Conditional Probability:

Conditional probability corresponds to a modified probability model that reflects partial information about the outcome of an experiment.
This modified model has a smaller sample space than the original model.

Example:
Suppose that we toss two dice and also suppose that each of the 36 possible outcomes are equally likely to occur and hence have probability $1/36$.
Let us assume that the first die is a four (4). Then, given this information, what is the probability that the sum of the two dice equals six?
Given that the initial die is a four, it follows that there can be at most six possible outcomes of our experiment, namely,

$$(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5)\ \&\ (4,6)$$

Since each of these outcomes originally had the same probability of occuring, they should still have equal probabilities. That is, given that the first die is a four, then the (conditional) probability of each of the outcomes (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) is $\frac{1}{6}$ while the conditional probability of the other 30 points in the sample space is 0.

Hence, the desired probability will be $1/6$.

Def^n:

The probability of an event A when it is known that some other event B has occured is called a conditional probability and is denoted by $P(A/B)$.

Thus, the conditional probability of the event A given the occurance of the event B is expressed as:

$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{P(A\cap B)}{P(B)}.$$

i.e $$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

Similarly, $$P(B/A) = \frac{P(B\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)}$$

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Here,

$$P(A \cap B) = P(A|B) \cdot P(B)$$

& $$P(B \cap A) = P(B|A) \cdot P(A)$$

Since, $P(A \cap B) = P(B \cap A)$

$$\boxed{\therefore \; P(A|B) \cdot P(B) = P(B|A) \cdot P(A)}$$

$$\boxed{\begin{aligned} \therefore \; P(A \cap B) &= P(A|B) \cdot P(B) = P(B) \cdot P(A|B) \\ \Rightarrow P(A \cap B) &= P(B|A) \cdot P(A) = P(A) \cdot P(B|A) \end{aligned}}$$

Law of compound probability / Multiplication Theorem / Multiplication Law

Def^n : For two events A and B, the probability of their simultaneous occurance is equal to the product of the unconditional probability of A and the conditional probability of B, given that A has actually occured.

$$\boxed{\text{i.e} \quad P(A \cap B) = P(A) \cdot P(B|A)}$$

Also, we can extend the above multiplication rule when the number of events is 3 or more.

For 3 events A_1, A_2 and A_3

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2)$$

For K events, the rule can be written as

$$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_K) = P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$$

$$\cdots \; P(A_K | A_1 \cap A_2 \cap \cdots \cap A_{K-1})$$

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(1) **Problem**: (Conditional Probability)

A pair of dice is thrown. Find the probability that sum of the points on the two dice is 10 or greater if a 5 appears on the first die.

Solⁿ: Let us first define the events as follows:

E: The event that sum of the points on the two dice is 10 or greater

F: The event that a 5 appears on the first die.

We have to find out the conditional probability

$P(E|F) = ?$

Here,

$E = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$

$F = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

$E \cap F = \{(5,5), (5,6)\}$

$n(E) = 6; \quad n(E \cap F) = 2;$

$n(F) = 6; \quad n(S) = 36;$

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

$$= \frac{n(E \cap F)/n(S)}{n(F)/n(S)}$$

$$= \frac{2/36}{6/36} = \frac{1}{3}$$

$\therefore P(E|F) = 1/3$

Alternatively, if F is considered as a reduced sample space, then only two sample points ((5,5), (5,6)) are favorable to the event that the sum is 10 or more. Since, there are 6 sample points in F, the required probability would be

$$P(E/F) = 2/6$$

$$= 1/3$$

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(2) Problem : (Conditional Probability)

A family has two children. What is the conditional probability that both are boys given that at least one of them is a boy?

Sol^n: Assume that the sample space S is given by $S = \{(b,b), (b,g), (g,b), (g,g)\}$ where all the outcomes are equally likely.
For instance, (b,g) means that the older child is a boy and the younger child is a girl.

Let us first define the events as follows:

E: The event that both children are boys

F: The event that at least one of them is a boy.

Here, $E = \{(b,b)\}$

$F = \{(b,b), (b,g), (g,b)\}$

$E \cap F = \{(b,b)\}$

$\therefore\ n(S) = 4;\quad n(E) = 1,\ n(F) = 3,\ n(E \cap F)$

Hence, $P(E/F) = \dfrac{P(E \cap F)}{P(F)}$

$$= \frac{n(E \cap F)/n(S)}{n(F)/n(S)}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

(3) **Problem: (Conditional Probability)**

The probability that a married man watches a certain TV show is 0.4 and that of his wife is 0.5. The probability that a man watches the show, given that his wife does is 0.7. Find the following probabilities:

(a) The probability that a married couple watches the show.

(b) The probability that a wife watches the show given that her husband does.

(c) The probability that at least one of the partners will watch the show.

Sol^n: let us first define the events as follows:

H: Husband watches the show

W: Wife watches the show

ATQ.
$$\left.\begin{aligned} P(H) &= 0.4 \\ P(W) &= 0.5 \\ P(H/W) &= 0.7 \end{aligned}\right\} \text{given}$$

(a) The probability that the couple watches the show is

$$P(H \cap W) = P(W \cap H) = P(W) \cdot P(H/W)$$
$$= 0.5 \times 0.7$$
$$= 0.35$$

(b) The conditional probability that a wife watches the show given that her husband also does is

$$P(W/H) = \frac{P(W \cap H)}{P(H)} = \frac{0.35}{0.4} = 0.87$$

(c) The probability that at least one (either H or W or both) watches the show is

$$P(W \cup H) = P(W) + P(H) - P(W \cap H)$$
$$= 0.5 + 0.4 - 0.35 = 0.55$$

(C. P)

(4) **Problem:** A box contains 7 red balls and 3 black balls. Three balls are drawn from the box one after the other. find the probability that the first two are red and the third is black if

(i) the balls are replaced before the next draw

(ii) the balls are not replaced.

Sol^n: Let, us define the events as follows:

R: The event of drawing red ball

B: The event of drawing black ball.

(i) Here, If a ball is replaced before the next draw, then the subsequent drawings are not affected as the number of balls in the bag remains the same. Hence, it is a case of unconditional probability.

R_1: The first ball is red

R_2: The second ball is red

B_3: The third ball is black.

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Therefore, the required probability is

$$P(R_1 \cap R_2 \cap B_3) = P(R_1) \cdot P(R_2) \cdot P(B_3)$$

$$= \left(\tfrac{7}{10}\right) \cdot \left(\tfrac{7}{10}\right) \cdot \left(\tfrac{3}{10}\right)$$

$$= \frac{147}{1000}$$

(ii) When the balls are not returned to the bag before the next draw, it is a case of conditional probability. For this case, the number of balls will decrease at each subsequent draws.

Therefore, the required probability is,

$$P(R_1 \cap R_2 \cap B_3) = P(R_1) \cdot P(R_2 \mid R_1) \cdot P(B_3 \mid R_1 \cap R_2)$$

$$= \left(\tfrac{7}{10}\right) \cdot \left(\tfrac{6}{9}\right) \cdot \left(\tfrac{3}{8}\right)$$

$$= \tfrac{7}{40}$$

C.P

(5) **Problem:** A coin is tossed until a head appears or it has been tossed three times. Given that the head does not appear on the first toss, what is the probability that the coin is tossed three times?

Sol^n: Let us consider the sample space of the experiment be

$$S = \{H, TH, TTH, TTT\}$$

Therefore,

$$P(H) = \frac{1}{2}$$

$$P(TH) = \frac{1}{4}$$

$$P(TTH) = \frac{1}{8} \quad \& \quad P(TTT) = \frac{1}{8}$$

Let us define two events as follows:

A: The event that the coin is tossed 3 times.

B: The event that no head appears on the first toss.

Hence,

~~P(A)=0~~ $A = \{TTH, TTT\}$

$$B = \{TH, TTH, TTT\}$$

$$A \cap B = \{TTH, TTT\}$$

Now,

$$P(A) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Hence, the required conditional probability is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1/4}{1/2} = \frac{1}{2}$$

Ans

C.P

(b) **Problem:** Suppose we have two Urns U_1 and U_2. The first urn contains 5 white balls and 7 red balls, while the second urn contains 3 white balls and 8 red balls. One ball is transferred from the first urn to the second urn unseen and then a ball is drawn from the second urn. What is the probability that this ball is red?

Sol^n: The ball to be drawn from the second urn (U_2) with a specified color is conditional upon the color of the ball to be transferred from U_1. If a white ball is transferred from U_1, then U_2 will consist of 4 white balls and 8 red balls. On the other hand, if the ball transferred from U_1 happens to be red, then U_2 will consist of 3 white balls and 9 red balls. Then the probability that the ball drawn from U_2 will be red is given by summing the probabilities of the following two events:

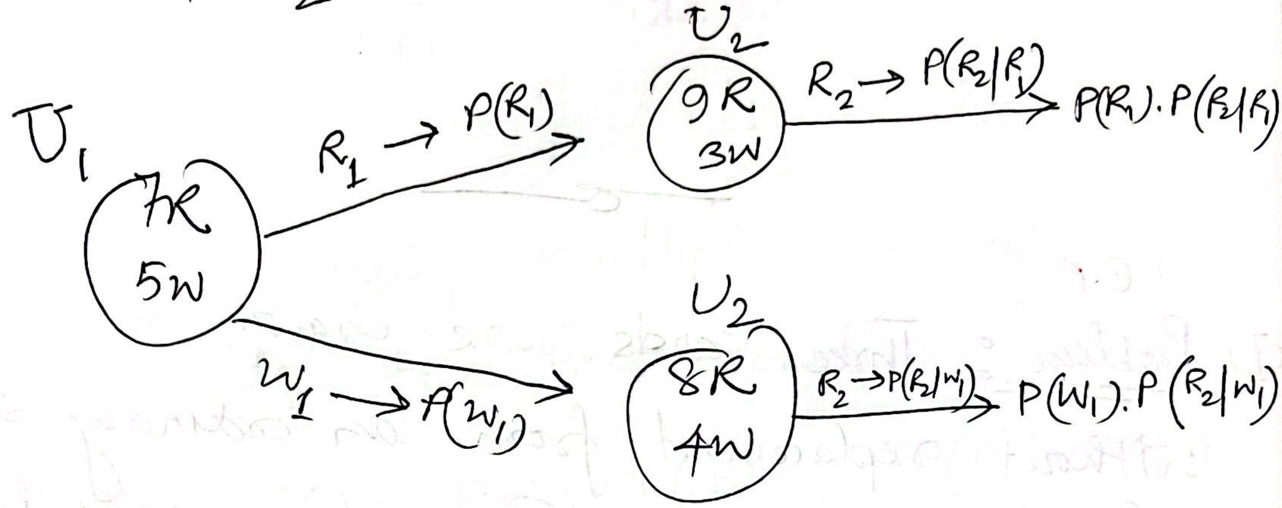
(a) When the ball transferred is red

(b) When the ball transferred is white.

Let, R_1, R_2 and W_1 represent, respectively the drawing of a red ball from U_1 and a white ball from U_2.

We are then interested in the union of two mutually exclusive events

$R_1 \cap R_2$ and $W_1 \cap R_2$.



$$P(\text{Red ball from } U_2) = P(R_1 \cap R_2) + P(W_1 \cap R_2)$$

Where,

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2|R_1)$$

$$= \left(\frac{7}{12}\right) \cdot \left(\frac{9}{12}\right)$$

$$\Rightarrow P(R_1 \cap R_2) = \frac{63}{144}$$

&

$$P(W_1 \cap R_2) = P(W_1) \cdot P(R_2|W_1)$$

$$= \left(5/12\right) \left(8/12\right)$$

$$= \frac{40}{144}$$

$\therefore$ P(Red ball from U_2) $= \frac{63}{144} + \frac{40}{144}$

$= \frac{103}{144}$

C.P

7) **Problem:** Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the first card is a red ace; the second card is a 10 or a jack and the third card is greater than 3 but less than 7.

Sol^n: let us define the events as follows:

A: The first card is a red ace

B: The second card is a 10 or a jack

C: The third card is greater than 3 but less than 7.

Now,

$$P(A) = \frac{2}{52} = \frac{1}{26}$$

$$P(B|A) = \frac{8}{51}$$

$$P\left(C \middle| (A \cap B)\right) = \frac{12}{50} = 6/25$$

Hence, employing multiplicative rule of probability for three events,

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

$$= \left(\frac{1}{26}\right)\left(\frac{8}{51}\right) \cdot \left(\frac{6}{25}\right)$$

$$= \frac{8}{5525}$$