1 Higher moments of the case reproductive number

$$C(\tau,\rho) = \beta \int_{t=\tau}^{t=\rho} x(t)dt,$$
$$\omega(\tau,\rho) = i(\tau)f(\rho-\tau),$$
$$i(\tau) = \beta x(\tau)y(\tau),$$

where $x(t)$ and $y(t)$ are the proportions of susceptible and infectious individuals, respectively, and $f(t)$ is the distribution of residence time in the infectious compartment.

The k^{th} raw moment of the expected case reproductive number $C(\tau,\rho)$ reads as

$$\begin{aligned} C_k &= \int\int i(\tau)f(\rho-\tau)d\tau d\rho\big(\int_\tau^\rho \beta x(s)ds\big)^k \\ &= \beta^k \int\int i(\tau)f(\rho-\tau)d\tau d\rho \underbrace{\int_\tau^\rho \cdots \int_\tau^\rho}_{k \text{ times}} x(t_1)\ldots x(t_k)dt_1\ldots dt_k \end{aligned}$$

The integral of the symmetric k-variable function $x(t_1)\ldots x(t_k)$ over the hybercube $[\tau,\rho]^k$ is equal to $k!$ times the integral of the function over the region $\tau < t_1 < t_2 < \ldots < t_k < \rho$. Hence, we have

$$\begin{aligned} &= k!\beta^k \int_{\tau<t_1<t_2<\ldots<t_k<\rho} \int \underbrace{\int\ldots\int}_{k \text{ times}} i(\tau)f(\rho-\tau)d\rho d\tau x(t_1)\ldots x(t_k)dt_1\ldots dt_k \\ &= k!\beta^k \int_{\tau<t_1<t_2<\ldots<t_k} \underbrace{\int\ldots\int}_{k \text{ times}} i(\tau)d\tau x(t_1)\ldots x(t_k)dt_1\ldots dt_k \underbrace{\int_{\rho=t_k}^{\infty} f(\rho-\tau)d\rho}_{F(t_k-\tau)} \end{aligned}$$

Using the Markovian property, we have $F(t_k-\tau) = F(t_k-t_{k-1}) \times F(t_{k-1}-t_{k-2}) \times \ldots \times F(t_1-\tau)$. By defining t_0 as τ, the term $\int_{t_l<t_{l+1}} dt_l i(t_l)F(t_{l+1}-t_l)$, for $l \in 0,\ldots,k-1$, is equal to $y(t_{l+1})$. The multiplication of $y(t_{l+1})$ and $\beta x(t_{l+1})$ equals $i(t_{l+1})$. By repeating the same procedure for k times, the remaining element would be $k!\int i(t_k)dt_k$.

2 A model with a more general incidence term

I'm not quite sure if the following relation between the incidence, i, and the expected reproductive number is always true

$$C(\tau,\rho) = \int_{t=\tau}^{t=\rho} h(t)dt,$$
$$\omega(\tau,\rho) = i(\tau)f(\rho-\tau),$$
$$i(\tau) = h(\tau)y(\tau),$$

where $h(t)$ is some function of states and, in turn, time, and $y(t)$ is the proportion of infectious individuals.

If so, it appears that the functional form of $h(t)$ does not matter. Please see below.

The second raw moment of the expected case reproductive number $C(\tau, \rho)$ reads as follows

$$
\begin{aligned}
C_2 &= \int_{\tau=0}^{\infty} \int_{\rho=\tau}^{\infty} i(\tau) f(\rho-\tau) d\rho d\tau \int_{t=\tau}^{\rho} \int_{s=\tau}^{\rho} h(t)h(s)dtds \\
&= 2\int_{\tau=0}^{\infty} \int_{\rho=\tau}^{\infty} \int_{t=s}^{\rho} \int_{s=\tau}^{t} i(\tau) f(\rho-\tau) d\rho d\tau h(t)h(s)dtds \\
&= 2\int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} \int_{s=\tau}^{t} i(\tau)h(t)h(s)ds \underbrace{\int_{\rho=t}^{\infty} f(\rho-\tau)d\rho}_{F(t-\tau)} \\
&= 2\int_{\tau=0}^{\infty} \int_{t=\tau}^{\infty} \int_{s=\tau}^{t} i(\tau)h(t)h(s)ds \quad \underbrace{F(t-\tau)}_{F(t-s)F(s-\tau)} \\
&= 2\int_{t=s}^{\infty} \int_{s=\tau}^{t} h(t)F(t-s)h(s)ds \underbrace{\int_{\tau=0}^{s} d\tau i(\tau) F(s-\tau)}_{y(s)} \\
&= 2\int_{t=0}^{\infty} \int_{s=0}^{t} h(t)dsF(t-s) \underbrace{h(s)y(s)}_{i(s)} \\
&= 2\int_{t=0}^{\infty} h(t) \underbrace{\int_{s=0}^{t} dsF(t-s)i(s)}_{y(t)} \\
&= 2\int_{t=0}^{\infty} i(t)dt
\end{aligned}
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