Inverse deformation gradient

1 Deformation gradient

Let $\phi(\cdot,t):\Omega^0\to\Omega^t$ be the flow map of our material. The deformation gradient $\mathbf{F}(\cdot,t):\Omega^0\to\mathbb{R}^{2\times 2}$ is defined as

$$\mathbf{F}(\mathbf{X},t)=\frac{\partial\phi}{\partial\mathbf{X}}(\mathbf{X},t),\quad \text{or}\quad F_{ij}(\mathbf{X},t)=\frac{\partial\phi_i}{\partial X_j}(\mathbf{X},t).$$

2 Inverse material mapping

If $\phi(\cdot,t):\Omega^0\to\Omega^t$ is the flow map of our material, let $\phi^{-1}(\cdot,t):\Omega^t\to\Omega^0$ be the inverse flow map. Then, by definition

$$\mathbf{X}=\phi^{-1}(\phi(\mathbf{X},t),t),\quad \forall\mathbf{X}\in\Omega^0,\ \forall t\geq 0$$

and

$$\mathbf{x}=\phi(\phi^{-1}(\mathbf{x},t),t),\quad \forall\mathbf{x}\in\Omega^t,\ \forall t\geq 0.$$

3 Inverse deformation gradient

The inverse of the deformation gradient $\mathbf{F}^{-1}(\cdot,t):\Omega^0\to\mathbb{R}^{2\times 2}$ is the mapping defined as

$$\mathbf{F}^{-1}(\mathbf{X},t)=(\mathbf{F}(\mathbf{X},t))^{-1}$$

In other words, at any point $\mathbf{X}\in\Omega^0$ and at any time $t\geq 0$, the function $\mathbf{F}^{-1}(\mathbf{X},t)$ is defined to be the inverse of the deformation gradient at that point and time.

4 Deformation gradient of inverse mapping

We can also define $\frac{\partial\phi^{-1}}{\partial\mathbf{x}}(\cdot,t):\Omega^t\to\mathbb{R}^{2\times 2}$ as a function over Ω^t. It is just the Jacobian of the inverse mapping. I sometimes use the notation

$$\frac{\partial\phi^{-1}}{\partial\mathbf{x}}=\frac{\partial\mathbf{X}}{\partial\mathbf{x}},\quad \text{or}\quad \frac{\partial\phi_i^{-1}}{\partial x_j}.$$

5 Inverse of the deformation gradient is the pull back of the inverse deformation gradient

I know that sounds confusing, but it is an important relation. That is, the inverse of the deformation gradient $\mathbf{F}^{-1}(\cdot,t):\Omega^0\to\mathbb{R}^{2\times 2}$ is the pull back of $\frac{\partial\phi^{-1}}{\partial\mathbf{x}}(\cdot,t):\Omega^t\to\mathbb{R}^{2\times 2}$. We can show this by differentiating the relation

$$\mathbf{X}=\phi^{-1}(\phi(\mathbf{X},t),t),\quad \forall\mathbf{X}\in\Omega^0,\ \forall t\geq 0$$

that is

$$\mathbf{I}=\frac{\partial\phi^{-1}}{\partial\mathbf{x}}(\phi(\mathbf{X},t),t)\frac{\partial\phi}{\partial\mathbf{X}}(\mathbf{X},t)$$

where $\mathbf{I}$ is the identity matrix. In index notation, this reads

$$\delta_{ij} = \frac{\partial \phi_i^{-1}}{\partial x_k}(\phi(\mathbf{X},t),t)\frac{\partial \phi_k}{\partial X_j}(\mathbf{X},t).$$

Now, since $\frac{\partial \phi}{\partial \mathbf{X}}(\mathbf{X},t) = \mathbf{F}(\mathbf{X},t)$ and $\mathbf{I} = \frac{\partial \phi^{-1}}{\partial \mathbf{x}}(\phi(\mathbf{X},t),t)\frac{\partial \phi}{\partial \mathbf{X}}(\mathbf{X},t)$, $\frac{\partial \phi^{-1}}{\partial \mathbf{x}}(\phi(\mathbf{X},t),t)$ must be the inverse of $\mathbf{F}(\mathbf{X},t)$. In other words,

$$\mathbf{F}^{-1}(\mathbf{X},t) = \frac{\partial \phi^{-1}}{\partial \mathbf{x}}(\phi(\mathbf{X},t),t), \quad \forall \mathbf{X} \in \Omega^0, \ t \geq 0.$$

I also sometimes use the notation

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \frac{\partial \phi_i^{-1}}{\partial x_j}.$$

Finally, of course this also implies that $\frac{\partial \phi^{-1}}{\partial \mathbf{x}}(\cdot,t) : \Omega^t \to \mathbb{R}^{2\times 2}$ is the push forward of $\mathbf{F}^{-1}(\cdot,t) : \Omega^0 \to \mathbb{R}^{2\times 2}$.