(3) = a_0 + 3a_1 + 9a_2 = 6$

$P_2(4) = a_0 + 4a_1 + 16a_2 = 4$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}$$

$\hookrightarrow$ (4×3) matrix - overdetermined system.

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \\ \quad U_1 \quad U_2 \quad U_3 \end{array}$$

<u>Step 1</u> : $k = 1$

$$P_1 = U_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$q_1 = \frac{P_1}{|P_1|} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Step 2 : $k = 2$.

$$P_2 = U_2 - (U_2^T q_1) q_1$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \left((1 \quad 2 \quad 3 \quad 4) \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} \right) \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{pmatrix}$$

$$q_2 = \frac{P_2}{|P_2|} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1.5 \\ -0.5 \\ 0.5 \\ 1.5 \end{pmatrix}$$

Step-3 : $k = 3$

$$P_3 = U_k - \sum_{i=1}^{2} (U_k^T q_i) q_i$$

$$P_3 = U_3 - \left[(U_3^T q_1) q_1 + (U_3^T q_2) q_2 \right]$$

$$U_3^T q_1 = (1 \quad 4 \quad 9 \quad 16) \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$U_3^T q_1 = 15.$$

$$U_3^T a_2 = \begin{pmatrix} 1 & 4 & 9 & 16 \end{pmatrix} \begin{pmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{pmatrix}$$

$$= 5\sqrt{5}.$$

$$P_3 = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix} - \left[\cancel{1.5}\, 15 \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} + 5\sqrt{5} \begin{pmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{pmatrix} \right]$$

$$P_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

$$a_3 = \frac{P_3}{\|P_3\|} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$Q = \left\{ \underbrace{\begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}}_{a_1}, \underbrace{\begin{pmatrix} -1.5/\sqrt{5} \\ -0.5/\sqrt{5} \\ 0.5/\sqrt{5} \\ 1.5/\sqrt{5} \end{pmatrix}}_{a_2}, \underbrace{\begin{pmatrix} +0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{pmatrix}}_{a_3} \right\}.$$

Now for R matrix,

$$R = \begin{bmatrix} U_1^T a_1 & U_2^T a_1 & U_3^T a_1 \\ 0 & U_2^T a_2 & U_3^T a_2 \\ 0 & 0 & U_3^T a_3 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}.$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -1.5/\sqrt{5} & 0.5/\sqrt{5} & 0.5/\sqrt{5} & 1.5/\sqrt{5} \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5.5 \\ 27\sqrt{5}/10 \\ -3.5 \end{bmatrix}$$

$$2a_2 = -3.5$$

$$a_2 = -1.75$$

$$\sqrt{5}a_1 + 5\sqrt{5}\,a_2 = \frac{27\sqrt{5}}{10}$$

$$a_1 = \frac{\frac{27\sqrt{5}}{10} - 5\sqrt{5}(-1.75)}{\sqrt{5}}$$

$$a_1 = 11.45.$$

$$2a_0 + 5a_1 + 15a_2 = 5.5$$

$$2a_0 = 5.5 - 5(11.45) - 15(-1.75)$$

$$a_0 = -12.75$$

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