1 Expected case reproductive number R_c

1.1 A model with a more general incidence term

Assume that the following relation between the incidence, i, and the expected reproductive number is true

$$C(\tau, \rho) = \int_{t=\tau}^{t=\rho} h(t)dt, \tag{1a}$$
$$\omega(\tau, \rho) = i(\tau)f(\rho - \tau), \tag{1b}$$
$$i(t) = h(t)y(t), \tag{1c}$$

where $h(t)$ is some function of states and, in turn, time, and $y(t)$ is the proportion of infectious individuals.

If so, it appears that the functional form of $h(t)$ does not matter. Please see below.

The second raw moment of the expected case reproductive number $C(\tau, \rho)$ reads as follows

$$C_2 = \int_{\tau=0}^{\infty}\int_{\rho=\tau}^{\infty} i(\tau)f(\rho-\tau)d\rho d\tau \int_{t=\tau}^{\rho}\int_{s=\tau}^{\rho} h(t)h(s)dtds \tag{2a}$$
$$= 2\int_{\tau=0}^{\infty}\int_{\rho=\tau}^{\infty}\int_{t=s}^{\rho}\int_{s=\tau}^{t} i(\tau)f(\rho-\tau)d\rho d\tau h(t)h(s)dtds \tag{2b}$$
$$= 2\int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty}\int_{s=\tau}^{t} i(\tau)h(t)h(s)ds \underbrace{\int_{\rho=t}^{\infty} f(\rho-\tau)d\rho}_{F(t-\tau)} \tag{2c}$$
$$= 2\int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty}\int_{s=\tau}^{t} i(\tau)h(t)h(s)ds \quad \underbrace{F(t-\tau)}_{F(t-s)F(s-\tau)} \tag{2d}$$
$$= 2\int_{t=s}^{\infty}\int_{s=\tau}^{t} h(t)F(t-s)h(s)ds \underbrace{\int_{\tau=0}^{s} d\tau i(\tau)F(s-\tau)}_{y(s)} \tag{2e}$$
$$= 2\int_{t=0}^{\infty}\int_{s=0}^{t} h(t)dsF(t-s)\underbrace{h(s)y(s)}_{i(s)} \tag{2f}$$
$$= 2\int_{t=0}^{\infty} h(t)\underbrace{\int_{s=0}^{t} dsF(t-s)i(s)}_{y(t)} \tag{2g}$$
$$= 2\underbrace{\int_{t=0}^{\infty} i(t)dt}_{Z} \tag{2h}$$
$$= 2Z \tag{2i}$$

1.2 Erlang infectious periods don't allow variance above one

The survival function of Erlang distribution is equal to

$$F(t) = \sum_{n=0}^{m-1} \frac{1}{n!} e^{-mt} (mt)^n.$$

It has been shown that the Erlang distribution has an increasing failure rate, i.e., $\ln F(t)$ is concave, which, in turn, implies the following property [1]

$$F(a+b) < F(a)F(b). \tag{3}$$

Given (1) and in view of (2c), the second raw moment of $C(\tau, \rho)$ equals

$$\begin{aligned} C_2 &= 2 \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} \int_{s=\tau}^{t} i(\tau) h(t) h(s) ds F(t-\tau) \\ &\overset{(3)}{<} 2 \int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} \int_{s=\tau}^{t} i(\tau) h(t) h(s) ds F(t-s) F(s-\tau) \\ &\xrightarrow{(2\mathrm{i})} < 2Z. \end{aligned}$$

Previously it has been shown that the mean of the expected case reproductive number, regardless of the infectious period distribution, is one yielding

$$\frac{C_2}{Z} - 1 < 1.$$

1.2.1 Simulations for Erlang Distribution of Infectious Period

The following model was simulated

$$\begin{aligned} \dot{x} &= -\beta(t) x^{\kappa+1} y + m\gamma z, \\ \dot{y}_1 &= \beta(t) x^{\kappa+1} y - m y_1, \\ &\vdots \\ \dot{y}_m &= m y_{m-1} - m y_m, \\ \dot{z} &= m y_m - m\gamma z \end{aligned}$$

where $\beta(t) = \beta_0 \exp(\beta_1 \sin(\omega t))$. The incidence is defined by

$$i(t) = \beta(t) x^{\kappa+1}(t) y(t),$$

and the expected case reproductive number of individuals got infected at time τ and recovered at time ρ is formulated as

$$C(\tau, \rho) = \int_{\tau}^{\rho} \beta(t) x^{\kappa+1} dt.$$

Table 1 reports the simulation result.

Table 1: Simulation results support the claim that the variance in the presence of Erlang distribution is less than one.

No.	step size	ω	m	κ	β_0	β_1	γ	μ	within	between	Var R_c	CV^2
1	2000	0.16	1	0	3.00	0.90	0.50	1.01	0.98	0.02	1.01	1.00
2	2000	0.16	2	0	3.00	0.90	0.50	1.00	0.49	0.01	0.51	0.51
3	2000	0.16	5	0	3.00	0.90	0.50	1.00	0.20	0.01	0.21	0.21
4	2000	0.16	10	0	3.00	0.90	0.50	1.00	0.10	0.01	0.11	0.11
5	2000	0.16	1	0	2.00	0.90	0.50	1.01	0.95	0.04	1.01	1.00
6	2000	0.16	2	0	2.00	0.90	0.50	1.00	0.48	0.03	0.52	0.51
7	2000	0.16	5	0	2.00	0.90	0.50	1.00	0.20	0.02	0.22	0.22
8	2000	0.16	10	0	2.00	0.90	0.50	1.00	0.10	0.02	0.12	0.12
9	2000	0.16	1	0	1.50	0.90	0.50	1.01	0.92	0.08	1.01	1.00
10	2000	0.16	2	0	1.50	0.90	0.50	1.01	0.47	0.06	0.53	0.53
11	2000	0.16	5	0	1.50	0.90	0.50	1.00	0.19	0.04	0.24	0.23
12	2000	0.16	10	0	1.50	0.90	0.50	1.00	0.10	0.04	0.14	0.13
13	2000	0.10	1	0	4	1	1.00	1.00	0.99	0.01	1.00	1.01
14	5000	0.10	1	0	4	1	0.50	1.00	0.54	0.46	1.00	1.00
15	2000	0.10	2	0	4	1	1.00	1.02	0.27	0.44	0.71	0.73
16	2000	0.10	1	0	4	2	1.00	1.00	0.97	0.03	1.00	1.00
17	2000	0.10	2	0	4	2	1.00	1.03	0.24	0.49	0.73	0.78
18	2000	0.10	3	0	4	1	1.00	1.02	0.13	0.55	0.69	0.72
19	6000	0.10	3	0	4	1	1.00	1.00	0.13	0.57	0.70	0.70
20	2000	0.00	1	0	2	0	0.00	1.00	0.83	0.16	1.00	1.00
21	2000	0.00	1	1	2	0	0.00	1.00	0.86	0.14	1.00	1.00
22	2000	0.00	1	0.50	2	0	0.00	1.00	0.85	0.15	1.00	1.00
23	2000	0.00	2	0	2	0	0.00	1.00	0.41	0.17	0.58	0.58
24	2000	0.00	2	0.50	2	0	0.00	1.00	0.42	0.15	0.58	0.58
25	2000	0.00	2	1	2	0	0.00	1.00	0.43	0.14	0.57	0.57

1.3 Higher moments of the case reproductive number

$$
\begin{aligned}
C(\tau,\rho) &= \beta \int_{t=\tau}^{t=\rho} x(t)dt,\\
\omega(\tau,\rho) &= i(\tau) f(\rho-\tau),\\
i(t) &= \beta x(t) y(t),
\end{aligned}
$$

where $x(t)$ and $y(t)$ are the proportions of susceptible and infectious individuals, respectively, $\omega(\tau,\rho)$ is the size of individuals infected at time point τ and recovered at time point ρ, $f(t)$ is the distribution of residence time in the infectious compartment, and $i(t)$ is the incidence at time point t.

The k^{th} raw moment of the expected case reproductive number $C(\tau,\rho)$ reads as

$$
\begin{aligned}
C_k &= \int\int i(\tau)f(\rho-\tau)d\tau d\rho\Big(\int_\tau^\rho \beta x(s)ds\Big)^k \\
&= \beta^k \int\int i(\tau)f(\rho-\tau)d\tau d\rho \underbrace{\int_\tau^\rho \cdots \int_\tau^\rho}_{k \text{ times}} x(t_1)\dots x(t_k)dt_1\dots dt_k
\end{aligned}
$$

The integral of the symmetric k-variable function $x(t_1)\dots x(t_k)$ over the hybercube $[\tau,\rho]^k$ is equal to $k!$ times the integral of the function over the region $\tau < t_1 < t_2 < \dots < t_k < \rho$. Hence, we have

$$
\begin{aligned}
&= k!\beta^k \int_{\tau<t_1<t_2<\dots<t_k<\rho} \int \underbrace{\int \dots \int}_{k \text{ times}} i(\tau)f(\rho-\tau)d\rho d\tau x(t_1)\dots x(t_k)dt_1\dots dt_k \\
&= k!\beta^k \int_{\tau<t_1<t_2<\dots<t_k} \underbrace{\int \dots \int}_{k \text{ times}} i(\tau)d\tau x(t_1)\dots x(t_k)dt_1\dots dt_k \underbrace{\int_{\rho=t_k}^\infty f(\rho-\tau)d\rho}_{F(t_k-\tau)}
\end{aligned}
$$

Using the Markovian property, we have $F(t_k-\tau) = F(t_k-t_{k-1}) \times F(t_{k-1}-t_{k-2}) \times \dots \times F(t_1-\tau)$. By defining t_0 as τ, the term $\int_{t_l<t_{l+1}} dt_l i(t_l)F(t_{l+1}-t_l)$, for $l \in 0,\dots,k-1$, is equal to $y(t_{l+1})$. The multiplication of $y(t_{l+1})$ and $\beta x(t_{l+1})$ equals $i(t_{l+1})$. By repeating the same procedure for k times, the remaining element would be $k!\int i(t_k)dt_k$.

2 Instantaneous case reproductive number

For the following SIR model,

$$
\begin{aligned}
\dot{x} &= -\beta(t)yx, \\
\dot{y} &= \beta(t)yx - y,
\end{aligned}
$$

we define the instantaneous case reproductive number associated with cohorts infected at time point τ and recovered at time point ρ as

$$R_i(\tau,\rho) = \beta(\tau)x(\tau)(\rho-\tau).$$

The size of the cohort infected at τ and recovered at ρ is equal to $w(\tau,\rho) = i(\tau)f(\rho-\tau)$. R_0 is defined as

$$
\begin{aligned}
R_0 &= \int_\tau\int_\rho d\tau d\rho w(\tau,\rho) \\
&= \int_\tau i(\tau) \\
&= Z.
\end{aligned}
$$

The first raw moment of R_i is then equal to

$$\begin{aligned} R_1 &= \int_\tau \int_\rho d\tau d\rho w(\tau,\rho)\beta(\tau)x(\tau)(\rho-\tau) \\ &= \int_\tau \int_\rho d\tau d\rho i(\tau) f(\rho-\tau)\beta(\tau)x(\tau)(\rho-\tau) \\ &= \int_\tau d\tau \beta(\tau)x(\tau)i(\tau) \underbrace{\int_{\rho>\tau} f(\rho-\tau)(\rho-\tau)d\rho}_{\text{mean of infectious period}=T_{\text{inf}}} \end{aligned}$$

In the case of a constant transmission rate ($\beta(t) = \beta$), R_1 will reduce to $\beta T_{\text{inf}} \int_\tau x(\tau)i(\tau)d\tau$. The mean of R_i is then

$$\mu = \frac{\beta T_{\text{inf}} \int_\tau x(\tau)i(\tau)d\tau}{Z}.$$

The second raw moment of R_i is equal to

$$\begin{aligned} R_2 &= \int_\tau \int_\rho d\tau d\rho i(\tau) f(\rho-\tau)\beta^2(\tau)x^2(\tau)(\rho-\tau)^2 \\ &= \int_\tau d\tau \beta^2(\tau)x^2(\tau)i(\tau) \underbrace{\int_{\rho>\tau} f(\rho-\tau)(\rho-\tau)^2 d\rho}_{\text{Var}_{\text{inf}}+T_{\text{inf}}^2}, \end{aligned}$$

where Var_{inf} is the variance of the infectious distribution. In the case of a constant transmission rate ($\beta(t) = \beta$), R_2 will reduce to $(\text{Var}_{\text{inf}}+T_{\text{inf}}^2)\beta^2 \int_\tau x(\tau)^2 i(\tau)d\tau$, and the variance of R_i reads as

$$\frac{(\text{Var}_{\text{inf}} + T_{\text{inf}}^2)\beta^2 \int_\tau x(\tau)^2 i(\tau)d\tau}{Z} - \frac{\beta T_{\text{inf}} \int_\tau x(\tau)i(\tau)d\tau}{Z}.$$

2.1 Simulation

Table 2: The mean and variance of R_i under an exponential distribution of the infectious period.

No.	β_0	β_1	γ	ω	Z	μ	within (CV^2)	between (CV^2)	Var R_c	(CV^2)
1	1.50		0.00	0.00	0.58	1.06	1.06	0.06	1.26	1.11
2	2.00		0.00	0.00	0.80	1.20	1.15	0.15	1.87	1.29
3	5.00		0.00	0.00	0.99	2.52	1.33	0.32	10.45	1.65
4	2.00		0.50	0.00	33.77	1.01	1.01	0.01	1.03	1.01
5	2.00	0.50	0.50	0.10	33.61	1.02	1.04	0.04	1.12	1.07
6	2.00	0.50	0.50	0.16	33.62	1.03	1.04	0.04	1.14	1.09

References

[1] Albert W Marshall and Frank Proschan. Classes of distributions applicable in replacement with renewal theory implications. In *Proc. 6th Berkeley Symp. Math. Statist. Prob*, volume 1, pages 395–415, 1972.