PUMPING LEMMA PRACTICE SHEET
AUTOMATA & COMPUTABILITY (CSE331)

Q: Prove they are Non - Regular.

1. {w ∈ {0,1,2}*: $0^n1^n2^n$ where $n \geq 0$ }
2. {w ∈ {0,1}*: $0^x1^y0^z$ where z > x+y and $x, y \geq 0$ }
3. {w ∈ {0,1}*: where w is a palindrome}
4. {w ∈ {0,1}*: where w is not a palindrome}
5. {w ∈ {a,b}*: number of a in w in a prime number }
6. {w ∈ {0}*: 0^{3^n} where $n \geq 0$} | { w = 1^n: n is a perfect cube (e.g., n= 1, 8, 27…) }
7. {w ∈ {0,1}* | ww where $n \geq 0$ }
8. {w ∈ {0}*: 1^{n^2} where $n \geq 0$ }
9. {w ∈ {0,1}*: 0^i1^j where $i > j$ }
10. { w = 1^n: n is a power of two }
11. { ww^R| w ε Σ* and w^R means w in reverse }
12. { w_1#w_2 such that length of w_1= length of w_2 }
13. { w ε Σ* | w = 0^i 1^j where i< 3j }
14. {w ∈ {0,1}*: $0^{2n}1^n$ where $n \geq 0$ }
15. {w ∈ {0,1}*: 0^n1^n where $n! = m$ }
16. {w ∈ {0,1,2,3}*: $1^n0^m3^n2^m$ where $n, m \geq 0$ }

There are a total of n problems. You have to solve all of them.

Problem 1 (CO5): Nonregular Language (25 points)

Use the pumping lemma to **demonstrate** that L_1 and L_2 are not regular.

(a) $L_1 = \{w \in \{0,1\}^* : w = 0^{n!} \text{ where } n \geq 0\}$ (5 points)

(b) $L_2 = \{w \in \{0,1\}^* : w = 0^a 1^b 1^c 0^d \text{ where } a+b = c+d \text{ and } a,b,c,d \geq 0\ \}$ (5 points)

(a) Assume for the sake of contradiction that L_1 is regular. Then let p be the pumping length for L_1. Now we take the string

$$w = 0^{p!} \in L_1.$$

Then the length of w is $|w| = p! \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_1$ for each $i \geq 0$. y consists of only 0s, so

$$xy^iz = 0^{p!+(i-1)|y|}$$

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p!+|y|}.$$

Now, $|y| \leq p < p \cdot p!$, hence,

$$p! < p! + |y| < p! + p \cdot p! = p!\,(1+p) = (p+1)!$$

So $p! < p! + |y| < (p+1)!$, and the length of xy^2z is strictly between two consecutive factorials. Hence, it cannot be a factorial. Thus we get a contradiction! Hence, L_1 is not a regular language.

(b) Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string

$$w = 0^p 110^p \in L_2.$$

Then the length of w is $|w| = 2p + 2 \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_2$ for each $i \geq 0$. $|xy| \leq p$, so y consists of only 0s, so

$$xy^iz = 0^{p+(i-1)|y|}110^p.$$

We choose $i = 4$, so that

$$xy^4z = 0^{p+3|y|}110^p.$$

Since this is in L_2, one can write it as $0^a1^b1^c0^d$ for $a + b = c + d$. By equating

$$0^{p+3|y|}110^p = 0^a1^b1^c0^d,$$

we get $a = p + 3|y|$, $d = p$ and $b + c = 2$. So $c - b \leq 2$. Furthermore,

$$c - b = a - d = 3|y| \geq 3,$$

as $|y| \geq 1$. So we get $c - b \leq 2$ and $c - b \geq 3$, which is a contradiction! Therefore, L_2 is not regular.

(b) **(Alternate Solutiion)** Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string

$$w = 0^p1^p1^p0^p \in L_2.$$

Then the length of w is $|w| = 4p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_2$ for each $i \geq 0$. $|xy| \leq p$, so y consists of only 0s, so

$$xy^iz = 0^{p+(i-1)|y|}1^p1^p0^p.$$

We choose $i = 2p + 2$, so that

$$xy^{2p+2}z = 0^{p+(2p+1)|y|}1^p1^p0^p.$$

Since this is in L_2, one can write it as $0^a1^b1^c0^d$ for $a+b=c+d$. By equating

$$0^{p+(2p+1)|y|}1^p1^p0^p = 0^a1^b1^c0^d,$$

we get $a = p+(2p+1)|y|$, $d = p$ and $b+c = 2p$. So $c-b \leq 2p$. Furthermore,

$$c-b = a-d = (2p+1)|y| \geq 2p+1,$$

as $|y| \geq 1$. So we get $c-b \leq 2p$ and $c-b \geq 2p+1$, which is a contradiction! Therefore, L_2 is not regular.

Problem 1 (CO5): Nonregular Language (25 points)

Use the pumping lemma to **demonstrate** that L_1, L_2, L_3, L_4 and L_5 is not regular.

(a) $L_1 = \{w \in \{0,1,2\}^* : 0^n1^n2^n \text{ where } n \geq 0\}$ (5 points)

(b) $L_2 = \{w \in \{0,1\}^* : 0^x1^y0^z \text{ where } z > x+y \text{ and } x, y \geq 0\}$ (5 points)

(c) $L_3 = \{w \in \{0,1\}^* : w \text{ is a palindrome}\}$ (5 points)

(d) $L_4 = \{w \in \{a,b\}^* : \text{ numbers of a in } w \text{ is a prime number}\}$ (5 points)

(e) $L_5 = \{w \in \{0\}^* : 0^{3^n} \text{ where } n \geq 0\}$ (5 points)

(a) Assume for the sake of contradiction that L_1 is regular. Then let p be the pumping length for L_1. Now we take the string
$$w = 0^p1^p2^p \in L_1.$$
Then the length of w is $|w| = 3p \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_1$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.
Then, for $i = 2$, xy^2z will be
$$xy^2z = xyyz = 0^{p+|y|}1^p2^p \notin L_1.$$
We have excess 0s in $xyyz$. Thus we get a contradiction! Hence, L_1 is not a regular language.

(b) Assume for the sake of contradiction that L_2 is regular. Then let p be the pumping length for L_2. Now we take the string
$$w = 0^p1^p0^{2p+1} \in L_2.$$
Then the length of w is $|w| = 4p+1 > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_2$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.
Then, for $i = 2$, xy^2z will be
$$xy^2z = xyyz = 0^{p+|y|}1^p0^{2p+1}.$$
This string is not in L_2, since $p+|y|+p \geq 2p+1$. Thus we get a contradiction! Hence, L_2 is not a regular language.

(c) Assume for the sake of contradiction that L_3 is regular. Then let p be the pumping length for L_3. Now we take the string

$$w = 0^p 1 0^p \in L_3.$$

Then the length of w is $|w| = 2p + 1 > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^i z \in L_3$ for each $i \geq 0$. Since $|xy| \leq p$, and the first p characters of w are all 0s, we can conclude that y consists of only 0s.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{p+|y|}10^p = 0^p 0^{|y|} 1 0^p.$$

This string is not a palindrome, so it is not in L_3. Thus we get a contradiction! Hence, L_3 is not a regular language.

(d) Assume for the sake of contradiction that L_4 is regular. Then let p be the pumping length for L_4. Now we take the string

$$w = \mathtt{a}^q \in L_4,$$

where q is a prime number greater than or equal to p. The length of w is $|w| = q \geq p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^i z \in L_4$ for each $i \geq 0$.

The length of $w = xyz$ is q. Then, for $i = 2$, the length of $xy^2z = xyyz$ will be $q + |y|$, which should be a prime. So, for any $i > 0$, the length of xy^iz will be $q + (i-1)|y|$.

So, for $i > 0$,

$$xy^iz = xyy^{i-1}z = \mathtt{a}^{q+(i-1)|y|}.$$

Since this string is in L_4, $q + (i-1)|y|$ is a prime number for each $i > 0$. But this is clearly not true, since choosing $i = q + 1$ gives

$$q + (i-1)|y| = q + q|y|,$$

which is divisible by q. Thus we get a contradiction! Hence, L_4 is not a regular language.

(e) Assume for the sake of contradiction that L_5 is regular. Then let p be the pumping length for L_5. Now we take the string

$$w = 0^{3^p} \in L_5.$$

Then the length of w is $|w| = 3^p > p$. So w can be split into xyz such that $|y| > 0$, $|xy| \leq p$, and $xy^iz \in L_5$ for each $i \geq 0$.

Then, for $i = 2$, xy^2z will be

$$xy^2z = xyyz = 0^{3^p+|y|}.$$

Since this string is in L_5, $3^p + |y|$ is a power of 3 (which is, of course, larger than 3^p). The next power of 3 larger than 3^p is 3^{p+1}. So we have

$$3^p + |y| \geq 3^{p+1} \implies |y| \geq 3^{p+1} - 3^p = 2 \cdot 3^p.$$

On the other hand, $|xy| \leq p$ gives us that $|y| \leq p$. So

$$p \geq |y| \geq 2 \cdot 3^p.$$

This is clearly false since $3^p > p$. Thus we get a contradiction! Hence, L_5 is not a regular language.