Independent Events:

If E and F are two events and if the occurrence of E does not affect, and is not affected by the occurrence of F, then E and F are said to be independent.

In otherwords, two events E and F are said to be independent if

$$P(E \wedge F) = P(EF) = P(E) \cdot P(F)$$

Using the definition of conditional probability,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \quad \text{———} \; (*)$$

we can say that, two events E and F are independent if

$$\left.\begin{aligned} P(E|F) &= P(E) \\ \text{or, } P(F|E) &= P(F) \end{aligned}\right\}$$

i.e., E and F are independent if knowledge that F has occurred does not affect the probability that E occurs.

So, the occurrence of E is independent of whether or not F occurs.

Problem :

Two ideal coins are tossed. Let A be the event "head on the first coin" and B be the event that "head on the second coin". Check whether the events are independent or not.

Sol^n : let us take,

A : Head on the first coin

B : ~~Head~~ on the second coin

A sample space for the given experiment is

$$S = \{HH, HT, TH, TT\}$$

Therefore,

$$A = \{HH, HT\}$$

$$B = \{HH, TH\} \quad \& \quad A \cap B = \{HH\}$$

Hence,

$$P(A) = 2/4 = 1/\cancel{4}\,2$$

$$P(B) = 2/4 = 1/\cancel{4}\,2$$

$$\& \quad P(A \cap B) = 1/4$$

Since,

$$P(A) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B)$$

$\therefore$ $P(A \cap B) = P(A) \cdot P(B)$, Thus A & B are independent.

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Problem: Three coins are tossed, show that the events "heads on the first coin" and the event "tails on the last two" are independent.

Sol^n: Let us define the events as follows:

A: Heads on the first coin

B: Tails on the last two coins

Then, we have a sample space for the experiment as

$$S = \left\{ \begin{array}{l} HHH, HHT, HTH, HTT \\ THH, THT, TTH, TTT \end{array} \right\}$$

	HH	HT	TH	TT
H	HHH	HHT	HTH	HTT
T	THH	THT	TTH	TTT

$$A = \{HHH, HHT, HTH, HTT\}$$

$$B = \{HTT, TTT\}$$

& $A \cap B = \{HTT\}$

Hence, $P(A) = 4/8 = 1/2$

$P(B) = 2/8 = 1/4$

$P(A \cap B) = 1/8$

$P(A) * P(B) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = P(A \cap B)$

So, A & B are independent.

Example: Suppose we toss two fair dice. Let E_1 denotes the event that the sum of the dice is six and F denote the event that the first die equals four.

Then,

$$P(E_1 \cap F) = P\{(4,2)\} = 1/36$$

while,

$$P(E_1) = P\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

$$= 5/36$$

$$\& \; P(F) = P\{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$$

$$= 6/36 = 1/6$$

$$\therefore P(E_1) \cdot P(F) = 5/36 * 1/6$$

$$= \frac{5}{216}$$

$$\therefore P(E_1) \cdot P(F) \neq P(E_1 \cap F)$$

Hence, E_1 and F are not independent.

Intuitively, the reason for this is clear for if we are interested in the possibility of throwing a six (with two die), then we will be quite happy if the first die lands four (or any other numbers 1, 2, 3, 4, 5) for then we will still have a possibility of getting a total of six.

On the other hand, if the first die landed six, then we would be unhappy as we would no longer have a chance of getting a total of six.

In otherwords, our chance of getting a total of six depends on the outcome of the first die and hence E_1 and F cannot be independent.

—o—

Now, let, E_2 be the event that the sum of dice equals seven.

$$P(E_2) = P\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$= 6/36 = 1/6$$

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$$P(E_2 \cap F) = P\{(4,3)\} = 1/36$$

$$\therefore P(E_2) \cdot P(F) = 1/6 \cdot 1/6$$

$$= 1/36$$

$$\therefore P(E_2 \cap F) = P(E_2) \cdot P(F)$$

Hence, E_2 and F are independent events

Problem (Independence):

A fire brigade has two fire engines operating independently. The probability that specific fire engine is available when needed is 0.99, then

(a) What is the probability that an engine is available when needed?

(b) What is the probability that neither is available when needed?

Solⁿ: Let A be the event that the first engine is available when needed and B be the event that the second engine is available when needed.

Then, $P(A) = P(B) = 0.99$.

Therefore,
The probability that both of them will be available when needed is

$P(A \cap B) = P(A) \cdot P(B)$ [since they operate independently]

$= 0.99 * 0.99$

$= 0.9801$

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.99 + 0.99 - 0.9801$

$= 0.9999$

(b) $P(A^c \cap B^c) = P((A \cup B)^c)$

$= 1 - P(A \cup B)$

$= 1 - 0.9999$

$= 0.0001$

Independence of more than two events :

The multiplication rule for independence events extends very simply to three or more independent events.

For three events, A, B, C when all are independent of each other

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

For n events $A_1, A_2, \ldots, A_n$

$$P(A_1 \cap A_2 \cap A_3 \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdots P(A_n) \quad (*)$$

Def^n :

The n events are said to be completely independent iff every combination of these events, taken any number at a time, is independent.

If every combination other than (eq^n $*$) is independent, then we say that the events are pairwise independent but not completely independent.

Problem :

Two coins are tossed. If A is the event "head on the first coin", B is the event "head on the second coin", and C is the event "coins fall alike", then show that the events A, B, and C are pairwise independent but not completely independent.

Sol^n : The sample space is

$$S = \{HH, HT, TH, TT\}$$

Let us define the events as follows.

$$A = \{HH, HT\}$$

$$B = \{HH, TH\}$$

$$C = \{HH, TT\}$$

$$A \cap B = \{HH\} \quad ; \quad \cancel{A \cap C}$$

$$A \cap C = \{HH\}$$

$$B \cap C = \{HH\} \quad \& \quad A \cap B \cap C = \{HH\}$$

Thus, the associated probabilities are

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$$

$$\therefore \quad P(A) \cdot P(B) = \frac{1}{4} = P(A \cap B)$$

$$P(B) \cdot P(C) = \frac{1}{4} = P(B \cap C)$$

$$P(A) \cdot P(C) = \frac{1}{4} = P(A \cap C)$$

Also,

$$P(A) \cdot P(B) \cdot P(C) = \frac{1}{8}$$

But, $P(A \cap B \cap C) = \frac{1}{4}$

Hence, $P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$

Therefore, the events are not independent when taken altogether, i.e., they are not completely independent but they are pairwise independent.

Conditional Probability and Partitions :

Let, S denotes the sample space of some random experiment and consider n events $A_1, A_2, A_3, \ldots, A_n$ such that they are mutually exclusive and collectively exhaustive, that is $A_i \cap A_j = \phi$ for all $i \neq j$ and

$$S = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n .$$

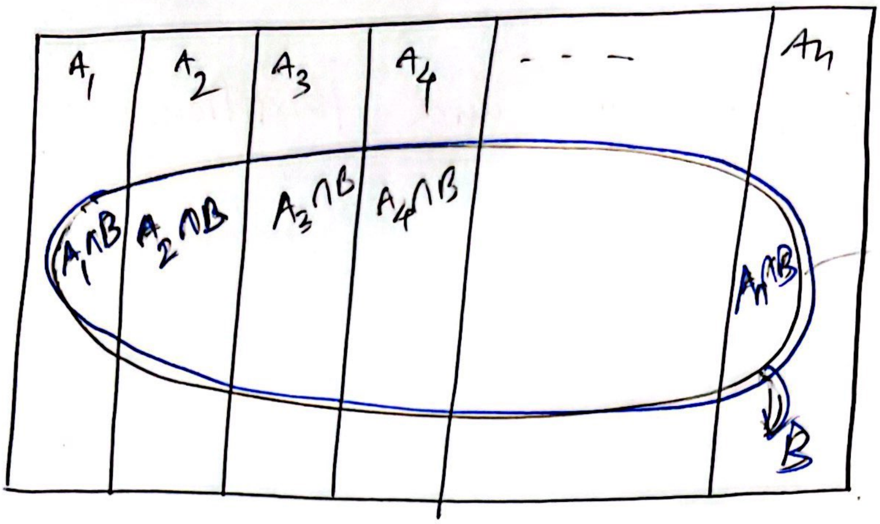
It is said that these events form a partition of S.

Total Probability Rule :

Suppose that the events $A_1, A_2, A_3, \ldots, A_n$ make partition of the sample space S and $P(A_j) > 0$ for $j = 1, 2, \ldots, n$.

Then for any event $B \in S$,

$$P(B) = \sum_{j=1}^{n} P(A_j) \cdot P(B/A_j)$$



The events $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ form a partition of the event B.

Hence, $B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B)$

Since, the n events ~~are~~ the right side of the above equation are mutually exclusive

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$\Rightarrow P(B) = \sum_{j=1}^{n} P(A_j \cap B)$$

Now, using multiplication rule, we have

$$P(A_j \cap B) = P(A_j) \cdot P(B|A_j)$$

Hence, $$\boxed{P(B) = \sum_{j=1}^{n} (A_j)\, P(B|A_j)}$$

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It is noted that the probability of B is the weighted average of the conditional probabilities $P(B/A_j)$ with weights $P(A_j)$.

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Background of Bayes' Theorem :

Very often we begin our probability analysis with initial or prior probability estimates for a specific event of interest. Then, from sources, such as a sample, a specific report or document, we obtain some additional information about the events.

Given this new information, we want to revise and up-date the prior probability values. The new and revised probabilities for the events are referred to as posterior probabilities.

Bayes' theorem, which will be dealt here provides a means of computing these revised probabilities.

Application :

Let us consider n mutually exclusive and collectively exhaustive events $A_1, A_2, A_3, \ldots, A_n$ and let B be any event.

If $P(B|A_1), P(B|A_2), \ldots, P(B|A_n)$ are known, then Bayes' Theorem is useful to compute the conditional probabilities of A_j events given B.

Bayes' Theorem :

Let, the events $A_1, A_2, A_3, \ldots, A_n$ form a partition of the sample space S, such that $P(A_j) > 0$ for $j = 1, 2, \ldots, n$ and let B be an event such that $P(B) > 0$.

Then,
$$P(A_i|B) = \frac{P(A_i) \cdot P(B|A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B|A_j)};$$

for $i = \overline{1, n}$

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Proof: Using the definition of conditional probability,

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \longrightarrow (1)$$

Since, the events $A_1, A_2, A_3, \ldots, A_n$ form a partition in S, and B is any event in S,

$$B = S \cap B$$

$$= (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) \cap B$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_n \cap B)$$

[using distributive law]

Thus,

$$P(B) = P\left[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)\right]$$

$$= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$$P(B) = \sum_{j=1}^{n} P(A_j \cap B) \longrightarrow (2)$$

[using Total probability law & they are mutually exclusive] using (2) into eqn (1)

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{\sum_{j=1}^{n} P(A_j \cap B)} \quad \text{——— (3)}$$

According to multiplicative law

$$P(A_j \cap B) = P(A_j) \cdot P(B \mid A_j) \quad \text{——— (4)}$$

Hence, from eqn (3), we have

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{\sum_{j=1}^{n} P(A_j \cap B)}$$

$$\Rightarrow P(A_i \mid B) = \frac{(A_i) \cdot P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j) \cdot P(B \mid A_j)} \quad [\text{by } (4)]$$

for $i = 1, 2, \ldots, n$

Proved

Problem: (Identifying the source of a defective item)

In a factory three different machines M_1, M_2 and M_3 were used for producing a large batch of similar manufactured items. Suppose that M_1, M_2 and M_3 produced respectively 25%, 35% and 40% of the total items. Suppose further that of their items, respectively 5%, 4% and 2% are defective. Finally suppose that one item is selected at random from the entire batch and it is found to be defective. Determine the probability that this item is produced by machine M_1; M_2 or M_3?

Sol^n: Let us define the events as follows:

A_1: The selected item was produced by M_1

A_2: The " " " " " M_2

A_3: The " " " " " M_3

B: The " " is defective.

Then we have,

$$\left.\begin{aligned} P(A_1) &= \frac{25}{100} = 0.25 \\ P(A_2) &= 35/100 = 0.35 \\ P(A_3) &= 40/100 = 0.40 \end{aligned}\right\}$$

Also,

$$P(B|A_1) = \frac{5}{100} = 0.05$$

$$P(B|A_2) = \frac{4}{100} = 0.04$$

$$P(B|A_3) = \frac{2}{100} = 0.02$$

Using Bayes' Theorem, the probability that the selected defective item is produced by machine M_1 is

(a) $$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{\sum_{j=1}^{3} P(A_j) \cdot P(B|A_j)}$$

$$= \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}$$

$$= \frac{0.25 * 0.05}{(0.25 * 0.05) + (0.35 * 0.04) + (0.40 * 0.02)}$$

$$= 0.36$$

The probability is 0.36 that a randomly selected defective item is produced by machine M_1.

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Interpretation: Approximately 36% of the defective items have been produced by machine M_1.

(b) $P(A_2 \cup A_3 | B) = P(A_2|B) + P(A_3|B)$

$$= \frac{P(A_2) \cdot P(B|A_2)}{\sum_{j=1}^{3} P(A_j) P(B|A_j)} + \frac{P(A_3) \cdot P(B|A_3)}{\sum_{j=1}^{3} P(A_j) P(B|A_j)}$$

$$= \frac{P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}{\sum_{j=1}^{3} P(A_j) P(B|A_j)}$$

$$= \frac{(0.35 * 0.04) + (0.40 * 0.02)}{\{(0.25 * 0.05) + (0.35 * 0.04) + (0.40 * 0.02)\}}$$

$$= \frac{0.014 + 0.008}{0.0345} = \frac{0.022}{0.0345} = 0.6377$$

The probability is 0.64 that a randomly selected defective item is produced by machine M_2 or machine M_3.

Approximately 64% of ~~the defective~~ defective items have been produced by M_2 or M_3.

Problem: (Identifying the sources of a defective sphere Parts)

A company produces sphere parts and supplies in packets. The company produces 2000 packets by plant -1 and 3000 packets by plant -2. Previous experience indicates that 10% produced by plant -1 are defective, while 15% are defective produced by plant - 2. One day a defective packet was identified. Compute the probability that the packet was produced by plant -1.

Soln: Let us define the events as follows:

A_1: The selected packet is produced by plant -1

A_2: " " " " " " " -2

B: The randomly selected packet is defective

Thus we have,

$$P(A_1) = \frac{2000}{2000+3000} = 0.40$$

$$P(A_2) = \frac{3000}{2000+3000} = 0.60.$$

$$\left.\begin{array}{l} P(B/A_1) = 0.10 \\ P(B|A_2) = 0.15 \end{array}\right\} \begin{array}{l} 10\% \\ 15\% \end{array}$$

Now, using the concept of Bayes' Th^m.

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)}$$

$$= \frac{0.40 * 0.10}{(0.40 * 0.10) + (0.60 * 0.15)}$$

$$= 0.3077 \approx 0.31$$

Approximately 31% of the defective packets are produced by plant-1.

Problem: (Selecting balls)

Three urns contain respectively as: urn-1 contains 2 red and 4 black balls, urn-2 contains 3 red and 1 black balls and urn-3 contains 3 red and 4 black balls. One urn is chosen at random and then two balls are randomly drawn from the selected urn. If the selected balls are different color, What is the probability that the ball come from urn 2? urn 2 or 3?

Sol^n: Urn-1 contains 2 red balls & 4 black balls. No. of total balls: 6

Urn-2 contains 3 red balls & 1 black ball. No. of total balls: 4

Urn-3 contains 3 red balls & 4 black balls

No. of total balls: 7.

Let, us define the events as follows

A_1: Urn-1 is chosen

A_2: Urn-2 is chosen

A_3: Urn-3 is chosen

B: The selected balls are different in color (i.e, one red & one black)

Now, $P(A_1) = 1/3$; $P(A_2) = P(A_3) = 1/3$;

$$P(B|A_1) = \frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2} = \frac{2\times 4}{15} = 8/15$$

$$P(B|A_2) = \frac{\binom{3}{1}\binom{1}{1}}{\binom{4}{2}} = \frac{3\times 1}{6} = \frac{1}{2}$$

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$$P(B|A_3) = \frac{\binom{3}{1}\binom{4}{1}}{\binom{7}{2}} = \frac{3\times 4}{21} = \frac{4}{7}$$

or,

$$P(B|A_3) = \left(\frac{3}{7}\times\frac{4}{6}\right) + \left(\frac{4}{7}\times\frac{3}{6}\right)$$

$$= \frac{4}{7}$$ এই নিয়মেও করা যাবে প্রতিটি ক্ষেত্রেই

(a) Using Bayes' Th^{m}, the probability of selecting urn-2 given that the selected balls are different in color is

$$P(A_2|B) = \frac{P(A_2)\cdot P(B|A_2)}{\sum_{j=1}^{3} P(A_j)\, P(B|A_j)}$$

$$= \frac{P(A_2)\cdot P(B|A_2)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{2}}{\left(\frac{1}{3}\times\frac{8}{15}\right) + \left(\frac{1}{3}\times\frac{1}{2}\right) + \left(\frac{1}{3}\times\frac{1}{7}\right)}$$

$$= \frac{105}{337} = 0.3116$$

(b) The probability of selecting urn-2 or urn-3 given that the selected balls are different in color is

$$A(A_2 \cup A_3 | B) = P(A_2|B) + P(A_3|B)$$

$$= \frac{P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)}{\sum_{j=1}^{3} P(A_j) \cdot P(B|A_j)}$$

$$= \frac{5/14}{337/630}$$

$$= \frac{225}{337} = 0.6677.$$

Problem: (Medical Diagnostics for Lung Cancer)

Assume that Oncology Dept. of DMC is giving a free medical test for lung disease. The test is 90% reliable in the following sense: If a person has lung cancer, there is 90% chance that the test will give a positive response; if the person does not have the disease, there is

5%. chance that the test will give a positive response; Data indicate that your chance of having the disease are only 1 in 10,000. Since it is a free medical test, and first and harmless, you decide to take the test. From the test report, you learn that the report come out positive. Now, What is the probability that you have actually lung cancer? i.e. What is the probability that you have the disease after you learn that the result of the test is positive?

Solⁿ: Let us define the events as follows:

A_1: You have the disease

A_2: You do not have the disease

B: The response of the test is positive.

Now, $P(A_1) = \frac{1}{10,000} = 0.0001$

$P(A_2) = 1 - 0.0001 = 0.9999$

$$P(B|A_1) = \frac{90}{100} = 0.90$$

$$P(B|A_2) = \frac{5}{100} = 0.05$$

Using Bayes' Th^{m}, the required probability is

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)}$$

$$= \frac{0.0001 * 0.9}{(0.0001 * 0.9) + (0.9999 * 0.05)}$$

$$= \frac{0.0009}{0.050085}$$

$$\approx 0.002$$

<u>Interpretation</u> : Out of every 1000 persons for whom the test gives a positive response, only 2 persons actually have cancer.

Problem: (Principal selection & introducing Co-education)

This year, suppose that there will be three candidates for the post of principal in NDC. They are Fr. Costa, Fr. Hemanta, & Fr. Adam. The chances that they get the post are 4:2:3. The probability that Fr. Costa if selected will introduce co-education in the college is 0.3. The probability of Fr. Hemanta & Fr. Adam doing the same are 0.5 & 0.8, respectively. What is the probability that there will be co-education in the college this year. If co-education will be introduced, What will be the chance that it will be introduced by the principal Fr. Costa?

Sol^n: Let us define the events as follows:

A_1: Fr. Costa will be selected as principal

A_2: Fr. Hemanta " " " " "

A_3: Fr. Adam " " " "

B: Co-education will be introduced.

Given that,

$$P(A_1) = \frac{4}{4+2+3} = \frac{4}{9}$$

$$P(A_2) = \frac{2}{9}$$

$$P(A_3) = 3/9$$

Also,

$$\left.\begin{aligned} P(B/A_1) &= 0.3 \\ P(B|A_2) &= 0.5 \\ P(B/A_3) &= 0.8 \end{aligned}\right\}$$

Now, we have to compute $P(B)$ & $P(A_1|B)$

Using Total probability Th^m.

$$P(B) = \sum_{j=1}^{3} P(A_j) \cdot P(B/A_j)$$

$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + P(A_3) \cdot P(B|A_3)$$

$$= \left(\frac{4}{9} * 0.3\right) + \left(\frac{2}{9} * 0.5\right) + \left(\frac{3}{9} * 0.8\right)$$

$$= \frac{23}{45}$$ Ans

Now, Using Bayes' Th^m we have,

$$P(A_1|B) = \frac{P(A_1) \cdot P(B|A_1)}{\sum_{j=1}^{3} P(A_j) \cdot P(B|A_j)}$$

$$= \frac{4/9 \times 0.3}{23/45}$$

$$= 6/23$$