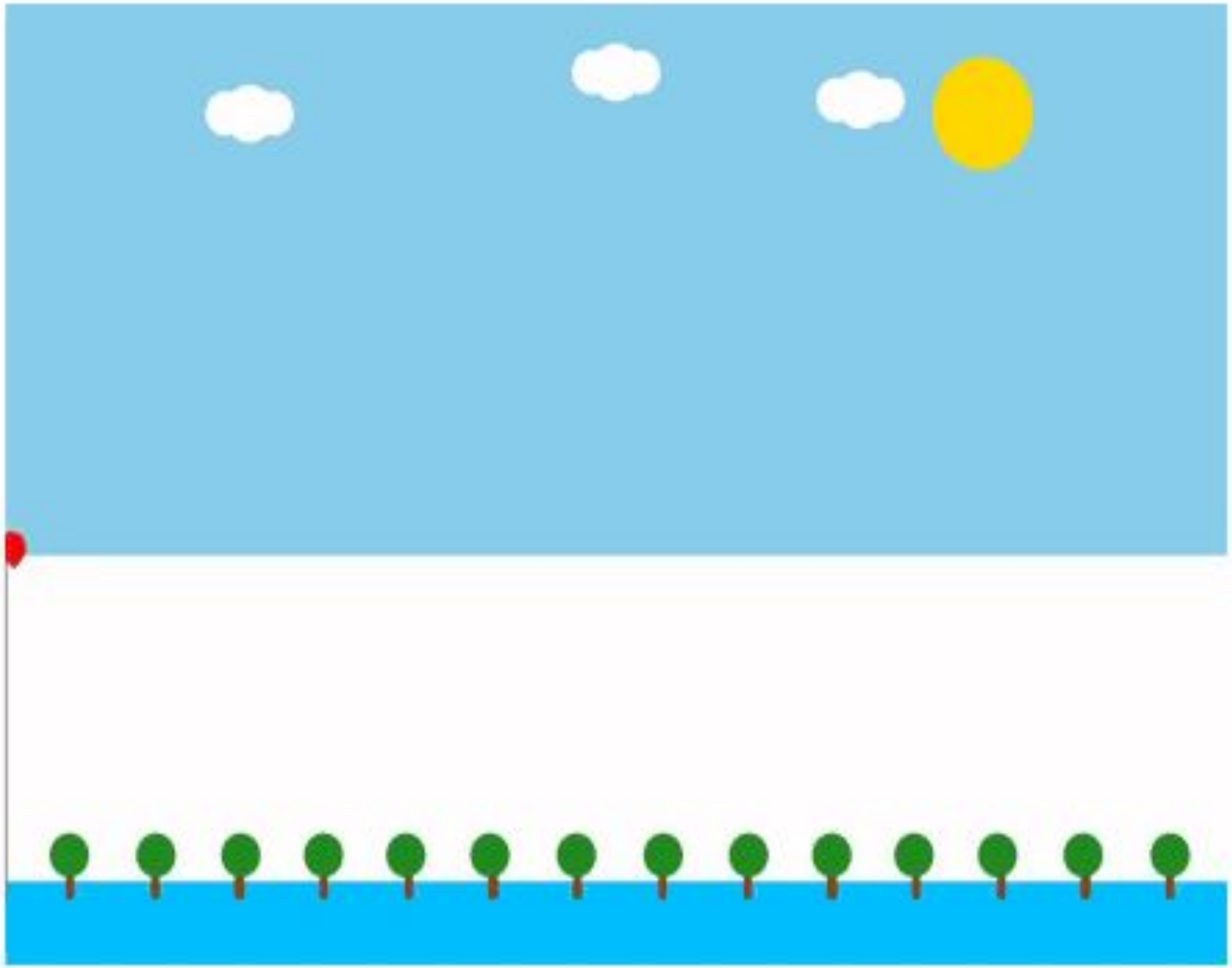
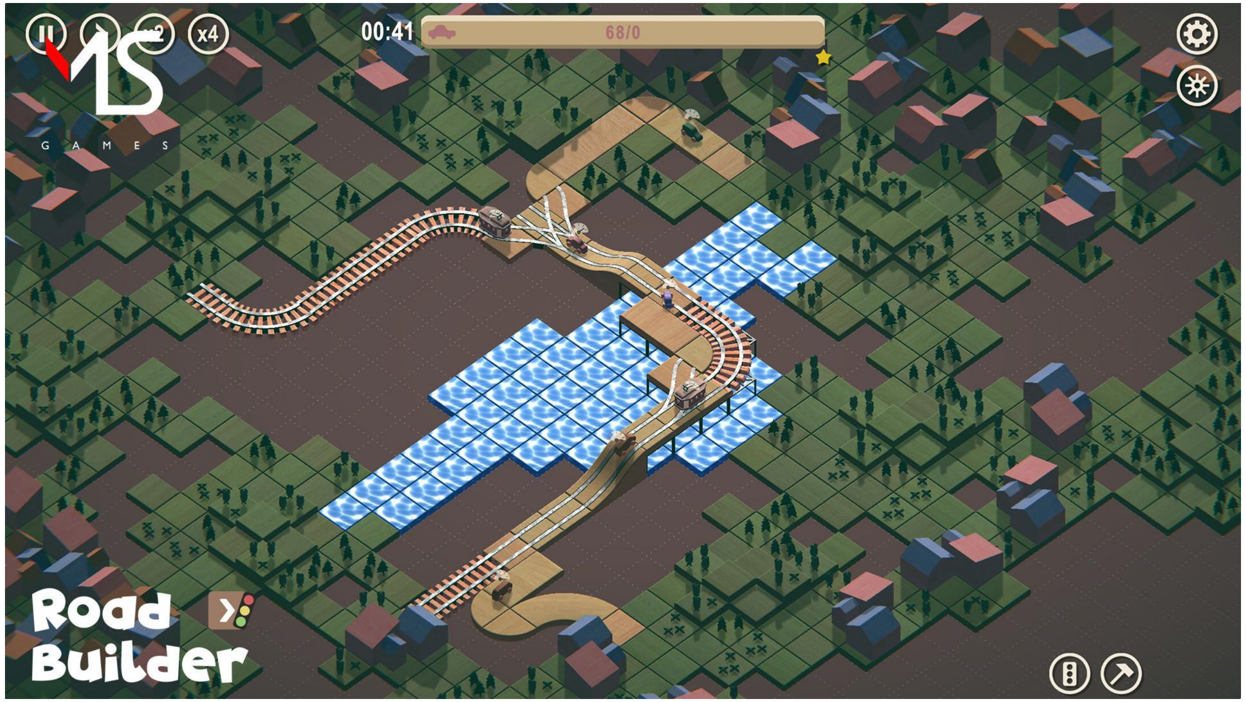
CURVES

Lecture Video Link

Link

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x4
Road
Builder


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Curve

- Intuition: A set of points drawn with a pen

- Mapping from time to place
- $f(t) = (x, y)$ for $t \in [0,1]$

Curve Representations (Types)

1. Explicit Representation
2. Implicit Representation
3. Parametric Representation
4. Subdivision Representation
5. Procedural Representation

Curve Representations (Overview)

- There are three ways to represent a curve
  - Explicit: y = f(x)

    $y = mx + b$ **$y = x^2$**

    (–) Must be a single valued function

    (–) Vertical lines, say x = d? No way to represent using single valued function
  - Implicit: f(x,y) = 0

    $x^2 + y^2 - r^2 = 0$

    (+) **y** can be multiple valued function of x

    (–) Continuity hard to detect
  - Parametric: (x, y) = (x(t), y(t))

    (x, y) = (cos*t*, sin*t*)

    (+) Easy to specify, modify and control

    (–) Extra hidden variable t, the parameter, non intuitive

Parametric Representation

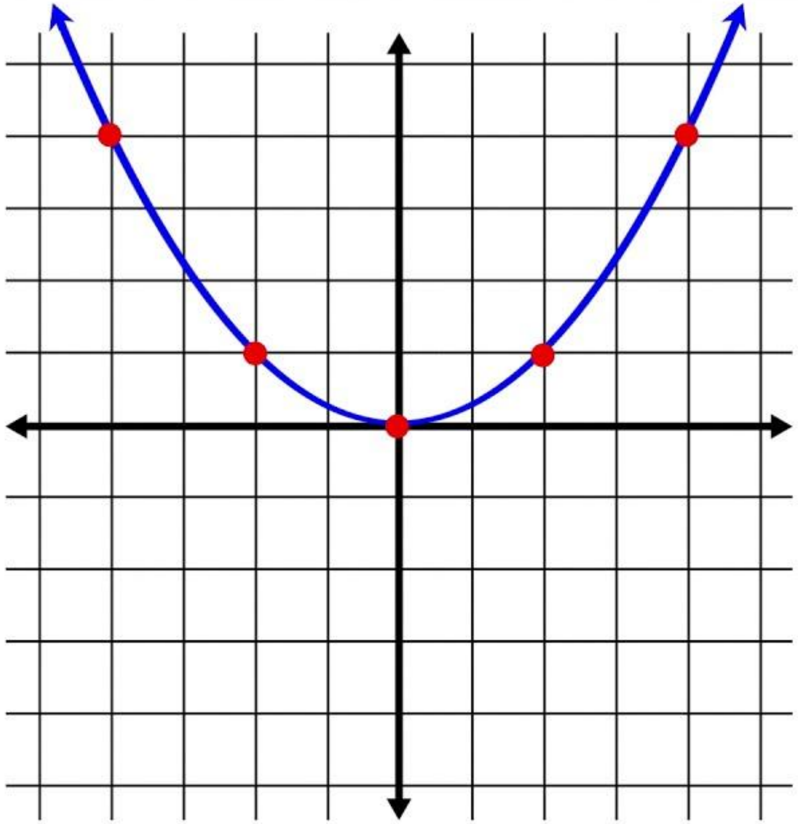
- Curves: single parameter t (e.g. time)
  - *x = x(t), y = y(t), z = z(t)*
- Circle:
  - *x = cos(t), y = sin(t), z = 0*
- Tangent described by derivative

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \qquad \frac{dp(t)}{dt} = \begin{bmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{bmatrix}$$

- Magnitude is “velocity”

$$f(t) = (2t, t^2)$$

Understanding Parametric Equations



$$\boxed{x = 2t \qquad y = t^2}$$

t	x	y
-2	-4	4
-1	-2	1
0	0	0
1	2	1
2	4	4

Same curve, different parameterization

- $f(t) = (t, 0)$
- $t \in [0, 1]$

$t = 0$ $t = 1$

- $f(t) = (1 - t, 0)$
- $t \in [0, 1]$

$t = 1$ $t = 0$

Parametric Curve Visualization

- 2D Curve Visualization
- https://www.geogebra.org/m/cAsHbXEU

- 3D Curve Visualization
- https://christopherchudzicki.github.io/MathBox-Demos/parametric_curves_3D.html

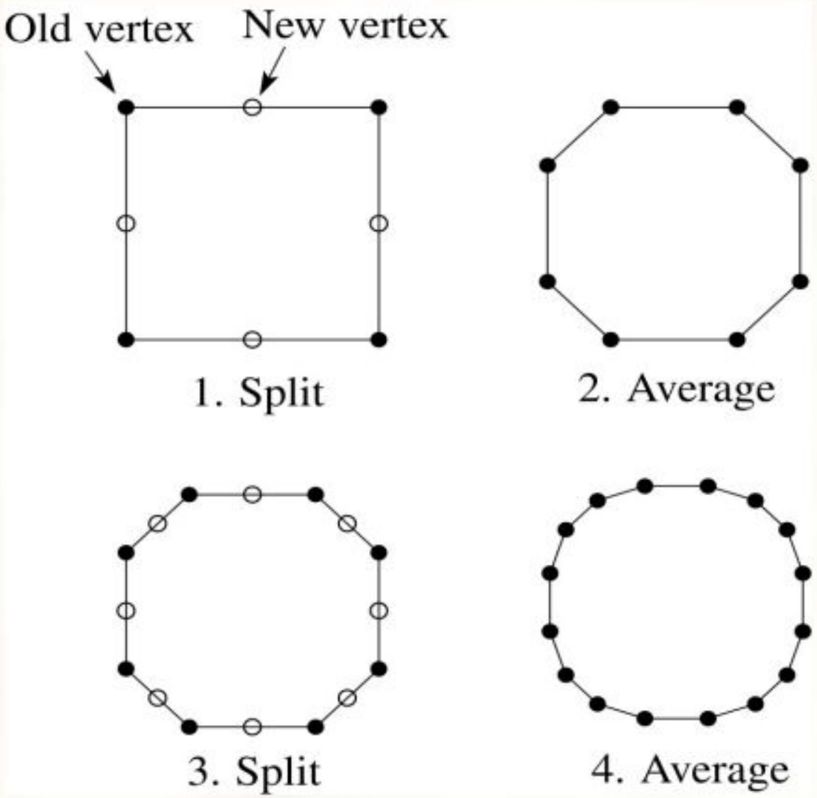
Subdivision Representation

- Start with a set of points
- Have a rule that adds points [possibly removing old ones]
- Repeat the rule

- Each subdivision makes the set of points closer to the intended curve

Example

- Start with a piecewise linear curve
- Insert new vertices at the midpoints (the **splitting step**)
- Average each vertex with the "next" (clockwise) neighbor (the **averaging step**)
- Go to the splitting step



Lagrange Polynomial

- Given n+1 points (x_0, y_0), (x_1, y_1) …….. (x_n, y_n)
- To construct a curve that passes through these points we can use Lagrange polynomial defined as follows:.

$$y = f(x) = \sum_{k=0}^{n} y_k L_{n,k}$$

$$L_{n,k} = \frac{(x - x_o)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_o)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}$$

Lagrange Polynomial (Example)

- Let's consider the following data points:

$$
\begin{aligned}
(x_0, y_0) &= (0, 2) \\
(x_1, y_1) &= (1, 3) \\
(x_2, y_2) &= (2, 5) \\
(x_3, y_3) &= (3, 10) \\
(x_4, y_4) &= (4, 20)
\end{aligned}
$$

Step 1: Define Each Basis Polynomial $L_j(x)$

Each basis polynomial $L_j(x)$ is defined as:

$$L_j(x) = \prod_{\substack{0 \le m \le 4 \\ m \ne j}} \frac{x - x_m}{x_j - x_m}$$

$L_0(x)$

$$L_0(x) = \frac{(x-1)(x-2)(x-3)(x-4)}{(0-1)(0-2)(0-3)(0-4)}$$

$$= \frac{(x-1)(x-2)(x-3)(x-4)}{(-1)(-2)(-3)(-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$L_1(x)$

$$L_1(x) = \frac{(x-0)(x-2)(x-3)(x-4)}{(1-0)(1-2)(1-3)(1-4)}$$
$$= \frac{x(x-2)(x-3)(x-4)}{(1)(-1)(-2)(-3)} = \frac{x(x-2)(x-3)(x-4)}{-6}$$

$L_2(x)$

$$L_2(x) = \frac{(x-0)(x-1)(x-3)(x-4)}{(2-0)(2-1)(2-3)(2-4)}$$
$$= \frac{x(x-1)(x-3)(x-4)}{(2)(1)(-1)(-2)} = \frac{x(x-1)(x-3)(x-4)}{4}$$

$L_3(x)$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)(x-4)}{(3-0)(3-1)(3-2)(3-4)}$$

$$= \frac{x(x-1)(x-2)(x-4)}{(3)(2)(1)(-1)} = \frac{-x(x-1)(x-2)(x-4)}{6}$$

$L_4(x)$

$$L_4(x) = \frac{(x-0)(x-1)(x-2)(x-3)}{(4-0)(4-1)(4-2)(4-3)}$$

$$= \frac{x(x-1)(x-2)(x-3)}{(4)(3)(2)(1)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

Step 2: Multiply Each $L_j(x)$ by y_j

Now build the full polynomial:

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x)$$

Plug in the y_j values:

$$P(x) = 2 \cdot L_0(x) + 3 \cdot L_1(x) + 5 \cdot L_2(x) + 10 \cdot L_3(x) + 20 \cdot L_4(x)$$

That is:

$$P(x) = 2 \cdot \frac{(x-1)(x-2)(x-3)(x-4)}{24} + 3 \cdot \frac{x(x-2)(x-3)(x-4)}{-6} + 5 \cdot \frac{x(x-1)(x-3)(x-4)}{4}$$

$$+10 \cdot \frac{-x(x-1)(x-2)(x-4)}{6} + 20 \cdot \frac{x(x-1)(x-2)(x-3)}{24}$$

Lagrange Polynomial (Problems)

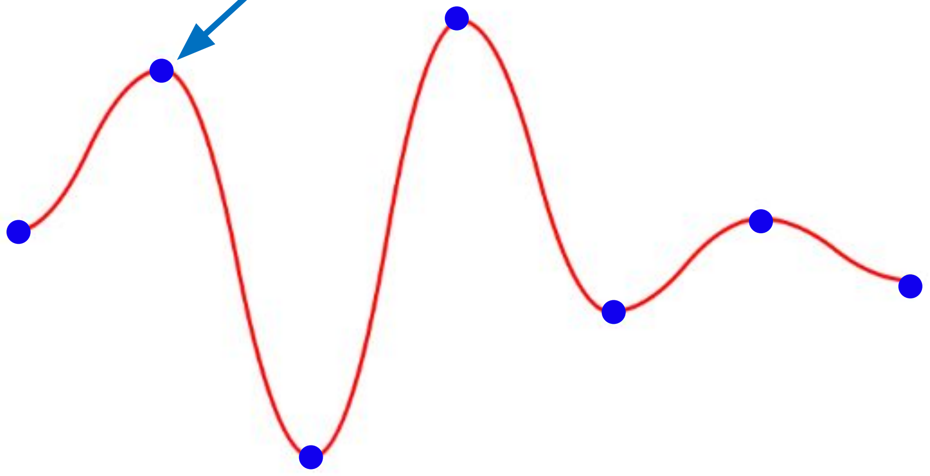
Problems:

- y=f(x), no multiple values
- Higher order functions tend to oscillate
- No local control (change any (xi, yi) changes the whole curve)
- Computationally expensive due to high degree.

How to draw complex curves

- Break them into “manageable” smaller curves
- These smaller curves are called **splines**

What happens on the joints?



Solution: Piecewise Linear Polynomial

- To overcome the problems with Lagrange polynomial
  - Divide given points into overlap sequences of 4 points
  - Construct **3rd degree** polynomial that passes through these points, p_0, p_1 , p_2 , p_3 then p_3, p_4 , p_5 , p_6 etc.
  - Then glue the curves so that they appear **sufficiently smooth** at joint points.

Piecewise Linear Polynomial

Questions:

- **Why 3rd Degree curves used?**
- **How to measure smoothness at joint point?**

Why Cubic Curves

Cubic polynomials are most often used for piecewise because:

(1) Lower-degree polynomials offer too little flexibility in controlling the shape of the curve.

(2) Higher-degree polynomials can introduce unwanted wiggles and also require more computation.

Why Cubic Curves

(3) No lower-degree representation allows a curve segment to be defined by two given endpoints with given derivative at each endpoints.

(4) No lower-degree curves are non planar in 3D.

Piecewise Linear Polynomial

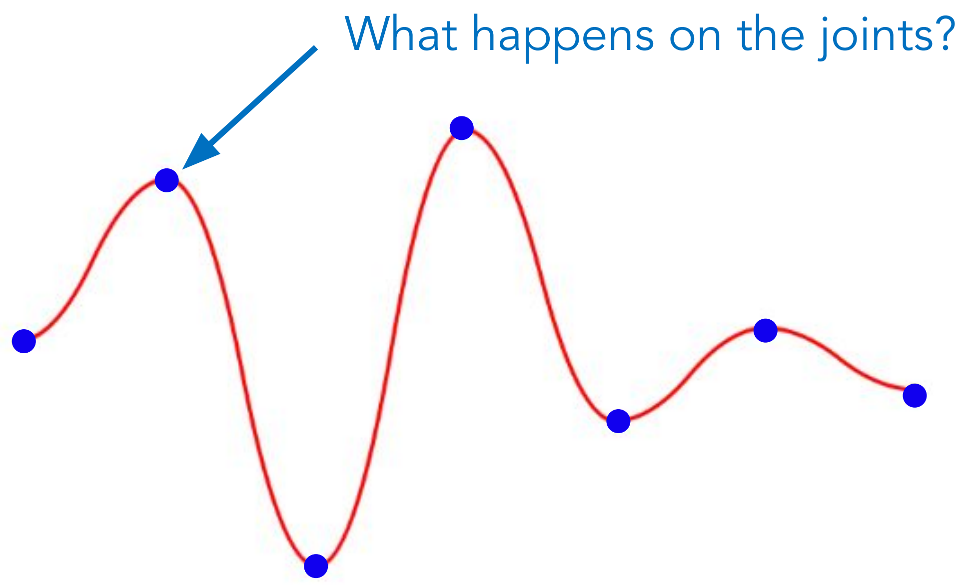
Questions:

- **Why 3rd Degree curves used?**
- **How to measure smoothness at joint point?**

Introducing Continuity

- AKA "Smoothness"
- Is there any gaps, sudden turns?

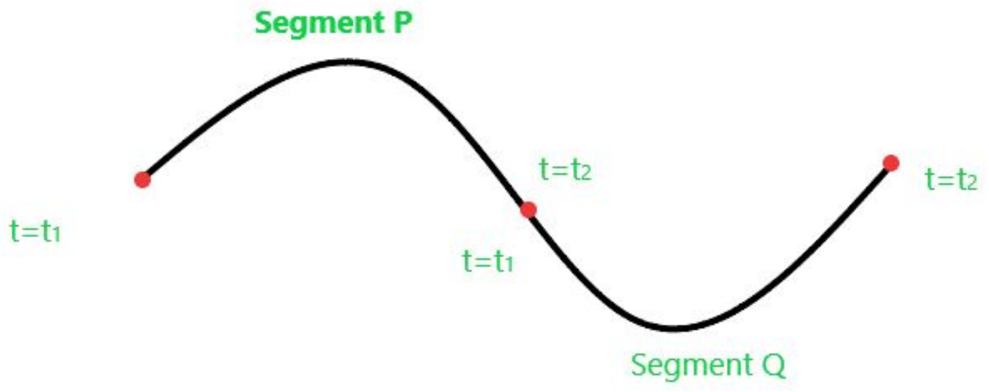
- Not same as "not wiggly"
- A curve can be wiggly but still smooth.



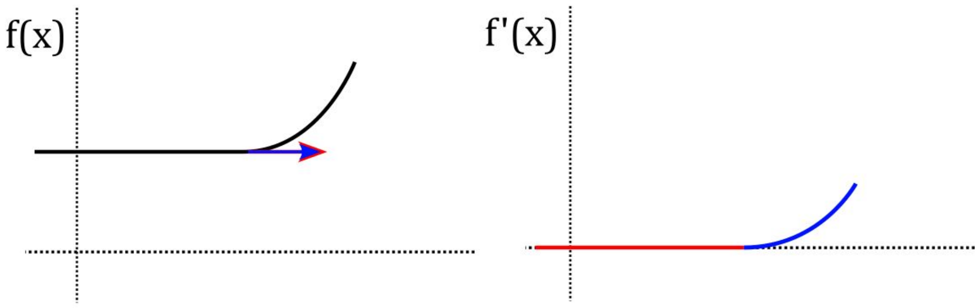
Continuity

- Are the points next to each other?
- Can we draw without lifting the pen?

- Approach a point from left and right, do we get the same point?

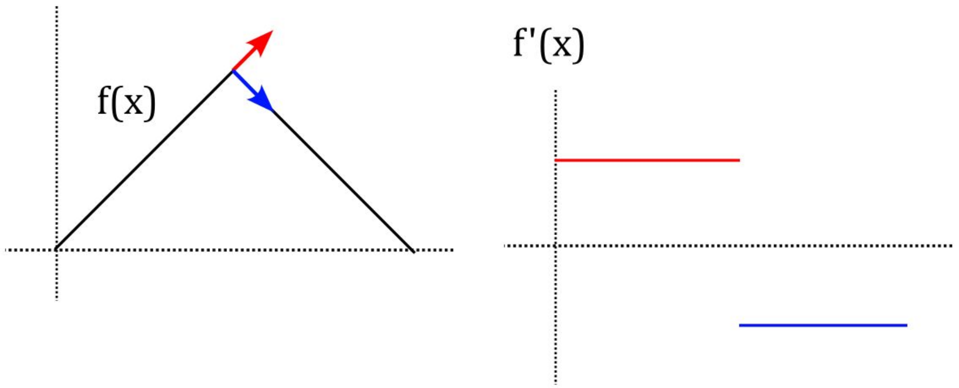


Continuity in Direction



First derivative same at joint

First derivative changes abruptly at joint



Derivative of a Curve

◦ $$f(t) = \left(f_x(t), f_y(t)\right)$$

$$f'(t) = \frac{\partial}{\partial t} f(t) = \left(\frac{\partial}{\partial t} f_x(t), \frac{\partial}{\partial t} f_y(t)\right)$$

◦ A velocity vector, tangent to the curve

Discontinuity Example

Piecewise line segments:

```
f(u) = if u<.5 then (u,0) else (u,1)
or
f(u) = (u<.5) ? (u,0) : (u,1)
```

Position discontinuity at `u=.5`

Discontinuity Example

Piecewise line segments:

```
f(u) = if u<.5 then (u,u) else (u,.5)
```

Tangent (first derivative) discontinuity at `u=.5`

Note: discontinuities happen when we switch

Parametric Continuity, C(n)

We say a curve is $C(n)$ continuous

If all its derivatives up to (and including) n are continuous

$C(0)$ – position

$C(1)$ – position and tangent (1st derivative)

$C(2)$ - position, tangent and 2nd derivative

How much continuity do we need

$C(0)$ – no gaps

$C(1)$ – no corners

$C(2)$ - looks smooth

Higher…

Important for airflow (airplane, car, boat design)

Important for reflections

Geometric Continuity, G(n)

Speed Matters?

```
f(u) = if u<0.5 then (u,0) else (2u-0.5,0)
```

It's a horizontal line

The pen doesn't change direction

It does change "speed" at the point

C and G continuity

$C(n)$ continuity – all derivatives up to n match

$G(n)$ continuity – direction of all derivatives up to n match

Measure of Smoothness

G^0 Geometric Continuity $\Leftrightarrow$ C^0 Parametric Continuity

If two curve segments join together.

G^1 Geometric Continuity

If the **directions** (but not necessarily the magnitudes) of the two segments' tangent vectors are equal at a join point.

C^1 Parametric Continuity

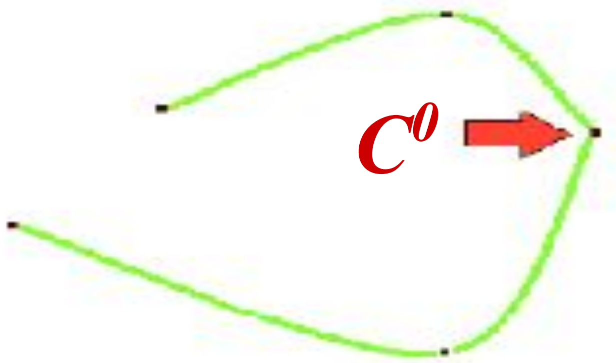
If the **directions and magnitudes** of the two segments' tangent vectors are equal at a join point.

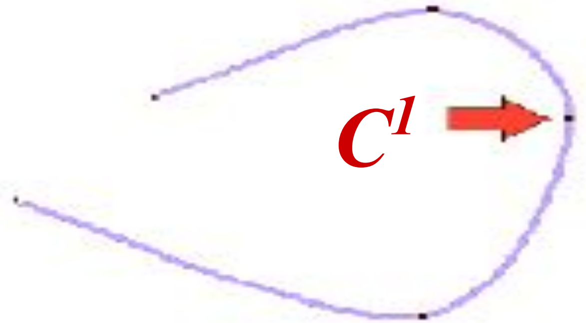
C^2 Parametric Continuity

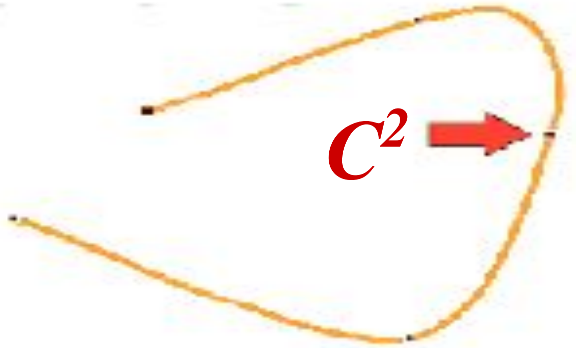
If the direction and magnitude of $Q^2(t)$ (curvature or **acceleration**) are equal at the join point.

C^n Parametric Continuity

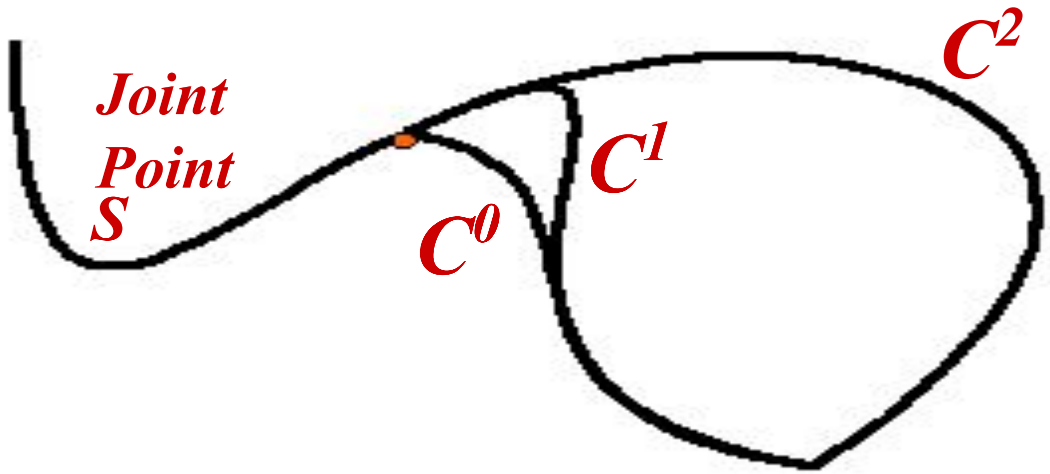
If the direction and magnitude of $Q^n(t)$ through the nth derivative are equal at the join point.





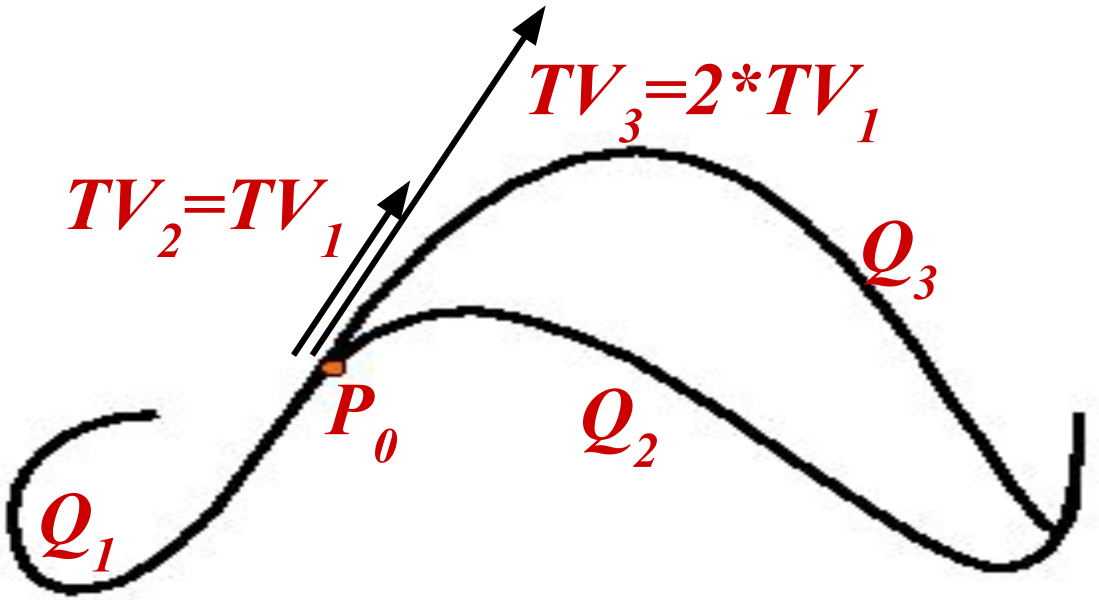


Measure of Smoothness



- By increasing parametric continuity we can increase smoothness of the curve.

- Q_1 & Q_2 are C^1 and G^1 continuous
- Q_1 & Q_3 are G^1 continuous only as Tangent vectors have different magnitude.
- Observe the effect of increasing in magnitude of TV



Line Segment Parameterization

- $f(t) = P_0 + t(P_1 - P_0)$
- $f(t) = \boldsymbol{a_0} + \boldsymbol{a_1} t$

- Two info to draw a line segment
  - Start point
  - End point / Velocity Vector

Linear Interpolation (or, Lerp)

- $$lerp(P_0, P_1, t) = P_0 + t(P_1 - P_0)$$

- Will be useful later

Cubic Curves

- $f(t) = \boldsymbol{a_0} + \boldsymbol{a_1}t + \boldsymbol{a_2}t^2 + \boldsymbol{a_3}t^3$
- Convenient enough to draw, complex enough to draw a wiggly shape

- Four info to draw a curve
  - Starting point, $f(0) = a_0$
  - Ending point, $f(1) = a_0 + a_1 + a_2 + a_3$
  - Direction at starting point, $f'(0) = a_1$
  - Direction at ending point, $f'(1) = a_1 + 2a_2 + 3a_3$

Desirable Properties of a Curve

- Simple control
  - lines need only two points
  - curves will need more (but not significantly more)
- Intuitive control
  - Physically meaningful quantities like position, tangent, curvature etc.
- Global Vs. Local Control
  - Portion of curve affected by a control point.
- General Parameterization
  - Handle multi-valued x-y mapping

BÉZIER CURVE

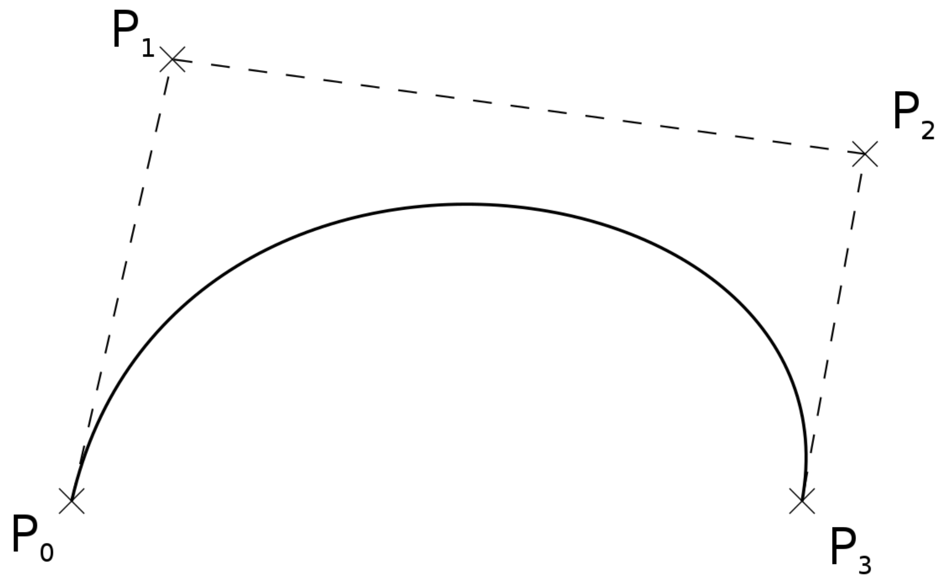
History

- Named after French engineer Pierre Bézier (1910–1999),
- First used it in the 1960s for designing curves for the bodywork of Renault cars.

Pierre Étienne Bézier was a French engineer and one of the founders of the fields of solid, geometric and physical modelling as well as in the field of representing curves, especially in computer-aided design and manufacturing systems. As an engineer at Renault, he becan

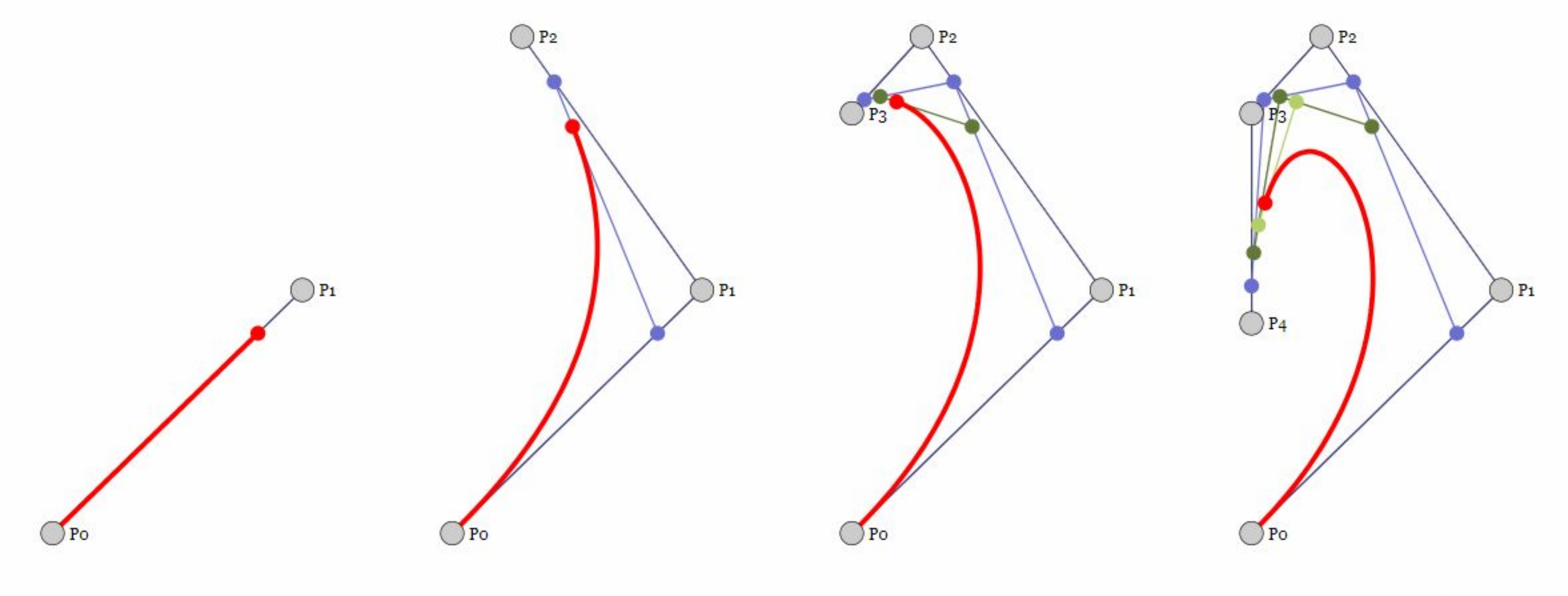
Bézier Curve

- Will have some control points
- Move around control points to change shape of the curve



- For example: 4 control points to build a cubic Bézier curve

Family of Bézier Curves



Linear Quadratic Cubic Quartic

Bézier Curve Demonstration

- Single Curve Graph
- https://www.desmos.com/calculator/d1ofwre0fr

- Family of Bezier Curve Animation
- https://www.jasondavies.com/animated-bezier/

How to draw Bézier Curves

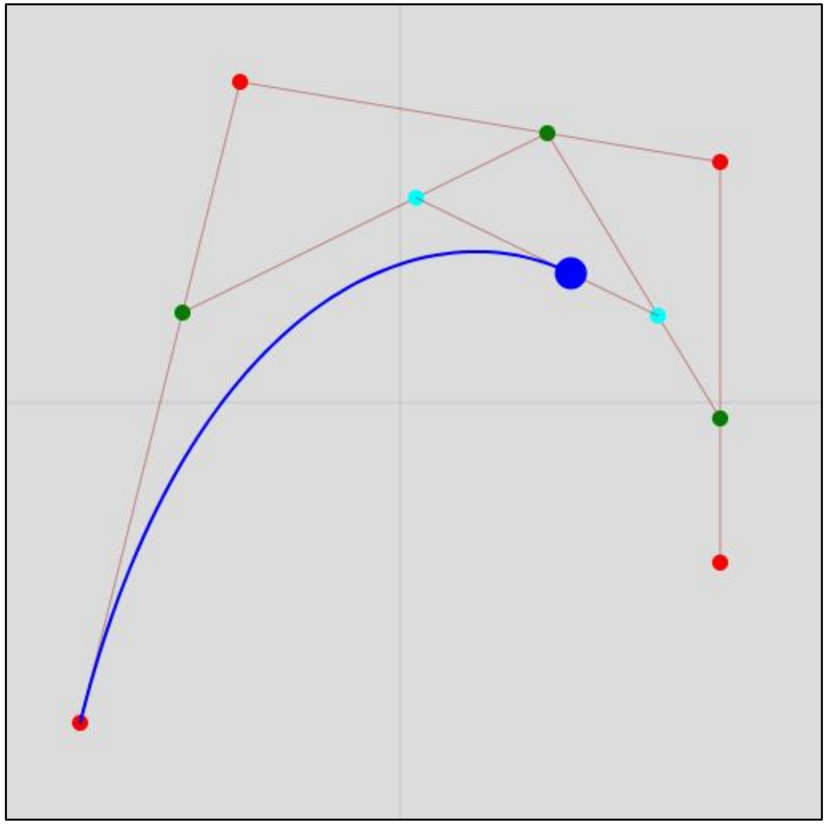
- Cubic Bezier Curve Code
- https://editor.p5js.org/shonku/sketches/vonrqpOhd

Draw Cubic Bézier Curves

```
Given 4 control points p1, p2, p3, p4
For t = 0 to 1 in proper interval:
  q1 = lerp(p1, p2, t)
  q2 = lerp(p2, p3, t)
  q3 = lerp(p3, p4, t)

  r1 = lerp(q1, q2, t)
  r2 = lerp(q2, q3, t)

  s = lerp(r1, r2, t)
  Draw(s)
```



Draw Bézier curves

- Process of recursively finding a point on a Bézier curve has a name
- “De Casteljau’s Algorithm”

Equation of a Cubic Bézier Curve

$s = r1 + (r2 - r2)t$

$r1 = q1 + (q2 - q1)t$

$r2 = q2 + (q3 - q2)t$

$q1 = p1 + (p2 - p1)t$

$q2 = p2 + (p3 - p2)t$

$q3 = p3 + (p4 - p3)t$

$$
\begin{aligned}
s &= r1 + (r2 - r1)t \\
&= q1 + (q2 - q1)t + (q2 + (q3 - q2)t - q1 - (q2 - q1)t)t \\
&= q1 + tq2 - tq1 + (q2 + tq3 - tq2 - q1 - tq2 + tq1)t \\
&= q1 + tq2 - tq1 + tq2 + t^2q3 - t^2q2 - tq1 - t^2q2 + t^2q1 \\
&= q1(1 - 2t + t^2) + q2(2t - 2t^2) + q3(t^2) \\
&= (p1 + (p2 - p1)t)(1 - 2t + t^2) + (p2 + (p3 - p2)t)(2t - 2t^2) + (p3 + (p4 - p3)t)(t^2) \\
&= (p1 + tp2 - tp1)(1 - 2t + t^2) + (p2 + tp3 - tp2)(2t - 2t^2) + (p3 + tp4 - tp3)(t^2) \\
&= (p1 - 2tp1 + t^2p1 + tp2 - 2t^2p2 + t^3p2 - tp1 + 2t^2p1 - t^3p1) + (2tp2 - 2t^2p2 + 2t^2p3 - 2t^3p3 - 2t^2p2 + 2t^3p2) + (t^2p3 + t^3p4 - t^3p3) \\
&= (1 - 3t + 3t^2 - t^3)p1 + (3t - 6t^2 + 3t^3)p2 + (3t^2 - 3t^3)p3 + t^3p4 \\
&= (1 - t)^3\, p1 + 3t(1 - t)^2\, p2 + 3t^2(1 - t)\, p3 + t^3\, p4
\end{aligned}
$$

Matrix Eq^n of a Cubic Bézier Curve

Dimensions

- $Q(t) = (1-3t+3t^2-t^3)p1 + (3t-6t^2+3t^3)p2 + (3t^2-3t^3)p3 + t^3p4$ [1]

$$Q(t) = \begin{bmatrix}(1-3t+3t^2-t^3) & (3t-6t^2+3t^3) & (3t^2-3t^3) & t^3\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\\P_4\end{bmatrix}$$

[1x4] [4x1]

$$Q(t) = \begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}-1 & 3 & -3 & 1\\3 & -6 & 3 & 0\\-3 & 3 & 0 & 0\\1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}P_1\\P_2\\P_3\\P_4\end{bmatrix}$$

[1x4][4x4][4x1]

$$Q(t) = T \cdot M_B \cdot G_B$$

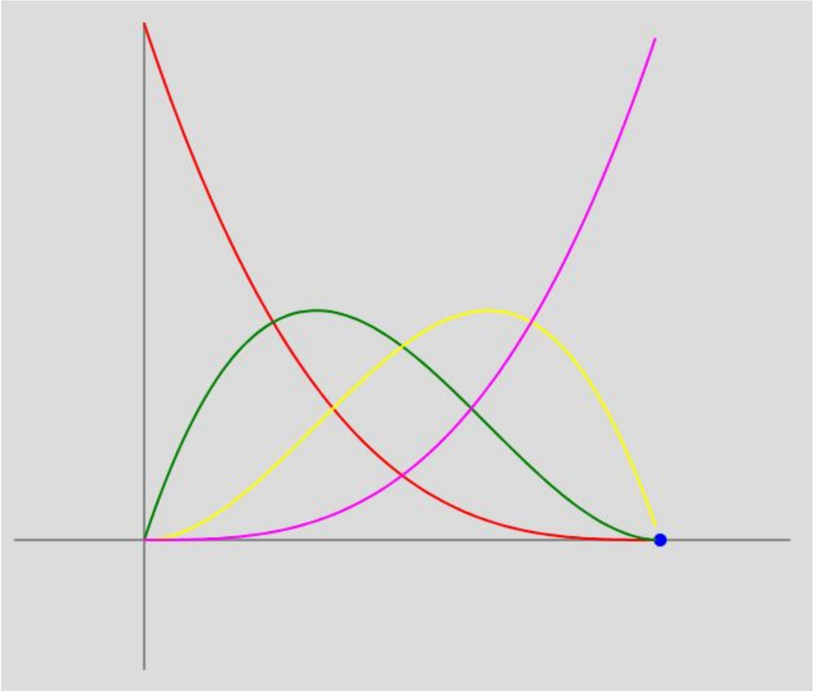
M_B: The basis Matrix of Bézier Curve

G_B: Geometric properties of Bézier Curve

The Bernstein Polynomials

- How much a point contributes to a certain position?

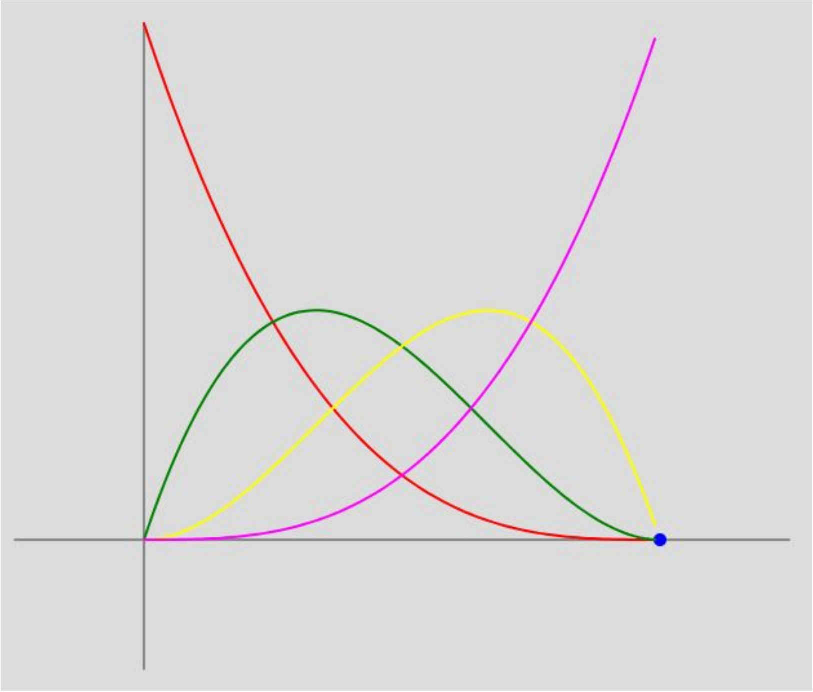
- $Q(t) = (1-3t+3t^2-t^3)p1 + (3t-6t^2+3t^3)p2 + (3t^2-3t^3)p3 + t^3p4$



The Bernstein Polynomials

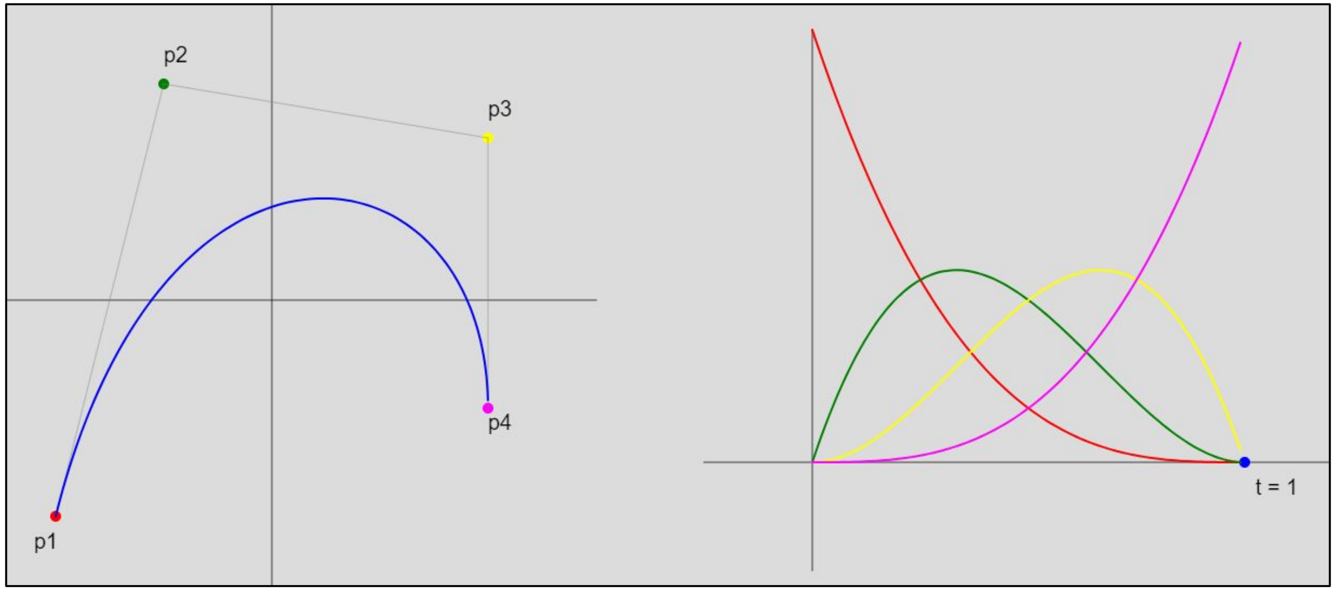
- $Q(t) = (1-3t+3t^2-t^3)p1 + (3t-6t^2+3t^3)p2 + (3t^2-3t^3)p3 + t^3p4$

- Sum of polynomials = 1
- So, a point on the curve is a ***convex combination*** of the control points



The Bernstein Polynomials

- Different Bernstein polynomial values for different values of t:
- https://editor.p5js.org/shonku/sketches/9dlzE16jW



Convex Hull Property

Convex combination of n points

$$P_1\lambda_1 + P_1\lambda_1 + \cdots + P_n\lambda_n$$

Where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$

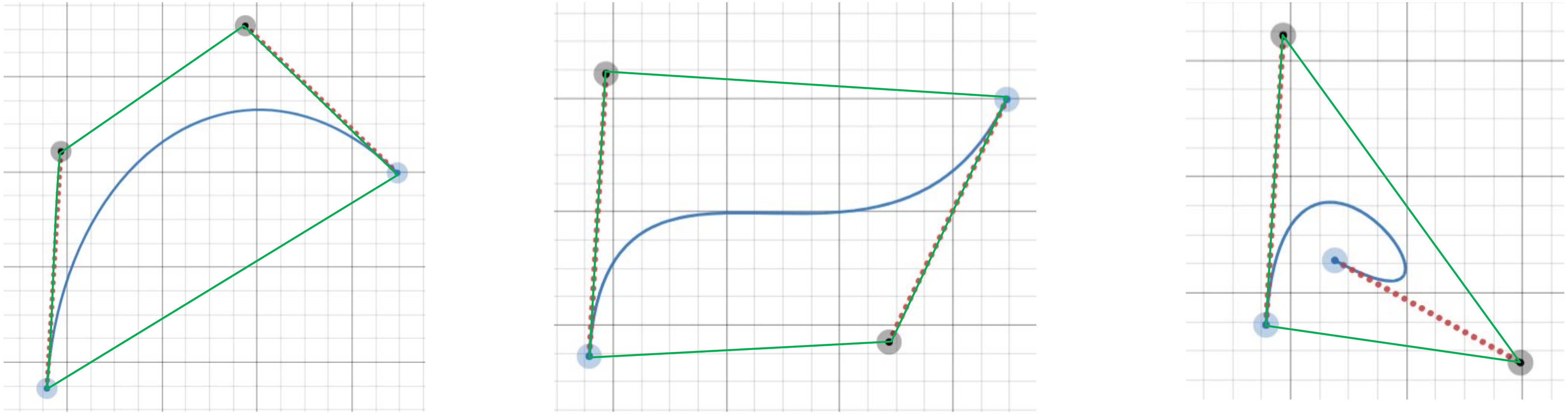
Any convex combination will lie inside the convex hull of the n points.

Not need for this course:
https://math.stackexchange.com/questions/229354/proof-that-the-convex-hull-of-a-finite-set-s-is-equal-to-all-convex-combinations

Convex Hull Property

As, $Q(t)$ is a convex combination of P_1, P_2, P_3 and P_4

Cubic Bézier curve must be bounded by convex hull of P_1, P_2, P_3 and P_4



Derivatives

Position: $Q(t) = (1 - 3t + 3t^2 - t^3)P_1 + (3t - 6t^2 + 3t^3)P_2 + (3t^2 - 3t^3)P_3 + t^3 P_4$

Velocity $Q'(t) = (-3 + 6t - 3t^2)P_1 + (3 - 12t + 9t^2)P_2 + (6t - 9t^2)P_3 + 3t^2 P_4$

Accelaration: $Q''(t) = (6 - 6t)P_1 + (-12 + 18t)P_2 + (6 - 18t)P_3 + 6tP_4$

Jolt: $Q'''(t) = (-6)P_1 + (18)P_2 + (-18)P_3 + 6P_4$

https://editor.p5js.org/shonku/sketches/XFv3T4ita

Derivatives

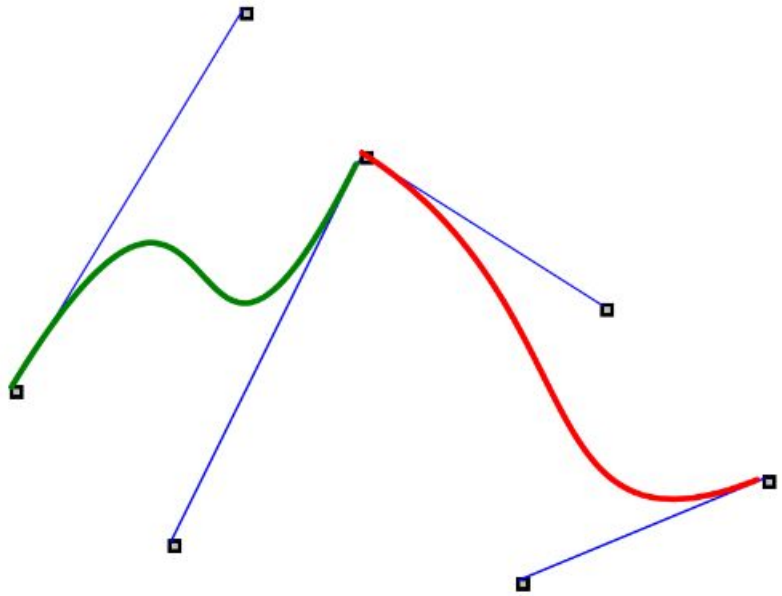
Velocity at $t = 0$ and $t = 1$

$$Q'(0) = (-3)P_1 + (3)P_2 = \mathbf{3(P_2 - P_1)}$$

$$Q'(1) = (-3 + 6 - 3)P_1 + (3 - 12 + 9)P_2 + (6 - 9)P_3 + 3P_4 = \mathbf{3(P_4 - P_3)}$$

Join Two Cubic Bézier Curve

- C(0) continuous
- Not enough

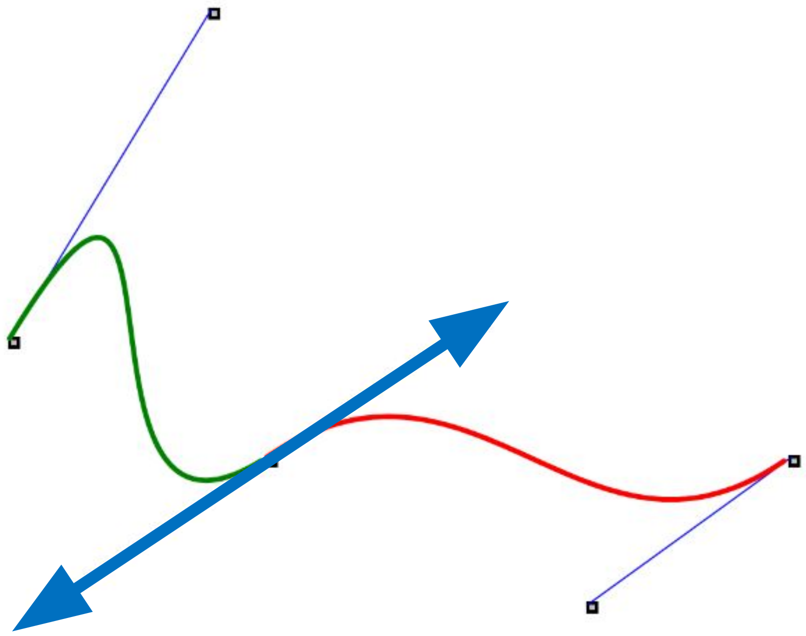


Join Two Cubic Bézier Curve

- How to ensure C(1)?

- See the smoothness in action:
- https://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

- Coding Example:
- https://editor.p5js.org/shonku/sketches/1wWKxEu0j



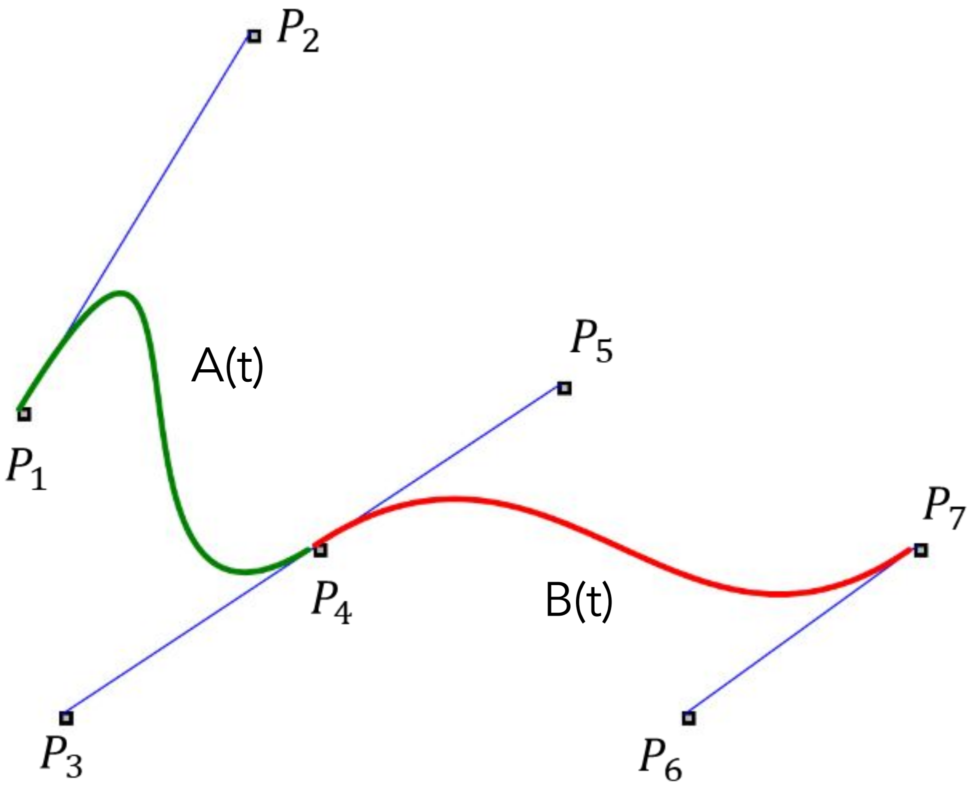
Join Two Cubic Bézier Curve

- For $C(1)$ continuity at point P_4
- Velocity of two curves at the joint must be equal

So, $A'(1) = B'(0)$

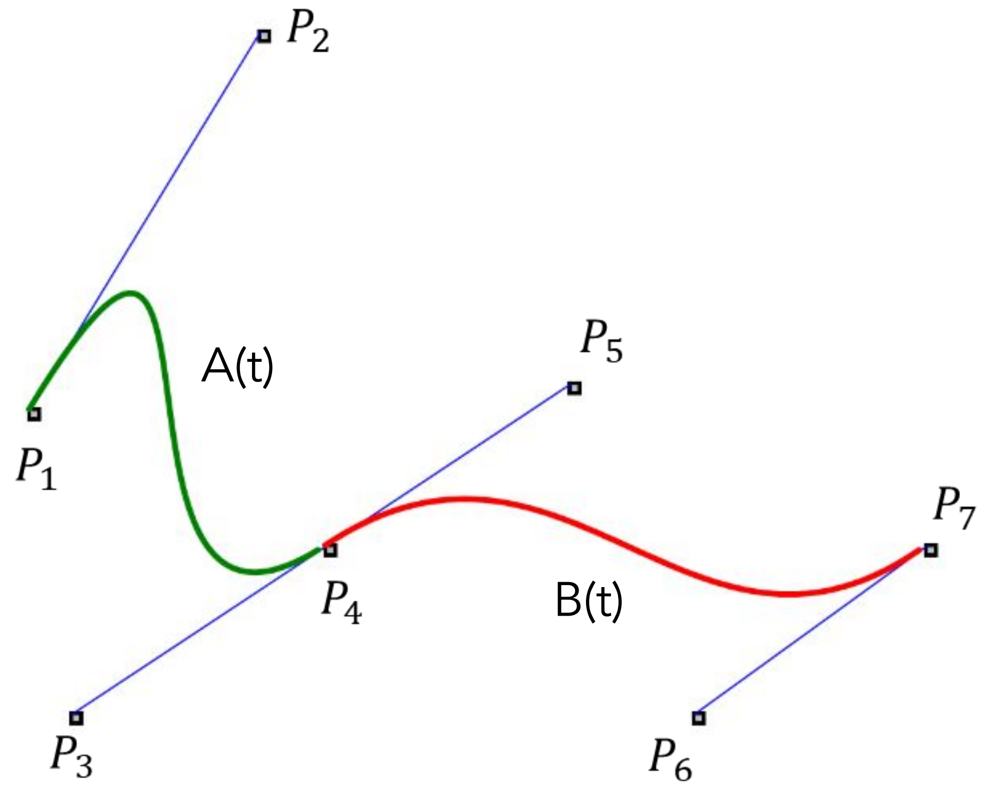
$\Rightarrow 3(P_4 - P_3) = 3(P_5 - P_4)$

$\Rightarrow P_5 = 2P_4 - P_3$



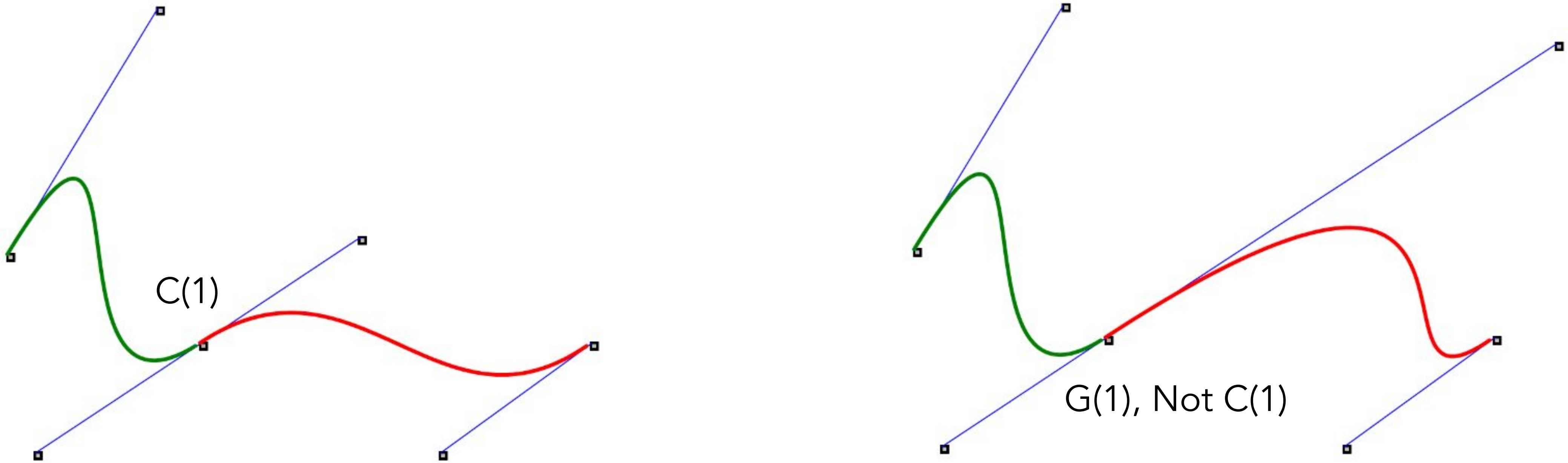
Join Two Cubic Bézier Curve

- $P_5 = 2P_4 - P_3$
- This condition must hold to be C(1) continuous
- Notice that we can no longer control P_5. It is decided by P_4 and P_3



Join Two Cubic Bézier Curve

- Difference of C and G continuity



That’s all

Thanks to…

- Prof. Michael Gleicher for his CS599 course
  https://pages.graphics.cs.wisc.edu/559-sp22/

- https://www.cs.utexas.edu/users/fussell/courses/cs384g-fall2011/lectures/lecture17-Subdivision_curves.pdf

Thanks to…

- Freya Holmér
- “The Beauty of Bézier Curves”
- https://www.youtube.com/watch?v=aVwxzDHniEw
- “The Continuity of Splines”
- https://www.youtube.com/watch?v=jvPPXbo87ds