Geometric Theory for Constrained Linear Systems

Martin Biel

1 Preliminaries

Definition 1. An $\mathcal{H}$-*polyhedron* in $\mathbb{R}^n$ is the intersection of a finite set of closed half-spaces

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq h\}$$

Definition 2. An $\mathcal{H}$-*polytope* in $\mathbb{R}^n$ is an $\mathcal{H}$-polyhedron that does not contain any ray $\{x + ty \mid x, y \in \mathcal{P}, t \geq 0\}$.

Definition 3. Given an $\mathcal{H}$-polyhedron $\mathcal{P} = \{x \in \mathbb{R}^n \mid Hx \leq h\}$ and a linear mapping $x \to Ax$, the *composition* of $\mathcal{P}$ and A is given by

$$\mathcal{P} \circ A = \{x \in \mathbb{R}^n \mid HAx \leq h\}$$

and the *composition* of A and $\mathcal{P}$ is given by

$$A \circ \mathcal{P} = \{y \in \mathbb{R}^m \mid \exists x \in \mathcal{P} \text{ s.t. } y = Ax\}$$

In computations, it is probable that one encounters intermediate polyhedra with redundant constraints. Consider a polyhedron

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i, i = 1, \dots, m\}$$

arising in some computation. For repeated computations to remain tractable it is important to reduce $\mathcal{P}$ to a minimal representation $\tilde{\mathcal{P}}$ by removing all redundant constraints. This can be done by applying Algorithm 1.

Algorithm 1 Computation of minimal representation of $\mathcal{P}$

for $i \in \mathcal{I} = 1, 2, \dots, m$ **do**
  $\beta \leftarrow \max a_i^T x \text{ s.t. } a_j^T x \leq b_j, \quad j \in \mathcal{I} \setminus \{i\}$
  if $\beta \leq b_i$ **then**
    Remove i from $\mathcal{I}$
  end if
end for

Another important computational tool is elimination. Consider polyhedra of the form

$$\mathcal{P} = \{x \in \mathbb{R}^n, y \in \mathbb{R} \mid a_i^T x + b_i y \leq c_i, i = 1, \dots, m\}$$

The projection onto the x-space is given by

$$\text{proj}_x(\mathcal{P}) = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R} \text{ s.t. } a_i^T x + b_i y \leq c_i, i = 1, \dots, m\}$$

The projected polyhedra can be obtained through for example Fourier-Motzkin elimination, as shown in Algorithm 2

Algorithm 2 Computation of projected polyhedron $\text{proj}_x(\mathcal{P})$ (Fourier-Motzkin)

```
proj_x(P) ← ∅
U ← ∅
L ← ∅
for i ∈ I = 1, 2, ..., m do
    if b_i > 0 then
        U_i(x) ← (1/b_i)(c_i − a_i^T x)
        AddUpper(U, U_i(x))
    else if b_i < 0 then
        L_i(x) ← (1/b_i)(c_i − a_i^T x)
        AddLower(L, L_i(x))
    else if b_i = 0 then
        AddHalfspace(proj_x(P), a_i^T x ≤ c_i)
    end if
end for
for L_i(x) ∈ L do
    for U_j(x) ∈ U do
        AddHalfspace(proj_x(P), L_i(x) − U_j(x) ≤ 0)
    end for
end for
```

Computational tools based on these techniques are available in many high-level languages. To name a few:

- `MPT3` in Matlab
- `Polyhedra.jl` in Julia

2 Autonomous Linear Systems

We begin by considering autonomous linear systems under state constraints, i.e.

$$x_{t+1} = Ax_t, \quad x_t \in \mathcal{X}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is some polyhedral set.

Definition 4. Given a set $\mathcal{S} \subset \mathbb{R}^n$, the *predecessor set* $\operatorname{Pre}(\mathcal{S})$ under $x_{t+1} = Ax_t$ is given by

$$\operatorname{Pre}(\mathcal{S}) = \{x \in \mathbb{R}^n \mid Ax \in \mathcal{S}\}$$

In other words, if $x_t \in \operatorname{Pre}(\mathcal{S})$ then $x_{t+1} \in \mathcal{S}$. In other words, $\operatorname{Pre}(\mathcal{S})$ is the set of states which evolve in the set $\mathcal{S}$ in one time step. It is readily verified that $\operatorname{Pre}(\mathcal{S}) = \mathcal{S} \circ A$, as shown in the following example.

Example 2.1. Consider the following linear system

$$x_{t+1} = \begin{pmatrix} 0.5 & 0 \\ 1 & -0.5 \end{pmatrix} x_t \tag{1}$$

under the state constraints

$$x_t \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq x \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \quad \forall t \geq 0$$

Represent $\mathcal{X}$ as an $\mathcal{H}$-polytope:

$$\mathcal{X} = \{x \in \mathbb{R}^2 \mid Hx \leq h\}$$

by introducing

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$$

Now, it holds that

$$\begin{aligned} \operatorname{Pre}(\mathcal{X}) &= \{x_t \in \mathbb{R}^2 \mid x_{t+1} \in \mathcal{X}\} \\ &= \{x_t \in \mathbb{R}^2 \mid Hx_{t+1} \leq h\} \\ &= \{x \in \mathbb{R}^2 \mid HAx \leq h\} \end{aligned}$$

and

$$HA = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \\ -0.5 & 0 \\ -1 & 0.5 \end{bmatrix}$$

The resulting set is shown together with $\mathcal{X}$ in Figure 1.

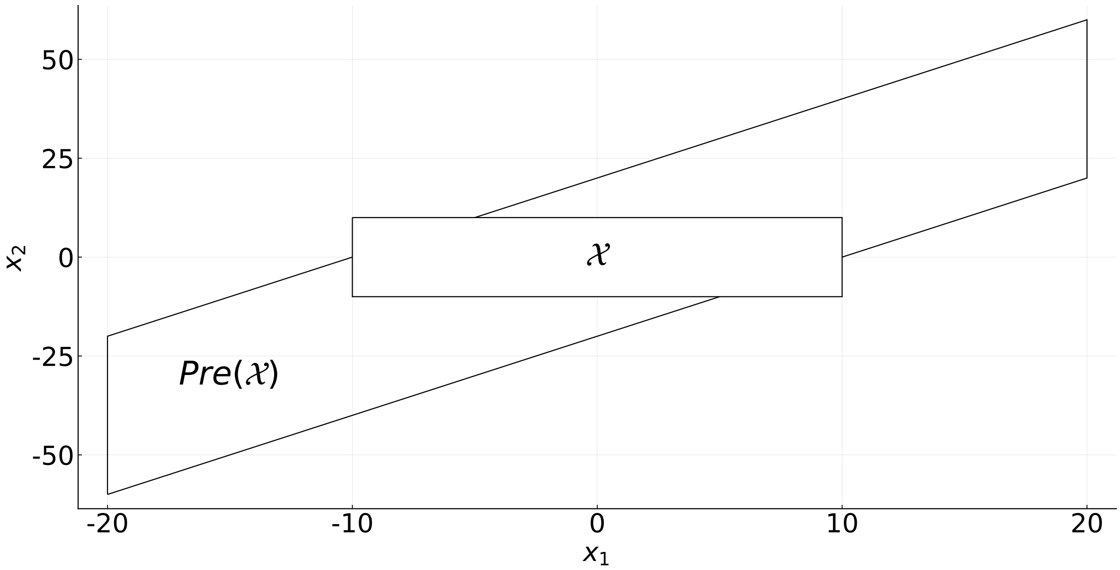

Figure 1: $\mathcal{X}$ and $\mathrm{Pre}(\mathcal{X})$ under the linear dynamics (1).

Notably, some states in $\mathcal{X}$ will evolve outside $\mathcal{X}$ in one time step.

Likewise, we can define the set of states that S can evolve to in one time step.

Definition 5. Given a set $\mathcal{S} \subset \mathbb{R}^n$, the *reachable set* $\mathrm{Reach}(\mathcal{S})$ under $x_{t+1} = Ax_t$ is given by

$$\mathrm{Reach}(\mathcal{S}) = \{x \in \mathbb{R}^n \mid \exists x_0 \in \mathcal{S} \text{ s.t. } x = Ax_0\}$$

In other words, if $x_t \in \mathcal{S}$ then $x_{t+1} \in \mathrm{Reach}(\mathcal{S})$. Note, that $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Reach}(\mathcal{S})$ are always defined in terms of some given dynamics.

We can now introduce the concept of invariance.

Definition 6. A set $\mathcal{O} \subseteq \mathcal{X}$ is positive invariant under $x_{t+1} = Ax_t$ if

$$x_t \in \mathcal{O} \Rightarrow x_k \in \mathcal{O} \quad \forall k \geq t$$

In other words, if the state x_t of the system enters $\mathcal{O}$ at time t, it will remain in $\mathcal{O}$ for all future time steps. This concept has imporant consequences for control design. For example, if the system state enters some positive invariant set $\mathcal{O} \subseteq \mathcal{X}$, then the state constraints $x_t \in \mathcal{X}$ will never be violated.

Example 2.2. Consider again the linear system

$$x_{t+1} = \begin{pmatrix} 0.5 & 0 \\ 1 & -0.5 \end{pmatrix} x_t$$

and the constraint set $\mathcal{X} = \{x \in \mathbb{R}^2 \mid x_2 - x_1 = 0\}$. Is $\mathcal{X}$ positive invariant?

Assume $x_t \in \mathcal{X}$. It follows that

$$x_2(t) - x_1(t) = 0$$

Now,

$$\begin{aligned} x_2(t+1) - x_1(t+1) &= x_1(t) - 0.5x_2(t) - 0.5x_1(t) \\ &= -0.5(x_2(t) - x_1(t)) \\ &= 0 \end{aligned}$$

Hence, it follows by induction that $x_t \in \mathcal{X} \Rightarrow x_{t+1} \in \mathcal{X}$. In conclusion, $\mathcal{X}$ is positive invariant under $x_{t+1} \in Ax_t$.

Now, we establish and prove a geometric condition for invariance.

Theorem 2.1. *A set $\mathcal{O} \subseteq \mathcal{X}$ is positive invariant under $x_{t+1} = Ax_t$ if and only if*

$$\mathcal{O} \subseteq \operatorname{Pre}(\mathcal{O})$$

Proof. First, assume that $\mathcal{O} \nsubseteq \operatorname{Pre}(\mathcal{O})$. It follows that $\exists \bar{x} \in \mathcal{O}$ such that $\bar{x} \notin \operatorname{Pre}(\mathcal{O})$. Hence, it holds that $A\bar{x} \notin \mathcal{O}$ so that $\mathcal{O}$ is not positive invariant. This proves necessity. Next, assume that $\mathcal{O}$ is not positive invariant, so that $\exists \bar{x} \in \mathcal{O}$ such that $A\bar{x} \notin \mathcal{O}$. It follows that $\bar{x} \notin \operatorname{Pre}(\mathcal{O})$. Hence, $\mathcal{O} \nsubseteq \operatorname{Pre}(\mathcal{O})$. This proves sufficiency. ■

Example 2.3. Consider again Example 2.1. It is readily verified in Figure 1 that $\mathcal{X} \nsubseteq \operatorname{Pre}(\mathcal{X})$. Hence, $\mathcal{X}$ is not positive invariant under the given dynamics. For example, the state $x_0 = \begin{pmatrix} 10 & -10 \end{pmatrix}$ evolves into $x_1 = \begin{pmatrix} 5 & 15 \end{pmatrix} \notin \mathcal{X}$.

An equivalent condition that we will see is more practical for computations follows immediately.

Corollary 2.2. *A set $\mathcal{O} \subseteq \mathcal{X}$ is positive invariant under $x_{t+1} = Ax_t$ if and only if*

$$\operatorname{Pre}(\mathcal{O}) \cap \mathcal{O} = \mathcal{O}$$

Now, there may exist multiple positive invariant subsets of $\mathcal{X}$. Hence, it is not immediately clear if the system will remain in $\mathcal{X}$ for a given starting state $x_0 \in \mathcal{X}$. This issue is alleviated by the following definition.

Definition 7. $\mathcal{O}_\infty(\mathcal{X}) \subseteq \mathcal{X}$ is the *maximal positive invariant set* under $x_{t+1} = Ax_t$ if

- $\mathcal{O}_\infty(\mathcal{X})$ is positive invariant under $x_{t+1} = Ax_t$

- $\mathcal{O} \subseteq \mathcal{X}$ positive invariant under $x_{t+1} = Ax_t \Rightarrow \mathcal{O} \subseteq \mathcal{O}_\infty(\mathcal{X})$

Hence, a system state trajectory x_t is only guaranteed to remain in the constraint set $\mathcal{X}$ if $x_0 \in \mathcal{O}_\infty(\mathcal{X})$. Given some $\mathcal{X}$, Corollary 3.5 can be used to formulate an efficient algorithm for computing the corresponding maximal positive invariant set $\mathcal{O}_\infty(\mathcal{X})$.

Algorithm 3 Computation of $\mathcal{O}_\infty(\mathcal{X})$

```
Ω_0 ← 𝒳
for j = 1, 2, … do
    Ω_j ← Pre(Ω_{j−1}) ∩ Ω_{j−1}
    if Ω_j = Ω_{j−1} then
        Ω_∞(𝒳) ← Ω_j
        return Ω_∞(𝒳)
    end if
end for
```

If Algorithm 3 terminates at some iteration k, then Corollary 3.5 implies that $\Omega_k = \mathcal{O}_\infty(\mathcal{X})$. Note, that if $\Omega_k = \emptyset$ at any iteration k, then it follows that $\mathcal{O}_\infty(\mathcal{X}) = \emptyset$.

Example 2.4. Consider again Example 2.1. It turns out that $\mathcal{O}_\infty(\mathcal{X})$ is obtained after a single step of Algorithm 3. In other words, $\mathrm{Pre}(\mathcal{X}) \cap \mathcal{X}$ is the largest positive invariant set contained in $\mathcal{X}$. $\mathcal{O}_\infty(\mathcal{X})$ is shown in Figure 2.

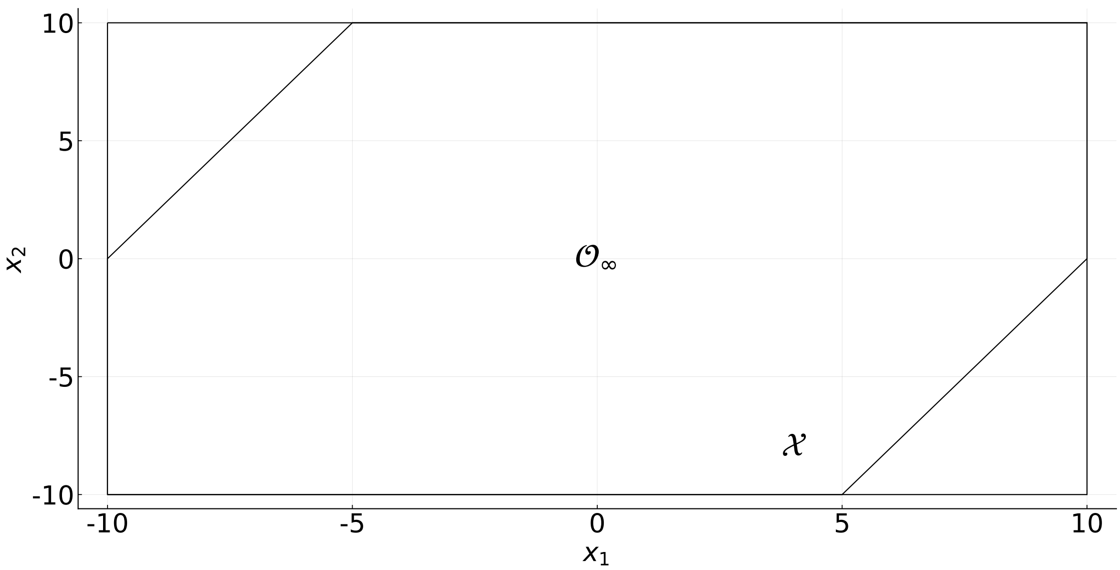

Figure 2: $\mathcal{X}$ and $\mathcal{O}_\infty(\mathcal{X})$ under the linear dynamics (1).

3 Controlled Linear Systems

Now, we consider systems subjected to external inputs, under both state and input constraints, i.e.

$$x_{t+1} = Ax_t + Bu_t, \quad x_t \in \mathcal{X}, \ u_t \in \mathcal{U}$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{U} \subseteq \mathbb{R}^m$ are polyhedral sets. We can now define useful sets analogous to the autonomous case.

Definition 8. Given a set $\mathcal{S} \subset \mathbb{R}^n$, the *one-step controllable set* $\mathrm{C}(\mathcal{S}; \mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t$ is given by

$$\mathrm{C}(\mathcal{S}; \mathcal{U}) = \{x \in \mathbb{R}^n \mid \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu \in \mathcal{S}\}$$

Example 3.1. Consider the following linear system

$$x_{t+1} = \begin{pmatrix} 1.5 & 0 \\ 1 & -1.5 \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t \tag{2}$$

subject to the input and state constraints

$$\begin{aligned} x_t \in \mathcal{X} &= \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \quad \forall t \ge 0 \\ u_t \in \mathcal{U} &= \{u \in \mathbb{R} \mid -5 \le u \le 5\}, \quad \forall t \ge 0 \end{aligned}$$

Represent $\mathcal{X}$ and $\mathcal{U}$ as $\mathcal{H}$-polytopes:

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^2 \mid Hx \le h\} = \left\{ x \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \le \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\} \\ \mathcal{U} &= \{u \in \mathbb{R} \mid H_u u \le h_u\} = \left\{ u \in \mathbb{R} \;\middle|\; \begin{pmatrix} 1 \\ -1 \end{pmatrix} u \le \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\} \end{aligned}$$

Now, it holds that

$$\begin{aligned} \mathrm{C}(\mathcal{X}; \mathcal{U}) &= \{x_t \in \mathbb{R}^2 \mid \exists u_t \in \mathcal{U} \text{ s.t. } x_{t+1} \in \mathcal{X}\} \\ &= \{x_t \in \mathbb{R}^2 \mid \exists u_t \in \mathcal{U} \text{ s.t. } Hx_{t+1} \le h\} \\ &= \left\{ x \in \mathbb{R}^2, u \in \mathbb{R} \;\middle|\; \begin{pmatrix} HA & HB \\ 0 & H_u \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \le \begin{pmatrix} h \\ h_u \end{pmatrix} \right\} \end{aligned}$$

Through Fourier-Motzkin elimination, u can be eliminated to derive half-spaces which define $\mathrm{C}(\mathcal{X};\mathcal{U})$ in the state-space.

$$\mathrm{C}(\mathcal{X};\mathcal{U}) = \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 0 \\ 1 & -1.5 \\ 1 & 0 \\ 1 & -1.5 \end{pmatrix} x \leq \begin{pmatrix} -10 \\ 10 \\ 10 \\ -10 \end{pmatrix} \right\}$$

The constrained state space $\mathcal{X}$ and the one-step controllable set $\mathrm{C}(\mathcal{X};\mathcal{U})$ are shown in Figure 3.

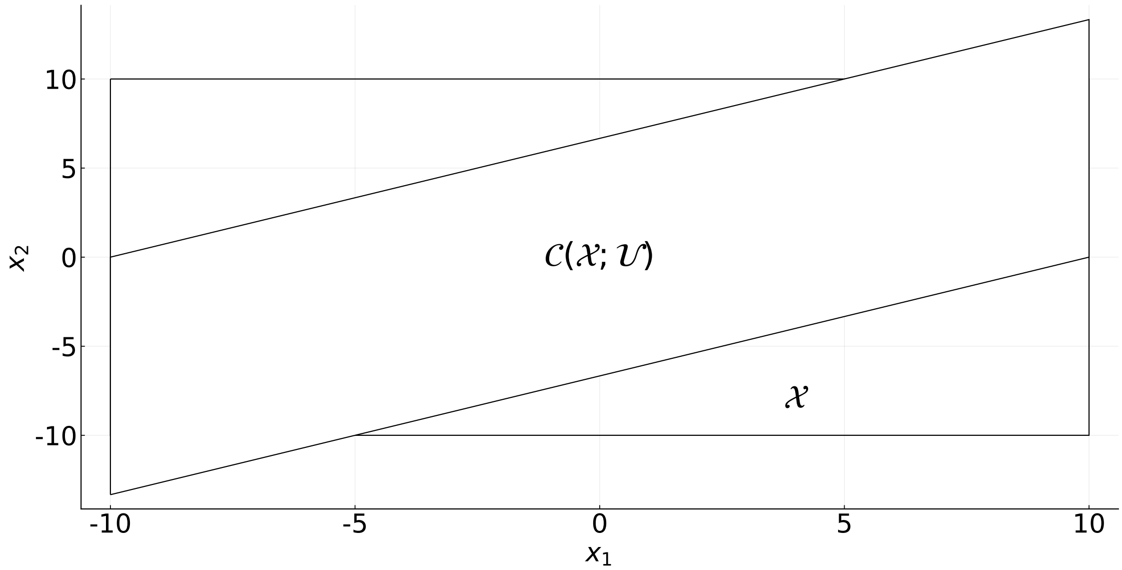

Figure 3: $\mathcal{X}$ and $\mathrm{C}(\mathcal{X};\mathcal{U})$ under the linear dynamics (3)

Notably, some states in $\mathcal{X}$ will evolve outside $\mathcal{X}$ in one time step, irregardless of any admissible control input applied to the system.

Again, we can define the analogous set of states that $\mathcal{S}$ can be driven to in one time step.

Definition 9. Given a set $\mathcal{S} \subset \mathbb{R}^n$, the *one-step reachable set* $\mathrm{R}(\mathcal{S};\mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t$ is given by

$$\mathrm{R}(\mathcal{S};\mathcal{U}) = \{x \in \mathbb{R}^n \mid \exists x_0 \in \mathcal{S}, u_0 \in \mathcal{U} \text{ s.t. } x = Ax_0 + Bu_0\}$$

Next, the concept of control invariance is introduced.

Definition 10. A set $\mathcal{C} \subseteq \mathcal{X}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$ if

$$x_t \in \mathcal{C} \Rightarrow \exists \{u_k\}_{k=t}^{\infty} \in \mathcal{U} \text{ s.t. } x_k \in \mathcal{C},\ \forall k \geq t$$

In other words, if the system state enters $\mathcal{C}$ at time t, it can be kept in $\mathcal{C}$ through some admissible control sequence $\{u_k\}_{k=t}^{\infty} \in \mathcal{U}$. Now, lets derive a geometric condition for control invariance, as in the autonomous case.

Theorem 3.1. *A set $\mathcal{C} \subseteq \mathcal{X}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$ if and only if*

$$\mathcal{C} \subseteq \mathrm{C}(\mathcal{C}; \mathcal{U})$$

Proof. First, assume that $\mathcal{C} \nsubseteq \mathrm{C}(\mathcal{C}; \mathcal{U})$. It follows that $\exists \bar{x} \in \mathcal{C}$ such that $\bar{x} \notin \mathrm{C}(\mathcal{C}; \mathcal{U})$. Therefore, there is no $u \in \mathcal{U}$ such that $A\bar{x} + Bu \in \mathcal{C}$. Hence, $\mathcal{C}$ is not controlled invariant. This proves necessity. Next, assume that $\mathcal{C}$ is not controlled invariant, so that there exists some state trajectory $\{x_t\}_{t=0}^{\infty}$ where for some t it holds that $x_t \in \mathcal{C}$ but $x_{t+1} \notin \mathcal{C}$. In other words, $Ax_t + Bu \notin \mathcal{C}$ for any $u \in \mathcal{U}$. It follows that $x_t \notin \mathrm{C}(\mathcal{C}; \mathcal{U})$. Hence, $\mathcal{C} \nsubseteq \mathrm{C}(\mathcal{C}; \mathcal{U})$. This proves sufficiency. ∎

Example 3.2. Consider again Example 3.1. It is readily verified in Figure 3 that $\mathcal{X} \nsubseteq \mathrm{C}(\mathcal{X}; \mathcal{U})$. Hence, $\mathcal{X}$ is not control invariant under the given dynamics. For instance, the state $x = \begin{pmatrix} 5 & -10 \end{pmatrix}$ will have $x_2 = 20$ in the next time step irregardless of any feasible control input u.

An interesting implication of control invariance is given in the next result.

Theorem 3.2. *$\mathcal{C} \subseteq \mathcal{X}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$, if and only if there exists an $m \times n$ matrix F such that $\mathcal{C}$ is positive invariant under $x_{t+1} = (A + BF)x_t$ and*

$$\mathcal{C} \subseteq \mathcal{U} \circ F$$

Proof. (The following proof is only valid when $\mathcal{X}$ and $\mathcal{U}$ are polytopes. The extension to polyhedrons is trivial, but left out for brevity.)

Let $c_1, \ldots, c_r$ denote the vertex points of $\mathcal{C}$ so that every $x \in \mathcal{C}$ can be represented by a convex combination $x = \sum_{i=1}^{r} \alpha_i c_i,\ \sum_{i=1}^{r} \alpha_i = 1$. Now, assume that $\mathcal{C}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$. It follows that for every c_i, there exists a $u_i \in \mathcal{U}$ such that

$$Ac_i + Bu_i \in \mathcal{C}$$

Introduce an $m \times n$ matrix F such that

$$Fc_i = u_i, \quad i = 1, \ldots, r$$

This is possible since $r \leq n$, but F is not unique if $r < n$. Now, for any $x \in \mathcal{C}$, it holds that

$$\begin{aligned}(A+BF)x &= (A+BF)\sum_{i=1}^{r}\alpha_i c_i \\ &= \sum_{i=1}^{r}\alpha_i(Ac_i + B(Fc_i)) \\ &= \sum_{i=1}^{r}\alpha_i(Ac_i + Bu_i) \in \mathcal{C}\end{aligned}$$

since $Ac_i + Bu_i \in \mathcal{C},\ i = 1, \dots, r$ by assumption and polytopes are closed under convex combinations. Hence, $x \in \mathrm{Pre}\,(\mathcal{C})$ under $x_{t+1} = (A+BF)x_t$ so that $\mathcal{C} \subseteq \mathrm{Pre}\,(\mathcal{C})$. By Theorem 2.1, $\mathcal{C}$ is positive invariant under $x_{t+1} = (A+BF)x_t$. Moreover, for any $x \in \mathcal{C}$ it holds that

$$\begin{aligned}Fx &= F\sum_{i=1}^{r}\alpha_i c_i \\ &= \sum_{i=1}^{r}\alpha_i u_i \in \mathcal{U}\end{aligned}$$

since $u_i \in \mathcal{U},\ i = 1, \dots, r$ and polytopes are closed under convex combinations. Hence,

$$\mathcal{C} \subseteq \mathcal{U} \circ F$$

This proves necessity. Next, assume that there exists an $m \times n$ matrix F such that $\mathcal{C}$ is positive invariant under $x_{t+1} = (A+BF)x_t$ and $\mathcal{C} \subseteq \mathcal{U} \circ F$. It follows that for all $x \in \mathcal{C}$ it holds that

$$Ax + BFx \in \mathcal{C}$$

Moreoever, $u := Fx \in \mathcal{U}$. Hence, $x \in \mathrm{C}\,(\mathcal{C}; \mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$ so that $\mathcal{C} \subseteq \mathrm{C}\,(\mathcal{C}; \mathcal{U})$. By Theorem 3.1, $\mathcal{C}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$. This proves sufficiency. ■

For the special case when $\mathcal{U} = \mathbb{R}^m$, the above result simplifies to the following corollary.

Corollary 3.3. *If $\mathcal{U} = \mathbb{R}^m$, then $\mathcal{C}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t$ if and only if there exists an $m \times n$ matrix F such that $\mathcal{C}$ is positive invariant under $x_{t+1} = (A+BF)x_t$*

The following alternative result can sometimes be more useful.

Theorem 3.4. *$\mathcal{C}$ is controlled invariant under $x_{t+1} = Ax_t + Bu_t$ if and only if*

$$\text{Reach}\,(\mathcal{C}) \subseteq \mathcal{C} + B\circ\mathcal{U}$$

where Reach $(\mathcal{C})$ *is under $x_{t+1} = Ax_t$.*

Proof. (Again, the proof is only valid when $\mathcal{X}$ and $\mathcal{U}$ are polytopes.) Assume $\mathcal{C}$ is controlled invariant. Take any $x \in \mathcal{C}$ and consider $y = Ax + Bu$. As $\mathcal{C}$ is controlled invariant there exists some $u \in \mathcal{U}$ so that $y \in \mathcal{C}$. It follows that

$$Ax = y - Bu$$

Now, $Ax \in \text{Reach}\,(\mathcal{C})$ under $x_{t+1} = Ax_t$, $y \in \mathcal{C}$ and $Bu \in B\circ\mathcal{U}$ (since $u \in \mathcal{U}$). As x was chosen arbitrarily from $\mathcal{C}$, it follows that Reach $(\mathcal{C}) \subseteq \mathcal{C} + B\circ\mathcal{U}$. This proves necessity. Next, assume that Reach $(\mathcal{C}) \subseteq \mathcal{C} + B\circ\mathcal{U}$ and let $c_1, \ldots, c_r$ be the vertex points for $\mathcal{C}$. It follows that

$$Ac_i = \tilde{c}_i + Bu_i, \quad i = 1, \ldots, r$$

for some $\tilde{c}_i \in \mathcal{C}$ and $u_i \in \mathcal{U}$. Now, introduce an $m \times n$ matrix F for which

$$Fc_i = -u_i, \quad i = 1, \ldots, r$$

(Note, that F is not unique if $r < n$.) It follows that

$$\begin{aligned} &Ac_i = \tilde{c}_i - BFc_i, \quad i = 1, \ldots, r \\ \Leftrightarrow\ &(A + BF)c_i = \tilde{c}_i \in \mathcal{C}, \quad i = 1, \ldots, r \\ \Rightarrow\ &\mathcal{C} \subseteq \text{Pre}\,(\mathcal{C}) \end{aligned}$$

under $x_{t+1} = (A + BF)x_t$. Moreover, for any $x \in \mathcal{C}$ it holds that

$$\begin{aligned} Fx &= F \sum_{i=1}^{r} \alpha_i c_i \\ &= \sum_{i=1}^{r} \alpha_i u_i \in \mathcal{U} \end{aligned}$$

since $u_i \in \mathcal{U},\ i = 1, \ldots, r$ and polytopes are closed under convex combinations. Hence,

$$\mathcal{C} \subseteq \mathcal{U}\circ F$$

By Theorem 3.2, $\mathcal{C}$ is controlled invariant. This proves sufficiency. ∎

For the special case when $\mathcal{U} = \mathbb{R}^m$, the following corollary is more useful to work with.

Corollary 3.5. *If $\mathcal{U} = \mathbb{R}^m$, then C is controlled invariant under $x_{t+1} = Ax_t + Bu_t$ if and only if*

$$\operatorname{Reach}(C) \subseteq C + \operatorname{Im} B$$

where $\operatorname{Reach}(C)$ *is under $x_{t+1} = Ax_t$.*

Example 3.3. Consider the linear system

$$x_{t+1} = \begin{pmatrix} 0.5 & 0 \\ 1 & -0.5 \end{pmatrix} x_t + \begin{bmatrix} 1 \\ -5 \end{bmatrix} u_t := Ax_t + Bu_t$$

and the constraint set $\mathcal{X} = \{x \in \mathbb{R}^2 \mid x_1 + x_2 = 0\}$. It is readily verified that $\mathcal{X}$ is not positive invariant under $x_{t+1} = Ax_t$. Is $\mathcal{X}$ controlled invariant?

First, note that $\mathcal{X}$ is spanned by $v = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$, and consider

$$Av = \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ -5 \end{bmatrix} \in \mathcal{X} + \operatorname{Im} B$$

Hence, by Theorem 3.4, $\mathcal{X}$ is controlled invariant. A feedback matrix F can be determined from $Fv = -u$. For example, take $F = \begin{bmatrix} 0.5 & 0 \end{bmatrix}$. It follows that any $x_0 \in \mathcal{X}$ will remain in $\mathcal{X}$ under the feedback law $u = Fx$.

The analogue to the maximal positive invariant set is another useful construct.

Definition 11. $C_\infty(\mathcal{X}; \mathcal{U}) \subseteq \mathcal{X}$ is the *maximal control invariant set* under $x_{t+1} = Ax_t + Bu_t$ if

- $C_\infty(\mathcal{X}; \mathcal{U})$ is control invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$.
- $C \subseteq \mathcal{X}$ control invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U} \Rightarrow C \subseteq C_\infty(\mathcal{X}; \mathcal{U})$.

An immediate consequence is that $\forall x_0 \in C_\infty(\mathcal{X}; \mathcal{U})$, $\exists \{u_t\}_{t=0}^{\infty} \in \mathcal{U}$ such that $x_t \in \mathcal{X}$ for all $t \geq 0$. In other words, $C_\infty(\mathcal{X}; \mathcal{U})$ is the set of starting states where the state trajectories can be kept in $\mathcal{X}$ through admissible control sequences. An algorithm for computing $C_\infty(\mathcal{X}; \mathcal{U})$ based on Theorem 3.1 is given in Algorithm 4.

Algorithm 4 Computation of $C_\infty(\mathcal{X}; \mathcal{U})$

```
Ω_0 ← X
for j = 1, 2, … do
    Ω_j ← C(Ω_{j−1}; U) ∩ Ω_{j−1}
    if Ω_j = Ω_{j−1} then
        C_∞(X; U) ← Ω_j
        return C_∞(X; U)
    end if
end for
```

Example 3.4. Consider again Example 3.1. $C_\infty(\mathcal{X};\mathcal{U})$ is obtained after 33 steps of Algorithm 4. C_∞ is shown in Figure 4.

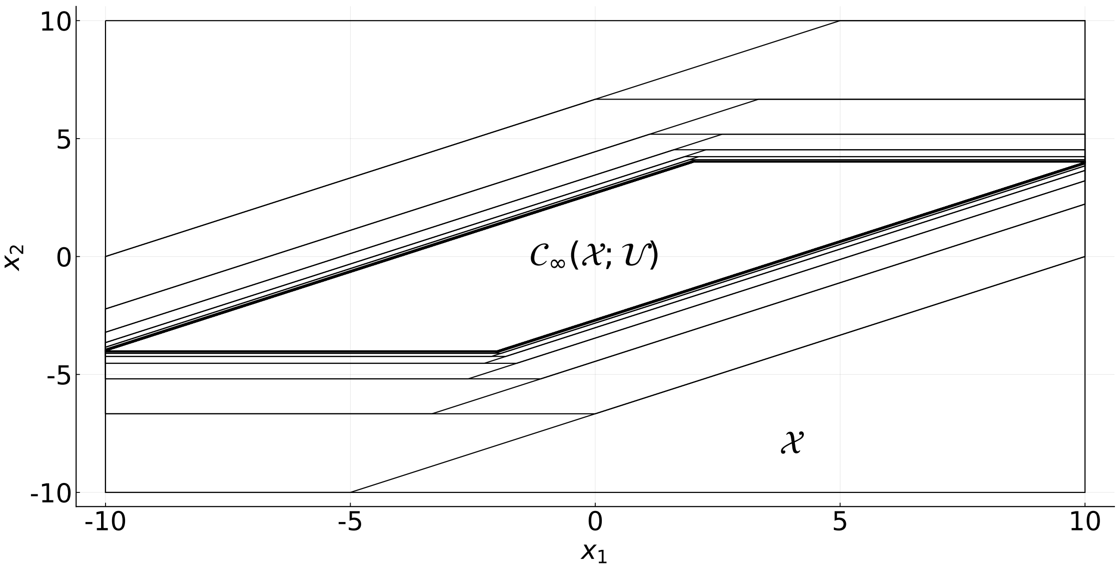

Figure 4: $\mathcal{X}$ and $C_\infty(\mathcal{X};\mathcal{U})$ under the linear dynamics (3).

So far, the main concern has been maintaining the state constraints. However, we are typically concerned with driving the system to some target set. Some more definitions are required to extend the framework to these types of problems.

Definition 12. For a given target set $\mathcal{T} \subseteq \mathcal{X}$, the *N-step controllable set* $\mathcal{K}_N(\mathcal{T};\mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$ is defined recursively as

$$\mathcal{K}_j(\mathcal{T};\mathcal{U}) = \mathrm{C}\,(\mathcal{K}_{j-1}(\mathcal{T});\mathcal{U}) \cap \mathcal{X}, \quad j = 1,\dots,N$$
$$\mathcal{K}_0(\mathcal{T};\mathcal{U}) = \mathcal{T}$$

where $\mathrm{C}\,(\mathcal{K};\mathcal{U})$ is under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$.

In other words, if $x_0 \in \mathcal{K}_N(\mathcal{T};\mathcal{U})$ then the system can be driven from x_0 to $\mathcal{T}$ in N steps. Typically, we want the target set to be controlled invariant so that the state can be kept there. The following definition is stated in terms of control invariant sets, and removes the dependency on N.

Definition 13. For a given control invariant set $\mathcal{C} \subseteq \mathcal{X}$, the *maximal stabilizable set* $\mathcal{K}_\infty(\mathcal{C};\mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$ is defined by

- $\mathcal{K}_N(\mathcal{C};\mathcal{U}) \subseteq \mathcal{K}_\infty(\mathcal{C}), \quad \forall N \in \mathbb{N}.$
- $\mathcal{K}_\infty(\mathcal{C};\mathcal{U})$ is control invariant under $x_{t+1} = Ax_t + Bu_t,\ u_t \in \mathcal{U}$.

In other words, if $x_0 \in \mathcal{K}_\infty(\mathcal{C};\mathcal{U})$, then the system can be driven from x_0 to $\mathcal{C}$, and be kept there, by an admissible control sequence. For any control invariant set $\mathcal{C} \subseteq \mathcal{X}$, it holds that $\mathcal{K}_\infty(\mathcal{C};\mathcal{U}) \subseteq \mathcal{C}_\infty(\mathcal{X};\mathcal{U})$. Moreover, $\mathcal{C}_\infty(\mathcal{X};\mathcal{U}) \setminus \mathcal{K}_\infty(\mathcal{C};\mathcal{U})$ includes all initial states from which it is not possible to steer the system to the stabilizable region $\mathcal{K}_\infty(\mathcal{C};\mathcal{U})$, and thereby to the target set $\mathcal{C}$. Trajectories that begin in $\mathcal{C}_\infty(\mathcal{X};\mathcal{U})$ can be kept in $\mathcal{X}$, but it is not necessarily possible to drive the state to some given target inside $\mathcal{X}$. An algorithm for computing $\mathcal{K}_\infty(\mathcal{C};\mathcal{U})$ is given by Algorithm 5.

Algorithm 5 Computation of $\mathcal{K}_\infty(\mathcal{T};\mathcal{U})$, where $\mathcal{T}$ is some given controlled invariant target set.

```
K_0 ← T
for j = 1, 2, ... do
    K_j ← C(K_{j-1}; U) ∩ X
    if K_j = K_{j-1} then
        K_∞(T; U) ← K_j
        return K_∞(T; U)
    end if
end for
```

Note, that Algorithm 5 does not necessarily terminate in finite time. Now, assume that we have designed a terminal set $\mathcal{X}_T$ in an MPC formulation. It follows that $\mathcal{K}_\infty(\mathcal{X}_T;\mathcal{U})$ is the set of initial states that will be admissible for the MPC controller.

Example 3.5. Consider again the linear system

$$x_{t+1} = \begin{pmatrix} 1.5 & 0 \\ 1 & -1.5 \end{pmatrix} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t \tag{3}$$

subject to the input and state constraints

$$\begin{aligned} x_t \in \mathcal{X} &= \left\{ x \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x \le \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \quad \forall t \ge 0 \\ u_t \in \mathcal{U} &= \{ u \in \mathbb{R} \mid -5 \le u \le 5 \}, \quad \forall t \ge 0 \end{aligned}$$

The aim is to determine optimal input sequences. To that end, consider the following constrained infinite-horizon optimal control formulation

$$\begin{aligned}
\underset{\{u_k\}_{k=0}^{\infty}}{\text{minimize}} \quad & \sum_{k=0}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k \\
\text{s.t.} \quad & x_{k+1} = \begin{pmatrix} 1.5 & 0 \\ 1 & -1.5 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, \quad k = 0, \dots, \infty \\
& x_k \in \mathcal{X}, \quad k = 0, \dots, \infty \\
& u_k \in \mathcal{U}, \quad k = 0, \dots, \infty \\
& x_0 = x
\end{aligned} \tag{4}$$

where the penalty matrices are given by

$$Q_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1.$$

As (4) is an infinite dimensional quadratic program, there is no immediate solution. We will approach this problem in several steps. First, consider the unconstrained variant

$$\begin{aligned}
\underset{\{u_k\}_{k=0}^{\infty}}{\text{minimize}} \quad & \sum_{k=0}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k \\
\text{s.t.} \quad & x_{k+1} = \begin{pmatrix} 1.5 & 0 \\ 1 & -1.5 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k \\
& x_0 = x
\end{aligned}$$

for which there is a closed form solution given by the discrete-time linear-quadratic regulator. The resulting discrete LQR controller is given by

$$u_t = -L_\infty x_t$$

where

$$L_\infty = \begin{pmatrix} 0.52 & 1.39 \end{pmatrix}$$

and the optimal cost is given by $x^T P x$, where

$$P = \begin{pmatrix} 29.94 & -28.74 \\ -28.74 & 47.24 \end{pmatrix}$$

It is not certain that the resulting input sequence is admissible for every starting point x_0. Hence, a good approach could be to characterize the state-space for which L_∞ feedback is optimal also for (4). Consider the following set

$$\mathcal{L} = (\mathcal{U} \circ L) \cap X = \{x \in \mathcal{X} \mid -5 \leq -L_\infty x \leq 5\}$$

In other words, $\mathcal{L}$ contains precisely those states in $\mathcal{X}$ that results in admissible inputs under the feedback L_∞. $\mathcal{L}$ is shown in Figure 5.

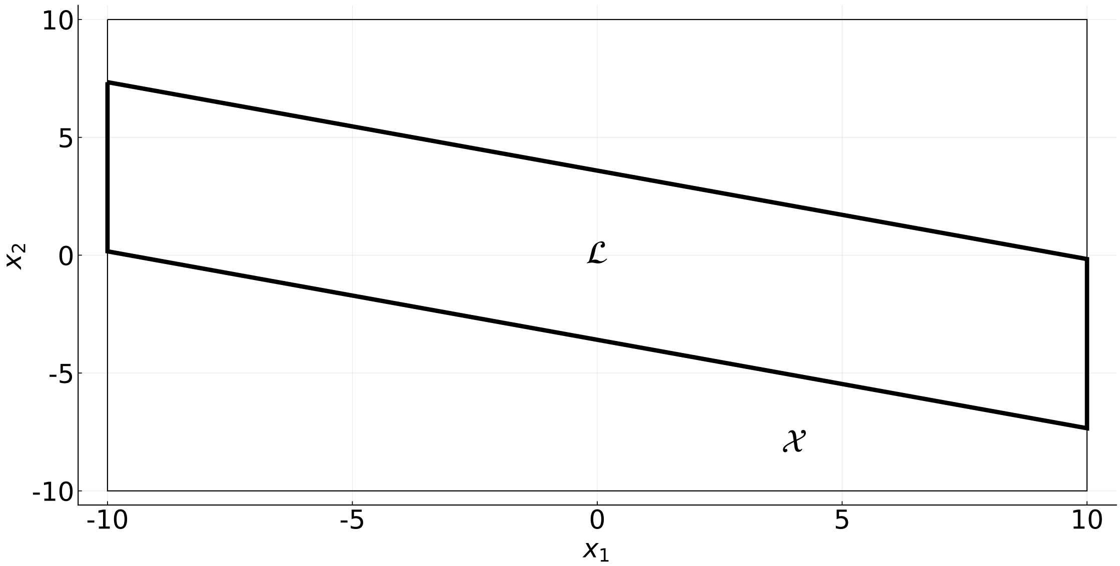

Figure 5: $\mathcal{L}$ and $\mathcal{X}$.

Hence, $\mathcal{L}$ could be a good candidate for a target set, as it would be possible to apply optimal feedback input to the system. However, it is not certain that the system will stay in $\mathcal{L}$ under the feedback L_∞. To that end, we compute $\mathcal{O}_\infty(\mathcal{L})$ under $x_{t+1} = (A - BL_\infty)x_t$ using Algorithm 3. The result is shown in Figure 6.

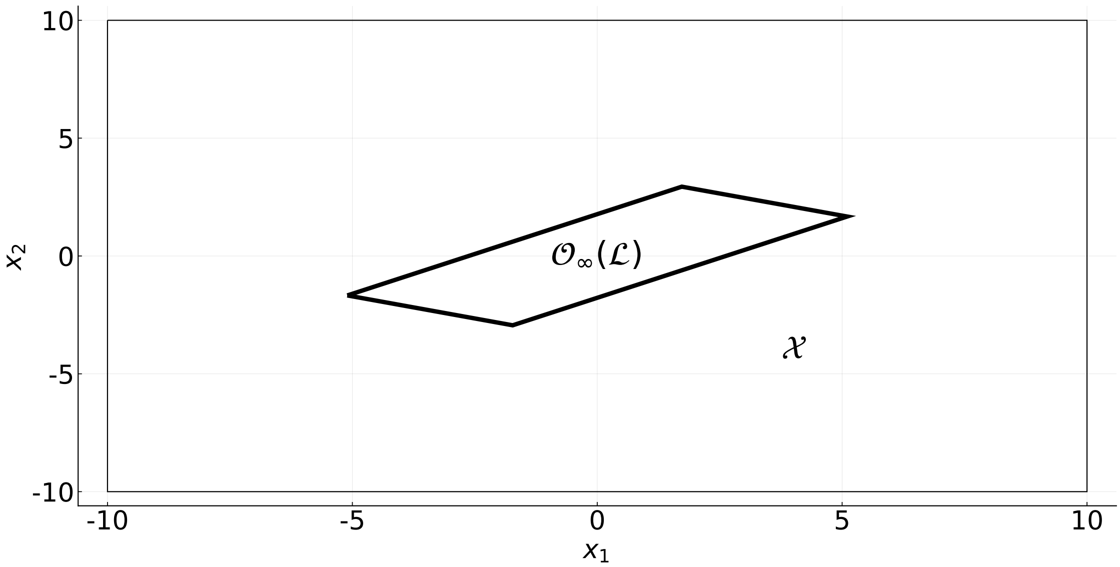

Figure 6: $\mathcal{O}_\infty(\mathcal{L})$ under $x_{t+1} = (A - BL_\infty)x_t$ and $\mathcal{X}$.

From the theory just presented, we know that if the system is driven to $\mathcal{O}_\infty(\mathcal{L})$ it can be kept there using the optimal feedback law $u_t = -L_\infty x_t$. In fact, it can be shown that the system will eventually be stabilized by this feedback. Hence, we introduce the terminal set

$$\mathcal{X}_f = \mathcal{O}_\infty(\mathcal{L})$$

Now, we reformulate (4) into a finite-horizon problem as follows

$$\begin{aligned}
\underset{u_0,\dots,u_{N-1}}{\text{minimize}} \quad & \sum_{k=0}^{N-1} x_k^T Q_1 x_k + u_k^T Q_2 u_k + x_N^T P x_N \\
\text{s.t.} \quad & x_{k+1} = \begin{pmatrix} 1.5 & 0 \\ 1 & -1.5 \end{pmatrix} x_k + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_k, \quad k = 0,\dots,N-1 \\
& x_k \in \mathcal{X}, \quad k = 0,\dots,N \\
& u_k \in \mathcal{U}, \quad k = 0,\dots,N-1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x
\end{aligned} \tag{5}$$

We have shown that $\mathcal{X}_f$ is input admissible and control invariant under $x_{t+1} = (A - BL_\infty)x_t$. Hence, it follows that

$$x_N^T P x_N = \sum_{k=N}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k$$

so that equivalence with (4) holds. The only remaining question is if (5) is feasible. Specifically, given a starting state $x_0 \in \mathcal{X}$, does there exists an admissible input sequence that steers the system to $\mathcal{X}_f$ without ever leaving $\mathcal{X}$? This notion is captured exactly by $\mathcal{K}_\infty(\mathcal{X}_f; \mathcal{U})$, which can be computed using Algorithm 5. It turns out that $K_\infty(\mathcal{X}; \mathcal{U}) = \mathcal{C}_\infty(\mathcal{X}; \mathcal{U})$. The target set is shown together with the set of feasible initial states in Figure 7.

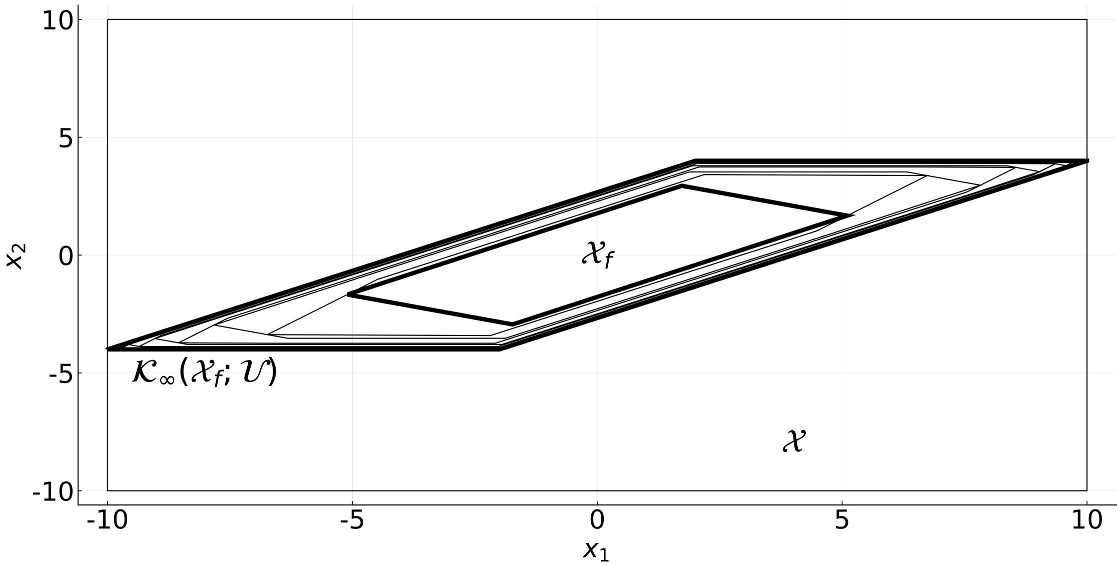

Figure 7: $\mathcal{X}_f$, $\mathcal{K}_\infty(\mathcal{X}_f; \mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t$ and $\mathcal{X}$.

It holds that (5) only has a feasible solution for $x_0 \in \mathcal{K}_\infty(\mathcal{X}; \mathcal{U})$. For instance, it is readily verified (using your favorite optimization modeling language) that (5) is infeasible if for instance $x_0 = \begin{pmatrix} 5 & -5 \end{pmatrix}$. In contrast, (5) is solvable if for instance $x_0 = \begin{pmatrix} -9 & -3.5 \end{pmatrix}$. A solution to (5) is shown for this particular x_0, and $N = 7$, in Figure 8.

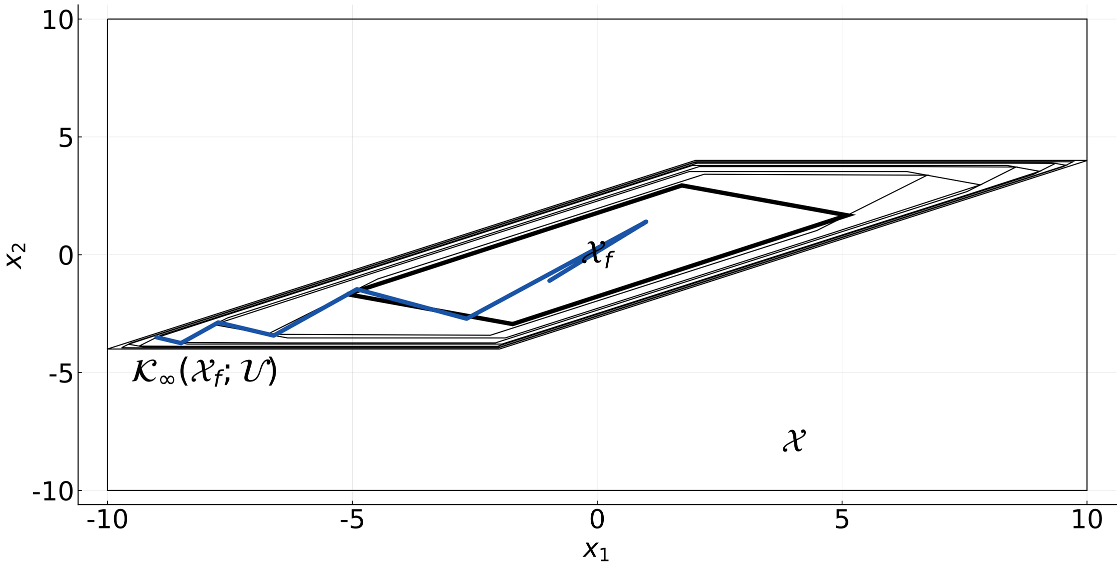

Figure 8: $\mathcal{X}_f$, $\mathcal{K}_\infty(\mathcal{X}_f; \mathcal{U})$ under $x_{t+1} = Ax_t + Bu_t$ and $\mathcal{X}$. Also, the optimal solution trajectory to (5) when $N = 7$ and $x_0 = (-9 - 3.5)$ is shown.