

**Problem**

Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , find a quadratic interpolant that passes through the data. Noting  $y = f(x)$ ,  $y_0 = f(x_0)$ ,  $y_1 = f(x_1)$ , and  $y_2 = f(x_2)$ , assume the quadratic interpolant  $f_2(x)$  is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

While deriving the above Newton's divided difference second order polynomial, several students will claim that the expression for  $b_2$  which is presented to them is *wrong* as it does not match theirs. They correctly do get

$$b_2 = \frac{\frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1}$$

but the expression presented to them instead is

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

What gives and why do we write the latter expression?

**Solution**

Both expressions for  $b_2$  are correct, but we choose the latter because it denotes a finite divided difference form. Writing it like this becomes the basis for a general form of Newton's divided difference polynomial for any order and becomes conducive to a computer algorithm. Many books, however, simply skip the steps to show how they got the second expression for  $b_2$ , while students think that there are typos in the text. So in this blog, we show these steps.

Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , pass a quadratic interpolant through the data. Noting  $y = f(x)$ ,  $y_0 = f(x_0)$ ,  $y_1 = f(x_1)$ , and  $y_2 = f(x_2)$ , assume the quadratic interpolant  $f_2(x)$  is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \tag{1}$$

At  $x = x_0$ ,

$$\begin{aligned} f_2(x_0) &= f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$\begin{aligned} b_0 &= f(x_0) \\ &= f[x_0] \end{aligned} \tag{2}$$

$f[x_0]$  is called a bracketed function for the zeroth divided difference and is given by  $f(x_0)$

At  $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$\begin{aligned} b_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= f[x_1, x_0] \end{aligned} \tag{3}$$

$f[x_1, x_0]$  is a bracketed function for the first divided difference and is given by  $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

At  $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\begin{aligned} b_2 &= \frac{f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} \\ &= \frac{\frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_1} \end{aligned} \tag{4}$$

But if we want to write this in the form where  $(x_2 - x_0)$  is in the denominator so as to express it in the divided difference form of  $f[x_2, x_1, x_0]$ , we need to do the following manipulations.

Add 0 in the form of  $\{-f(x_1) + f(x_1)\}$  to the numerator of equation (4)

$$b_2 = \frac{f(x_2) + \{-f(x_1) + f(x_1)\} - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Collecting  $\{f(x_1) - f(x_0)\}$  terms together

$$b_2 = \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\}\left(1 - \frac{x_2 - x_0}{x_1 - x_0}\right)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\} \left( \frac{x_1 - x_0 - x_2 + x_0}{x_1 - x_0} \right)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + \{f(x_1) - f(x_0)\} \left( \frac{x_1 - x_2}{x_1 - x_0} \right)}{(x_2 - x_0)(x_2 - x_1)}$$

Dividing the numerator and denominator by  $(x_2 - x_1)$

$$\begin{aligned} b_2 &= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} + \frac{\{f(x_1) - f(x_0)\}(x_1 - x_2)}{(x_1 - x_0)(x_2 - x_1)}}{x_2 - x_0} \\ &= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \quad (4) \\ &= f[x_2, x_1, x_0] \end{aligned}$$

$f[x_2, x_1, x_0]$  is a bracketed function for the second divided difference and is given by

$$\frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

The Newton's divided difference second order polynomial is

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

and from equations (1)-(3),

$$\begin{aligned} f_2(x) &= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1) \\ &= f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \end{aligned}$$