Model Predictive Control - EL2700

Assignment 1 : State Feedback Control Design

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Automatic control
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1 Task: Discrete-time linear design

Throughout these assignments, we will consider the design of a control system for an inverted pendulum on a cart, see Figure 1. The objective will be to move the cart from one position to the other while maintaining the pendulum upright. This system has challenging dynamics, similar to the ones that you can find when designing control systems for Segway vehicles, space rockets and many other interesting systems.

In this task, we will design a discrete-time linear state-feedback controller for our system. To this end, you will linearize the nonlinear continuous-time dynamics, compute a discrete-time model which describes the state evolution between sampling instances, and use this model to design a linear state feedback controller which achieves the desired control performance.

Nonlinear cart-pendulum model To develop the nonlinear model of the system dynamics, we will use the notation introduced in Figure 1: x is the cart position, θ is the pendulum angle, F is the force applied to the cart, and w is an external disturbance force (force due to wind on the pendulum). By classical mechanics, the system dynamics can be described by the following

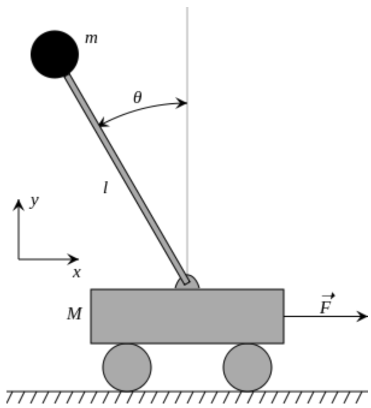

Figure 1: Cart and inverted pendulum schematics

differential equations.

$$\begin{aligned}(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta + b_c\dot{x} &= F - w \\ (I+ml^2)\ddot{\theta} + b_p\dot{\theta} - mgl\sin\theta - lw\cos\theta &= ml\ddot{x}\cos\theta\end{aligned} \tag{1}$$

Here, M and m denote the mass of the cart and pendulum, respectively, I is the inertia of the pendulum, l is the length of the pendulum, g denotes the gravitational constant, b_c and b_p represent the coefficient of friction for the cart and the inverted pendulum, respectively. By introducing the state variables $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \theta$, $x_4 = \dot{\theta}$, we obtain the following state-space form:

$$\begin{aligned}\dot{x}_1 &= x_2 & \dot{x}_2 &= f_1(x_2, x_3, x_4, F, w) \\ \dot{x}_3 &= x_4 & \dot{x}_4 &= f_2(x_2, x_3, x_4, F, w)\end{aligned} \tag{2}$$

where,

$$\begin{aligned}f_1(x_2, x_3, x_4, F, w) = \frac{1}{M + m - \frac{m^2l^2\cos^2 x_3}{I+ml^2}} \Bigg(F + \left(\frac{ml^2\cos^2 x_3}{I+ml^2} - 1\right) w - b_c x_2 - mlx_4^2 \sin x_3 \\ + \frac{m^2l^2 g \sin x_3 \cos x_3}{I+ml^2} - \frac{mlb_p x_4 \cos x_3}{I+ml^2}\Bigg)\end{aligned}$$

$$\begin{aligned}f_2(x_2, x_3, x_4, F, w) = \frac{1}{I + ml^2 - \frac{m^2l^2\cos^2 x_3}{M+m}} \Bigg(\frac{ml\cos x_3}{M+m} F + \frac{Ml\cos x_3}{M+m} w - b_p x_4 + mgl\sin x_3 \\ - \frac{m^2l^2x_4^2 \sin x_3 \cos x_3}{M+m} - \frac{mlb_c x_2 \cos x_3}{M+m}\Bigg)\end{aligned}$$

PART I: Analytical Task

Introduction to the analytical task In the first part of this assignment, we will perform analytical calculations to linearize and sample a continuous-time system, and explore the relationship between the locations of the continuous-time system poles and those of its discrete-time counterpart.

To facilitate analytical calculations, we will disregard the dynamics of the cart and consider the simpler dynamics of the inverted pendulum. We can derive a model of the pendulum dynamics from the second subsystem (2) by setting $x_1 = x_2 = 0$ (corresponding to the cart standing still at the initial position), and $w = 0$. In addition, we assume that the mass of the pendulum is much smaller than that of the cart, i.e. $M \gg m$. With these assumptions, it can be validated that the pendulum dynamics can be described by a first order ODE system by introducing $x_1 = \theta$, and $x_2 = \dot{\theta}$ as states and cart acceleration $u = \frac{F}{M}$ as input:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -a_0 \sin x_1 - a_1 x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 u \cos x_1 \end{bmatrix} \\ y &= x_1 \end{aligned} \tag{3}$$

where $a_0 = -\frac{mgl}{I+ml^2}$, $a_1 = \frac{b_p}{I+ml^2}$ and $b_0 = \frac{ml}{I+ml^2}$.

Linear model Verify that the upright pendulum position at rest, $\mathbf{x}_{ref} = [0\ 0]^T$ is an equilibrium point for (3). Linearize (3) around $\mathbf{x}_{ref}$ and write the model on the form

$$\begin{aligned} \frac{d}{dt}\Delta\mathbf{x} &= A_c \Delta\mathbf{x}(t) + B_c \Delta u(t) \\ \Delta y(t) &= C_c \Delta\mathbf{x}(t) \end{aligned} \tag{4}$$

where $\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_{ref}$, $\Delta u(t) = u(t) - u_{ref}$ and $\Delta y(t) = y(t) - C_c\mathbf{x}_{ref}$. Document your derivations in the report.

Discrete linear model Using zero-order-hold discretization of (4) with sample interval h to determine an equivalent discrete-time linear representation

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + Bu_t \\ y_t &= Cx_t \end{aligned} \tag{5}$$

Here, index t denotes the sampling instance, $\mathbf{x}_t = \Delta\mathbf{x}(th)$ and $y_t = \Delta y(th)$. Detail your derivation of A, B and C in the report.

Impact of sampling to the pole locations of the discrete-time system Evaluate the pole locations of the discrete time system for different values of the sampling interval h. Record in your report a plot of the pole locations of the discrete-time system for different values of the sampling interval h. Verify the relation between continuous-time poles s_i and discrete time poles z_i given by $z_i = e^{s_i h}$ Record another plot of continuous time and discrete time poles for different values of the sampling interval h in your report to verify this relation.

PART II: Design Task

Introduction to the design task In this section, we will consider the complete cart and pendulum system as derived in (2). The simulink model `cart_pend.slx` includes the non-linear cart-pendulum system dynamics. You will use this model to test the performance of the designed linear state feedback controller.

Design of the state feedback controller For designing the state feedback controller, we assume that the system to be controlled is described by a linear state space model. Therefore, the first step will be to compute linear continuous-time state space model of the non-linear cart-pendulum system in (2) by linearizing the system equations around the equilibrium point, $\mathbf{x}_{eq} = [0\ 0\ 0\ 0]^T$

ignoring the disturbance w. To this end, verify that the linerized continuous time model has the following form:

$$\begin{aligned}\dot{\mathbf{x}} &= A_c\mathbf{x} + B_c u \\ y &= C_c\mathbf{x} + D_c u\end{aligned} \tag{6}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -b_c v_2 & \frac{m^2l^2gv_2}{I+ml^2} & -\frac{mlb_pv_2}{I+ml^2} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mlb_cv_1}{m+M} & mglv_1 & -b_pv_1 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ \frac{mlv_1}{m+M} \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \qquad D_c = 0$$

$$v_1 = \frac{m+M}{I(m+M)+mMl^2} \qquad v_2 = \frac{I+ml^2}{I(m+M)+mMl^2}$$

Next, use the `c2d` command in MATLAB to discretize the continuous-time linear state space model (6) with given sampling time $h = 0.1$ to take the following form

$$\begin{aligned}\mathbf{x}_{t+1} &= A\mathbf{x}_t + Bu_t \\ y_t &= C\mathbf{x}_t + Du_t\end{aligned} \tag{7}$$

Now, based on this discretized linear model in (7), derive a state feedback controller of the form

$$u_t = -L\mathbf{x}_t + l_r r \tag{8}$$

where r is the reference position to which cart will move from the initial position and L and l_r are state feedback and feed forward gain respectively. Design state feedback gain L to place the closed loop poles at the desired location using `place` command in MATLAB. Validate that the feed forward gain

$$l_r = \frac{1}{C(I-(A-BL))^{-1}B} \tag{9}$$

achieves error-free tracking in stationarity. Design the feedback and feed forward gains so that the cart moves 90% of the distance between the initial position and the reference $r = 10$ within 10 seconds, keeping the pendulum within $\pm 10°$ around the upright position. In our simulation, we have already set $r = 10$ in `prepare_sim.m`.

Hint: Please note that the simulator uses nonlinear model. Hence, if you designed a controller that results in a large deviation from the steady state, the linear model approximation would not be valid anymore, and your controller may not work as the linear analysis suggests.

Performance in the presence of disturbance input Controllers based on state feedback (8) tracks the reference signal without error at steady state by careful calibration of the gains L, and l_r as you have already observed in the previous task. However, one of the primary objectives of state feedback regulator is to allow good performance in the presence of disturbances.

To investigate if our controller is robust to disturbances, add a small constant disturbance w to our cart-pendulum model in SIMULINK to mimic the presence of wind force on the inverted pendulum. With the disturbance input, the linear discretized state space model in (7) becomes

$$\begin{aligned}\mathbf{x}_{t+1} &= A\mathbf{x}_t + Bu_t + B_w w_t \\ y_t &= C\mathbf{x}_t + Du_t\end{aligned} \tag{10}$$

Comment on the performance of the state feedback regulator (8) including the disturbance input w in the Simulink model. Comment on the tracking ability of this state feedback regulator in the presence of disturbance. Verify that the steady state error in the output can be computed as

$$\tilde{y}_{ss} = \lim_{k\to\infty} r - y_k = -C[I-(A-BL)]^{-1}B_w w$$

PART III: Submitting report and MATLAB files

To complete this design project, you should upload a small report for the tasks in Part I and filled MATLAB skeleton file `assignment_1.m` to Canvas.

Good Luck!