CSE330

TOPIC NAME : Numerical Methods

DAY :

TIME : DATE : 8/2/25

Quiz 10% (4/6)

Assignment 15% (6)

Mid 20%

Final 30%

Attendance 5%

Lab 20%

Chapter 1 : Floating point Representation

Fixed point :

$$\left.\begin{array}{l}1.d_1d_2\\1.\_\,\_\end{array}\right\}2^2=4 \qquad \left.\begin{array}{l}1.d_1d_2d_3\\1.\_\,\_\,\_\end{array}\right\}2^3=8$$

1.00

1.11

1.01

1.10

conv1 / lecture note form :

$$\pm(0.\underbrace{d_1d_2d_3\cdots d_m}_{\text{mantissa/m/fraction}})_\beta \times \beta^e$$

Base

$\beta = 2$

Note : lecture note form এর ক্ষেত্রে $d_1 = 1$

* $m = 3,\ e = 4$; total কত number represent করতে পারে?

$(0 \cdot d_1 d_2 d_3)_2 \times 2^4$

$(0 \cdot 1 d_2 d_3)_2 \times 2^4$

Total number $= 2^2 \times 1 = 4 \times 1$ ← total exponent

$= 4 + 4$

$= 8$

$e = 4$

$0 \cdot 100 \times 2^4$

$0 \cdot 101 \times 2^4$

$0 \cdot 110 \times 2^4$

$0 \cdot 111 \times 2^4$

যতগুলো exponent থাকবে তা total number এর সাথে multiply.

if $e = 4, 5$

total $= 4 \times 2 = 8$

$= 8 + 8 = 16$

* $m = 3,\ e = [-2, 3]$

$(0 \cdot d_1 d_2 d_3)_2 \times 2^e$

$(0 \cdot 1 d_2 d_3)_2 \times 2^e$

total numbers $= 2^2 \times 6$

$= 4 \times 6$

$= 24 + 24$ (positive 24, neg 24)

$= 48$

[ ] → range of value

$[-2, 3]$

$= -2, -1, 0, 1, 2, 3$

$\{-2, 3\}$

$= -2, 3$

highest $= (0 \cdot 1 1 1)_2 \times 2^3$ (binary highest 1, $e_{max} = 3$)

lowest $= -(0 \cdot 1 1 1)_2 \times 2^3$

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positive lowest / minimum $= (0 \cdot 1\underline{0}\underline{0})_2 \times 2^{-2}$

Note: Positive lowest এ negative consider করবো না।

binary lowest 0, $e_{min} = -2$

Convention 2 / Normalised :

$$\pm (1 \cdot d_1 d_2 d_3 \cdots d_m)_2 \times 2^e$$

* $m = 3$, $e = [-2, 3]$. calculate total number, the highest, lowest and positive lowest number.

$$\pm (1 \cdot d_1 d_2 d_3)_2 \times 2^e$$

total number $= 2^3 \times 6 \times$

total exponent

$= 48 + 48$

$= 96$

highest binary ← highest exponent

Highest number $= (1 \cdot \underline{1}\underline{1}\underline{1})_2 \times 2^3$

lowest number $= -(1 \cdot 111)_2 \times 2^3$

lowest binary ← lowest exponent

positive lowest / minimum $= (1 \cdot \underline{0}\underline{0}\underline{0})_2 \times 2^{-2}$

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Convention 3 / Denormalised :

$$\pm (0.1 d_1 d_2 d_3 \ldots d_m)_\beta \times \beta^e$$

* $m = 3$, $e = [-2, 3]$

$$(0.1 d_1 d_2 d_3)_2 \times 2^e$$

total number $= 2^3 \times 6$

$= 48 + 48$

$= 96$

{ Denormalised works better than lecture note form in terms of total number representation.

Highest $= (0.1111)_2 \times 2^3$

lowest $= -(0.1111)_2 \times 2^3$

non-negative lowest $= (0.1000)_2 \times 2^{-2}$

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conv1 : $m = 3$, $e = \{4, 5, 6\}$, numbers we can represen

$$(0.d_1d_2d_3)_2 \times 2^e$$

$$(0.1d_2d_3)_2 \times 2^e$$

$$(0.100)_2 \times 2^4 = 0.5 \times 2^4 = 8$$
$$(0.101)_2 \times 2^4 = 0.625 \times 2^4 = 10 \quad 2$$
$$(0.110)_2 \times 2^4 = 0.75 \times 2^4 = 12 \quad 2$$
$$(0.111)_2 \times 2^4 = 14 \quad 2$$

$$(0.100)_2 \times 2^5 = 16$$
$$(0.101)_2 \times 2^5 = 20 \quad 4$$
$$(0.110)_2 \times 2^5 = 24 \quad 4$$
$$(0.111)_2 \times 2^5 = 28 \quad 4$$

$$(0.100)_2 \times 2^6 = 32$$
$$(0.101)_2 \times 2^6 = 40 \quad 8$$
$$(0.110)_2 \times 2^6 = 48 \quad 8$$
$$(0.111)_2 \times 2^6 = 56 \quad 8$$

not equidistant

e এর value 1 টা থাকলে equidistant,

e এর value more than 1 হলে not equidistant.

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IEEE Standard System

64 bit

$$\left(1.d_1 d_2 d_3 \cdots d_{52}\right)_2 \times 2^{e}$$

exponent bias

- 1 sign bit
- 11 bits exponent / e
- 52 bits mantissa / m

$2^{11} = 2048$

$\rightarrow e = [0, 2047]$

normalised $= [0-1023, 2047-1023]$

$= [-1023, 1024]$

0 ∞

smallest positive $= (1.00\cdots0)_2 \times 2^{0}$

$= 2^0 \times 2^0 / 2^{-2} / 2^{-3} / 2^{-4}$

$= 1 / 0.25 / 0.125 / 0.065$

The smaller the exponent is, the closer the ans is (~~real~~ near 0)

exponent bias $= 2^{n-1} - 1$

$= 2^{11-1} - 1 = 1023$

smallest positive $= (1.00\cdots0)_2 \times 2^{-1022}$ ($-1023+1$)

Highest value $= (1.111\cdots1)_2 \times 2^{1023}$ ($1024-1$)

Denormalised : $e \in [-1023+1, 1024+1]$

$= [-1022, 1025]$

0 ∞

smallest positive $= (0.100\cdots0)_2 \times 2^{-1021}$

Highest value $= (0.111\cdots1)_2 \times 2^{1024}$

Quiz 01
22 Feb, chapter 1
Assignment

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* 32 bit → 1 sign, 7 bit exponent, 24 bit mantissa using exponent biasing,

normalised $e = 2^{n-1} - 1$
$= 2^6 - 1$
$= 63$

$2^7 = 128$

$e = [0, 127]$
$= [0-63, 127-63]$
$= [-63, 64]$

Denormalised $e \in [-63+1, 64+1]$
$= [-62, 65]$

Rounding

79.6
79 — 80

79.56
79.5 — 79.6

79.567
79.56 — 79.57

* m = 3, lecture note form

$x = 0.1001$ → adjacent হলে left value copy করে last এ 1
↓ original value compare করতে

0.1001
0.100 — 0.101
nearest even number
middle value

roundup value = 0.100
$fl(x) = 0.100$

$y = 0.1001011$
$fl(y) = 0.101$

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Rounding error $= \dfrac{|fl(x) - x|}{|x|}$

Scale invariant rounding error/

rounding error; δ (delta) $\Rightarrow \dfrac{|fl(x) - x|}{|x|}$

maximum scale invariant rounding error/
maximum rounding error / maximum delta /
Machine epsilon

↓

conv1: $\frac{1}{2}\beta^{1-m}$

conv 2 $= \frac{1}{2}\beta^{-m}$

conv 3 $= \frac{1}{2}\beta^{-m}$

* $m = 4$, $e = [-3, 6]$; calculate max delta for conv 1

conv 1 :

max - delta $= \frac{1}{2}\beta^{1-m}$

$= \frac{1}{2} \times 2^{1-4}$

$= \frac{1}{2} \times 2^{-3}$

maximum rounding error, δ (delta) $= \dfrac{|fl(x) - x|_{max}}{|x|_{min}}$

Derivation

* convention 1

$m = 3$

0.100 ——|—— 0.101

$$\begin{array}{r} 0.101 \\ -\,0.100 \\ \hline 0.001 \end{array}$$

$\frac{1}{2} \times 0.001$ → decimal

$= \frac{1}{2} \times 2^{-3}$

$= \frac{1}{2} \times \beta^{-m} \times 2^{e} = |fl(x) - x|_{max}$

$|x|_{min} = 0.100 \times 2^{e}$

$= 2^{-1} \times 2^{e}$

$= \beta^{-1} \times 2^{e}$

$$\text{max delta} = \frac{\frac{1}{2}\beta^{-m} \times 2^{e}}{\beta^{-1} \times 2^{e}}$$

$= \frac{1}{2} \beta^{-m} \beta^{1}$

$= \frac{1}{2} \beta^{1-m}$

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convention 2

$m = 3$

$1.100 \quad 1.101$

$$\begin{array}{r} 1.101 \\ -\ 1.100 \\ \hline 0.001 \end{array}$$

$\frac{1}{2} \times 0.001$

$= \frac{1}{2} \times 2^{-3}$

$= \frac{1}{2} \times \beta^{-m} \times 2^{e}$

$= |fl(x) - x|_{max}$

$|x_{min}| = 1.000 \times 2^{e}$

$= 2^{0} \times 2^{e}$

$= |2^{e}$

$$\text{max delta} = \frac{\frac{1}{2}\beta^{-m} 2^{e}}{2^{e}} = \frac{1}{2}\beta^{-m}$$

convention 3

$m = 3$

$0.1000 \quad 0.10001 \quad 0.1001$

$$\begin{array}{r} 0.1001 \\ -\ 0.1000 \\ \hline 0.0001 \end{array}$$

$\frac{1}{2} \times 0.0001$

$= \frac{1}{2} \times 2^{-4} \times 2^{e}$

$= \frac{1}{2} \times 2^{-3} \times 2^{-1} \times 2^{e}$

$= \frac{1}{2} \beta^{-m} \beta^{-1} 2^{e}$

$= |fl(x) - x|_{max}$

$|x_{min}| = 0.1000 \times 2^{e}$

$= 2^{-1} \times 2^{e} = \beta^{-1} \times 2^{e}$

$$\text{max delta} = \frac{\frac{1}{2}\beta^{-m}\beta^{-1} 2^{e}}{\beta^{-1} 2^{e}} = \frac{1}{2}\beta^{-m}$$

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* $x = \frac{5}{8}$, $y = \frac{7}{8}$

$fl(x)$, $fl(y)$, $fl(x*y)$, $fl(fl(x) * fl(y))$

$x = \frac{5}{8} = \frac{1}{8} + \frac{4}{8} = \frac{1}{2^3} + \frac{1}{2^1} = 2^{-3} + 2^{-1}$ → 3rd position 1 → 1st pos 1

→ ২ এর power থাকলে এই নিয়ম use করবো

$= 0.101$

$fl(x) = 0.101$ $\left(\begin{array}{l} m = 3 \\ e \in [-3, 6] , \text{conv1} \end{array} \right)$

$= \frac{5}{8}$

$y = \frac{7}{8} = \frac{1}{8} + \frac{2}{8} + \frac{4}{8} = 2^{-3} + 2^{-2} + 2^{-1}$

$= 0.111$

$fl(y) = 0.111 = \frac{7}{8}$

$x * y = \frac{5}{8} \times \frac{7}{8} = \frac{35}{64} = \frac{32}{64} + \frac{2}{64} + \frac{1}{64}$

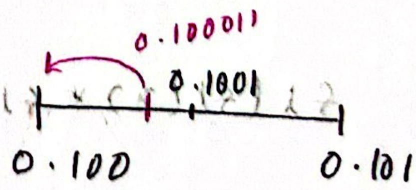
$= 2^{-1} + 2^{-5} + 2^{-6}$

$= 0.100011$

$fl(x*y) = 0.100$

$= 2^{-1}$

$= 0.5$



rounding error $= \frac{\left| 0.5 - \frac{35}{64} \right|}{\left| \frac{35}{64} \right|}$

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$$fl(\overset{same}{fl(x)} * \overset{same}{fl(y)}) = 0.5$$

$x = \frac{5}{8}$, $y = \frac{11}{16}$, $m = 3$, $e \in [-3, 1]$, conv 1

$$x = \frac{5}{8} = \frac{1}{8} + \frac{4}{8}$$
$$= 2^{-3} + 2^{-1}$$
$$= 0.101$$
$$fl(x) = 0.101 = \frac{5}{8}$$

$$y = \frac{11}{16}$$
$$= \frac{8}{16} + \frac{2}{16} + \frac{1}{16}$$
$$= 2^{-1} + 2^{-3} + 2^{-4}$$
$$= 0.1011$$

0.1011 → nearest even

0.101 —— 0.1011 —— 0.110

$$fl(y) = 0.110 = 2^{-1} + 2^{-2} = \frac{3}{4}$$

$$fl(x) * fl(y) = \frac{5}{8} \times \frac{3}{4} = \frac{15}{32}$$
$$= \frac{8}{32} + \frac{4}{32} + \frac{2}{32} + \frac{1}{32}$$
$$= 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5}$$
$$= 0.01111$$

0.011 —— 0.0111 —— 0.100

$$fl(fl(x) * fl(y)) = 0.100$$
$$= 2^{-1}$$
$$= 2$$

4 decimal → 0.0001 → 1 sig fig

5 decimal → 0.01056 → 4 sig fig

5 decimal → 0.10000 → 5 sig fig

Loss of significance : When we substract two close values the scale invariant rounding error get very high, we call this loss of significance.

* Quadratic formula for solving $x^2 - 56x + 1 = 0$.

To 4 s.f the roots are -

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{56 + \sqrt{(-56)^2 - 4 \cdot 1 \cdot 1}}{2(1)} = 28 + 3\sqrt{87}$$
$$= 28 + \sqrt{783}$$
$$= 55.98213716$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{56 - \sqrt{(-56)^2 - 4 \cdot 1 \cdot 1}}{2(1)} = 28 - \sqrt{783}$$
$$= 0.0178628407 3$$

$\sqrt{783} = 27.98213716$

$x_1 = 28.00 + 27.98 = 55.98$

$x_2 = 28.00 - 27.98 = 0.02000$

$$\delta_1 = \frac{|fl(x_1) - x_1|}{|x_1|} = \frac{|55.98 - 55.98213716|}{|55.98213716|}$$

$$= 0.00003817 \times 100$$
$$= 0.003817\ \%$$

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$$\delta_2 = \frac{|fl|x_2| - x_2|}{|x_2|}$$

$$= \frac{|0.02 - 0.0178628 4073|}{|0.01786284073|}$$

$$= 0.11964 \times 100$$

$$= 11.964\%$$

$$\alpha = x_1 = 55.98$$

$$\beta = x_2$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \beta = \frac{c}{a\alpha}$$

$$= \frac{1}{1 \times 55.98}$$

$$= 0.01786352269$$

$$\approx 0.01786$$

Chapter 2 : Polynomial Interpolation

$5x^2$ → power/exponent
→ term
variable
coefficient

Polynomial : collection of terms

$P_2(x) = 3 + 4x + 5x^2$; coefficient $= 3, 4, 5$

$P_7(x) = 3x + 4x^3 + 5x^7$; coeff $= 0, 3, 0, 4, 0, 0, 0, 5$

$P_n(x)$
→ degree

coefficient $= n+1$
term $= n+1$

$P_n(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$

$= a_0 + a_1 x^1 + a_2 x^2$

Taylor

$f(x) = \sin(x)$

error $= |f(x) - P_n(x)|$

* $x = 5$

$|f(5) - P_7(5)| =$ error

truncation error

$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \underbrace{+ \frac{x^9}{9!} - \dots \dots}_{\text{error}}$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

degree ↑ error ↓
→ Weierstrass theorem

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Maximum error bound / error bound :

$$= \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\text{error } \cancel{\text{ma}} \text{ bound} = \frac{|f^6(\xi)|}{6!}\left|(x - \overset{0}{x_0})^6\right|$$

$$= \frac{|f^6(\xi)|_{max}}{6!}|x^6|_{max} \rightarrow 2^6 = \frac{\sin(1.57)}{6!} \times 2^6$$

ξ [0, 2]

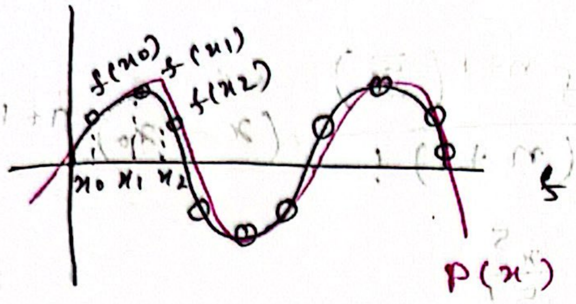
x [0, 2]

$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$f^{iv}(x) = \sin x$$
$$f^{v}(x) = \cos(x)$$
$$f^{vi}(x) = -\sin(x)$$

$$f^6(\xi) = -\sin(\xi)$$

$$|f^6(\xi)| = \sin(\xi)$$

↪ 1.57 ($\frac{\pi}{2}$) → sin ~~ঃ~~ highest possible value



$f(x) = \sin(x)$

n = number of nodes -1

node 5
$P_4(x)$

*

x		$f(x)$	
x_0	1	3	$f(x_0)$
x_1	3	8	$f(x_1)$
x_2	5	10	$f(x_2)$

$n = 3-1 = 2$

$P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$

$1^0 a_0 + a_1 1^1 + a_2 1^2 = 3 \quad \ldots\ldots (i)$

$3^0 a_0 + a_1 3^1 + a_2 3^2 = 8 \quad \ldots\ldots (ii)$

$5^0 a_0 + a_1 5^1 + a_2 5^2 = 10 \quad \ldots\ldots (iii)$

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Vandermonde

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \end{vmatrix} = \begin{vmatrix} 3 \\ 8 \\ 10 \end{vmatrix}$$

V $\quad$ a/x $\quad$ b/y

$$Va = b$$

$$\Rightarrow a = V^{-1} b = \begin{vmatrix} 2 \\ 9 \\ 16 \end{vmatrix}$$

$$P_2(x) = 2 + 9x + 16x^2$$

(ii) $x = 7$

error bound $= |f(7) - P_2(7)|$

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Lagrange

x	$f(x)$ / y
$x_0 = 1$	$3 = f(x_0)$
$x_1 = 3$	$5 = f(x_1)$
$x_2 = 5$	$9 = f(x_2)$

$n = 3 - 1 = 2$

$$P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2) + \cdots\cdots + l_n(x) f(x_n)$$

$l_n(x) \rightarrow$ Lagrange basis

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-3)(x-5)}{(1-3)(1-5)} = \frac{(x-3)(x-5)}{8}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-5)}{(3-1)(3-5)} = \frac{(x-1)(x-5)}{-4}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1)(x-3)}{(5-1)(5-3)} = \frac{(x-1)(x-3)}{8}$$

$$P_2(x) = \frac{(x-3)(x-5)}{8} \times 3 - \frac{(x-1)(x-5)}{4} \times 5 + \frac{(x-1)(x-3)}{8} \times 9$$

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Newton's Divided Difference

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) + \dots \dots$$

x	$f(x) / y$
$x_0 = 1$	$3 = f(x_0)$
$x_1 = 3$	$5 = f(x_1)$
$x_2 = 5$	$9 = f(x_2)$
$x_3 = 9$	$15 = f(x_3)$

$n = 4 - 1 = 3$

$n = 3 - 1 = 2$

$$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 3 + 1(x - 1) + \frac{1}{4}(x - 1)(x - 3)$$

$$P_3(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$

$$= 3 + 1(x - 1) + \frac{1}{4}(x - 1)(x - 3) - \frac{1}{24}(x - 1)(x - 3)(x - 5)$$

যে দুটো merge করবো, লাস্ট টা থেকে ফার্স্ট টা বিয়োগ

$x_0 = 1$ $\quad f[x_0] = 3$

$x_1 = 3$ $\quad f[x_1] = 5$

$x_2 = 5$ $\quad f[x_2] = 9$

$x_3 = 9$ $\quad f[x_3] = 15$

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{5-3}{3-1} = 1$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{9-5}{5-3} = 2$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{15-9}{9-5} = \frac{3}{2}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{2-1}{5-1} = \frac{1}{4}$$

$$f[x_1, x_2, x_3] = \frac{\frac{3}{2} - 2}{9-3} = -\frac{1}{12}$$

$$f[x_0, x_1, x_2, x_3] = \frac{-\frac{1}{12} - \frac{1}{4}}{9-1} = -\frac{1}{24}$$

Newton's Divided Difference Derivation

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots \dots + a_n(x-x_0)\dots(x-x_{n-1})$$

Here, $a_0 = P_n(x_0) = f(x_0)$

$$f(x_0) + a_1(x-x_0) = P_n(x_1) = f(x_1)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_i] = f(x_i)$$

The first divided difference

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

The second divided difference

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Similarly kth divided difference:

$$f[x_i, x_{i+1}, \dots x_{i+k-1}, x_{i+k}]$$

$$= \frac{f[x_{i+1}, x_{i+2}, \dots x_{i+k}] - f[x_i, x_{i+1}, \dots x_{i+k-1}]}{x_{i+k} - x_i}$$

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The process ends with the single nth divided difference,

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2 \ldots, x_n] - f[x_0, x_1 \ldots, x_{n-1}]}{x_n - x_0}$$

$$a_k = f[x_0, x_1, x_2 \ldots, x_k]$$

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1})$$

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Cauchy's theorem

error bound / maximum error bound = $\frac{|f^{n+1}(\xi)|_{max}}{(n+1)!} |(x-x_0)(x-x_1) \cdots \cdots (x-x_n)|_{max}$

* $f(x) = e^x \sin(x)$, given nodes $\overset{x_0}{0}, \overset{x_1}{1}, \overset{x_2}{2}$.

range ξ $[-1, 3]$, calculate error using cauchy's theorem.

error bound = $\frac{f^3(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2)$

যদি ξ এর range দেয়া না থাকে, আমাদের range create করে নিতে হবে।

$\xi \in$ [lowest value of node, highest value of node]

$= [0, 2]$

$f(x) = e^x \sin(x)$

$f'(x) = e^x \cos(x) + \sin(x) e^x$

$f''(x) = -e^x \sin(x) + e^x \cos(x) + e^x \cos(x) + \sin(x) e^x$

$= 2e^x \cos(x)$

$f'''(x) = 2(e^x \cos x - e^x \sin x)$

$= 2e^x \cos x - 2e^x \sin x$

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$$|f'''(x)| = |2e^x \cos x| + |-2e^x \sin x|$$

$$= 2e^3 \cos(0) + 2e^3 \sin\left(\tfrac{\pi}{2}\right)$$

maximize

$$= 80.34$$

$$w(x) = x(x-1)(x-2)$$
$$= (x^2 - x)(x-2)$$
$$= x^3 - 3x^2 + 2x$$

$$w'(x) = 3x^2 - 6x + 2$$

$$w'(x) = 0$$

$$\Rightarrow 3x^2 - 6x + 2 = 0$$

$$\therefore x_1 = 1 + \frac{1}{\sqrt{3}},$$

$$x_2 = 1 - \frac{1}{\sqrt{3}}$$

error bound $= \dfrac{|f^3(\xi)|}{3!} |(x-x_0)(x-x_1)(x-x_2)|$

$$= \frac{80.34}{3!} \times 6$$

x	$\lvert w(x) \rvert$
$1+\frac{1}{\sqrt{3}}$	0.384
$1-\frac{1}{\sqrt{3}}$	0.384
-1	6
3	6

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Hermite Interpolation

x	$f(x)$	$f'(x)$
$x_0 = -1$	1	2
$x_1 = 0$	0	2
$x_2 = 1$	1	0

$n = 3 - 1 = 2$

$$P_{2n+1}(x) = h_0(x)f(x_0) + h_1(x)f(x_1) + \cdots + h_n(x)f(x_n) + \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1) + \cdots + \hat{h}_n(x)f'(x_n)$$

$$P_5(x) = h_0(x)f(x_0) + h_1(x)\underbrace{f(x_1)}_{\to 0} + h_2(x)f(x_2) + \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1) + \hat{h}_2(x)\underbrace{f'(x_2)}_{\to 0}$$

$$= h_0(x)f(x_0) + h_2(x)f(x_2) + \hat{h}_0(x)f'(x_0) + \hat{h}_1(x)f'(x_1)$$

$$h_k(x) = \{1 - 2(x - x_k)\,l_k'(x_k)\}(l_k(x))^2$$

$$\hat{h}_k(x) = (x - x_k)(l_k(x))^2$$

$$h_0(x) = \{1 - 2(x - x_0)\,l_0'(x_0)\}(l_0(x))^2$$

$$= \{1 - 2(x+1)(-3/2)\}\left(\frac{x(x-1)}{2}\right)^2$$

$$\hat{h}_0(x) = (x - x_0)(l_0(x))^2$$

$$= (x+1)\left(\frac{x(x-1)}{2}\right)^2$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x(x-1)}{2} = \frac{x^2}{2} - \frac{x}{2}$$

$$l_0'(x) = x - \frac{1}{2}$$

$$l_0'(x_0) = -1 - \frac{1}{2} = -\frac{3}{2}$$

$h_2(x) = \{1-2(x-x_2) l_2'(x_2)\} (l_2(x))^2$

$= \{1-2(x-1)\frac{3}{2}\}\left(\frac{x(x+1)}{2}\right)^2$

$l_2(x) = \frac{(x+1)(x-0)}{(1+1)(1-0)}$

$= \frac{x(x+1)}{2}$

$= \frac{x^2}{2} + \frac{x}{2}$

$l_2'(x) = x + \frac{1}{2}$

$l_2'(x_2) = 1 + \frac{1}{2} = \frac{3}{2}$

$\hat{h}_1(x) = (x-x_1)(l_1(x))^2$

$= (x-0)(-x^2+1)^2$

$= x(-x^2+1)^2$

$l_1(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)}$

$= \frac{x^2-1}{-1}$

$= -x^2+1$

$P5(x) = h_0(x) f(x_0) + h_2(x) f(x_2) + \hat{h}_0(x) f'(x_0)$
$+ \hat{h}_1(x) f'(x_1)$

$= \{1-2(x+1)(-3/2)\}\left(\frac{x(x-1)}{2}\right)^2 (1)$

$+ \{1-2(x-1)(3/2)\}\left(\frac{x(x+1)}{2}\right)^2 (1)$

$+ 2(x+1)\left(\frac{x(x-1)}{2}\right)^2 + 2x(1-x^2)^2$

- why hermite better?

In hermite, using less amount of information, we can find less error function.

drawback: you need to calculate everything, if you add or drop any data.

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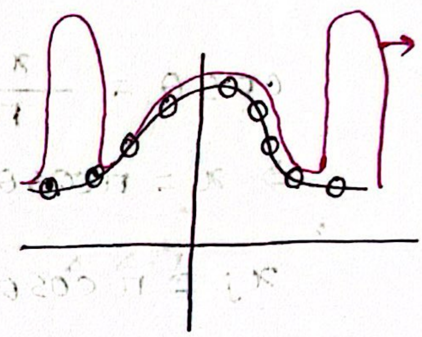
Runge Phenomenon

1. The function we try to be replicate need to be runge function.

2. The function has to be symmetric.

3. equally spaced nodes.

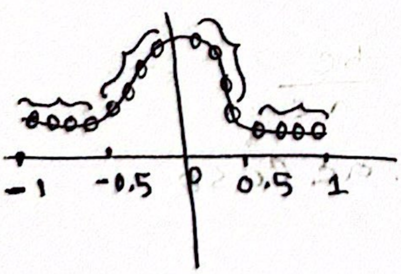
এই তিনটা rule maintain করে function create করলে corner এ error বাড়ে হয় → runge phenomenon



how to tackle?

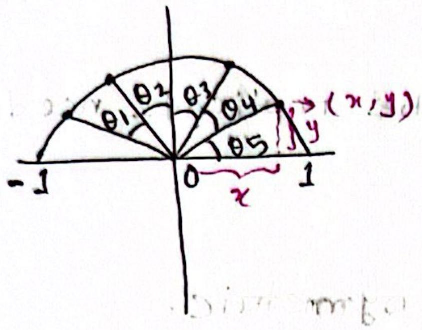
→ instead of equally spaced nodes, we use equally angle nodes. ↳ Chebyshev's nodes

piecewise solution :



one way of solving runge phenomenon.

Chebyshev's Nodes



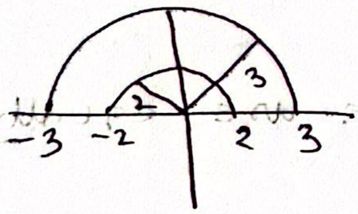
$\theta_1 = \theta_2 = \theta_3 = \theta_4$

$$\theta_j = \frac{(2j+1)\pi}{2(n+1)} \; ; \quad j = 0, 1, 2, \ldots, n$$

$$\cos\theta = \frac{x}{r}$$

$$\Rightarrow x = r\cos\theta$$

$$x_j = r\cos\theta_j + \text{centre}$$

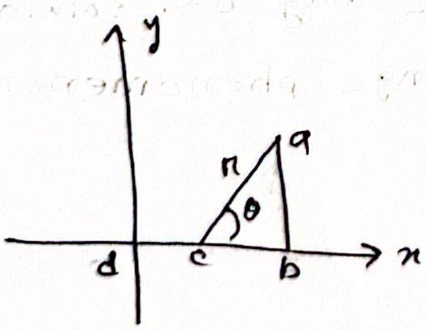


$[-a, b]$

centre $= 0$

<u>case 1</u>: $[-r, r]$, centre $= 0$, $r = r$

$bd = bc + dc$



$$\cos\theta = \frac{bc}{ac}$$

$$\Rightarrow bc = ac\cos\theta$$

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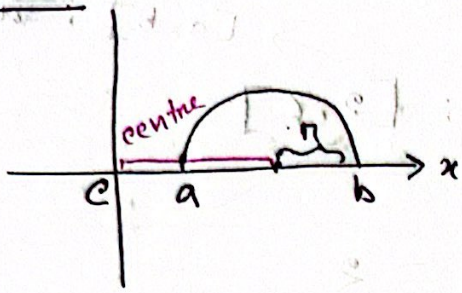
case 2



$$r = \frac{b-a}{2} \qquad [a, b] \quad a \neq b$$

$$\text{centre} = b - r = b - \frac{b-a}{2} = \frac{2b - b + a}{2} = \frac{a+b}{2}$$

$$x_j = r\cos\theta_j + \text{centre} = \frac{b-a}{2}\cos\theta_j + \frac{a+b}{2}$$

case 1 (math)

$$f(x) = \frac{1}{1+25x^2} \;;\; n = 3 \;,\; [-4, 4] \rightarrow [-r, r]$$

$r = 4$

centre $= 0$

$$x_0 = r\cos\theta_j + \text{centre} = 4\cos\frac{(2\times 0+1)\pi}{2(3+1)} + 0 = 4\cos\frac{\pi}{8}$$

$$x_1 = 4\cos\frac{(2\times 1+1)\pi}{2(3+1)} + 0 = 4\cos\frac{3\pi}{8}$$

$$x_2 = 4\cos\frac{(2\times 2+1)\pi}{2(3+1)} + 0 = 4\cos\frac{5\pi}{8}$$

$$x_3 = 4\cos\frac{(2\times 3+1)\pi}{2(3+1)} + 0 = 4\cos\frac{7\pi}{8}$$

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case 2 (math)

$\rightarrow [a,b]$; $a \neq b$

$$f(x) = \frac{1}{1+25x^2} \; ; \; n=3 \; ; \; [2,6]$$

$$r = \frac{b-a}{2} = \frac{6-2}{2} = 2$$

$$\text{centre} = \frac{a+b}{2} = \frac{2+6}{2} = 4$$

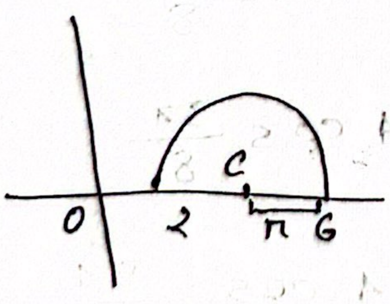
$$x_0 = r\cos\theta_0 + \text{centre} = 2\cos\frac{(2\times 0+1)\pi}{2(3+1)} + 4$$

$$= 2\cos\frac{\pi}{8} + 4$$

$$x_1 = 2\cos\frac{(2\times 1+1)\pi}{2(3+1)} + 4 = 2\cos\frac{3\pi}{8} + 4$$

$$x_2 = 2\cos\frac{5\pi}{8} + 4$$

$$x_3 = 2\cos\frac{7\pi}{8} + 4$$



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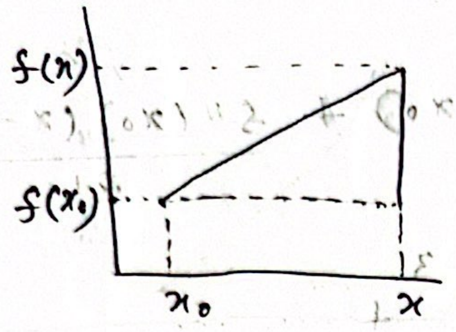

$$f'(x_0) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow \boxed{f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

Taylor series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!}$$

$$+ \frac{f'''(x_0)(x - x_0)^3}{3!} + \dots \dots$$

Proof of taylor series:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots \dots$$

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots \dots$$

$$f''(x) = 2a_2 + 3 \times 2a_3(x - x_0)$$

$$f'''(x) = 3 \cdot 2a_3$$

$x = x_0$

$f(x_0) = a_0$

$f'(x_0) = a_1$

$f''(x_0) = 2a_2$

$f'''(x_0) = 3 \cdot 2a_3$

$$a_2 = \frac{f''(x_0)}{2!}$$

$$a_3 = \frac{f'''(x_0)}{3!}$$

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$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \dots$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!}$$

$$+ \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots$$

$x_0 = 0 \rightarrow$ $$= \sin(0) + \cos(0)x - \sin(0) + \frac{(-\cos(0))x^3}{3!}$$

$$+ \sin(0) + \frac{\cos(0)\,x^5}{5!} + \dots$$

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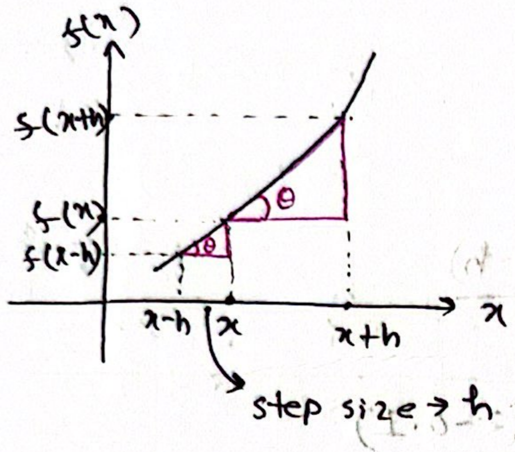
Chapter 3 – Differentiation



$$m \,/\, \tan\theta = \frac{f(x+h) - f(x)}{x+h-x}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{f'(x) = \frac{f(x+h) - f(x)}{h}}$$

↳ forward differentiation

↳ $n = 2-1 = 1$

Backward differentiation $= \dfrac{f(x) - f(x-h)}{x - x + h} = \dfrac{f(x) - f(x-h)}{h}$

↳ $n = 1$

Central differentiation $= \dfrac{f(x+h) - f(x-h)}{x+h-x+h} = \dfrac{f(x+h) - f(x-h)}{2h}$

Backward and forward D. এর ক্ষেত্রে $\boxed{T.E \propto h}$

central D. এর ক্ষেত্রে $\boxed{T.E \propto h^2}$

central D. এর ক্ষেত্রে T.E কম থাকবে।

for Backward and forward D. ⇒

Upper bound of error / error bound $= \dfrac{f''(\xi)}{2!} \times h$

for central D. ⇒

error bound $= \dfrac{f'''(\xi)}{3!} \times h^2$

$* \quad f(x) = \ln(x), \quad x = 2, \quad h = 0.1$

$$f'(x) = \frac{1}{x}$$

$$f'(2) = \frac{1}{2} = 0.5$$

$$\text{Backward D.} = \frac{f(x) - f(x-h)}{h}$$

$$= \frac{f(2) - f(2-0.1)}{0.1}$$

$$= \frac{\ln(2) - \ln(1.9)}{0.1}$$

$$= 0.51293$$

$$T.E = |0.5 - 0.512| = 0.012$$

$$\text{Forward D.} = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{f(2+0.1) - f(2)}{0.1}$$

$$= \frac{f(2.1) - f(2)}{0.1} = \frac{\ln(2.1) - \ln(2)}{0.1}$$

$$= 0.4879$$

$$T.E = |0.5 - 0.4879| = 0.012$$

$$\text{Central D.} = \frac{f(x+h) - f(x-h)}{2h} = \frac{f(2+0.1) - f(2-0.1)}{2 \times 0.1}$$

$$= \frac{\ln(2.1) - \ln(1.9)}{2 \times 0.1} = 0.500417$$

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$$T.E = |0.5 - 0.500417|$$

$$= 0.000417$$

* $h = 0.1$

x	1.1	1.2	1.3
$f(x)$	2.630	2.739	2.963

Using central D. calculate the value of $f'(1.2)$

$$\text{Central D.} = \frac{f(\overset{1.2+0.1}{x+h}) - f(\overset{1.2-0.1}{x-h})}{2h}$$

$$= \frac{f(1.3) - f(1.1)}{2 \times 0.1}$$

$$= \frac{2.963 - 2.63}{2 \times 0.1} = 1.665$$

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Richardson's Extrapolation

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{iv}(x)}{4!}h^4 + \frac{f^5(x)}{5!}h^5 + O(h^6)$$

$$f(x-h) = f(x) - f'(x)h + \frac{f^2(x)}{2!}h^2 - \frac{f^3(x)}{3!}h^3 + \frac{f^4(x)}{4!}h^4 - \frac{f^5(x)}{5!}h^5 \pm O(h^6)$$

$$CD/D_h = \frac{f(x+h) - f(x-h)}{2h}$$

central differentiation (CD) এর error কে minimize করে Richardson's Extrapolation.

$$f'(2) = \frac{2f'(x)h + \frac{2f'''(x)}{3!}h^3 + \frac{f^5(x)}{5!}h^5 + O(h^7)}{2h}$$

error কে বেশি contribute করে।

$$D_h = f'(x) + \underbrace{\frac{f^3(x)}{3!}h^2 + \frac{f^5(x)}{5!}h^4 + O(h^6)}_{\text{error}}$$

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Derivation

$$D_h = f'(x) + \frac{f^3(x)}{3!} h^2 + \frac{f^5(x)}{5!} h^4 + O(h^6)$$

$$D_{h/2} = f'(x) + \frac{f^3(x)}{3!} \left(\frac{h}{2}\right)^2 + \frac{f^5(x)}{5!} \left(\frac{h}{2}\right)^4 + O(h^6)$$

$$4D_{h/2} - D_h = 3f'(x) + 0 + \left(\frac{f^5(x)}{5!} \frac{h^4}{4} - \frac{f^5(x)}{5!} h^4\right) + O(h^6)$$

$$\frac{4D_{h/2} - D_h}{3} = f'(x) + \left(\frac{\frac{f^5(x)}{5!} \frac{h^4}{4} - \frac{f^5(x)}{5!} h^4}{3}\right) + O(h^6)$$

$$= D'_h \cancel{(h)}$$

$$\cancel{D'(h)}$$

$$\boxed{D'_h = \frac{4D_{h/2} - D_h}{3}}$$

Richardson's Extrapolation only works for central differentiation.

$$D''_h \cancel{(h)} = \frac{16 D'_{h/2} - D'_h}{15}$$

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* $f(x) = e^{2x} + 3x$

$h = 1.2, 0.6$

$f'(2)$ using RE.

$$D'_h = \frac{4D_{h/2} - D_h}{3}$$

$$D_{1.2} = \frac{f(2+1.2) - f(2-1.2)}{2 \times 1.2}$$

$$= 251.705$$

$$D_{0.6} = \frac{f(2+0.6) - f(2-0.6)}{2 \times 0.6}$$

$$= 140.356$$

$$D'_{1.2} = \frac{4D_{0.6} - D_{1.2}}{3}$$

$$= \frac{4 \times 140.356 - 251.705}{3}$$

$$= 103.23$$

$$f'(x) = 2e^{2x} + 3 = 2e^{4} + 3$$

$$f'(2) = 112.196$$

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*

h	0.1	0.2	
$f'(1)$	0.7	0.5	D_h
	$D_{0.1}$	$D_{0.2}$	

calculate $f'(1)$ using RE

$$D'_{0.2} = \frac{4D_{0.1} - D_{0.2}}{3} = \frac{4 \times 0.7 - 0.5}{3} = 0.767$$

$$\boxed{D''_{1.2} = \frac{16D'_{0.6} - D'_{1.2}}{15}}$$

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* $f(x) = e^{x} \sin(x)$

$h = 0.5$

$h = 0.25$

Calculate $f'(1)$ using R.E.

$$D_{0.5} = \frac{f(1+0.5) - f(1-0.5)}{2\times 0.5}$$

$$= \frac{e^{1.5}\sin(1.5) - e^{0.5}\sin(0.5)}{2\times 0.5}$$

$$= 3.68$$

$$D_{0.25} = \frac{f(1+0.25) - f(1-0.25)}{2\times 0.25}$$

$$= \frac{e^{1.25}\sin(1.25) - e^{1.25}\sin(1.25)}{2\times 0.25}$$

$$= 3.7385$$

$$D'_{0.5} = \frac{4D_{h/2} - D_h}{3}$$

$$= \frac{4\times 3.7385 - 3.68}{3} = 3.758$$

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*

x	$f(x)$
0.6	0.707178
0.8	0.8559892
0.9	0.925863
1.0	0.984007
1.1	1.033743
1.2	1.074575
1.3	1.127986

$h = 0.4$

$h = 0.2$

$h = 0.1$

$f'(1)$ using RE

$$D^2 h = \frac{2^4 D'(h/2) - D'(h)}{2^4 - 1}$$

h	Dh	$D'h$	$D^2 h$
0.4	0.52601		
		0.553	
0.2	0.5464		0.5359
		0.537	
0.1	0.5394		

$$D_h = \frac{f(1+0.4) - f(1-0.4)}{2 \times 0.4} = 0.52601$$

$$D_h = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2} = 0.5464645$$

$$D_h = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1} = 0.5394$$

$$D'h = \frac{4(Dh/2) - D(h)}{3} = \frac{4 \times 0.5464 - 0.52601}{3}$$

$$D'h = \frac{4(Dh/2) - D(h)}{3} = \frac{4 \times 0.5394 - 0.5464}{3}$$

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$$D^2 h = \frac{2^4 D'(h/2) - D'(h)}{2^4 - 1}$$

$$= \frac{16(0.537) - 0.553}{15} = 0.5359$$

*

x	1.0	1.2	1.4
$f(x)$	1.2717	1.3642	1.5496

$f'(1.2)$ using central difference

$h = 0.2$

$$f'(1.2) = \frac{f(1.2+0.2) - f(1.2-0.2)}{2 \times 0.2}$$

$$= \frac{1.5496 - 1.2717}{2 \times 0.2}$$

$$= 0.69475$$

$f(x) = x\sin(x) + x^2 \cos x$

Compute the truncation error / Error bound using ξ [1.0, 1.4]

$f(x) = x\sin x + x^2 \cos x$

$$f'(x) = \sin x + x\cos x + 2x\cos x - x^2 \sin x$$
$$= \sin x + 3x\cos x - x^2 \sin x$$

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$$f''(x) = \cos x + 3\cos x - 3x\sin x - 2x\sin x - x^2\cos x$$

$$= 4\cos x - 5x\sin x - x^2\cos x$$

$$f'''(x) = -4\sin x - 5\sin x - 5x\cos x - 2x\cos x + x^2\sin x$$

$$= -9\sin x - 7x\cos x + x^2\sin x$$

$$\text{error bound} = \left|\frac{f^3(\xi)\, h^2}{3!}\right|$$

$$= \left|\frac{(0.2)^2}{3!}\right| \left|9\sin(\xi) + 7(\xi)\cos(\xi) + \xi^2\sin(\xi)\right|$$

$$= \left|\frac{(0.2)^2}{3!}\right| \left|9\sin(1.4) + 7(1.4)\cos(1.0) + 1.4^2\sin(1.4)\right|$$