Similarly, consider a drug, on an average five doses of which is enough in order to get cured a certain disease. A company introduces a new drug and claims that in average only two doses of this new drug will need to get cure of the same disease. In order to verify the company's claim one has to formulate $H_0 : \mu = 5$ against the alternative $H_1 : \mu < 5$. The company's claim will be true if the null hypothesis is rejected. Such an alternative test will result in a one tailed test with the critical region in the left tail.

Again, suppose one wishes to test the inequality of income of two populations. In this case, the deviation of the income of one population to other may happen in any of the two sides, i.e., income of one population may be either more or less than other population. In such case, it is required to consider a two-tailed test.

16.2.9. Test Statistic. The decision about test of a hypothesis or the acceptance or rejection of a hypothesis is based on the statistical evidence from sample data. In testing procedure, it is desirable to select an appropriate statistic to be computed from sample data depending on the assumptions or available information or nature of population from which the sample has been drawn.

Definition. Test statistic. The statistic, which is used to provide evidence about the rejection or acceptance of null hypothesis, is called test statistic. The decision about the rejection or acceptance of a null hypothesis is taken comparing the observed (calculated) and theoretical (tabulated) value of test statistic.

16.2.10. Critical Region and Acceptance region. Sample space of experiment, which corresponds to the area under the sampling distribution curve of the test statistic, is divided into two mutually exclusive regions, such as acceptance region and rejection or critical region. If the value of the test statistic falls within the rejection region, the null hypothesis is rejected, and if the value of the test statistic falls within the acceptance region, we fail to reject it. In case of one-tailed test the critical region, the area of which is exactly equal to the level of significance, lies entirely on one extreme end of the curve depending on whether the test is right or left tailed, while in case of two-tailed test area of rejection is divided into two regions lie at both ends of the curve, area of each of these regions is usually exactly half of level of significance.

Definition. Critical region. The set of possible values of the test statistic, which provides evidence to contradict with null hypothesis and lead to the rejection of null hypothesis is called critical region. The set of values of the test statistic that support the alternative hypothesis and lead to rejecting the null hypothesis is called the rejection region.

Definition. Acceptance region. The set of values of the test statistic, which provides evidence to agree with the null hypothesis and lead to the acceptance of null hypothesis is called acceptance region.

Critical regions for different types of alternatives are displayed in following figures.

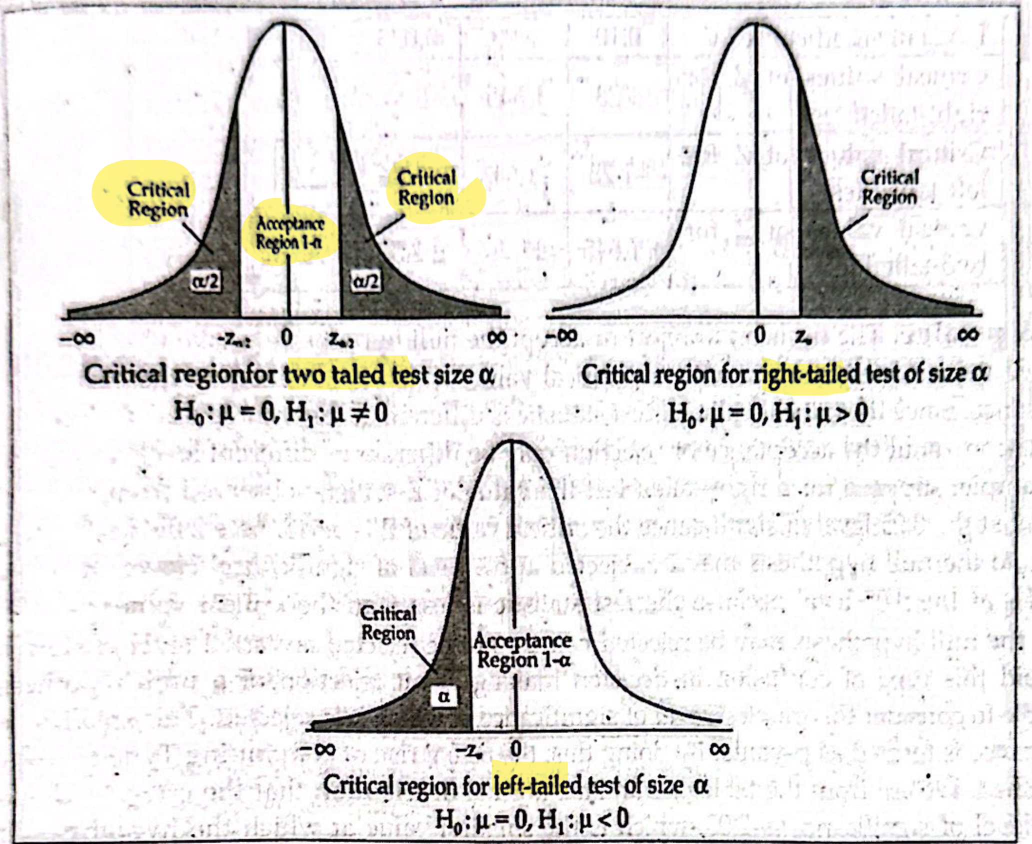

Fig. 16.1. *Areas of acceptance and rejection regions for one tailed and two-tailed test.*

16.2.11. Critical Value. A critical value is measured in the same units of measurement as the test statistics and identifies the value of the test statistic that would lead to the rejection or acceptance of null hypothesis at the specified level of significance. This is also called the theoretical value of statistic. The critical values are determined independently of the sample statistics from the sampling distribution under null hypothesis from the table. The critical value(s) for a hypothesis test is a threshold to which the value of the test statistic in a sample is compared to determine whether or not the null hypothesis is to be rejected. The critical value for any hypothesis test depends on (i) the significance level at which the test is carried out, and (ii) the type of test (one-sided or two-sided). There are two critical values for a two-tailed test, while one for a one-tailed test. The critical values of popular test statistics for different level of significance are available in Statistical tables.

Definition. Critical value. The value of the sample statistic that separates acceptance region and rejection region is called critical value. This is the value of the test statistic with which observed value is compared and decision regarding acceptance or rejection of null hypothesis is taken.

Some useful critical values of Z for both one-tailed and two-tailed tests at various levels of significance (negative value of Z stands for left-tailed test and positive value for right-tailed test) are presented in following table.

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Table 16.2. Critical values of Z statistic for different level of significance

Level of significance α	0.10	0.05	0.025	0.01	0.005	0.002
Critical values of Z for right-tailed test	1.28	1.645	1.96	2.33	2.58	2.88
Critical values of Z for left-tailed test	−1.28	−1.645	−1.96	−2.33	−2.58	−2.88
Critical values of Z for two-tailed test	± 1.645	±1.96	± 2.33	± 2.58	± 2.81	± 3.08

16.2.12. p-value. The decision to reject or accept the null hypothesis is taken by comparing the observed value of test statistic with the critical value, which is obtained on the basis of level of significance. Since the critical value of test statistic is different at different levels of significance, so the decision about the acceptance or rejection may be different at different levels of significance. For example, suppose for a right-tailed test the value of Z-statistic observed from the sample is 2.03 and at the 0.05 level of significance the critical value of Z is 1.645, here 2.03 lies in the critical region, so the null hypothesis may be rejected at 5% level of significance. However, we cannot reject H_0 at the 0.01 level because the test statistic is less than the critical value $z = 2.33$. That means the null hypothesis may be rejected or may not be rejected at varied level of significance. To avoid this type of confusion in decision making about rejection of a null hypothesis it is desirable to consider the smallest level of significance at which it is rejected. This smallest level of significance is termed as p-value. By doing this, the actual risk of committing Type I error can be established. We see from the table of standard normal distribution that the critical value of Z at 2.12% level of significance is 2.03, which is the smallest value at which the hypothesis may be rejected, so here the p value is 0.0212.

Definition. p-value. The p-value or observed significance level of a statistical test is the smallest value of the level of significance at which the null hypothesis can be rejected. It is the actual risk of committing a type I error, if H_0 is rejected based on the observed value of the test statistic. The p-value measures the strength of the evidence against H_0.

A small p-value indicates that the observed value of the test statistic lies far away from the hypothesized value. This presents strong evidence that H_0 is false and should be rejected. A large p-value indicates that the observed test statistic is not far from the hypothesized value and does not support rejection of H_0.

If the p-value is less than a pre-assigned significance level α, then the null hypothesis can be rejected, and we can report that the results are statistically significant at level α.

Many researchers classify p-values as follows :

i) If the p-value is less than 0.01, H_0 is rejected. The results are highly significant.

ii) If the p-value is between 0.01 and 0.05, H_0 is rejected. The results are statistically significant.

iii) If the p-value is between 0.05 and 0.10, H_0 is usually not rejected. The results are only tending toward statistically significant.

iv) If the p-value is greater than 0.10, H_0 is not rejected. The results are not statistically significant.

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In particular, for testing the null hypothesis $H_0 : \mu = \mu_0$, against $H_1 : \mu > \mu_0$, p-value is given by

$$\text{p-value} = P\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \geq z_{obs.} \middle| H_0 : \mu = \mu_0\right)$$

where, $z_{obs.}$ is the observed value of the test statistic associated with the smallest significance level at which the null hypothesis can be rejected.

It is to be noted here that all hypotheses testings are done under the assumption that the null hypothesis is true. It is also important to understand that the rejection of null hypothesis is to conclude that it is false, while fail to reject it does not necessarily mean that it is true. We fail to reject null hypothesis since we have no sufficient evidence to believe otherwise.

16.2.13. Assumption. The suppositions regarding population and/or sample, which are needed to take decision about the distribution of test statistic, are known as assumption. For example, the test statistic Z follows N(0, 1) under the assumption that the sample observations are drawn independently from normal population with known variance or the sample size is large. On the other hand, the test statistic t follows Student's t distribution under the assumption that the sample observations are independently drawn from a normal population with unknown variance and the sample size is small.

16.3. Survey of Important Test Statistics :

The important test statistics are 1) Z-test or normal test 2) t-test 3) χ^2-test and 4) F-test.

A brief survey of the important parametric test statistics is provided below. Applications of these test statistics have been described in section 16.5.

16.3.1. Z-test or Normal test. In a normal test we find U whose expected value E(U) is specified by the null hypothesis. The standard error $\sigma(U)$ of U is either known or estimated from the large sample. Then a statistic

$$Z = \frac{U - E(U)}{\sigma(U)}.$$

taken as a normal variable with mean 0 and standard deviation 1. It is symbolically expressed $Z \sim N(0, 1)$. If the distribution of U is normal and $\sigma(U)$ is known, then Z is exactly normally distributed. Frequently, however, the distribution of U is approximately normal or $\sigma(U)$ is estimated from a sample or perhaps both. When samples are large this approximation is usually quite satisfactory. That is why; normal test is often regarded as a large sample test.

In particular, when U is sample mean $\overline{X}$ then

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

The normal curve is symmetrical about mean; hence, the critical value at a particular level of significance at right side is the same as that of left side with a negative sign. Normal tests can be used in case of one-tailed as well as two-tailed tests.

16.3.2. t-tests (Student's t-test). In case of normal test, it is assumed that population variance is either known or estimated from large sample (usually $n > 30$), but very often we have to deal with small samples where population variance is unknown. In this situation the test statistic t is

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to be used instead of Z-statistic. The distribution of t contains a parameter ν (nu) known as degrees of freedom (d.f.). This is a positive integer and always less than n, the size of the sample. The relationship between ν and n depends on how $\sigma(U)$ calculated. The normal test can be regarded as a special case of t-test when ν is large. So, a t-test is also called a small sample test.

This statistic t is generally known as Student's t is defined in similar algebraic form of Z-statistic except that standard error is estimated from small sample. Thus the algebraic form of t is:

$$t = \frac{U - E(U)}{\text{Estimated } \sigma(U)} \quad \text{with} \quad \nu = n - 1$$

In particular, for sample mean $\overline{X}$, $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$

where $s^2 = \dfrac{1}{n-1}\Sigma(x - \overline{x})^2$ with $\nu = n - 1$.

Like the normal distribution, t distribution is also symmetrical about mean; hence, the critical value at a particular level of significance with certain degrees of freedom at right side is the same as that of left side with a negative sign. Like the normal test, the t-tests can also be one-tailed or two-tailed.

Another form of t-test is also used for testing the difference between two means in case of dependent or correlated or repeated samples (x, y). This is known as paired t-test, defined as

$$t = \frac{d - \overline{d}}{s(d)} \quad \text{with } \nu = n - 1, \text{ where, } d = x - y.$$

16.3.3. χ^2-tests. χ^2-tests are used mainly for testing hypothesis that specify the nature of one or more distributions as a whole. Thus, a hypothesis may define the mathematical form of a distribution or assert the two or more distributions are identical or two attributes are independent, etc. The elements common to the test statistics used for testing the above hypothesis is that each involves the comparison of an observed set of frequencies with a corresponding set of expected set of frequencies under null hypothesis. If O_i (i = 1, 2, ..., k) denotes the observed frequency, and E_i denotes the corresponding expected frequency, then the test statistic χ^2 is defined as

$$\chi^2(\nu) = \Sigma \frac{(O_i - E_i)^2}{E_i}$$

where ν denotes the degrees of freedom, the only parameter of the theoretical χ^2 distribution.

χ^2 is used for testing varying types of hypotheses and the value or definition of ν varies with type of hypothesis under consideration. For example,

i) χ^2- test for specific variance here: $\nu = n - 1$;

ii) χ^2- test for goodness of fit, here: $\nu = k - r$;

iii) χ^2- test for independence of attributes, here: $v = (r-1)(c-1)$.

Here n is the sample size, k is the number of cells or values of a variable, r is the number of restrictions imposed on the set of frequencies while calculating the expected frequencies, r is the number of rows and c is the number of columns. It is to be mentioned here that the first one is the parametric test while the next two are non-parametric. Tests of goodness of fit are beyond the scope of this book.

χ^2 is used for testing the hypothesis that normal population from which the sample is available has a specified variance. This is an exact and one can consider a one tailed as well as a two-tailed test. In this case χ^2 is defined as

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} \quad \text{with } n-1 \text{ degrees of freedom.}$$

However, for last two cases, χ^2 tests are approximate in the sense that the test statistic has an approximate χ^2 distribution under null hypothesis and these are one-tailed tests.

16.3.4. F-tests. R.A. Fisher originally devised this test and Snedecor called it F-test in honour of Fisher. Suppose s_1^2 and s_2^2 denote the sample variances of the same variance σ^2 of a normal population computed from two independent samples of sizes n_1 and n_2 respectively with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ degrees of freedom. Then the statistic F is defined as

$$F = \frac{s_1^2}{s_2^2} \quad \text{with } v_1 \text{ and } v_2 \text{ degrees of freedom, where, } s_1^2 > s_2^2.$$

The statistic F is so defined that s_1^2 in the numerator is expected to be larger than denominator s_2^2 when the null hypothesis is not true. Hence, only the upper tail of F-distribution is usually used as the critical region.

16.4. Steps in Hypothesis Testing

As it is clear from above discussion that in order to test the validity of claim or assumption about the population parameter, at first it is necessary to draw a sample from the respective population and then analyzed. Some assumptions regarding the population distribution may also be necessary for suitability of tests. The results of the analysis are used to decide whether the claim is true or not. Thus, the general procedure for hypothesis testing consists of following basic steps.

i) **State the null hypothesis and alternative hypothesis** ; Clear and precise formulation of the null and alternative hypothesis is the first and foremost step in the test of significance. Without formulation of this hypothesis, one cannot proceed to perform any test of significance. That means, at first it is required to state the assumed value of population parameter which is to be tested as null hypothesis. A justified alternative hypothesis along with null hypothesis is also to be established depending on the statement of problem. Care should be taken regarding the alternative whether it would

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be one-tailed or two-tailed. For example, suppose we want to test the hypothesis that the monthly average price of a commodity is Taka 100 per kg. In this case, the following null and alternative hypothesis are to be considered $H_0: \mu = 100$ against the alternative $H_1: \mu \neq 100$

ii) **Specify the level of significance (α) prior to sampling :** At the second step of test of hypothesis, it is required to specify the level of significance. Because, the risk of taking wrong decision depends on the nature of study, so maximum risk should be determined before drawing sample from the population. It is at the discretion of investigator to select its value. Although usually $\alpha = 0.05$ is considered, but value of α may vary depending on the sensitivity of the study. For example, 5% risk might be more for taking decision about the effectiveness of a drug, in that case, 1% or 0.1% level of significance may be considered.

iii) **Select the suitable test statistic :** Depending on the formulated hypothesis, assumption made about the population distribution, sample size, at this stage the appropriate test statistic is to be selected.

iv) **Establish the critical region :** At this stage the critical region is to be established on the basis of above steps. That means, the critical region is selected depending on alternative hypothesis whether it is one-tailed or two-tailed, the level of significance and the selected test statistic. For example, for the two tailed alternative $H_1: \mu \neq 100$ at 5% level of significance, suppose normal test statistic Z is selected for testing the hypothesis, then the critical region is the lower and upper 2.5% area of a standard normal distribution which are $Z < -1.96$ and $Z > 1.96$, hence the critical values are ± 1.96 (the values of z at certain level of significance can be obtained from the table 'Area under the normal curve available in statistical tables or in the appendix of almost all books on statistics).

v) **Collect sample and Compute the value of the test statistic:** After selecting the critical region, a sample of predetermined size n is collected. Then the value of selected test statistic is calculated from sample. In this case, it is assumed that the null hypothesis is true.

vi) **Compare observed and critical values:** The value of the test statistic computed in earlier step is compared with the critical value or values. The computed value is checked whether it falls within or beyond the critical region.

vii) **Make the decision:** The decision about the acceptance or rejection of hypothesis is taken on the basis of critical value. It is either "reject the null hypothesis" or "fail to reject the null hypothesis or not reject the null hypothesis". If the observed value of test statistic lies within the critical region, then "reject the null hypothesis", on the other hand if the observed value of test statistic falls beyond the critical region, then decision is taken as 'fail to reject null hypothesis'.

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For the convenience of the users, a set of decision rules for normal test for $\alpha = 1\%$, 5% and 10% are provided below:

Table 16.3. Decision rule for one-tailed and two-tailed test using Z-statistic

Alternative Hypothesis	Decision Rule		
	$\alpha = 0.01$ Reject H_0 if	$\alpha = 0.05$ Reject H_0 if	$\alpha = 0.10$ Reject H_0 if
$\mu \neq \mu_0$	$Z > 2.58$ or $Z < -2.58$	$Z > 1.96$ or $Z < -1.96$	$Z > 1.645$ or $Z < -1.645$
$\mu > \mu_0$	$Z > 2.53$	$Z > 1.645$	$Z > 1.28$
$\mu < \mu_0$	$Z < -2.53$	$Z < -1.645$	$Z < -1.28$

viii) **Draw conclusion:** This is a statement, which indicates the level of evidence (sufficient or insufficient) at given level of significance, and/or, whether the original claim is rejected or accepted. If decision is taken in favour of alternative hypothesis, then it is concluded that there is sufficient evidence to reject null hypothesis or accept the original claim.

16.5. Applications of Test Statistics

In this and the following sections we will discuss some specific applications of the test statistics viz. Z, t, χ^2 in testing various types of hypotheses related to business and management.

The applications are classified as follows:

Applications of Z-statistic:

i) Test of a single population mean;
ii) Test of equality of two population means;
iii) Test of a single population proportion;
iv) Test for difference between two population proportions;
v) Test of a specified correlation co-efficient;
vi) Test of equality of two-population correlation co-efficient.

Applications of t-statistic (small sample test):

i) Test of a single population mean;
ii) Test of difference between two population means;
iii) Test of significance of a correlation co-efficient with zero value;
iv) Test of a population of regression co-efficient with zero or specified value;
v) Test of difference between two population regressions co-efficient.

Applications of χ^2-statistic:

i) Test of a population variance with specific value;
ii) Test of equality of several variances;
iii) Test of equality of several correlation co-efficient;
iv) Test of equality of several population proportions;
v) Tests of independence of attributes;
vi) Test of goodness of fit.

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- ***Applications of F-statistic:***
  i) Test of significance of difference between two population variances;
  ii) Test of significance of several population means;
  iii) Test of significance of two or more regression co-efficient;

  The above-mentioned applications of the test statistics are discussed below.

16.6. Hypothesis Testing for Single Population Mean

Although it is difficult to draw a clear-cut line of demarcation between large and small samples, it is generally agreed that if the size of sample exceeds 29, then it may be regarded as a large sample. The test of significance used for large samples are different from that of small samples for the reasons that the assumptions we make in case of large samples do not hold for small samples. The following assumptions are to be made for Z-test.

- *For small sample:* The sample is randomly selected from a normally distributed population with known variance, and
- *For large sample:* The sample is randomly selected from a normally distributed population with unknown variance.

Some practical examples are cited below.

1. Population normal and variance known for any sample size (small and large) : Suppose X_1, X_2, ..., X_n be a random sample of size n drawn independently from a normal population with mean μ and variance σ^2. In this case,

$$X \sim N(\mu,\ \sigma^2),\ \text{then}\ \overline{X} \sim N(\mu,\ \sigma^2/n).$$

The following null and alternative hypothesizes may be considered for testing the population mean

i) $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ (for a two tailed alternative);
ii) $H_0 : \mu = \mu_0$ against $H_1 : \mu > \mu_0$ (for a right tailed alternative);
iii) $H_0 : \mu = \mu_0$ against $H_1 : \mu < \mu_0$ (for a left tailed alternative).

The test statistic for testing the null hypothesis H_0 for all the alternatives is

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}.$$

- **Decision rule.** (i) Suppose the level of significance is α. We find $z_{\alpha/2}$ from the standard normal integral table by using $P(|Z| > z_{\alpha/2}) = \alpha/2$. We reject the null hypothesis if the absolute observed value of z is greater than $z_{\alpha/2}$.

  (ii) For the second case, we find the z_α from the standard normal integral table by using $P(Z > z_\alpha) = \alpha$. We reject the null hypothesis if the observed z value is greater than z_α.

  (iii) For the third case, we find the $-z_\alpha$ from the standard normal integral table by using $P(Z < -z_\alpha) = \alpha$. We reject the null hypothesis if the observed z value is less than $-z_\alpha$

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The above three cases can be shown in the following table.

Table 16.4. Decision rule for a single mean test using Z statistic

Case No.	Type of test	Decision rule Reject H_0, if.
1	Two-tailed test $H_1: \mu \neq \mu_0$	$\|Z\| > z_{\alpha/2}$
2	Right-tailed test $H_1: \mu > \mu_0$	$Z > z_\alpha$
3	Left-tailed test $H_1: \mu < \mu_0$	$Z < -z_\alpha$

Example 16.6.1. The managing director of a firm claims that his firm produces 110 items on average daily. A random sample of 15 days gives the following data set.

110, 118, 130, 140, 142, 146, 112, 100, 95, 98, 96, 122, 123, 124, 130.

It is known that the number of items produced by the firm follows normal distribution with variance 300.

Can we conclude at 5% level of significance that the average daily production of items of that firm is

a) 110 items; b) More than 110 items; c) Less then 110 items?

d) Compute p-value for each case.

Solution. (a) Steps involved in testing the hypothesis will be followed in this case:

(i) First, we have to formulate null and alternative hypothesis. It is a two-tailed test. Since if the average number of items produced by the firm is more or less than 110 to some extent, then the claim of the managing director will be proved as false. In that case the claim would be rejected.

So, the null hypothesis and alternative hypothesis can be formulated as follows.

Null hypothesis: $H_0: \mu = 110$

Alternative hypothesis: $H_a: \mu \neq 110$.

(ii) Level of significance is: $\alpha = 0.05$.

(iii) Here, the sample is taken from a normal population with known variance. The appropriate test statistic is

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

where $\overline{X}$ = The sample mean;

μ = Population mean (to be tested);

σ = Standard deviation of the population;

$\sigma/\sqrt{n}$ = Standard error of the sample mean $\overline{X}$.

(iv) Here $\alpha = 0.05$ and it is a two tailed-test, the critical region will be on both sides of curve of Z in such a way that the critical region will comprise 2.5% or 0.025 area at the right end and 2.5% at the left end. From the table of area of standard normal distribution, we see that these

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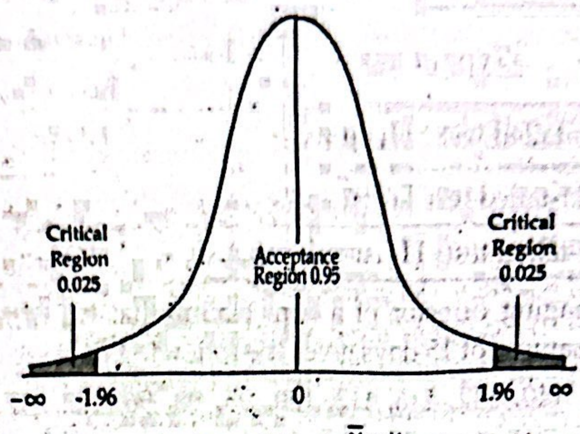
values of Z are ± 1.96, that means the critical regions are $Z<-1.96$ (at the left end) and $Z>1.96$ (at the right end). Here $|z_{0.025}|=1.96$.



(v) Under the null hypothesis, the value of Z is: $z=\dfrac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.

(vi) Here, $\bar{x}=\dfrac{\Sigma X}{n}=\dfrac{1786}{15}=119.07$, $\mu_0=110$,

$\sigma^2=300$, $\sigma=17.32$ and $n=15$. Substituting these values in the formula of Z, we have

$$z=\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}=\frac{119.07-110}{17.32/\sqrt{15}}=2.03.$$

(vii) It is found that the observed value of Z is 2.03, which is greater than the right tail critical value 1.96, hence it falls in the upper critical region.

(viii) Since the observed value of the test statistic falls in the critical region, so we fail to accept the null hypothesis at 5% level of significance.

- **Conclusion.** Hence, we cannot accept the claim of the managing director at 5% level of significance.
- **p-value.** From the table of the standard normal distribution, we find that $P(Z>2.03)=0.0212$ since it is a two tailed test, the p-value is $0.0212\times 2=0.0424$, that means the smallest level of significance at which the hypothesis may be rejected is approximately 4.34%.

(b) Null hypothesis is the same as (a)

Null hypothesis: $H_0: \mu=110$;

Alternative hypothesis: $H_a: \mu>110$;

Level of significance is: $\alpha=0.05$.

The appropriate test statistic is the same as (a). That is

$$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1).$$

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Here $\alpha = 0.05$ and it is one-sided right tailed-test, the critical value $z_{0.05}$ will be found in such a way that $P[Z > z_{0.05}] = 0.05$. It is found from the standard normal distribution that $z_{0.05} = 1.645$.

The calculated value of Z under the null hypothesis is 2.03 which is greater than 1.645. That is observed value of Z lies in the rejection region. Hence, we have no reason to accept the null hypothesis.

p-value. From the table of the standard normal distribution, we find that

$$p = P(Z > 2.03) = 0.0212.$$

That means the smallest level of significance at which the hypothesis may be rejected is approximately 2.12%.

(c) Null hypothesis is the same as (a)

Null hypothesis: $H_0 : \mu = 110$

Alternative hypothesis: $H_a : \mu < 110$

Level of significance is: $\alpha = 0.05$.

The appropriate test statistic is the same as (a). That is

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Here $\alpha = 0.05$ and it is one-sided left tailed-test, the critical value $z_{0.05}$ will be found in such way that $P[Z < -z_{0.05}] = 0.05$. It is found from the standard normal distribution that for left tail $z_{0.05} = -1.645$.

The calculated value of Z under the null hypothesis is 2.03 which is greater than 1.645. That is observed value of Z lies in the acceptance region. Hence, we have no reason to reject the null hypothesis, at 5% level of significance under the alternative hypothesis that the average production of the firm is less than 110.

p-value. From the table of the standard normal distribution, we find that $p = P(Z > 2.03) = 0.0212$

This is a left tailed test, so the p-value is

$$p = P(Z > -2.03) = 1 - 0.0212 = 0.9788$$

That means the smallest level of significance at which the hypothesis may be rejected is approximately 97.88%.

Note: In practice, if we are in a position to reject a null hypothesis, we compute p-value to find the exact level of significance. The p-value found in this case is quite unjustified.

Population normal, variance unknown and sample size is large ($n > 29$).

Example 16.6.2. Manager of a fertilizer factory claims that the average daily production of his factory follows normal distribution with mean production 880 kg. A random sample of 50 days

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shows that average production is 871 kg with standard deviation 21 kg. Test the significance of the claim of the manager at 5% level of significance. Also find p-value.

Solution. Here null and alternative hypotheses are.

$$H_0 : \mu = 880 \text{ and } H_1 : \mu \neq 880$$

Here population is normal but variance is unknown and the sample size is large. The sample standard deviation can be taken as a good estimate of the population standard to estimate the standard error of the sample mean $\bar{X}$. That is $\frac{\sigma}{\sqrt{n}}$ can be replaced by $\frac{s}{\sqrt{n}}$ for defining Z. The appropriate test statistic is $Z = \frac{\bar{X}-\mu}{s/\sqrt{n}}$, which is approximately a standard normal variable. The value of the test statistic Z under the null hypothesis is

$$z = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$$

Here s^2 is the estimate of population variance σ^2, defined as

$$s^2 = \frac{1}{n-1}\Sigma(x-\bar{x})^2.$$

It is a two-tailed test, so the critical region at 5% level of significance is

$$|Z| > 1.96.$$

We have, $\bar{x} = 871$, $s = 21$, $n = 50$, so the computed value of Z under null hypothesis is

$$z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{871-880}{21/50} = -3.03.$$

- **Decision.** Since the observed value of Z lies in the critical region, so we fail to accept null hypothesis. That means, the manager's claim is not justified.
- **p-value.** From the standard normal integral table, we find that $P(Z > 3.03) = 0.0005$ and $P(Z < -3.03) = 0.0005$ so the value of p is $0.0005 \times 2 = 0.001$ Since the p-value is far less than 0.01, the value of Z is highly significant.

3. Population is not normal, variance known and sample size is large.

Example 16.6.3. The producer of a company claims that the selling price of his product is very standard and it is TK. 1500 per unit with standard deviation TK. 45. There is some doubt of CAB (Consumers' Association of Bangladesh) regarding this price. They want to verify this price using statistical testing procedure. A random sample of the sailing prices of 100 products of this company from different areas were collected. The average price per unit was found Tk. 1510.

Can the CAB conclude at 5% level of significance that the average price of the product is standard? Also calculate p-value and 95% confidence interval for population mean.

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Solution. We have to test the hypothesis that the sailing price of the product is TK. 1500. So, the null hypothesis and alternative hypotheses are

Null hypothesis: $H_0: \mu = 1500$;

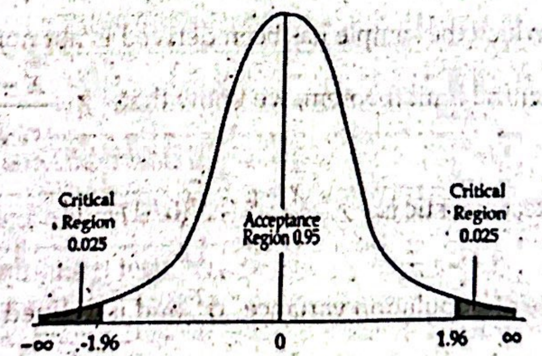
Alternative hypothesis: $H_a: \mu \neq 1500$;

Level of significance: $\alpha = 0.05$.

Here, we have to test the significance of a population mean with known population variance. But nothing is said about the form of distribution, but mean and variance exists. Since the sample size is large $(n = 100)$, according the central limit theorem, the sampling distribution of the mean is approximately normally distributed with mean μ and standard error $\frac{\sigma}{\sqrt{n}}$, so the appropriate test statistic is:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

Here $\alpha = 0.05$ (given), and it is a two tailed-test, the critical region will be on both sides of curve of z in such a way that the critical region will comprise 2.5% or 0.025 area at the right end and 2.5% at the left end. From the table of area of normal curve, we see that these values of Z are ± 1.96, that means the critical regions are $Z < -1.96$ (at the left end) and $Z > 1.96$ (at the right end).



Under the null hypothesis Z is given by: $z = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$.

Here, $\bar{x} = 1510$, $\mu_0 = 1500$, $\sigma = 45$ and $n = 100$, then;

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{1510 - 1500}{45/\sqrt{100}} = 2.22.$$

It is found that the observed value of Z is 2.22. It is greater than the critical value 1.96; hence it falls in the upper critical region. So we fail to accept the null hypothesis at 5% level of significance.

Conclusion. The claim of the producer is not right, that means, the price of the products as claimed by the producer is not standard.

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- **p-value.** From the table of area under the standard normal distribution we find that the value of $P(Z>2.22)=0.0132$ since it is a two tailed test, the p-value is $0.0132 \times 2 = 0.0264$, that means the smallest level of significance at which the hypothesis is rejected is 2.64%.
- **Confidence interval.** 95% confidence interval for μ is given by

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \text{here } z_{\alpha/2}=z_{0.025}=1.96,$$

So, the required CI is: $1510 \pm 1.96\frac{45}{\sqrt{100}} = (1501, 1519)$.

Thus, we may be 95% confident that the true mean price will be between TK. 1501 and TK. 1519.

4. Population is not normal; variance is unknown and the sample size is large.

Example 16.6.4. A manufacturer of fluorescent tubes claims that his tubes have a lifetime of 1950 burning hours. A random sample of 100 tubes is taken from a day's output and tested for burning life. It is found to have a mean burning lifetime of 1900 hours with a standard deviation of 150 hours. Can the claim of the manufacturer be accepted at 5% level of significance? Also find p-value.

Solution. Here nothing is said about population and the population variance is not known

$$H_0: \mu = 1900 \quad \text{and} \quad H_1: \mu \neq 1900$$

The population from which the sample has been derived is not normal, but the sample size large. So by the virtue of central limit theorem, we know that: $Z = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0, 1)$.

Thus the appropriate test statistic is: $Z = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0, 1)$.

Here s^2 is the estimate of population variance σ^2 and is defined as

$$s^2 = \frac{1}{n-1}\Sigma(x-\bar{x})^2.$$

It is a two-tailed test, so the critical region at 5% level of significance is

$$|Z| > 1.96.$$

Given, $\bar{x}=1900$, $s=150$, $n=100$, so the computed value of Z under null hypothesis is

$$z = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{1900-1950}{150/100} = -3.33.$$

- **Decision.** Since the observed value of Z lies in the critical region, so we fail to accept null hypothesis. That means, the manufacturer's claim is not accepted at 5% level of significance

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p-value. From the standard normal integral table, we find that $P(Z<-3.33)=0.0004$ so the value of p is $0.0004 \times 2 = 0.0008$ (approx). Since the p-value is far less than 0.01, the value of z is highly significant.

Population normal, variance unknown and sample size is small ($n<30$).

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn independently from a normal population with mean μ and unknown variance σ^2.

The following null and alternative hypothesizes may be considered.

i) $H_0: \mu=\mu_0$ against $H_1: \mu \neq \mu_0$ (for a two tailed alternative);

ii) $H_0: \mu=\mu_0$ against $H_1: \mu > \mu_0$ (for a right tailed alternative);

iii) $H_0: \mu=\mu_0$ against $H_1: \mu < \mu_0$ (for a left tailed alternative).

The test statistic for testing the null hypothesis H_0 for all the alternatives is

$$t=\frac{\bar{x}-\mu_0}{s/\sqrt{n}}, \quad \text{where} \quad s^2=\frac{1}{n-1}\Sigma(x-\bar{x})^2.$$

Here t follows Student's t-distribution with $(n-1)$ degrees of freedom.

The decision rules at 100α percent level of significance for hypothesis testing using t-statistic are as follows.

Table 16.5. Decision rule for t-test

Case No.	Type of test	Decision rule Reject H_0, if
1	Two-tailed test: $H_1: \mu \neq \mu_0$	$\lvert t \rvert > t_{\alpha/2;(n-1)}$
2	Right-tailed test: $H_1: \mu > \mu_0$	$t > t_{\alpha;(n-1)}$
3	Left-tailed test: $H_1: \mu < \mu_0$	$t < -t_{\alpha;(n-1)}$

Example 16.6.5. A wholesaler knows that on average sales in its store is 20% higher in December than in November. For the current year, a random sample of six stores was taken. Their percentage of sales increased in December was found to be 19.2, 18.4, 19.8, 20.2, 20.4, 19.0. Assuming that the sample has been drawn from a normal population with mean μ and unknown variance σ^2.

a) test the null hypothesis at 10% level of significance whether the true mean percentage sales increase is 20%, against the two-sided alternative.

b) do you think that the true mean percentage sales increase is more than 20% at 10% level of significance.

c) do you think that the true mean percentage sales increase is less than 20% at 10% level of significance.

Solution. Here the population variance is unknown and the sample size is small. The estimated standard error of sample mean $\bar{x}$ is given by

$$\hat{se}(\bar{x}) = s/\sqrt{n} \quad \text{where,} \quad s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1}.$$

(a) We want to test the null hypothesis

$H_0: \mu = \mu_0 = 20$ against the alternative $H_1: \mu = \mu_1 \neq 20$.

Since the sample size is small and population variance is unknown, the value of the test statistic under null hypothesis is

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

which is distributed as Student's t with $(n-1)$ degrees of freedom.

It is a two tailed test, so the decision rule is

Reject H_0 in favor of H_1 if $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} > t_{\alpha/2;(n-1)}$ or, $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < -t_{\alpha/2;(n-1)}$

The critical values of t at 10% level of significance with $n-1=5$ df are $\pm t_{n-1;\alpha/2} = \pm t_{5,05}$ $= \pm 2.015$ (From table of t-distribution).

Here, $n=6$, $\bar{x}=19.5$, $s^2=0.588$, and $s/\sqrt{n}=0.24$.

Then, we have, $t = \frac{19.5-20}{0.24} = -1.08$.

Since the observed value $t=-1.08$ lies between -2.015 and 2.015, hence we fail to reject the null hypothesis at 10% level of significance.

That means, the true mean sales is 20% higher in December than in November.

(b) In this case, we have to perform a one-tailed test, given by

$H_0: \mu = \mu_0 = 20$ against the alternative $H_1: \mu = \mu_1 > 20$.

The decision rule is to reject H_0 in favor of H_1 if $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} > t_{\alpha;n-1}$

The computed value of t is the same as before i,e, $t=-1.08$, the critical value of t at 10% level of significance is: $t_{0.105} = 1.476$

Since the observed value of $t=-1.08$ which is less than the critical value, we fail to reject the null hypothesis at 10% level of significance, which means the average sales increased by more than 20 percent is not evident from the given data.

(c) In this case, we have to perform a one-tailed test, given by

$H_0: \mu = \mu_0 = 20$ against the alternative $H_1: \mu = \mu_1 < 20$.

The decision rule is to reject H_0 in favor of H_1 if $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < -t_{\alpha;n-1}$.

The computed value of t is the same as before i.e., $t=-1.08$ the critical value of t at 10% level of significance is: $t_{0.10;5} = -1.476$.

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Since the observed value of $t=-1.08$ lies beyond the critical region, we fail to reject the null hypothesis at 10% level of significance, which means the average sales increased by less than 20 percent is not evident from the given data.

Example 16.6.6. (Large sample with unknown variance when parent population is not normal) A fertilizer factory manager claims that its average daily production is 912 kg. A random sample of 50 days shows that average production is 903 kg with standard deviation 21 kg. Test the significance of the claim of the manager at 5% level of significance.

Solution. Here, $H_0: \mu = 912$ and $H_1: \mu \neq 91.2$.

The population from which the sample has been derived is not normal, but the sample size is large. So by the virtue of central limit theorem, we know that $\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim N(0, 1)$.

Thus under the null hypothesis, the appropriate test statistic is

$$z = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$$

where s^2 is the estimate of population variance σ^2, defined as

$$s^2 = \frac{\Sigma(x-\bar{x})^2}{n-1}.$$

It is a two tailed test, so the critical region at 5% level of significance is

$$|Z| > 1.96.$$

Given, $\bar{x}=871$, $s=21$, $n=50$, so the computed value of Z under null hypothesis is

$$z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{903-912}{21/\sqrt{50}} = -3.03.$$

Decision. Since the observed value of Z lies in the critical region, so we fail to accept null hypothesis. That means, the manager's claim is not justified.

Example 16.6.7. (Two-tailed test for small sample with known variance) The yields of wheat from a random sample of six test plots are as 1.40, 1.80, 1.30, 1.90, 1.60 and 2.20 tons per acre, test whether the information supports the claim that the average yield for this kind of wheat is 1.5 tons/acre with standard deviation 0.43 tons/acre. Also find the p-value.

Solution. The null and alternative hypotheses for this test are

Null hypothesis: $H_0: \mu = 1.5$

Alternative hypothesis: $H_A: \mu \neq 1.5$.

Here, the sample size is small, the population variance is known and the sample is taken from normal population, so the appropriate test statistic is

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}.$$

Let the level of significance $\alpha = 0.05$ (if α is not given, it is considered as 0.05).

Thus, the critical values of Z are ± 1.96 that means the critical regions are

$$Z < -1.96 \text{ and } Z > 1.96.$$

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The average yield as calculated for the observations is 1.7 tons/acre, so the value of Z under null hypothesis is

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{1.7 - 1.5}{\frac{0.43}{\sqrt{6}}} = 1.14.$$

Since the observed value of Z is less than 1.96, that means the observed value lies in the acceptance region, so we fail to reject null hypothesis. Hence, the information supports that the average yield is 1.5 tons/acre.

- **p-value.** From the table of standard normal distribution, we find that $P(Z>1.14)=0.127$ and $P(Z<-1.14)=0.1271$ so, the p-value is $0.1271\times 2=0.2542$ that means the test will be significance at 25.42% level of significance.

Example 16.6.8. (Right tailed test for small sample size with known variance) A stenographer claims that she can take dictation at the rate of more than 100 words per minute with a standard deviation of 15 words. Can we reject the claim on the basis of 10 trials in which she demonstrates a mean of 102 words per minute? Use 5% level of significance.

Solution. If the stenographer can take dictation of even at the rate of 100 words per minute, her claim can not be accepted. So, the null hypothesis and alternative hypothesis to be considered are

Null hypothesis: $H_0: \mu = 100$

Alternative hypothesis: $H_1: \mu > 100.$

Since the population variance is known, the appropriate test statistic is

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

The critical region at 5% level of significance is: $Z>1.645$

Here, $\bar{x}=102$, $\sigma=15$, $n=10$, so the value of Z is

$$Z = \frac{102-100}{15/\sqrt{10}} = 0.42.$$

The computed value of Z does not exceed the critical value 1.645, so we fail to reject null hypothesis. So, stenographer's claim is not correct.

Example 16.6.9. (Left tailed test for small sample size with known variance) An automobile manufacturer company claims that a new model car achieves on an average 31.5 miles per gallon in highway driving. The distribution is known to be normal with standard deviation 2.4 miles per gallon. A random sample of sixteen automobiles provided an average of 30.6 miles per gallon in highway trials. Test the claim of company at the 5% level of significance against the population mean is less than 31.5 miles per gallon.

Solution. It is a left-tailed test, because, if the average coverage of distance is even equal to 31.5 miles, the company's claim will not be correct. Thus, the null and alternative hypotheses are

Null hypothesis: $H_0: \mu = 31.5$

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Alternative hypothesis: $H_A: \mu < 31.5$.

Since, the population variance is known, the appropriate test statistic is

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}.$$

The critical region at 5% level of significance is: $Z < -1.645$

Here, $\bar{x} = 30.6$, $\sigma = 2.4$, $n = 16$, so the value of Z is

$$Z = \frac{30.6 - 31.5}{2.4/\sqrt{16}} = -1.50.$$

The computed value of Z does not fall in the critical region, so we fail to the reject the null hypothesis that means, the average achievement of car is not less than 31.5 miles per gallon.

Example 16.6.10. (Two-tailed test for large sample with known variance) A large manufacturer of stereo components is concerned about the efficiency of many new employees hired during the last six months. The efficiency rating of all employees has been reasonably stable with mean rating 200 and a standard deviation of 20. A random sample of 75 new employees has been selected and their average efficiency rating is found as 197.5. Test the null hypothesis at 1% level of significance that the mean efficiency rating is still 200. Find 99% confidence interval for population mean.

Solution. It is a two-tailed test because if the mean efficiency rating obtained from sample is too high or too low in comparison with population mean rating, the null hypothesis would be rejected, so the null and alternative hypotheses to be considered are

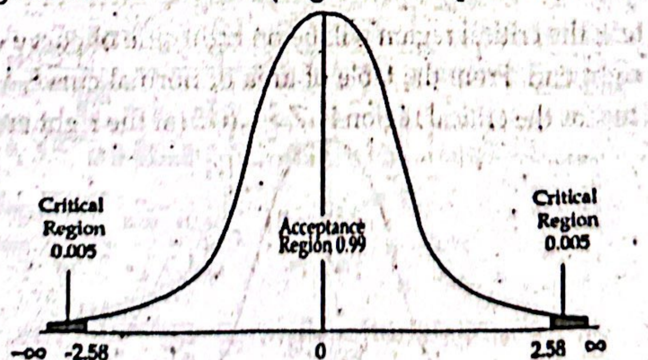
Null hypothesis: $H_0: \mu = 200$

Alternative hypothesis: $H_1: \mu \neq 200$.

We have to test the significance of a population mean with known population variance $\sigma^2 = 20^2$, and the sample size is also large, so the sampling distribution of the mean is normally distributed with standard error $\frac{\sigma}{\sqrt{n}}$, the appropriate test statistic for the selected hypothesis is

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}.$$

Here the level of significance is $\alpha = 0.01$ (as given in the problem).



As it is a two-tailed test, the critical region will be on both ends of curve of Z that will comprise of 0.005 area at the right end and 0.005 area at the left end. From the table of normal curve, a.f.

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