18/11/21

Muntasir Alam

PhD: Game theory / AI

Dept. of math: D.U

Topics

- Probability Law
- Set
- Probabilistic Model
- Conditional probability
- Independence
- Total probability theorem
- Bayés theorem.

First chapter.

↳ Discrete random variables :

Probability mass function. probability distribution function, Expectation and variance, Well known distributions : Uniform, binomial, bernoulli, ~~poison~~ poission.

↳ Continuous random variables :

Probability density function, ~~cumit~~ cumilative distribution function, Expectation variance.

↳ Joint random variables :

Joint pdf, pmf, expectation, variance, correlation, covariance, Independence of random var, Informal probability and stat model, Properties and samples.

Book : Probability and ~~statistics and~~ scientist, Academic press

— Sheldon M. Ross.

Probability and Stochastic process.

19/11/21

Set and set notation revisited

- A set should be well defined
- No repeat of elements.

* By well defined it means that, each element of the set should express the same thing.

* Universal set in probability:
  - A set of all possible outcomes.

* Null set → subset of all set.

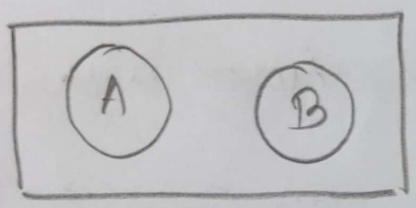
* Basic set operation:
  - Union
  - Intersection
  - Complement

  → To make new sets from existing ones.

Important properties of set :

A few important properties that a collection of set shows.

↳ 1) Mutually Exclusive :



$A \cap B = \phi$, $A \neq B$

↳ 2) Collectively Exhaustive :

A, B, C .... if they are sets and

$A \cup B \cup C \ldots = U$ (Universal set)

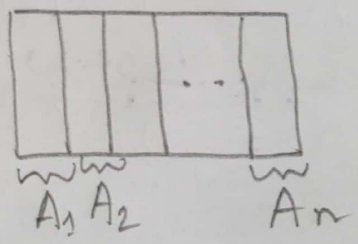
then they are collectively exhaustive.

$$\underbrace{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S}$$

$\bigcup_{i=1}^{n} A_i$ = কোনো কোনো সিট

↳ 3) Partition

A collection of sets $A_1, A_2, A_3 \ldots A_n$ is a partition if it is both mutually exclusive and collectively exhaustive.



↳ 4) De morgan's law:

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

Set theory in probability:

↳ An event is a set of outcomes.

Random experiment:

↳ An experiment that can result in different outcome on every repeat.

2 things are essential

- → Procedure (Monitor or commence experim
- → Observation (To approximate the result)
- → Model (To summerize the whole event)

Sample Space:

- → Discrete
- → Continuous

Set laws:

(i) $A \cup B = B \cup A$
$A \cap B = B \cap A$ } Communitive law

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$ } Associative law.

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ } Distributive law.

(iv) $A \cup \phi = A$
$A \cap \phi = \phi$
$A \cap S = A$
$A \cup S = S$ } Identity law.

S = Universal set.

(v) $A \cup A = A$
$A \cap A = A$ } Idempotent law.

(vi) $A \cap A' = \phi$

$A \cup A = S$

$(A')' = A$

~~(viii)~~

(vii) $P(S) = 1$

$\Rightarrow P(A \cup A^c) = 1$ $A^c = A$ complement.

$\Rightarrow P(A) + P(A^c) = 1$

জয়েন্ট Event:

একাধিক ইভেন্ট একসাথে হলে তারা Joint event আর তাদের সম্ভাবনা হলো Joint probability.

Exercice 1:

500 Random people are arranged based on gender and employment status.

Sex	Employed	Unemployed	
Male	255	20	275
Female	80	145	225
	335	165	500

$n(S)$ = Total = 500

$n(M)$ = Male = 275

$n(F)$ = Female = 225

$n(E)$ = Employed = 335

$n(U)$ = Unemployed = 165

$$P(M) = \frac{n(M)}{n(U)} = \frac{275}{500} = 0.55$$

Exam, 2

$n(S) = 100$

$n(E) = 75$

$n(B) = 50$

$n(B \cap E) = 40$

(a)

$$P(E) = \frac{75}{100}$$

$$= \frac{3}{4}$$

(b) For none who reads newspaper

$$n(B \cup E)' = (n(B) + n(E) - n(B \cap E))'$$

$$= (50 + 75 - 40)$$

$$= 85$$

Now, $n(B \cup E)' = n(S) - n(B \cup E)$

$$= 15 \text{ (Ans)}$$

(c) Reads English not Bengali

$\hookrightarrow E \cap B'$

$n(B)' = 50$ | $n(B') = n(S) - n(B)$

$= 100 - 50 = 50$

$$P(E \cap B') = \frac{n(E) - n(E \cap B)}{n(S)}$$

$$= \frac{75 - 40}{100} = 0.35$$

~~অন্যভাবে,~~

$$\cancel{P(E \cap B') = P(B) \cap P(B)'}$$

$$\cancel{= \frac{75}{100} \times \frac{50}{100} = 0.375}$$

Conditional Probability:

একটি Probability model যেখানে এক্সপেরিমেন্ট এর ~~কিছু~~ উপর ~~কিছু~~ কিছু তথ্য আগের কারণে Sample space কমানো আনা হয়।

অর্থাৎ, আগেকার Probability এর উপর নির্ভর করে একটি Probability।

The ~~probabit~~ probability of an Event E, when it is known that some other event F has occured is called a conditional probability of E and is defined as $P(E/F)$ and,

$$P(E/F) = \frac{P(E \cap F)}{P(F)} \qquad P(F) > 0$$

Similarly

$$P(F/E) = \frac{P(E \cap F)}{P(E)}$$

$P(E/F) \cdot P(F) = P(F/E) \cdot P(E)$

↳ Multiplication law

↳ Compound probability.

$P(E \cap F) = P(E) \cdot P(F/E)$

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

A family has two children. What is the probability that both are boys, given that at least one of them is a boy.

Sol^n:

Let, the samplespace is given by

$S = \{(b, g), (g, b), (g, g), (b, b)\}$

b = boy

g = girl.

let,

E = Event that both are boys.

F = Event that at least one of the children is a boy.

Here,

$E = \{(b,b)\}$

$F = \{(b,g), (g,b), (b,b)\}$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{n(E \cap F)}{n(S)}}{\frac{n(F)}{n(S)}}$$

$$= \frac{\frac{1}{4}}{3/4}$$

$$= 1/3$$

4) The probability that a married cooman watches a T.V show is 0.5 and that of her husband is 0.4. The probability that a man wathes the show, given that his wife does is 0.7%. Then, find the following

(a) The probability that the married couple watches the show.

(b) The probability that the married couple watches the show given that her husband does.

(c) At least one of the partner watches the show.

Sol^n:

(a) let,

H : Husband watches show = 0.4

W : wife watches show = 0.5

Husband watches show, given his ~~wif~~ wife does $H/W = 0.7$

(a) Both watches

$$P(W \cap H) = P(W) \cdot P(H/W)$$

$$= 0.5 \times 0.7$$

$$= 0.35$$

(Note; "given that" থাকলে Conditional থাকে, given that এর পরবর্তী হয় সব Conditional)

(b) Wife watches, given husband watches:

$$P(W/H) = \frac{P(W \cap H)}{P(H)} = \frac{0.35}{0.4}$$

(c) At least one watches

$$P(W \cup H) = P(W) + P(H) - P(H \cap W)$$

$$= 0.4 + 0.5 - 0.35$$

$$= 0.55$$

[4] A ball box contains 7 red ones and 3 black ones. Three balls are drawn from the box one after another.

(a) Find the probability the first two are red and the third one is black, if

(i) The balls are replaced before the next draw.

(ii) The balls are not replaced.

Soln:

Events ⇒

R: The event of drawing red balls

B: Event of drawing black balls.

(i) (This will be unconditional, as we are replacing the balls and not effecting the random event):

Now,

$R_1 = \frac{7}{10}$

$R_2 = \frac{7}{10}$

$B_3 = \frac{3}{10}$

∴ Probability of events:

$R_1 \cap R_2 \cap B_3 = \frac{3 \cdot 7^2}{10^3}$ (Ans)

(ii) (This is a conditional probability as not replacing the balls effects the field of ~~pr~~ results)

$R_1 = \frac{7}{10}$

$R_2 = \frac{6}{9}$

$B_3 = \frac{3}{8}$

The probability of three concequent conditional event

$P(A \cap B \cap C) =$

$P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

$P(A_1 \cap A_2 \cap \dots A_n)$

$= P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \dots P(A_n|(A_1 \cap A_2 \cap \dots A_n)$

$\therefore P(A \cap B \cap C) = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8}$

$= 7/40$

Ⅲ A coin is tossed until head has appeared or it has been tossed 3 times, given the head does not appear on the first toss. Now, what is the probability that the coin is tossed 3 times?

Soln.

$S = \{H, TH, TTH, TTT\}$

$P(H) = \frac{1}{2} = \frac{1}{2} = P(T)$

$P(TTH) = \frac{1}{8} = P(T) \cdot P(T) \cdot P(T)$

$P(TTT) = 1/8$

Now,

A = Coin is tossed 3 times

$= \{TTH, TTT\}$

B: No head on first trial.

$= \{TH, TTH, TTT\}$

$A \cap B = \{TTH, TTT\}$

$P(A) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

Three cards are drawn in succession without replacement from an ordinary deck of ~~plaw~~ playing card.

Find the probability that the first card is a red ace, the second card is 10 or jack and the third card is greater than 3 but less than 8.

Soln:

The first card is red ace : A

The second card is jack or 10 : B

The third card is <3 and >8 : C

$$P(A) = \frac{2}{52}$$

$$P(B|A) = \frac{8}{51} \quad (4 \text{ for } 10,\ 4 \text{ for Jack})$$

$$P(C|A \cap B) = \frac{16}{50}$$

$\therefore P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

$$= \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{16}{90}$$

Independent Event:

If E and F are two event and if the occurence of E does not affect F and is not affected by F, then E and F are independent.

$$P(E \cap F) = \cancel{\emptyset}\ P(E) \cap P(F)$$

$$P(E|F) = P(E)$$

[4] Two coins are tossed.

$S = \{HH, HT, TH, TT\}$

$A = \{HH, HT\}$

$B = \{HH, TH\}$

Three coins are tossed so that the events "~~hef~~ "head on the first coin" and "tails on the last two coins" are independent ?.

Sol^n: (নিজে করে দেখ)

Suppose, we toss two fair dice. let, E_1 denote the event that the sum of the dice is 6 and F denote the event that the first dice equals 4. Check whether the events are independent or not.

Sol^n:

Sample points of

$E = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$F = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$

$P(E) = \frac{5}{36}$

$P(F) = \frac{6}{36} = \frac{1}{6}$

$P(E \cap F) = \{(4,2)\}$

$P(E \cap F) = \frac{1}{36}$

$P(E) \cdot P(F) = \frac{5}{36} \cdot \frac{1}{36} = \frac{5}{216} \neq P(E \cap F)$

$\therefore$ They are not independent.

(কেননই Independent না যেহেতু sum টা, অন্য দুটোর মানের উপর নির্ভরশীল)

let,

E_2 = Sum হয় 7

$= \{(4,3)\}$

$P(E_2) = \frac{1}{36}$

$P(E_2 \cap F) = \{(4,3)\} = \frac{1}{36}$

$P(E_2) \cdot P(F) = \frac{1}{36} \cdot \frac{1}{6} =$

A fire brigrade has two engines operating independently. The probability that a specific engine is available when needed is 0.99. Then,

(a) What is the probability of an engine is available.

(b) What is the propability of neither engine is unavialable.

Solⁿ:

(a) let,

$P(E) = 0.99$

$P(F) = 0.99$

~~$P(E) \cap P(F$~~

$P(E \cap F) = 0.99 \times 0.99$

$= 0.9801$

Now,

$P(E') = 1 - 0.99 = 0.01$ (E is unavialable)

$P(F') = 1 - 0.99 = 0.01$ (E is unavailable)

$P(E' \cap F') = 0.01 \times 0.01 = 0.0001$ (Ans)

= None is available.

$P(E \cup F)$ = Any one is available

$= \underset{P(E)}{0.99} + \underset{P(F)}{0.99} - \underset{P(E \cap F)}{0.9801}$

$= 0.999$ (Ans)

⊞ Pairwise independent হয় কখন :

$$P(A_1 \cap A_2 \cap A_3) \neq P(A_1) \cdot P(A_2) \cdot P(A_3)$$

But

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$

$$P(A_3 \cap A_1) = P(A_3) \cdot P(A_1)$$

(H.W)

Two coins are tossed.

A: "Heads on first coin"

B: "Heads on second coin"

C: "Coins fall alike"

Check if they are independent or pairwise independent.

Partitions and Conditional Probability:

n Events $A_1, A_2 \ldots A_n$ such that they are mutually and mutually exclusive and collectively exhaustive (অর্থাৎ $A_i \cap A_j = \phi \ \forall \ i \neq j$ (for all $i \neq j$) এবং $S = A_1 \cup A_2 \cup \ldots A_n$)

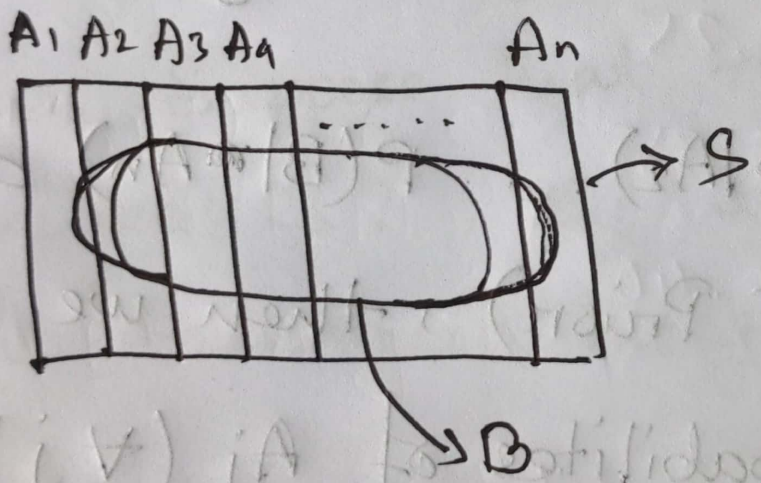
এই A ইভেন্টগুলো, S এর একটি Partition

Total probability rule:

Suppose that the events $A_1, A_2 \dots A_n$ make partition of the sample space S and $P(A_j) > 0 \ \forall j$.

Then, for any event $B \in S$

$$P(B) = \sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)$$



B, অসংখ্য Partition এ থাকতে।

Bayes theorem:

আগে থেকে জানা Probability দিয়ে পরের কোনো probability বের করা।

Let us consider n mutually exclusive and collectively exhaustive events $A_1, A_2 \ldots A_n$ and B be any event.

If $(B|A_1)$, $P(B|A_2)$ ... $P(B|A_n)$ are known (জানা Prior), then we can calculate probabilites of A_j $(\forall j)$, events given B.

(A_j মানে আসলে $P(A_1|B)$, $P(A_2|B)$ ..... $P(A_n|B)$)

$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)} \; ; \; i = 1 \text{ to } n$$

A factory, 3 different machines M_1, M_2, M_3 were used to produce a large batch of similar manufactured items. Suppose that m_1, m_2, m_3 produce 25%, 35% and 40% of the total items respectively. Suppose further that their items are 5%, 4% and 2% are deffective. Suppose, ~~to~~, an item is selected from the entire batch and it is found ~~that~~ to be deffective. Determine the probability of it being from machine (M_1)

Sol^n:

Let us define as follows:

A_1 : Selected item from M_1

A_2 : Selected item from M_2

A_3 : Selected item from M_3

B : The selected item is defective.

$P(A_1) = 25\% = 0.25$

$P(A_2) = 35\% = 0.35$

$P(A_3) = 40\% = 0.40$

$P(B|A_1) = 0.05 = 5\%$

$P(B|A_2) = 4\% = 0.04$

$P(B|A_3) = 2\% = 0.02$

The probability of the defective good from M_1:

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{\sum_{j=1}^{3} P(A_j) \cdot P(B/A_j)}$$

$$= \frac{25}{121} \, 0.36 \, 2 = \frac{25}{69} \text{ (Nice)}$$

Again, ∵ The good being from machine M_2 or M_3:

$$P(A_2 \cup A_3 | B) = P(A_2 \cup B) + P(A_3 \cup B)$$

$$= \frac{28}{69} + \frac{16}{69}$$

$$= \frac{32}{69} \text{ (Ans)}$$

中 $P_1 \rightarrow 2000$
$P_2 \rightarrow 3000$ } Production of P_1 and P_2 plant.

$P_1 \rightarrow 10\%$ defective

$P_2 \rightarrow 15\%$ defective.

3 urns contain :

$U_1 \Rightarrow R = 2, \; B = 4$

$U_2 \Rightarrow R = 3; \; B = 1$

$U_3 \Rightarrow R = 3; \; B = 4$

One urn is chosen at a random and two balls are chosen at a random. from that urn that are different colored, What is the probability of a ball being from Urn 2 ?

$U_1 \neq 6$

$U_2 =$

Sol^n:

let,

A_1 : U_1 is chosen

A_2 : U_2 is chosen

A_3 : U_3 is chosen.

$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$

B : Balls are different in color.

$$P(B|A_1) = \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{2\times4}{15} = \frac{8}{15}$$

$P(B|A_2) = \frac{1}{2}$

$P(B|A_3) = 4/7$

Suppose there are 3 candidates for the post of ~~principle~~ principal in NDC, The chances that they get the post are 4:2:3. The probability that Fr. Costa if get selected will introduce co-ed in college is 0.3%. The probability that Fr. Hemonto and Fr. Adam doing the same are 0.5 and 0.8. What is the chances of Co-ed in college this year. If co-ed will be introduced, what will be the chance that it will be introduced by Fr. Costa?

Soln:

A_1 : Fr. costa will be selected

A_2 : Fr Hemondo will be selected

A_3 : Fr Adam will be selected.

B : Co-ed will be introduced.

$P(A_1) = 4/9 \left[\left(\frac{4}{4+2+3}\right)\right]$

~~$P(A_2) = 2/$~~

$P(A_2) = 2/9$

$P(A_3) = 3/9$

$P(B|A_1) = 0.3$ (Co-ed introduced by Costa.

$P(B|A_2) = 0.5$ (" " " Hemonto)

$P(B|A_3) = \cancel{0.3}\ 0.8$ (" " " Adam)

$$P(B) = \sum_{i=1}^{3} P(A_i)\ P(B|A_i)$$

$$= \left(\frac{4}{9} \times 0.3\right) + \left(\frac{2}{9} \times 0.5\right) + \left(\frac{3}{9} \times 0.8\right)$$

$$= \frac{23}{45}$$

$P(A_1|B)$ = Probability of Co-ed being introduced by Costa

$$= \frac{P(A_1) \cdot P(B|A_1)}{P(B)}$$

$$= \frac{4/9 \times 0.3}{23/45} = \frac{6}{23}$$

Chapter 2 and 3

Random variable

↳ A variable, who's values are any definite number or quantity that ~~arises~~ arise as a chance factor such as they can't be predicted in advance.

* A r.v is a real value function defined over a sample space.

(Note: [illegible] Real value function [illegible] Domain [illegible])

২ প্রকার

- Descrete (DRV)
- Continuous ~~function~~ (CRV)

D.R.V এর Example:

- No. of phones recieved in a booth.
- No. of children in a family
- ~~Defect~~ No. of defective batch.

CRV এর Example: (অনুমান প্রমাণতে চেষ্টা হয়

- Currency values (
- Oil prices
- GPU prices
- Temperature

Probability distribution

↳ Frequency distribution এ যেমন কতগুলো ক্লাস নিয়ে ফ্রিকোয়েন্সি বের করা হয়েছিলো, এখানেও সেটাই হয়, কিন্তু এবারে মোট Event হবে, কতগুলো mutually exclusive & collectively exhaustive ক্লাসের নামে ক্লাস বের করা হয়।

Any statement of a function associating each of a set of mutually exclusively and collectively exhaustive classes or class intervals with its probability is a probability distribution.

Descrete probability distribution (DPD):

একটি D.R.V এর মান হয় সমান সমান নয়তো কিছু কম Probability এর হয়।

DR.V এর প্রতিটি মানের জন্য যে সংখ্যা, ~~distrib~~ তার Probability দেয়, সেটাই D.P.D.

Example:

একটি Coin. 3 বার উল্টানো

$S = \{HHH, HHT, HTT, HTH, THT, TTH, TTT\}$

x : A descrete R.V defining no of heads.

→ এটা D.R.V এর বিভিন্ন মান

$x : x$	0	1	2	3
$P(x=x) = f(x)$	1/8	3/8	3/8	1/8

→ DRV এর প্রতি মানের Probability

→ 0টি Head থাকার সম্ভাবনা 1/8

$(x, f(x)) = (0, 1/8), (1, 3/8), (2, 3/8), (3, 1/8)$

↳ Probability Mass function

$\frac{9}{21} + 12$

Verify if the given functions are PMF.

(a) $f(x) = \frac{2x-1}{8}$, $x = 0 \to 3$

(b) $f(x) = \frac{3x+6}{21}$ $x = 1, 2$

(c) $f(x) = \cancel{\frac{3x+6}{21}}$ $\frac{x+1}{16}$, $x = 0 \to 3$

Solⁿ:

PMF ~~এর~~ এর শর্ত

(1) $f(x) \geq 0$

(2) $\sum f(x) = 1$

(3) $P(X = x) = f(x)$

↳ সংকেতে সব মান এসেছে কিনা,

Total 7 teacher

Male = 4
Female = 3 → Form a committee of 2

X : A discrete R.V defining no. of males.

Events	Sequence of Events	$X = x$
E_1	Male, Male	2 (2 males)
E_2	Male, Female	1 (1 male)
E_3	Female, female	0 (0 males)

$$f(0) = P(X=0) = \frac{{}^4C_0 \cdot {}^3C_2}{{}^7C_2} = 3/21$$

$$f(1) = P(X=1) = \frac{{}^4C_1 \; {}^3C_1}{{}^7C_2} = 12/21$$

$$f(2) = P(X=2) = \frac{{}^4C_2 \; {}^3C_0}{{}^7C_2} = 6/21$$