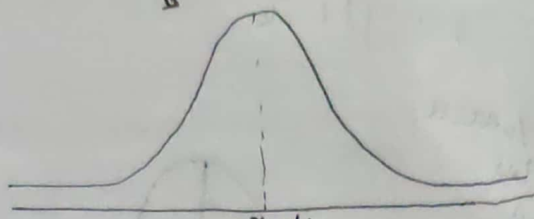
IMP

Continuous RV: Normal distribution:

graph:



always symmetric about $x = \mu$

Mean: μ

Variance: σ^2

A continuous random variable X having the bell shaped distribution as shown in the figure is called a normal random variable. The mathemetical equation of this variable depends on the two parameters μ and σ^2 known as the mean and varience respectively

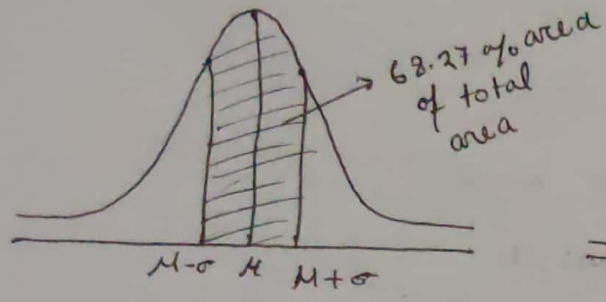
The normal random variable X is denoted by $N(\mu, \sigma^2)$ or $X \sim N(\mu, \sigma^2)$

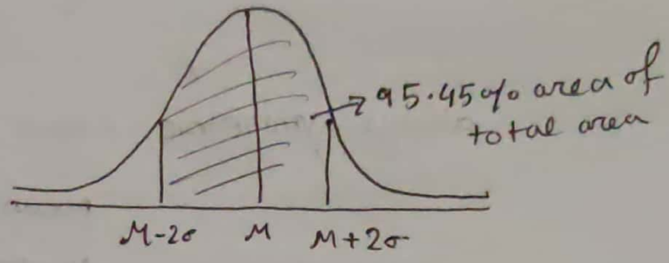
We will call the density function X as the normal probability density function and will denote it by $f(x; \mu, \sigma^2)$.

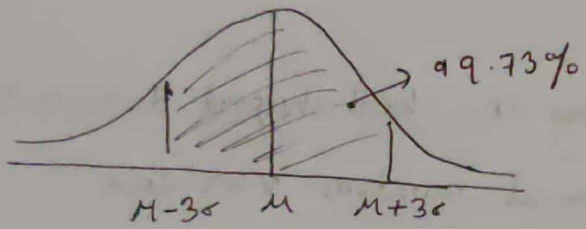
Def^n: A random variable X is said to have a normal distribution with mean μ and varience σ^2 $(-\infty < \mu < \infty$ and $\sigma^2 > 0)$ if X has a continuous distribution for which the probability density function is $f(x; \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \ -\infty < x < \infty$

A normal distribution is completely determined by mean and varience.

$\boxed{\text{Montegomery}}$ book







Problem: Verify that the area under the normal curve having the PDF $f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]; -\infty < x < \infty$ is 1.

Solution:

Here we need to prove $\int_{-\infty}^{\infty} f(x;\mu,\sigma^2)\,dx = 1$

We know, $f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

$$\int_{-\infty}^{\infty} f(x;\mu,\sigma^2)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \quad -①$$

Let, $y = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \Rightarrow dy = \frac{1}{2} \times 2\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right) dx$

$$\Rightarrow dy = \frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right) dx = \sqrt{2y}\,\frac{1}{\sigma}\,dx$$

$\therefore dx = \frac{\sigma}{\sqrt{xy}} dy$

x	$-\infty$	∞
y	$-\infty$	∞

$$① \Rightarrow \int_{-\infty}^{\infty} f(x;\mu,\sigma^2)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y} \frac{\sigma}{\sqrt{2y}} dy$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y} y^{-1/2} dy$$

$$= \frac{2}{2\sqrt{\pi}} \int_{0}^{\infty} y^{-1/2} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} y^{1/2 - 1} e^{-y} dy$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{\sqrt{\pi}} \times \sqrt{\pi}$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(x;\mu,\sigma^2)\,dx = 1$$

Even f^n:
$f(-x) = f(x)$ $\quad \int_{-\infty}^{\infty} f(x)dx = 2\int_{0}^{\infty} f(x)dx$

Odd f^n:
$f(-x) = -f(x)$ $\quad \int_{-\infty}^{\infty} f(x)dx = 0$

Gamma f^n:

$\Gamma(n) = \int_0^{\infty} y^{n-1} e^{-y} dy$

$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$

$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$\Gamma(n+1) = n\Gamma(n)$

Problem: Find the mean and varience of the normal distribution using its PDF.

We know that, the density function of a random variable X following a normal distribution is given by: $f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$

Now, Mean of $X = E[X] = \int_{-\infty}^{\infty} x f(x; \mu, \sigma^2)\, dx$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \quad \text{--- (1)}$$

Let, $y = \frac{x-\mu}{\sigma}$

$\Rightarrow x = \mu + \sigma y$

$\Rightarrow dx = \sigma dy$

x	$-\infty$	∞
y	$-\infty$	∞

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma y) \exp\left[-\frac{1}{2}y^2\right] \sigma\, dy$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy + \underbrace{\frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-\frac{1}{2}y^2} dy}_{0 \text{ (since, odd } f^n)}$$

$$= \frac{\mu}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} \left(e^{-\frac{1}{2}y^2} dy\right) + 0$$

$$E[X] = \frac{\sqrt{2}\,\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}y^2} dy \quad \text{--- (II)}$$

Let, $z = \frac{1}{2}y^2$ $\quad dz \Rightarrow \frac{1}{2} \times 2y\, dy = y\, dy \quad dy = \frac{1}{y} dz$

$$dy = \frac{1}{\sqrt{2z}} dz$$

y	0	∞
z	0	∞

② ⇒

$$E[x] = \sqrt{\frac{2}{\pi}} \times \mu \int_0^\infty e^{-z} \frac{1}{\sqrt{2z}} dz$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty z^{-1/2} e^{-z} dz$$

$$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty z^{1/2 - 1} e^{-z} dz$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{\mu}{\sqrt{\pi}} \cdot \sqrt{\pi} = \mu$$

$\therefore$ Mean of NRV, $E[X] = \mu$

$$V[X] = E[X^2] - \{E[X]\}^2$$

(*) $$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x; \mu, \sigma^2) dx$$

$$= \int_{-\infty}^{\infty} \{x(x-1) + x\} f(x; \mu, \sigma^2) dx = \int_{-\infty}^{\infty} x(x-1) f(x; \mu, \sigma^2) dx + \int_{-\infty}^{\infty} x f(x; \mu, \sigma^2) dx$$

Another process (*):

$$E[x^2] = E[(x-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x; \mu, \sigma^2) dx$$

Standand normal distribution:

If a random variable X has a normal distribution with mean μ and varience σ^2 (i.e. $X \sim N(\mu, \sigma^2)$), then the ~~variable~~ variate

$Z = \frac{X-\mu}{\sigma}$, will be called a standard normal variate (or z-value/ z-score) and its distribution is reffered to as the standard normal distribution having the following density function:

$$f(z;0,1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad ; -\infty < z < \infty$$

$E[Z] = 0$; $V[Z] = 1$

Since, $z = \frac{X-\mu}{\sigma}$

$$E[z] = E\left[\frac{X-\mu}{\sigma}\right] = E\left[\frac{X}{\sigma}\right] - E\left[\frac{\mu}{\sigma}\right] = \frac{1}{\sigma}E[X] - \frac{\mu}{\sigma}E[1]$$

$$= \frac{1}{\sigma}\mu - \frac{\mu}{\sigma} \times 1$$

$$= 0$$

$$V[z] = E[z^2] - \{E[z]\}^2 = E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] - 0^2$$

$$= \frac{1}{\sigma^2} E[(X-\mu)^2] = \frac{1}{\sigma^2} E[X^2 - 2\mu X + \mu^2]$$

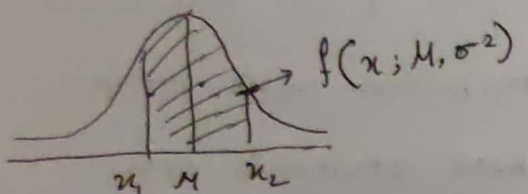
$$= \frac{1}{\sigma^2}\{E[X^2] - 2\mu E[X] + \mu^2 E[1]\}$$

$$= \frac{1}{\sigma^2}\{E[X^2] - 2\mu^2 + \mu^2\} \rightarrow E[X^2] = \sigma^2 + \mu^2$$

$$= \frac{1}{\sigma^2}\{E[X^2] - \mu^2\} \text{ — (1)} \qquad (1) \Rightarrow V[z] = \frac{1}{\sigma^2}\{\sigma^2 + \mu^2 - \mu^2\} = 1$$

Area under the normal curve:

The curve of any continuous probability distribution is constructed so that the area under the curve bounded by the two ordinates $x = x_1$ and $x = x_2$ equals the probability that the random variable x assumes a value between $x = x_1$ and $x = x_2$



That is, $P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x; \mu, \sigma^2)\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{x=x_1}^{x_2} \text{Exp}\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$

It is clear that the normal curve is completely dependent on the mean μ and the standard deviation σ of the distribution.

$$Z = \frac{X-\mu}{\sigma}$$

$$z_1 = \frac{x_1-\mu}{\sigma} \text{ and } z_2 = \frac{x_2-\mu}{\sigma}$$

$$P(x_1 < X < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{1}{2}z^2} dz$$

$$= \int_{z_1}^{z_2} f(z; 0, 1)\, dz = P(z_1 < Z < z_2)$$

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Montegomery – Normal D... (4.6)

$= P(z < z_2) - P(z < z_1)$

$= \phi(z_2) - \phi(z_1)$

$P(z < z) = \phi(z)$

$$\phi(z) = \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

Example: The GPA score of 80 students of the dept of SWE was found to follow the approximately a normal distribution with mean of 2.1 and S.D of 0.6. How many of these students are expected to have a score between 2.5 and 3.5?

Solution:

Given,

mean, $\mu = 2.1$

S.D, $\sigma = 0.6$

$x_1 = 2.5$

$x_2 = 3.5$

The corresponding z-values to the scores $x_1 = 2.5$ and $x_2 = 3.5$ can be obtained from the following equation:

$$z = \frac{x - \mu}{\sigma} \quad \text{—(1)}$$

From (1), $z_1 = \dfrac{x_1 - \mu}{\sigma} = \dfrac{2.5 - 2.1}{0.6} = 0.67$

$z_2 = \dfrac{x_2 - \mu}{\sigma} = \dfrac{3.5 - 2.1}{0.6} = 2.33$

Therefore, the required probability is

$$P(2.5 < X < 3.5) = P(0.67 < Z < 2.33)$$

$$= P(Z < 2.33) - P(Z < 0.67)$$

$$= \phi(2.33) - \phi(0.67)$$

$$= 0.9901 - 0.7486$$

$$= 0.2415$$

Hence 24.15% students approximately $0.2415 \times 80 = 20$ students out of 80 are expected to make a score between 2.5 and 3.5.

$$\text{Area} = \int_{z=0.67}^{2.3} f(z; 0, 1)\, dz = \int_{0.67}^{2.33} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= 0.2415$$

Problem: A company pays its employees an average wage of \$5.25/hr with a standard deviation of 60 cents. If the wage is approximately normally distributed,

(1) what percentage of the employee recieve wages bet^{n} \$4.75 and \$5.69/hr?

$\mu = \$5.25$ $\quad x_1 = 4.75$

$\sigma = 0.6$ $\quad x_2 = 5.69$

$$z_1 = \frac{4.75 - 5.25}{0.6} = -0.833$$

$$z_2 = \frac{5.69 - 5.25}{0.6} = 0.733$$

$$P(4.75 < X < 5.69) = P(-0.833 < z < 0.733)$$

$$= -P(z < -0.833) + P(z < 0.733)$$

$$= -\phi(-0.833) + \phi(0.73)$$

$$= -.2033 + 0.7673$$

$$= 0.564$$

$$= 56.4\%$$

Problem: If $Z \sim N(0, 1)$ then find the value of the constant Such that $P(0 \leq z \leq a) = 0.4525$

We know that, $P(0 \leq z \leq a) = P(z \leq a) - P(z \leq 0)$

$$0.4525 = P(z \leq a) - \phi(0)$$

$$\Rightarrow \phi(a) = 0.4525 + 0.5 = 0.9525$$

$P(z \leq a) = 0.9525$ But $P(z \leq 1.67) = 0.9525$ $\therefore a = 1.67$

Problem:

$X \sim N(25, 9)$ find k such that

i) 30% of the area under the normal curve lies on the left of the distribution.

ii) 15% of the area under the normal curve lies on the right of the distribution.

Given,

$\mu = 25 \quad \sigma = 3$

i) Since 30% of the area lies to the left of the distribution

$$P(X < k) = 0.30$$

Now transforming the x-value to z-value we have,

$$P(X<k) = P\left(Z < \frac{k-\mu}{\sigma}\right) = P\left(Z < \frac{k-25}{3}\right) = 0.30$$

From the table,

$$P(Z < -0.525) = 0.30$$

$$\frac{k-25}{3} = -0.525$$

$$k = 25 - 3 \times 0.525$$

$$= 23.925$$

~~From table~~

$$P(X > k) = P\left(Z > \frac{k-\mu}{\sigma}\right) = 0.15$$

$$\Rightarrow P\left(Z > \frac{k-\mu}{\sigma}\right) = 0.15$$

$$\Rightarrow 1 - P\left(Z < \frac{k-\mu}{\sigma}\right) = 0.15$$

$$\Rightarrow P\left(Z \leq \frac{k-\mu}{\sigma}\right) = 0.85$$

From table, $P(Z \leq 1.04) = 0.85$

$\frac{k-\mu}{\sigma} = 1.04 \Rightarrow k = 25 + (1.04 \times 3) = 28.12$

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Buses arrive at a specified stop at 15 mins. interval at 7 am. That is they arrive at 7.00, 7:15, 7:30, 7:45 and so on. If a passenger arrives at a stop at a line that is uniformly distanced between 7:00 and 7:30 find the probability that he waits

a) less than 5 mins for a bus

b) at least 12 minutes for a bus.

Soln:

Let X denotes the time in minutes past 7:00 am that the passenger arrives at the stop. Since X is a uniform random variable over the interval (0, 30) it follows the passenger will have to wait less than 5 minutes if he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence the desired probability for (problem a) is

$$P\{10 < X < 15\} + P\{25 < X < 30\} = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

Similarly he would have to wait at least 12 min if he arrives between 7 and 7:03 or between 7:15 and 7:18. Hence the required probability for problem b is

$$P\{0 < X < 3\} + P\{15 < X < 18\} = \frac{3}{30} + \frac{3}{30} = \frac{1}{5} \text{ (Ans)}$$

The current in a semiconductor diode is often measured by the Shockley equation $I = I_0 (e^{aV} - 1)$ where V is the voltage across the diode, I_0 is the reverse current, a is a constant and I is the resulting diode current. ~~and~~ Find $E[I]$ if $a=5$ $I_0 = 10^{-6}$ and V is uniformly distributed over $(1, 3)$.

Solution:

$$E[I] = E[I_0 (e^{aV} - 1)] = I_0 E[e^{aV} - 1] = I_0 (E[e^{aV} - 1])$$

$$= I_0 \int_1^3 \frac{e^{5V}}{3-1} dv - I_0$$

$$= \frac{I_0}{2} \int_1^3 e^{5V} dv - I_0$$

$$= \frac{10^{-6}}{2} \left[\frac{e^{5V}}{5}\right]_1^3 - 10^{-6}$$

$$= 10^{-7} [e^{15} - e^{5}] - 10^{-6}$$

$$= 0.326$$

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