Lecture 2: Discrete-time linear systems

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Outline

- Discrete-time linear system and sampling
- Stability of scalar and vector-valued systems
- Systems with inputs and outputs
- Observability and controllability
- State feedback and observers

Discrete-time linear systems

Discrete-time linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned} \tag{1}$$

Describes state evolution $\{x_0, x_1, \dots\}$ at discrete time instances.

Here

- $x_t \in \mathbb{R}^n$ is the *state vector*
- $u_t \in \mathbb{R}^m$ is the *control input*
- $y_t \in \mathbb{R}^p$ is the *system output*

while A, B, C and D are constant matrices of compatible dimensions.

Convenient to use (A, B, C, D) as short-hand notation for (1).

Autonomous linear systems

An autonomous linear system has no external input, evolves according to

$$x_{t+1} = Ax_t$$

Its state trajectory $\{x_t\}$ is given by

$$\begin{aligned} x_1 &= Ax_0 \\ x_2 &= Ax_1 = A^2x_0 \\ &\vdots \\ x_t &= A^t x_0 \end{aligned}$$

Its properties are determined by the matrix A (and the initial value x_0).

Properties of discrete-time linear systems

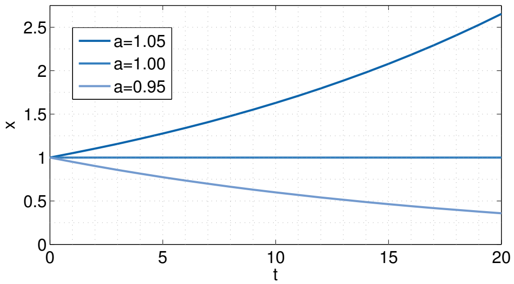
Example: money in the bank

$$x_{t+1} = a x_t$$

Can express state as function of initial value:

$$x_t = a^t x_0$$

converges to zero if $|a| < 1$, diverges of $|a| > 1$.



Stability of linear discrete-time systems

Definition. The linear discrete-time system (1) is *asymptotically stable* if the solution $\{x_t\}$ to

$$x_{t+1} = A x_t,$$

satisfies $\|x_t\| \to 0$ as $t \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Theorem. The discrete-time linear system (1) is asymptotically stable if and only if $|\lambda_i(A)| < 1$ for all $i = 1, \dots, n$.

Proof. See lecture notes.

Quiz

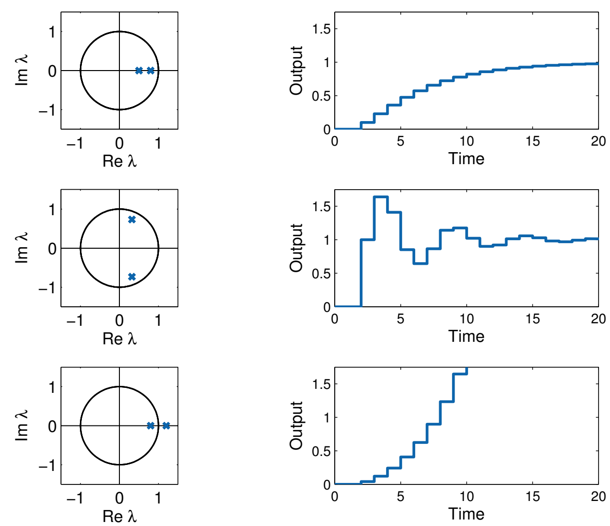
Quiz: for which values of the parameter θ is the system

$$x_{t+1} = \begin{bmatrix} \theta & 1/2 \\ 0 & 1/4 \end{bmatrix} x_t$$

asymptotically stable?

Stability boundary for discrete-time linear systems

"All eigenvalues have to be inside the unit circle in the complex plane".



Discrete-time systems may converge in finite time

Consider the discrete-time system

$$x_{t+1} = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} x_t$$

We then have that

$$x_2 = \mathbf{0}$$

for all x_0. Thus, the system converges in finite time.

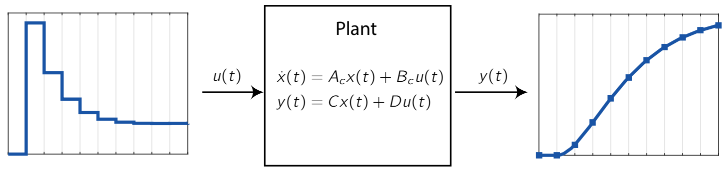
Finite-time convergence is impossible for continuous-time linear systems!

Note: a matrix with all eigenvalues at zero is called *nilpotent*.
(feedback that renders system matrix nilpotent is called *dead-beat*)

Discrete-time descriptions of continuous-time systems

Common operation of computer-controlled system:

- Output sampled every h seconds (h is called the *sampling time*)
- Control input held constant between samples



How does the state evolve between sampling instances?

Discrete-time descriptions of continuous-time systems

Recall from the basic course that

$$\dot{x}(t) = A_c x(t) + B_c u(t) \Rightarrow x(t+h) = e^{A_c h} x(t) + \int_{s=0}^{h} e^{A_c s} B_c u(s)\, ds$$

If u is constant during the sample interval, $u(t+s) = u(t)$ for $s \in [0, h)$

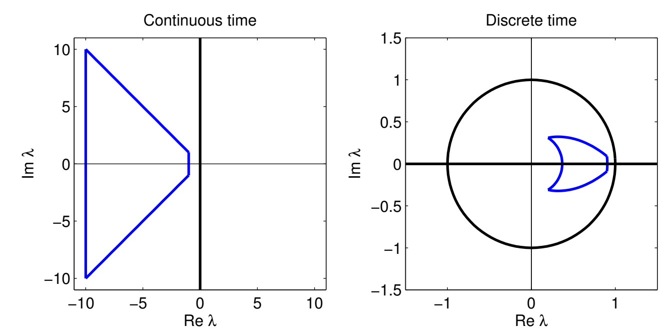
$$\begin{aligned} x(t+h) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with $A = e^{A_c h}$ and $B = \int_{s=0}^{h} e^{A_c s} B_c\, ds$. A discrete-time linear system!

An *exact* description of continuous-time system *at sampling instances*.

Good pole locations

Good pole locations for continuous and discrete-time systems



Outline

- Discrete-time linear system and sampling
- Stability of scalar and vector-valued systems
- Systems with inputs and outputs
- Observability and controllability
- State feedback and observers

Systems with inputs and outputs

Discrete-time linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t + Du_t \end{aligned} \tag{2}$$

with state $x_t \in \mathbb{R}^n$, input $u_t \in \mathbb{R}^m$ and output $y_t \in \mathbb{R}^p$.

Key questions:

- When can we find $\{u_0, u_1, \cdots, u_{t-1}\}$ so that $x_t = x_{\text{tgt}}$?
- When can we reconstruct x_0 from $\{y_0, y_1, \cdots, y_{t-1}\}$?
- How to design linear controllers and observers with desired dynamics?

Solution to system equations

Solution ("prediction equations")

$$\begin{aligned} x_1 &= Ax_0 + Bu_0 \\ x_2 &= A^2x_0 + ABu_0 + Bu_1 \\ &\vdots \\ x_N &= A^Nx_0 + \sum_{k=0}^{N-1} A^kBu_{N-1-k} \end{aligned}$$

State evolution is a *linear* function of the input sequence and initial state

$$X_N = \mathscr{h}_N + \mathscr{H}_N U_N$$

($\mathscr{H}_N$ also has an interesting structure: it is Toeplitz and lower triangular)

Reachability

Definition. The linear system (2) is *reachable* if for any target state x_{tgt}, it is possible to find $\{u_0, u_1, \ldots, u_{t-1}\}$ which drives the system state from $x_0 = 0$ to $x_t = x_{\text{tgt}}$ for some finite value of t.

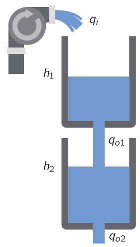
Theorem. The linear system (2) is reachable iff $\text{rank}(\mathscr{C}_n) = n$ where

$$\mathscr{C}_n = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \cdots & AB & B \end{bmatrix}$$

is the *controllability matrix*.

Quiz: controllability

Discrete-time linear model of double tank



$$x_{t+1} = \begin{bmatrix} 0.7047 & 0 \\ 0.2466 & 0.7047 \end{bmatrix} x_t + \begin{bmatrix} 0.7594 \\ 0.1252 \end{bmatrix} u_t$$

(a) which states are reachable from $(0, 0)$ in one time step?

(b) which states are reachable from $(0, 0)$ in two time steps?

(c) is there any reason to use control sequences of length ≥ 2?

Controllability

Definition. The system (2) is *controllable* if for any initial state x_0, it is possible to find $\{u_0, u_1, \dots, u_{t-1}\}$ so that $x_t = 0$ for some finite value of t.

Clearly, if (2) is reachable, it is also controllable; but there are discrete-time linear systems with are controllable but not reachable. One such example is

$$x_{t+1} = \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t$$

Although $\text{rank}(\mathcal{C}_2) = 1$, $u_t \equiv 0$ yields $x_t = 0$ no matter which x_0.

This distinction does not exist for continuous-time linear systems.

Observability

Definition. The system (2) is *observable* if there is a finite t such that knowledge about $\{u_0, \cdots, u_{t-1}\}$ and outputs $\{y_0, \cdots, y_{t-1}\}$ is sufficient for determining the initial state x_0.

Theorem. The linear system (2) is observable iff $\text{rank}(\mathcal{O}_\text{n}) = \text{n}$ where

$$\mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is the *observability matrix*.

(the proof is analogous to that of reachability)

The Popov-Belevitch-Hautus (PBH) tests

Theorem. The discrete-time linear system

$$x_{t+1} = Ax_t, \qquad y_t = Cx_t$$

is unobservable if and only if there exists a vector $v \neq 0$ such that

$$Av = \lambda v, \qquad Cv = 0 \tag{3}$$

Theorem. The discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t$$

is unreachable if and only if there exists $w \in \mathbb{R}^n$ with $w \neq 0$ such that

$$w^T A = \lambda w^T, \qquad w^T B = 0$$

Coordinate transformation

Often useful to change basis of state vector x by similarity transformation

$$z = Tx$$

for some invertible $T \in \mathbb{R}^{n\times n}$.

Dynamics in the new coordinates:

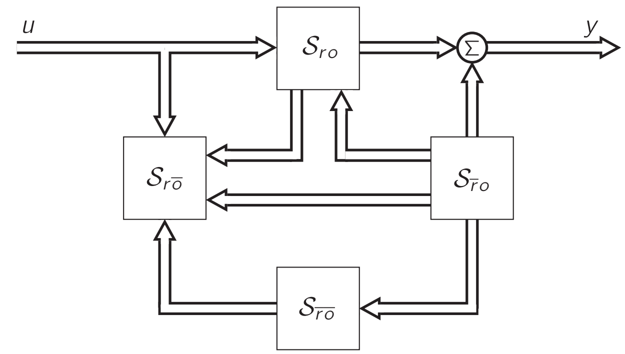
$$\begin{aligned} z_{t+1} &= Tx_{t+1} = T(Ax_t + Bu_t) &&= TAT^{-1}z_t + TBu_t \\ y_t &= Cx_t + Du_t &&= CT^{-1}z_t + Du_t \end{aligned}$$

Thus $z = Tx$ transforms (A, B, C, D) into $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}, \quad \tilde{D} = D$$

Kalman decomposition

Can decompose system in (un)controllable and (un)observable subsystems



$\mathcal{S}_{ro}$ is reachable and observable

$\mathcal{S}_{r\bar{o}}$ is reachable but not observable

$\mathcal{S}_{\bar{r}o}$ is not reachable but observable

$\mathcal{S}_{\bar{r}\bar{o}}$ is neither reachable nor observable

Kalman decomposition

Proposition. Can find similarity transform $z = Tx$ so that

$$z_{t+1} = \begin{bmatrix} A_{ro} & \tilde{A}_{22} & 0 & 0 \\ 0 & A_{\bar{r}o} & 0 & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & A_{r\bar{o}} & \tilde{A}_{34} \\ 0 & 0 & \tilde{A}_{43} & A_{\bar{r}\bar{o}} \end{bmatrix} z_t + \begin{bmatrix} B_{ro} \\ 0 \\ B_{r\bar{o}} \\ 0 \end{bmatrix} u_t$$
$$y_t = \begin{bmatrix} C_{ro} & C_{\bar{r}o} & 0 & 0 \end{bmatrix} z_t + Du_t$$

where

- $(A_{ro}, B_{ro}, C_{ro}, D)$ is reachable *and* observable
- $\left(\begin{bmatrix} A_{ro} & \tilde{A}_{22} \\ 0 & A_{\bar{r}o} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{ro} & C_{\bar{r}o} \end{bmatrix}, D \right)$ is observable
- $\left(\begin{bmatrix} A_{ro} & 0 \\ \tilde{A}_{31} & A_{r\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{ro} \\ B_{r\bar{o}} \end{bmatrix}, \begin{bmatrix} C_{ro} & 0 \end{bmatrix}, D \right)$ is reachable
- $(A_{\bar{r}\bar{o}}, 0, 0, D)$ is neither reachable nor observable.

State feedback

The linear state feedback

$$u_t = -Lx_t$$

results in the closed-loop system

$$x_{t+1} = (A - BL)x_t$$

When can we find L which assigns arbitrary eigenvalues to $A - BL$?

Theorem. For the linear system (2), there exists $L \in \mathbb{R}^{m\times n}$ such that the n eigenvalues of $A - BL$ can be assigned to arbitrary real or complex conjugate locations if and only if the system is reachable.

Observers

If x_t is not measurable, we can estimate it using an observer.

Basic idea: simulate the system and adjust the state estimate if the simulated output does not agree with the actual one.

$$\begin{aligned}\hat{x}_{t+1} &= \underbrace{\begin{matrix}A\hat{x}_t + Bu_t \\ C\hat{x}_t + Du_t\end{matrix}}_{\text{prediction}} \quad + \quad \underbrace{\begin{matrix}K(y_t - \hat{y}_t) \\ \end{matrix}}_{\text{correction}} \\ \hat{y}_t &= \end{aligned}$$

The dynamics of the state estimation error $e_t = x_t - \hat{x}_t$

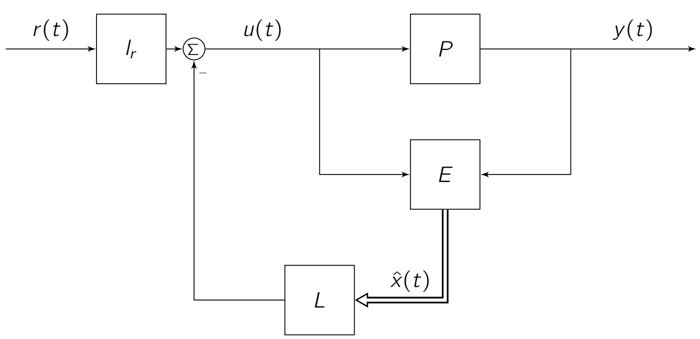
$$e_{t+1} = (A - KC)e_t \tag{4}$$

Observer design

Theorem. For the linear system (2), there exists $K \in \mathbb{R}^{n\times p}$ so that the n eigenvalues of $A - KC$ can be assigned to arbitrary real or complex conjugate locations if and only if the system is observable.

Output feedback

Combines state estimator and linear feedback from estimated states



Output feedback

The closed-loop dynamics are

$$\begin{aligned} x_{t+1} &= Ax_t + BL\hat{x}_t + Bl_r r_t \\ \hat{x}_{t+1} &= A\hat{x}_t + BL\hat{x}_t + KC(x_t - \hat{x}_t) \end{aligned}$$

In terms of x_t and $e_t = x_t - \hat{x}_t$:

$$\begin{bmatrix} x_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x_t \\ e_t \end{bmatrix} + \begin{bmatrix} Bl_r \\ 0 \end{bmatrix} r_t$$

Notes:

- the error dynamics are not reachable from r;
- the closed-loop eigenvalues are those of $A - BL$ and $A - KC$.

Summary

- Linear systems in discrete-time
  - sampling of continuous-time dynamics
  - eigenvalue conditions for asymptotic stability
  - similarities and differences with continuous-time systems
- Systems with inputs and outputs
  - the prediction equations
  - controllability and state transfer
  - observability and state reconstruction
- Linear state feedback and observers

Reading instructions

Read Chapter 1 of the lecture notes.

Appendix. Cayley-Hamilton Theorem

Theorem. Let $A \in \mathbb{R}^{n \times n}$ with characteristic polynomial

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0.$$

Then

$$p(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I = 0$$

"Every square real matrix satisfies its own characteristic polynomial"

Consequence:

$$A^n = -\alpha_{n-1}A^{n-1} - \cdots - \alpha_1 A - \alpha_0 I$$

(A^n is a linear combination of lower matrix powers)