Conditional Distribution :

The conditional distributions are exactly analogous to the conditional probabilities of the type $P(A|B)$ or $P(B|A)$, where A and B are two events in a sample space.

Using the definition of conditional probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

Replacing the events and B by the random variables X and Y, respectively, we can define the conditional probability of Y for given X as follows

$$P(Y=y \mid X=x) = \frac{\{P(X=x)\, P(Y=y)\}}{P(X=x)}$$

$$= \frac{f(x,y)}{g(x)}$$

Where X and Y are discrete random variables.

The function $f(x,y)/g(x)$ is a function of y with x held fixed. We call this function conditional probability distribution of the discrete random variable Y given $X=x$. It satisfies all the properties of a probability distribution. Writing this probability distribution as $f(y|x)$, we have

$$f(y|x) = \frac{f(x,y)}{\sum_{y} f(x,y)} = \frac{f(x,y)}{g(x)}$$

; for $g(x) > 0$

Similarly, the conditional probability distribution of the discrete random variable X, given $Y=y$ is defined as

$$f(x|y) = \frac{f(x,y)}{\sum_{x} f(x,y)} = \frac{f(x,y)}{h(y)}$$

; for $h(y) > 0$

If X and Y are continuous random variables, then their corresponding conditional probability distributions will similarly be defined as follows:

$$f(y|x) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dy} = \frac{f(x,y)}{g(x)} \; ; \; g(x)>0$$

$$\& \; f(x|y) = \frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,y)\,dx} = \frac{f(x,y)}{h(y)} \; , \; h(y)>0$$

If someone wishes to find the probability that the random variable X falls between a and b, when it is known that variable Y = y, we evaluate

$$P(a<X<b \mid Y=y) = \begin{cases} \sum_x f(x|y), & \text{if } X \text{ is discrete} \\ \int_a^b f(x|y)\,dx, & \text{if } X \text{ is continuous} \end{cases}$$

Example : Given the following joint distributions of the discrete random variables X and Y.

Y \ X	0	1	2
0	$3/8$	$9/28$	$3/28$
1	$6/28$	$6/28$	0
2	$1/28$	0	0

find $f(x|1)$, $f(y|1)$ & $P(X=0|Y=1)$.

Solⁿ : By definition,

$$f(x|1) = \frac{f(x,1)}{h(1)} \quad \& \quad f(y|1) = \frac{f(y,1)}{g(1)}$$

Now, $$h(1) = \sum_{x=0}^{2} f(x,1) = f(0,1) + f(1,1) + f(2,1)$$

$$= 6/28 + 6/28 + 0 = 3/7$$

Hence, the conditional distribution of X given $Y=1$ is

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{7}{3} f(x,1) \quad \text{for } x = 0,1,2$$

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Therefore,

$$f(0|1) = \left(\frac{7}{3}\right) f(0,1) = \frac{7}{3} * \frac{6}{28} = \frac{1}{2}$$

$$f(1|1) = \left(\frac{7}{3}\right) f(1,1) = \frac{7}{3} * \frac{6}{28} = \frac{1}{2}$$

$$f(2|1) = \left(\frac{7}{3}\right) f(2,1) = \frac{7}{3} * 0 = 0$$

Hence the conditional distribution of X, given $Y=1$ in tabular form is

x	0	1	2
$f(x\|1)$	1/2	1/2	0

Again, $g(1) = \sum_{y=0}^{2} f(1,y) = f(1,0) + f(1,1) + f(1,2)$

$= 9/28 + 6/28 + 0 = 15/28$

Hence, $f(y|1) = \dfrac{f(1,y)}{g(1)} = \dfrac{28}{15} f(1,y)$;

for $y = 0,1,2$

Therefore,

$$f(0|1) = \left(\frac{28}{15}\right) f(1,0) = \frac{28}{15} * \frac{9}{28} = 3/5$$

$$f(1|1) = \left(\frac{28}{15}\right) f(1,1) = \frac{28}{15} * \frac{6}{28} = 2/5$$

$$f(2|1) = \left(\frac{28}{15}\right) f(1,2) = \frac{28}{15} * 0 = 0$$

Hence, the conditional distribution of Y given X = 1 is

y:	0	1	2
f(y|1):	3/5	2/5	0

Also,

$$P(X=0 \mid Y=1) = \frac{f(0,1)}{h(1)} = f(0|1) = 1/2$$

$$\Rightarrow P(X=0 \mid Y=1) = \frac{6/28}{3/7}$$

$$= \frac{\cancel{6}^{2}}{\cancel{28}_{4}} \times \frac{7}{\cancel{3}}$$

$$= \frac{1}{2}$$

or, $P(X=0 \mid Y=1)$

$= f(0|1)$

$= \frac{1}{2}$.

Example: Consider the following probability distribution of X and Y.

$$f(x,y) = \frac{x+y}{21} \; ; \quad x = 1,2,3 \; \& \; y = 1,2$$

(i) Obtain the marginal and conditional distributions of X and Y.

(ii) Find $f(x|1)$ and

(iii) $P(X=2 | Y=1)$.

Sol^n: The marginal distributions of X and Y are

(i) $g(x) = \sum_{y=1}^{2} f(x,y) = \sum_{y=1}^{2} \frac{x+y}{21}$

$$\Rightarrow g(x) = \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21} \; ; \text{ for } x = 1,2,3$$

&

$h(y) = \sum_{x=1}^{3} f(x,y) = \sum_{x=1}^{3} \frac{x+y}{21}$

$$\Rightarrow h(y) = \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21} = \frac{3y+6}{21} \; ; \; y = 1,2$$

Thus the conditional distributions of X and Y are:

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$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{(x+y)/21}{(3y+6)/21}$$

$$\Rightarrow f(x|y) = \frac{x+y}{3y+6} \; ; \text{ for } x=1,2,3 \;\&\; y=1,2$$

And,

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{(x+y)/21}{(2x+3)/21}$$

$$\therefore f(y|x) = \frac{x+y}{2x+3} \; ; \text{ for } x=1,2,3 \;\&\; y=1,2.$$

(ii) The conditional distribution of X for $Y=1$ is

$$f(x|1) = \frac{x+1}{3\cdot 1+6} = \frac{x+1}{9} \; ; \; x=1,2,3$$

(iii) The conditional probability of X for given Y is

$$P(X=2 \mid Y=1) = \frac{2+1}{3\cdot 1+6} = \frac{3}{9} = \frac{1}{3}$$

(Ans)

Example: Two random variables X and Y have the following joint probability density function:

$$f(x,y) = \begin{cases} 1/2 \ ; & 0 \le x \le y \le 2 \\ 0 ; & \text{elsewhere} \end{cases}$$

(a) Find the marginal density of Y and hence the conditional density of X.

(b) Find also $P(X \le 0.5 \mid Y = 1.5)$

Sol^n: (a) The marginal density of Y is given by

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$\Rightarrow h(y) = \int_0^y \frac{1}{2}\,dx = \frac{1}{2}[x]_0^y = \frac{1}{2}y$$

Thus,

$$h(y) = \begin{cases} \frac{1}{2}y \ ; & 0 \le y \le 2 \\ 0 ; & \text{elsewhere} \end{cases}$$

Thus for any $0 < y \le 2$, the conditional density of X for given Y is

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{1/2}{1/2\, y}$$

$$\Rightarrow f(x|y) = 1/y \; ; \qquad 0 \le x \le y$$

(b) $$P(X \le 0.5 \mid y = 1.5) = \int_{-\infty}^{0.5} f(x|y=1.5)\,dx$$

$$= \int_{0}^{0.5} \frac{1}{1.5}\,dx$$

$$= \frac{1}{1.5}\left[x\right]_0^{0.5}$$

$$= \frac{1}{3}$$

Ans

Note :

$$\boxed{P(a < X < b \mid Y = y) = \int_a^b f(x|y)\,dx}$$

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Example : Find the conditional density of Y given X for the following distribution

$$f(x,y) = \frac{6-x-y}{8} \quad ; \quad 0<x<2,\ 2<y<4$$

$$= 0 \ ; \ \text{otherwise} .$$

Also compute $P(2<Y<3 \mid X=2)$

Sol^n: The conditional density of Y given X is

$$f(y|x) = \frac{f(x,y)}{g(x)} \ , \ \text{where}$$

$$g(x) = \int_{y=2}^{4} f(x,y)\,dy = \frac{1}{8}\int_{2}^{4}(6-x-y)\,dy$$

$$\Rightarrow g(x) = \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$\Rightarrow g(x) = \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right]$$

$$= \frac{1}{8}\left[16-4x-10+2x\right]$$

$$\therefore g(x) = \frac{1}{4}(3-x) \ ; \ 0<x<2$$

Thus,

$$f(y|x) = \frac{(6-x-y)/8}{(3-x)/4}$$

$$\Rightarrow f(y|x) = \frac{6-x-y}{2(3-x)}, \quad 0<x<2, \ 2<y<4$$

Now, when $x=2$,

$$P(2<Y<3 \mid X=2) = \int_{y=2}^{3} f(y|2)\,dy$$

$$= \int_{y=2}^{3} \frac{6-2-y}{2(3-2)}\,dy$$

$$= \frac{1}{2}\int_{y=2}^{3} (4-y)\,dy$$

$$= \frac{1}{2}\left[4y - \frac{y^2}{2}\right]_2^3$$

$$= \frac{1}{2}\left[\left(12-\frac{9}{2}\right) - (8-2)\right]$$

$$= \frac{1}{2}\left(6 - 9/2\right)$$

$$= \frac{3}{4} \ ; \quad 2<y<4$$

Ans

Independence of random variables:

Two random variables X and Y with marginal densities $g(x)$ and $h(y)$, respectively are said to be independent if and only if

$$f(x|y) = g(x)$$

$$\text{or } f(y|x) = h(y)$$

where $f(x|y)$ is the conditional density of X for a given Y and $f(y|x)$ is the conditional density of Y for a given X.

If X and Y are independent, then for any real numbers x and y it must be true that

$$f(x,y) = g(x) * h(y)$$

for all values of x & y.

Note:

$P(A|B) = P(A)$ or $P(B|A) = P(B)$ → Method-1

$P(A \cap B) = P(A) * P(B)$ → Method-2

Example:

Suppose X and Y have the following joint probability function

	X		
Y	2	4	Row sum
1	0.10	0.15	0.25
3	0.20	0.30	0.50
5	0.10	0.15	0.25
Column Sum	0.40	0.60	1.00

Check whether X and Y are independent or not?

Sol^n: (Method-1):

To check the independence of X and Y, we must compute $g(x)$ and $h(y)$ for all values of the random variables X and Y and equate to $f(x,y)$. These computations are shown below:

(i) $f(2,1) = 0.10$; $g(2) = 0.40$; $h(1) = 0.25$;

Hence, $g(2) * h(1) = 0.10 = f(2,1)$

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(ii) $f(4,1) = 0.15$; $g(4) = 0.60$; $h(1) = 0.25$;

Hence, $g(4) * h(1) = 0.15 = f(4,1)$

(iii) $f(2,3) = 0.20$; $g(2) = 0.40$; $h(3) = 0.50$;

Hence, $g(2) * h(3) = 0.20 = f(2,3)$

(iv) $f(4,3) = 0.30$; $g(4) = 0.60$; $h(3) = 0.50$;

Hence, $g(4) * h(3) = 0.30 = f(4,3)$

(v) $f(2,5) = 0.10$; $g(2) = 0.40$; $h(5) = 0.25$

Hence, $g(2) * h(5) = 0.10 = f(2,5)$

(vi) $f(4,5) = 0.15$; $g(4) = 0.60$; $h(5) = 0.25$;

Hence, $g(4) * h(5) = 0.15 = f(4,5)$

The above computations show that for all pairs of the values (x, y) of the random variables X and Y, $f(x,y) = g(x) * h(y)$.

Hence, the random variables are independent.

(Method-2):

An alternative way to check the independence of X and Y is done through computing the conditional probabilities of the type

$P(Y = i \mid X = j)$ for all i & j values.

Now

(i) $P(Y=1 \mid X=2) = f(1|2) = \dfrac{f(2,1)}{g(2)}$

~~⇒ P(Y|0~~

$= \dfrac{0.10}{0.40} = 0.25 = h(1)$

(ii) $P(Y=3 \mid X=2) = f(3|2) = \dfrac{f(2,3)}{g(2)} = \dfrac{0.20}{0.40} = 0.50 = h(3)$

(iii) $P(Y=5 \mid X=2) = f(5|2) = \dfrac{f(2,5)}{g(2)} = \dfrac{0.10}{0.40} = 0.25 = h(5)$

(iv) $P(Y=1 \mid X=4) = f(1|4) = \dfrac{f(4,1)}{g(4)} = \dfrac{0.15}{0.60} = 0.25 = h(1$

(v) $P(Y=3 \mid X=4) = f(3|4) = \dfrac{f(4,3)}{g(4)} = \dfrac{0.30}{0.60} = 0.50 = h(3)$

(vi) $P(Y=5 \mid X=4) = f(5|4) = \dfrac{f(4,5)}{g(4)} = \dfrac{0.15}{0.60} = 0.25 = h$

So, from the above computations we can say X and Y are independent random variables.

Example :

For the following probability distribution, show that X and Y are not independent.

$$f(x,y) = \frac{x+y}{8} \quad ; \quad 0<x<2,\ 0<y<2.$$

Sol^n : (Method - $\frac{1}{2}$):

$$g(x) = \int_{y=0}^{2} f(x,y)\,dy \quad ; \text{ for } 0<x<2$$

$$= \frac{1}{8}\int_{y=0}^{2} (x+y)\,dy$$

$$= \frac{1}{8}\left[xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{8}(2x+2)$$

$$\therefore g(x) = \frac{1}{4}(x+1) \quad ; \text{ for } 0<x<2$$

Also,

$$h(y) = \int_{x=0}^{2} f(x,y)\,dx \quad ; \text{ for } 0<y<2$$

$$= \frac{1}{8}\int_{x=0}^{2} (x+y)\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} + xy\right]_0^2$$

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$= \frac{1}{8}(2+2y)$

$\therefore \; h(y) = \frac{1}{4}(y+1)$; for $0<x<2$

Thus, $g(x) * h(y) = \frac{1}{4}(x+1) \cdot \frac{1}{4}(y+1)$

$\Rightarrow g(x) * h(y) = \frac{1}{16}(x+1)(y+1) \neq f(x, y)$

$\therefore \; g(x) * h(y) \neq f(x, y)$

Hence, X and Y are not independent.

(Method-2)

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{(x+y)/8}{(x+1)/4}$$

$$\Rightarrow f(y|x) = \frac{1}{2}\left(\frac{x+y}{x+1}\right) \neq h(y)$$

$$\& \; f(x|y) = \frac{f(x,y)}{h(y)} = \frac{(x+y)/8}{(y+1)/4}$$

$$\Rightarrow f(x|y) = \frac{1}{2}\left(\frac{x+y}{y+1}\right) \neq g(x)$$

Since the conditional distributions are not equal to the marginal distributions, so the variables are not independent.

Example: Given the following joint density function of X and Y as follows:

$$f(x,y) = \begin{cases} 6xy^2, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Verify if X and Y are independent.

Solⁿ: The marginal density of X is

$$g(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy = 6\int_0^1 xy^2\,dy\ ;\quad 0 \le x \le 1$$

$$\Rightarrow g(x) = 6\left[x\frac{y^3}{3}\right]_0^1 = 2x\ ;\quad 0 \le x \le 1$$

Similarly, the marginal density of Y is

$$h(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx = 6\int_0^1 xy^2\,dx\ ;\quad 0 \le y \le 1$$

$$\Rightarrow h(y) = 6\left[\frac{x^2}{2}y^2\right]_0^1 = 3y^2\ ;\quad 0 \le y \le 1$$

Hence,

$$g(x) * h(y) = 2x * 3y^2 = 6xy^2 = f(x,y)$$

$\therefore$ $f(x,y) = g(x) * h(y)$ for all real numbers

Therefore, X and Y are independent.

Example: Given the following joint density function of X and Y as

$$f(x,y) = \begin{cases} 2; & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0; & \text{otherwise.} \end{cases}$$

Show that X and Y are not independent.

Sol^n: The marginal density function of X is given by $g(x) = \int_{x=0}^{1} f(x,y)\,dy$; for $0 \le x \le 1$

$$= 2[y]_0^1 = \cancel{2x} = 2$$

$\therefore$ ~~$g(x) = 2y \; 2x$~~ $\therefore$ $g(x) = 2$

The marginal density function of Y is given by $h(y) = \int_0^1 f(x,y)\,dx = \int_0^1 2\,dx = 2$

$\therefore\ h(y) = 2$

Hence,

$$f(x, y) \neq g(x) * h(y)$$

Therefore, X and Y are not independent.