

## Probability:

The term probability is an estimate of the proportion of one or more uncertain experimental outcomes when the experiment is performed randomly.

i. What is the probability that the stock market will show an abrupt rise soon after the announcement of forthcoming budget?

\*What is the likelihood that the new vaccine will be more effective than the old one to control covid-19?

A review of set & set notations:

**Set:** → should be a collection of well defined objects.

→ must have distinct values/elements.

**Universal set:** The set which contains all possible outcomes.

$$U = \{x : x \text{ is the sum of points uppermost side on the two dice}\} = \{2, 3, 4, 5, 6, \dots, 12\}$$

**Null set:**  $\emptyset$  [subset of any set]

For any sets  $A$ ,  $\emptyset \subseteq A \therefore A \neq \emptyset$  also  $\emptyset$  is called an empty set.

**Venn diagram:**

**Set algebra:** Through which properties of set are defined.

**Set operations:** i) Union ii) Intersection iii) Complement

↳ used to prepare new sets from given states.

Union:

$x \in A \cup B$  iff  $x \in A$  or  $x \in B$  or  $x \in$  both  $A \& B$

Intersection:

$x \in A \cap B$  iff  $x \in A$  and  $x \in B$

Complement:

$x \in A^c$  iff  $x \notin A$

Two important properties of collection of sets while working with probability in real life:

1. Mutually Exclusive/Disjoint

A collection of sets  $A_1, A_2, \dots, A_n$  is mutually exclusive iff  $A_i \cap A_j = \emptyset$

where  $i \neq j$

2. Collectively exhaustive:

A collection of sets  $A_1, A_2, \dots, A_n$  is collectively exhaustive iff  $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$   
i.e.  $\bigcup_{i=1}^n A_i = \Omega$

Partition: A collection of sets  $A_1, A_2, \dots, A_n$  is a partition if it is both mutually exclusive and collectively exhaustive.

De Morgan's Law: i)  $(A \cap B)^c = A^c \cup B^c$

ii)  $(A \cup B)^c = A^c \cap B^c$

iii)  $(A \cap B)^c = A^c \cup B^c$

# Application of set theory in probability:

- An outcome is an observation
- An event is a set of outcomes

## Outcome:

An outcome of an experiment is any possible observation of that experiment.

## Random experiment/ Probabilistic experiment:

An experiment that can result in different outcomes even though it is repeated in the same manner. Things to consider for random experiments

Example

- Procedure: Monitor activity at a mobile store.
- Observation: Observe which type of mobile sets the next customer purchase.
- Model: Android and iOS are equally likely. The result of each purchase is unrelated to the results of previous purchase.

## Set theory      Probability theory

Set → Event

Universal set → Sample space

Element → Outcome

## Sample space:

→ Discrete Sample Space

→ Continuous Sample Space

Event has 3 operations which are Union, Intersection, Complement if  $E_1$  and  $E_2$  are two events and are mutually exclusive then,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

## Exioms of probability:

$S \rightarrow$  Sample Space  $E \rightarrow$  Event

i.  $P(S) = 1$

ii.  $0 \leq P(E) \leq 1$

iii. If mutually exclusive,  $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

## Laws of sets

- (i) For any sets A and B,  $A \cap B = B \cap A$  and  $A \cup B = B \cup A$  [commutative law]
- (ii) For any set A, B and C,  $(A \cup B) \cup C = A \cup (B \cup C)$  [Associative law]  
 $(A \cap B) \cap C = A \cap (B \cap C)$  [Law of double inclusion]
- (iii) For any set A, B and C  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  [Distributive law]  
 $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  [Law of absorption]
- (iv) For any given set A and any null set  $\emptyset$ , universal set  $U$   
 $A \cup \emptyset = A$   
 $A \cap \emptyset = \emptyset$   
 $A \cap U = A$   
 $A \cup U = U$  } identity law
- (v) For any given set A,  $A \cup A = A$  and  $A \cap A = A$  [idempotent law]
- (vi) For any subset A of a universal set U, there is one and only complement of A,  $A^c$  that follows,

$$A \cup A^c = U$$

$$A \cap A^c = \emptyset$$

$$(A^c)^c = A$$

$$P(A) + P(A^c) = 1 \quad P(S) = 1 \Rightarrow P(A \cup A^c) = 1 \Rightarrow P(A) + P(A^c) = 1$$

## Joint Probability:

Two or more events form a joint event if all of them occur simultaneously and the probability of these joint events are called joint probability.

A: represents the event smokers

B:      "                "        "        heart disease

$A \cap B$  = represents a joint event describing that a randomly chosen person is a smokers who is suffering from heart disease.

## Problem:

	Employment Status		
Sex	Employed (e)	Unemployed (u)	Total
Male (M)	235	20	275
Female (F)	80	145	225
Total	(335)	165	500

$$n(s) = 500 \quad n(M) = 275$$

Let us define the events as follows —

M: Randomly selected person is a male

$$n(F) = 225 \quad n(E) = 335 \quad n(V) = 165$$

E: " " " - employed

M & E: Randomly selected person is male and employed.

$M \cap U$ : " " " " " unemployed.  
 $F \cap E$ : " " " " " female " employed  
 $F \cap U$ : " " " " " unemployed.

$$P(M) = \frac{n(M)}{n(S)} = \frac{275}{500} = 0.55$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{335}{500} = 0.67$$

$$P(M \cap E) = \frac{n(M \cap E)}{n(S)} = \frac{255}{500} = 0.51$$

$$P(M \cap U) = \frac{n(M \cap U)}{n(S)} = 0.04$$

$$P(F \cap E) = \frac{n(F \cap E)}{n(S)} = \frac{80}{500} = 0.16$$

$$P(F \cap U) = \frac{n(F \cap U)}{n(S)} = \frac{145}{500} = 0.29$$

$$P(M) = P(M \cap E) + P(M \cap U) \quad [\text{sum of disjoint events}]$$

$$= 0.51 + 0.04 = 0.55$$

**Problem:**

$$n(S) = 100; n(E) = 75; n(B) = 50 \quad n(E \cap B) = 40$$

E: reads English newspaper

B: " Bengali " "

E ∩ B: " both "

a) Read English newspaper probability.

$$(a) P(E) = \frac{n(E)}{n(S)} = \frac{75}{100} = 0.75$$

b) Reads none of the newspaper.

$$(b) P(E^c \cap B^c) = \frac{85}{100} = 0.85$$

c) Reads English but not Bengali.

$$(c) P(E \cap B^c) = \frac{35}{100} = 0.35$$

	E	$E^c$	Total
B	$B \cap E = 40$	$B \cap E^c = 10$	50
$B^c$	$B^c \cap E = 35$	$B^c \cap E^c = 15$	50
Total	75	25	100

## Conditional Probability:

Conditional Probability correspond to a modified probability model that reflects partial information about the outcome of an experiment. This modified model has a reduced sample space than the original model.

**Definition:** The probability of an event  $E$  when it is known that some other event  $F$  has occurred is called a conditional probability of  $E$  and is denoted with  $P(E/F)$  and defined as

$$P(E/F) = \frac{P(E \cap F)}{P(F)}, P(F) > 0$$

$$\text{Similarly, } P(F/E) = \frac{P(F \cap E)}{P(E)} = \frac{P(E \cap F)}{P(E)}, P(E) > 0$$

$P(E/F) \cdot P(F) = P(F/E) \cdot P(E)$  ] multiplication theorem/multiplication law  
law of compound probability.

i.e.  $P(E \cap F) = \underbrace{P(E)}_{\text{unconditional}} \cdot \underbrace{P(F/E)}_{\text{conditional}}$  = The probability of two events taking place happening at the same time is equal to the product of the probability of unconditional and conditional probability.

$$= P(F) \cdot P(E/F)$$

$$\text{For 3 events } A, B, C: P(A \cap B \cap C) = P(A) \cdot P(B/A) \cdot P(C/A \cap B)$$

$$\text{For } n \text{ num of events, } A_1, A_2, \dots, A_n, P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1 \cap A_2) \dots$$

Problem:

A family has two children. What is their conditional probability that both are boys. Given that at least one of them is a boy?

Sol": Let, the sample space is given by:

$S = \{(b, b), (b, g), (g, b), (g, g)\}$  where all outcomes are equally likely.

E: The event that both children are boys.

F: The event that at least one of the children is a boy.

Here,  $E = \{(b, b)\}$   $F = \{(b, b), (b, g), (g, b)\}$

$$E \cap F = \{(b, b)\}$$

Hence the required probability is,  $P(E/F) = \frac{P(E \cap F)}{P(F)}$

$$\text{Required probability} = \frac{n(E \cap F)}{n(S)} = \frac{1}{4}$$

$$\text{Required probability} = \frac{n(F)}{n(S)} = \frac{3}{4}$$

$$(S \cap A)^c \cdot (A \cap E)^c \cdot (A)^c = (S \cap A)^c \cdot (A)^c \cdot (A \cap E)^c$$

Required probability =  $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{48}$

The probability that a married woman watches a TV show is 0.5 and that of her husband is 0.4.

The probability that a man watches the show given that his wife does is 0.7. Then find the following probabilities:

(a) The probability that the married couple watches the show.

(b) a wife watches the show given that her

husband does.

at least one of the partner watch the

(c)

show.

Soln: Let us define the event as follows:

H: Husband watches the show.

w: Wife

According to Question,

$$P(H) = 0.4 \quad P(w) = 0.5 \quad P(H/w) = 0.7$$

$$(a) P(H \cap w) = P(H/w) \cdot P(w) = 0.35$$

$$(b) P(w/H) = 0.35/0.4 = 0.875$$

$$(c) P(H \cup w) = P(H) + P(w) - P(w \cap H) = 0.9 - 0.35 \\ = 0.55$$

Problem:

Red = 7  
Black = 3

A box contains 7 red balls and 3 black balls. 3 balls are drawn from the box one after another. Find the probability that the 1st 2 balls are red and 3rd one is black if

(i) The balls are replaced before the next draw.

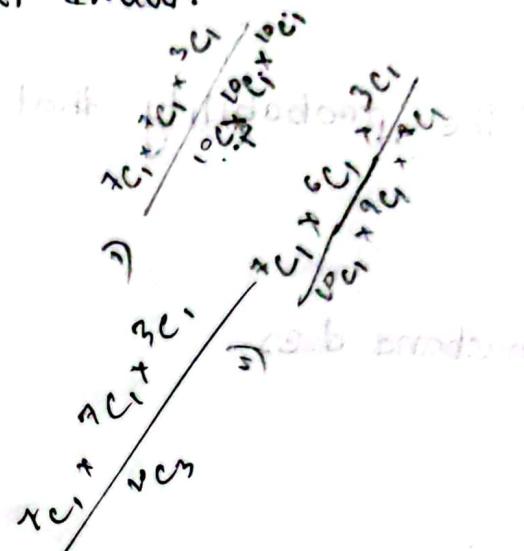
(ii) The balls are not replaced.

Solution:

Let us define the event as follows:

R: The event of drawing red balls.

B: The event of drawing black balls.



i) If a ball is replaced before the next draw, then the subsequent draws are not affected as the number of balls in the box remain the same. So this is a case of unconditional probability.

$$R_1: \text{the first ball is red.} \quad P(R_1 \cap R_2 \cap B_3) = P(R_1) \cdot P(R_2) \cdot P(B_3)$$

$$R_2: \text{the second ball is red.} \quad = \frac{7}{10} \cdot \frac{7}{10} \cdot \frac{3}{10}$$

B<sub>3</sub>: the third ball is black.

$$= \frac{147}{1000}$$

$$= 0.147$$

$$= 14.7\%$$

ii) Since the balls will not be replaced after each draw so the number of balls will decrease after each subsequent drawing.

So this is a conditional probability case.

$$P(R_1 \cap R_2 \cap R_3) = P(R_1) \cdot P(R_2 | R_1) \cdot P(R_3 | R_1 \cap R_2)$$

$$= \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} = \frac{7}{40}$$

A coin is tossed until a head appears on it has been tossed 3 times given that the head doesn't appear on the 1st toss then what is the probability that the coin is tossed 3 times?

Solution:

$$S = \{H, TH, TTH, TTT\} = \{H, TH, TTH, TTT\} \quad P(H) = P(T) = \frac{1}{2} \quad P(TH) = \frac{1}{4}$$

H: The event that head comes

$$P(TH) = \frac{1}{4}$$

T: The event that tail comes

A: coin is tossed 3 times

B: no head appears on the 1st toss.

$$B = \{TH, TTH, TTT\}$$

$$P(A) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$(A \cap B) = \{TTH, TTT\}$$

$$P(A \cap B) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

Problem: Three cards are drawn in succession without replacement from an ordinary deck of playing card. find the probability that the first card is a red ace, the second card is a 10 or a Jack and the third card is greater than 3 but less than 8.

Soln :

A: the first card is a red ace -

B: am second u u Jack on 10.

C: The third soft w part is greater than 3 but less than 8.

$$P(A) = \frac{2}{52} \quad P(B/A) = \frac{8}{51} \quad P(C/A \cap B) = \frac{16}{50}$$

$$P(A \cap B \cap C) = \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{16}{50} = \frac{32}{16575}$$

**Independent Events:** If E and F are two events and if the occurrence of E doesn't affect and isn't affected by the occurrence of F then E and F are said to be independent events.

$$P(P \cap F) = P(E) \cdot P(F)$$

From conditional property we know,

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \cdot P(F)}{P(F)} = P(E)$$

$$\boxed{P(E/F) = P(E)}$$

$$\boxed{P(F/E) = P(F)}$$



Two ideal coins are tossed:

A: head on the 1st coin

B: head on the 2nd coin.

Sol: Sample space,  $S = \{HH, HT, TH, TT\}$

$$A = \{HH, HT\}$$

$$B = \{HH, TH\} \quad P(A) = \frac{2}{4} = \frac{1}{2}$$

$$P(B) = \frac{2}{4} = \frac{1}{2}$$

$$\cancel{P(A)} - P(A \cap B) = \{HH\} \quad \text{zero chance} \quad P(A \cap B) = \frac{1}{4}$$

$$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$= P(A \cap B)$$

∴ The events are independent.

Problem:

Three coins are tossed, show that the events "head on the first coin" and the event "tails on the last two coins" are independent.

A: heads on the first coin

B: tails on the last two coins

	HH	HT	TH	TT
H	HHH	HHT	HTH	HTT
T	THH	THT	TTH	TTT

$$S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}$$

$$A = \{ HHH, HHT, HTH, HTT \}$$

$$B = \{ HTT, TTT \}$$

$$A \cap B = \{ HTT \}$$

$$P(A \cap B) = \frac{1}{8}$$

$$P(A) \times P(B) = \frac{4}{8} \times \frac{2}{8}$$

$$= \frac{1}{8}$$

The events are independent.

$$\text{Ans}$$

Problem:

Suppose we toss two fair dice. Let  $E_1$  denotes the event that the sum of the dice is six and  $F$  denotes the event that the first dice equals four. Check whether the events  $E_1$  &  $F$  are independent or not.

$$E_1 = \{(1, 5), (2, 4), (3, 3), (\underline{4, 2}), (5, 1)\}$$

$$P(E_1) = \frac{5}{36}$$

$$F = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\} \quad P(F) = \frac{1}{6}$$

$$E_1 \cap F = \{(4, 2)\} \quad P(E_1 \cap F) = \frac{1}{36}$$

$$P_1(E_1) \cdot P(F) = \frac{5}{36} \times \frac{1}{6}$$

$$= \frac{5}{216} \neq P(E_1 \cap F)$$

So the events are not independent.

$E_2$  = sum of two dices are seven

$$E_2 = \{\underline{(1, 6)}, \underline{(2, 5)}, \underline{(3, 4)}, \underline{(4, 3)}, \underline{(5, 2)}, \underline{(6, 1)}\}$$

$$P(E_2) = \frac{1}{6}$$

$$E_2 \cap F = \{(4, 3)\} \neq \emptyset$$

$$P(E_2 \cap F) = \frac{1}{36}$$

$$P(E_2) \cdot P(F) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Problem:

A fire brigade has 2 fire engines operating independently. The probability that a specific fire engine is available when needed is 0.99 then

- What is the probability an engine is available when needed?
- What is the probability neither engine is available when needed?

Soln: Let us define the events as follows:

A: The event that first engine is available when needed.

B: " " " second " " " " " "

$$\text{Given, } P(A) = P(B) = 0.99$$

$$P(A \cap B) = P(A) * P(B) = 0.99 \times 0.99 = 0.9801$$

$$\begin{aligned} \underline{(a)} \quad P(A \cup B) &= P(A) + P(B) - P(A \cap B) = 0.99 + 0.99 - 0.9801 \\ &= 0.9999 \end{aligned}$$

(b)

$$\begin{aligned} P(A^c \cap B^c) &= P(A^c \cup B^c)^c = 1 - P(A \cup B) \\ &= 1 - 0.9999 \end{aligned}$$

$$= 0.0001$$

$$P(A_1 \cap A_2 \cap A_3) \neq P(A_1) \cdot P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$$

pairwise independent

Two coins are tossed :-

A: "heads on the first coin"

B: "heads on the second coin"

C: "coins fall alike"

Check whether they are pairwise independent or completely independent.

$$A = \{(H, H), (H, T)\}$$

$$B = \{(H, H), (T, H)\}$$

$$C = \{(H, H), (T, T)\}$$

$$\begin{aligned} & P(A) \cdot P(B|A) \cdot P(C|A \cap B) \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \end{aligned}$$

$$P(A) = \frac{1}{2}$$

$$P(B) = \frac{1}{2}$$

$$P(C) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4}, \quad P(B \cap C) = \frac{1}{4}, \quad P(A \cap C) = \frac{1}{4}$$

$$P(A \cap B) = P(A) + P(B) \quad P(B \cap C) = P(B) \cdot P(C) \quad P(A \cap C) = P(A) \cdot P(C)$$

$$\text{but, } P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$$

$\therefore$  It is pairwise independent

## Conditional Probability and Partition:

Let  $S$  denotes the sample space of some random experiments and consider  $n$  events  $A_1, A_2, \dots, A_n$  such that they are mutually exclusive and collectively exhaustive i.e.  $A_i \cap A_j = \emptyset \quad \forall i \neq j$  and  $S = A_1 \cup A_2 \cup A_3 \dots \cup A_n$ .

It is said that these events form a partition of  $S$ .

## Total Probability Theorem/Rule:

Suppose that the events  $A_1, A_2, \dots, A_n$  make partition of the sample space  $S$ , and  $P(A_j) > 0$  for  $j = 1, 2, \dots, n$ . Then for any event  $B \in S$  :

$$P(B) = \sum_{j=1}^n P(A_j) \cdot P(B|A_j)$$

## Bayes' theorem:

Application: Let us consider  $n$  mutually exclusive and collectively exhaustive events  $A_1, A_2, \dots, A_n$  and  $B$  be any event,

If  $P(B|A_1), P(B|A_2), P(B|A_3), \dots, P(B|A_n)$  are known then Bayes' theorem is useful to complete the conditional probabilities of  $A_j$  (events) given  $B$ ,

$$P(A_1|B), P(A_2|B), \dots, P(A_n|B)$$

Priion probability: আগে একে জানা স্বতন্ত্র probability.

Posterior probability : priion probability use করে স্বতন্ত্র new probability যের ফল হবে

### Bayes' theorem:

$$P(A_i | B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^n P(A_j) \cdot P(B|A_j)}$$

Problem: In a factory 3 different machines  $M_1, M_2, M_3$  were used for producing large batch of similar manufactured items. Suppose that  $M_1, M_2$  and  $M_3$  produce 25%, 35% and 40% of the total items respectively. Suppose further that their items, respectively 5%, 4% and 2% are defective. Finally suppose that one item is selected at random from the entire batch and it is found to be defective. Determine the probability that this item is produced by machine  $M_3$  ( $M_2$  or  $M_1$ ).

Solution:

Let us define the events as follows:

$A_1$ : the selected item is produced by  $M_1$ .

$A_2$ : " " " produced by  $M_2$ .

$A_3$ : " " " produced by  $M_3$ .

$B$ : " " " defective.

$$\frac{0.25 \cdot 0.05 + 0.35 \cdot 0.04 + 0.40 \cdot 0.02}{0.25 + 0.35 + 0.40} = \frac{0.005 + 0.014 + 0.008}{0.90} = \frac{0.027}{0.90} = 0.03$$

Given that,

$$P(A_1) = 25\% = 0.25 \quad P(A_2) = 35\% = 0.35 \quad P(A_3) = 40\% = 0.4$$

Also,

$$P(B|A_1) = 5\% = 0.05 \quad P(B|A_2) = 4\% = 0.04 \quad P(B|A_3) = 2\% = 0.02$$

Using Bayes' theorem, the required probability that the selected defected item is produced by  $M_1$  is:

$$\begin{aligned} P(A_1|B) &= \frac{P(A_1) \cdot P(B|A_1)}{\sum_{j=1}^3 P(A_j) \cdot P(B|A_j)} = \frac{0.25 \times 0.05}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)} \\ &= \frac{0.0125}{0.0125 + 0.014 + 0.008} \\ &= \frac{0.0125}{0.0345} \\ &= 0.36 \end{aligned}$$

$$\begin{aligned} (b) P(A_2 \cup A_3|B) &= P(A_2|B) + P(A_3|B) \\ &\stackrel{\text{Total Probability}}{=} \frac{P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}{P(A_1) + P(A_2) + P(A_3)} \\ &= \frac{P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}{0.35 \times 0.04 + 0.4 \times 0.02} \\ &= \frac{0.35 \times 0.04 + 0.4 \times 0.02}{0.0345} = \frac{0.014 + 0.008}{0.0345} = 0.64 \end{aligned}$$

Problem:

$P_1$  থেকে produce করে 2000

$P_1$  produces 40%

$P_2$ ,  $P_3$  u u u 3000

60%

$P_1$ : 10% defective     $P_2$ : 15% defective

Problem: Three urns contain

$U_1$  has 2 red balls, 4 black balls

$U_2$  has 3 u u 1 u ball

$U_3$  has 3 u u 4 black balls

One urn is chosen at random and then two balls are randomly drawn from the selected urn. If the selected balls are different in colours, what is the probability that the balls come from urn 2?

$U_1$ : total 6

$A_1$ :  $U_1$  is chosen

B: the selected balls are

$U_2$  total 4

$A_2$ :  $U_2$  is chosen

different in colours

$U_3$ : total 7

$A_3$ :  $U_3$  is chosen

$\frac{A_1}{B}$

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$$

$$P(B|A_1) = \frac{\binom{2}{1} \binom{4}{1}}{\binom{6}{2}} = \frac{2 \times 4}{15} = \frac{8}{15}$$

$$P(B|A_2) = \frac{1}{2}$$

$$P(B|A_3) = \frac{4}{7}$$

$$P(A_2|B) = \frac{P(A_2) P(B|A_2)}{\sum_{j=1}^3 P(A_j) \cdot P(B|A_j)} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{8}{15} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{4}{7}} = \frac{\frac{1}{6}}{\frac{8}{45} + \frac{1}{6} + \frac{4}{21}} = \frac{\frac{105}{337}}{1} = 0.31$$

### Problem: Principal Selection

This year suppose that there will be three candidates for the post of principal in NDC. They are Father Costa, Father Hemonto, and Father Adam. The chances that they get the post are 4:2:3. The probability that Father Costa if gets selected will introduce co-education in college is 0.3. The probability that Fr. Hemonto and Fr. Adam doing the same are 0.5 and 0.8, respectively. What is the probability that there will be co-education in the college this year. If co-education will be introduced, what will be the chance that it will be introduced, what will be the chance that it will be introduced by Fr. Costa?

Ques. 8 (a) (i)

Solution:

$A_1$ : Fr. Costa will be selected as Principal

$A_2$ : Fr. Hemata u u u u u

$A_3$ : Fr. Adam u u u u u

B: Co-education will be introduced.

Given,

$$P(A_1) = \frac{4}{9}, P(A_2) = \frac{2}{9}, P(A_3) = \frac{3}{9}$$

$$\text{Also } P(B|A_1) = 0.3, P(B|A_2) = 0.5, P(B|A_3) = 0.8$$

Now,

$$\begin{aligned}
 P(B) &= \sum_{j=1}^3 P(A_j) \cdot P(B|A_j) \\
 &= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3) \\
 &= \left(\frac{4}{9} \times 0.3\right) + \left(\frac{2}{9} \times 0.5\right) + \left(\frac{3}{9} \times 0.8\right) \\
 &= \frac{23}{45} \quad (\text{Ans})
 \end{aligned}$$

Using Baye's theorem,

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(B)} = \frac{\frac{4}{9} \times 0.3}{\frac{23}{45}} = \frac{6}{23} \quad (\text{Ans})$$

## Chap-2 and 3

Random variable (R.v) / (n.v)

A ~~re~~ variable whose values are any definite numbers or quantities that arise as a result of chance factors such that they cannot be predicted in advance is called a random variable (n.v).

A ~~real~~ n.v is a real-valued function defined over a sample space. Examples.

Types of random variable:

Random variable

discrete random variable

continuous random variable

Examples: Discrete:

1. no. of telephone calls received in a telephone booth

2. no. of correct ans. in MCQ

3. no. of defective bulbs/firms

4. no. of children in a family

Continuous:  
① Height, Weight, Time of exam, Exam score, Temp.

② Rate of bank interest

③ Longevity of an electric bulb

④ Temperature recorded by meteorological dept

## Probability Distribution:

The idea of probability distribution exactly parallels to that of a frequency distribution. Each type of distributions is based on a set of mutually exclusive and collectively exhaustive measurement classes or class intervals.

Frequency Distribution: Class-interval  $\rightarrow$  frequency  $\{N \rightarrow\}$

Probability Distribution: Class-interval  $\rightarrow$  probability values  $\{1\}$

$\downarrow$  hypothetical distribution

based for our own definition theoretical distribution

## Probability Distribution:

Any statement of a function associating each of a set of mutual exclusive and collectively exhaustive classes or class intervals with its probability is a probability distribution. (P.d.).

s	s	l	o	x
0	1	2	3	4
0	1	2	3	4
0	1	2	3	4

A probability distribution will either be discrete or continuous according to the n.v. 2 types:-

## Discrete Probability Distribution:

A discrete random variable assumes each of its values on numbers with a certain probability. Sometimes these numbers are equally likely but sometimes some of the possible values of the n.v are more likely to occur than others.

In either case, the func. that gives the probability associated with each possible values of the n.v will result in a probability distribution of the n.v.

If a n.v.  $X$  has a discrete pn. distribution the pn. distribution of  $X$  is defined as  $f$  such that for any real no.  $x$ ,  $f(x) = P(X=x)$

Example:

A coin is tossed 3 times

$$S = \{ \text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THT}, \text{THH}, \text{TTH}, \text{TTT} \}$$

Individuals tosses of coin

$x$ : A discrete n.v. defining the no. of heads.

$x=x$	0	1	2	3
$P(x=x) = f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\text{When } P(x=2) = \frac{3}{8}$$

{No. of heads, probability of  $x\}$

$$\rightarrow (x, f(x)) = \{(0, \frac{1}{8}), (1, \frac{3}{8}), (2, \frac{3}{8}), (3, \frac{1}{8})\}$$

$\Rightarrow$  Probability Mass Function (PMF) of a discrete random variable.

for each outcome  $x$  to assign non-negative value  $f(x)$  such that  $\sum f(x) = 1$

## Introduction to Discrete Probability Distribution:

If a r.v.  $X$  has a discrete distribution, the probability distribution of  $X$  is defined as the function "f" such that for any real number  $x$ ,

$$f(x) = P(X=x)$$

It is important to note that all functions are not PMFs.

### Conditions for being a PMF:

1.  $f(x) \geq 0$

2.  $\sum_x f(x) = 1$

3.  $P(X=x) = f(x)$

### Problem:

Verify whether the following functions are PMFs or not:

a.  $f(x) = \frac{2x-1}{3}$ ;  $x=0, 1, 2, 3$

b.  $f(x) = \frac{x+1}{16}$ ;  $x=0, 1, 2, 3$

c.  $f(x) = \frac{3x+6}{21}$ ;  $x=1, 2$

### Solution:

a. Given,  $f(x) = \frac{2x-1}{3}$ ;  $x=0, 1, 2, 3$

i.  $f(x) \geq 0$   $f(0) = -\frac{1}{3} < 0$

but,  $\sum_x f(x) = f(0) + f(1) + f(2) + f(3)$   
 $= -\frac{1}{3} + \frac{1}{3} + \frac{3}{3} + \frac{5}{3} = 1$

1	shortest	longest
2	shortest	longest
3	shortest	longest

$X: x$	0	1	2	3
$P(X=x)$	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{3}{3}$	$\frac{5}{3}$

# Example of Random variable:

Ex : A school of :

Total  $T = 7$ ; Male = 4; Female = 3 teachers are there. We want to choose a Committee of 2 teachers.

Solution:

$x$ : A discrete random variable defining the number of males

Events	Sequence of events	$x = x$
$E_1$	Male, Male	2
$E_2$	Male, Female	1
$E_3$	Female, Female	0

$$\therefore f(0) = P(x=0)$$

$$= \frac{^4C_0 \times ^3C_2}{^7C_2} = \frac{3}{21}$$

$$f(1) = P(x=1)$$

$$= \frac{^4C_1 \times ^3C_1}{^7C_2} = \frac{9}{21}$$

$$f(2) = \frac{^4C_2 \times ^3C_0}{^7C_2} = \frac{6}{21}$$

Probability Mass Function :

Example:  $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$x$ : No. of heads

$(x=x)$	0	1	2	3
$P(x=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This is a PMF.

Example: A bag contains 10 balls of which 4 are black. If 3 balls are drawn at random without replacement, obtain the probability distribution for the number of black balls drawn.

Solution:

Let, the random variable  $X$  denotes the number of black balls drawn.

Clearly,  $X$  can assume the values 0, 1, 2, 3

To obtain the probability distribution of  $X$  we need to complete

the probabilities associated with 0, 1, 2, 3.

Since 3 balls are to be chosen the number of ways in which this choice can be made, is  ${}^{10}C_3$

$$\text{Thus, } f(0) = \frac{{}^4C_0 \times {}^6C_3}{{}^{10}C_3} = \frac{20}{120} = \frac{1}{6}$$

$$f(1) = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{60}{120} = \frac{1}{2}$$

$$f(2) = P(X=2) = \frac{{}^4C_2 \times {}^6C_1}{{}^{10}C_3} = \frac{36}{120} = \frac{9}{30} = \frac{3}{10}$$

$$P(3) = P(X=3) = \frac{{}^4C_3 \times {}^6C_0}{{}^{10}C_3} = \frac{4}{120} = \frac{1}{30}$$

$X=x$	0	1	2	3
$P(X=x)$	$\frac{20}{120}$	$\frac{60}{120}$	$\frac{36}{120}$	$\frac{4}{120}$

Now, if we wish to check whether the distribution is PMF or not, we have to check  $\sum_{x=0}^{\infty} f(x) = 1$ ;  $f(x) \geq 0$  for all  $x$ .

$$\sum_{x=0}^{\infty} f(x) = f(0) + f(1) + f(2) + f(3) = 1 \therefore \text{PMF}$$

Example:

The probability function of a d.n.v  $X$  is given by

$$f(x) = \alpha \cdot \left(\frac{3}{4}\right)^x, \quad x = 0, 1, 2, \dots, \infty$$

$$= 0 ; \text{ elsewhere}$$

Find  $\alpha$  and evaluate  $P(X \leq 3)$

Solution:

Since,

$f(x)$  is a probability function,  $\sum_{x=0}^{\infty} f(x) = 1$   $f(0) = \alpha \cdot \left(\frac{3}{4}\right)^0 = \alpha$

$$\Rightarrow \alpha + \alpha \cdot \frac{3}{4} + \alpha \cdot \left(\frac{3}{4}\right)^2 + \dots = \alpha \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots\right) = \alpha \left(\frac{4}{1-3/4}\right) = \alpha \cdot 16$$

$$\frac{1}{1-x} = \frac{1}{1-\frac{3}{4}} = 4$$

$$1+x+x^2+\dots$$

$$\Rightarrow \alpha \left[ 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots \right] = 1$$

$$\Rightarrow \alpha \cdot \left[ \frac{1}{1-\frac{3}{4}} \right] = 1 \Rightarrow \alpha = \frac{1}{16}$$

$$f(1) = \alpha \cdot \left(\frac{3}{4}\right)^1 = \alpha \cdot \frac{3}{4}$$

$$f(2) = \alpha \cdot \left(\frac{3}{4}\right)^2 = \alpha \cdot \frac{9}{16}$$

$$f(3) = \alpha \cdot \left(\frac{3}{4}\right)^3 = \alpha \cdot \frac{27}{64}$$

Hence,

the complete PMF of  $X$  is

$$f(x) = \frac{1}{4} \left(\frac{3}{4}\right)^x ; x = 0, 1, 2, \dots, \infty$$

$$= 0 ; \text{ elsewhere}$$

Also,

$$\begin{aligned} P(X \leq 3) &= f(0) + f(1) + f(2) + f(3) \\ &= \frac{1}{4} \left[ 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 \right] \\ &= \frac{1}{4} \left[ 1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} \right] \end{aligned}$$

$$= \frac{175}{256}$$

Example:

The dnv  $Y$  has a PMF as shown below:

$Y: Y$	-3	-2	-1	0	1
$P(Y=Y) = f(y)$	0.10	0.25	0.30	0.15	K

1. find the value of  $k$

$$1. f(-3) + f(-2) + f(-1) + f(0) + f(1) = 1$$

$$k = 0.2 \quad (\text{Ans})$$

2.  $P(-3 < Y < 0)$

$$2. f(-2) + f(-1) = 0.35 \quad (\text{Ans})$$

3.  $P(Y \geq -1)$

$$3. f(0) = 0.2 \quad f(-1) + f(0) + f(1)$$

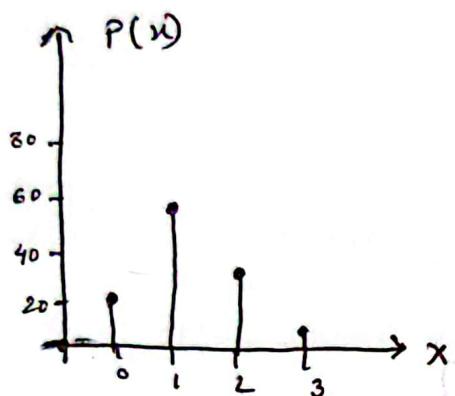
$$= 0.3 + 0.15 + 0.2 \quad \text{Ans}$$

$$= 0.65 \quad (\text{Ans})$$

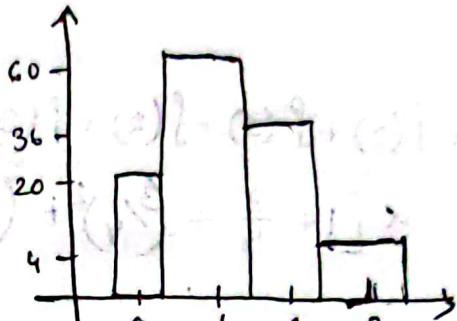
Example:

$X : \kappa$	0	1	2	3
$P(X = \kappa) = f(\kappa)$	$\frac{20}{120}$	$\frac{60}{120}$	$\frac{36}{120}$	$\frac{4}{120}$

Graphical representation:



using histogram:



Cumulative Distribution Function (CDF):

The cumulative distribution function CDF or simply the distribution function  $F(x)$  of a discrete random variable  $X$  with probability function  $f(x)$  defined over all real numbers  $x$  is the cumulative probability upto and including the point  $x$ .

Symbolically, it is defined as 
$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

$F(x)$  is always a monotonic increasing function. i.e.

$$F(a) \leq F(b) \text{ for } a \leq b$$

$$\left. \begin{aligned} \lim_{x \rightarrow -\infty} F(x) &= 0 \\ \lim_{x \rightarrow +\infty} F(x) &= 1 \end{aligned} \right\}$$

Example:

A coin is tossed 3 times. If  $X$  is the random variable representing the no. of heads obtained. find the probability distribution of  $X$  and hence obtain  $F(x)$ .

Solution:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$x : x$	0	1	2	3
$P(x=x) = f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$x : \text{no. of head obtained}$   
 $x = 0, 1, 2, 3$

$$F(x) = \sum_{t=0}^x f(t)$$

$$F(0) = f(0) = \frac{1}{8}$$

$$F(1) = f(0) + f(1) = F(0) + f(1) = \frac{4}{8}$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$F(2) = F(1) + f(2) = \frac{7}{8}$$

$$= 1$$

$$F(3) = 1$$

Therefore,

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{8} & , 0 \leq x \leq 1 \\ \frac{4}{8} & , 1 < x \leq 2 \\ \frac{7}{8} & , 2 < x \leq 3 \\ 1 & , x \geq 3 \end{cases}$$

Example

$$P(Y=y) = {}^y C_y \left(\frac{1}{2}\right)^y \left(\frac{1}{2}\right)^{2-y}$$

i) Check if  $P(Y)$  is a PMF.

ii) Find  $F(y)$  & hence find

$$F(-2) \& P(Y \leq 1) \Rightarrow F(1)$$

## Continuous Probability Distribution

The probability distribution of a continuous random variable cannot be presented in a tabular form, because a continuous variable can take on a non-innumerable, infinite set of values. More precisely, a continuous variable can take on any value in the given interval  $a \leq X \leq b$ . As a result a continuous random variable has a probability zero of assuming exactly one of its values,

$$\begin{aligned} P(a \leq x \leq b) &= P(x=a) + P(a < x < b) + P(x=b) \\ &= P(a < x < b) \end{aligned}$$

Probability Density Function (PDF): A PDF is a non-negative function and is constructed so that the area under its curve bounded by  $x$ -axis is equal to unity when computed over the range  $x$ , for which  $f(x)$  is defined.

Conditions for being a PDF:

- i.  $f(x) \geq 0$
- ii.  $\int_{-\infty}^{\infty} f(x) dx = 1$
- iii.  $P(a < x < b) = \int_a^b f(x) dx$

Example:

A random variable  $X$  has the following functional form:

$$f(x) = kx ; \quad 0 < x < 4 \\ = 0 , \text{ elsewhere}$$

- i. Find the value of  $k$
- ii. Find  $P(1 < x < 2)$  and  $P(x > 2)$

Soln:

i) For  $f(x)$  to be a PDF, we must have

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$= \int_{-\infty}^{\infty} kx^2 dx = kb(x+1) \Big|_{-\infty}^{\infty}$$

$$\Rightarrow \int_0^4 kx^2 dx = 1$$

$$\Rightarrow k \left[ \frac{x^3}{3} \right]_0^4 = 1$$

$$\Rightarrow 8k = 1 \therefore k = \frac{1}{8}$$

The complete PDF is  $f(x) = \frac{1}{8}x ; \quad 0 < x < 4$   
 $L = \int_{-\infty}^{\infty} f(x) dx = \frac{1}{8} \int_0^4 x dx = 0$ , elsewhere

$$\text{ii. } P(1 < x < 2) = \int_1^2 \frac{1}{8}x dx = \frac{1}{16} [x^2]_1^2 = \frac{3}{16} \quad (1)$$

$$P(x > 2) = \int_2^4 \frac{1}{8}x dx = \frac{1}{16} \int_2^4 x dx = \frac{1}{16} [x^2]_2^4 = \frac{3}{4}$$

$$= 1 - P(x \leq 2) = 1 - \int_0^2 \frac{1}{8}x dx = 1 - \frac{1}{16} [x^2]_0^2 = 1 - \frac{1}{16} (4 - 0) = \frac{15}{16} \quad (2)$$

Example: A continuous random variable  $X$  has the following density function

$$f(x) = \begin{cases} \frac{2}{27}(1+x) & ; 2 < x < 5 \\ 0 & ; \text{elsewhere} \end{cases} \quad \left. \begin{array}{l} \text{piecewise function} \\ \text{function} \end{array} \right\}$$

a) Verify that it satisfies the condition  $\int_{-\infty}^{\infty} f(x) dx = 1$

b) Find  $P(X < 4)$  and c) Find  $P(3 < X < 4)$

CDF a) 
$$\int_2^5 \frac{2}{27}(1+x) dx = \frac{2}{27} \int_2^5 (1+x) dx$$

$$= \frac{2}{27} \left[ x + \frac{x^2}{2} \right]_2^5$$

$$= \frac{1}{27} \left[ 2x + x^2 \right]_2^5$$

$$= \frac{1}{27} (35 - 8)$$

$$= \frac{1}{27} \times 27 = 1$$

b) 
$$\int_2^4 \frac{2}{27}(1+x) dx = \frac{2}{27} \left[ x + \frac{x^2}{2} \right]_2^4 = \frac{1}{27} \left[ 2x + x^2 \right]_2^4$$

$$= \frac{1}{27} [24 - 8] = \frac{16}{27}$$

c) 
$$\int_3^4 \frac{2}{27}(1+x) dx = \frac{1}{27} \left[ 2x + x^2 \right]_3^4 = \frac{1}{27} (24 - 15) = \frac{9}{27}$$

$$= \frac{1}{3}$$

CDF for continuous variable:

The cumulative distribution or distribution function,  $F(x)$  of a continuous random variable  $X$  with density function  $f(x)$  is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt, \text{ if the derivative of } F(x) \text{ exists, then}$$

$$f(x) = \frac{d}{dx}(F(x)) = F'(x)$$

The CDF,  $F(x)$  possesses the following properties,

$$\begin{array}{l} \text{PDF} \quad \text{CDF} \\ \downarrow \quad \downarrow \\ \text{f}(x)dx = d(F(x)) \end{array}$$

i.  $F'(x) = f(x) > 0$

ii.  $F(-\infty) = 0$

iii.  $F(+\infty) = 1$

iv.  $F(x)$  is defined at every point in a continuous range and is continuous.

$$\begin{aligned} P(a < x < b) &= \int_a^b f(x)dx = - \int_{-\infty}^b f(x)dx + \int_{-\infty}^a f(x)dx \\ &= F(b) - F(a) \end{aligned}$$

Cumulative distribution curves often have more or less the following shape pattern:

$$e^t = ab(t+1)^b \quad \left( \frac{d}{dt} e^t = ab(t+1)^{b-1} \right) = (b > a > 0) \quad \text{and}$$

Example:

If  $x$  has the density function

$$f(x) = \frac{2}{27}(1+x)^2, 2 \leq x \leq 5$$

= 0; elsewhere

Obtain the distribution function and hence find  $F(3)$  and  $F(4)$ .

Also verify that  $P(3 < x < 4) = F(4) - F(3)$

Solution:

The CDF of the given random variable is

$$F(x) = \frac{2}{27} \int_{-2}^x (1+t)^2 dt = \frac{2}{27} \left[ t + \frac{t^2}{2} \right]_2^x$$

$$F(x) = \frac{1}{27} \left[ x^2 + 2x - 8 \right]$$

$$F(3) = \frac{7}{27}, \quad F(4) = \frac{16}{27} \quad \text{Hence, } F(4) - F(3) = \frac{9}{27}$$

$$\text{Also, } P(3 < x < 4) = \int_3^4 f(x) dx = \frac{2}{27} \int_3^4 (1+x)^2 dx = \frac{1}{3}$$

Example: Suppose  $x$  is a random variable with density function.

$$f(x) = \frac{k}{(1+x)^2}, x > 0$$

= 0; elsewhere

a. Find the value of  $k$ .

b. Find  $F(x)$ .

c. Find  $P(x < 1)$

d. Find  $P(x > 1)$

Solution:

(a) Since,  $f(x)$  is a density function, it must satisfy

$$\int_0^\infty f(x)dx = 1$$

$$\Rightarrow \int_0^\infty \frac{k}{(1+x)^2} dx = 1$$

$$\Rightarrow k \left[ -\frac{1}{1+x} \right]_0^\infty = 1$$

$$\Rightarrow k \left[ \frac{1}{1+\infty} \right]_0^\infty = 1$$

$$\Rightarrow k(1-0) = 1$$

$$\Rightarrow k = 1$$

Thus, the complete density function is  $f(x) = \frac{1}{(1+x)^2}$ ;  $x > 0$

$$\begin{aligned} P(x) &= P(x < n) = \int_{t=0}^x \frac{1}{(1+t)^2} dt = \left[ -\frac{1}{1+t} \right]_0^x = \left[ \frac{-1}{1+t} \right]_0^x \\ &= 1 - \frac{1}{1+x} = \frac{x}{1+x} \end{aligned}$$

$$\textcircled{c} P(X < 1) = F(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$\textcircled{d} P(X > 1) = \int_1^\infty f(x) dx = \int_0^\infty f(x) dx - \int_0^1 f(x) dx$$

$$= \int_0^\infty \frac{1}{(1+x)^2} dx - \int_0^1 \frac{1}{(1+x)^2} dx$$

$$= \left[ -\frac{1}{1+x} \right]_0^\infty - \left[ \frac{1}{1+x} \right]_0^1$$

$$= \left[ \frac{1}{1+x} \right]_0^\infty - \frac{1}{2}$$

$$= 1 - 0 - \frac{1}{2}$$

$$= \frac{1}{2} \text{ (Answer)}$$

Example (HW):

find the constant  $k$  so that the following function may be a density function,  $f(x) = \frac{1}{k}; a \leq x \leq b$

$$= 0; \text{ elsewhere}$$

Find the CDF of the random variable  $X$ .

$$\int_0^\infty \frac{1}{k} dx = 1$$

$$\left[ \frac{x}{k} \right]_0^b = 1$$

$$\frac{b}{k} - \frac{a}{k} = 1 \Rightarrow k = b - a$$

$$f(x) = \frac{1}{b-a}$$

$$\int_{t=0}^x \frac{1}{b-a} dt = \left[ \frac{1}{b-a} t \right]_0^x = \frac{x}{b-a}$$

## Joint Probability Distribution:

In many instances it is necessary to find or consider

the properties of two or more random variables simultaneously.

This results in joint probability distributions and when these distributions involve two variables they are referred to as bivariate bivariate probability distributions. Clearly, bivariate probability distribution is a special case of joint probability distributions for two random variables.

### Joint probability distribution for discrete random variable:

Suppose that, a given experiment involves 2 random variables  $X$  and  $Y$  each of which has a discrete probability distribution. It is customary to refer to  $f(x,y)$  as the joint probability distribution of  $X$  and  $Y$ .

Since  $X$  and  $Y$  are discrete,  $f(x,y) = P(X=x, Y=y)$ . That is  $f(x,y)$  gives the probability that the outcomes  $x$  and  $y$  at the same time.

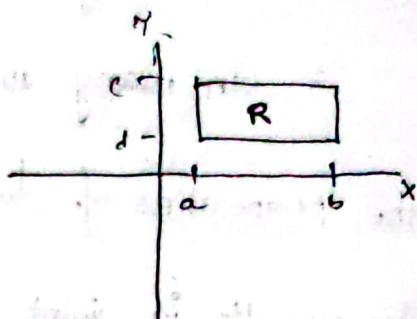
The function  $f(x,y)$  will be called a joint probability distribution of the random variables  $X$  and  $Y$  if it passes

the following properties:

$$i. f(x,y) \geq 0 \text{ for all } (x,y)$$

$$ii. \sum_{x,y} f(x,y) = 1$$

$$iii. P[(x,y) \in R] = \sum_{(x,y) \in R} f(x,y) \text{ for any region } R \text{ in the } xy\text{-plane.}$$



$$R = (a,b) \times (c,d)$$

Example: A coin is tossed three times. If  $X$  denotes the no. of heads and  $Y$  denotes the no. of tails in the last two tosses.

Find JPD of  $X$  and  $Y$ .

Solution:

The outcomes of the experiment and its associated probabilities are shown in the following table:

Outcome	$X$	$Y$	$P(X,Y)$
HHH	3	0	$\frac{1}{8}$
HHT	2	1	$\frac{1}{8}$
HTH	2	1	$\frac{1}{8}$
HTT	1	2	$\frac{1}{8}$
THH	2	0	$\frac{1}{8}$
THT	1	1	$\frac{1}{8}$
TTH	1	1	$\frac{1}{8}$
TTT	0	2	$\frac{1}{8}$

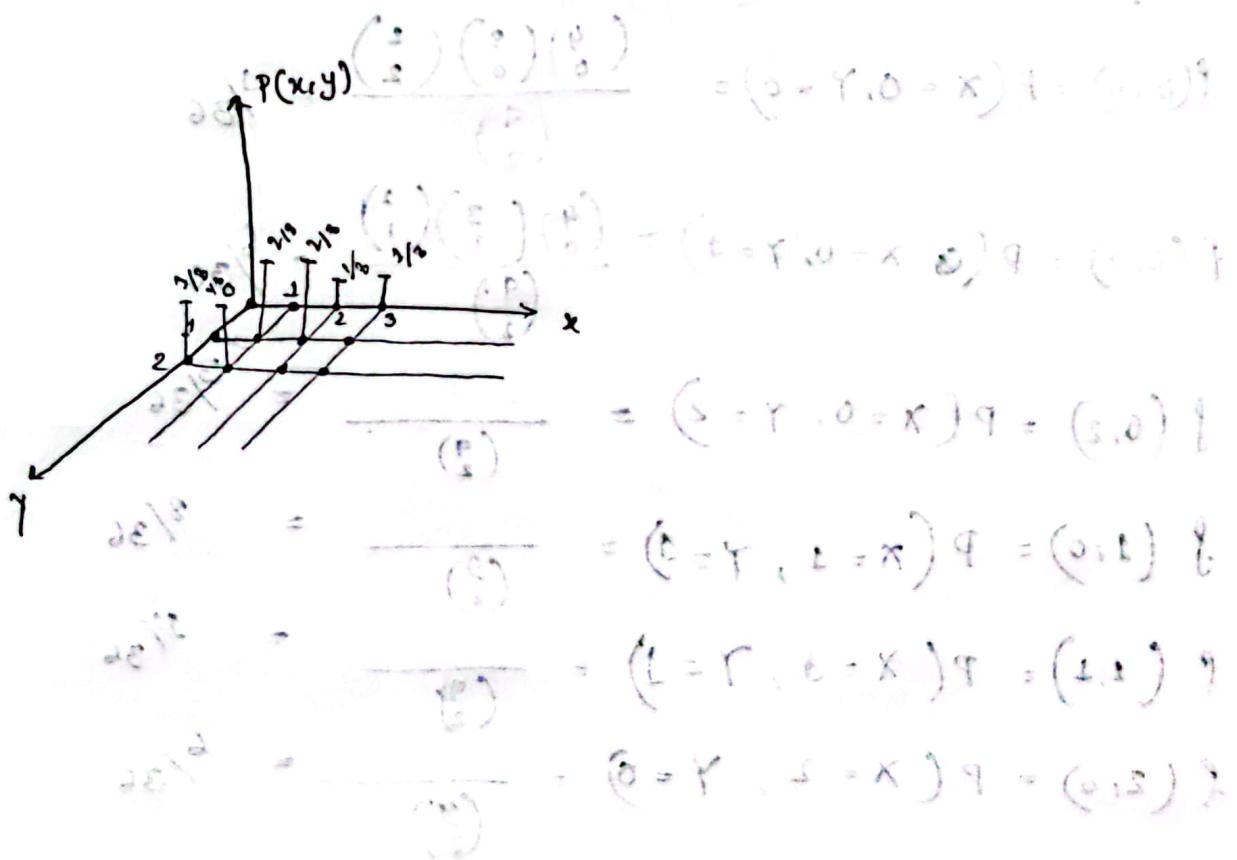
It is easy to observe that  $X$  assumes variable values ( $x = 0, 1, 2, 3$ )  
and  $Y$  assumes values ( $y = 0, 1, 2$ )

So, JPD can now be put in the following table:

Solution:

Y-values	X values				Row sum
	0	1	2	3	
0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$
1	0	$\frac{3}{8}$	$\frac{1}{8}$	0	$\frac{4}{8}$
2	$\frac{1}{8}$	$\frac{1}{8}$	0	0	$\frac{2}{8}$
Column sum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

Graphical representation:



Example:

A box contains 4 white balls, 3 black balls and 2 red balls. Two balls are to be drawn without replacement. Let,  $X$  denotes the no. of white balls and  $Y$  denotes the no. of black balls drawn. Find the JPD of  $(X, Y)$ . Find also the probability that  $X + Y \geq 3$ . Find the probability that  $X = 1$ .

Solution:

$X$ : no. of white balls drawn  $x = 0, 1, 2$

$Y$ : no. of black balls drawn  $y = 0, 1, 2$

The possible pairs of points are:  $(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (0, 2)$

$$f(0,0) = P(X=0, Y=0) = \frac{\binom{4}{0} \binom{3}{0} \binom{2}{0}}{\binom{9}{2}} = \frac{1}{36}$$

$$f(0,1) = P(X=0, Y=1) = \frac{\binom{4}{0} \binom{3}{1} \binom{2}{1}}{\binom{9}{2}} = \frac{6}{36}$$

$$f(0,2) = P(X=0, Y=2) = \frac{\binom{4}{0} \binom{3}{2} \binom{2}{2}}{\binom{9}{2}} = \frac{3}{36}$$

$$f(1,0) = P(X=1, Y=0) = \frac{\binom{4}{1} \binom{3}{0} \binom{2}{0}}{\binom{9}{2}} = \frac{8}{36}$$

$$f(1,1) = P(X=1, Y=1) = \frac{\binom{4}{1} \binom{3}{1} \binom{2}{1}}{\binom{9}{2}} = \frac{12}{36}$$

$$f(2,0) = P(X=2, Y=0) = \frac{\binom{4}{2} \binom{3}{0} \binom{2}{0}}{\binom{9}{2}} = \frac{6}{36}$$

The JPD can now be placed in the following table:

X values				
Y values	0	1	2	Rowsum
0	1/36	8/36	9/36	15/36
1	6/36	12/36	0	18/36
2	3/36	0	0	3/36
Columnsum	10/36	20/36	6/36	1

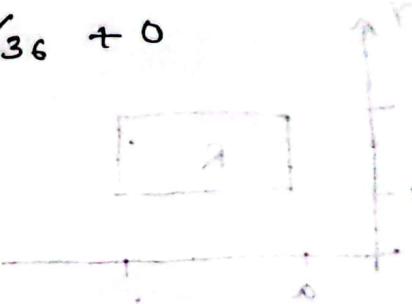
Now,

$$P(X+Y \geq 3) = f(1,2) + f(2,1) + f(2,2) = 0$$

Also,

$$\begin{aligned} P(X=1) &= \sum_{Y=0}^2 f(1, Y) = f(1,0) + f(1,1) + f(1,2) \\ &= 8/36 + 12/36 + 0 \end{aligned}$$

$$(1,0) + (1,1) = 20/36$$

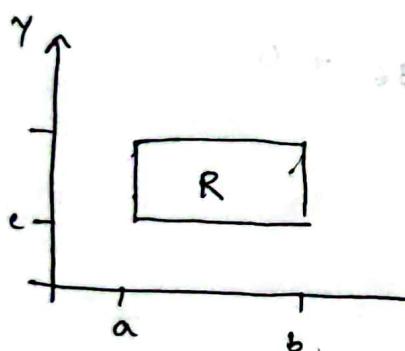


Joint distribution for continuous variables:

Let  $X$  and  $Y$  be two continuous random variables. If there exists a non-negative  $f(x,y)$  defined on the entire  $xy$  plane such that for every subset  $R$  of pairs of numbers.

$$P[(x,y) \in R] = \iint_R f(x,y) dx dy$$

then  $f(x,y)$  is called (the) joint density function (joint PDF) of  $(X,Y)$ .



$$R = (a, b) \times (c, d)$$

$$\iint_R f(x,y) dx dy = \int_{x=a}^b \int_{y=c}^d f(x,y) dy dx$$

# Like all other probability functions, the requirements for a continuous function  $f(x,y)$  to be a joint PDF are:

1.  $f(x,y) \geq 0, \forall (x,y) \in \mathbb{R}$  on  $-\infty < x < \infty, -\infty < y < \infty$
2.  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$
3.  $P[(x,y) \in R] = \iint_R f(x,y) dx dy = P(a \leq x \leq b, c \leq y \leq d)$   
 $= \int_a^b \int_c^d f(x,y) dy dx$

for any region  $R$  in the  $xy$  plane.

# The cumulative distribution function (CDF) for  $f(x,y)$  is defined by -

$$F(x,y) = P[x \leq x, Y \leq y] = \int_{-\infty}^x \int_{-\infty}^y f(x,y) dy dx$$

# the joint cumulative distribution function has the following properties -

$$1. 0 \leq F(x,y) \leq 1$$

$$2. \frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y), \text{ (where } F(x,y) \text{ is differential)}$$

$$3. F(x, -\infty) = F(-\infty, y) = 0$$

$$4. F(-\infty, \infty) = 1$$

Ex:

Let  $X$  and  $Y$  have the following distributions.

$$f(x,y) = x^2 + \frac{2y}{3} ; 0 \leq x \leq 1 \quad 0 \leq y \leq 2 \\ = 0 ; \text{ elsewhere}$$

check whether  $f(x,y)$  is a density function or not

Solution:

The function  $f(x,y)$  will be a joint PDF if

$$f(x,y) \geq 0 ; \forall (x,y) \in \mathbb{R}$$

$$\text{and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = 1$$

clearly  $f(x,y) \geq 0$  for values of  $x$  and  $y$  in the given range.

( $0 \leq x \leq 1$  and  $0 \leq y \leq 2$ )

$$\text{Now, } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = \int_{x=0}^1 \int_{y=0}^2 \left[ x^2 + \frac{2y}{3} \right] dy dx \\ = \int_{x=0}^1 \left[ x^2 y + \frac{xy^2}{6} \right]_0^2 dx$$

$$= \int_{x=0}^1 \left( 2x^2 + \frac{2x}{3} \right) dx \Rightarrow 2 \int_{x=0}^1 \left( x^2 + \frac{1}{3} x \right) dx \\ = 2 \left[ \frac{x^3}{3} + \frac{x^2}{6} \right]_0^1 = 2 \left( \frac{1}{3} + \frac{1}{6} \right) = 2 \times \frac{3}{6} = 1$$

Hence  $f(x,y)$  is a joint PDF.

Ex: Suppose that  $X$  and  $Y$  has the following density function  $f(x,y)$

$$= kx^2y^2 ; 0 \leq x \leq 2 ; 0 \leq y \leq 1$$

$$= 0 ; \text{elsewhere}$$

determine the value of the constant  $k$  and find  $P(X \leq 1)$

$$[k = 3/2 \text{ Ans}]$$

Solution:

$$\int_{x=0}^2 \int_{y=0}^1 f(x,y) dy dx = 1$$

Ex: Let the joint PDF of the continuous random variable  $(x, y)$  be

$$f(x, y) = \begin{cases} k(x^2 + 2xy); & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0; & \text{elsewhere} \end{cases}$$

a. Find the value of  $k$   $k = 6/5$  (Ans)

b. Find  $P(x \leq \frac{1}{2}, y \geq \frac{1}{2})$

c. What is the probability of the event  $(x \leq y)$ ?

Solution:

Hints:

$$\int_0^{1/2} \int_{\sqrt{y}}^y f(x, y) dx dy = 1/80 \text{ (Ans)}$$

$$\begin{aligned} c. P(x \leq y) &= \int_{y=0}^1 \int_{x=0}^y (x^2 + 2xy) dx dy \\ &= 2/5 \end{aligned}$$

### Marginal Distribution:

We have noted earlier that if the joint probability function  $f(x,y)$  of two random variables  $X$  and  $Y$  is known, the probability function of  $X$  and  $Y$  can be derived separately.

In that case when the distribution of  $X$  is derived from  $f(x,y)$  the resulting distribution is called marginal distribution of  $X$ .

Similarly in case of  $Y$  when the distribution of  $Y$  is derived from  $f(x,y)$  the resulting distribution is called marginal distribution of  $Y$ .

Hence we denote these two distributions by  $g(x)$  and  $h(y)$  respectively when they are derived from  $f(x,y)$ . Thus given the joint probability distribution  $f(x,y)$  of two discrete random variable  $X$  and  $Y$ , the probability distribution of  $X$  alone is

$$g(x) = \sum_y f(x,y)$$

And that of  $Y$  is,

$$h(Y) = \sum_x f(x, y)$$

For two continuous random variables  $X$  and  $Y$

$$g(x) = \int_{y=-\infty}^{\infty} f(x, y) dy; \quad h(y) = \int_{x=-\infty}^{\infty} f(x, y) dx$$

\*\* Marginal Distributions are indeed probability distributions;  
since they satisfy, all the properties of a probability distribution

For continuous case:

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dy dx = 1$$

$$\text{and } P[a < X < b]$$

$$= P[a < X < b, -\infty < Y < \infty]$$

$$= \int_{x=a}^b \int_{y=-\infty}^{\infty} f(x, y) dy dx = \int_a^b g(x) dx$$

((Y) d. (x) d.)  $P$  has  $x$   $\int$  on  $y$  indicates happening and both

between two probabilities from values of  $y$

**Definition:** Let  $F(x, y)$  be the joint CDF of  $(x, y)$ . Then the marginal CDF of  $x$  is

$$F_1(x) = \lim_{y \rightarrow \infty} F(x, y)$$

Similarly, the marginal CDF of  $y$  is

$$F_2(y) = \lim_{x \rightarrow \infty} F(x, y)$$

The probability function or PDF of  $x$  associated with the marginal CDF of  $x$  is called the marginal PDF of  $x$ .

Ex:

Suppose  $x$  and  $y$  have the following joint distribution

	0	1	2	3	Rowsum
0	0	$\frac{1}{3}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{4}{8}$
Columnsum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$	$\frac{1}{8}$	1

Find the marginal distribution of  $x$  and  $y$  ( $g(x), h(y)$ )

Hints: [Rowsum and columnsum are results]

Solution:

For random variable  $X$  towards continuous distribution function

For the random variable  $X$

$$g(0) = P(X=0) = \sum_{y=0}^0 f(0,y) = f(0,0) + f(0,1) = 0 + \frac{1}{8} = \frac{1}{8}$$

$$g(1) = P(X=1) = \sum_{y=1}^1 f(1,y) = f(1,0) + f(1,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$g(2) = P(X=2) = \sum_{y=2}^2 f(2,y) = f(2,0) + f(2,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$g(3) = P(X=3) = \sum_{y=3}^3 f(3,y) = f(3,0) + f(3,1) = 0 + 0 = 0$$

Similarly for the n.v.  $Y$  for continuous distribution function

$$h(0) = P(Y=0) = \sum_{x=0}^3 f(x,0) = f(0,0) + f(1,0) + f(2,0) + f(3,0)$$
$$= 0 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$
$$= \frac{3}{8}$$

$$h(1) = P(Y=1) = \sum_{x=0}^3 f(x,1) = f(0,1) + f(1,1) + f(2,1) + f(3,1)$$

$\Rightarrow X \sim f_0$  discrete distribution with probability of

$$\{P(X=x) = \frac{1}{8}, \forall x=0,1,2,3\}$$

The marginal distributions derived above may be written as

MD of  $x$ :

$X:x$	0	1	2	3	Sum
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

MD of  $Y$ :

$Y:y$	0	1	Sum
$f(y)$	$\frac{4}{8}$	$\frac{4}{8}$	1

Example:

Find the marginal densities of  $x$  and  $y$  from the following joint density function and verify that marginal distributions are also probability distributions.

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y) & \text{for } 0 < x < 2, 2 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Also compute  $P(X+Y < 3)$  and  $P(X < 1.5, Y < 2.5)$

Solution:

By definition, the marginal density of  $x$  is

$$g(x) = \frac{1}{8} \int_{y=2}^4 (6-x-y) dy = \frac{1}{8} (3x - 12); \text{ for } 0 < x < 2$$

Similarly the marginal density of  $Y$  is

$$h(y) = \int_{x=0}^2 f(x,y) dx = \frac{1}{3} \int_{x=0}^2 (6-x-y) dx = \frac{1}{y} (5-y); \text{ for } 2 \leq y \leq 4$$

We need to verify that  $g(x)$  and  $h(y)$  are probability distributions.

It is clear that in the given range of the variables  $X$  and  $Y$ ,

$$g(x) > 0 \text{ and } h(y) > 0$$

Also,

$$\int_{x=0}^2 g(x) dx = \frac{1}{4} \int_0^2 (3-x) dx = 1$$

$$\int_{y=2}^4 h(y) dy = \frac{1}{4} \int_2^4 (5-y) dy = 1$$

Thus  $g(x)$  and  $h(y)$  satisfy all the conditions of being a probability density function.

and  $P(X < \frac{3}{2}, Y < \frac{5}{2})$

$$= \int_{x=0}^{\frac{3}{2}} \int_{y=2}^{\frac{5}{2}} f(x,y) dy dx = \frac{1}{8} \int_{x=0}^{\frac{3}{2}} \int_{y=2}^{\frac{5}{2}} (6-x-y) dy dx = \frac{9}{32}$$

Example:

The random probability function of a discrete bivariate r.v. is as follows:

$$f(x,y) = \begin{cases} \frac{1}{36} & \text{for } 1 \leq x = y \leq 6 \\ \frac{2}{36} & \text{for } 1 \leq y < x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Find  $g(x)$  and  $h(y)$

$$\left( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) = \frac{6}{6} = 1$$

$$\left( \frac{2}{6} + \frac{2}{6} + \frac{2}{6} \right) = \frac{6}{6} = 1$$