Continuous Probability Distribution :

The probability distribution of a continuous random variable cannot be presented in a tabular form, because a continuous variable can take on a non-innumerably infinite set of values. More precisely, a continuous variate can take on any value in the given interval $a \leq X \leq b$. As a result, a continuous random variable has a probability zero of assuming exactly any of its values.

This implies that

$$P(a \leq X \leq b) = P(X=a) + P(a < X < b) + P(X=b)$$
$$= P(a < X < b)$$

In continuous probability distribution, we assign probabilities to variate values within intervals, while in discrete distribution, we associate a finite probability to each variate value.

Thus a probability distribution in the case of a continuous distribution takes the form of an algebraic expression or formula and the probability that X assumes values between a and b is the area bounded by the interval (a, b).

As before, we shall designate the probability distribution of a continuous random variable X by the functional notation $f(x)$. In dealing with continuous variable, $f(x)$ is usually called a probability density function (PDF) or simply density function.

Def^n: (PDF)

A probability density function is a non-negative function and is constructed so that the area under its curve bounded by x-axis is equal to unity when computed over the range of x, for which $f(x)$ is defined.

The above definition leads to conclude that a PDF is one ~~of~~ that possesses the following properties:

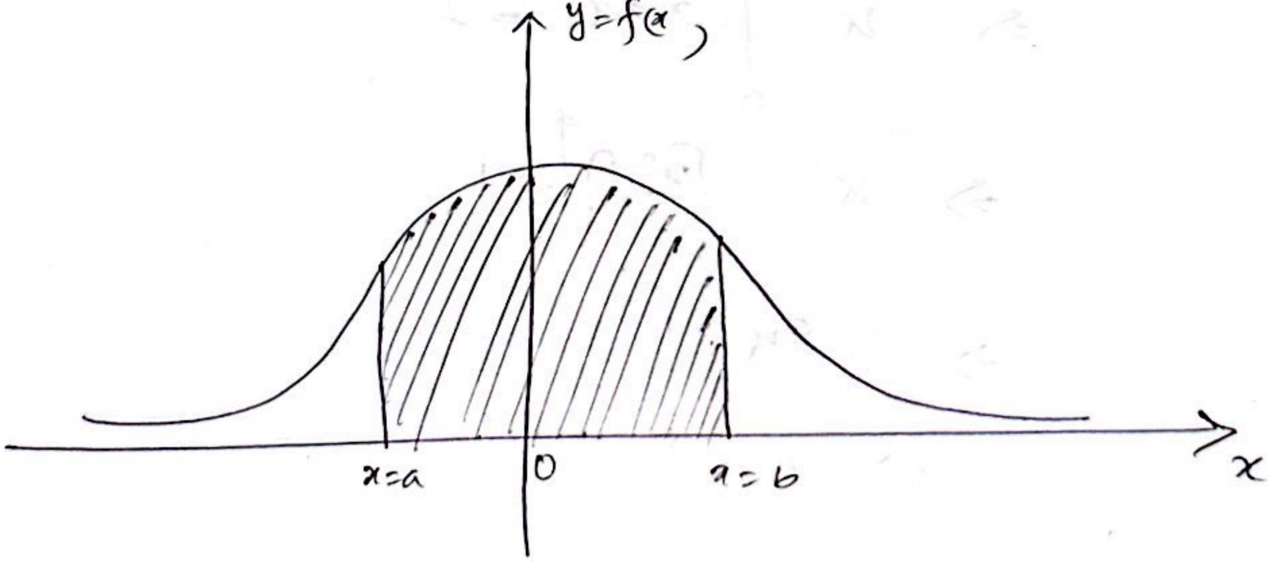
① $f(x) \geq 0$

② $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

③ $\displaystyle P(a < X < b) = \int_a^b f(x)\,dx$

The third property above states that the random variable X falling in any interval (a, b) represents the area under the curve $y = f(x)$ between $X = a$ and $X = b$.

A typical probability ~~density function~~ distribution is sketched in the figure below, where the total area under the curve is 1 and the value of $P(a < X < b)$ is equal to the shaded region.



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Example: A random variable X has the following functional form:

$$f(x) = Kx; \quad 0 < x < 4$$
$$= 0; \quad \text{elsewhere}$$

(i) Determine K for which f(x) is a PDF

(ii) Find $P(1 < X < 2)$ and $P(X > 2)$.

Solⁿ: (i) For f(x) to be a PDF, we must have $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Thus, $\int_0^4 f(x)\,dx = \int_0^4 Kx\,dx = 1$

$$\Rightarrow K \int_0^4 x\,dx = 1$$

$$\Rightarrow K \cdot \left[\frac{x^2}{2}\right]_0^4 = 1$$

$$\Rightarrow 8K = 1$$

$$\Rightarrow K = 1/8.$$

The complete PDF is thus

$$f(x) = \begin{cases} \frac{x}{8}; & 0<x<4 \\ 0; & \text{elsewhere} \end{cases}$$

(ii) Again,

$$P(1<X<2) = \frac{1}{8}\int_1^2 x\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{8}\times\frac{1}{2}\left[2^2-1^2\right]$$

$$= \frac{3}{16}$$ Ans

and,

$$P(X>2) = \frac{1}{8}\int_2^4 x\,dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{16}\left(4^2-2^2\right)$$

$$= \frac{3}{4}$$ (Ans)

Example:

A continuous random variable X has the following density function:

$$f(x) = \frac{2}{27}(1+x) ; \quad 2 < x < 5$$

$$0 ; \text{ elsewhere}$$

(a) Verify that it satisfies the condition.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(b) Find $P(X<4)$ and

(c) Find $P(3<X<4)$.

Sol^n:

(a) $$\int_2^5 f(x)\,dx = \int_2^5 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^5$$

$$= \frac{2}{27}\left[\left(5+\frac{25}{2}\right) - \left(2+\frac{4}{2}\right)\right]$$

$$= \frac{2}{27}\left[5+\frac{25}{2} - 4\right]$$

$$= \frac{2}{27}\left[1+\frac{25}{2}\right] = 1$$

since, $\int_{2}^{5} f(x)dx = 1$, thus the given function is a density function.

since, the lower limit is 2, Integrate between 2 & 4.

(b) $P(X<4) = \frac{2}{27}\int_{2}^{4}(1+x)\,dx$

$$= \frac{2}{27}\left[x+\frac{x^2}{2}\right]_2^4$$

$$= \frac{16}{27}$$

(c) Evaluating the integral between 3 & 4 we obtain $P(3<X<4)$:

$$P(3<X<4) = \frac{2}{27}\int_{3}^{4}(1+x)\,dx$$

$$= \frac{2}{27}\left[x+\frac{x^2}{2}\right]_3^4$$

$$= \frac{2}{27}\left[\left(4+\frac{16}{2}\right)-\left(3+\frac{9}{2}\right)\right]$$

$$= \frac{2}{27}\left[12-\frac{15}{2}\right] = \frac{2}{27}\times\frac{9}{2}$$

$$= \frac{1}{3}.$$ Ans

Continuous Distribution Function :

Exactly analogous to the distribution function of a discrete random variable, a continuous random variable has also a distribution function.

Defn :

The cumulative distribution or distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

If the derivative of $F(x)$ exists, then

$$f(x) = \frac{d}{dx} F(x) = F'(x)$$

Thus, $\boxed{f(x)\,dx = dF(x)}$

The distribution function $F(x)$ possesses the following properties:

(i) $F'(x) = f(x) > 0$

(ii) $F(-\infty) = 0$

(iii) $F(+\infty) = 1$

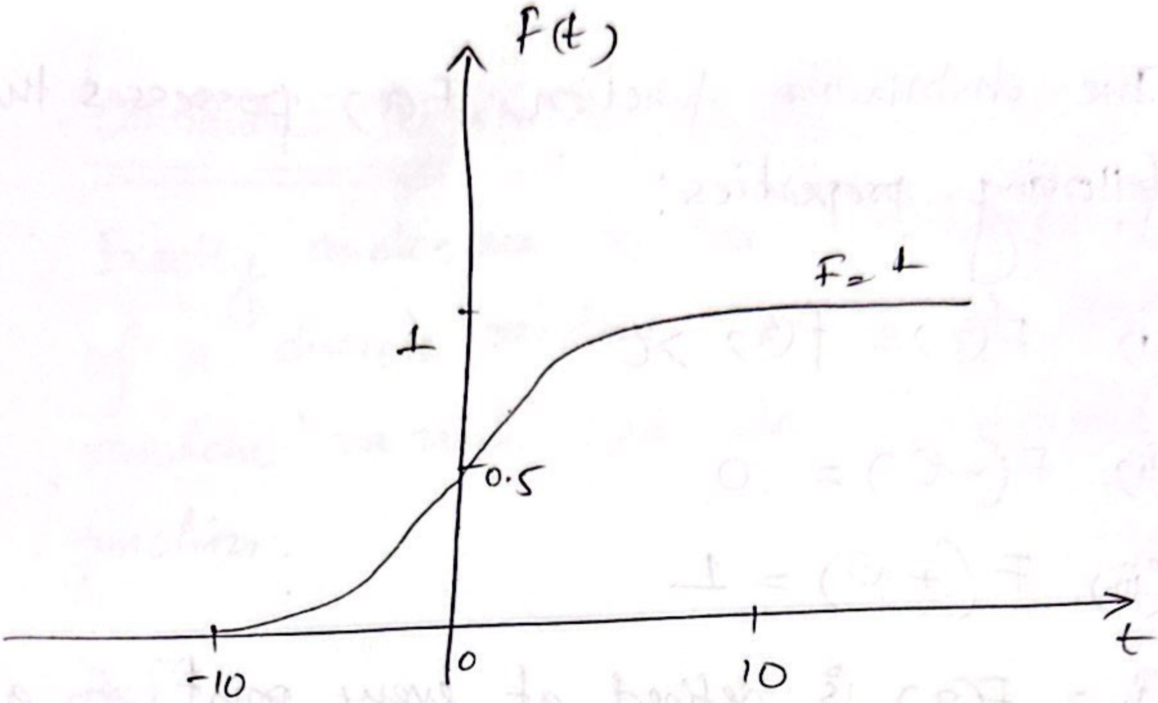
(iv) $F(x)$ is defined at every point in a continuous range and is continuous.

One can easily verify that the distribution function has the following interesting and useful result:

$$P(a < X < b) = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$$

$$= F(b) - F(a);$$

Cumulative distribution curves often have more or less the S-shape pattern. A typical curve appears below:



Here, the horizontal axis shows possible values of the variable T. For any point on the axis, t the height of the curve F(t) is the probability that T is less than or equal to t

Example :

A box contains good and defective items. If an item drawn is good, the number 1 is assign to the drawing; otherwise the number 0 is assigned. Let p be the probability of drawing a good item at random. Then

$$f(x) = P(X=x) = \begin{cases} 1-p \ ; & x=0 \\ p \ ; & x=1 \end{cases}$$

consequently, the distribution function (CDF) of X is

$$F(x) = P(X \leq x) = \begin{cases} 0 \ ; & x<0 \\ 1-p \ ; & 0 \leq x < 1 \\ 1 \ , & x \geq 1 \end{cases}$$

Example : If X has the density function

$$f(x) = \frac{2}{27}(1+x) \; ; \quad 2<x<5$$

$$= 0 \; ; \text{ elsewhere}$$

Obtain the distribution function and hence find $F(3)$ & $F(4)$.

Also verify that $P(3<X<4) = F(4) - F(3)$.

Sol^n : The distribution function is

$$F(x) = \frac{2}{27}\int_2^x (1+t)\,dt$$

$$= \frac{2}{27}\left[t + \frac{t^2}{2}\right]_2^x$$

$$= \frac{2}{27}\left[\left(x + \frac{x^2}{2}\right) - (2+2)\right]$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2} - 4\right]$$

$$\Rightarrow F(x) = \frac{1}{27}\left[x^2 + 2x - 8\right]$$

For $x = 3$ & 4

$$F(3) = \frac{1}{27}\left[3^2 + 6 - 8\right] = 7/27$$

$$F(4) = \frac{1}{27}\left[4^2 + 8 - 8\right]$$

$$\Rightarrow F(4) = \frac{16}{27}$$

Hence,

$$F(4) - F(3) = \frac{16}{27} - \frac{7}{27} = \frac{1}{3}$$

Also,

$$P(3 < X < 4) = \int_3^4 f(x)\,dx$$

$$= \int_3^4 \frac{2}{27}(1+x)\,dx$$

$$= \frac{2}{27}\left[x + \frac{x^2}{2}\right]_3^4$$

$$= \frac{2}{27}\left[(4+8) - \left(3 + \frac{9}{2}\right)\right]$$

$$= \frac{2}{27}\left[12 - 3 - \frac{9}{2}\right]$$

$$= \frac{2}{27}\left[9 - \frac{9}{2}\right]$$

$$= \frac{2}{27} \times \frac{9}{2}$$

$$= \frac{1}{3}$$

Hence,

$$\cancel{P(3X)}\ P(3 < X < 4) = \underline{F(4) - F(3)}$$

Example: Let X be a continuous random variable with the following density function:

$$f(x) = \frac{x}{2} \; ; \quad 0 < x < 2$$

$$0 \; ; \quad \text{otherwise}$$

Find the distribution function & $P(X<1)$.

Sol^n: $F(x) = P(X < x) = \int_0^x f(t)\,dt$

$$= \int_0^x \frac{t}{2}\,dt$$

$$= \frac{1}{2}\left[\frac{t^2}{2}\right]_0^x$$

$$\Rightarrow F(x) = \frac{1}{4}x^2$$

Thus,

$$F(x) = \begin{cases} 0 \; ; & x < 0 \\ \frac{x^2}{4} \; ; & 0 \leq x < 2 \\ 1 \; ; & x \geq 2 \end{cases}$$

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and $P(X<1) = F(1) = \frac{1}{4}$

Ans

Example:

Given that the CDF of a random variable X is as follows:

$$F(x) = \begin{cases} 0, & x<0 \\ x, & 0 \le x \le 1 \\ 1, & x>1 \end{cases}$$

Find the probability density function of X.

Sol^n: Since the density function $f(x)$ is the derivative of the distribution function $F(x)$, when the derivative exists.

$$f(x) = \frac{d}{dx}(F(x)) = \begin{cases} \frac{d}{dx}(0) = 0, & x<0 \\ \frac{d}{dx}(x) = 1, & 0<x<1 \\ \frac{d}{dx}(1) = 0, & x>1 \end{cases}$$

and $f(x)$ is undefined at $x=0$ & $x=1$.

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Note that, $F(x)$ is a continuous function of x, while $f(x)$ is discontinuous at the points $x = (0, 1)$.

In general, the distribution function for a continuous random variable must be continuous, but the density function need not to be continuous everywhere.

Example: Suppose X is a random variable with density function

$$f(x) = \frac{k}{(1+x)^2}, \quad x > 0$$
$$= 0, \quad \text{elsewhere}$$

(a) Find the value of k

(b) Find $F(x)$

(c) Find $P(X<1)$

(d) Find $P(X>1)$

Sol^n: Since $f(x)$ is a density function, it must satisfy

$$\int_0^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad k\int_0^{\infty} \frac{1}{(1+x)^2}dx = 1$$

$$\Rightarrow \left[\frac{(1+x)^{-2+1}}{-2+1}\right]_0^{\infty} = 1/k$$

$$\Rightarrow \left[\frac{1}{1+x}\right]_{\infty}^{0} = 1/k$$

$$\Rightarrow \quad [1-0] = 1/k$$

$$\Rightarrow 1 = 1/k$$

$$\Rightarrow k = 1 \; .$$

Thus, the complete density function of X is

$$f(x) = \frac{1}{(1+x)^2} \; ; \; x>0$$

(b) $F(x) = P(X \leq x) = \int_0^x \frac{dx}{(1+x)^2} = \frac{x}{1+x}$

(c) $P(X<1) = F(1) = \frac{1}{2}$

(d) $P(X>1) = \int_1^{\infty} f(x)dx = \int_0^{\infty} f(x)dx - \int_0^1 f(x)dx$

$$= \int_0^{\infty} \frac{dx}{(1+x)^2} - \int_0^{1} \frac{dx}{(1+x)^2}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2} \quad \underline{\text{Ans}}$$

Example : Find the constant K so that the following function may be a density function :

$$f(x) = 1/K, \quad a \le x \le b$$

$$= 0, \quad \text{elsewhere}$$

Find also the cumulative distribution of the random variable X defined above.

Soln : If $K > 0$, $f(x) \geq 0$ for every x.

Hence, for $f(x)$ to be a density function we have,

$$1 = \int_a^b f(x)\,dx = \int_a^b 1/K \, dx$$

$$\Rightarrow 1 = \frac{1}{K}\int_a^b dx = \frac{1}{K}[x]_a^b$$

$$\Rightarrow K = b-a$$

Hence, the complete density function is

$$f(x) = \frac{1}{b-a} \; ; \quad a \le x \le b$$

$$= 0 \; ; \quad \text{elsewhere}$$

The distribution function $F(x)$ is defined as follows:

$$F(x) = \int_a^x \frac{dx}{b-a} \; ; \quad a \le x \le b$$

$$= 0 \; ; \quad x > b$$

Example: The continuous random variable X has the following distribution function $F(X)$. Find the density function $f(x)$.

$$F(x) = \begin{cases} 0, & x \le 0 \\ \frac{x^3}{27}, & 0 \le x \le 3 \\ 1, & x \geq 3 \end{cases}$$

Solⁿ: By definition,

$$f(x) = \frac{d}{dx}\left(F(x)\right)$$

$$= \frac{d}{dx}\left(\frac{x^3}{27}\right) = \frac{1}{27} \cdot 3x^2 = \frac{x^2}{9}$$

Hence, $f(x) = \frac{x^2}{9}; \quad 0 \leq x \leq 3$

$= 0$; otherwise

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III. Joint Probability Distribution:

In many instances, it is necessary to consider the properties of two or more random variables simultaneously. This results in joint probability distributions and when these distributions involve two variables, they are referred to as bi-variate probability distributions. Clearly, a bi-variate probability distribution is a special case of joint probability distributions for two random variables.

Joint probability distribution for discrete random variables:

Suppose that a given experiment involves two random variables X and Y, each of which has a discrete probability distribution. It is customary to refer to $f(x,y)$ as the joint probability distribution of X and Y.

Since X and Y are discrete, $f(x,y) = P(X=x, Y=y)$

That is $f(x,y)$ gives the probability that the outcomes x and y occur at the same time.

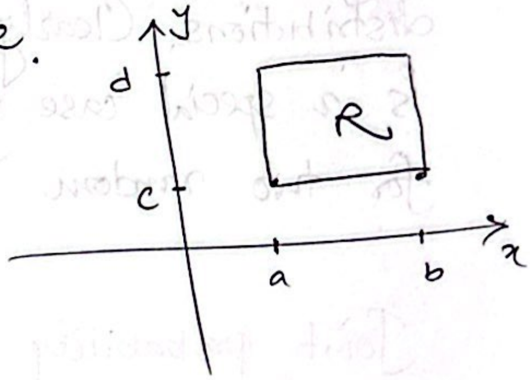
The function $f(x,y)$ will be called a joint probability distribution of the discrete random variables X

and Y if it possesses the following properties:

(1) $f(x,y) \geq 0$ for all (x,y)

(2) $\sum_{x} \sum_{y} f(x,y) = 1$

(3) $P[(X,Y) \in R] = \sum\sum_{(x,y) \in R} f(x,y)$ for any region R in the XY plane.



Example:

A coin is tossed three times. If X denotes the number of heads and Y denotes the number of tails in the last two tosses, find the joint probability distribution of X and Y.

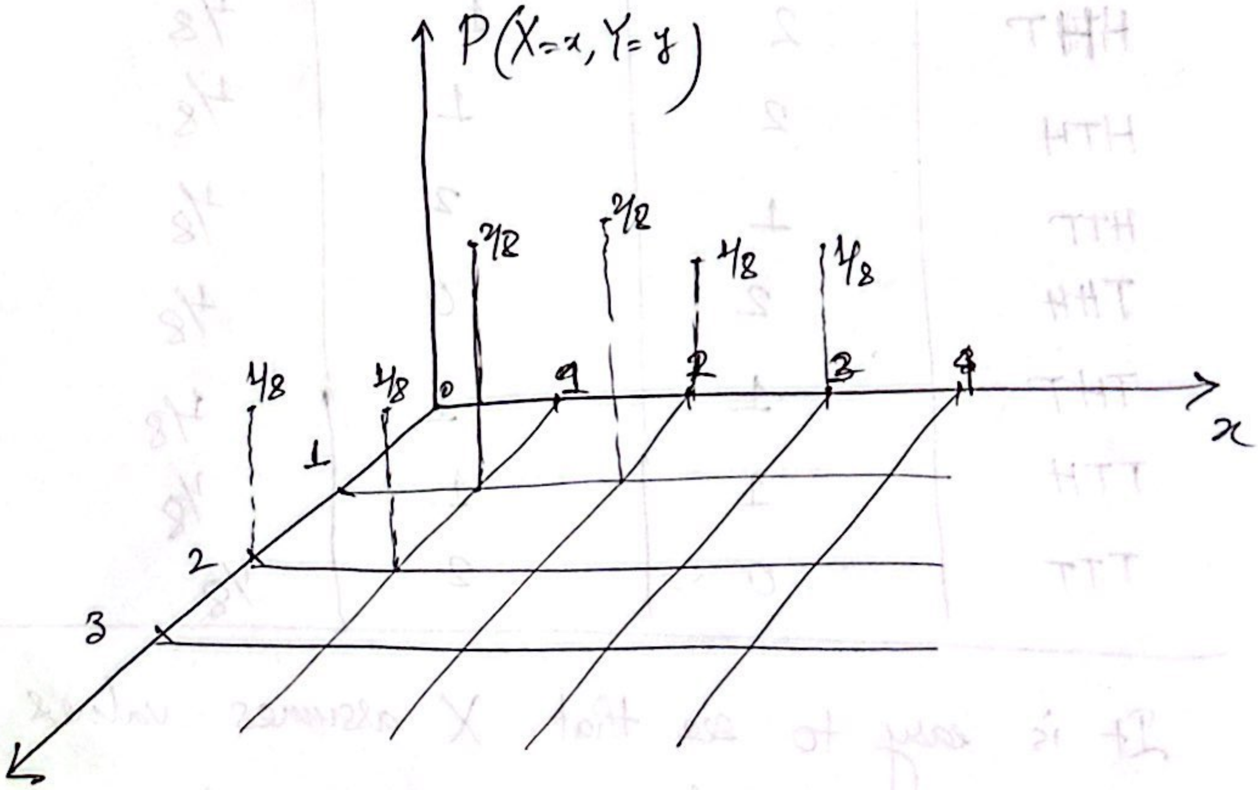
Sol^n: The outcomes of the experiment and the associated probabilities are shown below:

Outcome	X	Y	$P(X,Y)$
HHH	3	0	1/8
HHT	2	1	1/8
HTH	2	1	1/8
HTT	1	2	1/8
THH	2	0	1/8
THT	1	1	1/8
TTH	1	1	1/8
TTT	0	2	1/8

It is easy to see that X assumes values 0, 1, 2 and 3, while Y assumes values 0, 1, and 2. The joint probability distribution can now be put in the following form:

	X values				
Y Values	0	1	2	3	Row sum
0	0	0	1/8	1/8	2/8
1	0	2/8	2/8	0	4/8
2	1/8	1/8	0	0	2/8
Column Sum	1/8	3/8	3/8	1/8	1

A graphical representation of the above probability distribution is show below.



Example:

A box contains 4 white balls, 3 black balls and 2 red balls. Two balls are to be drawn without replacement. Let, X denotes the number of white balls and Y denotes the number of black balls drawn. find the joint probability function of (X, Y) find the probability that $X+Y \geq 3$. Also find the probability that $X = 1$.

Solⁿ: In this case, X takes only the values 0, 1, and 2; and Y takes only the values 0, 1, and 2.

The possible pair of values are (0,0), (0,1), (0,2) (1,0), (1,1), (2,0)

$$f(0,0) = P(X=0, Y=0) = \frac{\binom{4}{0}\binom{3}{0}\binom{2}{2}}{\binom{9}{2}} = 1/36$$

$$f(0,1) = P(X=0, Y=1) = \frac{\binom{4}{0}\binom{3}{1}\binom{2}{1}}{\binom{9}{2}} = 6/36$$

$$f(0,2) = P(X=0, Y=2) = \frac{\binom{4}{0}\binom{3}{2}\binom{2}{0}}{\binom{9}{2}} = 3/36$$

$$f(1,0) = P(X=1, Y=0) = \frac{\binom{4}{1}\binom{3}{0}\binom{2}{1}}{\binom{9}{2}} = \frac{8}{36}$$

$$f(1,1) = P(X=1, Y=1) = \frac{\binom{4}{1}\binom{3}{1}\binom{2}{0}}{\binom{9}{2}} = \frac{12}{36}$$

$$f(2,0) = P(X=2, Y=0) = \frac{\binom{4}{2}\binom{3}{0}\binom{2}{0}}{\binom{9}{2}} = 6/36$$

The joint probability distribution can now be put in the following table:

	X values			
Y values	0	1	2	Row sum
0	1/36	8/36	6/36	15/36
1	6/36	12/36	0	18/36
2	3/36	0	0	3/36
Column sum	10/36	20/36	6/36	1

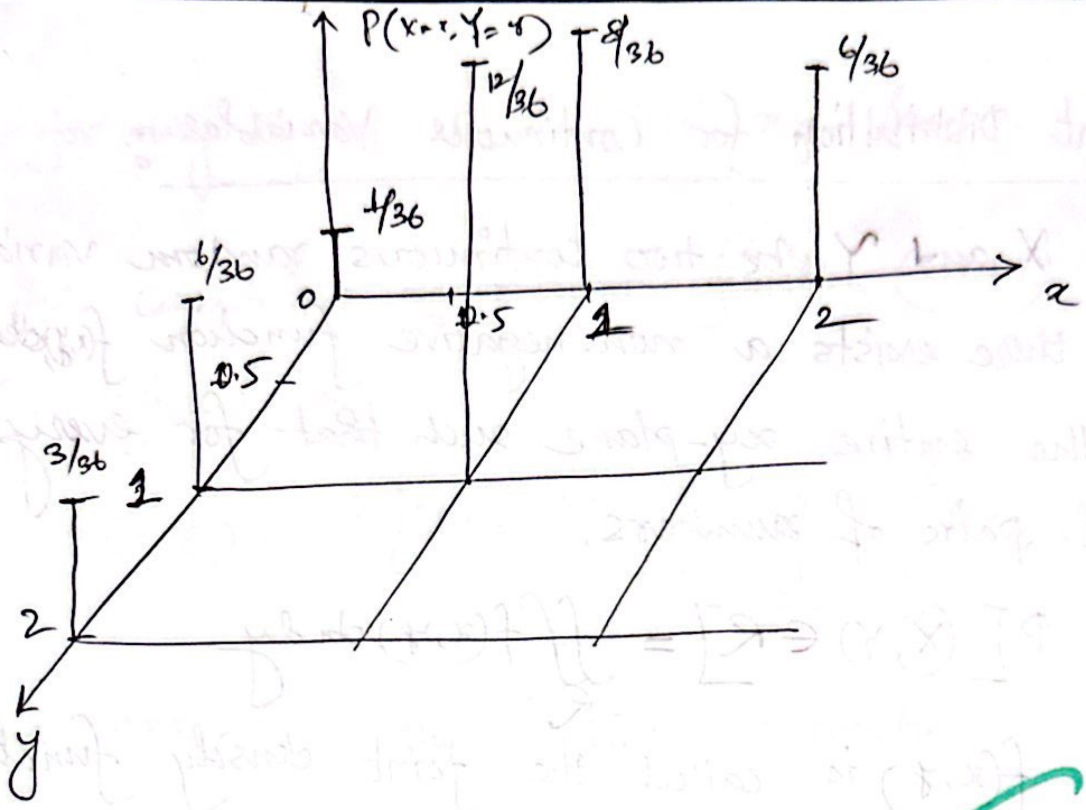
Now, $P(X+Y \geq 3) = f(1,2) + f(2,1) + f(2,2)$

$= 0+0+0$

$= 0$

Also, $P(X=1) = \sum_{y=0}^{2} f(1,y) = f(1,0) + f(1,1) + f(1,2)$

$= \frac{8}{36} + \frac{12}{36} + 0 = \frac{20}{36}.$


P(X=x, Y=y)
12/36
8/36
6/36
4/36
6/36
3/36
0
0.5
1
2
x
0.5
1
2
y

Joint Distribution for Continuous Variables :

Let, X and Y be two continuous random variables. If there exists a non-negative function $f(x,y)$ defined on the entire xy-plane such that for every subset R of pairs of numbers,

$$P[(X,Y) \in R] = \iint_R f(x,y)\,dx\,dy$$

then $f(x,y)$ is called the joint density function (joint PDF) of (X,Y).

Like all other probability functions, the requirements for a continuous function $f(x,y)$ to be joint PDF are :

① $f(x,y) \geq 0$; $\forall (x,y) \in R$; $-\infty < x, y < +\infty$

② $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

③ $$P[(X,Y) \in R] = \iint_R f(x,y)\,dx\,dy$$

$$= P(a \leq X \leq b,\ c \leq Y \leq d)$$

$$= \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dx\,dy.$$

for any region R in the xy-plane

The cumulative distribution function (CDF) for $f(x, y)$ is defined by

$$F(x, y) = P[X \leq x, Y \leq y] = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dx\, dy$$

The CDF has the following properties :

(i) $0 \leq F(x, y) \leq 1$

(ii) $\dfrac{\partial^2 F(x, y)}{\partial x \partial y} = f(x, y)$, whenever $F(x, y)$ is differentiable

(iii) $F(x, -\infty) = F(-\infty, y) = 0$

(iv) $F(-\infty, \infty) = 1$.

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Example : Let X and Y have the following distribution

$$f(x,y) = x^2 + \frac{xy}{3} \; ; \quad 0 \le x \le 1, \; 0 \le y \le 2$$

$$= 0 \; ; \quad \text{elsewhere}$$

Check whether $f(x,y)$ is a density function or not

Sol^n: The function $f(x,y)$ will be a joint PDF if

$f(x,y) \geq 0 \;, \; \forall (x,y) \in \mathbb{R}$

and $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Clearly, $f(x,y) \geq 0$ for values of x and y in the given range. $(0 \le x \le 1, \; 0 \le y \le 2)$

Now,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \left(x^2 + \frac{xy}{3}\right) dx\,dy$$

$$= \int_0^2 \left[\int_0^1 \left[x^2 + \frac{xy}{3}\right] dx\right] dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^1 dy$$

$$= \int_0^2 \left[\frac{1}{3} + \frac{1}{6}y\right] dy$$

$$= \left[\frac{1}{3}y + \frac{1}{12}y^2\right]_0^2$$

$$= \frac{2}{3} + \frac{4}{12}$$

$$= \frac{2}{3} + \frac{1}{3} = 1$$

Hence, $f(x, y)$ is a joint PDF.

Example: Suppose that X and Y have the following density function:

$$f(x,y) = Ky^2; \quad 0 \le x \le 2, \quad 0 \le y \le 1$$
$$= 0; \quad \text{elsewhere}$$

Determine the constant K and find $P(X \le 1)$.

Sol^n: To determine K, we must have

$$\int_{x=0}^{2}\int_{y=0}^{1} f(x,y)\,dy\,dx = 1$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{1} Ky^2\,dy\,dx = 1$$

$$\Rightarrow K\int_{x=0}^{2}\left[\frac{y^3}{3}\right]_0^1 dx = 1$$

$$\Rightarrow \frac{K}{3}\left[x\right]_0^2 = 1$$

$$\Rightarrow \frac{2}{3}K = 1$$

$$\therefore K = 3/2.$$

Hence, the complete density function is

$$f(x,y) = \frac{3}{2} y^2 \; ; \; 0 \le x \le 2, \; 0 \le y \le 1$$

Now,

$$P(X \le 1) = \frac{3}{2} \int_{x=0}^{1} \int_{y=0}^{1} y^2 \, dy \, dx$$

$$= \frac{3}{2} \int_0^1 \left[\frac{y^3}{3} \right]_0^1 dx$$

$$= \frac{3}{2} \times \frac{1}{3} \left[x \right]_0^1 = \frac{1}{2}$$

Ans

Example:

Let the joint PDF of the continuous random variable (X, Y) be

$$f(x,y) = \begin{cases} K(x^2 + 2xy) \; ; & 0 \le x \le 1, \; 0 \le y \le 1 \\ 0 \; ; & \text{elsewhere} \end{cases}$$

(a) Determine the value of K,

(b) find $P\left(X \le \frac{1}{2}, Y \ge \frac{1}{2}\right)$,

(c) What is the probability of the event $(X \le Y)$?

Sol^n: (a) Since, $f(x,y)$ is a density function,

then $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Thus, $$\int_{x=0}^{1}\int_{y=0}^{1} f(x,y)\,dy\,dx = 1$$

$$\Rightarrow \int_{x=0}^{1}\left[\int_{y=0}^{1} K(x^2+2xy)\,dy\right]dx = 1$$

$$\Rightarrow K\int_{x=0}^{1}\left[x^2y + xy^2\right]_0^1 dx = 1$$

$$\Rightarrow K\int_{x=0}^{1}(x^2+x)\,dx = 1$$

$$\Rightarrow K\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow K\left(\frac{1}{3}+\frac{1}{2}\right) = 1$$

$$\Rightarrow \left(\frac{2+3}{6}\right)K = 1$$

$$\Rightarrow K = 6/5.$$

(b) $P\left(X \le \frac{1}{2}, Y \ge \frac{1}{2}\right) = \frac{6}{5}\int_{x=0}^{1/2}\int_{y=\frac{1}{2}}^{1}(x^2+2xy)\,dy\,dx$

$$= \frac{6}{5}\int_{x=0}^{1/2}\left[x^2y+xy^2\right]_{1/2}^{1}dx$$

$$\cancel{= \frac{6}{5}\int_{x=0}^{1/2}(x^2+x^2)-\left(\frac{x^2}{2}+x^2\right)}$$

$$= \frac{6}{5}\int_{x=0}^{1/2}\left[(x^2+x)-\left(\frac{1}{2}x^2+\frac{x}{4}\right)\right]dx$$

$$= \frac{6}{5}\int_{0}^{1/2}\left(x^2+x-\frac{x^2}{2}-\frac{x}{4}\right)dx$$

$$= \frac{6}{5}\int_{0}^{1/2}\left(\frac{x^2}{2}+\frac{3x}{4}\right)dx$$

$$= \frac{6}{5}\left[\frac{x^3}{6}+\frac{3x^2}{8}\right]_{0}^{1/2}$$

$$= \frac{6}{5}\left[\frac{1}{48}+\frac{3}{32}\right]$$

$$= \frac{6}{5}\left[\frac{2+9}{96}\right]$$

$$= \frac{\cancel{6}}{5}\times\frac{11}{\cancel{96}_{16}} = \frac{11}{80}$$ Ans

(c) $P(X \le Y) = \frac{6}{5} \int_{y=0}^{1} \left[\int_0^y (x^2 + 2xy)\, dx \right] dy$

$$= \frac{6}{5} \int_0^1 \left[\frac{x^3}{3} + x^2 y \right]_0^y dy$$

$$= \frac{6}{5} \int_0^1 \left(\frac{y^3}{3} + y^3 \right) dy$$

$$= \frac{6}{5} \cdot \frac{4}{3} \int_0^1 y^3\, dy$$

$$= \frac{24}{15} \left[\frac{y^4}{4} \right]_0^1 = \frac{6}{15} [1] = \frac{2}{5}.$$

(Ans)