

CSE330

TOPIC NAME: Numerical Methods

DAY:

TIME:

DATE: 8 / 3 / 25

Quiz 10% (1/6)

Assignment 15% (6)

Mid 20%

Final 30%

Attendance 5%

Lab 20%

Chapter 1: Floating point Representation

Fixed point:

$$\left\{ \begin{array}{l} 1. d_1 d_2 d_3 \\ 1. - - - \end{array} \right\} 2^2 = 4$$

$$\left\{ \begin{array}{l} 1. d_1 d_2 d_3 \\ 1. - - - \end{array} \right\} 2^3 = 8$$

1. 00

1. 11

1. 01

1. 10

conv/lecture note form:

$$\pm (0.d_1 d_2 d_3 \dots d_m)_{\beta} \times \beta^e$$

$\beta = 2$

montissa/m
fraction

Base

Note: lecture note form $0.\overline{d_1 d_2 d_3 \dots d_m} = 1 + 2^{-m}$

TOPIC NAME
8

* $m = 3, e = 4$; total ~~কর্তৃত~~ number represent কর্তৃত সংখ্যা

$$(0.d_1d_2d_3)_2 \times 2^e$$

$$(0.1d_2d_3)_2 \times 2^e$$

$$\begin{aligned} \text{Total number} &= 2^3 \times 1 = 4 \times 1^{\substack{\leftarrow \text{total} \\ \text{exponent}}} \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

$$e = 4$$

$$0.100 \times 2^4$$

$$0.101 \times 2^4$$

$$0.110 \times 2^4$$

$$0.111 \times 2^4$$

কর্তৃত exponent

কর্তৃত or total

number $\times 2^e$ multiply.

if $e \leq 4, 5$

$\text{total} = 4 \times 2 = 8$

$= 8 + 8 = 16$

* $m = 3, e = [-2, 3]$

$$(0.d_1d_2d_3)_2 \times 2^e$$

$$(0.1d_2d_3)_2 \times 2^e$$

$$\text{total numbers} = 2^3 \times 6 \quad \text{start from } -2 \text{ to } 3$$

$$= 4 \times 6$$

$$= 24 + 24$$

$$= 48$$

$[-2, 3] \rightarrow$ range of

value

$$[-2, 3]$$

$$= -2, -1, 0, 1, 2, 3$$

$$\text{highest} = (0.111)_2 \times 2^3 \quad (\text{binary highest } 1, e_{\max} = 3)$$

$$\text{lowest} = -(0.111)_2 \times 2^3$$

TOPIC NAME:

DAY:

TIME:

DATE: / /

$$\text{positive lowest / minimum} = (0.1\underset{2}{00})_2 \times 2^{-2}$$

Note: Positive lowest & negative consider করবে না,

$$\text{binary lowest } 0, e_{\min} = -3$$

Convention 2 / Normalised:

$$\pm (1.d_1d_2d_3 \dots d_m)_2 \times 2^e$$

* $m=3, e=[-2, 3]$. calculate total number, the

highest, lowest and positive lowest number.

or suppose Monk softed arrow bellow mentioned

$$\pm (1.d_1d_2d_3)_2 \times 2^e$$

$$\text{total number} = 2^3 \times 6 \times 3(111 \cdot 0) = +36$$

total exponent = 6 × 3 = 18

$$= 48 + 48$$

$$= 96$$

$$\text{Highest number} = (1.\underset{\substack{\leftarrow \text{highest binary}}}{{111}})_2 \times 2^{\underset{\substack{\leftarrow \text{highest exponent}}}{3}}$$

$$\text{lowest number} = (1.\underset{\substack{\leftarrow \text{lowest binary}}}{{111}})_2 \times 2^{\underset{\substack{\leftarrow \text{lowest exponent}}}{3}}$$

$$\text{positive lowest / minimum} = (1.\underset{2}{000})_2 \times 2^{\underset{\substack{\leftarrow \text{lowest exponent}}}{-2}}$$

TOPIC NAME

DAY:

TIME:

DATE:

Convention 3 / Denormalised:

format: $\pm (0.d_1d_2d_3d_4d_5d_6)_{10} \times 2^e$

* m = 3, e = [-2, 3]

$(0.d_1d_2d_3)_{10} \times 2^e$

total number = $2^3 \times 6 = 48$

$$= 48 + 48$$

$$= 96$$

{ Denormalised works better than lecture note
form in terms of total (number) representation.

Highest = $(0.1111)_2 \times 2^3 = 5$ (maximum benefit)

lowest = $-(0.1111)_2 \times 2^3$

non-negative lowest = $(0.81000)_2 \times 2^{-2}$

GOOD LUCK

TOPIC NAME : _____

DAY : _____

TIME : _____

DATE : / /

conv1 : $m = 3, e = \{4, 5, 6\}$, number system we can represent

$$(0.111)_2 \times 2^e$$

L.d. R.

R.d. R.

$$(0.111)_2 \times 2^e$$

$$(0.111)_2 \times 2^4 = 0.5 \times 2^4$$

$$(0.111)_2 \times 2^4 = 0.625 \times 2^4 = 10$$

$$(0.111)_2 \times 2^4 = 0.75 \times 2^4 = 12$$

$$(0.111)_2 \times 2^4 = 14$$

diff results soft & difference soft, same DNT

$$(0.110)_2 \times 2^5 = 16$$

$$(0.101)_2 \times 2^5 = 20$$

$$(0.110)_2 \times 2^5 = 24$$

$$(0.111)_2 \times 2^5 = 28$$

$$(0.100)_2 \times 2^6 = 32$$

$$(0.101)_2 \times 2^6 = 40$$

$$(0.110)_2 \times 2^6 = 48$$

$$(0.111)_2 \times 2^6 = 56$$

[16, 32, 48, 56] \rightarrow not equidistant

not equidistant

e.g. value 1 is not equidistant,

e.g. value more than 1 is not equidistant.

value 1 is not equidistant

IEEE Standard System

64 bit

{ 1 sign bit

{ 11 bits exponent / e

{ 52 bits mantissa / ms × 2^e = (1.d₁d₂d₃...d₅₂)₂ × 2^e

$$2^{11} = 2048$$

$$\rightarrow e = [0, 2047]$$

exponent bias

$$\text{smallest positive} = (1.00\ldots 0)_2 \times 2^0$$

$$= 2^0 \times 2^{-1}/2^{-2}/2^{-3}/2^{-4}$$

$$= 1/0.25/0.125/0.065$$

The smaller the exponent is, the closer the number is to zero (near 0)

$$0.001 = 2^{-1} \times 2^{-2} \times 2^{-3} \times 2^{-4}$$

$$\text{exponent bias} = 2^{11-1} - 1 = 1023$$

- 1023 + 1

$$\text{smallest positive} = (1.00\ldots 0)_2 \times 2^{-1023}$$

$$\text{highest value} = (1.11\ldots 1)_2 \times 2^{1023}$$

Denormalised: $e \in [-1023+1, 1024+1]$

$$= [-1022, 1025]$$

format bias = 1.0 [1023, 1024, 1025]

↓ ↓

$$\text{smallest positive} = (0.100\ldots 0)_2 \times 2^{-1023}$$

$$\text{highest value} = (0.111\ldots 1)_2 \times 2^{1024}$$

TOPIC NAME:

DAY:

TIME:

DATE: / /

- * 32 bit \rightarrow 1 sign, 7-bit exponent, 24-bit mantissa using exponent biasing

$$2^7 = 128$$

$$\text{normalised } e = 2^{n-1} - 1 \\ = 2^6 - 1$$

$$e = [0, 127]$$

$$x = (e)(m) \quad (\text{sign}) \times (\text{mantissa}) = 630$$

$$= [0 - 63, 127 - 63]$$

$$= [-63, 654]$$

Denormalised $e \in [-63+1, 654+1]$

$= [-62, 655]$ (middle)

Rounding

$$\begin{array}{c} 79.6 \\ \hline 79 \quad 80 \end{array}$$

$$\begin{array}{c} 79.56 \\ \hline 79.5 \quad 79.6 \end{array}$$

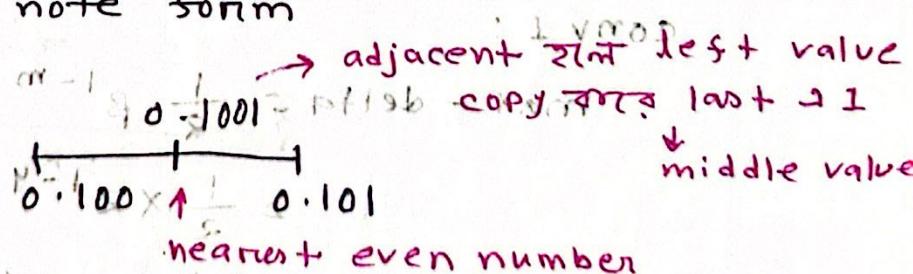
$$\begin{array}{c} 79.567 \\ \hline 79.56 \quad 79.57 \end{array}$$

Actual round off value is $[3, 8] = 3$, $n = m = 2$

* $m = 3$, lecture note form

$$x = 0.1001$$

original value
compare with



round up value = 0.101
 $f(x) = 0.100$

$$y = 0.1001011$$

$$f(y) = 0.101$$

DAY:

TIME:

DATE: 20/2/25

Rounding error = $\frac{|f(x) - x|}{|x|}$ where $x \leftarrow$ add some error

Let $x = 10^{10}$ & $f(x) = 10^9$

$$10^{10} \rightarrow 10^{10.5}$$

Scale invariant rounding error /

rounding error, $\delta(f(\Delta)) = \frac{|f(x) - x|}{|x|}$

$$\text{Machine Epsilon}$$

maximum scale invariant rounding error /

maximum rounding error / maximum delta /

Machine epsilon

$$\downarrow \\ \text{conv1: } \frac{1}{2} \beta^{1-m}$$

$$\text{conv 2} = \frac{1}{2} \beta^{-m}$$

$$\text{conv 3} = \frac{1}{2} \beta^{-m}$$

* $m=4$, $e=[-3, 6]$; calculate max delta for conv1

conv1:

$$\text{max-delta} = \frac{1}{2} \beta^{1-m}$$

$$= \frac{1}{2} \times 2^{1-4}$$

$$= \frac{1}{2} \times 2^{-3}$$

\downarrow
using formula

TOPIC NAME:

DAY: _____

TIME: _____

DATE: / /

$$\text{maximum rounding error}, \delta(\text{delta}) = \frac{|f(x) - x|_{\max}}{|x|_{\min}}$$

Derivation* convention 1

$$m = 3$$

$$\begin{array}{r}
 + \\
 0.100 \\
 + \\
 0.101 \\
 \hline
 \end{array}
 \quad
 \begin{array}{r}
 - \\
 0.100 \\
 - \\
 0.101 \\
 \hline
 0.001
 \end{array}$$

$$\frac{1}{2} \times 0.001 \xrightarrow{\text{decimal}}$$

$$= \frac{1}{2} \times 2^{-3}$$

$$= \frac{1}{2} \times \beta^{-m} x_2^e = |f_1(x) - x|_{\max}$$

$$|x|_{\min} = 0.100 \times 2^e$$

$$\epsilon_{\text{min}} = 2^{-1} \times 2^e$$

$$\Rightarrow \beta^{-1} \times 2^e$$

$$\begin{aligned}
 \text{max delta} &= \frac{\frac{1}{2} \beta^{-m} x_2^e}{\beta^{-1} \times 2^e} \\
 &= \frac{1}{2} \beta^{-m} \beta^{-1} \\
 &= \frac{1}{2} \beta^{1-m}
 \end{aligned}$$

TOPIC NAME:

DAY:

TIME:

DATE:

Convention 2

$$\frac{1}{2} \times 0.001$$

$$= \frac{1}{2} \times 2^{-3}$$

$$= \frac{1}{2} \times \beta^{-m} \times 2^e$$

$$= |f(x) - x|_{\max}$$

$$\begin{array}{r} & \text{min bias} \\ \hline 1.100 & 1.101 \\ 1.0999999999999999 & \end{array}$$

$$\begin{array}{r} 1.101 \\ - 1.100 \\ \hline 0.001 \end{array}$$

$$|x_{\min}| = 1.000 \times 2^e$$

$$= 2^0 \times 2^e$$

$$= |2^e(0.001)| = 2^{e \cdot m} - 2^e \times \frac{1}{2} =$$

$$\text{max delta} = \frac{\frac{1}{2} \beta^{-m} 2^e}{2^e} = \frac{1}{2} \beta^{-m}$$

Convention 3

$$\frac{1}{2} \times 0.0001$$

$$= \frac{1}{2} \times 2^{-4} \times 2^e$$

$$= \frac{1}{2} \times 2^{-3} \times 2^{-1} \times 2^e$$

$$= \frac{1}{2} \beta^{-m} \beta^{-1} 2^e$$

$$= |f(x) - x|_{\max}$$

$$|x_{\min}| = 0.1000 \times 2^e$$

$$= 2^{-1} \times 2^e = \beta^{-1} \times 2^e$$

$$\text{max delta} = \frac{\frac{1}{2} \beta^{-m} \beta^{-1} 2^e}{\beta^{-1} 2^e} = \frac{1}{2} \beta^{-m}$$

$$\begin{array}{r} & m = 3 \\ \hline 0.1000 & 0.1001 \\ 0.09999999999999999 & \end{array}$$

$$\begin{array}{r} 0.1001 \\ - 0.1000 \\ \hline 0.0001 \end{array}$$

TOPIC NAME :

DAY:

TIME:

DATE: / /

$$* x = \frac{5}{8}, y = \frac{7}{8}$$

$f_1(x), f_1(y), f_1(x+y), f_1(f_1(x) + f_1(y))$

$$\text{LHS} x, f_1\left(\frac{5}{8}\right) \stackrel{\text{def}}{=} \frac{1}{2^3} + \frac{4}{2^4} \quad \left\{ \begin{array}{l} \xrightarrow{\text{3rd position 1}} \\ \xrightarrow{\text{1st pos 1}} \end{array} \right.$$

$$\frac{1}{2^3} + \frac{1}{2^4} \quad \xrightarrow{\text{2 एवं power}} \quad \text{भाग्यल एवं नियम उपयोग से} \quad = 0.101$$

$$f_1(x) = 0.101$$

$$\frac{1}{2^3} + \left(\frac{3}{8} \text{ ele } [3, 6] \text{, conv1} \right)$$

$$= 0.101$$

$$P = S + E = 0.101$$

$$1101.0 =$$

$$y = \frac{7}{8} = \frac{1}{2^3} + \frac{3}{2^4} + \frac{1}{2^5} = 2^{-3} + 2^{-2} + 2^{-1}$$

$$= 0.111 \quad \frac{1101.0}{101.0}$$

$$f_1(y) = 0.111 = \frac{7}{8}$$

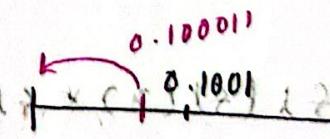
$$x * y = \frac{5}{8} * \frac{7}{8} \stackrel{\text{def}}{=} \frac{35}{64} = \frac{32}{64} + \frac{3}{64} + \frac{1}{64}$$

$$= 2^{-1} + 2^{-5} + 2^{-6}$$

$$= 0.100011$$

$$f_1(x * y) \stackrel{\text{def}}{=} 0.100011$$

$$= 2^{-1}$$



$$0.100 \quad 0.100011 \quad 0.101$$

$$= 0.5$$

$$\text{rounding error} = \frac{|0.5 - \frac{35}{64}|}{|\frac{35}{64}|}$$

TOPIC NAME

PAGE

TIME

DATE

$$\text{same} \quad \text{same}$$

$$f_1(f_1(x) + f_1(y)) = 0.5 \quad \frac{2}{8} = \frac{1}{4}$$

$$(x+y+xy)x^2 + (y+x)xy - xy^2 = 0.101 = 0.101$$

~~101010~~

$$x = \frac{5}{8}, y = \frac{11}{16}, m = 3, e \in E-3, \{3\}, \text{conv 1}$$

$$y = \frac{11}{16}$$

$$= \frac{8}{16} + \frac{2}{16} + \frac{1}{16}$$

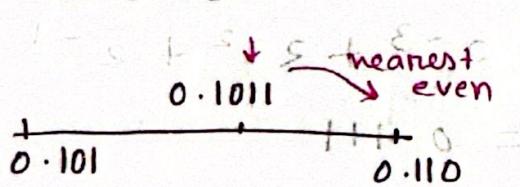
$$= 2^{-1} + 2^{-3} + 2^{-4}$$

$$= 0.1011$$

$$x = \frac{5}{8} = \frac{1}{8} + \frac{4}{8}$$

$$= 2^{-3} + 2^{-1}$$

$$f_1(x) = 0.101 = \frac{5}{8}$$



$$f_1(y) = 0.110 = 2^{-1} + 2^{-2} = \frac{3}{4}$$

$$f_1(x) * f_1(y) = \frac{5}{8} \times \frac{3}{4} = \frac{15}{32}$$

$$= \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32}$$

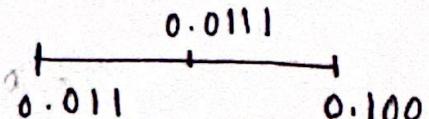
$$= 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5}$$

$$= 0.0111$$

$$f_1(f_1(x) * f_1(y)) = 0.100$$

$$= 2^{-1}$$

$$= 2$$



4 decimal $\rightarrow 0.0001 \rightarrow 1$ sig fig

5 decimal $\rightarrow 0.01056 \rightarrow 4$ sig fig

5 decimal $\rightarrow 0.10000 \rightarrow 5$ sig fig

Loss of significance: When we subtract two close values the scale invariant rounding error get very high, we call this loss of significance.

* Quadratic formula for solving $x^2 - 56x + 1 = 0$.

To 4 s.f. the roots are -

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{56 + \sqrt{(-56)^2 - 4 \cdot 1 \cdot 1}}{2(1)} = 28 + 3\sqrt{783}$$

$$= 28 + \sqrt{783}$$

$$= 55.98213716$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{56 - \sqrt{(-56)^2 - 4 \cdot 1 \cdot 1}}{2(1)} = 28 - \sqrt{783}$$

$$= 0.01786284073$$

$$\sqrt{783} = 27.98213716$$

$$x_1 = 28.00 + 27.98 = 55.98$$

$$x_2 = 28.00 - 27.98 = 0.02000$$

$$\delta_1 = \frac{|f_1(x_1) - x_1|}{|x_1|} = \frac{|55.98 - 55.98213716|}{|55.98213716|}$$

$$= 0.00003817 \times 100$$

$$= 0.003817\%$$

TOPIC NAME

DAY

TIME:

DATE:

$$\delta_2 = \frac{|51|x_2 - x_2|}{|x_2|}$$

$$= \frac{|0.02 - 0.01786284073|}{|0.01786284073|}$$

$$= 0.11964 \times 10^0$$

$$= 11.964\%$$

$$\alpha = x_1 = 55.98$$

$$\beta = x_2$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \beta = \frac{c - \alpha x}{a}$$

$$= \frac{1}{1 \times 55.98}$$

$$= 0.01786352269$$

$$\approx 0.01786$$

Chapter 2: Polynomial Interpolation

$5x^2$ → power/exponent
 → term
 variable
 coefficient

Polynomial: collection of terms

$$P_2(x) = 3 + 4x + 5x^2; \text{ coefficient} = 3, 4, 5$$

$$P_7(x) = 3x + 4x^3 + 5x^7; \text{ coeff} = 0, 3, 0, 4, 0, 0, 0, 5$$

$P_n(x)$
 → degree

coefficient = $n+1$

term = $n+1$

$$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

$$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$= a_0 + a_1 x^1 + a_2 x^2$$

Taylor

$$f(x) = \sin(x)$$

$$\text{error} = |f(x) - P_n(x)|$$

$$* x = 5$$

$$|f(5) - P_4(5)| = \text{error}$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$P_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \quad \text{error}$$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

degree ↑ error ↓
 ↳ Weierstrass theorem

TOPIC NAME : _____

DAY: _____

TIME: _____

DATE: / /

Maximum error bound / error bound:

$$= \frac{f^{n+1}(\bar{x})}{(n+1)!} (x - x_0)^{n+1}$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\begin{aligned} \text{error max bound} &= \frac{|f^6(\bar{x})|}{6!} |(x - x_0)^6| \\ &= \frac{|f^6(\bar{x})|_{\max}}{6!} |x^6|_{\max} \xrightarrow{\text{as shown}} 2^6 = \frac{\sin(1.57)}{6!} x^6 \end{aligned}$$

$$\bar{x} [0, 2]$$

$$f(x) = \sin(x)$$

$$x [0, 2]$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f''''(x) = \sin(x)$$

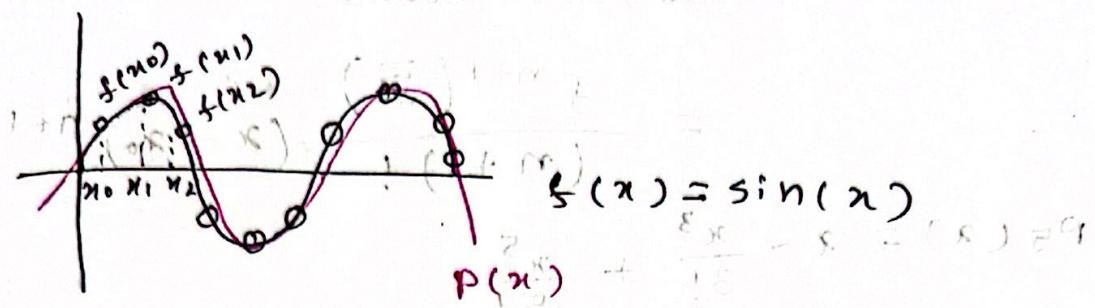
$$f''''(x) = \cos(x)$$

$$f''''(x) = -\sin(x)$$

$$f^6(x) = -\sin(x)$$

$$|f^6(\bar{x})| = \sin(\bar{x})$$

$$\hookrightarrow 1.57 \left(\frac{\pi}{2}\right) \rightarrow \sin \frac{\pi}{2} \text{ highest possible value}$$



$n = \text{number of modes} - 1$ bond off nodes

node 5
 $\rightarrow P_5(x) =$

* $(x)_m = (x) +$

$$\frac{x}{x_0 - 1} \cdot \frac{f(x)}{(x)_2} = (x)_2 - (x)_1$$

$$x_1 \quad 3(x)_2 - 8 f(x_1) + 8$$

$$x_2 \quad 5(x)_2 - 10 f(x_2) + 10$$

$$n = 3 - 1 = 2$$

$$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$(x)_0 = 1 \quad (x)_1 = 2$$

$$1^{\circ} a_0 + a_1 1^1 + a_2 1^2 = 3 \quad \dots (i)$$

$$3^{\circ} a_0 + a_1 3^1 + a_2 3^2 = 8 \quad \dots (ii)$$

$$5^{\circ} a_0 + a_1 5^1 + a_2 5^2 = 10 \quad \dots (iii)$$

TOPIC NAME : _____

DAY : _____

TIME : _____

DATE : / /

Vandermonde

SOLUTION

$$\begin{array}{|ccc|c|} \hline & 1 & 1 & 1 & a_0 \\ & 1 & 3 & 9 & a_1 \\ & 1 & 5 & 25 & a_2 \\ \hline & v & & & \end{array} \quad \begin{array}{|c|c|} \hline b/x & 3 \\ \hline b/y & 8 \\ \hline b/z & 10 \\ \hline \end{array}$$

$\therefore x = 1 - 2 = 0$

$$\begin{matrix} Va = b \\ \Rightarrow a = V^{-1} b \end{matrix} \quad \begin{array}{|c|c|} \hline 2 & \\ \hline 9 & \\ \hline 16 & \\ \hline \end{array}$$

round off

$$P_2(x) = 2 + 9x + 16x^2$$

$$(ii) x = \frac{7(\varepsilon-1)(\varepsilon-\kappa)}{(\varepsilon-1)(\varepsilon-\kappa)}$$

$$\text{error bound} = |f(7) - P_2(7)|$$

$$= \frac{(7-\varepsilon)(7-\kappa)}{(7-\varepsilon)(7-\kappa)} \cdot \frac{(\varepsilon-\kappa)(\varepsilon-\kappa)}{(\varepsilon-\kappa)(\varepsilon-\kappa)} = (\varepsilon-\kappa)$$

$$= \frac{(\varepsilon-\varepsilon)(7-\kappa)}{\varepsilon} \cdot \frac{(\varepsilon-\kappa)(7-\kappa)}{(\varepsilon-\kappa)(7-\kappa)} = (7-\kappa)$$

$$= \frac{(\varepsilon-\kappa)(7-\kappa)}{8} + \varepsilon \cdot \frac{(\varepsilon-\kappa)(7-\kappa)}{1} = 8 \cdot \frac{(\varepsilon-\kappa)(7-\kappa)}{8} = (\varepsilon-\kappa)$$

Lagrange

x	$f(x) / y$				
$x_0 = 1$	3 = $f(x_0)$	10	1	1	1
$x_1 = 3$	5 = $f(x_1)$	10	8	8	1
$x_2 = 5$	9 = $f(x_2)$	10	6.5	6.5	1
$n = 3 - 1 = 2$					

$$P_n(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) \\ + \dots + l_n(x)f(x_n)$$

↓
Lagrange basis

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-3)(x-5)}{(1-3)(1-5)} = \frac{(x-3)(x-5)}{8}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-5)}{(3-1)(3-5)} = \frac{(x-1)(x-5)}{-4}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \frac{(x-1)(x-3)}{8}$$

$$P_2(x) = \frac{(x-3)(x-5)}{8} x_3 - \frac{(x-1)(x-5)}{4} x_5 + \frac{(x-1)(x-3)}{8} x_9$$

TOPIC NAME : _____

derivation
from book

DAY: _____

TIME: _____

DATE: 01/03/2025

Newton's Divided Difference

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2]$$

$$(x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)$$

$$(x - x_2) + \dots \dots$$

$$\begin{array}{c|c} x & f(x) / y \\ \hline x_0 = 1 & 3 = f(x_0) \end{array}$$

$$x_1 = 3 \quad 5 = f(x_1)$$

$$x_2 = 5 \quad 9 = f(x_2)$$

$$x_3 = 9 \quad 15 = f(x_3) \quad n = 4 - 1 = 3$$

$$n = 3 - 1 = 2$$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 3 + 1(x - 1) + \frac{1}{4}(x - 1)(x - 3)$$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= 3 + 1(x - 1) + \frac{1}{4}(x - 1)(x - 3) - \frac{1}{24}(x - 1)(x - 3)(x - 5)$$

GOODLUCK

$$x_0 = 1$$

$$x_1 = 3$$

$$x_2 = 5$$

$$x_3 = 9$$

$$f[x_0] = 3$$

$$f[x_1] = 5$$

$$f[x_2] = 9$$

$$f[x_3] = 15$$

$$f[x_4] = 21$$

$$f[x_5] = 28$$

$$f[x_6] = 36$$

$$f[x_7] = 45$$

$$f[x_8] = 55$$

$$f[x_9] = 66$$

$$f[x_{10}] = 78$$

$$f[x_{11}] = 91$$

$$f[x_{12}] = 105$$

$$f[x_{13}] = 120$$

$$f[x_{14}] = 136$$

$$f[x_{15}] = 153$$

$$f[x_{16}] = 171$$

$$f[x_{17}] = 190$$

$$f[x_{18}] = 210$$

$$f[x_{19}] = 231$$

$$f[x_{20}] = 253$$

$$f[x_{21}] = 276$$

$$f[x_{22}] = 300$$

$$f[x_{23}] = 325$$

$$f[x_{24}] = 351$$

$$f[x_{25}] = 378$$

$$f[x_{26}] = 406$$

$$f[x_{27}] = 435$$

$$f[x_{28}] = 465$$

$$f[x_{29}] = 500$$

$$f[x_{30}] = 536$$

$$f[x_{31}] = 573$$

$$f[x_{32}] = 611$$

$$f[x_{33}] = 650$$

$$f[x_{34}] = 689$$

$$f[x_{35}] = 729$$

$$f[x_{36}] = 769$$

$$f[x_{37}] = 810$$

$$f[x_{38}] = 852$$

$$f[x_{39}] = 895$$

$$f[x_{40}] = 940$$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{5 - 3}{3 - 1} = 1$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{9 - 5}{5 - 3} = 2$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{15 - 9}{9 - 5} = \frac{3}{2}$$

$$f[x_3, x_4] = \frac{f[x_4] - f[x_3]}{x_4 - x_3} = \frac{21 - 15}{15 - 9} = \frac{6}{6} = 1$$

$$f[x_4, x_5] = \frac{f[x_5] - f[x_4]}{x_5 - x_4} = \frac{28 - 21}{21 - 15} = \frac{7}{6}$$

$$f[x_5, x_6] = \frac{f[x_6] - f[x_5]}{x_6 - x_5} = \frac{36 - 28}{28 - 21} = \frac{8}{7}$$

$$f[x_6, x_7] = \frac{f[x_7] - f[x_6]}{x_7 - x_6} = \frac{45 - 36}{36 - 28} = \frac{9}{8}$$

$$f[x_7, x_8] = \frac{f[x_8] - f[x_7]}{x_8 - x_7} = \frac{55 - 45}{45 - 36} = \frac{10}{9}$$

$$f[x_8, x_9] = \frac{f[x_9] - f[x_8]}{x_9 - x_8} = \frac{66 - 55}{55 - 45} = \frac{11}{10}$$

$$f[x_9, x_{10}] = \frac{f[x_{10}] - f[x_9]}{x_{10} - x_9} = \frac{78 - 66}{66 - 55} = \frac{12}{11}$$

$$f[x_{10}, x_{11}] = \frac{f[x_{11}] - f[x_{10}]}{x_{11} - x_{10}} = \frac{91 - 78}{78 - 66} = \frac{13}{12}$$

$$f[x_{11}, x_{12}] = \frac{f[x_{12}] - f[x_{11}]}{x_{12} - x_{11}} = \frac{105 - 91}{91 - 78} = \frac{14}{13}$$

$$f[x_{12}, x_{13}] = \frac{f[x_{13}] - f[x_{12}]}{x_{13} - x_{12}} = \frac{120 - 105}{105 - 91} = \frac{15}{14}$$

$$f[x_{13}, x_{14}] = \frac{f[x_{14}] - f[x_{13}]}{x_{14} - x_{13}} = \frac{136 - 120}{120 - 105} = \frac{16}{15}$$

$$f[x_{14}, x_{15}] = \frac{f[x_{15}] - f[x_{14}]}{x_{15} - x_{14}} = \frac{153 - 136}{136 - 120} = \frac{17}{16}$$

$$f[x_{15}, x_{16}] = \frac{f[x_{16}] - f[x_{15}]}{x_{16} - x_{15}} = \frac{171 - 153}{153 - 136} = \frac{18}{17}$$

$$f[x_{16}, x_{17}] = \frac{f[x_{17}] - f[x_{16}]}{x_{17} - x_{16}} = \frac{190 - 171}{171 - 153} = \frac{19}{18}$$

(ii) ~~for merge operation~~
for merge operation

$$f[x_0, x_1, x_2] = \frac{f[x_1] - f[x_0]}{x_2 - x_0} = \frac{5 - 3}{9 - 1} = \frac{2}{8} = \frac{1}{4}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2] - f[x_1]}{x_3 - x_1} = \frac{9 - 5}{15 - 3} = \frac{4}{12} = \frac{1}{3}$$

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_3] - f[x_0]}{x_3 - x_0} = \frac{15 - 3}{21 - 1} = \frac{12}{20} = \frac{3}{5}$$

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_4] - f[x_0]}{x_4 - x_0} = \frac{21 - 3}{28 - 1} = \frac{18}{27} = \frac{2}{3}$$

$$f[x_0, x_1, x_2, x_3, x_4, x_5] = \frac{f[x_5] - f[x_0]}{x_5 - x_0} = \frac{28 - 3}{36 - 1} = \frac{25}{35} = \frac{5}{7}$$

$$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6] = \frac{f[x_6] - f[x_0]}{x_6 - x_0} = \frac{36 - 3}{45 - 1} = \frac{33}{44} = \frac{3}{4}$$

$$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] = \frac{f[x_7] - f[x_0]}{x_7 - x_0} = \frac{45 - 3}{55 - 1} = \frac{42}{54} = \frac{7}{9}$$

$$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] = \frac{f[x_8] - f[x_0]}{x_8 - x_0} = \frac{55 - 3}{66 - 1} = \frac{52}{65} = \frac{8}{13}$$

DATE :
TIME :

TOP NAME

DAY

DATE :

TIME :

Newton's Divided Difference Derivation

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ \dots + \dots + a_n(x - x_0)\dots(x - x_{n-1})$$

Here, $a_0 = P_n(x_0) = f(x_0)$

$$f(x_0) + a_1(x - x_0) = P_n(x_1) = f(x_1)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_i] = f(x_i)$$

The first divided difference

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The second divided difference

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Similarly kth divided difference:

$$f[x_i, x_{i+1}, \dots, x_{i+k-1}, x_{i+k}]$$

$$= \frac{f[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - f[x_i, x_{i+1}, \dots, x_{i+k-1}]}{x_{i+k} - x_i}$$

TOPIC NAME:

DAY: 1

TIME:

DATE: 1/1/2024

The process ends with the single nth divided difference,

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

$$a_k = f[x_0, x_1, x_2, \dots, x_k]$$

$$P_n(x) = f[x_0] + \sum_{k=1}^n f[x_0, x_1, \dots, x_k] \frac{(x-x_0) \dots (x-x_{k-1})}{(x-x_k)}$$

These divided differences are called divided differences of f.

Example:

Find the divided differences of $f(x) = x^2$ at $x_0 = 1, x_1 = 2, x_2 = 3$.

Solution: We have $f(x) = x^2$, $x_0 = 1, x_1 = 2, x_2 = 3$.

Divided differences of $f(x) = x^2$ at $x_0 = 1, x_1 = 2, x_2 = 3$ are given by

$f[x_0] = f[1] = 1^2 = 1$

$f[x_0, x_1] = f[1, 2] = \frac{f[2] - f[1]}{2 - 1} = \frac{4 - 1}{2 - 1} = 3$

$f[x_0, x_1, x_2] = f[1, 2, 3] = \frac{f[3] - f[2]}{3 - 2} = \frac{9 - 4}{3 - 2} = 5$

$f[x_0, x_1, x_2, x_3] = f[1, 2, 3, 4] = \frac{f[4] - f[3]}{4 - 3} = \frac{16 - 9}{4 - 3} = 7$

$f[x_0, x_1, x_2, x_3, x_4] = f[1, 2, 3, 4, 5] = \frac{f[5] - f[4]}{5 - 4} = \frac{25 - 16}{5 - 4} = 9$

$f[x_0, x_1, x_2, x_3, x_4, x_5] = f[1, 2, 3, 4, 5, 6] = \frac{f[6] - f[5]}{6 - 5} = \frac{36 - 25}{6 - 5} = 11$

$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6] = f[1, 2, 3, 4, 5, 6, 7] = \frac{f[7] - f[6]}{7 - 6} = \frac{49 - 36}{7 - 6} = 13$

$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7] = f[1, 2, 3, 4, 5, 6, 7, 8] = \frac{f[8] - f[7]}{8 - 7} = \frac{64 - 49}{8 - 7} = 15$

$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] = f[1, 2, 3, 4, 5, 6, 7, 8, 9] = \frac{f[9] - f[8]}{9 - 8} = \frac{81 - 64}{9 - 8} = 17$

$f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] = f[1, 2, 3, 4, 5, 6, 7, 8, 9, 10] = \frac{f[10] - f[9]}{10 - 9} = \frac{100 - 81}{10 - 9} = 19$

*Imp for
TOPIC Mid*

DAY: _____

TIME: _____

DATE: / /

Cauchy's theorem

$$\text{error bound / maximum error bound} = \frac{|f^{n+1}(\xi)|}{(n+1)!} \left| \frac{(x-x_0)(x-x_1) \dots}{(x-x_n)} \right|_{\max}$$

* $f(x) = e^x \sin(x)$, given nodes x_0, x_1, x_2 .

range $\xi \in [-1, 3]$, calculate error using Cauchy's theorem.

$$\text{error bound} = \frac{|f'''(\xi)|}{3!} |(x-x_0)(x-x_1)(x-x_2)|$$

यदि ये दो नोड्स का बीच में एक तीसरा नोड्स जोड़ा जाए तो इसका विवरण यह होगा कि यह दो नोड्स के बीच में बहुप्रतीक्षित है।

$$\begin{aligned} \xi &\in [\text{lowest value of node}, \text{highest value of node}] \\ &= [0, 2] \end{aligned}$$

$$f(x) = e^x \sin(x)$$

$$f'(x) = e^x \cos(x) + \sin(x) e^x$$

$$\begin{aligned} f''(x) &= -e^x \sin(x) + e^x \cos(x) + e^x \cos(x) + \sin(x) e^x \\ &= 2e^x \cos(x) \end{aligned}$$

$$f'''(x) = 2(e^x \cos x - e^x \sin x)$$

$$= 2e^x \cos x - 2e^x \sin x$$

TOPIC NAME

DAY

TIME

DATE

$$|f'''(x)| = |2e^x \cos x| + |-2e^x \sin x|$$

$$= 2e^3 \cos(0) + 2e^3 \sin\left(\frac{\pi}{2}\right)$$

\rightarrow
maximize

$$= 80.34$$

$$w(x) = x(x-1)(x-2)$$

$$= (x^2 - x)(x - 2)$$

$$= x^3 - 3x^2 + 2x$$

$$w'(x) = 3x^2 - 6x + 2$$

$$w'(x) = 0$$

$$\Rightarrow 3x^2 - 6x + 2 = 0$$

$$\therefore x_1 = 1 + \frac{1}{\sqrt{3}},$$

$$x_2 = 1 - \frac{1}{\sqrt{3}}$$

error bound = $\frac{|f''(3)|}{3!}$

$$|(x-x_0)(x-x_1)(x-x_2)|$$

$$= \frac{80.34}{3!} \times 6$$

x	$ w(x) $
$1 + \frac{1}{\sqrt{3}}$	0.384
$1 - \frac{1}{\sqrt{3}}$	0.384
-1	6
3	6

TOPIC NAME :

DAY :

TIME :

DATE : 06/3/25

Hermite Interpolation(x₀, f(x₀), f'(x₀))

<u>x</u>	<u>f(x)</u>	<u>f'(x)</u>
x ₀ = -1	1	2
x ₁ = 0	0	0
x ₂ = 1	1	0

$$n = 3 - 1 = 2$$

$$P_{2n+1}(x) = h_0(x)f(x_0) + h_1(x)f(x_1) + \dots + h_n(x)f(x_n) + h_0(x)f'(x_0) + h_1(x)f'(x_1) + \dots + h_n(x)f'(x_n)$$

$$P_5(x) = h_0(x)f(x_0) + h_1(x)f(x_1) + h_2(x)f(x_2) + h_0(x)f'(x_0) + h_1(x)f'(x_1) + h_2(x)f'(x_2)$$

$$= h_0(x)f(x_0) + h_2(x)f(x_2) + \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1)$$

$$h_K(x) = \left\{ 1 - 2(x - x_K) \right\} J_K^2(x_K) (J_K(x))^2$$

$$\hat{h}_K(x) = ((x - x_K)(J_K(x)))^{2/(k+1)}$$

$$h_0(x) = \left\{ 1 - 2(x - x_0) \right\} J_0^2(x_0) (J_0(x))^2$$

$$= \left\{ 1 - 2(x - x_0) \right\} \left(\frac{x(x-1)}{2} \right)^2$$

$$\hat{h}_0(x) = (x - x_0) (J_0(x))^2$$

$$= (x+1) \left(\frac{x(x-1)}{2} \right)^2$$

$$J_0(x) = \frac{(x-x_0)(x-x_1)}{(x_0-x_1)(x_1-x_0)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2}$$

$$= \frac{x^2}{2} - \frac{x}{2}$$

$$J_0'(x) = x - \frac{1}{2}$$

$$= -\frac{3}{2}$$

$$h_2(x) = \{1 - 2(x - x_2)x_2' (x_2)\} (l_{20}(x))^2$$

$$= \{1 - 2(x - 1)\frac{3}{2}\} \left(\frac{x(x+1)}{2}\right)^2$$

$$l_{20}(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)}$$

$$= \frac{x(x+1)}{2}$$

$$= \frac{x^2}{2} + \frac{x}{2}$$

$$x_2'(x) = x + \frac{1}{2}$$

$$l_{20}'(x_2) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$h_1(x) = (x - x_1) (l_1(x))^2$$

$$= (x - 0) (-x^2 + 1)^2$$

$$l_1(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)}$$

$$= \frac{x^2 - 1}{-1} = x^2 - 1$$

$$P_5(x) = h_0(x) f(x_0) + h_2(x) f(x_2) + h_0(x) f'(x_0)$$

$$+ (h_1(x) f'(x_1)) =$$

$$= \{1 - 2(x+1)(-3/2)\} \left(\frac{x(x+1)}{2}\right)^2 (1) =$$

$$+ \{1 - 2(x-1)(3/2)\} \left(\frac{x(x+1)}{2}\right)^2 (1)$$

$$+ 2(x+1) \left(\frac{x(x-1)}{2}\right)^2 + 2x(-1+x^2)^2$$

- why hermite better?

In hermite, using less amount of information, we can find less error function.

drawback: you need to calculate everything, if you add or drop any data.

TOPIC NAME : _____

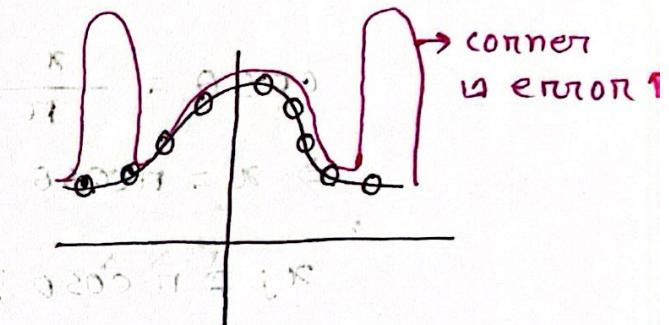
DAY : _____
TIME : _____ DATE : 7/3/25

Runge Phenomenon

1. The function we try to replicate need to be runge function.
2. The function has to be symmetric.
3. equally spaced nodes

এই font rule maintain
করে function create করল
corner in error তৈরি
 \rightarrow runge phenomenon

$$\frac{R(1-x)}{(1+x)^2}$$

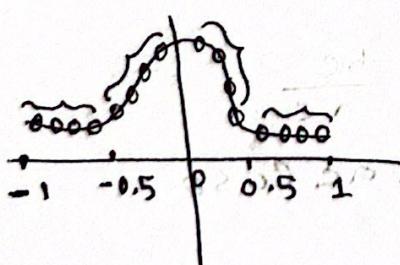


how to tackle?

instead of ~~equally spaced nodes~~
 \rightarrow ~~equally spaced nodes~~, we use ~~equally~~ angle nodes.

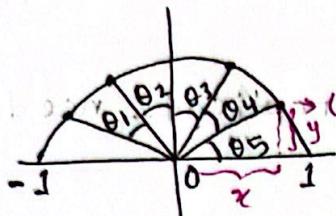
\hookrightarrow Chebyshev's nodes

piecewise solution :



one way of solving
runge phenomenon.

Chebyshev's Nodes



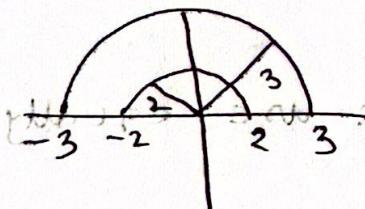
$$\theta_1 = \theta_2 = \theta_3 = \theta_4$$

$$\theta_j = \frac{(2j+1)\pi}{2(n+1)} ; j \in \{0, 1, 2, \dots, n\}$$

$$\cos \theta = \frac{x}{n}$$

$$\Rightarrow x = n \cos \theta$$

$$x_j = n \cos \theta_j + \text{centre}$$

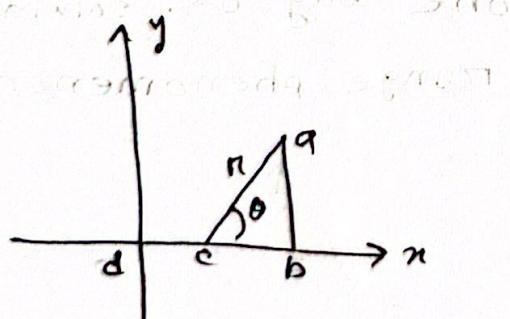


[−a, b]

centre = 0

case 1: [−π, π], centre = 0, n = π

$$bd = bc + dc$$



$$\cos \theta = \frac{bc}{ae}$$

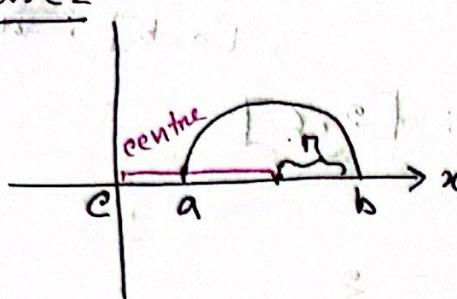
$$\Rightarrow bc = ac \cos \theta$$

TOPIC NAME : _____

DAY : _____

TIME : / /

DATE : / /

case 2

$$r = \frac{b-a}{2} \quad (d \text{ from } 2 \rightarrow [a, b])$$

$a \neq b$

$$\begin{aligned} \text{centre} &= b - r \\ &= b - \frac{b-a}{2} = \frac{2b-b+a}{2} \\ &= \frac{a+b}{2} \end{aligned}$$

$$x_j = r \cos \theta_j + \text{centre}$$

$$= \frac{b-a}{2} \cos \theta_j + \frac{a+b}{2}$$

case 1 (math)

$$f(x) = \frac{1}{1+25x^2} \quad ; n = 3, [-4, 4] \rightarrow [-n, n]$$

$$r = 4$$

$$\text{centre} = 0$$

$$x_0 = r \cos \theta_j + \text{centre}$$

$$= 4 \cos \frac{(2 \times 0 + 1)\pi}{2(3+1)} + 0 = 4 \cos \frac{\pi}{8}$$

$$x_1 = 4 \cos \frac{(2 \times 1 + 1)\pi}{2(3+1)} + 0 = 4 \cos \frac{3\pi}{8}$$

$$x_2 = 4 \cos \frac{(2 \times 2 + 1)\pi}{2(3+1)} + 0 = 4 \cos \frac{5\pi}{8}$$

$$x_3 = 4 \cos \frac{(2 \times 3 + 1)\pi}{2(3+1)} + 0 = 4 \cos \frac{7\pi}{8}$$

TOPIC NAME

DAY:

TIME

DATE:

case 2 (math)

$\rightarrow [a, b] ; a \neq b$

$$f(x) = \frac{1}{1+25x^2} ; n=3 ; [2, 6]$$

$$r = \frac{b-a}{2} = \frac{6-2}{2} = 2$$

$$\text{centre} = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

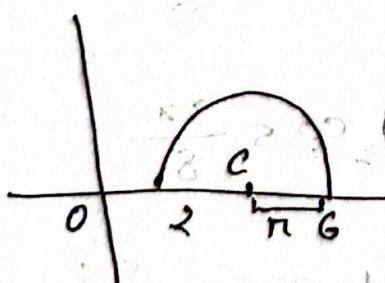
$$x_0 = \pi \cos \theta_0 + \text{centre} = 2 \cos \frac{(2 \times 0 + 1)\pi}{2(3+1)} + 4$$

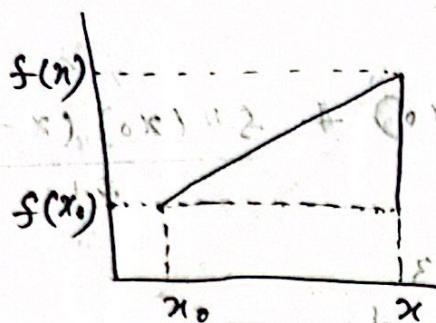
$$= 2 \cos \frac{\pi}{8} + 4$$

$$x_1 = 2 \cos \frac{(2 \times 1 + 1)\pi}{2(3+1)} + 4 = 2 \cos \frac{3\pi}{8} + 4$$

$$x_2 = 2 \cos \frac{5\pi}{8} + 4$$

$$x_3 = 2 \cos \frac{7\pi}{8} + 4$$





$$f'(x_0) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f(x) = f'(x_0)(x - x_0) + f(x_0)$$

Taylor Series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \dots$$

Proof of Taylor Series:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

$$f''(x) = 2a_2 + 3 \times 2a_3(x - x_0)$$

$$f'''(x) = 3 \cdot 2a_3$$

$$x = x_0$$

$$a_2 = \frac{f''(x_0)}{2!}$$

$$f(x_0) = a_0$$

$$a_3 = \frac{f'''(x_0)}{3!}$$

$$f'(x_0) = a_1$$

$$f''(x_0) = 2a_2$$

$$f'''(x_0) = 3 \cdot 2a_3$$

TOPIC NAME

NAME _____

TIME :

DATE :

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!}$$

$$+ \frac{f'''(x_0)(x - x_0)^3}{3!} + \dots$$

$$x_0 = 0 \rightarrow \sin(x) = \sin(0) + \cos(0)x - \frac{\sin(0)}{3!} + \frac{(-\cos(0))x^3}{3!}$$

$$+ (\sin(0) + \frac{\cos(0)x^2}{2!}) + \frac{(-\sin(0))x^5}{5!} + \dots$$

comes out to be

$$+ (-\sin(0))x + (\cos(0)x) - \sin(0) + \cos(0)x^3$$

$$+ (\sin(0)x^2) + (\cos(0)x^4) + \dots$$

(cos x) cos x sin x + (sin x) cos x + ... = cos x

$$= \cos x + \cos x \sin^2 x$$

$$(\cos x)^2 = \cos^2 x$$

$$(\cos x)^3 = \cos^3 x$$

$$\sin x = \cos x$$

$$\sin x = \cos x \cdot x$$

$$\sin x = \cos x \cdot x$$

$$\sin x = \cos x \cdot x$$

GOOD LUCK

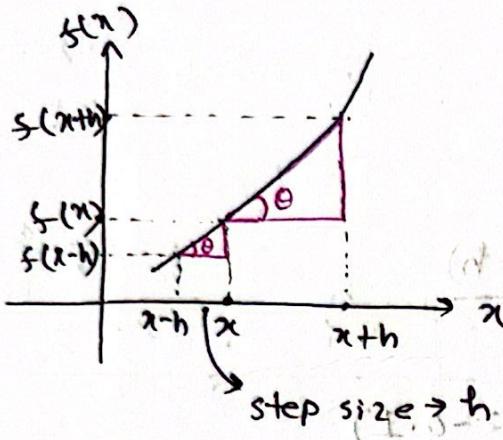
TOPIC NAME :

DAY:

TIME:

DATE: 8 / 3 / 25

Chapter 3 - Differentiation



$$m / \tan \theta = \frac{f(x+h) - f(x)}{x+h - x}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

↳ forward differentiation
↳ n = 2 - 1 = 1

$$\text{Backward differentiation} = \frac{f(x) - f(x-h)}{x - x+h} = \frac{f(x) - f(x-h)}{h}$$

↳ n = 1

$$\text{Central differentiation} = \frac{f(x+h) - f(x-h)}{x+h - x+h} = \frac{f(x+h) - f(x-h)}{2h}$$

Backward and forward D. এর তারিখ T.E $\propto h$

Central D. এর তারিখ T.E $\propto h^2$

Central D. এর তারিখ T.E কর্মা ঘনক

For Backward and Forward D. \Rightarrow

Upper bound of error / error bound = $\frac{f''(3)}{2!} \times h$

For Central D. \Rightarrow

error bound = $\frac{f'''(3)}{3!} \times h^2$

$f'''(3) = 1.812$

$(1.812)(2)^2 = 7.248$ ≈ 7.25 (approx)

1.812 ≈ 1.816 (approx)

1.816 ≈ 1.816 (approx)

TOPIC NAME:

DATE:

TIME:

DATE:

$$* f(x) = \ln(x), x = 2 \text{ with } h = 0.1 \rightarrow \text{mid-point}$$

$$\text{Forward D.} \quad f'(x) = \frac{1}{x}$$

$$f'(2) = \frac{1}{2} \approx 0.5$$

$$\text{Backward D.} = \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(2) - f(2-0.1)}{0.1}$$

$$= \frac{\ln(2) - \ln(1.9)}{0.1}$$

$$= 0.51293$$

$$T.E = |0.5 - 0.51293| = 0.012$$

$$\text{Forward D.} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{f(2+0.1) - f(2)}{0.1}$$

$$= \frac{f(2.1) - f(2)}{0.1} = \frac{\ln(2.1) - \ln(2)}{0.1}$$

$$= 0.4879$$

$$T.E = |0.5 - 0.4879| = 0.012$$

$$\text{Central D.} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2+0.1) - f(2-0.1)}{2 \times 0.1}$$

$$= \frac{\ln(2.1) - \ln(1.9)}{2 \times 0.1} = 0.500417$$

GOOD LUCK!!

TOPIC NAME : _____

DAY : _____

TIME : _____

DATE : / /

$$T.E = |0.5 - 0.500417| \text{ (from the above table)}$$

$$e_{\bar{x}} = 0.000417 \text{ (from the above table)}$$

$$(x+0.1) \approx x + 0.1$$

$$* h = 0.1$$

x	1.1	1.2	1.3
$f(x)$	2.63	2.739	2.863

Using central D. calculate the value of $f'(1.2)$

$$\text{Central D.} = \frac{f(x+h) - f(x-h)}{2h}$$

$$\frac{f(1.3) - f(1.1)}{2 \times 0.1}$$

$$(f(1.3) - f(1.1)) \approx (2.863 - 2.63) / 2 \times 0.1 = 1.665$$

$$\frac{2.863 - 2.63}{2 \times 0.1} = 1.665$$

$$(f(1.3) - f(1.1)) \approx (2.863 - 2.63) / 0.2 = 1.665$$

ANSWER

Richardson's Extrapolation

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f''''(x)}{4!}h^4 + \frac{f''''''(x)}{5!}h^5 + O(h^6)$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f''''(x)}{4!}h^4 - \frac{f''''''(x)}{5!}h^5 + O(h^6)$$

$$CD/D_h = \frac{f(x+h) - f(x-h)}{2h}$$

+ A forward period
- A backward period

Central differentiation (CD) \Rightarrow error term

minimize \rightarrow Richardson's Extrapolation.

$$f'(x) = \frac{2f'(x)h + \frac{12f'''(x)}{3!}h^3 + \frac{f''''(x)}{5!}h^5 + O(h^7)}{2h}$$

$$D_h = f'(x) + \underbrace{\frac{f''''(x)}{3!}h^2 + \frac{f''''''(x)}{5!}h^4}_{\text{error}} + O(h^6)$$

TOPIC NAME : _____ DAY : _____

TIME : _____ DATE : / /

Derivation

$$D_h = f'(x) + \frac{f'''(x)}{3!} h^2 + \frac{f''''(x)}{5!} h^4 + O(h^6)$$

$$D_{h/2} = f'(x) + \frac{f'''(x)}{3!} \left(\frac{h}{2}\right)^2 + \frac{f''''(x)}{5!} \left(\frac{h}{2}\right)^4 + O(h^6)$$

$$4D_{h/2} - D_h = 3f'(x) + 0 + \left(\frac{f'''(x)}{5!} \frac{h^4}{4} - \frac{f''''(x)}{5!} h^4 \right) + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{3} = f'(x) + \left(\frac{\frac{f'''(x)}{5!} \frac{h^4}{4} - \frac{f''''(x)}{5!} h^4}{3} \right) + O(h^6)$$

$$= D'_h(\cancel{h})$$

$$D'(h)$$

$$D'_h = \frac{4D_{h/2} - D_h}{3}$$

Richardson's Extrapolation only works for central differentiation.

$$D''_h = \frac{16D_{h/2} - D_h}{15}$$

TOPIC NAME

DAY

TIME

DATE

$$* f(x) = e^{2x} + 3x$$

$$D_h = \frac{f(x+h) - f(x-h)}{2h}$$

1) $f'(2)$ using RE.

$$D'_h = \frac{4D_{h/2} - D_h}{3}$$

$$D_{1.2} = \frac{f(2+1.2) - f(2-1.2)}{2 \times 1.2}$$

$$= 251.705$$

$$D_{0.6} = \frac{f(2+0.6) - f(2-0.6)}{2 \times 0.6}$$

$$= 140.356$$

$$D'_{1.2} = \frac{4D_{0.6} - D_{1.2}}{3}$$

$$= \frac{4 \times 140.356 + 251.705}{3}$$

$$= 103.23$$

$$f'(x) = 2e^{2x} + 3 = 2e^1 + 3$$

$$f'(2) = 112.196$$

TOPIC NAME : _____

DAY : _____

TIME : _____

DATE : / /

*	h	0.1	0.2	
	$f'(1)$	0.7	0.5	D_h
		$D_{0.1}$	$D_{0.2}$	

calculate $s'(1)$ using RE (Lagrange method)

$$D'_{0.2} = \frac{4D_{0.1} + D_{0.2}}{3} = \frac{4 \times 0.7 - 0.5}{3} = 0.8767$$

$$D''_{1.2} = \frac{16D'_{0.6} - D'_{1.2}}{15}$$

$$(s+1)^{-8} - (s+1)^{-2} = 0.009$$

$$0.8 \times 3$$

$$\frac{(s+1)^{-8} - (s+1)^{-2}}{0.8 \times 3} = 0.003$$

$$0.003 \times 3$$

$$0.009 = 0.009$$

$$0.009 = 0.009$$

$$0.009 = 0.009$$

TOPIC NAME

DAY:

TIME:

DATE:

$$* f(x) = e^x \sin(x)$$

$$h = 0.5$$

$$h = 0.25$$

Calculate $f'(1)$ using R.E.

$$D_{0.5} = \frac{f(1+0.5) - f(1-0.5)}{2 \times 0.5}$$

$$= \frac{e^{1.5} \sin(1.5) - e^{0.5} \sin(0.5)}{2 \times 0.5}$$

$$= 3.68$$

$$D_{0.25} = \frac{f(1+0.25) - f(1-0.25)}{2 \times 0.25}$$

$$= \frac{e^{1.25} \sin(1.25) - e^{1.25} \sin(1.25)}{2 \times 0.25}$$

$$= 3.7385$$

$$D'_{0.5} = \frac{4D_{h/2} - Dh}{3}$$

$$= \frac{4 \times 3.7385 - 3.68}{3} = 3.758$$

TOPIC NAME :

DAY: _____
TIME: _____ DATE: / /

*	x	$s(x)$	$(s)(x) - (s)(x-h)$	$\frac{h}{2} = 0.4$
	0.6	0.707178		$h = 0.2$
	0.8	0.8559892		$h = 0.1$
	0.9	0.925863		using RE
	1.0	0.984007		
	1.1	1.033713		
	1.2	1.074575		
	1.3	1.127986		

$$D^2 h = \frac{2^4 D(h/2) - D(h)}{2^4 - 1}$$

Dh $D'h$ $D^2 h$

0.4	0.52601	0.15359
0.2	0.5464	
0.1	0.5394	

$$D_h = \frac{s(1+0.4) - s(1-0.4)}{2 \times 0.4} = 0.52601$$

$$D_h = \frac{s(1+0.2) - s(1-0.2)}{2 \times 0.2} = 0.5464645$$

$$D_h = \frac{s(1+0.1) - s(1-0.1)}{2 \times 0.1} = 0.5394$$

$$D'h = \frac{4(D_h/2) - D(h)}{3} = \frac{4 \times 0.5464 - 0.52601}{3}$$

$$D'h = \frac{4(D_h/2) - D(h)}{2} = \frac{4 \times 0.5394 - 0.5464}{3}$$

GOOD LUCK

TOPIC NAME:

DATE:

TIME:

DATE:

$$D^2 h = \frac{2^4 D(h/2) - D(h)}{2^4 - 1}$$

$$\text{Given } h = 0.2 \quad \therefore h/2 = 0.1 \quad \therefore D(h/2) = f(1.2)$$

$$D(h) = f(1.0)$$

$$\therefore D^2 h = \frac{2^4 f(1.2) - f(1.0)}{2^4 - 1} = \frac{16(0.537) - 0.553}{15} = 0.5359$$

Ques. Calculate $f'(1.2)$ using
central difference

*	x	1.0	1.2	1.34	1.4
	$f(x)$	1.2717	1.3643	1.5496	

$f'(1.2)$ using central difference

$$f'(1.2) = \frac{f(1.2+0.2) - f(1.2-0.2)}{2 \times 0.2}$$

$$= \frac{1.5496 - 1.2717}{2 \times 0.2}$$

$$h = 0.2$$

$$= 0.69475 \approx 0.695$$

$$f(x) = x \sin x + x^2 \cos x$$

Compute the truncation error + 1) Error bound

using $\xi \in [1.0, 1.4]$

$$f(x) = x \sin x + x^2 \cos x - (1.34)^2$$

$$f'(x) = \sin x + x \cos x + 2x \cos x - (1.34)^2 \sin x$$

$$= -\sin x + 3x \cos x - x^2 \sin x$$

$$f''(x) = -\cos x + 3 \cos x - 6x \sin x + x^2 \cos x$$

TOPIC NAME : _____

DAY: _____

TIME: _____

DATE: / /

$$f''(x) = \cos x + 3\cos x - 13x \sin x - 2x \sin x - x^2 \cos x$$

$$(1.4 - \varepsilon)(1.4 + \varepsilon) \sin(\varepsilon) < (1.4 - \varepsilon)^2 + \varepsilon^2 = (1.4)^2$$

$$(1.4 - \varepsilon)^2 + \varepsilon^2 = 4\cos x + 5x \sin x - x^2 \cos x$$

$$f'''(x) = -4\sin x - 5\sin x - 5x \cos x - 3x \cos x$$

$$(1.4)^3 = (1.4)^2 + (1.4 - \varepsilon)^2 + (1.4)^2 + x^2 \sin x$$

$$= -9\sin x - 7x \cos x + x^2 \sin x$$

$$\text{error bound} = \left| \frac{f'''(\varepsilon) h^3}{3!} \right|$$

$$\left| \frac{(0.2)^3}{3!} \left| 9\sin(\varepsilon) + 7(\varepsilon) \cos(\varepsilon) + \varepsilon^2 \sin(\varepsilon) \right| \right|$$

$$= \left| \frac{(0.2)^3}{3!} \left| 9\sin(1.4) + 7(1.4) \cos(1.4) + 1.4^2 \sin(1.4) \right| \right|$$

general approximation formulae

Final answer

approximate value

GOOD LUCK