Grinshpan

ERROR AND SIGNIFICANT DIGITS

Let x be the true value of some quantity and $\tilde{x}$ be an approximation to x. The error of $\tilde{x}$ is

$$\operatorname{err}(\tilde{x}) = x - \tilde{x},$$

the absolute error of $\tilde{x}$ is $|x - \tilde{x}|$, and the relative error of $\tilde{x}$ is

$$\operatorname{rel}(\tilde{x}) = \frac{x - \tilde{x}}{x}.$$

The relative error is only defined for $x \neq 0$.

EXAMPLE If $x = 5$ and $\tilde{x} = 5.1$, then the error is -0.1, the absolute error is 0.1 and the relative error is -0.02.

The relative error is invariant under scaling,

$$1 - \frac{\tilde{x}}{x} = 1 - \frac{10^3 \cdot \tilde{x}}{10^3 \cdot x} = 1 - \frac{0.01 \cdot \tilde{x}}{0.01 \cdot x},$$

whereas the regular error is not: $x - \tilde{x}$ is directly proportional to the scalar.

Let x and $\tilde{x}$ be written in decimal form. The number of significant digits tells us to about how many positions x and $\tilde{x}$ agree. More precisely, we say that $\tilde{x}$ has m significant digits of x if the absolute error $|x - \tilde{x}|$ has zeros in the first m decimal places, counting from the leftmost nonzero (leading) position of x, followed by a digit from 0 to 4. Note that the tail portion of the form $5000\ldots = 4999\ldots$ is still allowed.

$$\begin{array}{cccccccccc}
 & 1 & 2 & & m-1 & m & m+1 & & & \\
|x - \tilde{x}| \leq & \boxed{0} & \boxed{0} & \cdots & \boxed{0} & \boxed{0} & \boxed{5} & \boxed{0} & \boxed{0} & \cdots \\
 & \uparrow & & & & & & & &
\end{array}$$

leading position of x

EXAMPLES 5.1 has 1 significant digit of 5: $|5 - 5.1| = \mathbf{0}.1$
0.51 has 1, not 2, significant digits of 0.5: $|0.5 - 0.51| = 0.\mathbf{0}1$
4.995 has 3 significant digits of 5: $5 - 4.995 = \mathbf{0.00}5$
4.994 has 2, not 3, significant digits of 5: $5 - 4.994 = \mathbf{0.0}06$
0.5 has all significant digits of 0.5
1.4 has 0 significant digits of 2: $2 - 1.4 = 0.6$

The way that significant digits are counted is motivated by the scientific (exponential) representation of $x \neq 0$,

$$x = \square.\square\square\ldots\square \times 10^n,$$

where the leading digit is nonzero. Thus $\tilde{x}$ has m digits of x if

$$|x - \tilde{x}| \leq 5 \times 10^{n-m},$$

where n the leading power of 10 in the decimal expansion of x.

The number of significant digits is invariant under scaling by an integer power of 10.

Let us suppose for definiteness that $x = \pm\, a.\square\square...\square... \times 10^n$, where $a = 1, 2, ..., 8,$ or 9.

Then $a \times 10^n \le |x| \le (a+1) \times 10^n$.

So the bound $|x - \tilde{x}| \le 5 \times 10^{n-m}$ implies that

$$\left|\frac{x-\tilde{x}}{x}\right| \le \frac{5 \cdot 10^{n-m}}{a \cdot 10^n} = \frac{5}{a} \times 10^{-m} \le 5 \times 10^{-m},$$

which means that the relative error agrees with 0.0 to at least m decimal places.

Conversely, if the magnitude of the relative error is at most 5×10^{-m}, then

$$|x - \tilde{x}| \le 5|x| \cdot 10^{-m} \le 5(a+1) \times 10^{n-m}.$$

Hence $\tilde{x}$ has at least $(m-1)$ (but not necessarily m) digits of x.

EXAMPLE 5.1 has 1 digit of 5, but $|\ \mathrm{rel}(5.1)| = 0.02 < 5 \times 10^{-2}$.

Our discussion may be summarized as follows.

PROPOSITION Let m be a nonnegative integer and β be positive.

- If $|x| \ge \beta$, then $\tilde{x} = x(1+\varepsilon)$, where $|\varepsilon| = |-\mathrm{rel}(\tilde{x})| \le |x - \tilde{x}|/\beta$.
- If $\tilde{x}$ has m significant digits of x, then $|\ \mathrm{rel}(\tilde{x})| \le 5 \times 10^{-m}$.
- If $|\ \mathrm{rel}(\tilde{x})| \le 5 \times 10^{-m}$, then $\tilde{x}$ has at least $(m-1)$ significant digits of x.

Observe, in conclusion, that $-\log_{10}(|\mathrm{rel}(\tilde{x})|)$ gives us an approximate number of significant digits, a crude estimate of accuracy.