# ***MAT215*** *COMPLETE PRACTICE SHEET SOLVE*

# Practice Sheet # 1

## Part A

1. Perform each of the indicated operations:

(i) $(i-2)\{2(1+i)-3(i-1)\}$

$= (i-2)\{(2+2i)-3i+3\}$

$= (i-2)(5-i)$

$= 5i - i^2 - 10 + 2i$

$= -i^2 + 7i - 10$

$= 1 + 7i - 10$

$= 7i - 9$

(ii) $\frac{8-i}{1-i}$

$= \frac{(8-i)(1+i)}{1-i^2}$

$= \frac{8+8i-i-i^2}{1+1}$

$= \frac{8+7i+1}{2}$

$= \frac{9+7i}{2}$

(ii) $\frac{(2+i)(3-2i)(1-i)}{(1-i)^2}$

$= \frac{(6-4i+3i-2i^2)(1-i)}{(1-i)^2}$

$= \frac{-2i^2-i+6}{1-i}$

$= \frac{-2(-1)-i+6}{1-i}$ ....

(iii) $(2i-1)^2 \left\{ \frac{4}{1-i} + \frac{2-i}{1+i} \right\}$

$= \left\{ \frac{4+4i+2-2i-i+i^2}{1-i^2} \right\} (2i-1)^2$

$= \frac{5+i}{1+1} \left(4i^2-4i+1\right)$

$= \frac{(5+i)(-4-4i+1)}{2}$

$= \frac{(5+i)(-3-4i)}{2}$

$= \frac{-15-20i-3i-4i^2}{2}$

$= \frac{-15+4-23i}{2}$

$= \frac{-11-23i}{2}$

(iv) $3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$

$= 3\left(\frac{1+2i+i^2}{1-2i+i^2}\right) - 2\,\frac{(1-i)^2\cdot(1-i)}{(1+i)^2\cdot(1+i)}$

$= 3\left(\frac{1+2i-1}{1-2i-1}\right) - 2\left(\frac{1-2i-1}{1+2i-1}\right)\left(\frac{1-i}{1+i}\right)$

$= 3\left(\frac{2i}{-2i}\right) - 2\,\frac{-2i(1-i)}{2i(1+i)}$

$= -3 + 2\,\frac{1-i}{1+i}$

$= \frac{2-2i-3-3i}{1+i} = \frac{-1-5i}{1+i}$

$= \frac{(-1-5i)(1-i)}{1-i^2}$

$= \frac{-1+i-5i+5i^2}{1+1}$

$= \frac{-1-4i-5}{2} = \frac{-6-4i}{2}$

$= -3-2i$

Ans.

(v) $\dfrac{3i^{10} - i^{19}}{2i - 1}$

$= \dfrac{3(i^2)^5 - i^{10} \cdot i^9}{2i - 1}$

$= \dfrac{-3 - (i^2)^5 (i^3)^3}{2i - 1}$

$= \dfrac{-3 + (-i)^3}{2i - 1}$

$= \dfrac{-3 - i^3}{2i - 1}$

$= \dfrac{(-3 + i)(2i + 1)}{(2i - 1)(2i + 1)}$

$= \dfrac{-6i - 3 + 2i^2 + i}{4i^2 - 1}$

$= \dfrac{-5i - 3 - 2}{-4 - 1}$

$= \dfrac{-5i - 5}{-5}$

$= i + 1$

(vi) $\dfrac{i^4 - i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

$= \dfrac{(i^2)^2 - (i^3)^3 + (i^2)^{16}}{2 - i^2 \cdot i^3 + (i^2)^5 - (i^3)^5}$

$= \dfrac{1 - (-i)^3 + 1}{2 - i - 1 - (-i)^5}$

$= \dfrac{1 - i + 1}{1 - i + i^2 \cdot i^3}$

$= \dfrac{2 - i}{1 - i + i}$

$= 2 - i$

1. $|z| = |\bar{z}|$

$i = \sqrt{-1}$

2. $\bar{\bar{z}} = z$

$i^2 = -1$

3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$i^3 = i^2 \cdot i$

$= -1 \cdot i = -i$

4. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$i^4 = i^2 \cdot i^2 = 1$

5. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

6. $z + \bar{z} = 2 \cdot Re(z)$

7. $z \cdot \bar{z} = |z|^2 = x^2 + y^2$

② Show that the followings illustrate the associative low of addition.

(i) $(5+3i)+\{(-1+2i)+(7-5i)\}$

$= (5+3i)+(6-3i)$

$= 11$

(ii) $\{(5+3i)+(-1+2i)\}+(7-5i)$

$= (4+5i)+(7-5i)$

$= 11$

$\therefore$ (i) & (ii) illustrate the associative law of addition.

③ If $z_1 = 1-i$, $z_2 = -2+4i$, $z_3 = \sqrt{3}-2i$, evaluate each of the followings.

(i) $|2z_2 - 3z_1|^2 = |2(-2+4i) - 3(1-i)|^2$

$= |-4+8i-3+3i|^2$

$= |-7+11i|^2$

$= \left(\sqrt{(-7)^2 + (11)^2}\right)^2$

$= \left(\sqrt{170}\right)^2$

$= 170$

(ii) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$

$= \left|\dfrac{1-i-2+4i+1}{1-i+2-4i+i}\right|$

$= \left|\dfrac{3i}{3-4i}\right|$

$= \dfrac{\sqrt{(3)^2}}{\sqrt{3^2+(-4)^2}}$

$= \dfrac{3}{\sqrt{9+16}}$

$= \dfrac{3}{\sqrt{25}}$

$= \dfrac{3}{5}$

(iii) $\overline{(z_2+z_3)(z_1-z_3)}$

$= \overline{(-2+4i+\sqrt{3}-2i)(1-i-\sqrt{3}+2i)}$

$= \overline{(\sqrt{3}-2+2i)(1-\sqrt{3}+i)}$

$= \overline{\sqrt{3}-3+i\sqrt{3}-2+2\sqrt{3}-2i+2i+i2\sqrt{3}+2i^2}$

$= \overline{-5+3\sqrt{3}+i3\sqrt{3}-2}$

$= 3\sqrt{3}-7+3\sqrt{3}i$

(iv) $\mathrm{Re}\left\{2z_1^3+3z_2^3-5z_3^2\right\}$

$= 2(1-i)^3+3(-2+4i)^3-5(\sqrt{3}-2i)^2$

$= 2(1-3i-3i^2-i^3)+3(-8+48i+96$
$\quad -64i)-5(3-4\sqrt{3}i-4)$

$= 2-6i-6-2i-24+144i+288-192i$
$\quad -15-20\sqrt{3}i+20$

$= 265-56i-20\sqrt{3}i$

$= 265$

Ans.

(v) $\mathrm{Im}\left\{\dfrac{z_1 z_2}{z_3}\right\}$

$= \dfrac{(1-i)(-2+4i)}{\sqrt{3}-2i}$

$= \dfrac{-2+4i+2i-4i^2}{\sqrt{3}-2i}$

$= \dfrac{(2+6i)(\sqrt{3}+2i)}{(\sqrt{3})^2-(2i)^2}$

$= \dfrac{2\sqrt{3}+4i+6\sqrt{3}i+12i^2}{3+4}$

$= \dfrac{2\sqrt{3}-12+4i+6\sqrt{3}i}{7}$

$= \dfrac{4+6\sqrt{3}}{7}$

(vi) $z_1^2+2z_1-3$

$= (1-i)^2+2(1-i)-3$

$= 1-2i-1+2-2i-3$

$= -1-4i$

(vii) $|z_1\bar{z}_2 + z_2\bar{z}_1|$

$= |(1-i)(-2-4i)+(-2+4i)(1+i)|$

$= |-2-4i+2i+4i^2-2-2i+4i+4i^2|$

$= |-4+8i^2|$

$= |-4-8|$

$= |-12|$

$= 12$

(ix) $(z_3-\bar{z}_3)^5$

$= \{\sqrt{3}-2i-(\sqrt{3}+2i)\}^5$

$= (-4i)^5$

$= -1024i^5$

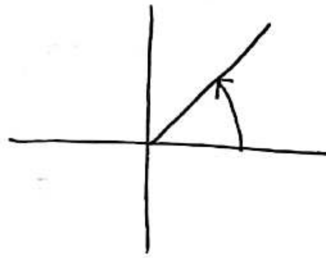
$= -1024\, i^2 \cdot i^3$

$= -1024i$

[If $z = x+iy$

then $\bar{z} = x-iy$ ]

(4.) Express each of the following complex number in polar form and show them graphically.



(i) $2 + 2\sqrt{3}i$

$r = \sqrt{x^2 + y^2}$

$= \sqrt{2^2 + (2\sqrt{3})^2}$

$= \sqrt{4 + 12}$

$= \sqrt{16} = 4$

$\tan\theta = \frac{y}{x}$

$= \frac{2\sqrt{3}}{2}$

$= \sqrt{3}$

$= \tan 60^\circ = \tan\frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3}$

$\therefore \operatorname{Arg}(2 + 2\sqrt{3}i) = \frac{\pi}{3}$

$\left[ r e^{i\theta} = r \operatorname{cis}(\theta + 2\pi n) \right]$

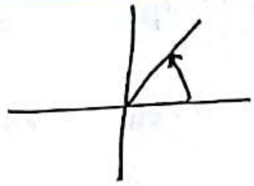
$\therefore \arg(2 + 2\sqrt{3}i) = \frac{\pi}{3} + 2n\pi$

$\therefore 2 + 2\sqrt{3}i = r \operatorname{cis}(\theta + 2n\pi)$

$= 4 \operatorname{cis}\left(\frac{\pi}{3} + 2n\pi\right)$ ; $n = 0, \pm 1, \pm 2, \ldots$

or, $= 4e^{i\pi/3}$

(ii) $2\sqrt{2} + 2\sqrt{2}i$

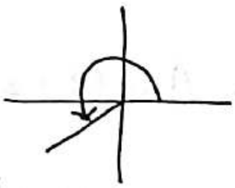


$$\therefore \pi = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2}$$
$$= \sqrt{8+8}$$
$$= \sqrt{16}$$
$$= 4$$

$$\tan\theta = \frac{2\sqrt{2}}{2\sqrt{2}}$$
$$= 1$$
$$= \tan 45^\circ = \tan \pi/4$$
$$\therefore \theta = \frac{\pi}{4}$$

$$2\sqrt{2} + 2\sqrt{2}i = 4 \operatorname{cis}\left(\frac{\pi}{4} + 2n\pi\right) \quad ; \quad n = 0, \pm 1, \pm 2, \dots$$

(iii) $-2\sqrt{3} - 2i$

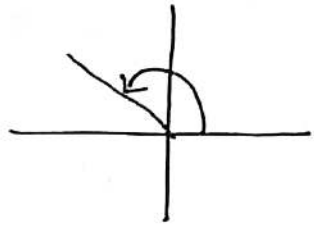


$$\pi = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$$
$$= \sqrt{12+4}$$
$$= \sqrt{16}$$
$$= 4$$

$$\tan\theta = \frac{-2}{-2\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}$$
$$= \tan 30^\circ = \tan\frac{\pi}{6}$$
$$\therefore \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\therefore -2\sqrt{3} - 2i = 4 \operatorname{cis}\left(\frac{7\pi}{6} + 2n\pi\right) \quad ; n = 0, \pm 1, \pm 2, \dots$$

(iv) $z = -1 + \sqrt{3}\,i$

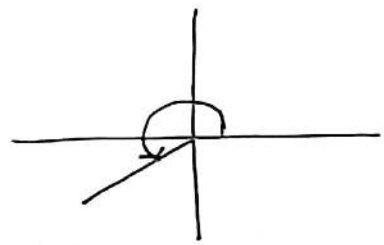


$$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$$
$$= \sqrt{4}$$
$$= 2$$

$$\tan\theta = \frac{\sqrt{3}}{-1}$$
$$= -\sqrt{3}$$
$$= -\tan 60^\circ = -\tan\frac{\pi}{3}$$
$$\therefore \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore -1 + \sqrt{3}\,i = 2\,\text{cis}\left(\frac{2\pi}{3} + 2n\pi\right) \quad ; \; n = 0, \pm 1, \pm 2, \dots$$

(v) $-\sqrt{6} - \sqrt{2}\,i$



$$r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$$
$$= \sqrt{6+2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

$$\tan\theta = \frac{-\sqrt{2}}{-\sqrt{6}}$$
$$= \frac{1}{\sqrt{3}}$$
$$= \tan 30^\circ = \tan \pi/6$$
$$\therefore \theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\therefore -\sqrt{6} - \sqrt{2}\,i = 2\sqrt{2}\,\text{cis}\left(\frac{7\pi}{6} + 2n\pi\right) \quad ; \; n = 0, \pm 1, \pm 2, \dots$$

or $2\sqrt{2}\,e^{i\frac{7\pi}{6}}$

5. Prove the followings:

[ If $z = x + iy$

then, $\bar{z} = x - iy$ ]

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\Rightarrow$ Let, $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$$LHS = \overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2}$$

$$= x_1 + x_2 - i(y_1 + y_2)$$

$$= (x_1 - iy_1) + (x_2 - iy_2)$$

$$= \bar{z}_1 + \bar{z}_2 = R.H.S.$$

(ii) $|z_1 z_2| = |z_1||z_2|$

$$L.H.S. = |z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)|$$

$$= |x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2|$$

$$= |x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)|$$

$$= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2}$$

$=$

$$= \sqrt{x_1^2 x^2 - 2x_1x_2y_1y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2x_1x_2y_1y_2 + x_2^2 y_1^2}$$

$$= \sqrt{x_1^2 (x_2^2 + y_2^2) + y_1^2 (x_2^2 + y_2^2)}$$

$$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$= \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$$

$$= |z_1| \cdot |z_2|$$

= R.H.S.

$\therefore$ L.H.S. = R.H.S.

[Proved]

(iii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

$|z_1 + z_2|^2 = (z_1 + z_2) \cdot (\overline{z_1 + z_2})$ $\qquad [\because |z|^2 = z \cdot \bar{z}]$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1 \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 z_2 + z_2 \bar{z}_2$

$= |z_1|^2 + \overline{\bar{z}_1 z_2} + \bar{z}_1 z_2 + |z_2|^2$ $\qquad [\because \bar{\bar{z}} = z]$

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 \bar{z}_2)$ $\qquad [\because z + \bar{z} = 2\,\mathrm{Re}(z)]$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1 \bar{z}_2|$ $\qquad [\because \mathrm{Re}(z) \leq |z|]$

$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\qquad [\because |z| = |\bar{z}|]$

$\leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| + |z_2|$ [Proved]

Replacing $z_2$ by $-z_2$ we get,

$|z_1 - z_2| \leq |z_1| + |-z_2|$

$\leq |z_1| + |z_2|$ $\qquad [\because |z| = |-z|]$

[Proved]

(iv) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2}) \qquad [\because |z|^2 = z \cdot \bar{z}]$$

$$= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$$

$$= |z_1|^2 - (\overline{\bar{z}_1 z_2} + \bar{z}_1 z_2) + |z_2|^2 \qquad [\because \bar{\bar{z}} = z]$$

$$= |z_1|^2 - 2\,\mathrm{Re}\,|\bar{z}_1 z_2| + |z_2|^2 \qquad [\because z + \bar{z} = 2R(z)]$$

$$\geq |z_1|^2 - 2|z_1 z_2| + |z_2|^2$$

$$\geq |z_1|^2 - 2|z_1| \cdot |z_2| + |z_2|^2$$

$$\geq (|z_1| - |z_2|)^2$$

$\therefore |z_1| - |z_2| \leq |z_1 - z_2|$ or, $|z_1 - z_2| \geq |z_1| - |z_2|$

[Proved]

Replacing $z_2$ by $-z_2$ we get

$$|z_1 + z_2| \geq |z_1| - |-z_2|$$

$$\geq |z_1| - |z_2| \qquad [\because |z| = |-z|]$$

[Proved]

(6) State and prove the Dè Moivre's Theorem

⇒ Statement: For $n \in \mathbb{Z}$,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof: The statement is true for $n=1$

Suppose the statement is true for $n=K$

i.e; $(\cos\theta + i\sin\theta)^K = \cos(K\theta) + i\sin(K\theta) \quad \ldots \text{(i)}$

We will show that the statement is true for $n=K+1$

$$\begin{aligned}(\cos\theta + i\sin\theta)^{K+1} &= (\cos\theta + i\sin\theta)^K (\cos\theta + i\sin\theta) \\ &= (\cos(K\theta) + i\sin(K\theta)) \cdot (\cos\theta + i\sin\theta) \\ &= \cos(K\theta + \theta) + i\sin(K\theta + \theta) \\ &= \cos(K+1)\theta + i\sin(K+1)\theta\end{aligned}$$

Therefore, by the principle of mathematical induction the statement is true for $n \in \mathbb{N}$.

[Proved]

(7) Evaluate each of the following by De Moivre's Theorem:

(i) $\frac{(8 \text{cis } 40^\circ)^3}{(2 \text{cis } 60^\circ)^4} = \frac{8^3(\cos 3(40^\circ) + i\sin 3(40^\circ))}{2^4 (\cos 4(60^\circ) + i\sin 4(60^\circ))}$

$$= \frac{2^5 (\cos 120^\circ + i\sin 120^\circ)}{(\cos 240^\circ + i\sin 240^\circ)}$$

$$= \frac{2^5 (-\frac{1}{2} + i\,\sqrt{3}/2)}{(-\frac{1}{2} - i\sqrt{3}/2)}$$

$$= \frac{2^5 (-\frac{1}{2} + i\,\sqrt{3}/2)^2}{(-\frac{1}{2})^2 - (i\,\sqrt{3}/2)^2}$$

$$= \frac{2^5 (-\frac{1}{2} + i\,\sqrt{3}/2)^2}{1/4 + 3/4}$$

$$= 2^5 \left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)^2$$

$\frac{(8 \text{ cis } 40^\circ)^3}{(2 \text{ cis } 60^\circ)^4} = \frac{512 \text{ cis } 3\times40^\circ}{16 \text{ cis } 4\times60^\circ}$

$= 32 \text{ cis}(120 - 240)$

$= 32 \text{ cis}(-120)$

$= 32\{\cos(-120) + i\sin(-120)\}$

$= 32 (\cos 120 - i\sin 120)$

$= 32 \left(\frac{-1}{2} - i\,\frac{\sqrt{3}}{2}\right)$

$= (-16 - 16\sqrt{3}\,i)$

(ii) $$\frac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$$

$$= \frac{36\, e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3}}}{16\, e^{\frac{4\pi i}{3}}}$$

$$= \frac{9}{4} e^{\frac{\pi i}{6} - \frac{5\pi i}{4} + \frac{5\pi i}{3} - \frac{4\pi i}{3}}$$

$$= \frac{9}{4} e^{\frac{2\pi i - 15\pi i + 20\pi i - 16\pi i}{12}}$$

$$= \frac{9}{4} e^{-\frac{9}{12}\pi i} = \frac{9}{4} e^{-\frac{3}{4}\pi i}$$

$$= \frac{9}{4}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$= \frac{9}{4}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \frac{9}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{9}{4}\left(-\frac{1}{\sqrt{2}}\right)(1+i) = \frac{-9}{4\sqrt{2}}(1+i) = \frac{-9\sqrt{2}}{8}(1+i)$$

(iii) $(5\ cis 20^\circ)(3\ cis 40^\circ)$

$= 5(\cos 20^\circ + i\sin 20^\circ)\, 3(\cos 40^\circ + i\sin 40^\circ)$

$= 15\ (\cos 20^\circ \cos 40^\circ + \cos 20^\circ\, i \sin 40^\circ + i \sin 20^\circ \cos 40^\circ + i^2 \sin 20^\circ \sin 40^\circ)$

$= 15\ \{(\cos 20^\circ \cos 40^\circ - \sin 20^\circ \sin 40^\circ) + i\,(\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ)\}$

$= 15\ \{\cos(20^\circ + 40^\circ) + i \sin(20^\circ + 40^\circ)\}$

$\left[\begin{array}{l} \sin(A+B) = \sin A \cos B + \cos A \sin B \\ \cos(A+B) = \cos A \cos B - \sin A \sin B \end{array}\right.$

$= 15\ (\cos 60^\circ + i \sin 60^\circ)$

$= 15\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

(iv) $(2\ cis\ 50^\circ)^6$

$= 2^6\ \{\cos 6(50^\circ) + i \sin 6(50^\circ)\}$

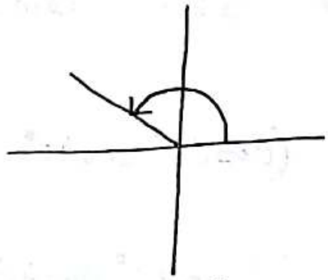
$= 2^6\ (\cos 300^\circ + i \sin 300^\circ)$

$= 2^6\ \left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= 2^6\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

(8) Find all the roots of the following equations

(i) $(-1+i)^{1/3} = z$



Sol: $r = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1)$

$= \tan^{-1}\left(\tan(3\pi/4)\right)$

$= \tan^{-1}\tan\left(\pi - \pi/4\right)$

$= \frac{3\pi}{4}$

$(-1+i)^{1/3} = (\sqrt{2})^{1/3}\left\{\cos\left(\frac{3\pi}{4} + 2\pi n\right) + i\sin\left(\frac{3\pi}{4} + 2\pi n\right)\right\}^{1/3}$ ; $n = 0, 1, 2$

$= 2^{1/6}\left\{\cos\left(\frac{3\pi}{4} + 2\pi n\right)\left(\frac{1}{3}\right) + i\sin\left(\frac{3\pi}{4} + 2\pi n\right)\left(\frac{1}{3}\right)\right\}$

$= 2^{1/6}\left\{\text{cis}\left(\frac{\pi}{4} + \frac{2}{3}\pi n\right) = 2^{1/6}\ \text{cis}\left(\frac{3\pi + 8\pi n}{12}\right)\right.$

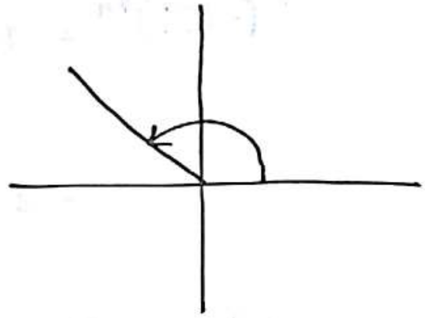
If $n = 0$, $z_1 = 2^{1/6}\ \text{cis}\left(\frac{\pi}{4}\right)$

If $n = 1$, $z_2 = 2^{1/6}\ \text{cis}\left(\frac{11\pi}{12}\right)$

If $n = 2$, $z_3 = 2^{1/6}\ \text{cis}\left(\frac{19\pi}{12}\right)$

The roots are $\sqrt[6]{2}\ \text{cis}\,45^\circ$, $\sqrt[6]{2}\ \text{cis}\,165^\circ$ & $\sqrt[6]{2}\ \text{cis}\,285^\circ$

(ii) $z^5 = -4 + 4i$



Sol: $r = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1) = \tan^{-1}\tan\left(-\frac{\pi}{4}\right)$

$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore (-4+4i)^{1/5} = (4\sqrt{2})^{1/5}\left(\text{cis}\ \frac{3\pi}{4} + 2\pi n\right)^{1/5}$ where, $n = 0,1,2,3,4$

$= (\sqrt{32})^{1/5}\left\{\text{cis}\left(\frac{3\pi}{4} + 2\pi n\right)\frac{1}{5}\right\} = \sqrt[5]{32}\ \text{cis}\left(\frac{3\pi + 8\pi n}{20}\right)$

The roots are.

If $n=0$, $z_1 = \sqrt[5]{32}\ \text{cis}\left(\frac{3\pi}{2}\right)$

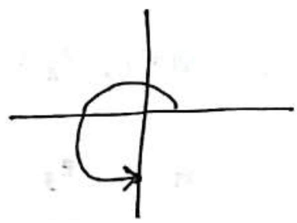
$n=1$, $z_2 = \sqrt[5]{32}\ \text{cis}\left(\frac{11\pi}{20}\right)$

$n=2$, $z_3 = \sqrt[5]{32}\ \text{cis}\left(\frac{19\pi}{20}\right)$

If $n=3$, $z_4 = \sqrt[5]{32}\ \text{cis}\left(\frac{27\pi}{20}\right)$

$n=4$, $z_5 = \sqrt[5]{32}\ \text{cis}\left(\frac{35\pi}{20}\right)$

(iii) $z^4 = -16i$



$\Rightarrow r = \sqrt{0^2 + (-16)^2} = 16$

$\theta = \tan^{-1}\left(\frac{-16}{0}\right) = \tan^{-1}\infty = \tan^{-1}\tan\left(\frac{\pi}{2}\right)$

$= \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

$$(-16i)^{1/4} = 16^{1/4}\left\{\cos\left(\frac{3\pi}{2}+2\pi n\right) + i\sin\left(\frac{3\pi}{2}+2\pi n\right)\right\}^{1/4}$$

$$= 2\ \text{cis}\left(\frac{3\pi+4\pi n}{8}\right) \qquad \text{where, } n=0,1,2,3$$

If $n=0$, $z_1 = 2\ \text{cis}\left(\frac{3\pi}{8}\right)$

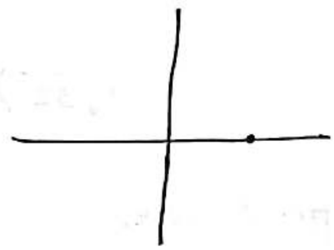
$n=1$, $z_2 = 2\ \text{cis}\left(\frac{7\pi}{8}\right)$

If $n=2$, $z_3 = 2\ \text{cis}\left(\frac{11\pi}{8}\right)$

$n=3$, $z_4 = 2\ \text{cis}\left(\frac{15\pi}{8}\right)$

(iv) $z^6 = 64$

$\Rightarrow r = \sqrt{64^2+0^2} = 64$



$\theta = \tan^{-1}\left(\frac{0}{64}\right) = \tan^{-1} 0 = \tan^{-1}\tan 0 = 0$

$$\therefore (64)^{1/6} = (64)^{1/6}\left\{\cos(0+2\pi n) + i\sin(0+2\pi n)\right\}^{1/6}$$

$$= 2\ \text{cis}\left(\frac{\pi n}{3}\right) \qquad \text{where, } n=0,1,2,3,4,5$$

If $n=0$, $z_1 = 2\ \text{cis}\,0^\circ$

$n=1$, $z_2 = 2\ \text{cis}\left(\frac{\pi}{3}\right)$

$n=2$, $z_3 = 2\ \text{cis}\left(\frac{2\pi}{3}\right)$

If $n=3$, $z_4 = 2\ \text{cis}(\pi)$

$n=4$, $z_5 = 2\ \text{cis}\left(\frac{4\pi}{3}\right)$

$n=5$, $z_6 = 2\ \text{cis}(2\pi)$

(iv) $z^4 + z^2 + 1 = 0$ (Polynomial eq$^n$)

$\Rightarrow$ Let, $x = z^2$

$\therefore x^2 + x + 1 = 0$

$\therefore x = \dfrac{-1 \pm \sqrt{1^2 - 4\cdot1\cdot1}}{2\cdot1}$

$\therefore z^2 = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\therefore z = \left(\dfrac{-1 \pm \sqrt{-3}}{2}\right)^{1/2}$

$z_1 = \left(\dfrac{-1}{2} + \dfrac{i\sqrt{3}}{2}\right)^{1/2}$

$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\theta = \tan^{-1}\left(\dfrac{\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\tan\left(-\frac{\pi}{3}\right)$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$z_1 = 1^{1/2}\left\{\text{cis}\left(\frac{2\pi}{3} + 2\pi n\right)\right\}^{1/2}$

$= \text{cis}\left(\dfrac{\pi + 3\pi n}{3}\right)$

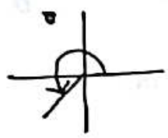
When, $n = 0$, $z = \text{cis}\left(\frac{\pi}{3}\right)$

$n = 1$, $z = \text{cis}\left(\frac{4\pi}{3}\right)$

$z_2 = \left(\dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}\right)^{1/2}$

$r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = 1$

$\theta = \tan^{-1}\left(\dfrac{-\sqrt{3}/2}{-1/2}\right) = \tan^{-1}(\sqrt{3})$



$= \tan^{-1}\tan\left(\frac{\pi}{3}\right)$

$= \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$z_2 = 1^{1/2}\left\{\text{cis}\left(\frac{4\pi}{3} + 2\pi n\right)\right\}^{1/2}$

$= \text{cis}\left(\dfrac{2\pi + 3\pi n}{3}\right)$

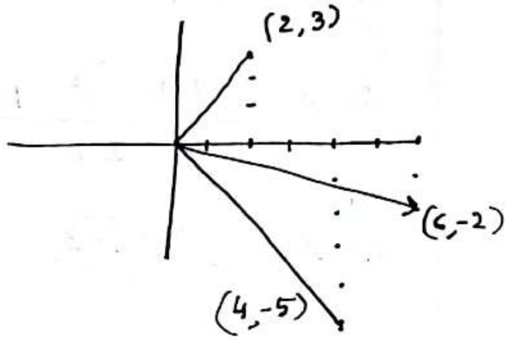
When $n = 0$, $z = \text{cis}\left(\frac{2\pi}{3}\right)$

$n = 1$, $z = \text{cis}\left(\frac{5\pi}{3}\right)$

# Part B

1. Perform the indicated operations analytically and graphically.

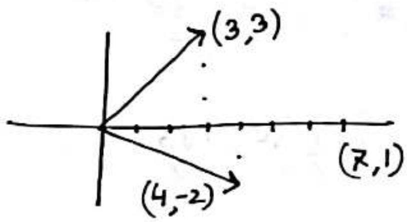
(a) $(2+3i)+(4-5i)$




$\Rightarrow$ Let, $z_1 = 2+3i$ & $z_2 = 4-5i$

$\therefore z_1 + z_2 = 2+3i+4-5i = 6-2i$

(b) $(7+i)-(4-2i)$




$\Rightarrow$ Let, $z_1 = 7+i$ & $z_2 = 4-2i$

$\therefore z = z_1 - z_2 = 7+i-4+2i = 3+3i$

2. Describe geometrically the set of points $z$ satisfying the following conditions:

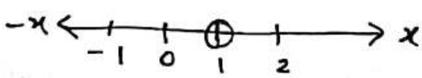
(a) $Re(z) > 1$

$\Rightarrow$ Now, $z = (x+iy)$

$\therefore Re(z) = x$

$\therefore Re(z) > 1$

$\Rightarrow x > 1$



(b) $|2z+3| > 4$

Here,

$$z = x + iy$$

$$\therefore |2(x+iy)+3| > 4$$

$$\Rightarrow |2x + 2iy + 3| > 4$$

$$\Rightarrow \sqrt{(2x+3)^2 + (2y)^2} > 4$$

$$\Rightarrow 4x^2 + 12x + 9 + 4y^2 > 16$$

$$\Rightarrow 4x^2 + 12x + 4y^2 > 7$$

$$\therefore x^2 + 3x + y^2 > 7/4$$

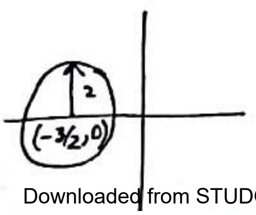
$$\Rightarrow x^2 + 2 \cdot x \cdot \frac{3}{2} + \left(\frac{3}{2}\right)^2 + y^2 > 7/4 + \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 > 7/4 + \frac{9}{4}$$

$$\Rightarrow (x + 3/2)^2 + y^2 > 4$$

$$\therefore \left(x + \frac{3}{2}\right)^2 + y^2 > 2^2$$

Now, $\left(x + \frac{3}{2}\right)^2 + y^2 = 2^2$




(c) $Re\left(\frac{1}{z}\right) > 1$

Here, $z = x + iy$

$$\Rightarrow \frac{1}{z} = \frac{1}{x+iy}$$

$$= \frac{x - iy}{x^2 + y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$\therefore Re\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}$$

Now,

$$Re\left(\frac{1}{z}\right) > 1$$

$$\Rightarrow \frac{x}{x^2+y^2} > 1$$

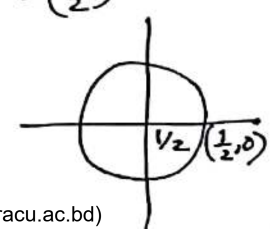
$$\Rightarrow x > x^2 + y^2$$

$$\Rightarrow 0 > x^2 + y^2 - x$$

$$\Rightarrow x^2 - 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + y^2 < \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 + y^2 < \left(\frac{1}{2}\right)^2$$

Now, $\left(x + \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$



(3.) Using the properties of conjugate and modulus, show that:

(i) $\overline{\bar{z} + 3i} = z - 3i$

$\Rightarrow$ Here, $z = x + iy$

$$\text{L.H.S.} = \overline{\bar{z} + 3i}$$

$$= \overline{\overline{x + iy} + 3i}$$

$$= \overline{x - iy + 3i}$$

$$= \overline{x - i(y-3)}$$

$$= x + i(y-3)$$

$$= x + iy - 3i$$

$$= z - 3i$$

$$= \text{R.H.S.}$$

[Proved]

(iii) $\left|(2\bar{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}\,|2z+5|$

$$\text{L.H.S.} = \left|\left\{2\left(\overline{x+iy}\right)+5\right\}(\sqrt{2}-i)\right|$$

$$= \left|\left\{2(x-iy)+5\right\}(\sqrt{2}-i)\right|$$

$$= \left|(2x-2iy+5)(\sqrt{2}-i)\right|$$

$$= \left|(2x+5)-2iy\right| \cdot \left|\sqrt{2}-i\right|$$

$$= \sqrt{(2x+5)^2+(-2y)^2} \cdot \sqrt{(\sqrt{2})^2+(-1)^2}$$

$$= \sqrt{(2x+5)^2+(+2y)^2} \cdot \sqrt{2+1}$$

$$= |2x+5+2yi| \cdot \sqrt{3}$$

$$= \sqrt{3} \cdot |2(x+iy)+5|$$

$$= \sqrt{3}\,|2z+5|$$

$$= \text{R.H.S.}$$

(iv) $|2z+3\bar{z}| \leq 4|\text{Re}(z)| + |z|$

$$\text{L.H.S} = |2z+3\bar{z}|$$

$$= \left|2(x+iy)+3\left(\overline{x+iy}\right)\right|$$

$$= |2x+2iy+3(x-iy)|$$

$$= |2x+2iy+3x-3iy|$$

$$= |5x-iy|$$

$$= |4x+(x-iy)|$$

$$\leq |4x| + |x-iy|$$

$$[\because |z_1+z_2| \leq |z_1|+|z_2|]$$

$$\leq 4|x| + |x-iy|$$

$$\leq 4|\text{Re}(z)| + |\bar{z}|$$

$$\leq 4|\text{Re}(z)| + |z|$$

$$\leq \text{R.H.S.}$$

(4.) Find the modulus and argument of the following complex number

(i) $\dfrac{2-i}{2+i}$

For Numerator

$r = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

$\theta = \tan^{-1}(-1/2)$

$= 360° - 26.57°$

$= 333.43°$

$\therefore (2-i) = \sqrt{5}\ \text{cis}\,(333.43°)$

For denominator

$r = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$

$\theta = \tan^{-1}(1/2)$

$= 26.57°$

$\therefore (2+i) = \sqrt{5}\ \text{cis}\,(\cancel{333}\ 26.57°)$

$\therefore \dfrac{2-i}{2+i} = \dfrac{\sqrt{5}\ \text{cis}\,(333.43°)}{\sqrt{5}\ \text{cis}(26.57°)}$

$= \text{cis}(333.43° - 26.57°)$

$= 1 \cdot \text{cis}\,(307°)$

$\therefore$ Modulus $= 1$ & Argument $= 307°$

4. (ii) $\frac{\sqrt{3}+i}{\sqrt{3}-i}$

(iii) $\frac{(1+\sqrt{3}i)^2}{(1-\sqrt{3}i)^2}$

$= \frac{(\sqrt{3}+i)^2}{(\sqrt{3})^2 - i^2}$

$= \frac{3 + 2\sqrt{3}i - 1}{3+1}$

$= \frac{2 + 2\sqrt{3}i}{4}$

$= \frac{1+\sqrt{3}i}{2}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$= \frac{\pi}{3}$

$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\therefore$ Arg $= \frac{\pi}{3}$, Mod $= 1$

(iii) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^2$

For numerator:

$r = \sqrt{1^2+(\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\sqrt{3}/1\right)$

$= 60^\circ$

$(1+\sqrt{3}i) = 2\ cis 60^\circ$

For denominator:

$r = \sqrt{1^2+(-\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(-\sqrt{3}/1\right)$

$= 360^\circ - 30^\circ = 330^\circ$

$(1-\sqrt{3}i) = 2\ cis 330^\circ$

$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^2 = \left(\frac{2\ cis 60^\circ}{2\ cis 330^\circ}\right)^2$

$= \left\{cis(60^\circ - 330^\circ)\right\}^2$

$= cis\left\{(-270^\circ)\times 2\right\}$

$= cis(-540^\circ)$

Modulus = 1 & Argument = $-540^\circ$

(5.) Prove that $|z-i| = |z+i|$ represents a straight line.

$\Rightarrow |x+iy-i| = |x+iy+i|$ $\quad [\because z = x+iy]$

$\Rightarrow |x+i(y-1)| = |x+i(y+1)|$.

$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$

$\Rightarrow y^2-2y+1 = y^2+2y+1$

$\Rightarrow 4y = 0$

$y = 0$

$\because y = 0 \quad \therefore$ x-axis is a straight line. [Proved]

(6.) Prove that $|z+2i| + |z-2i| = 6$ represents an ellipse.

$\Rightarrow |x+iy+2i| + |x+iy-2i| = 6$

$\Rightarrow |x+i(y+2)| + |x+i(y-2)| = 6$

$\Rightarrow \sqrt{x^2+(y+2)^2} + \sqrt{x^2+(y-2)^2} = 6$

$$\Rightarrow \left(\sqrt{x^2+(y+2)^2}\right)^2 = \left\{6-\sqrt{x^2+(y-2)^2}\right\}^2$$

$$\Rightarrow x^2+(y+2)^2 = 36-12\sqrt{x^2+(y-2)^2}+x^2+(y-2)^2$$

$$\Rightarrow y^2+4y+4 = 36-12\sqrt{x^2+(y-2)^2}+y^2-4y+4$$

$$\Rightarrow 8y = 36-12\sqrt{x^2+(y-2)^2}$$

$$\Rightarrow 2y = 9-3\sqrt{x^2+(y-2)^2}$$

$$\Rightarrow \left\{3\sqrt{x^2+(y-2)^2}\right\}^2 = (9-2y)^2$$

$$\Rightarrow 9\left\{x^2+(y-2)^2\right\} = 81-36y+4y^2$$

$$\Rightarrow 9x^2+9y^2-36y+36 = 81-36y+4y^2$$

$$\Rightarrow 9x^2+5y^2 = 45$$

$$\therefore \left(\frac{x^2}{5}+\frac{y^2}{9}\right) = 1$$

This an equation of ellipse.

[Proved]

(7.) Find an equation of circle center at (2,3) with radius 3.

$\Rightarrow$ We know,

$$|z - z_0| = R$$

Here,

$R = 3$

$z_0 = 2 + 3i$

$\therefore |z - z_0| = 3$

$\Rightarrow |z - 2 - 3i| = 3$

(8) Sketch the region in x plane represented by the ~~fto~~ following set of points.

(i) $Re(\bar{z} - 1) = 2$

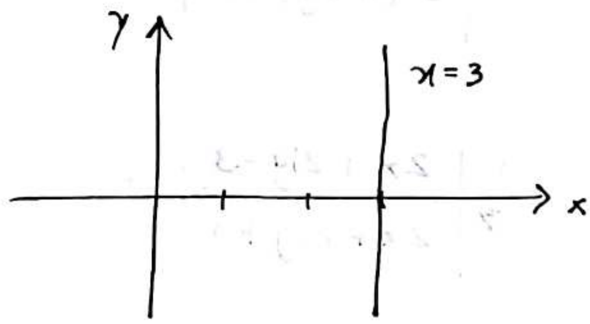
Now, $(\overline{x+iy} - 1) = (x - iy - 1)$

$= (x-1) - iy$

$\therefore Re(\bar{z} - 1) = x - 1$

$\therefore x - 1 = 2$

$\therefore x = 3$



(ii) $Im(z^2) = 4$

$\Rightarrow z^2 = (x+iy)^2$

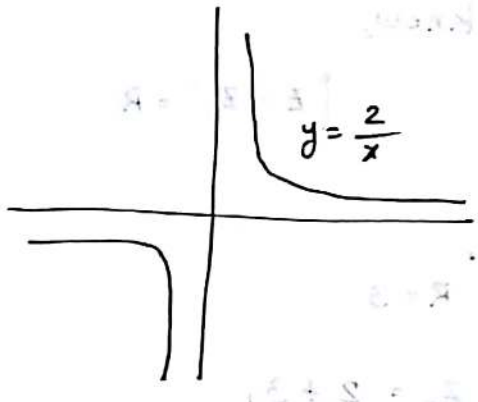
$= x^2 + 2ixy + y^2 i^2$

$= (x^2 - y^2) + 2ixy$

$\therefore Im(z^2) = 2xy$

$\therefore 2xy = 4$

$\therefore y = \frac{2}{x}$




(iii) $\left|\frac{2z-3}{2z+3}\right| = 1$

$\Rightarrow \left|\frac{2(x+iy)-3}{2(x+iy)+3}\right| = 1$

$\Rightarrow \left|\frac{2x+2iy-3}{2x+2iy+3}\right| = 1$

$\Rightarrow \left|\frac{(2x-3)+2iy}{(2x+3)+2iy}\right| = 1$

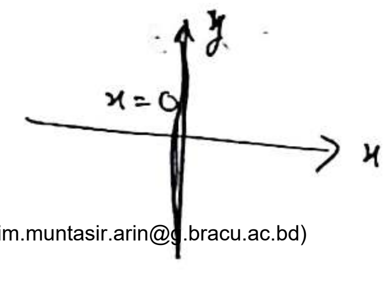
$\Rightarrow \frac{\sqrt{(2x-3)^2+(2y)^2}}{\sqrt{(2x+3)^2+(2y)^2}} = 1$

$\Rightarrow (2x-3)^2 + 4y^2 = (2x+3)^2 + 4y^2$

$\Rightarrow 4x^2 - 12x + 9 = 4x^2 + 12x + 9$

$\Rightarrow 24x = 0$

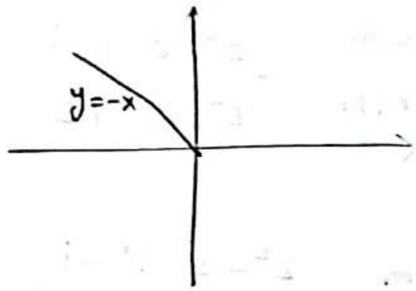
$\therefore x = 0$



(iv) $Re(z) + Im(z) = 0$

$\Rightarrow$ $Re(z) = x$

$Im(z) = y$




$Re(z) + Im(z) = 0$

$\therefore x + y = 0$

$\therefore y = -x$

# Practice Sheet # 2

① Evaluate the following limits:

(i) $$\lim_{z \to 1+i} \frac{z^2 - z + 1 - i}{z^2 - 2z + 2}$$

$$= \lim_{z \to 1+i} \frac{z^2 - z - i^2 - i}{z^2 - 2z + 1 + 1}$$

$$= \lim_{z \to 1+i} \frac{z^2 - i^2 - z - i}{(z-1)^2 + 1}$$

$$= \lim_{z \to 1+i} \frac{(z+i)(z-i) - (z+i)}{(z-1)^2 - i^2}$$

$$= \lim_{z \to 1+i} \frac{(z+i)\cancel{(z-i-1)}}{(z-1+i)\cancel{(z-1-i)}}$$

$$= \lim_{z \to 1+i} \frac{z+i}{z-1+i}$$

$$= \frac{1+i+i}{1+i-1+i} = \frac{1+2i}{2i}$$

$$= \frac{-i^2+2i}{-i^2 \cdot 2i} = \frac{-i(i-2)}{-i \cdot 2i^2}$$



$= \frac{i-2}{\ }$ Ans

(ii) $$\lim_{z \to 1+i} \left(\frac{z-1-i}{z^2 - 2z + 2}\right)^2$$

$$= \lim_{z \to 1+i} \left(\frac{z-1-i}{z^2 - 2z + 1 + 1}\right)^2$$

$$= \lim_{z \to 1+i} \left\{\frac{z-1-i}{(z-1)^2 - i^2}\right\}^2$$

$$= \lim_{z \to 1+i} \left\{\frac{\cancel{(z-1-i)}}{(z-1+i)\cancel{(z-1-i)}}\right\}^2$$

$$= \lim_{z \to 1+i} \left(\frac{1}{z-1+i}\right)^2$$

$$= \left(\frac{1}{1+i-1+i}\right)^2$$

$$= \left(\frac{1}{2i}\right)^2$$

$$= \frac{-1}{4}$$ Ans.

(iii) $\lim_{z\to i} \dfrac{z^2+1}{z^6+1}$

$= \lim_{z\to i} \dfrac{z^2-i^2}{z^6-i^6}$

$= \lim_{z\to i} \dfrac{z^2-i^2}{(z^2)^3-(i^2)^3}$

$= \lim_{z\to i} \dfrac{(z^2-i^2)}{(z^2-i^2)(z^4+z^2\cdot i^2+i^4)}$

$= \lim_{z\to i} \dfrac{1}{z^4-z^2+1}$

$= \dfrac{1}{i^4-i^2+1}$

$= \dfrac{1}{1+1+1}$

$= \dfrac{1}{3}$ Ans.

(iv) $\lim_{z\to i/2} \dfrac{(2z-3)(4z+i)}{(iz-1)^2}$

$= \dfrac{(i-3)(2i+i)}{(i^2/2-1)^2}$

$= \dfrac{(i-3)\cdot 3i}{(-1/2-1)^2}$

$= \dfrac{3i^2-9i}{9/4}$

$= \dfrac{-3-9i}{9/4}$

$= \dfrac{-3(1+3i)\cdot 4}{9}$

$= -\dfrac{4}{3}(1+3i)$

(2) If $f(z) = \dfrac{2z-1}{3z+2}$, prove that $\lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h} = \dfrac{7}{(3z_0+2)^2}$

provided $z_0 \neq -\frac{2}{3}$.

$\Rightarrow$ $\because f(z) = \dfrac{2z-1}{3z+2}$

$\therefore f(z_0) = \dfrac{2z_0-1}{3z_0+2}$ & $f(z_0+h) = \dfrac{2(z_0+h)-1}{3(z_0+h)+2}$

$$f(z_0+h) - f(z_0) = \frac{2(z_0+h)-1}{3(z_0+h)+2} - \frac{2z_0-1}{3z_0+2}$$

$$= \frac{(2z_0+2h-1)(3z_0+2) - (2z_0-1)(3z_0+3h+2)}{(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{6z_0^2 + 4z_0 + 6hz_0 + 4h - 3z_0 - 2 - 6z_0^2 - 6hz_0 - 4z_0 + 3z_0 + 3h + 2}{(3z_0+3h+2)(3z_0+2)}$$

$$= \frac{7h}{(3z_0+3h+2)(3z_0+2)}$$

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h \to 0} \frac{7h}{h(3z_0+3h+2)(3z_0+2)}$$

$$= \lim_{h \to 0} \frac{7}{(3z_0+h+2)(3z_0+2)}$$

$$= \frac{7}{(3z_0+2)(3z_0+2)}$$

$$= \frac{7}{(3z_0+2)^2}$$ [Proved]

## Continuity

If the limit point and the function point of a function is equal then the function is continuous at that point.

$f(z_0)$ is continuous at $z_0$ if,

(i) $\lim_{z \to z_0} f(z)$ exist.

(ii) $f(z)$ is defined.

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$

(3.) Let $f(z) = \frac{z^2+4}{z-2i}$ if $z \neq 2i$, while $f(2i) = 3+4i$, is $f(z)$ continuous at $z = 2i$.

$\Rightarrow$

$$f(z) \begin{cases} \frac{z^2+4}{z-2i}, & z \neq 2i \\ 3+4i, & z = 2i \end{cases}$$

$$\lim_{z \to 2i} f(z) = \lim_{z \to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z \to 2i} \frac{z^2 - 4i^2}{z-2i}$$

$$= \lim_{z \to 2i} \frac{(z+2i)(z-2i)}{(z-2i)}$$

$$= \lim_{z \to 2i} (z+2i)$$

$$= 4i$$

But $f(2i) = 3+4i$

$$\therefore \lim_{z \to 2i} f(z) \neq f(2i)$$

$f(z)$ is not continuous at $z = 2i$.

(4.) Find all points of discontinuity for the function $f(z) = \dfrac{2z-3}{z^2+2z+2}$

$\Rightarrow$ The function will be discontinuous at those points where

$$(z^2+2z+2) = 0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2 - 4.1.2}}{2.1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\cdot\sqrt{-1}}{2} = \frac{2(-1 \pm \sqrt{-1})}{2} = (-1 \pm i)$$

Function will be discontinuous at $(-1+i)$ and $(-1-i)$ points.

## Differentiability

The derivative of $f(z)$ is denoted by $f'(z)$ or $\frac{d}{dz} f(z)$ or $\frac{dw}{dz}$

and defined as,
$$f(z) = \lim_{\Delta z \to 0} \left\{ \frac{f(z+\Delta z) - f(z)}{\Delta z} \right\}$$

If $f(z)$ is differentiable at $z = z_0$

$$f'(z) = \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right\}$$

(5) Using the defination, find the derivative of each function at the indicated points.

(i) $f(z) = \dfrac{2z - i}{z + 2i}$ at $z = -i$

$\Rightarrow$ Here, $z_0 = -i$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(-i) = \lim_{\Delta z \to 0} \frac{f(-i + \Delta z) - f(-i)}{\Delta z} \quad \ldots\ldots (i)$$

Now,

$$f(-i + \Delta z) = \frac{2(-i + \Delta z) - i}{-i + \Delta z + 2i}$$

$$= \frac{-3i + 2\Delta z}{i + \Delta z} \quad \text{--- (ii)}$$

Again,

$$f(-i) = \frac{2(-i) - i}{-i + 2i}$$

$$= \frac{-3i}{i} = -3 \quad \ldots (iii)$$

Substituting the value from (ii) & (iii) in equation (i) we get,

$$f'(-i) = \lim_{\Delta z \to 0} \frac{\frac{-3i + 2\Delta z}{i + \Delta z} - (-3)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3i + 2\Delta z + 3i + 3\Delta z}{i + \Delta z} \times \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{5 \Delta z}{(i + \Delta z) \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{5}{i + \Delta z}$$

$$= \frac{5}{i} = \frac{5i}{i^2} = -5i$$

(ii) $f(z) = 3z^{-2}$ at $z = 1+i$

$\Rightarrow$ Here,
$z_0 = 1+i$

Now, $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\therefore f'(1+i) = \lim_{\Delta z \to 0} \frac{f(1+i+\Delta z) - f(1+i)}{\Delta z} \quad \ldots\ldots (i)$$

Now,

$$f(1+i+\Delta z) = 3(1+i+\Delta z)^{-2}$$

$$= \frac{3}{\{(1+i)+\Delta z\}^2} = \frac{3}{1+2i+i^2+2(1+i)\Delta z + \Delta z^2}$$

$$= \frac{3}{2i + 2\Delta z + 2\Delta z i + \Delta z^2} \quad \text{-----} \; (ii)$$

Again,

$$f(1+i) = 3(1+i)^{-2}$$

$$= \frac{3}{(1+i)^2} = \frac{3}{1+2i+i^2} = \frac{3}{2i} \quad \text{-----} (iii)$$

~~Substituting the values from (ii) & (iii) in equation (i) we~~ get.

~~$f'(1+i) =$ before that~~ (ii) − (iii) ⇒

$$f(1+i+\Delta z) - f(1+i) = \frac{3}{2i + 2\Delta z + 2\Delta z i + \Delta z^2} - \frac{3}{2i}$$

$$= \frac{6i - 6i - 6\Delta z - 6\Delta z i - 3\Delta z^2}{2i(2i + 2\Delta z + 2\Delta z i + \Delta z^2)}$$

$$= \frac{-3\Delta z(2 + 2i + \Delta z)}{2i(2i + 2\Delta z + 2\Delta z i + \Delta z^2)}$$

From $eq^n$ (i) we get,

$$f'(1+i) = \lim_{\Delta z \to 0} \frac{-3\Delta z(2+2i+\Delta z)}{2i(2i + 2\Delta z + 2\Delta z i + \Delta z^2)} \times \frac{1}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3(2+2i+\Delta z)}{2i(2i+2\Delta z+2\Delta z i+\Delta z^2)}$$

$$= \frac{-3(2+2i)}{2i \cdot 2i}$$

$$= \frac{-6-6i}{4i^2}$$

$$= \frac{-2(3+3i)}{-4}$$

$$= \frac{3}{2}(1+i) \quad \underline{\text{Ans.}}$$

6. Evaluate the following Limits using L' Hospital's rule.

(i) $$\lim_{z \to 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i}$$

$$= \lim_{z \to 2i} \frac{2z}{4z+3-4i} \quad [\text{L' Hospital}]$$

$$= \frac{2 \cdot 2i}{4 \cdot 2i + 3 - 4i}$$

$$= \frac{4i}{8i+3-4i}$$

$$= \frac{4i}{4i+3}$$

$$= \frac{4i(4i-3)}{(4i)^2 - 3^2}$$

$$= \frac{16i^2 - 12i}{-16-9}$$

$$= \frac{-16-12i}{-25}$$

$$= \frac{16}{25} + \frac{12}{25} i$$

(ii) $$\lim_{z \to 0} \frac{z - \sin z}{z^3}$$

$$= \lim_{z \to 0} \frac{1 - \cos z}{3z^2} \quad [\text{L' Hospital}]$$

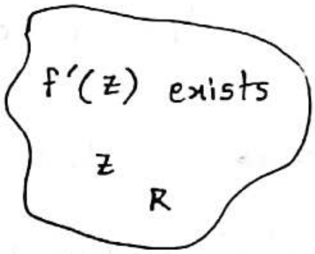
$$= \lim_{z \to 0} \frac{\sin z}{6z} \quad [\text{L' Hospital}]$$

$$= \lim_{z \to 0} \frac{\cos z}{6} \quad [\quad .. \quad]$$

$$= \frac{\cos 0^\circ}{6}$$

$$= \frac{1}{6}$$ Ans.

**Analytic Function:** If $f'(z)$ exists at all points $z$ of a region $R$, then $f(z)$ is said to be analytic in $R$.




[ $f'(z)$ is differentiable ]

$f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at $z_0 = (x_0 + y_0)$ then 1st order partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$ and they must ~~st~~ satisfy.

$$\left.\begin{aligned} u_x &= V_y \\ u_y &= -V_x \end{aligned}\right\} \text{known as Cauchy - Reimann equation}$$

This is known as necessary condition.

Also $f'(z)$ can be written as at $z_0$,

$$f'(z_0) = u_x(x_0, y_0) + i.v_x /x_0, y_0)$$

$\therefore f'(z_0) = (V_y - iu_y)$ which is known as sufficient condition.

**Harmonic function:** A real-valued function $u$ of two real variable $x$ and $y$ is said to be harmonic in a given domain in the $xy$-plane if through out that domain it has continuous partial derivatives of 1st & 2nd order and satisfies the partial deffirential equation.

$$U_{xx}(x,y) + U_{yy}(x,y) = 0 \longrightarrow \text{Laplace equation.}$$

(7) Determine $u$ is harmonic function. Find the conjugate harmonic function $v$ and express $u+iv$ as an analytic function of $z$.

(i) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$\Rightarrow$ Now,

$$u_x = 6xy + 4x$$

$$u_{xx} = 6y + 4$$

Again,

$$u_y = 3x^2 - 3y^2 - 4y$$

$$u_{yy} = -6y - 4$$

$$\therefore U_{xx} + u_{yy} = 6y + 4 - 6y - 4 = 0$$

$\therefore U$ is a harmonic function.

Now,

$$U_x = (6xy + 4x) = V_y \quad \text{----} \quad (i)$$

$$u_y = (3x^2 - 3y^2 - 4y) = -V_x \quad \text{---} \quad (ii)$$

Keeping $x$ constant & integrating both sides of eq$^n$ (i) with respect to $y$ we ~~we~~ get,

$$\int V_y \, dy = \int (6xy + 4x) \, dy$$

$$\Rightarrow V = 6x\frac{y^2}{2} + 4xy + \phi(x)$$

$$\therefore V = 3xy^2 + 4xy + \phi(x) \quad \text{......} \quad (iii)$$

Differentiating eq$^n$ (iii) with respect to $x$ we get,

$$V_x = 3y^2 + 4y + \phi'(x)$$

$$\Rightarrow -(3x^2 - 3y^2 - 4y) = 3y^2 + 4y + \phi'(x) \qquad [\text{From eq}^n\text{(ii)}]$$

$$\Rightarrow -3x^2 + 3y^2 + 4y = 3y^2 + 4y + \phi'(x)$$

$$\Rightarrow \phi'(x) = -3x^2$$

$$\Rightarrow \int \phi'(x) \, dx = -3\int x^2 \, dx$$

$$\therefore \phi(x) = -3\frac{x^3}{3} + c = -x^3 + c$$

Substituting the value of $\phi(x)$ in eq$^n$ (iii) we get,

$$V = 3xy^2 + 4xy - x^3 + c$$

This is the conjugate harmonic function.

And analytic function of $z$ is,

$$f(z) = (u + iv)$$

$$= (3x^2y + 2x^2 - y^3 - 2y^2) + i(3xy^2 + 4xy - x^3 + c)$$

(ii) $u = xe^x \cos y - ye^x \sin y$

$\Rightarrow$ Now,

$$u_x = e^x \cos y + xe^x \cos y - ye^x \sin y$$

$$u_{xx} = e^x \cos y + e^x \cos y + xe^x \cos y - ye^x \sin y$$

Again,

$$u_y = -xe^x \sin y - e^x \sin y - ye^x \cos y$$

$$u_{yy} = -xe^x \cos y - e^x \cos y - e^x \cos y + ye^x \sin y$$

$$U_{xx} + V_{yy} = 2e^x \cos y + xe^x \cos y - ye^x \sin y - xe^x \cos y$$
$$- 2e^x \cos y + ye^x \sin y$$

$\therefore U_{xx} + V_{yy} = 0$

$\therefore$ u is harmonic function.

Now,

$$u_x = e^x \cos y + xe^x \cos y - ye^x \sin y = V_y \quad \ldots (i)$$

$$u_y = -xe^x \sin y - e^x \sin y - ye^x \cos y = -V_x \quad \ldots (ii)$$

By integrating both sides of equation (i) with respect to y,

$$\int V_y \, dy = \int (e^x \cos y + xe^x \cos y - ye^x \sin y) \, dy$$

$$\Rightarrow V = e^x \sin y + xe^x \sin y - e^x \left[ y \int \sin y \, dy - \int \left\{ \frac{d}{dy}(y) \int \sin y \, dy \right\} dy \right]$$

$$= e^x \sin y + xe^x \sin y - e^x \left[ -y \cos y - \int -\cos y \, dy \right]$$

$$= e^x \sin y + xe^x \sin y - e^x (-y \cos y + \sin y) + \phi(x)$$

$$= \cancel{e^x \sin y} + xe^x \sin y + e^x y \cos y - \cancel{e^x \sin y} + \phi(x) \quad \ldots (iii)$$

$$= xe^x \sin y + e^x \cos y + \phi(x) \ldots$$

Differentiating equation (iii) with respect to $x$ we get,

$$V_x = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x)$$

$$\Rightarrow xe^x \sin y + e^x \sin y + ye^x \cos y = xe^x \sin y + e^x \sin y + e^x y \cos y + \phi'(x)$$

[From $eq^n$ (ii)]

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x) = \int 0 \cdot dx$$

$$\Rightarrow \phi(x) = c$$

Substituting the value of $\phi(x)$ in $eq^n$ (iii) we get,

$V = xe^x \sin y + y \cdot e^x \cos y + c$; which is the conjugate harmonic

Analytic function, $f(z) = (u + iv)$

$$= xe^x \cos y - ye^x \sin y + i(xe^x \sin y + ye^x \cos y + c$$

(iii) $u = e^{-x}(x \sin y - y \cos y)$

$$\Rightarrow u = e^{-x} x \sin y - e^{-x} y \cos y$$

$$u_x = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y$$

$$u_{xx} = e^{-x} x \sin y - e^{-x} \sin y - e^{-x} \sin y - e^{-x} y \cos y$$

$u_y = e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y$

$u_{yy} = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} \sin y + e^{-x} y \cos y$

$\therefore u_{xx} + u_{yy} = xe^{-x} \sin y - y.e^{-x} \cos y - 2e^{-x} \sin y - e^{-x} x \sin y$
$+ 2e^{-x} \sin y + e^{-x} y \cos y$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore$ u is harmonic.

Now,

$u_x = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y = V_y \;\cdots\cdots (i)$

$u_y = e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y = -V_x \;\cdots\cdots (ii)$

Integrating equation (i) with respect to y, we get,

$\int V_y \, dy = \int (-e^{-x}. x \sin y + e^{-x} \sin y + e^{-x} y \cos y) \, dy$

$\Rightarrow V = xe^{-x} \cos y - e^{-x} \cos y + e^{-x} \left[ y \int \cos y \, dy - \int \left\{ \frac{d}{dy}(y) \int \cos y \, dy \right\} dy \right]$

$= xe^{-x} \cos y - e^{-x} \cos y + e^{-x} \left( y \sin y - \int \sin y \, dy \right)$

$= xe^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cos y + \phi(x)$

$\therefore V = xe^{-x}\cos y + e^{-x}y\sin y + \phi(x) \; \ldots\ldots \; (iii)$

Differentiating (iii) with respect to $x$ we get,

$V_x = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y + \phi'(x)$

$\Rightarrow -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y = e^{-x}\cos y - xe^{-x}\cos y - e^{-x}y\sin y + \phi'(x)$

[From eq$^n$ (ii)]

$\Rightarrow \phi'(x) = 0$

$\Rightarrow \int \phi'(x)\,dx = \int 0\,dx$

$\Rightarrow \phi(x) = C$

Substituting the value of $\phi(x)$ in equation (iii) we get,

$V = xe^{-x}\cos y + e^{-x}y\sin y + c$ ; which is the conjugate harmonic funct

Analytic function, $f(z) = (u + iv)$

$= (e^{-x}x\sin y - e^{-x}y\cos y) + i(xe^{-x}\cos y + e^{-x}y\sin y + c)$

# Singular point: A point at which $f(z)$ fails to be analytic is called singular point or singularity of $f(z)$.

# Isolated singularities: The function $f(z)$ has an isolated singularity at $z_0$, if we can find $\delta > 0$, such that the circle $|z - z_0| = \delta$ enclosed no singular point of $f(z)$ other than $z_0$.

# Poles: If $z_0$ is an islated singularity than we can find a positive integer $n$ such that,

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$$

then $z_0$ is called a pole of order $n$. If $n = 1$, $z_0$ is called a simple pole.

(8) For the following function locate and name the singularities in the finite z-plane and determine whether they are isolated singularities or not.

(a) $f(z) = \dfrac{z}{(z^2+4)^2}$

Sol: Here,

$$(z^2+4)^2 = 0$$

$$\Rightarrow z^2 - 4i^2 = 0$$

$$\Rightarrow z^2 - (2i)^2 = 0$$

$$\Rightarrow (z+2i)(z-2i) = 0$$

$$\therefore z = 2i, -2i$$

$\therefore$ Isolate singular points are $2i, -2i$

Here, $n = 2$

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

When $z_0 = 2i$

$$\lim_{z \to 2i} (z-2i)^2 \frac{z}{(z^2+4)^2}$$

$$= \lim_{z \to 2i} (z-2i)^2 \frac{z}{(z^2 - 4i^2)^2}$$

$$= \lim_{z \to 2i} (z-2i)^2 \frac{z}{(z+2i)^2(z-2i)^2}$$

$$= \lim_{z \to 2i} \frac{z}{(z+2i)^2}$$

$$= \frac{2i}{(2i+2i)^2}$$

$$= \frac{2i}{(4i)^2}$$

$$= \frac{2i}{-16}$$

$$= \frac{1}{8i} \neq 0$$

$\therefore z = 2i$ is a pole of order 2.

When $z_0 = -2i$

$$\lim_{z \to -2i} (z+2i)^2 \frac{z}{(z^2+4)^2}$$

$$= \lim_{z \to -2i} (z+2i)^2 \frac{z}{(z^2 - 4i^2)^2}$$

$$= \lim_{z \to -2i} (z+2i)^2 \frac{z}{\{(z+2i)(z-2i)\}^2}$$

$$= \lim_{z \to -2i} \frac{z}{(z-2i)^2}$$

$$= \frac{-2i}{(-2i-2i)^2}$$

$$= \frac{-2i}{(-4i)^2}$$

$$= \frac{-2i}{16i}$$

$$= \frac{-1}{8i} \neq 0$$

$\therefore z = -2i$ is a pole of order 2.

(b) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+4)^4}$

Now,

$$(z^2+2z+4)^4 = 0$$

$$\Rightarrow z^2+2z+4 = 0$$

$$\Rightarrow z = \frac{-2 \pm \sqrt{2^2 - 4.1.4}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$= -1 \pm \sqrt{3}i$$

Again,

$$\ln(0) = \text{undefined}$$

$$\therefore \ln(z-2) = 0$$

$$\Rightarrow z-2 = 0$$

$$\therefore z = 2$$

Isolated singular points are $(-1+\sqrt{3}i)$, $(-1-\sqrt{3}i)$, $2$.

Here, $n = 4$

$$\lim_{z \to z_0} (z-z_0)^n f(z) = \lim_{z \to (-1+\sqrt{3}i)} (z+1-\sqrt{3}i)^4 \frac{\ln(z-2)}{(z^2+2z+4)^4}$$

$$= \lim_{z \to (-1+\sqrt{3}i)} \{z+1-\sqrt{3}i\}^4 \cdot \frac{\ln(z-2)}{[\{z-(-1+\sqrt{3}i)\}\{z-(-1-\sqrt{3}i)\}]^4}$$

$$= \lim_{z \to (-1+\sqrt{3}i)} (z+1-\sqrt{3}i)^4 \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4(z+1+\sqrt{3}i)^4}$$

$$= \lim_{z \to (-1+\sqrt{3}i)} \frac{\ln(z-2)}{(z+1+\sqrt{3}i)^4}$$

$$= \frac{\ln(-1+\sqrt{3}i-2)}{(-1+\sqrt{3}i+1+\sqrt{3}i)^4}$$

$$= \frac{\ln(\sqrt{3}i-3)}{(2\sqrt{3}i)^4} \neq 0$$

$\therefore z = (-1+\sqrt{3}i)$ is a pole of order 4.

Again, here, $n = 4$

$$= \lim_{z \to (-1-\sqrt{3}i)} \{z - (-1-\sqrt{3}i)\}^4 \frac{\ln(z-2)}{\left[\{z-(-1+\sqrt{3}i)\}\{z-(-1-\sqrt{3}i)\}\right]^4}$$

$$= \lim_{z \to (-1-\sqrt{3}i)} (z+1+\sqrt{3}i)^4 \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4 (z+1+\sqrt{3}i)^4}$$

$$= \lim_{z \to (-1-\sqrt{3}i)} \frac{\ln(z-2)}{(z+1-\sqrt{3}i)^4}$$

$$= \frac{\ln(-1-\sqrt{3}i-2)}{(-1-\sqrt{3}i+1-\sqrt{3}i)^4}$$

$$= \frac{\ln(-3-\sqrt{3}i)}{(-2\sqrt{3}i)^4} \neq 0$$

$\therefore z = (-1-\sqrt{3}i)$ is a pole of order 4.

Ans.

# Exponential function of Complex variable

$$f(z) = e^{z} = e^{(x+iy)}$$

$$= e^{x} \cdot e^{iy}$$

$$= e^{x} (\cos y + i \sin y) \qquad [\because (\cos\theta + i\sin\theta) = e^{i\theta}]$$

$$\therefore |f(z)| = |e^{z}| = |e^{x} (\cos y + i \sin y)|$$

$$= |e^{x}| \cdot |\cos y + i \sin y|$$

$$= e^{x} \sqrt{\cos^2 y + \sin^2 y}$$

$$= e^{x} \cdot 1$$

$$\therefore |f(z)| = e^{x} > 0 \quad \text{for all } x \in R$$

$$e^{(z + 2\pi i)} = e^{z} \cdot e^{2\pi i}$$

$$= e^{z} (\cos 2\pi + i \sin 2\pi)$$

$$= e^{z} (1 + 0)$$

$$= e^{z}$$

$$z = r \cdot e^{i\theta}$$

$$\therefore z = r \cdot e^{i(\theta + 2\pi n)}$$

# Practice Sheet # 3

(1) Show that:

(i) $\text{Exp}(2 \pm 3\pi i) = -e^2$

Now,

$$e^{2+3\pi i} = e^2 \cdot e^{3\pi i}$$

$$= e^2(\cos 3\pi + i\sin 3\pi)$$

$$= e^2(-1+0)$$

$$= -e^2$$

Again,

$$e^{2-3\pi i} = e^2 \cdot e^{-3\pi i}$$

$$= -e^2$$

[Showed]

(ii) $\text{Exp}\left(\frac{2+\pi i}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$

Now,

$$e^{\frac{2+\pi i}{4}} = e^{1/2} \cdot e^{\pi i/4}$$

$$= e^{1/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= e^{1/2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{e}}{\sqrt{2}}(1+i)$$

(iii) $\text{Exp}(z + \pi i) = -e^z$

Now,

$$e^{z+\pi i} = e^z \cdot e^{\pi i}$$

$$= e^z(\cos\pi + i\sin\pi)$$

$$= e^z(-1+0)$$

$$= -e^z$$

[Showed]

(2.) Find all the values of $z$ such that:

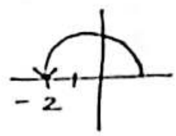
(i) $e^z = -2$

Here, $x = -2$ & $y = 0$

$r = \sqrt{(-2)^2 + 0} = 2$

$\theta = \tan^{-1}(0/-2)$

$= \tan^{-1}(0)$




$= \tan^{-1} \tan(\pi)$

$= \pi$

$\therefore e^z = r \cdot e^{i(\theta + 2\pi n)}$

$= 2\, e^{i(2\pi n + \pi)}$

$\Rightarrow \ln e^z = \ln \left\{ 2e^{i(2\pi n + \pi)} \right\}$

$\Rightarrow z = \ln 2 + \ln e^{(2\pi n + \pi)i}$

$= \ln 2 + (2\pi n + \pi)i$

$= \ln 2 + (2n+1)\pi i$

Ans.

(ii) $e^z = 1 + \sqrt{3}i$

Here, $x = 1$ & $y = \sqrt{3}$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(\sqrt{3}/1)$

$= \tan^{-1} \tan\left(\frac{\pi}{3}\right)$

$= \frac{\pi}{3}$

$\because (1 + \sqrt{3}i) = r \cdot e^{i(\theta + 2\pi n)}$

$\therefore e^z = 2\, e^{i\left(\frac{\pi}{3} + 2\pi n\right)}$

$\Rightarrow \ln e^z = \ln \left\{ 2e^{i\left(\frac{\pi}{3} + 2\pi n\right)} \right\}$

$\Rightarrow z = \ln 2 + \ln e^{i\left(\frac{\pi}{3} + 2\pi n\right)}$

$= \ln 2 + i\left(\frac{\pi}{3} + 2\pi n\right)$ ; $n = 0, \pm 1, \pm 2 \ldots$

Ans.

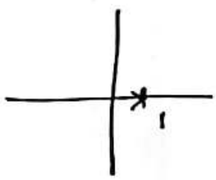
(iii) $e^{2z-1} = 1$

Here,

$x = 1$ & $y = 0$

$r = \sqrt{1^2 + 0^2} = 1$

$\theta = \tan^{-1}(0/1)$




$= \tan^{-1}(0)$

$= 0$

$\because 1 = r \cdot e^{i(\theta + 2\pi n)}$

$\therefore e^{2z-1} = 1 \cdot e^{i(0 + 2\pi n)}$

$\Rightarrow \ln e^{2z-1} = \ln(e^{2\pi n i})$

$\Rightarrow 2z - 1 = 2\pi n i$

$\Rightarrow 2z = 2\pi n i + 1$

$\therefore z = \frac{1}{2} + i\pi n$

Ans.

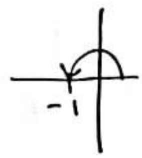
(iv) $e^{z} = -1$

Here, $x = -1$ & $y = 0$

$r = \sqrt{(-1)^2 + 0^2} = 1$

$\theta = \tan^{-1}(0/-1)$

$= \tan^{-1} \tan(\pi)$




$= \pi$

$\because -1 = r \cdot e^{i(\theta + 2\pi n)}$

$\therefore e^{z} = 1 \cdot e^{i(\pi + 2\pi n)}$

$\Rightarrow \ln e^{z} = \ln\{e^{i(\pi + 2\pi n)}\}$

$\Rightarrow z = i(\pi + 2\pi n)$

$\therefore z = (2n + 1)\pi i$

Ans.

## Trigonometric Functions

$e^{\pm 2\pi i} = 1$

Here,

$e^{iz} = \cos z + i\sin z$ ; $e^{-iz} = \cos z - i\sin z$

$\therefore e^{iz} + e^{-iz} = 2\cos z$

$\therefore \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ .

$\therefore e^{iz} - e^{-iz} = 2i\sin z$

$\therefore \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

(3) Prove that:

(i) $\sin(z + 2\pi) = \sin z$

L.H.S. $= \sin(z + 2\pi)$

$= \frac{1}{2i}\left\{e^{i(z+2\pi)} - e^{-i(z+2\pi)}\right\}$

$= \frac{1}{2i}\left\{e^{iz}.e^{2\pi i} - e^{-iz}.e^{-2\pi i}\right\}$

$= \frac{1}{2i}\left(e^{iz}.1 - e^{-iz}.1\right)$

$= \sin z$

$=$ R.H.S.

(ii) $\cos(z + 2\pi) = \cos z$

L.H.S. $= \cos(z + 2\pi)$

$= \frac{1}{2}\left\{e^{i(z+2\pi)} + e^{-i(z+2\pi)}\right\}$

$= \frac{1}{2}\left\{e^{iz}.e^{2\pi i} + e^{-iz}.e^{-2\pi i}\right\}$

$= \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

$= \cos z$

$=$ R.H.S.

# Hyperbolic Function

1. $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$
2. $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$
3. $\cosh^2 z - \sinh^2 z = 1$
4. $\frac{d}{dz}(\sinh z) = \cosh z$
5. $\frac{d}{dz}(\cosh z) = \sinh z$

# Prove that:

(a) $\sin z = (\sin x \cdot \cosh y + i \cos x \cdot \sinh y)$

(b) $\cos z = (\cos x \cdot \cosh y - i \sin x \cdot \sinh y)$

Sol: We know, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$$\therefore \sin iz = \frac{1}{2i}(e^{i^2 z} - e^{-i^2 z})$$

$$= \frac{1}{2i}(e^{-z} - e^{z})$$

$$= \frac{-1}{2i}(e^{z} - e^{-z})$$

$$= \frac{-i}{2i^2}(e^{z} - e^{-z})$$

$$= \frac{-i}{-2}(e^{z} - e^{-z})$$

$\because \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$$\therefore \cos iz = \frac{1}{2}(e^{i^2 z} + e^{-i^2 z})$$

$$= \frac{1}{2}(e^{-z} + e^{z})$$

$$= \frac{1}{2}(e^{z} + e^{-z})$$

$$= \cosh z \quad \text{.... (ii)}$$



$$= i \frac{1}{2}(e^{z} - e^{-z}) = i \sinh z \quad \text{(i)}$$

(a) L.H.S. $= \sin z$

$= \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cdot \cosh y + \cos x \cdot i \sinh y$

$= \sin x \cdot \cosh y + i \cos x \cdot \sinh y$

$=$ R.H.S

(b) L.H.S. $= \cos z$

$= \cos(x+iy)$

$= \cos x \cdot \cos iy - \sin x \cdot \sin iy$

$= \cos x \cdot \cosh y - \sin x \cdot i \sinh y$

$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$

$=$ R.H.S.

[Proved]

---

4.(i) L.H.S. $= \sin hz$

$= -i \sin iz$ [*]

$= -i \sin i(x+iy)$

$= -i \sin(ix - y)$

$= -i(\sin ix \cdot \cos y - \cos ix \cdot \sin y)$

$= -i \sin ix \cdot \cos y + i \cos ix \cdot \sin y$

$= -i^2 \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

$= \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

$=$ R.H.S.

4(ii) L.H.S. $= \cos hz$

$= \cos iz$

$= \cos i(x+iy)$

$= \cos(ix - y)$

$= \cos ix \cdot \cos y + \sin ix \cdot \sin y$

$= \cos hx \cdot \cos y + i \sin hx \cdot \sin y$

$=$ R.H.S.

[*] $\because i \sin hx = \sin iz$

$\therefore i^2 \sin hx = i \sin iz$

$\Rightarrow -\sin hx = i \sin iz$

$\therefore \sin hx = -i \sin iz$

$$= -i$$

(5.) Show that:

(a) $\sin^{-1} z = -i \ln\left[iz \pm (1-z^2)^{1/2}\right]$

$\Rightarrow$ Let, $u = \sin^{-1} z$

$$\Rightarrow z = \sin u$$

$$\Rightarrow z = \frac{1}{2i}\left(e^{iu} - e^{-iu}\right)$$

$$\Rightarrow 2iz = e^{iu} - e^{-iu}$$

$$\Rightarrow 2iz \cdot e^{iu} = e^{2iu} - 1$$

$$\Rightarrow e^{2iu} - 2ize^{iu} - 1 = 0$$

$$\Rightarrow e^{iu} = \frac{-(-2iz) \pm \sqrt{(-2iz)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$= \frac{2iz \pm \sqrt{4i^2z^2 + 4}}{2}$$

$$= \frac{2iz \pm 2\sqrt{-z^2+1}}{2}$$

$$= iz \pm \sqrt{1-z^2}$$

$$\Rightarrow \ln e^{iu} = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

$$\Rightarrow iu = \ln\left(iz \pm \sqrt{1-z^2}\right) \quad [*]$$

(b) $\cos^{-1} z = -i \ln\left[z \pm i(1-z^2)^{1/2}\right]$

Let, $u = \cos^{-1} z$

$$\Rightarrow z = \cos u$$

$$\Rightarrow z = \frac{1}{2}\left(e^{iu} + e^{-iu}\right)$$

$$\Rightarrow 2z = e^{iu} + e^{-iu}$$

$$\Rightarrow 2z \cdot e^{iu} = e^{2iu} + 1$$

$$\Rightarrow (e^{iu})^2 - 2ze^{iu} + 1 = 0$$

$$\Rightarrow e^{iu} = \frac{-(-2z) \pm \sqrt{(-2z)^2 - 4\cdot 1\cdot 1}}{2\cdot 1}$$

$$= \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$

$$= \frac{2z \pm 2\sqrt{z^2-1}}{2}$$

$$= z \pm \sqrt{-i^2z^2 + i^2}$$

$$= z \pm i\sqrt{-z^2+1}$$

$$\Rightarrow \ln e^{iu} = \ln\left(z \pm i\sqrt{1-z^2}\right)$$

$$\Rightarrow iu = \ln\left(z \pm i\sqrt{1-z^2}\right)$$

6. Solve the equation:

(i) $\cosh z = \frac{1}{2}$

$\Rightarrow \frac{1}{2}(e^{z} + e^{-z}) = \frac{1}{2}$

$\Rightarrow e^{z} + e^{-z} = 1$

$\Rightarrow e^{z}.e^{z} + e^{-z}.e^{z} = e^{z}$

$\Rightarrow e^{2z} + 1 = e^{z}$

$\Rightarrow (e^{z})^2 - e^{z} + 1 = 0$

$\therefore e^{z} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.1}}{2.1}$

$= \frac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1 \pm \sqrt{3}i}{2}$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Now,

$e^{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$r = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\because \frac{1}{2} + \frac{\sqrt{3}}{2}i = r.e^{i(\theta + 2\pi n)}$

$\Rightarrow e^{z} = 1.e^{i(\frac{\pi}{3} + 2\pi n)}$

$\therefore z = i\left(\frac{\pi}{3} + 2\pi n\right)$

Again,

$e^{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$r = \sqrt{(\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right)$

$= \tan^{-1}(-\sqrt{3}) = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$\because \frac{1}{2} - \frac{\sqrt{3}}{2}i = r.e^{i(\theta + 2\pi n)}$

$\therefore e^{z} = 1.e^{i(\frac{5\pi}{3} + 2\pi n)}$

$\therefore z = i\left(\frac{5\pi}{3} + 2\pi n\right)$ ; $n = 0, \pm 1, \pm 2 \ldots$

Ans.

(6) (ii) $\sinh z = i$

$$\Rightarrow \frac{1}{2}(e^{z} - e^{-z}) = i$$

$$\Rightarrow e^{z} - e^{-z} = 2i$$

$$\Rightarrow (e^{z})^2 - 1 = 2i e^{z}$$

$$\Rightarrow (e^{z})^2 - 2ie^{z} - 1 = 0$$

Now,

$$e^{z} = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$= \frac{2i \pm \sqrt{4i^2 + 4}}{2}$$

$$= \frac{2i \pm 2\sqrt{-1+1}}{2} = i$$

$$\therefore r = \sqrt{0^2 + 1^2} = 1$$

$$\therefore \theta = \tan^{-1}(1/0)$$

$$= \tan^{-1} \alpha$$

$$= \tan^{-1} \tan(\pi/2)$$

$$= \frac{\pi}{2}$$

$$\therefore i = r\cdot e^{i(\theta + 2\pi n)}$$

$$\therefore e^{z} = 1\cdot e^{i(\frac{\pi}{2} + 2\pi n)}$$

$$\therefore z = i\left(\frac{\pi}{2} + 2\pi n\right) \quad ; \quad n = 0, \pm 1, \pm 2, \pm 3 \dots$$

(7) Show that:

Logarithmic Function

(i) $\ln(-1+\sqrt{3}i) = \ln 2 + 2(n+\frac{1}{3})\pi i$

Sol: From $(-1+\sqrt{3}i)$

$x = -1$ & $y = \sqrt{3}$

$\therefore r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\tan(\pi - \frac{\pi}{3}) = \frac{2\pi}{3}$

$\therefore (-1+\sqrt{3}i) = r.e^{i(\theta+2\pi n)}$

$\Rightarrow (-1+\sqrt{3}i) = 2\, e^{i(\frac{2\pi}{3}+2\pi n)}$

$\Rightarrow \ln(-1+\sqrt{3}i) = \ln 2 + \ln e^{i(\frac{2\pi}{3}+2\pi n)}$

$= \ln 2 + 2(n+\frac{1}{3})\pi i$

[Showed]

(ii) $\ln(1-i) = \frac{1}{2}\ln 2 + (2n+\frac{7}{4})\pi i$

Sol: From $(1-i)$

$x = 1$ & $y = -1$

$\therefore r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}(-1/1)$

$= \tan^{-1}(-1)$

$= \tan^{-1}\tan(2\pi - \frac{\pi}{4}) = \frac{7\pi}{4}$

$\therefore (1-i) = r.e^{i(\theta+2\pi n)}$

$\Rightarrow (1-i) = \sqrt{2}\, e^{i(\frac{7\pi}{4}+2\pi n)}$

$\Rightarrow \ln(1-i) = \ln 2^{1/2} + \ln e^{(\frac{7}{4}+2n)\pi i}$

$= \frac{1}{2}\ln 2 + (2n+\frac{7}{4})\pi i$

[Showed]

(iii) $\ln\left(i^{1/2}\right) = \left(n + \frac{1}{4}\right)\pi i$

Sol: From $i^{1/2}$, $x = 0$ & $y = 1$

$\therefore\ r = \sqrt{0^2 + 1^2} = 1$

$\therefore\ \theta = \tan^{-1}\left(1/0\right) = \tan^{-1}\alpha$

$= \tan^{-1}\left(\tan \pi/2\right) = \frac{\pi}{2}$

$\therefore\ i = r \cdot e^{i(\theta + 2\pi n)}$

$\therefore\ i = 1 \cdot e^{i\left(\frac{\pi}{2} + 2\pi n\right)}$

$\Rightarrow \ln i = \ln e^{\left(2n + \frac{1}{2}\right)\pi i}$

$\Rightarrow \ln i = \left(2n + \frac{1}{2}\right)\pi i$

$\Rightarrow \frac{1}{2}\ln i = \frac{1}{2}\left(2n + \frac{1}{2}\right)\pi i$

$\therefore\ \ln\left(i^{1/2}\right) = \left(n + \frac{1}{4}\right)\pi i$

[showed]

# Practice Sheet # 4

(1.) Evaluate $\displaystyle\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along

(a) The curve $y = x^2+1$

Sol: $\Rightarrow$ $dy = 2x\,dx$

$$\int_0^2 \left[(3x+x^2+1)\,dx + 2x(2x^2+2-x)\,dx\right]$$

$$= \left[\frac{3x^2}{2} + \frac{x^3}{3} + x + \frac{4x^4}{4} + \frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2$$

$$= \frac{3\times 4}{2} + \frac{8}{3} + 2 + 16 + 8 - \frac{2\times 8}{3} = \frac{88}{3} \quad \text{Ans.}$$

(b) The straight line joining $(0,1)$ and $(2,5)$

$\Rightarrow$ Equation of the straight line $\dfrac{x-0}{0-2} = \dfrac{y-1}{1-5}$

$$\Rightarrow \frac{x}{-2} = \frac{y-1}{-4}$$

$$\Rightarrow 4x = 2y-2$$

$$\Rightarrow y = 2x+1$$

$$\therefore dy = 2dx$$

$$= \int_0^2 [(3x + 2x + 1)\,dx + (4x + 2 - x)\,2\,dx]$$

$$= \left[\frac{3x^2}{2} + x^2 + x + \frac{2 \cdot 3x^2}{2} + 4x\right]_0^2$$

$$= \frac{3 \times 4}{2} + 4 + 2 + \frac{3 \times 4}{2} + 4 \times 2 = 32$$

(c) The straight line from (0,1) to (0,5) and then from (0,5) to (2,5)

⇒ For (0,1) to (0,5)

$x = 0$

$\therefore dx = 0$

$$\int_1^5 0 + (2y - 0)\,dy$$

$$= \left[\frac{2y^2}{2}\right]_1^5$$

$$= (25 - 1) = 24$$

Along (0,5) to (2,5)

$y = 5$

$\therefore dy = 0$

$$\int_0^2 (3x + 5)\,dx + 0$$

$$= \left[\frac{3x^2}{2} + 5x\right]_0^2$$

$$= \frac{3}{2} \cdot 4 + 10 = 16$$

The required value is $= 24 + 16 = 40$

(d) The straight lines from (0,1) to (2,1) and then from (2,1) to (2,5)

⇒ Along (0,1) to (2,1)

$y = 1$

$\therefore dy = 0$

$$\int_0^2 (3x+1)\,dx + 0$$

$$= \left[\frac{3x^2}{2} + x\right]_0^2$$

$$= \frac{3}{2}.4 \times 2 = 8$$

Along (2,1) to (2,5)

$x = 2$

$\therefore dx = 0$

$$\int_1^5 0 + (2y-2)\,dy$$

$$= \left[y^2 - 2y\right]_1^5$$

$$= (25-10) - (1-2) = 16$$

The required value is $= 8 + 16 = 24$ Ans.

(2) Evaluate $\oint_C (x+2y)\,dx + (y-2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta \leq 2\pi$ if C is described in a counterclockwise direction.

Sol ⇒ Given,

$x = 4\cos\theta$

$dx = -4\sin\theta\,d\theta$

& $y = 3\sin\theta$

$\therefore dy = 3\cos\theta\,d\theta$

$$\oint_C (x+2y)\,dx + (y-2x)\,dy$$

$$= \int_0^{2\pi} \left[ (4\cos\theta + 6\sin\theta)(-4\sin\theta\, d\theta) + (3\sin\theta - 8\cos\theta)(3\cos\theta\, d\theta) \right]$$

$$= \int_0^{2\pi} (-16\sin\theta\cos\theta\, d\theta - 24\sin^2\theta\, d\theta + 9\sin\theta\cos\theta\, d\theta - 24\cos^2\theta\, d\theta)$$

$$= \int_0^{2\pi} (-7\sin\theta\cos\theta\, \cancel{d\theta} - 24\, \cancel{d\theta})$$

$$= \cancel{7\sin 2\pi \cos} \; - 24\int_0^{2\pi} d\theta - \frac{7}{2}\int_0^{2\pi} 2\sin\theta\cos\theta\, d\theta$$

$$= -24[\theta]_0^{2\pi} - \frac{7}{2}\int_0^{2\pi} [\sin 2\theta]_\theta$$

$$= -48\pi - \frac{7}{2} \; \cancel{\sin 4\pi - \sin 0} \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -48\pi - \frac{7}{2}(\cos 4\pi - \cos 0)$$

$$= -48\pi - \frac{7}{2}(1-1)$$

$$= -48\pi$$

Ans.

(3.) Evaluate $\int_C (x^2 - iy^2)\, dz$ along

(a) the parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$

$\Rightarrow$ Here, $y = 2x^2$ And, $z = x + iy$

$\therefore dy = 4x\, dx$ $dz = dx + i\,dy = dx + 4ix\, dx$

$$\int_1^2 (x^2 - i4x^4)(dx + 4ix\, dx)$$

$$= \int_1^2 (x^2 dx + 4ix^3 dx - 4ix^4 dx - 16i^2 x^5 dx)$$

$$= \left[\frac{x^3}{3} + ix^4 - \frac{4ix^5}{5} + \frac{16x^6}{6}\right]_1^2$$

$$= \left(\frac{8}{3} + 16i - \frac{128i}{5} + \frac{416}{3}\right) - \left(\frac{1}{3} + i - \frac{4i}{5} + \frac{8}{3}\right)$$

$$= \frac{511}{3} - \frac{49i}{5}$$ Ans.

(b) The straight line (1,1) to (1,8) then (1,8) to (2,8)

$\Rightarrow$ Along (1,1) to (1,8)

$x = 1$

$\therefore \ dx = 0$

And.

$z = x + iy$

$dz = dx + idy = idy$

$$\int_1^8 (1 - iy^2)\, i\, dy$$

$$= \int_1^8 (i - i^2 y^2)\, dy$$

$$= \left[ iy + \frac{y^3}{3} \right]_1^8$$

$$= 8i + \frac{512}{3} - i - \frac{1}{3}$$

$$= 7i + \frac{511}{3}$$

Along. (1,8) to (2,8)

$y = 8$

$dy = 0$

$\therefore \ dz = dx$

$$\int_2^1 (x^2 - i64)\, dx$$

$$= \left[ \frac{x^3}{3} - i64x \right]_2^1$$

$$= \left(\frac{1}{3} - 64i\right) - \left(\frac{8}{3} - 128i\right) = \frac{7}{3} - 64i$$

The required line value is $7i + \frac{511}{3} + \frac{7}{3} - 64i = \frac{518}{3} - 57i$

Ans.

(c) the straight line from $(1,1)$ to $(2,8)$

$\Rightarrow$ Equation of the line $\dfrac{x-1}{1-2} = \dfrac{y-1}{1-8}$

$$\Rightarrow \frac{x-1}{-1} = \frac{y-1}{-7}$$

$$\Rightarrow 7x - 7 = y - 1$$

$$\Rightarrow y = 7x - 6$$

$$dy = 7\,dx$$

Again,

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$= dx + i7dx$$

$$= (1+7i)\,dx$$

$$\int_1^2 \{x^2 - i(7x-6)^2\}\,(1+7i)\,dx$$

$$= (1+7i)\int_1^2 (x^2 - 49i x^2 + 84xi - 36i)\,dx$$

$$= (1+7i)\left[\frac{x^3}{3} - \frac{49x^3 i}{3} + \frac{42x^2 i}{\cancel{42}} - 36ix\right]_1^2$$

$$= (1+7i)\left[\left(\frac{8}{3} - \frac{392i}{3} + \frac{42\cdot 4i}{\cancel{42}} - 72i\right) - \left(\frac{1}{3} - \frac{49i}{3} + \frac{42i}{\cancel{42}} - 36i\right)\right]$$

$$= (1+7i)\left(\frac{7}{3} - \frac{73i}{3}\right)$$

$$= \frac{7}{3} - \frac{73i}{3} + \frac{49i}{3} - \frac{511i}{3} = \frac{518}{3} - 8i$$

Ans.

(4.) Evaluate $\oint_C |z|^2 dz$ around the square with vertices at $(0,0), (1,0), (1,1), (0,1)$

$z = \sqrt{x^2+y^2}$ $\quad |z|^2 = x^2+y^2$

$dz = dx + i\,dy$

Sol ⇒ Along $(0,0)$ to $(1,0)$

$y = 0$

$\therefore dy = 0$

$$\int_0^1 (x^2+0)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Along $(1,0)$ to $(1,1)$

$x = 1$

$\therefore dx = 0$

$$\int_0^1 (1+y^2)\,i\,dy$$

$$= \left[iy - \frac{iy^3}{3}\right]_0^1 = i + \frac{i}{3} = \frac{4i}{3}$$

Along $(1,1)$ to $(0,1)$

$y = 1$

$dy = 0$

$$\int_1^0 (x^2+1)\,dx$$

$$= \left[\frac{x^3}{3}+x\right]_1^0 = -\left(\frac{1}{3}+1\right) = -\frac{4}{3}$$

Along $(0,1)$ to $(0,0)$

$x = 0$

$\therefore dx = 0$

$$\int_1^0 (0+y^2)\,i\,dy$$

$$= i\left[y^3/3\right]_1^0 = -i\left[\frac{1}{3}\right] = \frac{-i}{3}$$

The required value $= \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3} = (i-1)$ Ans.

(5.) (a) Evaluate $\int_C (z^2 + 3z)\,dz$ along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction.

Sol $\Rightarrow$ Equation of the line $\frac{x-2}{2-0} = \frac{y-0}{0-2}$

$$\Rightarrow -x+2 = y$$

$$\Rightarrow dy = -dx$$

Again,

$$z = x + iy$$

$$dz = dx + i\,dy$$

$$= dx - i\,dx$$

$$= (1-i)\,dx$$

$$\int_2^0 \left\{(x+iy)^2 + 3(x+iy)\right\}(1-i)\,dx$$

$$= (1-i)\int_2^0 \left(x^2 + 2ixy - y^2 + 3x + 3iy\right) dx$$

$$= (1-i)\int_2^0 \left\{x^2 + 3x + 2ix(-x+2) - (-x+2)^2 + 3i(-x+2)\right\} dx$$

$$= -(1-i)\int_0^2 \left(x^2 + 3x - 2ix^2 + 4ix - 4 + 4x - x^2 - 3ix + 6i\right) dx$$

$$= -(1-i)\left[\frac{x^3}{3} + \frac{3x^2}{2} - \frac{2ix^3}{3} + 2ix^2 - 4x + 2x^2 - \frac{x^3}{3} - \frac{3ix^2}{2} + 6ix\right]_0^2$$

$$= -(1-i)\left(\cancel{\frac{+8}{3}} + 6 - \frac{16i}{3} + 8i - 8 + 8 \cancel{-\frac{8}{3}} - 6i + 12i\right)$$

$$= -(1-i)\left(6 + \frac{26i}{3}\right) = -\frac{4}{3}(11 + 2i)$$ Ans.

5.(b) Evaluate $\int_c (z^2+3z)\,dz$ the straight line from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$.

$dz = dx + i\,dy$

Sol ⇒ Along $(2,0)$ to $(2,2)$

$x = 2$

$dx = 0$

$$\int_0^2 \{(x+iy)^2 + 3(x+iy)\}\, i\,dy$$

$$= \int_0^2 (4 + 4iy + i^2y^2 + 6 + 3iy)\, i\,dy$$

$$= \left[4yi + \cancel{2i^2y^2} - \frac{iy^3}{3} + 6yi + \cancel{\frac{3i^2y^2}{2}}\right]_0^2$$

$$= 8i + \cancel{8i} - \frac{8i}{3} + 12i + \cancel{6i} - 6i$$

$$= i\int_0^2 (4 + 4iy + i^2y^2 + 6 + 3iy)\,dy$$

$$= i\int_0^2 (10 + 7iy - y^2)\,dy$$

$$= i\left[10y + \frac{7iy^2}{2} - \frac{y^3}{3}\right]_0^2$$

$$= 20i + 14i^2 - \frac{8}{3}i$$

$$= \frac{52i}{3} - 14$$

Along $(2,2)$ to $(0,2)$

$y = 2$

$dy = 0$

$$\int_2^0 \{(x+iy)^2 + 3(x+iy)\}\,dx$$

$$= -\int_0^2 (x^2 + 4ix + 4i^2 + 3x + 6i)\,dx$$

$$= -\left[\frac{x^3}{3} + 2ix^2 - 4x + \frac{3x^2}{2} + 6ix\right]_0^2$$

$$= -\left[\frac{8}{3} + 8i - 8 + 6 + 12i\right]$$

$$= -\left[\frac{2}{3} + 20i\right]$$

∴ The required value =

$$= \frac{52i}{3} - 14 - \frac{2}{3} - 20i = -\frac{44}{3} - \frac{8i}{3}$$

Ans.

6. (a) Evaluate $\int_{i}^{2-i} (3xy + iy^2)\, dz$ along the straight line joining $z = i$ and $z = 2-i$

Sol ⇒ Here, $z_1 = i = 0 + 1\cdot i ; (0,1)$

$z_2 = 2 - i = 2 + i(-1) ; (2,-1)$

And, $z = x + iy$

$dz = dx + i\,dy$

$= -dy + i\,dy = (i-1)dy$

Along $(0,1)$ to $(2,-1)$

$\frac{x-0}{0-2} = \frac{y-1}{1+1} \Rightarrow 2x = -2(y-1)$

$\Rightarrow x = -y + 1$

$\Rightarrow y = 1 - x$

$\Rightarrow dy = -dx \qquad \Rightarrow dx = -dy$

$$\int_{1}^{-1} \{3y(-y+1) + iy^2\}(i-1)\,dy$$

$$= \int_{1}^{-1} (-3y^2 + 3y + iy^2)(i-1)\,dy$$

$$= (i-1)\left[-y^3 + \frac{3y^2}{2} + \frac{iy^3}{3}\right]_{1}^{-1}$$

$$= (i-1)\left[\left\{-(-1) + \frac{3}{2} - \frac{i}{3}\right\} - \left(-1 + \frac{3}{2} + \frac{i}{3}\right)\right]$$

$$= (i-1)\left(2 - \frac{2i}{3}\right) = \frac{8i}{3} - \frac{4}{3} \text{ Ans.}$$

(7)(a) Evaluate $\oint (\bar{z})^2 dz$ around the circles $|z|=1$

Sol ⇒ $|z| = 1$

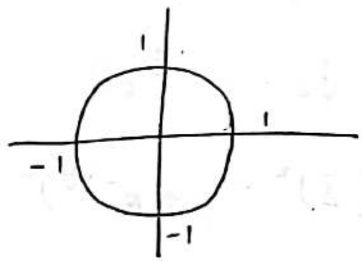
$\Rightarrow z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta} d\theta$

$\because z = e^{i\theta}$

$\therefore \bar{z} = e^{-i\theta}$

$\therefore (\bar{z})^2 = (e^{-i\theta})^2 = e^{-2i\theta}$




$$\int_0^{2\pi} (e^{-2i\theta})\, ie^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{i\theta - 2i\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta}\, d\theta$$

[Limit of circle is always 0 to $2\pi$]

$$= i \left[ \frac{e^{-i\theta}}{-i} \right]_0^{2\pi}$$

$$= \frac{i}{-i} \left[ e^{-i2\pi} - e^0 \right]$$

$$= -1\,(1-1)$$

$$= 0$$

Ans.

(7.) (b) Evaluate $\oint_C (\bar{z})^2 \, dz$ around the circles $|z-1| = 1$.

$\Rightarrow$ Here, $|z-1| = 1$

$$\Rightarrow z - 1 = e^{i\theta}$$

$$\Rightarrow z = 1 + e^{i\theta}$$

$$\bar{z} = 1 + e^{-i\theta}$$

$$dz = i e^{i\theta} d\theta$$

$$(\bar{z})^2 = (1 + e^{-i\theta})^2$$

$$= 1 + 2e^{-i\theta} + e^{-2i\theta}$$

$$\int_0^{2\pi} (1 + 2e^{-i\theta} + e^{-2i\theta}) \, i e^{i\theta} d\theta$$

$$= i \int_0^{2\pi} (e^{i\theta} + 2e^{i\theta - i\theta} + e^{i\theta - 2i\theta}) d\theta$$

$$= i \int_0^{2\pi} (e^{i\theta} + 2 + e^{-i\theta}) d\theta$$

$$= i \left[ \frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i} \right]_0^{2\pi}$$

$$= (e^{2\pi i} + 4\pi i - e^{-2\pi i}) - (e^0 + 0 - e^{-0}) = (1 + 4\pi i - 1) - (1 - 1)$$



$$= 4\pi i$$

Ans

(8) Evaluate $\oint_C \frac{dz}{z-2}$ around (a) the circle $|z-2|=4$, (b) $|z-7|=9$

Sol:⇒ (a) $|z-2|=4$

$\Rightarrow z-2 = 4e^{i\theta}$

$\Rightarrow z = 4e^{i\theta}+2$

$\therefore dz = 4ie^{i\theta}\,d\theta$

$$\int_0^{2\pi} \frac{4ie^{i\theta}\,d\theta}{4e^{i\theta}}$$

$$= i\int_0^{2\pi} e^{i\theta-i\theta}\,d\theta$$

$$= i\int_0^{2\pi} d\theta$$

$$= i\left[\theta\right]_0^{2\pi}$$

$$= 2\pi i$$

Ans.

(b) $|z-1|=9$

$\Rightarrow z-1 = 9e^{i\theta}$

$\Rightarrow z = 9e^{i\theta}+1$

$\Rightarrow dz = 9ie^{i\theta}\,d\theta$

$$\int_0^{2\pi} \frac{9ie^{i\theta}\,d\theta}{9e^{i\theta}+1-2}$$

$$= \int_0^{2\pi} \frac{9ie^{i\theta}\,d\theta}{9e^{i\theta}-1}$$

Let, $9e^{i\theta}-1 = u$

$\therefore du = 9ie^{i\theta}d\theta$

If $\theta = 0$, then, $u = 9-1 = 8$

If $\theta = 2\pi$ then, $u = 9-1 = 8$

$$= \int_8^8 \frac{du}{u}$$

$$= \left[\ln u\right]_8^8$$

$$= \ln 8 - \ln 8 = 0$$

Ans.

(9.) Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around the circle $|z|=1$.

$\Rightarrow$ Here, $|z|=1$

$$z = e^{i\theta}$$

$$dz = ie^{i\theta}d\theta$$

$$\int_0^{2\pi} (5e^{4i\theta} - e^{3i\theta} + 2)\, ie^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} (5e^{5i\theta} - e^{4i\theta} + 2e^{i\theta})\, d\theta$$

$$= i\left[\frac{5e^{5i\theta}}{5i} - \frac{e^{4i\theta}}{4i} + \frac{2e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= \left[e^{5i\theta} - \frac{e^{4i\theta}}{4} + 2e^{i\theta}\right]_0^{2\pi}$$

$$= \cancel{e^{10i}}\left[\cos 5\theta + i\sin 5\theta\right]_0^{2\pi} - \frac{1}{4}\left[\cos 4\theta + i\sin 4\theta\right]_0^{2\pi} + 2\left[\cos\theta + i\sin\theta\right]_0^{2\pi}$$

$$= (1+0-1+0) - \frac{1}{4}(1+0-1+0) + 2(1+0-1+0)$$

$$= 0 - 0 + 0$$

$$= 0$$

Ans.

# Cauchy's Integral Formula

(1.) Evaluate (a) $\oint \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$ where $C$ is the circle $|z|=3$

Sol: Now,

$$\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \oint_C \left[\frac{1}{z-2} - \frac{1}{z-1}\right]\left(\sin \pi z^2 + \cos \pi z^2\right)$$

$$= \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i f(2) - 2\pi i f(1) \quad \ldots \text{(i)}$$

Now,

$$f(z) = \sin(\pi z^2) + \cos(\pi z^2)$$

$$\therefore f(2) = \sin 4\pi + \cos 4\pi = 0 + 1 = 1 \qquad \& \qquad f(1) = \sin \pi + \cos \pi = 0 - 1 = -1$$

From $eq^n$ (i) $\Rightarrow$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i - 2\pi i(-1) = 4\pi i$$

Ans.

[Formula: $f^n(z_0) = \dfrac{n!}{2\pi i} \oint \dfrac{f(z)}{(z-z_0)^{n+1}}\, dz$ ]

1. (b) Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$

Sol: Here, $(n+1) = 4$ and $f(z) = e^{2z}$ $\therefore z_0 = -1$

We know, cauchy's integral formula,

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\Rightarrow f^3(z_0) = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!} f^3(-1)$$

$$= \frac{2\pi i}{3!} \cdot 8e^{-2}$$

$$= \frac{8\pi i e^{-2}}{3} \quad \text{Ans.}$$

$\because f(z) = e^{2z}$

$\therefore f'(z) = 2e^{2z}$

$\Rightarrow f''(z) = 4e^{2z}$

$\Rightarrow f'''(z) = 8e^{2z}$

$\therefore f'''(-1) = 8e^{-2}$

3. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ were C is the circle $|z-1| = 4$.

Sol: Here, $z_0 = \pi i$

$f(z) = e^{3z}$ & $f(z_0) = e^{3\pi i}$ $= \cos(3\pi) + i\sin(3\pi)$

$= -1 + 0$

$= -1$

We know, Cauchy's Integral Formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)} dz$$

$$\Rightarrow (-1) = \frac{1}{2\pi i} \oint_C \frac{e^{3z}}{(z - \pi i)} dz$$

$$\therefore \oint_C \frac{e^{3z}}{z - \pi i} dz = -2\pi i$$

Ans.

(4.) Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and $C$ is the circle $|z| = 3$

Sol: Here, $(n+1) = 2 \quad \therefore n = 1$

$$\frac{1}{z^2+1} = \frac{1}{z^2 - i^2} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2+1} dz = \frac{1}{2\pi i} \oint_C \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right] e^{zt} dz$$

$$= \frac{1}{4\pi i^2}\left[\oint_C \frac{e^{zt}}{z-i} dz - \oint_C \frac{e^{zt}}{z+i}\right]$$

$$= \frac{1}{4\pi i^2}\left[2\pi i f(i) - 2\pi i f(-i)\right] = \frac{1}{2i}\left[f(i) - f(-i)\right]$$

Now,

$$\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^{1+1}} dz = \frac{1}{2i} \left[ f'(i) - f'(-i) \right] \quad \ldots (i)$$

$\because f(z) = e^{zt}$

$\therefore f'(z) = te^{zt}$

$\therefore f'(i) = te^{it}$ & $f'(-i) = te^{-it}$

From equation (i) we get,

$$\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^2} dz = \frac{1}{2i} \left[ te^{it} - te^{-it} \right]$$

$$= \frac{t}{2i} \left[ (\cancel{\cos t} + i\sin t) - (\cancel{\cos t} - i\sin t) \right]$$

$$= \frac{t}{2i} \cdot 2i\sin t$$

$$= t.\sin t$$

Ans.

# Practice Sheet # 5

① Expand each of the following functions in a Taylor series about the indicated points.

(i) $e^{-z}$ at $z = 0$

$\Rightarrow$

| | |
|---|---|
| $f(z) = e^{-z}$ | $f(0) = e^{-0} = 1$ |
| $f'(z) = -e^{-z}$ | $f'(0) = -e^{-0} = -1$ |
| $f''(z) = e^{-z}$ | $f''(0) = e^{-0} = 1$ |
| $f'''(z) = -e^{-z}$ | $f'''(0) = -e^{-0} = -1$ |
| $f^{iv}(z) = e^{-z}$ | $f^{iv}(0) = e^{-0} = 1$ |

According to Taylor's theorem,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

$$\therefore f(z) = f(0) + f'(0)(z-0) + \frac{f''(0)}{2!}(z-0)^2 + \frac{f'''(0)}{3!}(z-0)^3 + \frac{f^{iv}(0)}{4!}(z-0)^4$$

$$= 1 + (-1)z + \frac{1}{2!}\cdot z^2 + \frac{-1}{3!}\cdot z^3 + \frac{1}{4!}z^4$$

$$= 1 + (-1)\frac{z}{1!} + \frac{z^2}{2!} + (-1)\frac{z^3}{3!} + \frac{z^4}{4!} \qquad = \sum_{n=0}^{\infty} \frac{z^n}{n!}(-1)^n$$

Ans.

(ii) $\cos z$ at $z = \frac{\pi}{2}$

Sol: $f(z) = \cos z$ | $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f'(z) = -\sin z$ | $f'(\pi/2) = -\sin\frac{\pi}{2} = -1$

$f''(z) = -\cos z$ | $f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(z) = \sin z$ | $f'''(\pi/2) = \sin \pi/2 = 1$

$f^{iv}(z) = \cos z$ | $f^{iv}(\pi/2) = \cos \pi/2 = 0$

$f^{v}(z) = -\sin z$ | $f^{v}(\pi/2) = -\sin \pi/2 = -1$

According to Taylor series,

$$f(z) = f(\pi/2) + \frac{f'(\frac{\pi}{2})}{1!}\left(z - \frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2!}\left(z - \frac{\pi}{2}\right)^2 + \frac{f'''(\pi/2)}{3!}\left(z - \frac{\pi}{2}\right)^3$$

$$+ \frac{f^{iv}(\pi/2)}{4!}\left(z - \frac{\pi}{2}\right)^4 + \frac{f^{v}(\pi/2)}{5!}\left(z - \frac{\pi}{2}\right)^5$$

$$= 0 + (-1)\left(z - \frac{\pi}{2}\right) + 0 + \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^3 + 0 + \frac{(-1)}{5!}\left(z - \frac{\pi}{2}\right)^5$$

$$= (-1)\frac{1}{1!}\left(z - \frac{\pi}{2}\right)^1 + (-1)^2 \cdot \frac{1}{3!}\left(z - \frac{\pi}{2}\right)^3 + (-1)^3 \cdot \frac{1}{5!}\left(z - \frac{\pi}{2}\right)^5$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (-1)^{n+1} \left(z - \frac{\pi}{2}\right)^{2n+1}$$

Ans.

(iii.) $z^3 - z^2 + 4z + 2$ at $z = 2$

Sol: $f(z) = z^3 - z^2 + 4z + 2$

$f'(z) = 3z^2 - 2z + 4$

$f''(z) = 6z - 2$

$f'''(z) = 6$

$f^{iv}(z) = 0$

$f(2) = 14$

$f'(2) = 12$

$f''(2) = 10$

$f'''(2) = 6$

$f^{iv}(2) = 0$

From Taylor Series,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^n(a)}{n!}(z-a)^n + \cdots$$

$$\therefore f(z) = f(2) + f'(2)(z-2) + \frac{f''(2)}{2!}(z-2)^2 + \frac{f'''(2)}{3!}(z-2)^3 + \frac{f^{iv}(2)}{4!}(z-2)^4$$

$$= 14 + 12(z-2) + \frac{10}{2!}(z-2)^2 + \frac{6}{3!}(z-2)^3 + 0$$

$$= 14 + 12(z-2) + 5(z-2)^2 + (z-2)^3$$

Ans.

(iv.) $ze^{2z}$ at $z=-1$

Sol: $f(z) = ze^{2z}$

$f'(z) = z\cdot 2\cdot e^{2z} + e^{2z}\cdot 1$

$f''(z) = 4ze^{2z} + 2e^{2z} + 2e^{2z}$

$= 4(ze^{2z} + e^{2z})$

$f'''(z) = 4(2ze^{2z} + e^{2z} + 2e^{2z})$

$= 8ze^{2z} + 12e^{2z}$

$f(-1) = -e^{-2}$

$f'(-1) = -2e^{-2} + e^{-2} = -e^{-2}$

$f''(-1) = 0$

$f'''(-1) = -8e^{-2} + 12e^{-2}$

$= 4e^{-2}$

From Taylor's Series,

$$f(z) = f(-1) + f'(-1)(z+1) + \frac{f''(-1)}{2!}(z+1)^2 + \frac{f'''(-1)}{3!}(z+1)^3$$

$$= -e^{-2} - e^{-2}(z+1) + 0 + \frac{4e^{-2}}{3!}(z+1)^3$$

$$= e^{-2}\left(-1-(z+1) + \frac{4}{3!}(z+1)^3\right)$$

Ans.

(2) Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

(i) $|z| < 1$

Sol: $\dfrac{z}{(z-1)(2-z)} = \dfrac{A}{z-1} + \dfrac{B}{2-z}$

$\Rightarrow z = A(2-z) + B(z-1)$

When, $z = 1$

$1 = A(2-1) + B(1-1)$

$\therefore A = 1$

When, $z = 2$

$2 = A(2-2) + B(2-1)$

$\therefore B = 2$

$\therefore f(z) = \dfrac{1}{z-1} + \dfrac{2}{2-z} = \dfrac{1}{-(z+1)} + \dfrac{2}{2(1-z/2)}$

$= -(1-z)^{-1} + (1-z/2)^{-1}$

$= -[1 + z + z^2 + z^3 + \cdots] + [1 + \frac{z}{2} + (\frac{z}{2})^2 + (\frac{z}{2})^3 + \cdots]$

$= -[z + z^2 + z^3 + \cdots] + [\frac{z}{2} + (\frac{z}{2})^2 + (\frac{z}{2})^3 + \cdots]$

$= \sum_{n=0}^{\alpha} (-1) z^n + \sum_{n=0}^{\alpha} (\frac{z}{2})^n = \sum_{n=0}^{\alpha} (-1 + \frac{1}{2^n}) z^n$

Ans.

(ii.) $1 < |z| < 2$

Sol: Here, $|z| > 1$ & $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1$ $\Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\dots\right]+\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right]$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

Ans.

(iii) $|z| > 2$

Here, $\frac{1}{|z|} < \frac{1}{2}$

$\therefore \frac{2}{|z|} < 1$

Sol: $$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{\cancel{z(1-z/2)}\,z\left(\frac{2}{z}-1\right)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{-2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

Ans.

(iv.) $|z-1| > 1$

Sol: Let, $z-1 = w$

$$\Rightarrow w+1 = z$$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$f(w) = \frac{1}{w+1-1} + \frac{2}{2-w-1}$$

$$= \frac{1}{w} + \frac{2}{1-w}$$

$$= \frac{1}{w} + \frac{-2}{w(1-1/w)}$$

$$= \frac{1}{w} - \frac{2}{w}\sum_{n=0}^{\infty}\left(\frac{1}{w}\right)^n$$

$$\therefore f(z) = \frac{1}{z-1} - \frac{2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n$$

Ans.

(v) $0 < |z-2| < 1$

Sol: Let, $z-2 = w$

$$\therefore z = w+2$$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$f(w) = \frac{1}{w+2-1} + \frac{2}{2-w-2}$$

$$= \frac{1}{w+1} + \frac{2}{-w}$$

$$= \sum_{n=0}^{\infty}(-w)^n - \frac{2}{w}$$

$$\therefore f(z) = \sum_{n=0}^{\infty}(-z+2)^n - \frac{2}{z-2}$$

Ans.

(3). Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series valid for

(i) $0 < |z| < 2$ & (ii) $|z| > 2$

Sol: $\dfrac{1}{z(z-2)} = \dfrac{A}{z} + \dfrac{B}{z-2}$

$\Rightarrow 1 = A(z-2) + Bz$

When, $z = 0$

$1 = -2A$

$\therefore A = \dfrac{-1}{2}$

When, $z = 2$

$1 = 0 + 2B$

$\therefore B = \dfrac{1}{2}$

(i) $f(z) = \dfrac{1}{z(z-2)} = \dfrac{-1/2}{z} + \dfrac{1/2}{z-2}$

$= \dfrac{1}{2z-4} - \dfrac{1}{2z}$

$= \dfrac{1}{-4(1-z/2)} - \dfrac{1}{2z}$

$= \dfrac{-1}{4} \sum_{n=0}^{\alpha} \left(\dfrac{z}{2}\right)^n - \dfrac{1}{2z}$

Ans.

(ii) $f(z) = \dfrac{-1}{2z} + \dfrac{1}{2z-4}$

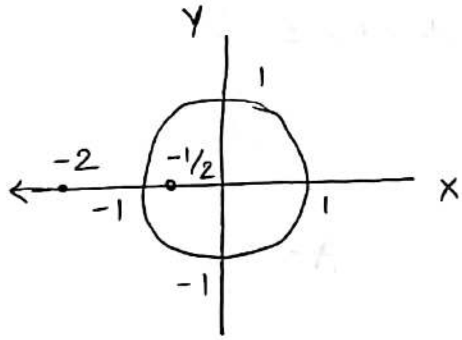
$= \dfrac{1}{2z(1-2/z)} - \dfrac{1}{2z}$

$= \dfrac{1}{2z} \sum_{n=0}^{\alpha} \left(\dfrac{2}{z}\right)^n - \dfrac{1}{2z}$

Ans.

④ Evaluate $\oint_C \frac{z^2}{2z^2+5z+2}\,dz$ using the residue at the poles where C is the unit circle $|z|=1$.

Sol: Here,




$$2z^2+5z+2=0$$

$$\Rightarrow 2z^2+4z+z+2=0$$

$$\Rightarrow 2z(z+2)+1(z+2)=0$$

$$\Rightarrow (z+2)(2z+1)=0$$

$\because |z|=1$

$\therefore z+2=0$ or, $2z+1=0$

$\therefore z=-2$ $\qquad$ $\therefore z=-\frac{1}{2}$

$\therefore z=-\frac{1}{2}$ is valid for $|z|=1$.

$\therefore$ Residue at $z=-\frac{1}{2}$

Here, $a=-\frac{1}{2}$ & $m=1$

Acording to Cauchy's-Residue Theorem,

$$\frac{1}{2\pi i}\oint f(z) = \lim_{z\to a} \frac{1}{(m-1)!}\cdot\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}$$

$$\therefore R = \lim_{z \to -1/2} \frac{1}{0!} \cdot 1 \cdot \left\{ \cancel{\left(z+\frac{1}{2}\right)}^{1} \frac{z^2}{(z+2)\cancel{\left(z+\frac{1}{2}\right)}} \right\}$$

$$= \lim_{z \to -1/2} \frac{z^2}{z+2}$$

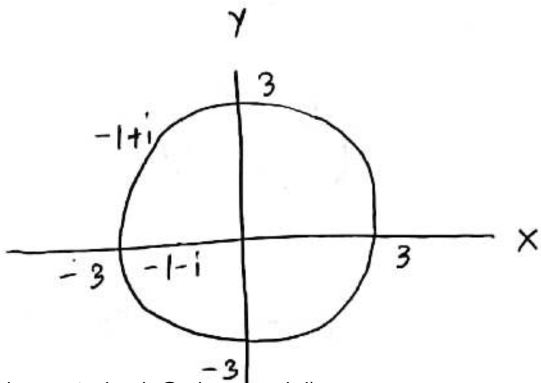
$$= \frac{1/4}{3/2}$$

$$= \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$$

$$\therefore \oint_C \frac{z^2}{2z^2+5z+2}\, dz = 2\pi i \cdot R$$

$$= 2\pi i \times \frac{1}{6} = \frac{\pi i}{3} \quad \underline{\text{Ans.}}$$

(5) Evaluate $\oint_C \frac{z^2+4}{z^3+2z^2+2z}\, dz$ using the residue at the poles, around the circle $|z| = 3$.




<u>Sol:</u> Here,

$$z^3 + 2z^2 + 2z = 0$$

$$\Rightarrow z(z^2 + 2z + 2) = 0$$

$$\Rightarrow z\,(z^2+2z+1+1)=0$$

$$\Rightarrow z\left\{(z+1)^2+1\right\}=0$$

$$\Rightarrow z\left\{(z+1)^2-i^2\right\}=0$$

$$\Rightarrow z\,(z+1+i)(z+1-i)=0$$

$$\therefore z=0,\ (-1-i),\ (-1+i)$$

Since all poles are inside the circle, so all are valid at $|z|=3$

Residue at $z=0$

Here, $a=0$ & $m=1$

$$\therefore R_1=\lim_{z\to 0}\frac{1}{0!}\frac{d^{m+1}}{dz^{m+1}}\left\{(z-a)^m f(z)\right\}$$

$$=\lim_{z\to 0}1\cdot 1\cdot \cancel{(z-0)}\frac{z^2+4}{\cancel{z}(z+1+i)(z+1-i)}$$

$$=\frac{0+4}{(1+i)(1-i)}$$

$$=\frac{4}{1-i^2}=\frac{4}{2}=2$$

$$R_2 = \lim_{z \to -1-i} \frac{\cancel{(z+1+i)}(z^2+4)}{z\cancel{(z+1+i)}(z+1-i)}$$

$$= \lim_{z \to -1-i} \frac{z^2+4}{z(z+1-i)}$$

$$= \frac{(-1-i)^2+4}{\cancel{z}(-1-i)(-1-i+1-i)}$$

$$= \frac{1+2i+i^2+4}{(-1-i)(-2i)}$$

$$= \frac{2i+4}{2i+2i^2}$$

$$= \frac{(2i+4)(2i+2)}{(2i\cancel{+}-2)(2i+2)}$$

$$= \frac{4i^2+4i+8i+8}{(2i)^2-2^2}$$

$$= \frac{12i+4}{-8} = \frac{4(3i+1)}{-8}$$

$$= -\frac{1}{2} - \frac{3i}{2}$$

$$R_3 = \lim_{z \to -1+i} \frac{(z+1-i)(z^2+4)}{z(z+1+i)(z+1-i)}$$

$$= \lim_{z \to -1+i} \frac{z^2+4}{z(z+1+i)}$$

$$= \frac{(-1+i)^2+4}{(-1+i)(-1+i+1+i)}$$

$$= \frac{1-2i+i^2+4}{(-1+i)\,2i}$$

$$= \frac{4-2i}{2i^2-2i}$$

$$= \frac{(4-2i)(-2+2i)}{(-2-2i)(-2+2i)}$$

$$= \frac{-8+8i+4i-4i^2}{(-2)^2-(2i)^2}$$

$$= \frac{12i-4}{8}$$

$$= -\frac{1}{2} + \frac{3i}{2}$$

$$\oint_C \frac{z^2+4}{z^3+2z^2+2z}\, dz = 2\pi i\,(R_1+R_2+R_3)$$

$$= 2\pi i\left(2-\frac{1}{2}-\frac{3i}{2}-\frac{1}{2}+\frac{3i}{2}\right)$$

$$= 2\pi i\,(2-1)$$

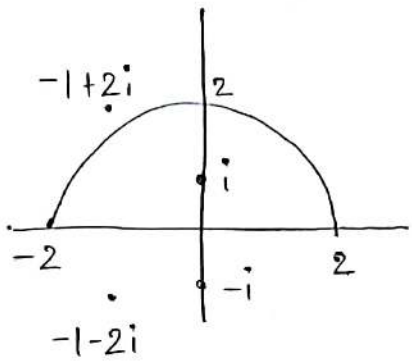
$$= 2\pi i$$

Ans.

(6.) Evaluate $\oint_C \frac{z e^{i\pi z}}{(z^2+2z+5)(z^2+1)^2}\, dz$ using the residue at the poles where $C$ is the upper half circle of equation $|z|=2$.

Sol: Considering the denominator,

$$(z^2+2z+5)(z^2+1)^2$$




$$= (z^2+2z+1+4)(z^2-i^2)^2$$

$$= \{(z+1)^2+2^2\}\{(z+i)(z-i)\}^2$$

$$= \{(z+1)^2-(2i)^2\}\,(z+i)^2(z-i)^2$$

$$= (z+1+2i)(z+1-2i)\,(z+i)^2(z-i)^2$$

$\therefore z = (-1-2i), (-1+2i), -i, +i$

Here,

$f(z)$ has simple poles at $z = (-1+2i), (-1-2i)$

& $f(z)$ has pole of order 2 at $z = -i, i$

Since $|z| = 2$, so only $z = +i, -i$ is valid for $|z| = 2$

$\therefore z = +i$ is valid for ~~uppter~~ upper half circle of $|z| = 2$

Now, Residue at $z = i$

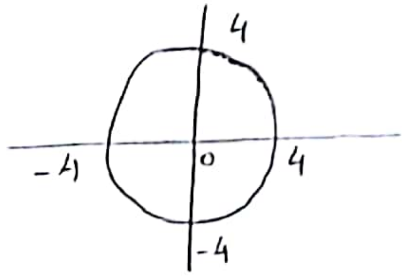
Here, $a = i$ & $m = 2$

$$R = \lim_{z \to a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$= \lim_{z \to i} \frac{1}{1!} \frac{d}{dz} \left\{ \cancel{(z-i)^2} \frac{z e^{i\pi z}}{(z+1+2i)(z+1-2i)(z+i)^2 \cancel{(z-i)^2}} \right\}$$

(7.) Evaluate $\frac{1}{2\pi i} \oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz$ using the residue at the poles, around the circle C with the equation $|z| = 4$

Sol: Here, the denominator,




$$z^4 + 10z^2 + 9$$

$$= z^4 + 9z^2 + z^2 + 9$$

$$= z^2(z^2+9) + 1(z^2+9)$$

$$= (z^2+9)(z^2+1)$$

$$= \{z^2 - (3i)^2\}\{z^2 - i^2\}$$

$$= (z+3i)(z-3i)(z+i)(z-i)$$

$\therefore\ z = +3i, -3i, +i, -i$

Since all the poles are ~~insides~~ the circle

So all poles are valid for $|z| = 4$

We know, Residue Theorem,

$$\frac{1}{2\pi i} \oint f(z) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$R_1 = \lim_{z\to -i} \frac{(z+i)(z^2-z+2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z\to -i} \frac{z^2-z+2}{(z-i)(z+3i)(z-3i)}$$

$$= \frac{-1+i+2}{(-2i)\,2i(-4i)}$$

$$= \frac{1+i}{-16i} = \frac{-1}{16i} - \frac{i}{16i}$$

$$= \frac{i}{16} - \frac{1}{16}$$

$$R_2 = \lim_{z\to i} \frac{(z-i)(z^2-z+2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z\to i} \frac{z^2-z+2}{(z+i)(z+3i)(z-3i)}$$

$$= \frac{-1-i+2}{2i\cdot 4i\,(-2i)}$$

$$= \frac{1-i}{16i} = \frac{1}{16i} - \frac{i}{16i}$$

$$= \frac{-i}{16} - \frac{1}{16}$$

$$R_3 = \lim_{z\to -3i} \frac{(z+3i)(z^2-z+2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z\to -3i} \frac{z^2-z+2}{(z+i)(z-i)(z-3i)}$$

$$= \frac{-9+3i+2}{(-2i)(-4i)(-6i)}$$

$$= \frac{3i-7}{+48i} = \frac{-7}{48i} + \frac{3}{48}$$

$$= \frac{+7i}{48} + \frac{1}{16}$$

$$R_4 = \lim_{z\to 3i} \frac{(z-3i)(z^2-z+2)}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$= \lim_{z\to 3i} \frac{z^2-z+2}{(z+i)(z-i)(z+3i)}$$

$$= \frac{-9-3i+2}{4i\cdot 2i\cdot 6i}$$

$$= \frac{-7-3i}{-48i} = \frac{3}{48} + \frac{7}{48i}$$

$$= \frac{1}{16} - \frac{7i}{48}$$

$$\frac{1}{2\pi i}\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\,dz = R_1 + R_2 + R_3 + R_4$$

$$= \frac{i}{16} - \frac{1}{16} - \frac{i}{16} - \frac{1}{16} + \frac{1}{16} + \frac{7i}{48} + \frac{1}{16} - \frac{7i}{48}$$

$$= 0$$

Ans.

## Practice Sheet #6

① Show that $\displaystyle\int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$

Sol: Lets consider, $\displaystyle\int_C \frac{\ln(z+i)}{z^2+1}\,dz$

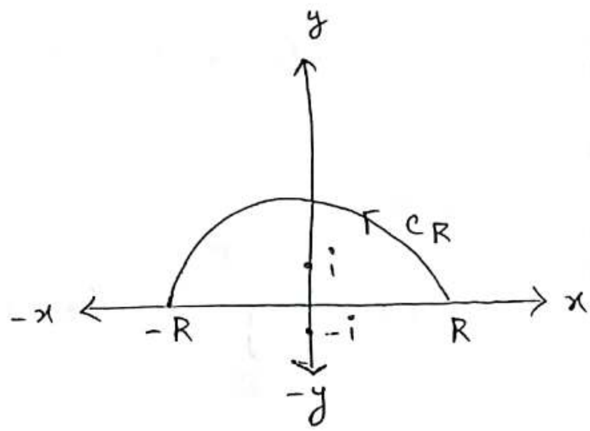
$\therefore f(z) = \dfrac{\ln(z+i)}{z^2+1}$

$z^2+1=0$

$\Rightarrow z^2 = -1$

$\Rightarrow z^2 = i^2$

$\therefore z = \pm i$




Only $z=i$ is valid for above region.

$\therefore$ Residue at $z=i$

$$R = \lim_{z\to i} (z-i)\frac{\ln(z+i)}{(z+i)(z-i)} = \frac{\ln(2i)}{2i}$$

$$\int_c f(z)\,dz = 2\pi i \cdot R$$

$$= 2\pi i \,\frac{\ln(2i)}{2i}$$

$$= \pi \ln(2i)$$

$$= \pi\ln 2 + \pi \ln i$$

$$= \pi \ln 2 + \pi \ln e^{i\pi/2}$$

$$= \pi\ln 2 + \pi\left(\frac{i\pi}{2}\right)$$

$$= \pi \ln 2 + \frac{i\pi^2}{2}$$

Here, $x = 0,\ y = 1\ \&\ r = 1$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{0}\right)$$

$$= \tan^{-1}(\infty) = \tan^{-1}\tan\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}$$

$$\because z = re^{i\theta}$$

$$\therefore i = e^{i\frac{\pi}{2}}$$

We know,

$$\int_c f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz$$

$$\therefore \pi\ln 2 + \frac{i\pi^2}{2} = \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{C_R} f(z)\,dz$$

$$= \int_{0}^{R} \frac{\ln(i-x)}{x^2+1}\,dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1}\,dx + \int_{C_R} f(z)\,dz$$



[ Replacing $x$ by $(-x)$ ]

Here,

$$\ln(i+x) + \ln(i-x) = \ln(i+x)(i-x)$$

$$= \ln(i^2 - x^2)$$

$$= \ln(-1-x^2)$$

$$= \ln\{(-1)(1+x^2\}$$

$$= \ln(-1) + \ln(x^2+1)$$

$$= \ln e^{2\pi i} + \ln(x^2+1)$$

$$= 2\pi i + \ln(x^2+1)$$

$\because x = 1,\ y = 0$

$\&\ r = 1$

$\therefore \theta = \tan^{-1}(0/1)$

$= \tan^{-1}\tan(\pi)$

$= \pi$

$\because z = e^{i\pi}$

$\therefore i = e^{i\pi}$

$\therefore i^2 = e^{2\pi i}$

$$\therefore \int_c f(z)\, dz = \int_0^R \frac{1}{x^2+1}\{2\pi i + \ln(x^2+1)\}\, dx + \int_{C_R} f(z)\, dz$$

By equating the real part of the equation we get,

$$\pi \ln 2 = \int_0^R \frac{\ln(x^2+1)}{x^2+1}\, dx$$

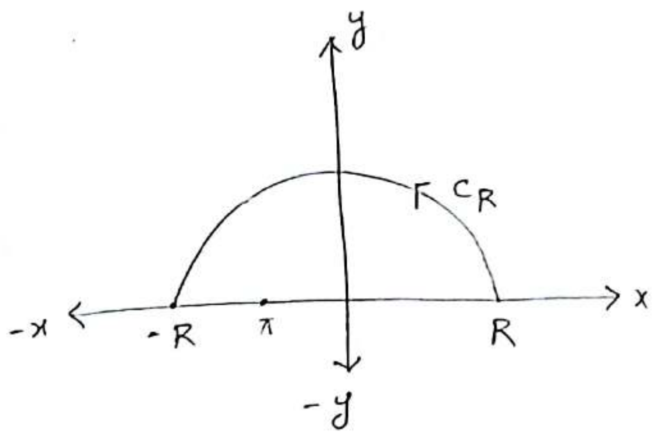
$$\Rightarrow \lim_{R\to\alpha} \int_0^R \frac{\ln(x^2+1)}{x^2+1}\, dx = \pi \ln 2$$

$$\therefore \int_0^\alpha \frac{\ln(x^2+1)}{x^2+1}\, dx = \pi \ln 2$$

Ans.

(2.) Show that $\int_0^{\alpha} \frac{1}{x^4+x^2+1}\,dx = \frac{\pi\sqrt{3}}{6}$

<u>Sol</u>: Let's consider, $\oint_C \frac{1}{z^4+x^2+1}\,dz$




Here,

$$f(z) = \frac{1}{z^4+z^2+1}$$

For poles,

$$z^4+z^2+1=0$$

$$\Rightarrow z^4+2z^2+1-z^2=0$$

$$\Rightarrow (z^2+1)^2-z^2=0$$

$$\Rightarrow (z^2+1+z)(z^2+1-z)=0$$

$\therefore\ z^2+z+1=0$ or, $z^2-z+1=0$

$$\Rightarrow z = \frac{-1\pm\sqrt{1^2-4.1.1}}{2\cdot 1} \qquad \Rightarrow z = \frac{1\pm\sqrt{(-1)^2-4.1.1}}{2\cdot 1}$$

$$= \frac{-1\pm\sqrt{3}i}{2} \qquad = \frac{1\pm\sqrt{-3}}{2}$$

$$= \frac{-1}{2}\pm\frac{i\sqrt{3}}{2} \qquad = \frac{1}{2}\pm\frac{i\sqrt{3}}{2}$$

So, the poles are $z = \left(-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right), \left(\frac{1}{2}\pm\frac{i\sqrt{3}}{2}\right)$ ~~lie inside~~ C.

But only $z = \frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ lies inside c.

$$R_1 = \lim_{z \to \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{\left(z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}{\left(z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z^2 + z + 1\right)}$$

$$= \lim_{z \to \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{1}{\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z^2 + z + 1\right)}$$

$$= \frac{1}{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left\{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + 1\right\}}$$

$$= \frac{1}{i\sqrt{3}\left\{\left(\frac{1}{4} + \frac{i\sqrt{3}}{2} - \frac{3}{4}\right) + \frac{1}{2} + \frac{i\sqrt{3}}{2} + 1\right\}}$$

$$= \frac{1}{i\sqrt{3}\,(i\sqrt{3}+1)} = \frac{1 - i\sqrt{3}}{i\sqrt{3}(1 + i\sqrt{3})(1 - i\sqrt{3})} = \frac{1 - i\sqrt{3}}{i\sqrt{3}\,(1+3)} = \frac{1 - i\sqrt{3}}{4\sqrt{3}i}$$

$$= \frac{1}{i\sqrt{3} - 3} \checkmark$$

$$R_2 = \lim_{z \to -\frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{\left(z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)}{\left(z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z^2 - z + 1\right)}$$

$$= \lim_{z \to -\frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{1}{\left(z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z^2 - z + 1\right)}$$

$$= \frac{1}{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left\{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + 1\right\}}$$

$$= \frac{1}{i\sqrt{3}\left\{\left(\frac{1}{4} - \frac{i\sqrt{3}}{2} - \frac{3}{4}\right) + \frac{1}{2} - \frac{i\sqrt{3}}{2} + 1\right\}}$$

$$= \frac{1}{i\sqrt{3}\,(1 - i\sqrt{3})} = \frac{1 + i\sqrt{3}}{i\sqrt{3}(1 - i\sqrt{3})(1 + i\sqrt{3})} = \frac{1 + i\sqrt{3}}{i\sqrt{3}\,(1 + 3)} = \frac{1 + i\sqrt{3}}{4\sqrt{3}i}$$

$$= \frac{1}{i\sqrt{3} + 3} \checkmark$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\,(R_1 + R_2) = 2\pi i \left(\frac{1 - i\sqrt{3} + 1 + i\sqrt{3}}{4\sqrt{3}i}\right)$$

$$= 2\pi i \left(\frac{2}{4\sqrt{3}i}\right)$$

$$= \frac{\pi}{\sqrt{3}}$$

We know,

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$\Rightarrow \frac{\pi}{\sqrt{3}} = \int_{-\alpha}^{\alpha} \frac{1}{x^4+x^2+1}\,dx + 0$$

$$= \int_{-\alpha}^{0} \frac{dx}{x^4+x^2+1} + \int_{0}^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$= \int_{\alpha}^{0} \frac{(-)\,dx}{x^4+x^2+1} + \int_{0}^{\alpha} \frac{dx}{x^4+x^2+1} \quad [\text{Replacing } x \text{ by } (-x)]$$

$$= \int_{0}^{\alpha} \frac{dx}{x^4+x^2+1} + \int_{0}^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$= 2\int_{0}^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$\Rightarrow \int_{0}^{\alpha} \frac{dx}{x^4+x^2+1} = \frac{\pi}{2\sqrt{3}} = \frac{\pi\sqrt{3}}{6} \quad [\text{Showed}]$$

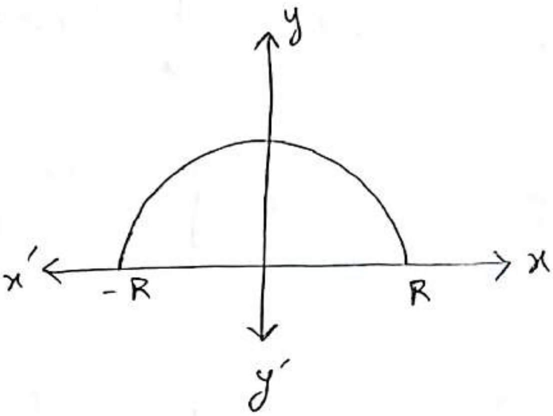
(3.) Show that $\int_0^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}$

Sol: Let's consider, $\int_C \frac{dz}{z^4+1}$ where $c$ is the contour.

For pole, $z^4+1=0$

$\Rightarrow z^4 = -1$

$\therefore z = (-1)^{1/4}$




Here,
$x=-1$ & $y=0$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-1)$

$= \tan^{-1}(0)$

$= \tan^{-1}\tan(\pi)$

$= \pi$

We know,
$z = e^{i(\theta+2\pi n)}$

$\therefore (-1) = e^{i(\pi+2\pi n)}$

$\Rightarrow (-1)^{1/4} = e^{i(\pi+2\pi n)\frac{1}{4}}$

$= e^{i\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)}$ ; $n=0,1,2,3$

$n=0$ : $z_1 = e^{\pi i/4} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$n=1$ : $z_2 = e^{i\frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$n=3:\ z_3 = e^{i\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)} = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$n=2:\ z_1 = e^{i\left(\frac{5\pi}{4}\right)} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$\therefore\ z = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ poles are inside c.

$$R_1 = \lim_{z \to \frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} \frac{1}{\left(z+\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\left(z-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\left(z+\frac{1}{2}+\frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}+\cancel{\frac{i}{\sqrt{2}}}+\frac{1}{\sqrt{2}}-\cancel{\frac{i}{\sqrt{2}}}\right)\left(\cancel{\frac{1}{\sqrt{2}}}+\frac{i}{\sqrt{2}}-\cancel{\frac{1}{\sqrt{2}}}+\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}+\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}\,(2i/\sqrt{2})\,(\sqrt{2}+2i/\sqrt{2})}$$

$$= \frac{1}{2i\cdot\sqrt{2}\left(1+\frac{2i}{2}\right)}$$

$$= \frac{1}{2\sqrt{2}i(1+i)}$$

$$= \frac{1}{2\sqrt{2}i - 2\sqrt{2}} = \frac{1}{2\sqrt{2}(i-1)}$$

$$R_2 = \lim_{z \to -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}} \frac{1}{\left(z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(z + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\left(-\frac{1}{\sqrt{2}} + \cancel{\frac{i}{\sqrt{2}}} - \frac{1}{\sqrt{2}} - \cancel{\frac{i}{\sqrt{2}}}\right)\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$$

$$= \frac{1}{(-\sqrt{2})(-\sqrt{2} + i\sqrt{2})(i\sqrt{2})}$$

$$= \frac{1}{-2i \cdot \sqrt{2}(i-1)}$$

$$= \frac{1}{-2\sqrt{2}i(i-1)}$$

$$= \frac{1}{+2\sqrt{2} + 2\sqrt{2}i}$$

$$= \frac{+1}{2\sqrt{2}(i+1)}$$

$$\therefore \oint_c \frac{dz}{z^4+1} = 2\pi i[R_1 + R_2] = 2\pi i\left[\frac{1}{2\sqrt{2}(i-1)} + \frac{1}{2\sqrt{2}(i+1)}\right]$$

$$= \frac{2\pi i}{2\sqrt{2}}\left[\frac{1}{i-1} + \frac{1}{i+1}\right]$$

$$= \frac{\pi i}{\sqrt{2}} \left[ \frac{i+1+i-1}{(i+1)(i-1)} \right]$$

$$= \frac{2\pi i^2}{\sqrt{2}\,(i^2-1^2)}$$

$$= \frac{-2\pi}{-2\sqrt{2}}$$

$$= \cancel{\frac{-\pi i}{\sqrt{2}}} \quad \frac{\pi}{2}$$

We know,

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$\Rightarrow \frac{\pi}{2} = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

$$= \int_{-\alpha}^{\alpha} \frac{dx}{x^4+1} + 0$$

$$= \int_{-\alpha}^{0} \frac{dx}{x^4+1} + \int_{0}^{\alpha} \frac{dx}{x^4+1} \quad = \int_{\alpha}^{0} \frac{-dx}{x^4+1} + \int_{0}^{\alpha} \frac{dx}{x^4+1}$$

[Replacing $x$ by $(-x)$]

$$= \int_0^{\alpha} \frac{dx}{x^4+1} + \int_0^{\alpha} \frac{dx}{x^4+1}$$

$$= 2\int_0^{\alpha} \frac{dx}{x^4+1}$$

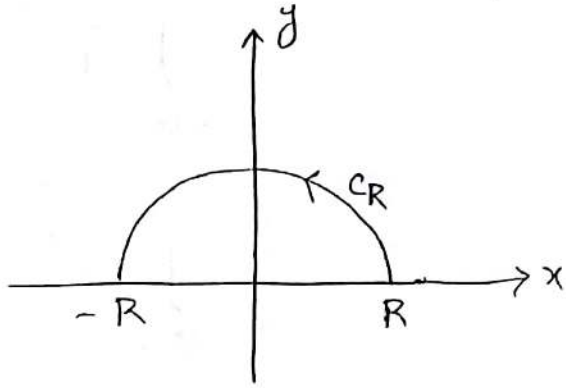
$$\therefore \int_0^{\alpha} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$ [Showed]

(4.) Show that $\int_0^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1}\, dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi\sqrt{3}}$

Sol: Lets consider $\int_C \frac{e^{i2\pi z}}{z^4+z^2+1}\, dz$

$$\therefore f(z) = \frac{e^{i2\pi z}}{z^4+z^2+1}$$

For poles,




$$(z^4+z^2+1) = 0$$

$$\Rightarrow z^4+2z^2+1-z^2 = 0$$

$$\Rightarrow (z^2+1)^2 - z^2 = 0$$



$$\Rightarrow (z^2+1+z)(z^2+1-z) = 0$$

$\therefore\ z^2+z+1=0$

$\Rightarrow z=\dfrac{-1\pm i\sqrt{3}}{2}$

$=-\dfrac{1}{2}\pm\dfrac{i\sqrt{3}}{2}$

or, $z^2-z+1=0$

$\Rightarrow z=\dfrac{-(-1)+\sqrt{(-1)^2-4\cdot1\cdot1}}{2\cdot1}$

$\Rightarrow z=\dfrac{1\pm i\sqrt{3}}{2}=\dfrac{1}{2}\pm\dfrac{i\sqrt{3}}{2}$

$\therefore z=\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right),\left(\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)$ poles are inside c.

$$R_1=\lim_{z\to-\frac{1}{2}+\frac{i\sqrt{3}}{2}}\frac{e^{i2\pi z}}{\left(z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(z-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\left(z-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}$$

$$=\frac{e^{i2\pi\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}}{\left(\cancel{-\frac{1}{2}}+\frac{i\sqrt{3}}{2}+\cancel{\frac{1}{2}}+\frac{i\sqrt{3}}{2}\right)\left(-\frac{1}{2}+\cancel{\frac{i\sqrt{3}}{2}}-\frac{1}{2}-\cancel{\frac{i\sqrt{3}}{2}}\right)\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}$$

$$=\frac{e^{-\pi i}\cdot e^{-\sqrt{3}\pi}}{i\sqrt{3}\cdot(-1)\left(i\sqrt{3}-1\right)}=\frac{(-1)e^{-\pi\sqrt{3}}}{-i^2\cdot3+i\sqrt{3}}=\frac{-e^{-\pi\sqrt{3}}}{i\sqrt{3}+3}$$

$$=\frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}\left(i+\sqrt{3}\right)}$$

$$R_2 = \lim_{z \to \frac{1}{2} + \frac{i\sqrt{3}}{2}} \frac{e^{i2\pi z}}{\left(z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}$$

$$= \frac{e^{i2\pi\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)}}{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2} + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} + \cancel{\frac{i\sqrt{3}}{2}} + \frac{1}{2} - \cancel{\frac{i\sqrt{3}}{2}}\right)\left(\cancel{\frac{1}{2}} + \frac{i\sqrt{3}}{2} - \cancel{\frac{1}{2}} + \frac{i\sqrt{3}}{2}\right)}$$

$$= \frac{e^{i\pi} \cdot e^{-\sqrt{3}\pi}}{(1 + i\sqrt{3}) \cdot 1 \cdot i\sqrt{3}} = \frac{(-1)\, e^{-\sqrt{3}\pi}}{i\sqrt{3} - 3} = \frac{-e^{-\sqrt{3}\pi}}{\sqrt{3}(i - \sqrt{3})}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\left[R_1 + R_2\right]$$

$$= 2\pi i\left[\frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}(i + \sqrt{3})} + \frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}(i - \sqrt{3})}\right]$$

$$= 2\pi i \times \frac{-e^{-\pi\sqrt{3}}}{\sqrt{3}} \cdot \left[\frac{1}{i + \sqrt{3}} + \frac{1}{i - \sqrt{3}}\right]$$

$$= \frac{-2\pi i e^{-\pi\sqrt{3}}}{\sqrt{3}}\left[\frac{i - \sqrt{3} + i + \sqrt{3}}{i^2 - (\sqrt{3})^2}\right]$$

$$= \frac{-2\pi i e^{-\pi\sqrt{3}}}{\sqrt{3}}\left[\frac{2i}{-1 - 3}\right] = \frac{-4\pi i^2 e^{-\pi\sqrt{3}}}{-4\sqrt{3}}$$



$$= \frac{-\pi e^{-\pi\sqrt{3}}}{}$$

Equating the real parts,

$$\oint_C \frac{\cos 2\pi z}{z^4+z^2+1}\,dz = -\frac{\pi}{\sqrt{3}} e^{-\pi\sqrt{3}} = \oint_C f(z)\,dz$$

We know,

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$\Rightarrow -\frac{\pi}{\sqrt{3}} e^{-\sqrt{3}\pi} = \lim_{R\to\alpha} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

$$= \int_{-\alpha}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1}\,dx + 0$$

$$= 2\int_{0}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1}\,dx$$

$$\therefore \int_{0}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1}\,dx = \frac{-\pi e^{-\sqrt{3}\pi}}{2\sqrt{3}} \qquad [\text{Showed}]$$

## Laplace Transformation of some elementary functions:

| | $F(t)$ | $L\{F(t)\} = f(s)$ |
|---|---|---|
| 1. | $1$ | $\frac{1}{s}$ |
| 2. | $t$ | $\frac{1}{s^2}$ |
| 3. | $t^n$ | $\frac{n!}{s^{n+1}}$ |
| 4. | $e^{at}$ | $\frac{1}{s-a}$ |
| 5. | $\sin at$ | $\frac{a}{s^2+a^2}$ |
| 6. | $\cos at$ | $\frac{s}{s^2+a^2}$ |
| 7. | $\sin hat$ | $\frac{a}{s^2-a^2}$ |
| 8. | $\cos hat$ | $\frac{s}{s^2-a^2}$ |

9. $\frac{t^n}{(n+1)!}$ $\qquad$ $\frac{1}{s^{n+1}}$ ; $n > -1$

# Laplace Transformation    Practice Sheet # 7

① Find the laplace transformation of each of following function:

(a) $3e^{-2t}$

Sol: $L\{3e^{-2t}\}$

$= 3L\{e^{-2t}\}$

$= 3 \cdot \frac{1}{s-(-2)}$

$= \frac{3}{s+2}$

(b) $4t^3 - e^{-t}$

$4L\{t^3\} - L\{e^{-t}\}$

$= 4 \cdot \frac{3!}{s^{3+1}} - \frac{1}{s-(-1)}$

$= \frac{24}{s^4} - \frac{1}{s+1}$

(c) $7\sin 2t - 3\cos 2t$

$= 7L\{\sin 2t\} - 3L\{\cos 2t\}$

$= 7 \frac{2}{s^2+2^2} - 3 \cdot \frac{s}{s^2+2^2}$

$= \frac{14}{s^2+4} - \frac{3s}{s^2+4} = \frac{14-3s}{s^2+4}$

(d) $(t^2+1)^2$

$= t^4 + 2t^2 + 1$

$= L\{t^4\} + 2L\{t^2\} + L\{1\}$

$= \frac{4!}{s^{4+1}} + 2\frac{2!}{s^{2+1}} + \frac{1}{s}$

$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$

(e) $(4e^{2t}-2)^3$ $\quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$= 64e^{6t} - 3\cdot 16e^{4t}\cdot 2 + 3\cdot 4e^{2t}\cdot 4 - 8$

$= 64e^{6t} - 96e^{4t} + 48e^{2t} - 8$

$= 64\, L\{e^{6t}\} - 96\, L\{e^{4t}\} + 48\, L\{e^{2t}\} - 8\, L\{1\}$

$= 64\cdot \frac{1}{s-6} - 96\cdot \frac{1}{s-4} + 48\, \frac{1}{s-2} - 8\cdot \frac{1}{s}$

$= \frac{64}{s-6} - \frac{96}{s-4} + \frac{48}{s-2} - \frac{8}{s}$ Ans.

2. Evaluate each of the following:

(i) $L\{t^3 e^{-3t}\}$

$= f(s+3)$

$= \frac{6}{(s+3)^4}$

Ans.

Here,

$f(s) = L\{t^3\}$

$= \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$\therefore f(s-(-3)) = f(s+3) = \frac{6}{(s+3)^4}$

(ii) $L\{5e^{3t}\sin 4t\}$

$= 5\,L\{e^{3t}\sin 4t\}$

$= 5\,f(s-3)$

$= 5\cdot\dfrac{4}{(s-3)^2+16}$

$= \dfrac{20}{s^2-6s+25}$ Ans.

Here,

$f(s) = L\{\sin 4t\} = \dfrac{4}{s^2+4^2}$

$f(s-3) = \dfrac{4}{(s-3)^2+16}$

(iii) $L\{(t+2)^2e^t\}$

$= L\{(t^2+4t+4)e^t\}$

$= L\{t^2e^t\} + L\{4te^t\} + 4\,L\{e^t\}$

$= \dfrac{2}{(s-1)^3} + \dfrac{4}{(s-1)^2} + \dfrac{4}{s-1}$

Ans.

Here,

(i) $f(s) = L\{t^2\} = \dfrac{2!}{s^{2+1}} = \dfrac{2}{s^3}$

$\therefore f(s-1) = \dfrac{2}{(s-1)^3}$

(ii) $f(s) = L\{t\} = \dfrac{1}{s^2}$

$\therefore f(s-1) = \dfrac{1}{(s-1)^2}$

(iii) $f(s) = L\{1\} = \dfrac{1}{s}$

$\therefore f(s-1) = \dfrac{1}{s-1}$

(iv) $L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

$= 3\,L\{e^{-t}\sinh 2t\} - 5\,L\{e^{-t}\cosh 2t\}$

$= 3 \cdot \dfrac{2}{(s+1)^2-4} - 5\,\dfrac{s+1}{(s+1)^2-4}$

$= \dfrac{6}{(s+1)^2-4} - \dfrac{5s+5}{(s+1)^2-4}$

$= \dfrac{6-5s+5}{(s+1)^2-4}$

$= \dfrac{1-5s}{s^2+2s-3}$ Ans.

Here,

(i) $f(s) = L\{\sinh 2t\}$

$= \dfrac{2}{s^2-2^2}$

$\therefore f(s-(-1)) = f(s+1)$

$= \dfrac{2}{(s+1)^2-4}$

(ii) $f(s) = L\{\cosh 2t\}$

$= \dfrac{s}{s^2-2^2}$

$\therefore f(s+1) = \dfrac{s+1}{(s+1)^2-4}$

(v) $L\{e^{-4t}\cosh 2t\}$

Here,

$f(s) = L\{\cosh 2t\} = \dfrac{s}{s^2-2^2}$

$\therefore f(s-(-4)) = f(s+4) = \dfrac{s+4}{(s+4)^2-4}$

$\therefore L\{e^{-4t}\cosh 2t\} = \dfrac{s+4}{s^2+8s+12}$ Ans.

(vi) $L\{e^{2t}(3\sin 4t - 4\cos 4t)\}$

$= 3L\{e^{2t}\sin 4t\} - 4L\{e^{2t}\cos 4t\}$

Here,

(i) $f(s) = L\{\sin 4t\}$

$= \dfrac{4}{s^2+4^2}$

$\therefore f(s-2) = \dfrac{4}{(s-2)^2+16}$

(ii) $f(s) = L\{\cos 4t\}$

$= \dfrac{s}{s^2+4^2}$

$\therefore f(s-2) = \dfrac{s-2}{(s-2)^2+16}$

Now,

$3L\{e^{2t}\sin 4t\} - 4L\{e^{2t}\cos 4t\} = \dfrac{12}{(s-2)^2+16} - \dfrac{4s-8}{(s-2)^2+16}$

$= \dfrac{12-4s+8}{(s-2)^2+16}$

$= \dfrac{20-4s}{s^2-4s+20}$ Ans.

3. Determine each of the following:

(i) $L^{-1}\left\{\frac{12}{4-3s}\right\}$

$= L^{-1}\left\{\frac{12}{-3(s-4/3)}\right\}$

$= L^{-1}\left\{\frac{-4}{s-4/3}\right\}$

$= -4\, L^{-1}\left\{\frac{1}{s-4/3}\right\}$

$= -4e^{4/3\, t}$

Ans.

(ii) $L^{-1}\left\{\frac{2s-5}{s^2-9}\right\}$

$= L^{-1}\left\{\frac{2s}{s^2-3^2}\right\} - L^{-1}\left\{\frac{5}{s^2-3^2}\right\}$

$= 2\, L^{-1}\left\{\frac{s}{s^2-3^2}\right\} - \frac{5}{3} L^{-1}\left\{\frac{3}{s^2-3^2}\right\}$

$= 2\cos 3t - \frac{5}{3}\sin 3t$

Ans.

(iii) $L^{-1}\left\{\frac{23s-15}{s^2+8}\right\}$

$= L^{-1}\left\{\frac{23s}{s^2+(2\sqrt{2})^2}\right\} - L^{-1}\left\{\frac{15}{s^2+(2\sqrt{2})^2}\right\}$

$= 23\, L^{-1}\left\{\frac{s}{s^2+(2\sqrt{2})^2}\right\} - \frac{15}{2\sqrt{2}} L^{-1}\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$

$= 23\cos 2\sqrt{2}t - \frac{15}{2\sqrt{2}}\sin 2\sqrt{2}t$

Ans.

(vi) $L^{-1}\left\{\frac{1}{s^4}\right\} = L^{-1}\left\{\frac{1}{s^{3+1}}\right\} = \frac{t^3}{3!} = \frac{t^3}{6}$ Ans.

(iv) $L^{-1}\left\{\frac{1}{s^{3/2}}\right\}$

$= L^{-1}\left\{\frac{1}{s^{\frac{1}{2}+1}}\right\}$

$= \frac{t^{1/2}}{\Gamma\left(\frac{1}{2}+1\right)}$

$= \frac{t^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$

$= \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}}$

$= 2\sqrt{\frac{t}{\pi}}$ Ans.

$\left[\because \frac{1}{s^{n+1}} ; n>-1 = \frac{t^n}{(n+1)!}\right]$

$(n+1)! = \Gamma(n+1) = n\,\Gamma(n) = \cancel{n!}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(v) $L^{-1}\left\{\frac{s+1}{s^{4/3}}\right\}$

Here, $\frac{s+1}{s^{4/3}} = \frac{s}{s^{4/3}} + \frac{1}{s^{4/3}} = \frac{1}{s^{1/3}} + \frac{1}{s^{4/3}} = \frac{1}{s^{-2/3+1}} + \frac{1}{s^{1/3+1}}$

$L^{-1}\left\{\frac{s+1}{s^{4/3}}\right\} = L^{-1}\left\{\frac{1}{s^{-\frac{2}{3}+1}}\right\} + L^{-1}\left\{\frac{1}{s^{\frac{1}{3}+1}}\right\}$

$= \frac{t^{-2/3}}{\left(-\frac{2}{3}+1\right)!} + \frac{t^{1/3}}{\left(\frac{1}{3}+1\right)!} = \frac{t^{-2/3}}{\Gamma\frac{1}{3}} + \frac{t^{1/3}}{\Gamma\frac{4}{3}}$

Ans.

(vii) $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\}$

$= L^{-1}\left\{\frac{1}{\sqrt{2(s+3/2)}}\right\}$

$= \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{(s+3/2)^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}} e^{-3/2\, t} \quad L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$

$= \frac{e^{-3t/2}}{\sqrt{2}} \quad L^{-1}\left\{\frac{1}{s^{-\frac{1}{2}+1}}\right\}$

$= \frac{e^{-\frac{3t}{2}}}{\sqrt{2}} \quad \frac{t^{-\frac{1}{2}}}{(\frac{1}{2})!}$

$= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{3t}{2}}$

Ans.

(viii) $L^{-1}\left\{\frac{1}{(s+4)^{5/2}}\right\}$

$= e^{-4t} \quad L^{-1}\left\{\frac{1}{s^{5/2}}\right\}$

$= e^{-4t} \quad L^{-1}\left\{\frac{1}{s^{3/2+1}}\right\}$

$= e^{-4t} \cdot \frac{t^{3/2}}{(\frac{3}{2}+1)!}$

$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2}\Gamma\frac{3}{2}} \quad \left[\Gamma(n+1) = n\Gamma(n)\right]$

$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}$

$= e^{-4t} \frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$

$= \frac{4}{3} e^{-4t} \quad \frac{t^{3/2}}{\pi}$ Ans

4. Evaluate each of the following using partial fraction:

(i) $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$=L^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$

$=6L^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\}+2L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$

$=6e^{2t}\cos 4t + 2e^{2t}\sin 4t$

$=2e^{2t}(3\cos 4t+\sin 4t)$ Ans.

(ii) $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

$=L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$

$=4L^{-1}\left\{\frac{s+4}{(s+4)^2}\right\}-4L^{-1}\left\{\frac{1}{(s+4)^2}\right\}$

$=4L^{-1}\left\{\frac{1}{s+4}\right\}-4e^{-4t}\cdot t$

$=4e^{-4t}-4te^{-4t}=4e^{-4t}(1-t)$

(iii) $L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Sol: $\frac{2s^2-4}{(s+1)(s-2)(s-3)}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{s-3}$

$\therefore 2s^2-4=A(s-2)(s-3)+B(s-3)(s+1)+C(s+1)(s-2)$

$s=2:\quad 2\cdot 2^2-4 = B\cdot 3\cdot(-1) \Rightarrow 8-4=-3B \Rightarrow B=\frac{-4}{3}$

$s=3:\quad 2\cdot 3^2-4 = C\cdot 4\cdot 1 \Rightarrow 18-4=4C \Rightarrow C=\frac{7}{2}$

$s=-1:\quad 2\cdot 1-4 = A(-3)(-4) \Rightarrow 2-4=12A \Rightarrow A=\frac{-1}{6}$

$$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)} = -\frac{1}{6(s+1)} - \frac{4}{3(s-2)} + \frac{7}{2(s-3)}$$

$$L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = -\frac{1}{6}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{7}{2}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Ans.

(v) $L^{-1}\left\{\dfrac{3s+1}{(s^2+1)(s-1)}\right\}$

Sol: $$\frac{3s+1}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{A}{s-1} + \frac{Bs}{s^2+1} + \frac{C}{s^2+1}$$

$\therefore\ 3s+1 = A(s^2+1) + Bs(s-1) + C(s-1)$

$s=1:\quad 4 = 2A \quad \Rightarrow A = 2$

$s=0:\quad 1 = A - C \quad \Rightarrow C = 2-1 = 1$

$s=-1:\quad -3+1 = 2A + 2B - 2 \Rightarrow B = \dfrac{-2-4+2}{2} = -2$

$$\frac{3s+1}{(s^2+1)(s-1)} = \frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore L^{-1}\left\{\frac{3s+1}{(s^2+1)(s-1)}\right\} = 2L^{-1}\left\{\frac{1}{s-1}\right\} - 2L^{-1}\left\{\frac{s}{s^2+1^2}\right\} + L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$= 2e^t - 2\cos t + \sin t$$

Ans.

(vii) $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

Sol: Here, $$\frac{s^2+2s+3}{\{(s^2+2\cdot s\cdot 1+1^2)+1\}\{(s^2+2\cdot s\cdot 1+1^2)+2^2\}}$$

$$= \frac{s^2+2s+3}{\{(s+1)^2+1^2\}\{(s+1)^2+2^2\}}$$

Now, $$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\therefore s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 +$$

$$2Cs + Ds^2 + 2Ds + D$$

$$= s^3(A+C) + (2A+B+2C+D)s^2 + s(5A+2B+2C+2D) + (5B+2D)$$

By equating both sides we get,

$$A + C = 0 \quad \ldots\ldots\ldots (i)$$

$$2A + B + 2C + D = 1 \quad \ldots\ldots (ii)$$

$$5A + 2B + 2C + 2D = 2 \quad \ldots\ldots (iii)$$

$$5B + 2D = 3 \quad \ldots\ldots (iv)$$

From $eq^n$ (ii) $\Rightarrow 2(A+C) + B + D = 1$

$$\Rightarrow B + D = 1 \quad \ldots\ldots (v)$$

(iv) $- 2\times$(v) $\Rightarrow$

$$\begin{array}{r} 5B + 2D = 3 \\ -2B - 2D = -2 \\ \hline 3B = 1 \end{array}$$

$$\therefore B = \frac{1}{3}$$

From $eq^n$ (v) $\Rightarrow D = 1 - B$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

Substituting the value of B, D & $A = -C$ in $eq^n$ (iii) we get,

$$-5C + 2C + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = 2$$

$$\Rightarrow -3C + \frac{2}{3} + \frac{4}{3} - 2 = 0$$

$$\Rightarrow \frac{2+4-6}{3} = 3C$$

$$\Rightarrow 0 = 3C$$

$$\therefore C = 0$$

From $eq^n$ (i) $\Rightarrow A + 0 = 0$

$$\therefore A = 0$$

$$\therefore L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{0+\frac{1}{3}}{s^2+2s+2} + \frac{0+\frac{2}{3}}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

Ans.

(iv) $L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$

Sol: Here,

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$\therefore 5s^2-15s-11 = A(s-2)^3 + B(s-2)^2(s+1) + C(s+1)(s-2) + D(s+1)$$

$$= A\left(s^3 - 3s^2\cdot 2 + 3s\cdot 2^2 - 2^3\right) + B\left(s^2 - 2\cdot s\cdot 2 + 2^2\right)(s+1)$$

$$+ C(s^2 - 2s + s - 2) + D(s+1)$$

$$= A\left(s^3 - 6s^2 + 12s - 8\right) + B\left(s^3 - 4s^2 + 4s + s^2 - 4s + 4\right)$$

$$+ C(s^2 - s - 2) + D(s+1)$$

$$= As^3 - 6As^2 + 12As - 8A + Bs^3 - 3Bs^2 + 4B + Cs^2 - Cs$$

$$- 2C + Ds + D$$

$$= s^3(A+B) + s^2(-6A-3B+C) + s(12A-C+D) + (-8A+4B-2C+D)$$

By equating both sides we get,

$A+B=0$ ....... (i)

$-6A-3B+C=5$ ..... (ii)

$12A-C+D=-15$ ..... (iii)

$-8A+4B-2C+D=-11$ .... (iv)

Now,

From $eq^n$ (i) $\Rightarrow B = -A \ldots$ (v)

$eq^n$ (ii) $\Rightarrow -6A - 3(-A) + C = 5$

$\Rightarrow -6A + 3A + C = 5$

$\Rightarrow C = 5 + 3A \ldots$ (vi)

From (iii) $\Rightarrow 12A - 5 - 3A + D = -15$

$\Rightarrow D = -9A - 10 \ldots$ (vii)

(iv) $\Rightarrow -8A - 4A - 2(3A+5) - 9A - 10 = -11$

$\Rightarrow -21A - 6A - 10 - 10 = -11$

$\Rightarrow -27A = 9$

$\therefore A = -1/3$

(v) $\Rightarrow B = \frac{1}{3}$

(vi) $\Rightarrow C = 5 + (-1) = 4$

(viii) $\Rightarrow D = 3 - 10 = -7$

$$L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$-7L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4te^{2t} - 7e^{2t}L^{-1}\left\{\frac{1}{s^{2+1}}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4te^{2t} - \frac{7}{2}t^2e^{2t}$$

Ans.

5. Solve the given differential equation:

(i) $Y'' - 3Y' + 2Y = 4e^{2t}$ , $Y(0) = -3$, $Y'(0) = 5$

Sol: $L\{Y'' - 3Y' + 2Y\} = L\{4e^{2t}\}$

$\Rightarrow L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$

$\Rightarrow [s^2L\{Y\} - s^1Y(0) - s^0Y'(0)] - 3[sL\{Y\} - s^0Y(0)] + 2[L\{Y\}]$

$$= 4\left(\frac{1}{s-2}\right)$$

$\Rightarrow [s^2y - sy(0) - y'(0)] - 3[sy - y(0)] + 2y = \frac{4}{s-2}$ $[\text{Let, } L\{Y\} = y]$

$$\Rightarrow \left[s^2 y - s(-3) - 5\right] - 3\left[sy - (-3)\right] + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2 y + 3s - 5 - 3sy - 9 + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2 y - 3sy + 2y = \frac{4}{s-2} - 3s + 14$$

$$\Rightarrow y\,(s^2 - 3s + 2) = \frac{4 - 3s^2 + 6s + 14s - 28}{s-2}$$

$$\Rightarrow y = \frac{20s - 3s^2 - 24}{(s^2 - 3s + 2)(s-2)} = \frac{20s - 3s^2 - 24}{(s^2 - 2s + 1 - s + 1)(s-2)}$$

$$= \frac{20s - 3s^2 - 24}{\{(s-1)^2 - (s-1)\}(s-2)}$$

$$= \frac{20s - 3s^2 - 24}{(s-1)(s-1-1)(s-2)}$$

$$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} \quad \cdots \; (i)$$

Now,

$$\frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} \quad \cdots \; (ii)$$

$$\Rightarrow 20s - 3s^2 - 24 = A(s-2)^2 + B(s-2)(s-1) + C(s-1) \quad \cdots \; (iii)$$

$s=2$ : $-12+40-24 = 0+0+C(2-1) \Rightarrow C=4$.

$s=1$ : $-3+20-24 = A \Rightarrow A=-7$

$s=0$ : $-24 = 4A+2B-C = -28+2B-4 \Rightarrow B = \frac{8}{2} = 4$

From equation (ii) we get,

$$\frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

From eq$^n$(i) $\Rightarrow y = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

$$\Rightarrow L\{Y\} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\Rightarrow Y = L^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\}$$

$$= -7L^{-1}\left\{\frac{1}{s-1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= -7e^{t} + 4e^{2t} + 4te^{2t}$$

Ans.

(ii) $Y'' + 9Y = \cos 2t, \quad Y(0) = 1, \; Y(\pi/2) = -1$

Sol: Here, $Y'(0)$ is unknow, so assume $Y'(0) = C$

$\therefore L\{Y'' + 9Y\} = L\{\cos 2t\}$

$\Rightarrow L\{Y''\} + 9L\{Y\} = \frac{s}{s^2+2^2}$

$\Rightarrow [s^2 L\{Y\} - sY(0) - Y'(0)] + 9y = \frac{s}{s^2+4}$ $\quad$ [Let, $L\{Y\} = y$]

$\Rightarrow s^2 y - s - c + 9y = \frac{s}{s^2+4}$

$\Rightarrow y(s^2+9) = \frac{s}{s^2+4} + s + c$

$\Rightarrow y = \frac{s}{(s^2+9)(s^2+4)} + \frac{s}{(s^2+9)} + \frac{c}{s^2+9}$

$\Rightarrow L\{Y\} = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{s^2+9} + \frac{c}{s^2+9}$

$\Rightarrow Y = L^{-1}\left\{\frac{s}{5(s^2+4)} + \frac{4}{5}\frac{s}{(s^2+9)} + \frac{c}{s^2+9}\right\}$

$Y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t$

$\therefore Y\left(\frac{\pi}{2}\right) = \frac{4}{5}(0) + \frac{c}{3}(-1) + \frac{1}{5}(-1)$

$\Rightarrow -1 = -\frac{c}{3} - \frac{1}{5}$

$\Rightarrow c = \left(1 - \frac{1}{5}\right) \times \frac{1}{3} = \frac{12}{5}$

$\therefore Y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t$ Ans.

(iii) $Y'' + 2Y' + 5Y = e^{-t}\sin t$ , $Y(0) = 0$, $Y'(0) = 1$

Sol: $L\{Y''\} + 2L\{Y'\} + 5L\{Y\} = L\{e^{-t}\sin t\}$

$\Rightarrow [s^2 L\{Y\} - sY(0) - Y'(0)] + 2[sL\{Y\} - Y(0)] + 5y = \frac{1}{(s+1)^2+1}$

$\Rightarrow [s^2 y - s.0 - 1] + 2[sy - 0] + 5y = \frac{1}{(s^2+2s+1)+1}$

$\Rightarrow s^2 y - 1 - 2sy + 5y = \frac{1}{s^2+2s+2}$

$\Rightarrow y(s^2 - 2s + 5) = \frac{1}{s^2+2s+2} + 1$

$\Rightarrow y = \frac{1 + s^2 + 2s + 2}{(s^2+2s+2)(s^2-2s+5)}$

$$L\{Y\} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \quad \cdots\cdots (i)$$

[Same as 4(vii)]

Now,

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \quad \cdots (ii)$$

$$\Rightarrow s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B$$
$$Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$$

$$= s^3(A+C) + s^2(2A+B+2C+D) + s(5A+2B+2C+2D)$$
$$+ (5B+2D)$$

Equating both sides of the equation we get,

$A + C = 0 \quad \cdots\cdots (iii)$

$2A + B + 2C + D = 1 \quad \cdots\cdots (iv)$

$5A + 2B + 2C + 2D = 2 \quad \cdots (v)$

$5B + 2D = 3 \quad \cdots\cdots (vi)$

Now, (iii) $\Rightarrow A = -C \quad \cdots (vii)$

$(vi) \Rightarrow 5B + 2D = 3$

$\cancel{(vi)} \Rightarrow B = \frac{3-2D}{5} \ldots\ldots (viii)$

$(iv) \Rightarrow 2(-C) + \frac{3-2D}{5} + 2C + D = 1$

$\Rightarrow \frac{3-2D}{5} + D = 1$

$\Rightarrow 3 + 3D = 5$

$\therefore D = \frac{2}{3}$

$(viii) \Rightarrow B = \frac{3 - \frac{4}{3}}{5} = \frac{9-4}{3} \times \frac{1}{5} = \frac{1}{3}$

$(v) \Rightarrow 5(-C) + 2 \cdot \frac{1}{3} + 2C + 2 \cdot \frac{2}{3} = 2$

$\Rightarrow -3C + \frac{2}{3} + \frac{4}{3} = 2$

$\Rightarrow \frac{2+4-6}{3} = 3C$

$\Rightarrow C = \frac{0}{9}$

$\therefore C = 0$



$\therefore A = 0$

Substituting the value of A, B, C & D in eq$^n$ (ii)

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{0\cdot s+\frac{1}{3}}{s^2+2s+2} + \frac{0\cdot s+\frac{2}{3}}{s^2+2s+5}$$

From eq$^n$ (i)

$$Y = L^{-1}\left\{\frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{1}{3}L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$

$$= \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

Ans.

(iv) $Y''' - 3Y'' + 3Y' - Y = e^t t^2$ $\qquad Y(0)=0,\ Y'(0)=0,\ Y''(0)=-2$

Sol: Let, $L\{Y\} = y$

$\therefore L\{Y'''\} - 3L\{Y''\} + 3L\{Y'\} - L\{Y\} = L\{e^t t^2\}$

$\Rightarrow [s^3L\{Y\} - s^2Y(0) - sY'(0) - Y''(0)] - 3[s^2L\{Y\} - sY(0) - Y'(0)]$

$$+3\left[s L\{Y\} - Y(0)\right] - y = \frac{2!}{(s-1)^{2+1}}$$

$$\Rightarrow \left[s^3 y + 2\right] - 3\left[s^2 y - 0 - 0\right] + 3\left[sy - 0\right] - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3 y + 2 - 3s^2 y + 3sy - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow y(s^3 - 3s^2 + 3s - 1) = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow \cancel{y = \frac{2 - 2(s-1)^3}{(s^3 - 3s^2 + 3s - 1)(s-1)^3}}$$

$$\Rightarrow y(s-1)^3 = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow y = \frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}$$

$$\Rightarrow L\{Y\} = \frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}$$

$$\Rightarrow Y = 2L^{-1}\left\{\frac{1}{(s-1)^6}\right\} - 2L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2e^t \frac{t^5}{5!} - 2e^t \frac{t^2}{2!}$$

$$= \frac{e^t t^5}{60} - e^t t^2$$

Ans.

(v) $tY'' + 2Y' + tY = 0 \qquad Y(0)=1,\ Y(t)=0,\ Y'(0)=c$

Sol: $L\{tY''\} + 2L\{Y'\} + L\{tY\} = L\{0\}$

$\Rightarrow -\frac{d}{ds}\left[s^2 L\{Y\} - sY(0) - Y'(0)\right] + 2\left[sL\{Y\} - Y(0)\right]$

$- \frac{d}{ds}(y) = 0$

$\Rightarrow -\frac{d}{ds}\left[s^2 y - s - c\right] + 2(sy - 1) - \frac{dy}{ds} = 0$

$\Rightarrow -\left[s^2 \frac{dy}{ds} + 2sy - 1\right] + 2sy - 2 - \frac{dy}{ds} = 0$

$\Rightarrow -s^2 \frac{dy}{ds} - \frac{dy}{ds} - 2sy + 1 + 2sy - 2 = 0$

$\Rightarrow -\frac{dy}{ds}\left[s^2+1\right] = 1$

$\Rightarrow \frac{dy}{ds} = \frac{-1}{s^2+1}$

$\therefore y = -\tan^{-1}s + c_1$

$= -\tan^{-1}s + \frac{\pi}{2}$

$= +\tan^{-1}\left(\frac{1}{s}\right)$

$-\tan^{-1}(\infty) + c_1 = 0$

$\Rightarrow -\pi/2 + c_1 = 0$

$\therefore c_1 = \frac{\pi}{2}$

$\therefore Y = L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} = \frac{\sin t}{t}$