ELSEVIER

Brief paper

Robust output feedback model predictive control of constrained linear systems ☆

D.Q. Mayne[a], S.V. Raković[a,∗], R. Findeisen[b], F. Allgöwer[b]

[a]*Department of Electrical and Electronic Engineering, Imperial College London*
[b]*Institute for Systems Theory and Automatic Control (IST), University of Stuttgart, Germany*

Abstract

This paper provides a solution to the problem of robust output feedback model predictive control of constrained, linear, discrete-time systems in the presence of bounded state and output disturbances. The proposed output feedback controller consists of a simple, stable Luenberger state estimator and a recently developed, robustly stabilizing, tube-based, model predictive controller. The state estimation error is bounded by an invariant set. The tube-based controller ensures that all possible realizations of the state trajectory lie in a simple uncertainty tube the 'center' of which is the solution of a nominal (disturbance-free) system and the 'cross-section' of which is also invariant. Satisfaction of the state and input constraints for the original system is guaranteed by employing tighter constraint sets for the nominal system. The complexity of the resultant controller is similar to that required for nominal model predictive control.
© 2006 Elsevier Ltd. All rights reserved.

Keywords: Output feedback; Robust model predictive control; Observer; Robust control invariant tubes

1. Introduction

Model predictive control has received considerable attention driven largely by its ability to handle hard constraints as well as nonlinearity. An inherent problem is that model predictive control normally requires full knowledge of the state (Findeisen, Imsland, Allgöwer, & Foss, 2003; Mayne, Rawlings, Rao, & Scokaert, 2000). Whereas, in many control problems, not all states can be measured exactly. In practice this problem is often overcome by employing 'certainty equivalence'. For linear systems not subject to disturbances that employ an observer and linear control, stability of the closed-loop can be guaranteed by the separation principle. However, when state and output disturbances are present and the system or its controller is

nonlinear (as is the case with model predictive control of constrained systems), stability of the closed-loop cannot, in general, be ensured by simply combining a stable estimator with a stable state feedback controller (Atassi & Khalil, 1999; Teel & Praly, 1995). Robust stability is obtained if the nominal system is inherently robust and the estimation errors are sufficiently small; however, predictive controllers are not always inherently robust (Grimm, Messina, Teel, & Tuna, 2004; Scokaert, Rawlings, & Meadows, 1997).

An appealing approach for overcoming this drawback is to use robust controller design methods that take the state estimation error directly into account. In the late sixties, Witsenhausen (1968a, 1968b) studied robust control synthesis and set-membership estimation of linear dynamic systems subject to bounded uncertainties; this work was followed by Schweppe (1968), Bertsekas and Rhodes (1971a,1971b), Glover and Schweppe (1971) and by related work on viability (Aubin, 1991; Kurzhanski & Vályi, 1997; Kurzhanski & Filippova, 1993). Further results include: a paper by Blanchini (1990) that deals with linear, constrained, uncertain, discrete-time systems, an anti-windup scheme that employs invariant sets in a similar fashion to their use in this

paper (Shamma, 2000), output feedback control for uncertain dynamic systems with scalar controls (Shamma & Tu, 1998), recursive dynamic programming equations for optimal control of dynamic systems subject to uncertainty and imperfect measurements (Moitié, Quincampoix, & Veliov, 2002), and optimal control of systems subject to both observable and unknown disturbances (Quincampoix & Veliov, 2005). Several authors, e.g. Bemporad and Garulli (2000) advocate set-membership estimation for model predictive control; to reduce complexity in such controllers, Chisci and Zappa (2002) employs efficient computational methods. For a similar system class, Löfberg (2002) proposes joint state estimation and feedback using min-max optimization. For linear systems with input constraints, the method in Lee and Kouvaritakis (2001) achieves stability by using invariant sets for an augmented system. By constructing an invariant set for the observer error, Kouvaritakis, Wang, and Lee (2000) adapts the model predictive controller in Cannon, Kouvaritakis, Lee, and Brooms (2001) to ensure closed-loop asymptotic stability. In Fukushima and Bitmead (2005) the authors use a comparison system to establish ultimate boundedness of the trajectories of a closed-loop system consisting of an observer and a model predictive controller.

In this paper we consider the output feedback problem for constrained linear discrete-time systems subject to state and measurement disturbances. The basic idea is to consider the state estimation error as an additional, unknown but bounded uncertainty that can be accounted for in a suitably modified robust model predictive controller that controls the nominal observer state rather than the unknown but bounded system state. The state estimation error is bounded by a simple, precomputed, invariant set. The controller uses a tube, the center of which is obtained by solving a conventional (disturbance-free) model predictive control problem that yields the center of the tube; the tube is obtained by 'adding' to its center another simple, pre-computed, invariant set. Tighter constraints in the optimal control problem solved online ensure that all realizations of the state trajectory satisfy the state and control constraints.

In comparison to previous work on *output model predictive control*, simplicity is achieved by using a Luenberger observer (that is less complex than a set-membership state estimator), an invariant set that bounds the estimation error, as in Shamma (2000), and a simple tube-based robust model predictive controller that is nearly as simple to implement as a nominal linear model predictive controller. Robust exponential stability of a "minimal" robust invariant set is established.

Nomenclature: In the following $\mathbb{N} \triangleq \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ \triangleq \{1, 2, \ldots\}$ and $\mathbb{N}_q \triangleq \{0, 1, \ldots, q\}$. A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is the closed and bounded polyhedron. Given two sets $\mathscr{U}$ and $\mathscr{V}$, such that $\mathscr{U} \subset \mathbb{R}^n$ and $\mathscr{V} \subset \mathbb{R}^n$, Minkowski set addition is defined by $\mathscr{U} \oplus \mathscr{V} \triangleq \{u + v \mid u \in \mathscr{U}, v \in \mathscr{V}\}$ and Minkowski (Pontryagin) set difference by $\mathscr{U} \ominus \mathscr{V} \triangleq \{x \mid x \oplus \mathscr{V} \subseteq \mathscr{U}\}$. A set $\mathscr{U} \subset \mathbb{R}^n$ is a C set if it is a compact, convex set that contains the origin in its (non-empty) interior. Let $d(z, X) \triangleq \inf\{|z - x|_p \mid x \in X\}$ where $|\cdot|_p$ de-

notes the p vector norm. Let $\rho(A)$ denote spectral radius of a given matrix $A \in \mathbb{R}^{n \times n}$.

2. Proposed control methodology

We consider the following uncertain discrete-time linear time-invariant system:

$$x^+ = Ax + Bu + w, \quad y = Cx + v, \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state, $u \in \mathbb{R}^m$ is the current control action, x^+ is the successor state, $w \in \mathbb{R}^n$ is an unknown state disturbance, $y \in \mathbb{R}^p$ is the current measured output, $v \in \mathbb{R}^p$ is an unknown output disturbance, $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$, the couple (A, B) is assumed to be controllable and the couple (A, C) observable. The state and additive disturbances w and v are known only to the extent that they lie, respectively, in the C sets $\mathbb{W} \subset \mathbb{R}^n$ and $\mathbb{V} \subset \mathbb{R}^p$: $(w, v) \in \mathbb{W} \times \mathbb{V}$. Let $\phi(i; x, \mathbf{u}, \mathbf{w})$ denote the solution of (1) at time i if the initial state (at time 0) is x and the control and disturbance sequences are, respectively, $\mathbf{u} \triangleq \{u(0), u(1), \ldots\}$ and $\mathbf{w} \triangleq \{w(0), w(1), \ldots\}$. System (1) is subject to the following set of hard state and control constraints:

$$(x, u) \in \mathbb{X} \times \mathbb{U}, \tag{2}$$

where $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are polyhedral and polytopic sets, respectively, and both contain the origin as an interior point.

To estimate the state a simple Luenberger observer is employed (as in Shamma, 2000):

$$\hat{x}^+ = A\hat{x} + Bu + L(y - \hat{y}), \quad \hat{y} = C\hat{x}, \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the current observer state, $u \in \mathbb{R}^m$ is the current control action, $\hat{x}^+$ is the successor state of the observer system, $\hat{y} \in \mathbb{R}^p$ is the current observer output and $L \in \mathbb{R}^{n \times p}$. The estimated state $\hat{x}$ satisfies the following uncertain difference equation:

$$\hat{x}^+ = A\hat{x} + Bu + LC\tilde{x} + Lv, \tag{4}$$

while the state estimation error $\tilde{x} \triangleq x - \hat{x}$ satisfies

$$\tilde{x}^+ = A_L\tilde{x} + (w - Lv), \quad A_L \triangleq A - LC, \tag{5}$$

where $L \in \mathbb{R}^{n \times p}$ satisfies $\rho(A_L) < 1$.

We control the observer (3) in such a way that the real state $x = \hat{x} + \tilde{x}$ always satisfies the state constraint and the associated control always satisfies the control constraint. To achieve this, we introduce a third dynamic system, the nominal system obtained from (1) by neglecting the disturbances w and v:

$$\bar{x}^+ = A\bar{x} + B\bar{u}, \tag{6}$$

where $\bar{x} \in \mathbb{R}^n$ denotes the nominal system state and $\bar{u} \in \mathbb{R}^m$ is the input to the nominal system. Let $\bar{\phi}(i; \bar{x}, \bar{\mathbf{u}})$ denote the state of the nominal system (6) at time i if its initial state (at time 0) is $\bar{x}$ and if the control sequence is $\bar{\mathbf{u}} \triangleq \{\bar{u}(0), \bar{u}(1), \ldots\}$. Choosing an initial state $\bar{x}$ and a nominal control sequence $\bar{\mathbf{u}}$ yields a nominal state sequence $\bar{\mathbf{x}} \triangleq \{\bar{\phi}(i; \bar{x}, \bar{\mathbf{u}}), i = 0, 1, \ldots\}$; this sequence is the center of a tube that we shall shortly construct.

In order to counteract the disturbances we combine in the control u a feed-forward part, given by the tube based model predictive controller, and a feedback part

$$u = \bar{u} + Ke, \tag{7}$$

where $e \triangleq \hat{x} - \bar{x}$ and $K \in \mathbb{R}^{m \times n}$ satisfies $\rho(A + BK) < 1$. With this control, the closed-loop observer state satisfies

$$\hat{x}^+ = A\hat{x} + B\bar{u} + BKe + LC\tilde{x} + Lv, \tag{8}$$

while the error e between the observer state and the nominal system satisfies the difference equation

$$e^+ = A_K e + (LC\tilde{x} + Lv), \quad A_K \triangleq A + BK. \tag{9}$$

We choose an initial state $\bar{x}(0)$ and a nominal control sequence $\bar{\mathbf{u}}$ to obtain the nominal state sequence $\bar{\mathbf{x}}$. By definition, the observer state differs from the nominal state by e ($\hat{x} = \bar{x} + e$), and the actual state differs from the observer state by $\tilde{x}$ ($x = \hat{x} + \tilde{x}$), so that

$$x = \bar{x} + e + \tilde{x}. \tag{10}$$

Providing we can bound e and $\tilde{x}$, we can choose $\bar{x}(0)$ and $\bar{\mathbf{u}}$ so that the actual (unknown) state and control satisfy the original constraints.

3. Bounding the estimation and control errors

We recall the following standard definitions (Blanchini, 1999; Kolmanovsky & Gilbert, 1998):

Definition 1. A set $\Omega \subset \mathbb{R}^n$ is *positively invariant* for the system $x^+ = f(x)$ and the constraint set $\mathbb{X}$ if $\Omega \subseteq \mathbb{X}$ and $f(x) \in \Omega$, $\forall x \in \Omega$. A set $\Omega \subset \mathbb{R}^n$ is *robust positively invariant* for the system $x^+ = f(x, w)$ and the constraint set $(\mathbb{X}, \mathbb{W})$ if $\Omega \subseteq \mathbb{X}$ and $f(x, w) \in \Omega$, $\forall w \in \mathbb{W}$, $\forall x \in \Omega$.

Our next step is to establish that the estimation and control errors (respectively, $\tilde{x}$ and e) can be bounded by robust positively (control) invariant sets. The difference equation (5) can be rewritten in the form

$$\tilde{x}^+ = A_L\tilde{x} + \tilde{\delta}, \quad \tilde{\delta} \triangleq w - Lv, \tag{11}$$

where the 'disturbance' $\tilde{\delta}$ lies in the C set $\tilde{\Delta}$ defined by

$$\tilde{\Delta} \triangleq \mathbb{W} \oplus (-L\mathbb{V}). \tag{12}$$

Since $\rho(A_L) < 1$, there exists a C set $\tilde{S}$ that is finite time computable and robust positively invariant (Raković, Kerrigan, Kouramas, & Mayne, 2005; Raković, 2005) for the system (11) and constraint set $(\mathbb{R}^n, \tilde{\Delta})$. The set $\tilde{S}$ satisfies $A_L\tilde{S} \oplus \tilde{\Delta} \subseteq \tilde{S}$. Since $\tilde{x}(0) \in \tilde{S}$ implies $\tilde{x}(i) \in \tilde{S}$ for all $i \in \mathbb{N}$, an obvious consequence is:

Proposition 1. *If the initial system and observer states, $x(0)$ and $\hat{x}(0)$, respectively, satisfy $\tilde{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$, then $x(i) \in \hat{x}(i) \oplus \tilde{S}$ for all $i \in \mathbb{N}$, and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$.*

The assumption that $\tilde{x}(0) \in \tilde{S}$ is a steady state assumption for the observer error. If the observer state $\hat{x}$ lies in the tightened constraint set $\mathbb{X} \ominus \tilde{S}$ the original state x is guaranteed to lie in $\mathbb{X}$. We bound the control error e by an invariant set. The difference equation (9) can be rewritten in the form:

$$e^+ = A_K e + \bar{\delta}, \quad \bar{\delta} \triangleq LC\tilde{x} + Lv, \tag{13}$$

where (since $\tilde{x}$ is bounded by $\tilde{S}$) the " artificial disturbance" $\bar{\delta}$ lies in the set $\bar{\Delta}$ defined by

$$\bar{\Delta} \triangleq LC\tilde{S} \oplus L\mathbb{V}. \tag{14}$$

Because $\rho(A_K) < 1$, there exists a C set $\bar{S}$ that is finite time computable and robust positively invariant (Raković et al., 2005; Raković, 2005) for system (13) and constraint set $(\mathbb{R}^n, \bar{\Delta})$. The set $\bar{S}$ satisfies $A_K\bar{S} \oplus \bar{\Delta} \subseteq \bar{S}$. Since $e(0) \in \bar{S}$ implies $e(i) \in \bar{S}$ for all $i \in \mathbb{N}$, an obvious consequence is

Proposition 2. *If the initial observer and nominal system states, $\hat{x}(0)$ and $\bar{x}(0)$, respectively, satisfy $e(0) = \hat{x}(0) - \bar{x}(0) \in \bar{S}$, then $\hat{x}(i) \in \bar{x}(i) \oplus \bar{S}$ for all $i \in \mathbb{N}$, all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$.*

Defining $S \triangleq \tilde{S} \oplus \bar{S}$, we obtain:

Proposition 3. *Suppose $\hat{x}(0)$, $\bar{x}(0)$ and $\bar{\mathbf{u}} = \{\bar{u}(0), \bar{u}(1), \ldots\}$ are given. If the initial system, observer, and nominal system states satisfy $\tilde{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$ and $e(0) = \hat{x}(0) - \bar{x}(0) \in \bar{S}$, and $u(i) = \bar{u}(i) + K(\hat{x}(i) - \bar{x}(i))$ for all $i \in \mathbb{N}$, then the system state satisfies $x(i) \in \hat{x}(i) \oplus \tilde{S} \subseteq \bar{x}(i) \oplus S$ for all $i \in \mathbb{N}$ and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$ ($\bar{x}(i) \triangleq \bar{\phi}(i; \bar{x}(0), \bar{\mathbf{u}})$).*

The control constraint $u \in \mathbb{U}$ induces a constraint on $\bar{u}$ via the control law $u = \bar{u} + Ke$; to ensure $u \in \mathbb{U}$, $\bar{u}$ must satisfy the tighter constraints $\bar{u} \in \mathbb{U} \ominus K\bar{S}$. In order to ensure that the unknown state satisfies state constraints $x \in \mathbb{X}$ we must ensure that $\hat{x} \in \mathbb{X} \ominus \tilde{S}$ and that $\bar{x} \in \mathbb{X} \ominus (\tilde{S} \oplus \bar{S})$. Summarizing, to employ our procedure, we assume:

A1. There exist a controller K and an output injection L such that $\rho(A_K) < 1$, $A_K = A + BK$ and $\rho(A_L) < 1$, $A_L = A + LC$ and the associated robust positively invariant sets $\tilde{S}$ and $\bar{S}$ satisfy $S = \tilde{S} \oplus \bar{S} \subset \mathbb{X}$ and $K\bar{S} \subset \mathbb{U}$.

The assumptions on $\bar{S}$ and $\tilde{S}$, which are satisfied if $\mathbb{W}$ and $\mathbb{V}$ are sufficiently small, can be verified by solving a linear program as shown in Raković et al. (2005), Raković (2005), robustness to large uncertainty cannot necessarily be achieved. Combining Propositions 2 and 3 and the discussion above, we obtain:

Theorem 1. (i) *Suppose the initial system, observer, and nominal system states all lie in $\mathbb{X}$ and satisfy $\tilde{x}(0) = x(0) - \hat{x}(0) \in \tilde{S}$ and $e(0) = \hat{x}(0) - \bar{x}(0) \in \bar{S}$. Then (i) $x(0) \in \bar{x}(0) \oplus S$. If, in addition, initial state $\bar{x}(0)$ and control sequence $\bar{\mathbf{u}}$ of the nominal system satisfy the tighter constraints $\bar{x}(i) = \bar{\phi}(i; \bar{x}(0), \bar{\mathbf{u}}) \in \mathbb{X} \ominus S$ and $\bar{u}(i) \in \mathbb{U} \ominus K\bar{S}$ for all $i \in \mathbb{N}$, then (ii) the state*

$x(i)$ and the control $u(i)$ of the controlled system $x(i+1) = Ax(i) + Bu(i) + w(i)$, $u(i) = \bar{u}(i) + K(\hat{x}(i) - \bar{x}(i))$ satisfy the original constraints $x(i) \in \mathbb{X}$ and $u(i) \in \mathbb{U}$ for all $i \in \mathbb{N}$ and all admissible disturbance sequences $\mathbf{w}$ and $\mathbf{v}$.

4. Robust output feedback MPC

Theorem 1 provides the ingredients we require to obtain output model predictive control of the original system by employing the model predictive controller in Mayne, Seron, and Raković (2005) to control robustly the observer system (4):

$$\hat{x}^+ = A\hat{x} + Bu + \bar{\delta}, \quad \bar{\delta} \triangleq LC\tilde{x} + Lv, \tag{15}$$

where $\bar{\delta} \in \bar{\Delta} = LC\tilde{S} \oplus L\mathbb{V}$. Let the cost $V_N(\bar{x}, \bar{\mathbf{u}})$ be defined by

$$V_N(\bar{x}, \bar{\mathbf{u}}) \triangleq \sum_{i=0}^{N-1} \ell(\bar{x}(i), \bar{u}(i)) + V_f(\bar{x}(N)), \tag{16}$$

where N is the horizon and $V_f(\cdot)$ is the terminal cost function and $\ell(\cdot)$ is the stage cost defined by

$$\ell(x, u) \triangleq (1/2)[x'Qx + u'Ru], \quad V_f(x) = (1/2)x'Px, \tag{17}$$

where P, Q and R are assumed to be positive definite. Note that the results in this paper are also applicable to the case when $\ell(\cdot)$ and $V_f(\cdot)$ are defined using polytopic vector norms (e.g. ℓ_1, ℓ_∞). The state of the nominal system $\bar{x}$ and control applied to the nominal system $\bar{u}$ are subject to the tighter constraints defined by:

$$(\bar{x}, \bar{u}) \in \bar{\mathbb{X}} \times \bar{\mathbb{U}}, \quad \bar{\mathbb{X}} \triangleq \mathbb{X} \ominus S, \quad \bar{\mathbb{U}} \triangleq \mathbb{U} \ominus K\bar{S}. \tag{18}$$

The initial and final states $\bar{x}$ and $\bar{x}(N)$ of the nominal system are required to satisfy, respectively,

$$\hat{x} \in \bar{x} \oplus \bar{S}, \quad \bar{x}(N) \in X_f. \tag{19}$$

The set X_f is the terminal constraint set for the nominal system. The initial nominal state $\bar{x}$ is, with the nominal control sequence, $\mathbf{u}$, a decision variable; given $\bar{x}$, the set of admissible control sequences is

$$\mathscr{U}_N(\bar{x}) = \{\bar{\mathbf{u}} \mid \bar{u}(i) \in \bar{\mathbb{U}}, \ \bar{\phi}(i; \bar{x}, \bar{\mathbf{u}}) \in \bar{\mathbb{X}}, \ \forall i \in \mathbb{N}_{N-1},$$
$$\bar{\phi}(N; \bar{x}, \mathbf{u}) \in X_f\}. \tag{20}$$

Let $\bar{X}_N \triangleq \{\bar{x} \mid \mathscr{U}_N(\bar{x}) \neq \emptyset\}$. The nominal optimal control problem $P_N(\hat{x})$, solved on-line is:

$$V_N^*(\hat{x}) = \min_{\bar{x}, \bar{\mathbf{u}}}\{V_N(\bar{x}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \mathscr{U}_N(\bar{x}), \ \hat{x} \in \bar{x} \oplus \bar{S}\},$$

$$(\bar{x}^*(\hat{x}), \bar{\mathbf{u}}^*(\hat{x})) = \arg \min_{\bar{x}, \bar{\mathbf{u}}}\{V_N(\bar{x}, \bar{\mathbf{u}}) \mid \bar{\mathbf{u}} \in \mathscr{U}_N(\bar{x}),$$
$$\hat{x} \in \bar{x} \oplus \bar{S}\}. \tag{21}$$

The terminal cost $V_f(\cdot)$ and the terminal constraint set X_f satisfy the usual stabilizing conditions (Mayne et al., 2000):

A2. $A_{K_f}X_f \subset X_f$, $X_f \subset \bar{\mathbb{X}}$, $K_f X_f \subset \bar{\mathbb{U}}$,

A3. $V_f(A_{K_f}x) + \ell(x, K_f x) \leqslant V_f(x) \ \forall x \in X_f$,

where K_f is not necessarily the same as K. The solution of $P_N(\hat{x})$ is $(\bar{x}^*(\hat{x}), \bar{\mathbf{u}}^*(\hat{x}))$ from which, as shown in Mayne et al. (2005), the model predictive control law $\kappa_N(\cdot)$ is obtained

$$\kappa_N(\hat{x}) \triangleq \bar{u}_0^*(\hat{x}) + K(\hat{x} - \bar{x}^*(\hat{x})), \tag{22}$$

where $\bar{u}_0^*(\hat{x})$ is the first element in the sequence $\bar{\mathbf{u}}^*(\hat{x})$.

Definition 2. A set $\mathscr{R}$ is robustly exponentially stable (Lyapunov stable and exponentially attractive) for the system $x^+ = f(x, \kappa(x), w)$, $w \in \mathbb{W}$, with a region of attraction $\mathscr{X}$ if there exists a $c > 0$ and a $\gamma \in (0, 1)$ such that any solution $x(\cdot)$ of $x^+ = f(x, \kappa(x), w)$ with initial state $x(0) \in \mathscr{X}$, and admissible disturbance sequence $w(\cdot)$ ($w(i) \in \mathbb{W}$ for all $i \in \mathbb{N}$) satisfies $d(x(i), \mathscr{R}) \leqslant c\gamma^i d(x(0), \mathscr{R})$ for all $i \in \mathbb{N}$.

We can now establish our second main result:

Theorem 2. Suppose $\bar{X}_N$ is bounded. Then, (i) the set $\bar{S} \times \tilde{S}$ is robustly exponentially stable for the composite system $\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}) + \bar{\delta}$, $\tilde{x}^+ = A_L\tilde{x} + \tilde{\delta}$ with a region of attraction $\hat{X}_N \times \tilde{S}$ where $\hat{X}_N \triangleq \bar{X}_N \oplus \bar{S}$ and, (ii) any state $x(0) = \hat{x}(0) + \tilde{x}(0)$ where $(\hat{x}(0), \tilde{x}(0)) \in \hat{X}_N \times \tilde{S}$ is robustly steered to S exponentially fast while satisfying the state and control constraints.

Proof. (i) Since $\tilde{x}(0) \in \tilde{S}$, it follows from Proposition 1 that $\tilde{x}(i) \in \tilde{S}$ for all $i \in \mathbb{N}$. It follows that the uncertain observer system (15) is identical to the uncertain system considered in Mayne et al. (2005) with $\hat{x}$ replacing x, $\bar{\delta}$ replacing w, and $\bar{\Delta}$ replacing W, and that problem $P_N(\hat{x})$ is, with these substitutions, identical to problem $P_N^*(x)$ considered in Mayne et al. (2005). It follows from Theorem 1 in Mayne et al. (2005) that the set $\bar{S}$ is robustly exponentially stable for the controlled (uncertain) observer system $\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}) + \bar{\delta}$ which establishes (i). Part (ii) of the theorem follows from the fact that $x(i) \in \hat{x}(i) \oplus \tilde{S}$ for all $i \in \mathbb{N}$. $\square$

Note that, from Mayne et al. (2005) and Theorem 2, $\bar{x}^*(\hat{x}(i)) \to 0$, $d(\hat{x}(i), \bar{S}) \to 0$ and $d(x(i), S) \to 0$ robustly and exponentially fast as $i \to \infty$. For each j, let $\bar{X}_j \triangleq \{\bar{x} \mid \mathscr{U}_j(\bar{x}) \neq \emptyset\}$ where $\mathscr{U}_j(\bar{x})$ is defined by (20), j replacing N. If **A2** holds, the set sequences $\{\bar{X}_j\}$, $\{\bar{X}_j \oplus \bar{S}\}$ and $\{\bar{X}_j \oplus S\}$ are monotonically non-decreasing, i.e. $\bar{X}_j \subseteq \bar{X}_{j+1}$ for all $j \in \mathbb{N}$. This guarantees the recursive feasibility of the problem $P_N(\hat{x})$. Problem $P_N(\hat{x})$ is a quadratic program only marginally more complex than that required for standard linear model predictive control if the sets $\tilde{S}$, $\bar{S}$, S and X_f, which can be precomputed, are polytopic.

5. Brief example

Our illustrative example is a double integrator:

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w, \quad y = [1 \quad 1]x + v$$

with additive disturbances $(w, v) \in \mathbb{W} \times \mathbb{V}$ where $\mathbb{W} \triangleq \{w \in \mathbb{R}^2 \mid \|w\|_\infty \leqslant 0.1\}$ and $\mathbb{V} \triangleq \{v \in \mathbb{R} \mid |v| \leqslant 0.05\}$. The state and
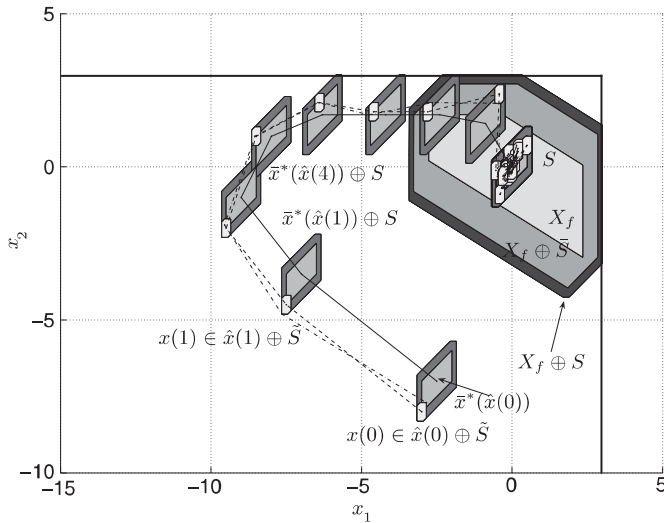
Fig. 1. The robust output-feedback MPC tube $\{\bar{x}^*(\hat{x}(i)) \oplus S\}$.

control constraints are $(x, u) \in \mathbb{X} \times \mathbb{U}$ where $\mathbb{X} \triangleq \{x \in \mathbb{R}^2 \,|\, x^1 \in [-50, 3], \, x^2 \in [-50, 3]\}$ and $\mathbb{U} \triangleq \{u \in \mathbb{R} \,|\, |u| \leqslant 3\}$ (x^i is the ith coordinate of a vector x). The control matrix K and the output injection matrix L are $K = [1\ 1]$, $L = [1\ 1]'$. The cost function is defined by (16)–(17) with $Q = I$, $R = 0.01$; the terminal cost $V_f(x)$ is the value function $(1/2)x'P_f x$ for the unconstrained optimal control problem for the nominal system $\bar{x}^+ = A\bar{x} + B\bar{u}$ and $u = K_f x$ is the associated LQR controller. The robust positively invariant sets $\tilde{S}$, $\bar{S}$ and S are precomputed using results in Raković et al. (2005); Raković (2005). The terminal constraint set X_f satisfies **A2** and **A3** and is the maximal positively invariant set (Blanchini, 1999; Gilbert & Tan, 1991) for system $\bar{x}^+ = (A + BK_f)\bar{x}$ under the tighter constraints $\bar{\mathbb{X}} = \mathbb{X} \ominus S$ and $\bar{\mathbb{U}} = \mathbb{U} \ominus K\bar{S}$. The horizon is $N = 13$. The sets $\bar{X}_j$ expand into the bottom left quadrant of $\mathbb{R}^2$ as j increases. The tube $\{\bar{x}^*(\hat{x}(i)) \oplus S\}$ that bounds the unknown state trajectory, for a random initial state $x_0 \in \hat{x}_0 \oplus \tilde{S}$ where $\hat{x}_0 = (-3, -8)'$ and a random sequence of extreme admissible state and output disturbances, is shown in Fig. 1.

6. Conclusions

This paper presents a simple output feedback model predictive controller for constrained linear systems with input and output disturbances. Simplicity is achieved by using a Luenberger observer (that is less complex than a set-membership state estimator), an invariant set that bounds the estimation error and a tube-based robust model predictive controller that is almost as simple to implement as a nominal linear model predictive controller. Robust exponential stability is established.

References

Atassi, A. N., & Khalil, H. K. (1999). A separation principle for the stabilization of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 44(9), 1672–1687.

Aubin, J. P. (1991). *Viability theory. Systems & control: Foundations & applications*. Boston, Basel, Berlin: Birkhauser.

Bemporad, A., & Garulli, A. (2000). Output-feedback predictive control of constrained linear systems via set-membership state estimation. *International Journal of Control*, 73(8), 655–665.

Bertsekas, D. P., & Rhodes, I. B. (1971a). On the minimax reachability of target sets and target tubes. *Automatica*, 7, 233–247.

Bertsekas, D. P., & Rhodes, I. B. (1971b). Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, 16, 117–128.

Blanchini, F. (1990). Control synthesis of discrete time systems with control and state bounds in the presence of disturbance. *Journal of Optimization Theory and Applications*, 65(1), 29–40.

Blanchini, F. (1999). Set invariance in control. *Automatica*, 35, 1747–1767 (survey paper).

Cannon, M., Kouvaritakis, B., Lee, Y. I., & Brooms, A. C. (2001). Efficient non-linear model based predictive control. *International Journal of Control*, 74(4), 361–372.

Chisci, L., & Zappa, G. (2002). Feasibility in predictive control of constrained linear systems: The output feedback case. *International Journal of Robust and Nonlinear Control*, 12(5), 465–487.

Findeisen, R., Imsland, L., Allgöwer, F., & Foss, B. A. (2003). State and output feedback nonlinear model predictive control: An overview. *European Journal of Control*, 9(2–3), 190–206 (Survey paper).

Fukushima, H., & Bitmead, R. B. (2005). Robust constrained predictive control using comparison model. *Automatica*, 41(1), 97–106.

Gilbert, E. G., & Tan, K. T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36, 1008–1020.

Glover, J. D., & Schweppe, F. C. (1971). Control of linear dynamic systems with set constrained disturbances. *IEEE Transactions on Automatic Control*, 16(5), 411–423.

Grimm, G., Messina, M. J., Teel, A. R., & Tuna, S. (2004). Examples when model predictive control is nonrobust. *Automatica*, 40, 1729–1738.

Kolmanovsky, I., & Gilbert, E. G. (1998). Theory and computation of disturbance invariance sets for discrete-time linear systems. *Mathematical Problems in Engineering: Theory, Methods and Applications*, 4, 317–367.

Kouvaritakis, B., Wang, W., & Lee, Y. I. (2000). Observers in nonlinear model-based predictive control. *International Journal of Robust and Nonlinear Control*, 10(10), 749–761.

Kurzhanski, A. B., & Filippova, T. F. (1993). On the theory of trajectory tubes: A mathematical formalism for uncertain dynamics, viability and control. In A. B. Kurzhanski (Ed.), *Advances in nonlinear dynamics and control: A report from Russia, ser. PSCT 17*. Boston, Basel, Berlin: Birkhauser.

Kurzhanski, A., & Vályi, I. (1997). *Ellipsoidal calculus for estimation and control. Systems & control: Foundations & applications*. Boston, Basel, Berlin: Birkhauser.

Lee, Y. I., & Kouvaritakis, B. (2001). Receding horizon output feedback control for linear systems with input saturation. *IEE Control Theory and Applications*, 148(2), 109–115.

Löfberg, J. (2002). Towards joint state estimation and control in minimax MPC. In *Proceedings of 15th IFAC world congress*, Barcelona, Spain, 2002. Paper ID 1449 on CD-ROM.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36, 789–814 (Survey paper).

Mayne, D. Q., Seron, M., & Raković, S. V. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41, 219–224.

Moitié, R., Quincampoix, M., & Veliov, V. M. (2002). Optimal control of discrete-time uncertain systems with imperfect measurement. *IEEE Transactions on Automatic Control*, 47(11), 1909–1914.

Quincampoix, M., & Veliov, V. M. (2005). Optimal control of uncertain systems with incomplete information for the disturbances. *SIAM*, 43(4), 1373–1399.

Raković, S. V. (2005). *Robust control of constrained discrete time systems: Characterization and implementation*. Ph.D. thesis, Imperial College London, London, UK.

Raković, S. V., Kerrigan, E. C., Kouramas, K. I., & Mayne, D. Q. (2005). Invariant approximations of the minimal robustly positively invariant sets. *IEEE Transactions on Automatic Control*, 50(3), 406–410.

Schweppe, F. C. (1968). Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Transactions on Automatic Control*, *13*(1), 22–28.

Scokaert, P. O. M., Rawlings, J. B., & Meadows, E. S. (1997). Discrete-time stability with perturbations: Application to model predictive control. *Automatica*, *33*(3), 463–470.

Shamma, J. S. (2000). Anti-windup via constrained regulation with observers. *System & Control Letters*, *40*, 261–268.

Shamma, J. S., & Tu, K. Y. (1998). Output feedback control for systems with constraints and saturations: Scalar control case. *System & Control Letters*, *35*, 1–11.

Teel, A., & Praly, L. (1995). Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, *33*(5), 1443–1488.

Witsenhausen, H. S. (1968a). A minimax control problem for sampled linear systems. *IEEE Transactions on Automatic Control*, *13*(1), 5–21.

Witsenhausen, H. S. (1968b). Sets of possible states of linear systems given perturbed observations. *IEEE Transactions on Automatic Control*, 556–558.

David Mayne received the Ph.D. and D.Sc degrees from the University of London, and the degree of Doctor of Technology, honoris causa, from the University of Lund, Sweden. He has held posts at the University of the Witwatersrand, the British Thomson Houston Company, University of California, Davis and Imperial College London where he is now Senior Research Fellow. His research interests include optimization, optimization based design, nonlinear control and model predictive control.

Saša V. Raković received the B.Sc. degree in Electrical Engineering from the Technical Faculty Čačak, University of Kragujevac (Serbia and Montenegro), the M.Sc degree in Control Engineering and the Ph.D. degree in Control Theory from Imperial College London. He is currently employed as a Research Associate in the Control and Power Research Group at Imperial College London. His research interests include set invariance, robust model predictive control and optimization based design.

Rolf Findeisen received the M.Sc. degree in Chemical Engineering from the University of Wisconsin, Madison and the Diploma degree in Engineering Cybernetics and the Dr.-Ing. from the University of Stuttgart. He is currently Habilitand (assistant professor) at the University of Stuttgart. His research interests include nonlinear system theory and control, particularly predictive and optimization based control, with applications in mechatronics, process control, and biology.

Frank Allgöwer is the director of the Institute for Systems Theory and Automatic Control at the University of Stuttgart. He studied Engineering Cybernetics and Applied Mathematics in Stuttgart and at UCLA, respectively, and received his Ph.D. degree in chemical engineering from the University of Stuttgart. He is an editor of Automatica, associated editor of a number of journals and organizer and co-organizer of several international conferences. His main areas of interest are in nonlinear, robust and predictive control, with application to a wide range of fields.