Polynomial

$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x_n$.

↳ n represents degree of the polynomial.

highest kon power porjonto jacchey polynomial.

$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

↳ degree = 3

$P_3(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3$

coefficients = 4. [value with x].

degree = n, coefficient = (n+1)

$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

Degree = 5, coefficient = 6.

$P_{27}(x)$ degree = 27, coefficient = 28.

Vector space: A region where we can
add vectors
multiply with scalars

$(1 + x + x^2 + x^3)$ $\quad$ $(1 + x + x^2) * 5$

$5 + 5x + 5x^2$.

** **Basis**: Basis is a set of vectors that spans the space.

$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

basis = $\{1, x, x^2, x^3\}$ → $2 + 3x + 4x^2 + 5x^3$

→ $1 + 2x + 9x^2 + 8x^3$

Dimensional space = 4

$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

basis = $\{1, x, x^2, x^3, x^4, x^5\}$ → elements

Dimensional space of basis = 6.

Dimensional space is 1 more than degree.

$P_{58}(x)$ Dimensional space = 59
coefficient = 59.
$\downarrow$
degree

Functional space : $f(x)$ can go up to infinity.

$$f(x) = 2 + 3x + 10x^2 + 14x^3 + 25x^4 + \ldots$$
$$\hookrightarrow P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

We replicate $f(x)$ to $P_n(x)$ because we can consider a finite no. of terms for $P_n(x)$ and we can manipulate it.

$f(x) \in V^\alpha \longrightarrow$ belongs to a infinite vector space of dimension.

$P_n(x) \in V^{n+1} \longrightarrow$ dimension
$\hookrightarrow$ vector space.

$$f(x) = 2 + 3x + 4x^2 + 5x^3 + 8x^4 + 9x^5 + \ldots$$
$$P_2(x) = a_0 + a_1 x + a_2 x^2$$
Error

Polynomial can replicate upto x^2 term.

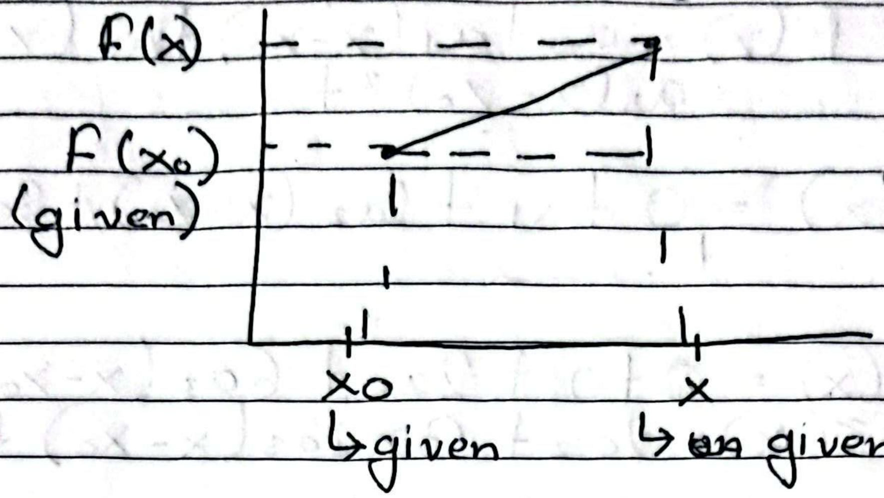
If we increase the degree, the error decreases.

$$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5.$$
$\hookrightarrow$ less error

Weierstrass Approximation Theorem

$$|f(x) - P_n(x)|$$

Taylor Series : [Previously seen]



determine f(x)

$f'(x_0) \rightarrow$ given

Calulate gradient $\dfrac{y_2 - y_1}{x_2 - x_1} = f'(x_0)$

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\boxed{f(x) = f'(x_0)(x - x_0) + f(x_0)}$$

Actual Taylor Series → infinitely large

Taylor Series

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + f'''(x_0)\frac{(x - x_0)^3}{3!} + \ldots$$

$2(x-3) = 2x-6 = 2$

Proof of Taylor Series :

$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \ldots$

$f'(x) = 0 + a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \ldots$

$f''(x) = 0 + 0 + 2a_2 + 6a_3(x-x_0) + \ldots$

$f''(x) = 2a_2 + 3\times 2a_3(x-x_0) + \ldots$

$f'''(x) = 3 \cdot 2 a_3 \cancel{(x-x_0)}$

$x = x_0$

$f(x_0) = a_0 + a_1(x_0-x_0) + a_2(x_0-x_0)^2 + a_3(x_0-x_0)^3 + \ldots$

$f(x_0) = a_0$

$f'(x_0) = a_1 + 2a_2(x_0-x_0) + 3a_3(x_0-x_0)^2 + \ldots$

$f'(x_0) = a_1$

$f''(x_0) = 2a_2 \qquad a_2 = \dfrac{f''(x_0)}{2!}$

$f'''(x_0) = 3 \cdot 2 \cdot a_3$

$a_3 = \dfrac{f'''(x_0)}{3!} \longrightarrow (3\times 2\times 1)$

$f(x) = f(x_0) + f'(x_0)(x-x_0) + \dfrac{f''(x_0)(x-x_0)^2}{2!} + \dfrac{f'''(x_0)(x-x_0)^3}{3!} + \ldots$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!}$$
$$+ \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f^4(x_0)}{4!}(x-x_0)^4$$
$$+ \frac{f^5(x_0)(x-x_0)^5}{5!} + \ldots$$

We always consider $f(x_0) = 0$

$$f(x) = \sin(0) + \overbrace{\cos(0)(x)}^{x} + (-\sin(0)) + \underbrace{\frac{(-\cos(0))(x)^3}{3!}}_{-\frac{x^3}{3!}} + \sin(0)(x^4) + \underbrace{\cos(0)\frac{x^5}{5!}}_{\frac{x^5}{5!}}$$

e^x in Taylor Series

e^{-x} in Taylor Series

1st term $\rightarrow f(0.1) \approx 0.1$

2 terms $\rightarrow f(0.1) \approx 0.1 - \frac{0.1^3}{6} = 0.099833$

3 terms $\rightarrow f(0.1) \approx 0.1 - \frac{0.1^3}{5!} = 0.0998334$

exact answer = $\underbrace{0.0998334}_{\hookrightarrow 6\text{ sf}}$

Vandermonde Matrix

	House size (m^2)	Rent ($)	
nodal points/ nodes →	20	200	$f(x)$
	30	250	[Actual values]
[given values]	40	275	

For House Size 25 (m^2) = Rent?

We can generate a Polynomial in this case.

$$P_n(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$$

<u>Assume</u>:

Nodes

$$f(x) = P_n(x).$$

→ Suppose in a polynomial we give 30, it needs to show 250

$$P_n(30) = 250.$$

	Age	Salary
nodes	25	25k
	30	30k
	45	~~7~~45k

Predict the salary of a person of Age = 35?

We can generate a Polynomial.

$$P_n(x) = a_0 + a_1x + a_2x^2 + a_nx^n.$$

P_n(nodes) = f(nodes)

$P_n(25) = 25k$

How can we set up the degree 2.

$$P_2(x) = a_0 + a_1(x) + a_2(x)^2$$

We need to determine the coefficients.
For that, we have multiple value methods.

Vandermonde Matrix

Lagrange

Hermite Interpolation.

3 unknown values (a_0, a_1, a_2) we need to set up 3 equations.

$$P_2(25) = a_0 + a_1 x + a_2 x^2$$

$$P_2(25) = a_0 + 25a_1 + 625a_2 = 25k$$

$$P_2(30) = a_0 + 30a_1 + 900a_2 = 30k$$

$$P_2(45) = a_0 + 45a_1 + 2025a_2 = 70k$$

n degree, (n+1) coefficients

data points = 3, degree = n.

(n+1) nodes, n degree.

8 data points, 7 degree.

Time	Velocity
0	0
10	227.4
15	362.8
20	517.35

data points = 4, degree = 3,
(a_0, a_1, a_2, a_3)

$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
↳ Vandermonde Matrix

We usually use it for degree = 1. We do not even use for degree = 2/3.

	x_0	$f(x_0)$
$x_1 \longrightarrow$	2	5
$x_2 \longrightarrow$	3	6

degree = 1.

$P_1(x_1) = a_0 + a_1 x_1 = f(x_1)$
$P_2(x_2) = a_0 + a_1 x_2 = f(x_2)$

Coefficient Matrix

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix}$$

	Time	Velocity
$x_1 \rightarrow$	15	362.8
$x_2 \rightarrow$	20	517.35

degree = 1.

$P_1(15) = a_0 + 15a_1 = 362.8$

$P_1(20) = a_0 + 20a_1 = 517.35$

$$\begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.35 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.35 \end{bmatrix} \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix}^{-1}$$

If determinent is 0, it is not possible to inverse the matrix. So we cannot solve the matrix.

$$= \frac{1}{\text{determinent}} \begin{bmatrix} 20 & -15 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 362.8 \\ 517.35 \end{bmatrix}$$

$$\text{determinent} = (20 \times 1) - (15 \times 1)$$
$$= 5.$$

$$= \frac{1}{5}\begin{bmatrix} 20 & -15 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 362.8 \\ 517.35 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -1/5 & 1/5 \end{bmatrix}\begin{bmatrix} 362.8 \\ 517.35 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} -100.85 \\ 30.91 \end{bmatrix}.$$

In case of higher degree polynomials, the matrix becomes very large.

Lagrange Polynomial : updated version of Vandermonde Matrix because in Vandermonde Matrix the matrix becomes very large and the time complexity is also large.

	x	$f(x)$
x_0	2	30
Nodes x_1	5	48
x_2	7	25

We can establish a relationship between x and $f(x)$ and set up a polynomial to find any unknown value.

degree = 2

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

Lagrange : Represent a polynomial function in terms of lagrange basis

$$P_2(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

l_0, l_1, l_2 are known as Lagrange basis.

We need to compute l_0, l_1, l_2.

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

	Time(s)	Velocity (m/s)
x_0	15	227.04
x_1	20	~~360~~ 362.78
x_2	22.5	517.35

Set up a Polynomial using Lagrange basis/ Lagrange form to calculate $P_n(17) = ?$

$$P_2(x) = l_0(x)\,f(x_0) + l_1(x)\,f(x_1) + l_2(x)\,f(x_2)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$l_0(x) = \frac{(x-20)(x-22.5)}{(15-20)(15-22.5)}$$

$$l_1(x) = \frac{(x-15)(x-22.5)}{(20-15)(20-22.5)}$$

$$l_2(x) = \frac{(x-15)(x-20)}{(22.5-15)(22.5-20)}$$

$$l_0(x) = \frac{2}{75}(x-20)(x-22.5)$$

$$l_1(x) = \frac{-2}{25}(x-15)(x-22.5)$$

$$l_2(x) = \frac{4}{75}(x-15)(x-20)$$

$$P_2(x) = \left(\frac{2}{75} * 227.04\right)(x-20)(x-22.5) + \left(\frac{-2}{25} * 362.78\right)(x-15)(x-22.5) + \left(\frac{4}{75} * 517.35\right)(x-15)(x-20)$$

$P_2(17) =$ Final answer. There will be some errors.

$f(x) = \cos(x)$ $x_0 = -\pi/4$, $x_1 = 0$, $x_2 = \pi/4$.

Important thing to note: Convert the calulator mode to rad mode from degree mode. Since none of the calulations will be in degree mode. (radian mode)

$P_n(x) = 2$

$$P_2(x) = l_0(x)\,f(x_0) + l_1(x)\,f(x_1) + l_2(x)\,f(x_2)$$

$$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)}\frac{(x - x_2)}{(x_0 - x_2)}$$

$$= \frac{x}{-\pi/4}\frac{(x - \pi/4)}{-\pi/4 - \pi/4} = \frac{8x}{\pi^2}\left(x - \frac{\pi}{4}\right)$$

$$l_1(x) = \frac{(x-x_0)\,(x-x_2)}{(x_1-x_0)\,(x_1-x_2)}$$

$$= \frac{(x+\pi/4)}{\pi/4} \frac{(x-\pi/4)}{(0-\pi/4)}$$

$$= \frac{-16}{\pi^2}(x+\pi/4)(x-\pi/4)$$

$$l_2(x) = \frac{(x-x_0)\,(x-x_1)}{(x_2-x_0)\,(x_2-x_1)}$$

$$= \frac{8}{\pi^2}\left(x+\frac{\pi}{4}\right)$$

$$f(x_0) = \frac{1}{\sqrt{2}} \quad f(x_1) = 1 \quad f(x_2) = \frac{1}{\sqrt{2}}$$

$$P_2(x) = \frac{8}{\pi^2} x \left(x-\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{-16}{\pi^2}\right)\left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) +$$

$$\frac{8}{\pi^2}\left(x+\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right)$$

	Time	Velocity
x_0	0	0
x_1	10	227.4
x_2	15	362.8
x_3	20	517.35
x_4	22.5	602.97
x_5	30	901.67

16

6

1

16

4

6.5

$P_5(x)$

↳ degree = 5

The Question asked me to find $P_2(16)$.

$P_2(x)$ = We can consider first 3 data points

$P_2(x)$ at $x = 16$ / $P_2(16)$. So we need to consider the data points that are closer to 16.

l_0, l_1, l_2

~~$l_0(x) = (x - x$~~

$P_3(16) = ?$ Home Task

x_0	$f(x_0)$	Degree = 1
x_1	$f(x_1)$	

$P_1(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$

$l_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \rightarrow l_0(x_0) = 1, \; l_0(x_1) = 0$

$l_1(x) = \frac{(x - x_0)}{(x_1 - x_0)} \rightarrow l_1(x_0) = 0, \; l_1(x_1) = 1$.

$l_0(x_1) = \frac{(x_1 - x_1)}{(x_0 - x_1)} = 0$.

So $l_1(x_0) = \frac{(x_0 - x_0)}{(x_1 - x_0)} = 0$.

$l_0(x_i) \begin{cases} 0 & i = 1 \\ 1 & i = 0 \end{cases}$ $\quad l_1(x_i) \begin{cases} 0 & i = 0 \\ 1 & i = 1 \end{cases}$.

$l_n(x_j) = \delta_{nj} \rightarrow$ kroneker delta

$\boxed{n = 0, 1}$ $\boxed{j = i}$

$\delta_{nj} = 0 \quad n \neq j$

$\delta_{nj} = 1 \quad n = j$

Expleein this expression :

$l_1(x_0)$

$n = 1, \; j = 0$

Newton's Divided Difference & Cauchy's Theorem

$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$
$+ a_3(x-x_0)(x-x_1)(x-x_2) + \dots +$
$a_n(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})$

$a_0 = f[x_0]$
$a_1 = f[x_0, x_1]$
$a_2 = f[x_0, x_1, x_2]$
$a_3 = f[x_0, x_1, x_2, x_3]$

$P_3(x) = f[x_0] + f[x_0, x_1](x-x_0) +$
$f[x_0, x_1, x_2](x-x_0)(x-x_1) +$
$f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

	x	$f(x)$
x_0	-1	5
x_1	0	1
x_2	1	3
x_3	2	11
New data point x_4	4	20

$P_3(x) = f[x_0] + f[x_0, x_1](x-x_0) +$
$f[x_0, x_1, x_2](x-x_0)(x-x_1) +$
$f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

$P_3(x) = 5 + (-4)(x+1) + (3)(x+1)(x) +$
$0(x+1)(x)(x-1)$

$P_3(x) = 5 - 4(x+1) + 3x(x+1)$.

$x_0 = -1 \quad f[x_0] = 5$

$x_1 = 0 \quad f[x_1] = 1$

$x_2 = 1 \quad f[x_2] = 3$

$x_3 = 2 \quad f[x_3] = 11$

$x_4 = 4 \quad f[x_4] = 20$

$$f[x_0, x_1] = \frac{1-5}{0+1} = -4$$

$$f[x_1, x_2] = \frac{3-1}{1-0} = 2$$

$$f[x_3, x_2] = \frac{11-3}{2-1} = 8$$

$$f[x_4, x_3] = \frac{20-11}{4-2} = \frac{9}{2}$$

$$f[x_0, x_1, x_2] = \frac{2-(-4)}{1+1} = 3$$

$$f[x_1, x_2, x_3] = \frac{8-2}{2-0} = \cancel{\frac{6}{10}} = 3$$

$$f[x_2, x_3, x_4] = \frac{\frac{9}{2}-8}{4-1} = -\frac{7}{6}$$

$$f[x_0, x_1, x_2, x_3] = \frac{3-3}{2+1} = 0$$

$$f[x_1, x_2, x_3, x_4] = \frac{-\frac{7}{6}-3}{4-0} = -\frac{25}{24}$$

$$f[x_0, x_1, x_2, x_3, x_4] = \frac{-\frac{25}{4}-0}{4-(-1)} = -\frac{5}{24}$$

$P_4(x) = f[x_0] + f[x_0, x_1](x - x_0) +$
$f[x_0, x_1, x_2](x - x_0)(x - x_1) +$
$f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$
$+ f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3)$

$P_4(x) = 5 + (-4)(x+1) + 3(x+1)(x) + 0 +$
$\left(\frac{-5}{24}\right)(x+1)(x)(x-1)(x-2)$

Advantage: New data points can be incorporated easily.

	Time	Velocity
x_0	10	227.4
x_1	15	362.8
x_2	20	517.35
x_3	30	901.67

(a) Find: $P_n(x)$
(b) Find $P_n(17)$
(c) Add new data point and find the new polynomial.

(a) $P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) +$
$f[x_0, x_1, x_2](x - x_0)(x - x_1)$

$x_0 = 10$ $f[x_0] = 227.4$
$f[x_0, x_1] = 27.08$
$x_1 = 15$ $f[x_1] = 362.8$
$f[x_0, x_1, x_2] = 0.383$
$f[x_1, x_2] = 30.91$
$x_2 = 20$ $f[x_2] = 517.35$
$f[x_1, x_2, x_3] = 0.501$
$f[x_2, x_3] = 38.432$
$f[x_0, x_1, x_2, x_3] = 5.9 \times 10^{-3}$
$x_3 = 30$ $f[x_3] = 901.67$

(a) $P_2(x) = 227.4 + (27.08)(x-10) + 0.383(x-10)(x-15)$

(b) $P_2(17) = 227.4 + (27.08)(17-10) + 0.383(17-10)(17-15)$

(c) $P_3(x) = 227.4 + (27.08)(x-10) + 0.383(x-10)(x-15) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$

$(5.9 \times 10^{-3})(x-10)(x-15)(x-20)$

Cauchy's Theorem :

$$\underset{\substack{\uparrow \\ \text{Error}}}{\left| f(x) - P_n(x) \right|} = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1) \ldots (x-x_n) \right|$$

Example : $f(x) = \cos(x)$ $\left\{ \overset{x_0}{-\frac{\pi}{4}}, \overset{x_1}{0}, \overset{x_2}{\frac{\pi}{4}} \right\}$

Calculate the upper bound of error using Cauchy's Theorem.

The range of ξ can be given or not.

$x / \xi \in [-1, 1]$

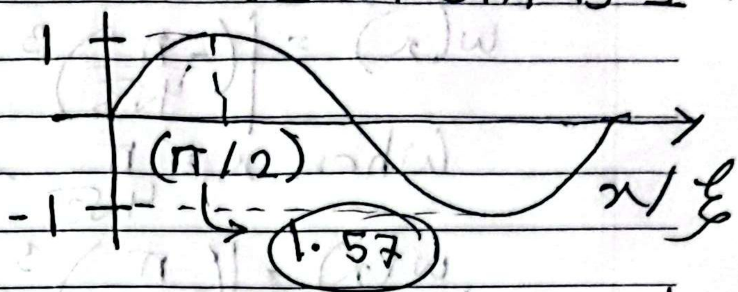
$$\left| f(x) - P_2(x) \right| = \left| \frac{f^3(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$$

$$\boxed{\begin{aligned}\cos(x) &\Rightarrow -\sin(x) \\ &\Rightarrow -\cos(x) \\ &\Rightarrow -(-\sin(x)) = \sin(x)\end{aligned}}$$ - 3rd derivative

$$= \left|\frac{\sin(\xi)}{3!}\right| \left|\left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)\right|$$

upper bound:

$$\left|\frac{\sin(\xi)}{3!}\right| \rightarrow$$ Max value of sin is 1.



$$\left|\frac{\sin(1)}{3!}\right|.$$

We cannot input ξ as 1.57 because it is outside the range $\xi \in (-1,1)$ so we will consider ξ as 1.

$$w(x) = \left(x+\frac{\pi}{4}\right)(x)\left(x-\frac{\pi}{4}\right)$$

$$= (x)\left(x^2 - \frac{\pi^2}{16}\right)$$

$$w(x) = x^3 - \frac{\pi^2}{16}x$$

~~$y = x^2$~~

$$w'(x) = 3x^2 - \frac{\pi^2}{16}.$$

$\frac{dy}{dx} = 0$

$y = x^2 + 3x + 1$

$\frac{dy}{dx} = 2x + 3.$

$\frac{dy}{dx} = 0$

$\hookrightarrow$ y max

$w'(x) = 0$

$3x^2 - \frac{\pi^2}{16} = 0$

$3x^2 = \frac{\pi^2}{16}$

$x = \pm \frac{\pi}{4\sqrt{3}}$

When $x = -\pi/4\sqrt{3}$,

$w(x) = \left|\left(\frac{-\pi}{4\sqrt{3}}\right)^3 - \frac{\pi^2}{16}\left(\frac{-\pi}{4\sqrt{3}}\right)\right| = 0.186$

When $x = \frac{\pi}{4\sqrt{3}}$,

$w(x) = \left|\left(\frac{\pi}{4\sqrt{3}}\right)^3 - \frac{\pi^2}{16}\left(\frac{\pi}{4\sqrt{3}}\right)\right| = 0.186$.

When $x = -1$,

$w(x) = \left|(-1)^3 - \frac{\pi^2}{16}(-1)\right| = 0.383$

When $x = 1$

$w(x) = \left|(1)^3 - \frac{\pi^2}{16}(1)\right| = 0.383$.

$|f(x) - P_2(x)| \le \left|\frac{\sin(1)}{3!}\right| \times 0.383$

↳ upper bound.