Mathematical Expectation: / Expected value of a r.v.

Def^n: If X is a discrete random variable having a PMF $f(x)$, then the expected value of X or the mathematical expectation of X is denoted with $E(X)$ and is defined by as

$$E(X) = \sum_x x f(x)$$

[In other words, the expected value of X is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes that value]

On the other hand, if X is a continuous random variable having a PDF $f(x)$, then the excepted value of X is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

Expected Value = Mean

$E(X) =$ Mean

Theorem :

Let, X be a discrete random variable with probability function $f(x)$ and c be a constant. Then $E(c) = c$.

Proof : By definition,

$$E(c) = \sum_x c f(x)$$

$$= c \sum_x f(x)$$

$$= c \cdot 1 \qquad \left[\because \sum_x f(x) = 1 \text{ for PMF}\right]$$

$$\therefore E(c) = c$$

(Proved)

Defn :

Let, X be a random variable with probability distribution $f(x)$, then the expected value of the function $w(X)$ of the random variable X is

$$E[w(X)] = \sum_x w(x) f(x); \text{ if } X \text{ is discrete.}$$

$$\& \quad E[w(X)] = \int_{-\infty}^{\infty} w(x) f(x)\, dx; \text{ if } X \text{ is continuous.}$$

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Variance : Let X be a random variable with finite mean $\mu = E(X)$, then the variance of X is denoted by $V(X)$, and is defined as

$$V(X) = E(X-\mu)^2$$

or, $$V(X) = E\left[(X - E(X))^2\right]$$

Standard Deviation : The positive square root of the variance is known as the standard deviation

i.e $\sigma = \sqrt{V(X)} = \sqrt{E(X-\mu)^2}$

So, we can write,

$$V(X) = \sigma^2$$

Theorem : Let X be a discrete random variable with probability mass function $f(x)$; then

$$V(X) = \sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2$$

Proof : By definition.

$$V(X) = \sigma^2 = E(X-\mu)^2 ; \text{ where } \mu = E(X)$$

$$\Rightarrow \sigma^2 = E(X^2 - 2X\mu + \mu^2)$$

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$$\Rightarrow \sigma^2 = E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu.\mu + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$\therefore \sigma^2 = E(X^2) - \mu^2$$

where, $\boxed{E(X^2) = \sum_x x^2 f(x)}$

Th^m :

The expected value of the sum of two random variables X and Y is the sum of the expected values of the variables. symbolically,

$$E[X+Y] = E(X) + E(Y)$$

Th^m :

Let X be a random variable with a finite mean. Then for any numerical constants a and b,

$$E(aX+b) = a E(X) + b$$

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Th^m :

The expected value of the two random variables X and Y is equal to the product of their expected values, only when the variables are independent, i.e.,

$$E(XY) = E(X).E(Y)$$

In other words, the expected value of the product of two random variables is equal to the product of their expectations.

Moment Generating Function (MGF) :

Let, X be a random variable with probability function $f(x)$. Then the function $M_X(t)$ is called the moment generating function (MGF) of the random variable X and is defined by

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x) \quad ; \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad ; \text{ If } X \text{ is continuous}$$

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Problem:

A discrete random variable X has a probability function as shown in the following table:

Values of $X: x$	-3	-2	0	1	2
$P(X=x)=f(x)$	0.10	0.30	0.15	0.40	0.05

Find $E(X)$ and $V(X)$.

Sol^n:

By definition we know

$$\mu = E(X) = \sum_{x=-3}^{2} x f(x)$$

$$= (-3)(0.10) + (-2)\cdot(0.30) + (0)(0.15) + (1)(0.40) + (2)(0.05)$$

$$= -0.4$$

$$V(X) = E(X-\mu)^2 = \sum_{x=-3}^{2} (X-\mu)^2 f(x)$$

$$= (-3+0.4)^2 * (0.1) + (-2+0.4)^2 * (0.30) + (0+0.4)^2 * (0.15) + (1+0.4)^2 * (0.40) + (2+0.4)^2 * (0.05) = 2.54$$

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Problem :

Using the formula $V(X) = E(X^2) - \mu^2$ find the variance & standard deviation (SD) of the previous probability function (see previous example)

Sol^n :

By definition we know,

$$E(X^2) = \sum_x x^2 f(x)$$

$$= (-3)^2 * (0.10) + (-2)^2 * (0.30) + (0)^2 * (0.15) + (1)^2 * (0.40) + (2)^2 * (0.05)$$

$$= 2.7$$

Now,

$$V(X) = E(X^2) - \mu^2$$

$$= 2.7 - (-0.4)^2$$

$$= 2.54 \quad \text{Ans.}$$

Hence,

$$SD, \sigma = \sqrt{V(X)}$$

$$= \sqrt{2.54}$$

$$= 1.59$$

Ans

Problem:

In a coin-tossing game, a man is promised to receive TK. 5 if he gets all heads or all tails when three coins are tossed and he pays off (loses) TK. 3 if either one or two heads appear. How much is he expected to gain in the long run?

Sol^n:

The random variable X here is the amount of money (in TK.) the man can win. Here, the r.v X will take on a value 5 when the coins show all heads or all tails and -3, otherwise

The table below shows the outcomes of the experiment, value of X and its associated probabilities:

Outcome :	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$X: x$	5	-3	-3	-3	-3	-3	-3	5
$P(X=x)=f(x)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

It appears from the above table that the variable X assumes values -3 and $+5$ with probabilities 6/8 and 2/8, respectively. Since the value -3 occurs 6 times and 5 occurs 2 times, the expected value of X is

$$E(X) = \sum_x x f(x) = -3(6/8) + 5(2/8)$$

$$= -\frac{18}{8} + \frac{10}{8}$$

$$= -1$$

Thus the man is expected to lose TK. 1 in the long run.

Now let us examine what happens if the man receives TK. 5 for all heads or all tails, TK. 0 for 2 heads and pays off TK. 3 for 1 head. Now, the random variable X will take on values 5, 0 and -3 with associated probabilities 2/8, 3/8, and 3/8, respectively.

The expected value in this case will be

$$E(X) = \sum_x x f(x)$$

$$= 5\left(\frac{2}{8}\right) + 0(3/8) + (-3)\left(\frac{3}{8}\right)$$

$$= \frac{10}{8} + 0 - \frac{9}{8}$$

$$= \frac{1}{8} = 0.125$$

This shows that the man will be marginally gainer winning only 12.5 paisa if the payment is made as designed above.

Problem:

A life insurance company in Bangladesh offers to sell a TK. 25000 one-year term life insurance policy to a 25 year-old man for a premium of TK. 2500. According to Bangladesh life table, the probability of surviving one year for a 25-year-old man is 0.97 and of his dying is 0.03. What is the company's expected gain in the long-run?

Sol^n:

The gain X is a random variable that may take on the values TK. 2500, if the man survives or 2500 − 25000 = −TK 22500 if he dies. Consequently, the probability distribution of X is as follows:

$X : x$	2500	−22500
$f(x)$	0.97	0.03

$\therefore E[X] = 2500 * 0.97 - 22500 * 0.03$

$= 1750$

Thus the company's ultimate gain is Tk. 1750.

Problem:

Let, X denotes the number of spots showing on the face of a well-balanced die after it is rolled once. If $Y = X^2 + 2X$, find

(a) $E[X]$, $E[Y]$, $E[X^2]$, $E[Y^2]$

(b) $V[X]$ and $V[Y]$.

Soln:

The random variables X and Y together with their probability distributions are shown in the following table:

$X: x$	1	2	3	4	5	6
$Y: y$	3	8	15	24	35	48
$f(x) = f(y)$	1/6	1/6	1/6	1/6	1/6	1/6