25. Determine **C(1) and G(1)** continuity of the following functions at the given points:

a. At **t = 2π**,

$$(x(t), y(t)) = \begin{cases} (t, \sin t) & \text{for } t \leq 2\pi \\ (t, 1 - \cos t) & \text{for } t > 2\pi \end{cases}$$

b. At **t = π/4**,

$$(x(t), y(t)) = \begin{cases} (t, \sin t) & \text{if } t \leq \frac{\pi}{4} \\ (t, 1 - \cos t) & \text{if } t > \frac{\pi}{4} \end{cases}$$

c. At **t = 1**,

$$(x(t), y(t)) = \begin{cases} (6t, t^3) & \text{if } t \leq 1 \\ (t^4 + 5, t^2) & \text{if } t > 1 \end{cases}$$

d. At **t = 1**,

$$(x(t), y(t)) = \begin{cases} (t, t^2) & \text{for } t \leq 1 \\ (t, t^2 + (t-1)^3) & \text{for } t > 1 \end{cases}$$

25a) To check for C^1 continuity, first we need to check C^0

At $t = 2\pi$ in $(t, \overset{1st}{\sin t})$ & $(t, \overset{2nd}{1 - \cos t})$

1st function → $(2\pi, 0)$

2nd function → $(2\pi, 1-1) = (2\pi, 0)$ [Equal/same coordinate at $t = 2\pi$ hence C^0] & G^0

1st order derivative → For C^1

$$\frac{d\,(x(t), y(t))}{dt} = \begin{matrix} (1, \cos t) \\ (1, \sin t) \end{matrix}$$

When $t = 2\pi$, $(1, 1)$ $\rightarrow$ Not equal hence not $\underline{\underline{C^1}}$

$(1, 0)$

To check for G^1 continuity: take unit vectors of the 2 velocity vectors & check if equal

$\bar{v}_1 = (1, 1) \rightarrow$ Magnitude $= \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\hat{v}_1 = \frac{1i + 1j}{\sqrt{2}}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$\bar{v}_2 = (1, 0) \rightarrow$ Already a unit vector

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \neq (1, 0)$ Hence not $\underline{\underline{G^1}}$

b) Same piecewise functions, as part $\underline{a}$

$\underline{\text{For } C^0}$ At $t = \pi/4$ $\left(\pi/4, \frac{\sqrt{2}}{2}\right)$

$\left(\pi/4, 1 - \frac{\sqrt{2}}{2}\right) = \left(\frac{\pi}{4}, \frac{2-\sqrt{2}}{2}\right)$

Not the same hence $\underline{\text{not}}$ $\underline{\underline{C^0}}$

If not C^0 then it $\underline{\underline{\text{cannot be } C^1 \text{ continuous}}}$

If not C^0 then $\underline{\underline{\text{not}}}$ G^0 hence $\underline{\underline{\text{not } G^1}}$

c) For C^0,

$$(x(t), y(t)) = \begin{cases} (6t, t^3) & \text{if } t \leq 1 \\ (t^4+5, t^2) & t > 1 \end{cases}$$

At $t=1$,

1st function = $(6, 1)$

2nd function = $(1^4+5, 1^2) = (6, 1)$ Equal hence C^0 & G^0

For C^1,

$$\frac{dx(t), dy(t)}{dt} = \begin{cases} (6, 3t^2) \\ (4t^3, 2t) \end{cases}$$

At $t=1$,

1st function = $(6, 3)$ Not equal hence not C^1

2nd function = $(4, 2)$

For G^1,

$v_1 = (6, 3)$

$$\hat{v}_1 = \frac{(6,3)}{\sqrt{6^2+3^2}} = \frac{6i+3j}{3\sqrt{5}} = \frac{3(2i+j)}{3\sqrt{5}}$$

$$\hat{v}_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

$v_2 = (4, 2)$

$$\hat{v}_2 = \frac{(4,2)}{\sqrt{4^2+2^2}} = \frac{4i+2j}{2\sqrt{5}} = \frac{2(2i+j)}{2\sqrt{5}}$$

$$\hat{v}_2 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

Equal! Hence G^1

d) $$(x(t), y(t)) = \begin{cases} (t, t^2) & t \leq 1 \\ (t, t^2 + (t-1)^3) & t > 1 \end{cases}$$

For C^0

At $t = 1$

1st function = $(1, 1)$

2nd function = $(1, 1^2 + (1-1)^3) = (1, 1)$

Equal! Hence C^0 & G^0

For C^1

$$\frac{d\,(x(t),\ y(t))}{dt} = \begin{Bmatrix} (1, 2t) \\ (1, 2t + 3(t-1)^2 \end{Bmatrix}$$

At $t = 1$

1st function = $(1, 2)$

2nd function = $(1, 2 + 3(1-1)^2) = (1, 2)$

Equal! Hence C^1 & if C^1 then automatically G^1

29. Given the four control points in 3D:

$P0 = (0,0,0)$, $P1 = (3,6,0)$, $P2 = (6,6,6)$, $P3 = (9,0,6)$

Find the point on the **cubic Bézier** curve at **t = 0.5**.

Let's use the matrix notation:

$$Q(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P0 \\ P1 \\ P2 \\ P3 \end{bmatrix}$$

At $t = 0.5$

$$= \begin{bmatrix} 0.125 & 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Multiply

$$= \begin{bmatrix} 0.125 & 0.375 & 0.375 & 0.125 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= 0.125 P_0 + 0.375 P_1 + 0.375 P_2 + 0.125 P_3$$

$$= 0.125(0,0,0) + 0.375(3,6,0) + 0.375(6,6,6) + 0.125(9,0,6)$$

$$= (0 + 1.125 + 2.25 + 1.125,\ 0 + 2.25 + 2.25 + 0,\ 0 + 0 + 2.25 + 0.75)$$

$$= (4.5, 4.5, 3)$$

31. Given the first three control points of a cubic **Bézier** curve:

P0 = (2, 1), P1 = (3, 4), P2 = (5, 6)

and the point on the curve at t = 0.5:

f(0.5) = (4, 5)

Find the **fourth control point**, P3 = (x3, y3).

$$f(t) = (1-3t+3t^2-t^3)P_0 + (3t-6t^2+3t^3)P_1 + (3t^2-3t^3)P2 + t^3 P_3$$

$\therefore$ When $t = 0.5$,

$f(0.5) = (4,5)$ Substituting $t=0.5$ here

$$\therefore (4,5) = (0.125)P_0 + (0.375)P_1 + (0.375)P2 + 0.125P_3$$

$$(4,5) = 0.125(2,1) + 0.375(3,4) + 0.375(5,6) + 0.125(x_3, y_3)$$

$\therefore$ Writing the eqn in terms of x:

$$4 = (0.125 \times 2) + (0.375 \times 3) + (0.375 \times 5) + 0.125x_3$$

$$x_3 = \frac{0.75}{0.125} = \underline{6}$$

$\therefore$ Writing the eqn in terms of y:

$$5 = (0.125 \times 1) + (0.375 \times 4) + (0.375 \times 6) + 0.125y_3$$

$$y_3 = \frac{1.125}{0.125} = \underline{9}$$

32. You are going to draw 3 cubic **Bézier** curves joined together to form a single smooth composite curve. You have already decided upon the control points for the first and last Bézier curves:

- First Bézier curve (Curve A):
  A0 = (0, 0), A1 = (1, 2), A2 = (2, 2), A3 = (3, 0)
- Third Bézier curve (Curve C):
  C0 = (6, 0), C1 = (7, -2), C2 = (8, -2), C3 = (9, 0)

You want to insert a **Bézier curve (Curve B)** between them such that the entire 3-curve segment is **C(1) continuous**.

Find the **4 control points**- B0, B1, B2, B3 of the **middle Bézier** curve (Curve B).

For the spline to be C^1 continuous, we have to first ensure C^0 continuity

Hence, all 3 curve segments need to be connected!

Wherever curve A will end, B needs to start from there

$\therefore$ End point of A, A3 = Starting point of B, B0

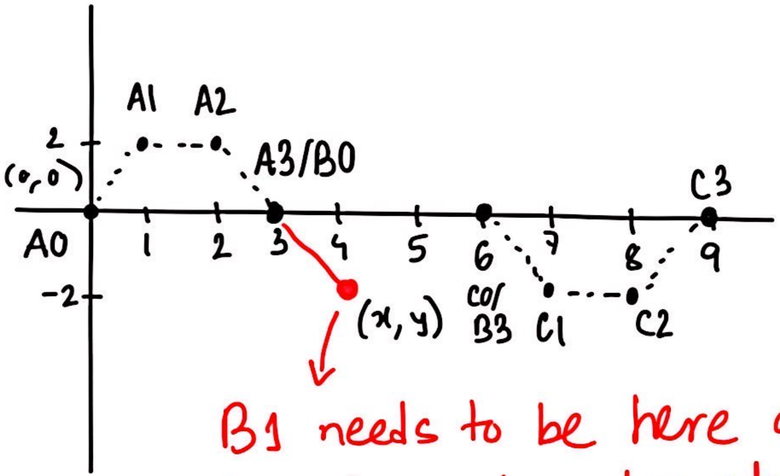
$\therefore$ B0 = (3,0)

Similarly, wherever curve B will end, C will start from there

$\therefore$ End point of B, B3 = Starting point of C, C0

$\therefore$ B3 = (6,0)

For B1 & B2:



B1 needs to be here as to maintain C^1 continuity, the tangent vector at A3 must be equal on both sides.

$\therefore$ A3 is the mid point of the line segment from A2 to B1

$$\therefore (3,0) = \left(\frac{x+2}{2}, \frac{y+2}{2}\right)$$

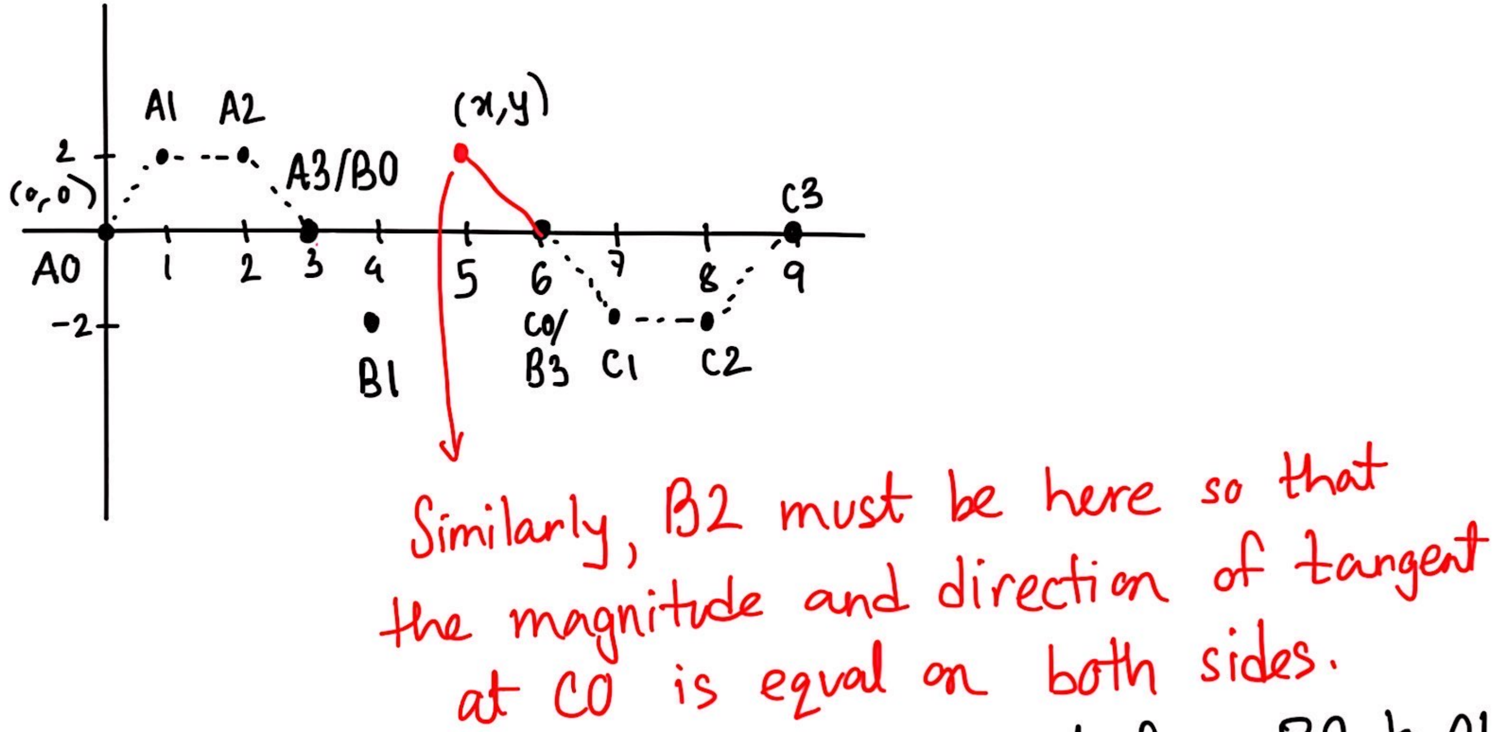
$$\Rightarrow \frac{x+2}{2} = 3$$

$$\therefore x = 6-2 = 4$$

$$\Rightarrow \frac{y+2}{2} = 0$$

$$\therefore y = 0-2 = -2 \qquad \therefore B1\ (4,-2)$$

Similarly,



$\therefore$ C0 is the midpoint of the line segment from B2 to C1

$$\therefore (6,0) = \left(\frac{x+7}{2}, \frac{y-2}{2}\right)$$

$$\Rightarrow \frac{x+7}{2} = 6$$

$$\therefore x = 12-7 = 5$$

$$\Rightarrow \frac{y-2}{2} = 0$$

$$\therefore y = 0+2 = 2$$

$\therefore$ B2 $(5,2)$

[4] Alternate Method to find B1 & B2

We know, for any 2 curve segments to maintain C^1 continuity,

$$P_5 = 2P_4 - P_3$$

Now if we compare the points,

$P_3 = A2$ (3^{rd} control point of A curve)

$P_4 = A_3$ (4^{th} control point of A curve)

$P_5 = B1$ (2^{nd} control point of B curve)

$$\therefore B1 = 2(3,0) - (2,2)$$
$$= (6,0) - (2,2)$$
$$= (4,-2)$$

Similarly, same condition to use for B & C curves' C^1 continuity

$P_5 = 2P_4 - P_3$

$\therefore$ $P_3 = B2$ (3^{rd} control point of B curve)

$P_4 = B3$ (4^{th} control point of B curve)

$P_5 = C1$ (2^{nd} control point of C curve)

$$\therefore C1 = 2(6,0) - (x,y)$$
$$\Rightarrow (7,-2) = (12,0) - (x,y)$$
$$\Rightarrow (x,y) = (12,0) - (7,-2)$$
$$= (12-7, 0+2)$$
$$= (5,2)$$