Offset-free tracking in

Model Predictive Control

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Preliminaries

When a system is subjected to disturbances of some sort, or when there is a mismatch between the MPC model and the actual system, then the nominal system model will be an incorrect representation of the dynamics. An improvement can be achieved by augmenting the system to include disturbance terms on the state and output

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_d d(t) \\ y(t) &= Cx(t) + C_d d(t), \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $B_d \in \mathbb{R}^{n\times n_d}$, $C_d \in \mathbb{R}^{p\times n_d}$. The augmented system is observable if (A, C) is observable and

$$\operatorname{rank}\begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix} = n + n_d, \tag{1}$$

where $n_d \leq p$. Let the augmented state be $\xi_t = \begin{bmatrix} x_t^T & d_t^T \end{bmatrix}^T \in \mathbb{R}^{n+n_d}$. The augmented observer is now given by

$$\begin{aligned} \hat{\xi}_t^{pred} &= \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \hat{\xi}_t t - 1 + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{t-1} = A_a \hat{\xi}_{t-1} + B_a u_{t-1} \\ \hat{\xi}_t &= \hat{\xi}_t^{pred} + \begin{bmatrix} K_x \\ K_d \end{bmatrix} \left(y_t - \begin{bmatrix} C & C_d \end{bmatrix} \hat{\xi}_t^{pred} \right) = \hat{\xi}_t^{pred} + K_a(y_t - C_a \hat{\xi}_t^{pred}) \end{aligned}$$

For offset-free reference tracking, we require that

$$\operatorname{rank}\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} = n + p \tag{2}$$

where $p \leq m$.

In summary, according to theorem 5.5.2 we have that if (A, B) is controllable, (C, A) is observable and (1) and (2) are fulfilled, then if the observer gains K_x and K_d are chosen such that the observer is asymptotically stable, the steady state output will go to the reference value.

Exercise 1)

Consider an augmented state/disturbance observer with $n_d = p$

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_d d(t) \\ y(t) &= Cx(t) + C_d d(t), \end{aligned}$$

with (A, B) controllable, (C, A) observable and

$$\operatorname{rank}\begin{bmatrix} A-I & B \\ C & 0 \end{bmatrix} = n + p.$$

(a) With K being a stable observer gain, show that if $(A - KCA)$ is Hurwitz, then

$$B_d = K, \quad C_d = I - CK$$

are valid choices for the disturbance matrices.

(b) With the above disturbance matrices, and with the augmented observer gain $K_x = K, K_d = I$, derive expressions for $\hat{x}_t$ and $\hat{d}_t$ where you show that both estimators are independent of $\hat{d}_{t-1}$.

(c) Define the vectors

$$\begin{aligned} d_t^x &= \hat{x}_t - (A\hat{x}_{t-1} + Bu_{t-1}) && \text{(=difference between \textit{estimated} and \textit{predicted} state)} \\ d_t^y &= y_t - C\hat{x}_t && \text{(=difference between \textit{measured} and \textit{estimated} output)} \end{aligned}$$

and show that these are the disturbances added to the system ($B_d d_t$ and $C_d d_t$). Finally, give an interpretation of the result.

Exercise 2)

This exercise is done in MATLAB. A MIMO system is described by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

with the following equations and matrices

$$A = \begin{bmatrix} 0.97 & 0 & 0 & 0 \\ 0 & 0.41 & 0 & 0 \\ 0 & 0 & 0.85 & 0 \\ 0 & 0 & 0 & 0.93 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 0 \\ 0 & 0.50 \\ 0 & 0.50 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.23 & 0.0 & 0.87 & 0.0 \\ 0.0 & 0.18 & 0.0 & 0.29 \end{bmatrix}$$

(a) Assuming that there is no disturbance acting on the system, design an observer for the system with poles in $\{0.50, 0.51, 0.52, 0.53\}$.

(b) We wish to add reference tracking to the system. Formulate an MPC with reference preview. Use the identity matrix as output weight ($Q = I$) and ignore the input weight ($R = 0$). The input is constrained by

$$\begin{aligned} -2 \le u_1(t+i) \le 2 \\ -2 \le u_2(t+i) \le 2 \end{aligned}$$

(c) To get smoother control inputs, add the control increment

$$\delta u = u_t - u_{t-1}.$$

to the cost function with weight $\tilde{R} = I \cdot 0.1$.

(d) Finally, we add disturbances acting directly on the system states. To still achieve offset-free tracking, we augment the system dynamics in the model predictive controller using the augmented system presented in the previous task. Formulate the MPC and the observer. Is the value of K computed in subtask (a) still a reasonable choice? Show in simulation that the augmented system achieves error-free tracking at steady state!

Solutions

Exercise 1)

(a) We need to show that

$$\begin{bmatrix} A-I & B_d \\ C & C_d \end{bmatrix} \tag{3}$$

has full rank. With the matrices inserted, the condition can equivalently be tested by considering the nullspace of the matrix

$$\begin{bmatrix} A-I & K \\ C & I-CK \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives the following two equations

$$(A-I)x + Ky = 0, \qquad Cx + (I-CK)y = 0$$

must be satisfied. Setting $Ky = (I-A)x$ in the second equation gives

$$(A-I)x + Ky = 0, \qquad y + CAx = 0$$

which when combined gives

$$(A - KCA - I)x = 0 \implies x = 0 \implies y = 0$$

where the first implication is due to the condition that $(A - KCA)$ is a strictly stable matrix. This means that the unique solution is $\begin{bmatrix} x & y \end{bmatrix}^T = \bar{0}$ and that the matrix (3) is full rank.

(b) Given that $B_d = K$, $C_d = I - CK$, $K_x = K$ and $K_d = I$

$$\begin{aligned} C_a \hat{\xi}^{pred}_{t+1} &= C(A\hat{x}_t + K\hat{d}_t + Bu_t) + (I-CK)\hat{d}_t \\ &= CA\hat{x}_t + CBu_t + \hat{d}_t \end{aligned}$$

The observer dynamics can now be written as

$$\hat{\xi}_t = \begin{bmatrix} A & K \\ 0 & I \end{bmatrix} \hat{\xi}_{t-1} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{t-1} + \begin{bmatrix} K \\ I \end{bmatrix} \left(y_t - CA\hat{x}_{t-1} - CBu_{t-1} - \hat{d}_{t-1} \right)$$

This implies that

$$\begin{aligned} \hat{x}_t &= A\hat{x}_{t-1} + K\hat{d}_{t-1} + Bu_{t-1} + K\left(y_t - CA\hat{x}_{t-1} - CBu_{t-1} - \hat{d}_{t-1} \right) \\ &= A\hat{x}_{t-1} + Bu_{t-1} + K\left(y_t - C(A\hat{x}_{t-1} + Bu_{t-1}) \right) \\ \hat{d}_t &= \hat{d}_{t-1} + y_t - CA\hat{x}_{t-1} - CBu_{t-1} - \hat{d}_{t-1} = y_t - C(A\hat{x}_{t-1} + Bu_{t-1}) \end{aligned}$$

(c) The individual disturbances acting on the state and output can be computed as

$$\begin{aligned} B_d\hat{d}_t &= K(y_t - C(A\hat{x}_{t-1} + Bu_{t-1})) = \hat{x}_t - (A\hat{x}_{t-1} + Bu_{t-1}) = d^x_t \\ C_d\hat{d}_t &= (I-CK)(y_t - C(A\hat{x}_{t-1} + Bu_{t-1})) \\ &= y_t - C(A\hat{x}_{t-1} + Bu_{t-1}) - CK(y_t - C(A\hat{x}_{t-1} + Bu_{t-1})) \\ &= y_t - C(A\hat{x}_{t-1} + Bu_{t-1}) - C(\hat{x}_t - A\hat{x}_{t-1} - Bu_{t-1}) = y_t - C\hat{c}_t = d^y_t \end{aligned}$$

With these definitions, the disturbance vector can be written as

$$\hat{d}_t = d^y_t + Cd^x_t.$$

This means that we can interpret this observer as the same one that the system without a disturbance has

$$\hat{x}_t = A\hat{x}_{t-1} + Bu_{t-1} + K(y_k - A\hat{x}_{t-1} - Bu_{t-1})$$

and that the disturbances are simply computed with

$$\begin{aligned} \hat{d}^x_t &= \hat{x}_t - (A\hat{x}_{t-1} + Bu_{t-1}) \\ \hat{d}^y_t &= y_t - C\hat{x}_t, \end{aligned}$$

that is, they are computed as the current difference between the predicted/measured value and the estimated one.

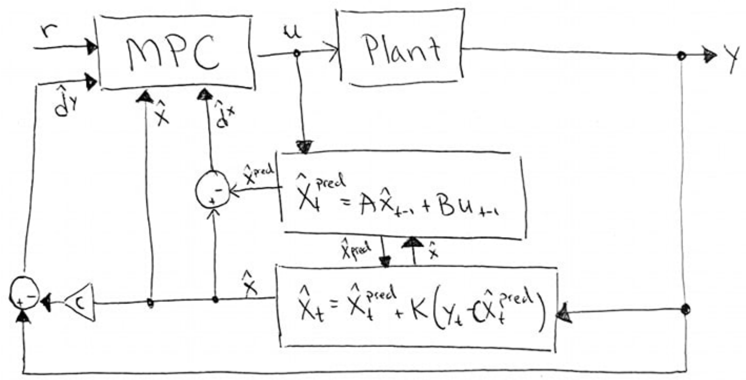

Figure 1: The resulting extended system when using $K_x = K$, $K_d = I$, $B_d = K$, $C_d = I - CK$ can be seen as three separate observers - one for the state and two for the error in state and output respectively.

Exercise 2)

(a) The poles of the observer will be in

$$\mathrm{eig}(A - KC) = \mathrm{eig}\begin{bmatrix} 0.97 & 0 & 0 & 0 \\ 0 & 0.41 & 0 & 0 \\ 0 & 0 & 0.85 & 0 \\ 0 & 0 & 0 & 0.93 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \\ k_7 & k_8 \end{bmatrix} \begin{bmatrix} 0.23 & 0.0 & 0.87 & 0.0 \\ 0.0 & 0.18 & 0.0 & 0.29 \end{bmatrix}$$

Using the MATLAB place command to place the poles correctly, we get the following gain

$$K = \begin{bmatrix} 7.1739 & 0 \\ 0 & -0.0962 \\ -1.0115 & 0 \\ 0 & 1.1976 \end{bmatrix}$$

(b) We want the output of the system y to follow a given reference r. Since $y = Cx$, this can be written as

$$\begin{array}{lll} \text{minimize} & \sum_{i=0}^{N-1}(Cx_{t+i|t} - r_{t+i})^T(Cx_{t+i|t} - r_{t+i}) & \\ \text{subject to} & x(t+i) = Ax(t) + Bu(t) & \text{for } i = 0, \dots, N-1 \\ & -1 \leq u_{t+i} \leq 1 & \text{for } i = 0, \dots, N-1 \end{array}$$

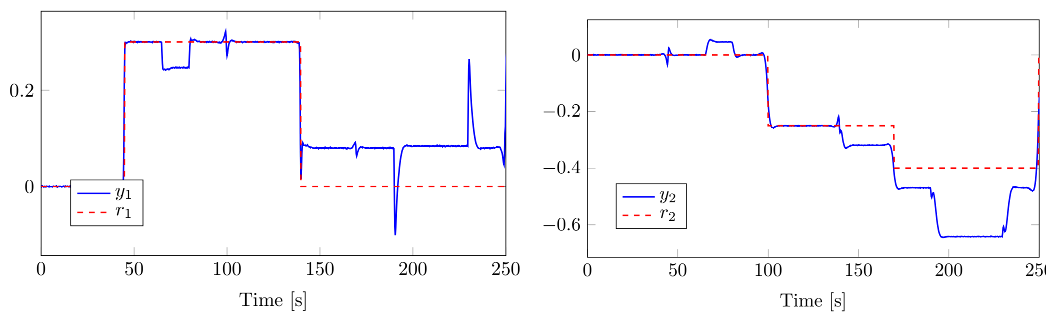

Figure 2: The system response when no augmented system is used. The system is not able to track the reference without a steady state error.

(c) A cost is added for the control increment

$$\begin{array}{lll} \text{minimize} & \sum_{i=0}^{N-1}(Cx_{t+i|t} - r_{t+i})^T(Cx_{t+i|t} - r_{t+i}) + (u_{t+i|t} - u_{t+i-1|t})^T\tilde{R}(u_{t+i|t} - u_{t+i-1|t}) & \\ \text{subject to} & x(t+i) = Ax(t) + Bu(t) & \text{for } i = 0, \dots, N-1 \\ & -1 \leq u_{t+i} \leq 1 & \text{for } i = 0, \dots, N-1. \end{array}$$

Note that the previous control input u_{t-1} needs to be saved for this implementation to work. The response to this controller is shown in Figure 2. Note how the controller is not able to compensate for disturbances acting on the system.

(d) In exercise 1, we had to assume that $(A - KCA)$ was a strictly stable matrix. Using the previously computed K gives

$$(A - KCA) = \begin{bmatrix} -0.6305 & 0 & -5.3051 & 0 \\ 0 & 0.4171 & 0 & 0.0259 \\ 0.2257 & 0 & 1.5980 & 0 \\ 0 & -0.0884 & 0 & 0.6070 \end{bmatrix} \implies \text{full rank}$$

this means that this choice of K works also in the case of the disturbed system. Now, the optimization problem is written as

$$\begin{array}{lll} \text{minimize} & \sum_{i=0}^{N-1}\big((Cx_{t+i|t} + \hat{d}_t^y - r_{t+i})^T(Cx_{t+i|t} + \hat{d}_t^y - r_{t+i}) & \\ & \quad +(u_{t+i|t} - u_{t+i-1|t})^T\tilde{R}(u_{t+i|t} - u_{t+i-1|t})\big) & \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t + \hat{d}_t^x & \\ & -2 \leq u_{t+i} \leq 2 & \text{for } i = 0, \dots, N-1 \\ & x_t = \hat{x}_t & \end{array}$$

Where the observer is given by

$$\begin{aligned} \hat{x}_t &= \hat{x}_t^{pred} + K\left(y_t - C\hat{x}_t^{pred}\right) \\ \hat{d}_t^x &= \hat{x}_t - \hat{x}_t^{pred} \\ \hat{d}_t^y &= y_t - C\hat{x}_t, \end{aligned}$$

This results in a system which can follow a reference signal and compensate for disturbances, as shown in Figure 3.

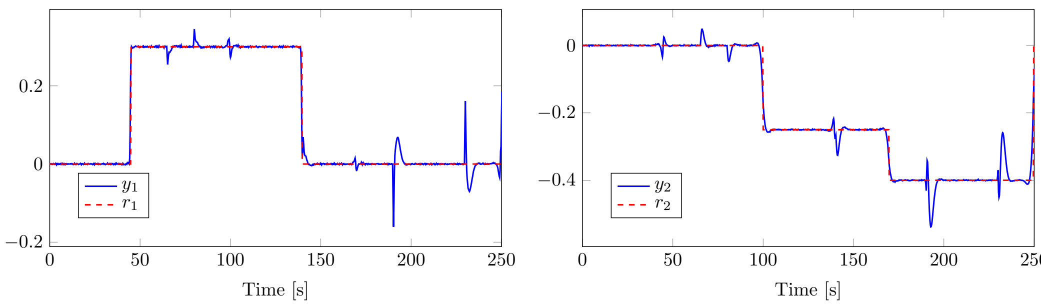

Figure 3: The system response when an augmented system is used. The system is able to track the reference without a steady state error.

References

[1] F. Borrelli, and M. Morari. *Offset Free Model Predictive Control.* Proceedings of the 46th IEEE Conference on Decision and Control 2007.

[2] G. Pannocchia, *Offset-free tracking MPC: A tutorial review and comparison of different formulations.* Proceedings of the 2015 European Control Conference (ECC) 2015.

Implementation of the controller in YALMIP:

```
Q = eye(p);
R = eye(m)*0.1;

x = sdpvar(n, N+1); % from x(1) = 0 to x(N+1) = N
r = sdpvar(p, N+1);
u = sdpvar(m, N+1); % from u(1) = -1 to u(N+1) = N-1

dx = sdpvar(n, 1);
dy = sdpvar(p, 1);

cost = 0;
constraints = [];

for i = 1:N
    cost = cost + ((C*x(:,i)+dy)-r(:,i))'*Q*((C*x(:,i)+dy)-r(:,i));
    cost = cost + (u(:,i+1)-u(:,i))'*R*(u(:,i+1)-u(:,i));

    % Input constraints
    constraints = [constraints, -2 <= u(1, i) <= 2];
    constraints = [constraints, -2 <= u(2, i) <= 2];

    % State evolution
    constraints = [constraints, x(:, i+1) == A*x(:, i) + B*u(:, i) + dx];
end

options = sdpsettings('verbose', 0, 'solver', 'quadprog');
controller = optimizer(constraints, cost, options, ...
                       [x(:,1); u(:,1); dx; dy; r(1,:)'; r(2,:)'], ...
                       [u(1,2)',u(2,2)']);
```

Implementation of the observer and the disturbance estimators:

```
% Observer
xpred = A*xhat + B*uprev;
xhat = xpred + K*(y_of(:,i) - C*xpred);
dx = xhat - xpred;
dy = y(:,i) - C*xhat;

% Get reference
references = reference(t);

% Controller
result = controller{[xhat; u(:,i); dx; dy; references]};

% Next input to apply
u(:,i+1) = result;
```