Chapter 07.08
Simpson 3/8 Rule for Integration

After reading this chapter, you should be able to

1. *derive the formula for Simpson's 3/8 rule of integration,*
2. *use Simpson's 3/8 rule it to solve integrals,*
3. *develop the formula for multiple-segment Simpson's 3/8 rule of integration,*
4. *use multiple-segment Simpson's 3/8 rule of integration to solve integrals,*
5. *compare true error formulas for multiple-segment Simpson's 1/3 rule and multiple-segment Simpson's 3/8 rule, and*
6. *use a combination of Simpson's 1/3 rule and Simpson's 3/8 rule to approximate integrals.*

Introduction

The main objective of this chapter is to develop appropriate formulas for approximating the integral of the form

$$I = \int_a^b f(x)dx \tag{1}$$

Most (if not all) of the developed formulas for integration are based on a simple concept of approximating a given function $f(x)$ by a simpler function (usually a polynomial function) $f_i(x)$, where i represents the order of the polynomial function. In Chapter 07.03, Simpsons 1/3 rule for integration was derived by approximating the integrand $f(x)$ with a 2$^{\text{nd}}$ order (quadratic) polynomial function. $f_2(x)$

$$f_2(x) = a_0 + a_1x + a_2x^2 \tag{2}$$

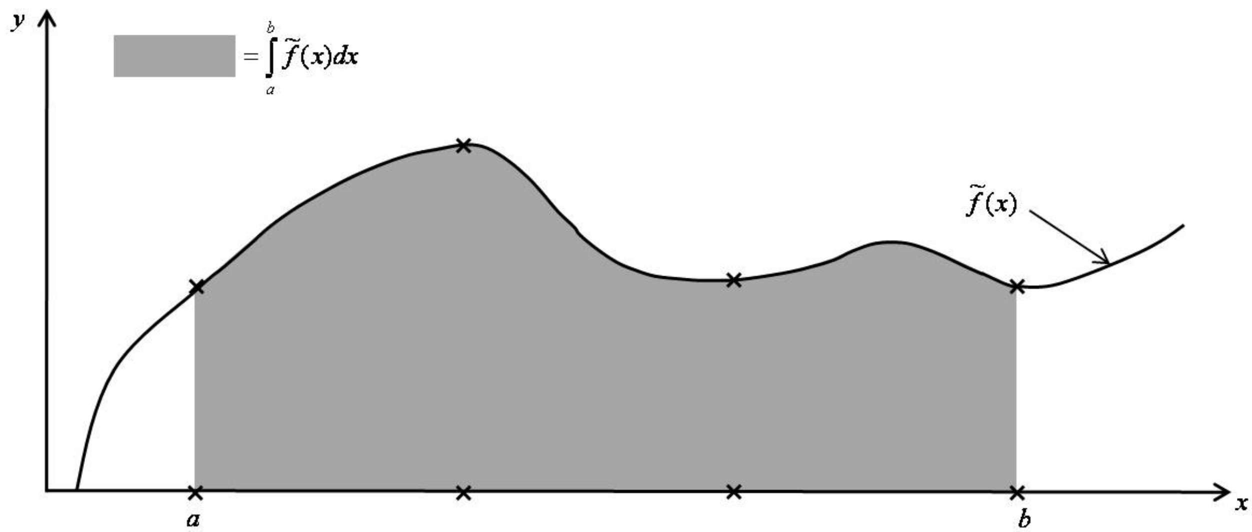

Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \tag{3}$$

which can also be symbolically represented in Figure 1.

Method 1

The unknown coefficients a_0, a_1, a_2 and a_3 in Equation (3) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}$ into Equation (3) as follows.

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2 \end{aligned}\right\} \tag{4}$$

Equation (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \tag{5}$$

The above Equation (5) can symbolically be represented as

$$[A]_{4\times 4}\vec{a}_{4\times 1} = \vec{f}_{4\times 1} \tag{6}$$

Thus,

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A]^{-1} \times \vec{f} \tag{7}$$

Substituting Equation (7) into Equation (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \tag{8}$$

As indicated in Figure 1, one has

$$\left.\begin{aligned} x_0 &= a \\ x_1 &= a + h \\ &= a + \frac{b-a}{3} \\ &= \frac{2a+b}{3} \\ x_2 &= a + 2h \\ &= a + \frac{2b-2a}{3} \\ &= \frac{a+2b}{3} \\ x_3 &= a + 3h \\ &= a + \frac{3b-3a}{3} \\ &= b \end{aligned}\right\} \tag{9}$$

With the help from MATLAB [Ref. 2], the unknown vector $\vec{a}$ (shown in Equation 7) can be solved for symbolically.

<u>Method 2</u>

Using Lagrange interpolation, the cubic polynomial function $f_3(x)$ that passes through 4 data points (see Figure 1) can be explicitly given as

$$f_3(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\times f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\times f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\times f(x_3)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\times f(x_3) \quad (10)$$

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$I=\int_a^b f(x)dx$$
$$\approx\int_a^b f_3(x)dx$$
$$=(b-a)\times\frac{\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\}}{8} \quad (11)$$

Since

$$h=\frac{b-a}{3}$$
$$b-a=3h$$

and Equation (11) becomes

$$I\approx\frac{3h}{8}\times\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\} \quad (12)$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.
The true error in Simpson 3/8 rule can be derived as [Ref. 1]

$$E_t=-\frac{(b-a)^5}{6480}\times f''''(\zeta)\text{ , where } a\le\zeta\le b \quad (13)$$

Example 1

The vertical distance in meters covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$s=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution

$$h = \frac{b-a}{n}$$
$$= \frac{b-a}{3}$$
$$= \frac{30-8}{3}$$
$$= 7.3333$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$I \approx \frac{3h}{8} \times \{f(t_0) + 3f(t_1) + 3f(t_2) + f(t_3)\}$$

$$t_0 = 8$$

$$f(t_0) = 2000\ln\left(\frac{140000}{140000-2100\times 8}\right) - 9.8\times 8$$
$$= 177.2667$$

$$\left\{\begin{aligned} t_1 &= t_0 + h \\ &= 8 + 7.3333 \\ &= 15.3333 \\ f(t_1) &= 2000\ln\left(\frac{140000}{140000-2100\times 15.3333}\right) - 9.8\times 15.3333 \\ &= 372.4629 \end{aligned}\right.$$

$$\left\{\begin{aligned} t_2 &= t_0 + 2h \\ &= 8 + 2(7.3333) \\ &= 22.6666 \\ f(t_2) &= 2000\ln\left(\frac{140000}{140000-2100\times 22.6666}\right) - 9.8\times 22.6666 \\ &= 608.8976 \end{aligned}\right.$$

$$\begin{cases} t_3 = t_0 + 3h \\ \quad = 8 + 3(7.3333) \\ \quad = 30 \\ f(t_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 \\ \quad\quad = 901.6740 \end{cases}$$

Applying Equation (12), one has

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$

$$= 11063.3104\, m$$

The exact answer can be computed as

$$I_{exact} = 11061.34\, m$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal segments, the width h can be defined as

$$h = \frac{b-a}{n} \tag{14}$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.

The integral shown in Equation (1) can be expressed as

$$I = \int_a^b f(x)dx$$

$$\approx \int_a^b f_3(x)dx$$

$$\approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \ldots\ldots + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \tag{15}$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{\begin{matrix} f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) + f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6) \\ + \ldots.. + f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n) \end{matrix}\right\} \tag{16}$$

$$= \frac{3h}{8}\left\{f(x_0) + 3\sum_{i=1,4,7,..}^{n-2} f(x_i) + 3\sum_{i=2,5,8,..}^{n-1} f(x_i) + 2\sum_{i=3,6,9,..}^{n-3} f(x_i) + f(x_n)\right\} \tag{17}$$

Example 2

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$s=\int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Use Simpson 3/8 multiple segments rule with six segments to estimate the vertical distance.

Solution

In this example, one has (see Equation 14):

$$f(t)=2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$h=\frac{30-8}{6}=3.6666$$

$$\{t_0, f(t_0)\}=\{8,177.2667\}$$

$$\{t_1, f(t_1)\}=\{11.6666,270.4104\}\text{where } t_1=t_0+h=8+3.6666\ =11.6666$$

$$\{t_2, f(t_2)\}=\{15.3333,372.4629\}\text{ where } t_2=t_0+2h=15.3333$$

$$\{t_3, f(t_3)\}=\{19,484.7455\}\text{ where } t_3=t_0+3h=19$$

$$\{t_4, f(t_4)\}=\{22.6666,608.8976\}\text{ where } t_4=t_0+4h=22.6666$$

$$\{t_5, f(t_5)\}=\{26.3333,746.9870\}\text{ where } t_5=t_0+5h=26.3333$$

$$\{t_6, f(t_6)\}=\{30,901.6740\}\text{ where } t_6=t_0+6h=30$$

Applying Equation (17), one obtains:

$$I=\frac{3}{8}(3.6666)\left\{177.2667+3\sum_{i=1,4,..}^{n-2=4}f(t_i)+3\sum_{i=2,5,..}^{n-1=5}f(t_i)+2\sum_{i=3,6,..}^{n-3=3}f(t_i)+901.6740\right\}$$

$$=(1.3750)\begin{Bmatrix}177.2667+3(270.4104+608.8976)\\+3(372.4629+746.9870)+2(484.7455)+901.6740\end{Bmatrix}$$

$$=11{,}601.4696\,m$$

Example 3

Compute

$$I=\int_{8}^{30}\left\{2000\ln\left(\frac{140000}{140000-2100t}\right)-9.8t\right\}dt,$$

using Simpson 1/3 rule (with $n_1=4$), and Simpson 3/8 rule (with $n_2=3$).

Solution

The segment width is

$$h=\frac{b-a}{n}$$

$$=\frac{b-a}{n_1+n_2}$$

$$= \frac{30-8}{(4+3)}$$
$$= 3.1429$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$\left.\begin{aligned} t_0 &= a = 8 \\ t_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\ t_2 &= t_0 + 2h = 8 + 2(3.1429) = 14.2857 \\ t_3 &= t_0 + 3h = 8 + 3(3.1429) = 17.4286 \\ t_4 &= t_0 + 4h = 8 + 4(3.1429) = 20.5714 \end{aligned}\right\} \text{Simpson's 1/3 rule}$$

$$t_5 = t_0 + 5h = 8 + 5(3.1429) = 23.7143$$
$$t_6 = t_0 + 6h = 8 + 6(3.1429) = 26.8571$$
$$t_7 = t_0 + 7h = 8 + 7(3.1429) = 30$$

Now

$$f(t_0 = 8) = 2000\ln\left(\frac{140{,}000}{140{,}000 - 2100 \times 8}\right) - 9.8 \times 8$$
$$= 177.2667$$

Similarly:

$$f(t_1) = 256.5863$$
$$f(t_2) = 342.3241$$
$$f(t_3) = 435.2749$$
$$f(t_4) = 536.3909$$
$$f(t_5) = 646.8260$$
$$f(t_6) = 767.9978$$
$$f(t_7) = 901.6740$$

For multiple segments $(n_1 = \text{first 4 segments})$, using Simpson 1/3 rule, one obtains (See Equation 19):

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(t_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f(t_i) + 2\sum_{i=2,\ldots}^{n_1-2=2} f(t_i) + f(t_{n_1}) \right\}$$

$$= \left(\frac{h}{3}\right)\{ f(t_0) + 4(f(t_1) + f(t_3)) + 2f(t_2) + f(t_4) \}$$

$$= \left(\frac{3.1429}{3}\right)\{177.2667 + 4(256.5863 + 435.2749) + 2(342.3241) + 536.3909\}$$

$$= 4364.1197$$

For multiple segments $\left(n_2 = \text{last } 3 \text{ segments}\right)$, using Simpson 3/8 rule, one obtains (See Equation 17):

$$I_2 = \left(\frac{3h}{8}\right)\left\{f\left(t_0\right) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f\left(t_i\right) + 3\sum_{i=2,\ldots}^{n_2-1=2} f\left(t_i\right) + 2\sum_{i=3,6,\ldots}^{n_2-3=0} f\left(t_i\right) + f\left(t_{n_1}\right)\right\}$$

$$= \left(\frac{3h}{8}\right)\left\{f\left(t_0\right) + 3f\left(t_1\right) + 3f\left(t_2\right) + 2(\text{no contribution}) + f\left(t_3\right)\right\}$$

$$= \left(\frac{3h}{8}\right)\left\{f\left(t_4\right) + 3f\left(t_5\right) + 3f\left(t_6\right) + f\left(t_7\right)\right\}$$

$$= \left(\frac{3}{8}\times 3.1429\right)\left\{536.3909 + 3(646.8260) + 3(767.9978) + 901.6740\right\}$$

$$= 6697.3663$$

The mixed (combined) Simpson 1/3 and 3/8 rules give

$$I = I_1 + I_2$$

$$= 4364.1197 + 6697.3663$$

$$= 11061m$$

Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880}\times f''''(\zeta) \tag{18}$$

With Simpson 3/8 rule (See Equation 12), it seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significantly higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2, 4, 6, 8, … (any even numbers) for

$$I = \int_a^b f(x)dx$$

$$\approx \left(\frac{h}{3}\right)\left\{f\left(x_0\right) + 4f\left(x_1\right) + f\left(x_2\right) + f\left(x_2\right) + 4f\left(x_3\right) + f\left(x_4\right) + \ldots\ldots + f\left(x_{n-2}\right) + 4f\left(x_{n-1}\right) + f\left(x_n\right)\right\}$$

$$= \left(\frac{h}{3}\right)\left\{f\left(x_0\right) + 4\sum_{i=1,3,\ldots}^{n-1} f\left(x_i\right) + 2\sum_{i=2,4,6\ldots}^{n-2} f\left(x_i\right) + f\left(x_n\right)\right\} \tag{19}$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be certain integers that are multiples of 3).

If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments) would be appropriate.

Computer Algorithm for Mixed Simpson 1/3 and 3/8 Rule for Integration

Based on the earlier discussion on (single and multiple segments) Simpson 1/3 and 3/8 rules, the following "pseudo" step-by-step mixed Simpson rules for estimating

$$I = \int_a^b f(x)dx$$

can be given as

Step 1

User inputs information, such as

$f(x)$ = integrand

n_1 = number of segments in conjunction with Simpson 1/3 rule (a multiple of 2 (any even numbers)

n_2 = number of segments in conjunction with Simpson 3/8 rule (a multiple of 3)

Step 2

Compute

$$n = n_1 + n_2$$

$$h = \frac{b-a}{n}$$

$$x_0 = a$$

$$x_1 = a + 1h$$

$$x_2 = a + 2h$$

.

.

$$x_i = a + ih$$

.

.

$$x_n = a + nh = b$$

Step 3

Compute result from multiple-segment Simpson 1/3 rule (See Equation 19)

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,...}^{n_1-1} f(x_i) + 2\sum_{i=2,4,6...}^{n_1-2} f(x_i) + f(x_{n_1})\right\} \quad \text{(19, repeated)}$$

Step 4

Compute result from multiple segment Simpson 3/8 rule (See Equation 17)

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(x_0) + 3\sum_{i=1,4,7...}^{n_2-2} f(x_i) + 3\sum_{i=2,5,8...}^{n_2-1} f(x_i) + 2\sum_{i=3,6,9,...}^{n_2-3} f(x_i) + f(x_{n_2})\right\} \quad \text{(17, repeated)}$$

Step 5

$$I \approx I_1 + I_2 \quad (20)$$

and print out the final approximated answer for I .

SIMPSON'S 3/8 RULE FOR INTEGRATION	
Topic	Simpson 3/8 Rule for Integration
Summary	Textbook Chapter of Simpson's 3/8 Rule for Integration
Major	General Engineering
Authors	Duc Nguyen
Date	July 9, 2017
Web Site	http://numericalmethods.eng.usf.edu