4. Linear-quadratic control

In this chapter, we will derive optimal control laws for linear systems with quadratic cost functions in the state and controls. A significant effort is made to convey insight into key properties of the optimal control law and how the loss function can be altered to achieve desired properties of the closed-loop system. In addition, we consider the dual problem of estimating the system state vector of a discrete-time linear system in a recursive manner in a way that minimizes a specific least-squares cost, leading up to a deterministic version of the Kalman filter. Output feedback control combining the linear qudratic regulator and the Kalman filter will also be discussed.

4.1 The linear-quadratic regulator

The linear quadratic control problem considers a linear system

$$x_{t+1} = Ax_t + Bu_t \tag{4.1}$$

and aims at finding the control sequence $\{u_0, u_1, \dots, u_{N-1}\}$ which minimizes the quadratic cost

$$J(u_0, \dots, u_{N-1}; x_0) = x_N^T Q_f x_N + \sum_{t=0}^{N-1} (x_t^T Q_1 x_t + u_t^T Q_2 u_t)$$

for given matrices $Q_1 \succeq 0$, $Q_2 \succ 0$ and $Q_f \succeq 0$. The stage cost encourages driving the system state to zero (making $x_t^T Q_1 x_t$ small) and using small energy inputs (making $u_t^T Q_2 u_t$ small). By tuning the weight matrices, we can choose to put more emphasis on achieving efficient regulation or energy-efficient control, effectively adjusting the closed-loop bandwidth.

4.1.1 The finite-horizon optimal controller

Let us first consider the problem over a finite horizon N. The optimal control law, summarized below, can then be computed via dynamic programming.

Theorem 4.1.1 Consider the linear system (4.1). The control $\{u_0, u_1, \dots, u_{N-1}\}$ which minimizes the cost

$$J_N = x_N^T Q_f x_N + \sum_{t=0}^{N-1} x_t^T Q_1 x_t + u_t^T Q_2 u_t$$

where $Q_1 \succeq 0$, $Q_2 \succ 0$ and $Q_f \succeq 0$ is the time-varying linear state-feedback

$$u_t = -L_t x_t$$

with

$$L_t = (B^T P_{t+1} B + Q_2)^{-1} B^T P_{t+1} A$$

and where P_t satisfies the Riccati recursion

$$P_t = Q_1 + A^T P_{t+1} A - A^T P_{t+1} B (B^T P_{t+1} B + Q_2)^{-1} B^T P_{t+1} A \tag{4.2}$$

with boundary condition $P_N = Q_f$. The minimal value of the loss function is $J_N^\star = x_0^T P_0 x_0$.

Proof. We will derive the optimal control law using dynamic programming. To this end, we use induction to show that the cost-to-go function is quadratic

$$v_t(x_t) = x_t^T P_t x_t.$$

and derive an explicit expression on how P_{t-1} depends on P_t.

First, notice that the induction hypothesis is satisfied for $t = N$ with $P_N = Q_f$. Now, for an arbitrary stage t the dynamic programming recursion (3.7) gives that

$$\begin{aligned} v_t(x_t) &= \min_{u_t} \left\{ x_t^T Q_1 x_t + u_t^T Q_2 u_t + v_{t+1}(Ax_t + Bu_t) \right\} = \\ &= \min_{u_t} \left\{ u_t^T (B^T P_{t+1} B + Q_2) u_t + 2u_t^T B^T P_{t+1} A x_t + x_t^T (A^T P_{t+1} A + Q_1) x_t \right\} \end{aligned}$$

By completion-of-squares, Lemma 3.1.1, the optimal action is

$$u_t^\star = -(B^T P_{t+1} B + Q_2)^{-1} B^T P_{t+1} A x_t := -L_t x_t$$

and the associated cost-to-go becomes

$$v_t = x_t^T \left(Q_1 + A^T P_{t+1} A - A^T P_{t+1} B (B^T P_{t+1} B + Q_2)^{-1} B^T P_{t+1} A \right) x_t := x_t^T P_t x_t$$

We notice that the cost-to-go function is indeed quadratic, and by induction, remains quadratic for all $t = N, N-1, \dots, 1, 0$. The proof is complete. ■

4.1.2 The infinite-horizon case

It is natural to ask if the solution presented in Theorem 4.1.1 remains valid when the horizon tends to infinity. Can we ascertain that the optimal cost is finite? Will the Riccati recursion and the associated state feedback gains converge?

We can immediately observe that without additional assumptions, the LQ-cost may indeed grow unbounded. For example, the system

$$x_{t+1} = 2x_t + 0u_t$$

will clearly have an unbounded LQ cost as $N \to \infty$. If (A,B) is reachable, on the other hand, then we know that there exists a bounded control sequence $\{u_0, u_1, \dots, u_{n-1}\}$ that drives any initial state to the origin in n steps. Once at the state has reached the origin, we can apply $u_t = 0$ for all future times to ensure that x_t remains at rest at the origin. This control sequence has a finite LQ-cost, so the *optimal* infinite-horizon cost must also be finite.

For the infinite-horizon case, the terminal cost will not matter, and we will consider problems with cost functions on the form

$$J_\infty = \sum_{t=0}^{\infty} x_t^T Q_1 x_t + u_t^T Q_2 u_t \tag{4.3}$$

with $Q_2 \succ 0$. The associated value function $v : \mathbb{R}^n \mapsto \mathbb{R}$ is

$$v(z) = \min_{u_0, u_1, \dots} \left\{ \sum_{t=0}^{\infty} x_t^T Q_1 x_t + u_t^T Q_2 u_t \mid x_{t+1} = Ax_t + Bu_t,\ x_0 = z \right\}$$

Contrary to the finite-horizon case, the value-function does not depend on time since the remaining horizon is always infinite. Hence, the DP recursion reduces to the Bellman equation

$$v(z) = \min_u \left\{ z^T Q_1 z + u^T Q_2 u + v(Az + Bu) \right\}$$

It turns out that the cost-to-go is quadratic in the current state, *i.e.* $v(z) = z^T P z$ for some positive definite matrix P. Hence, it satisfies

$$\begin{aligned} z^T P z &= \min_u \left\{ z^T Q_1 z + u^T Q_2 u + (Az + Bu)^T P (Az + Bu) \right\} = \\ &= \min_u \left\{ z^T (Q_1 + A^T P A) z + 2 z^T A^T P B u + u^T (Q_2 + B^T P B) u \right\} = \\ &= z^T \left(Q_1 + A^T P A - A^T P B (Q_2 + B^T P B)^{-1} B^T P A \right) z \end{aligned}$$

where the last step follows from the completion-of-squares Lemma. Since this relationship holds for all z, P must satisfy the *algebraic Riccati equation (ARE)*

$$P = Q_1 + A^T P A - A^T P B (Q_2 + B^T P B)^{-1} B^T P A \tag{4.4}$$

We also note that the minimizing u in the Bellman equation is $u = -Lx$ where

$$L = (Q_2 + B^T P B)^{-1} B^T P A.$$

As discussed above, if (A,B) is reachable, the optimal LQR cost is finite. However, this does not necessarily imply that the closed-loop closed loop is stable, since Q_1 only needs to be positive semidefinite and certain (linear combinations of) states may not be penalized by the cost function. We can get additional insight into the potential stability problems by re-writing the ARE (4.4) as

$$\begin{aligned} P &= Q_1 + A^T P A - L^T (Q_2 + B^T P B) L = \\ &= Q_1 + (A - BL)^T P (A - BL) - L^T Q_2 L + 2(A - BL)^T P B L = \\ &= (A - BL)^T P (A - BL) + Q_1 + L^T Q_2 L \end{aligned} \tag{4.5}$$

where the last inequality follows from the identity $(A - BL)^T P B L = L^T Q_2 L$. We notice that the solution P to the ARE satisfies a Lyapunov equation, but closed-loop stability cannot be ensured with the given assumptions, since $Q_1 + L^T Q_2 L$ can only be guaranteed to be positive semidefinite. However, the next lemma shows that if $(A, Q_1^{1/2})$ observable, then Theorem 2.2.5 applies and closed-loop stability is guaranteed.

Proposition 4.1.2 Let $Q_2 \in \mathbb{R}^{m\times m}$ be positive definite and let $L \in \mathbb{R}^{m\times n}$. If $(A, Q_1^{1/2})$ is observable, then so is $(A - BL, (Q_1 + L^T Q_2 L)^{1/2})$

Proof. If $(A - BL, (Q_1 + L^T Q_2 L)^{1/2})$ would be unobservable, then there would exist $v \neq 0$ such that

$$(A - BL)v = \lambda v \qquad (Q_1 + L^T Q_2 L)^{1/2} v = 0$$

The second equality would then imply that

$$v^T (Q_1 + L^T Q_2 L)v = \|Q_1^{1/2} v\|_2^2 + \|Q_2^{1/2} L v\|_2^2 = 0,$$

i.e. that $Q_1^{1/2} v = 0$ and $Lv = 0$. Thus,

$$(A - BL)v = Av = \lambda v \qquad Q_1^{1/2} v = 0$$

which contradicts that $(A, Q_1^{1/2})$ is observable. □

We are now ready to summarize our results for infinite-horizon LQR.

Theorem 4.1.3 Consider the linear system (4.1) and the infinite-horizon cost

$$J_\infty = \sum_{t=0}^{\infty} x_t^T Q_1 x_t + u_t^T Q_2 u_t$$

with $Q_1 \succeq 0$ and $Q_2 \succ 0$. The optimal control is a static linear feedback $u_t = -Lx_t$ where

$$L = (Q_2 + B^T P B)^{-1} B^T P A \tag{4.6}$$

and P satisfies the algebraic Riccati equation

$$P = Q_1 + A^T P A - A^T P B (Q_2 + B^T P B)^{-1} B^T P A \tag{4.7}$$

If (A, B) is reachable, then the cost is bounded. If, in addition, $(A, Q_1^{1/2})$ is observable, then (4.7) admits a unique positive definite solution and the closed loop is asymptotically stable.

It turns out that Theorem 4.1.3 also holds when (A, B) is stabilizable and $(A, Q_1^{1/2})$ is detectable. However, the proof becomes a little bit more technical, so we refer to, *e.g.*, [**AnM:71**] for details.

Loss functions with cross-terms between control and state penalties*

As we will see later in this chapter, it is sometimes useful to allow for cross-terms between the control and the state in LQR design. We are thus interested in cost functions on the form

$$J = \sum_{t=0}^{N} x_t^T Q_1 x_t + u_2^T Q_2 u_2 + 2x_t^T Q_{12} u_t = \sum_{t=0}^{N} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

In addition to the assumption that $Q_2 \succ 0$, we will also need to assume that $Q_1 - Q_{12} Q_2^{-1} Q_{12}^T \succeq 0$. By the completion-of-squares lemma, it holds that

$$\begin{aligned} x_t^T Q_1 x_t + u_t^T Q_2 u_t + 2x_t^T Q_{12} u_t &= (u_t + Q_2^{-1} Q_{12}^T x_t)^T Q_2 (u_t + Q_2^{-1} Q_{12}^T x_t) + x_t^T (Q_1 - Q_{12} Q_2^{-1} Q_{12}^T) x_t = \\ &= \tilde{u}_t^T Q_2 \tilde{u}_t + x_t^T \tilde{Q}_1 x_t \end{aligned}$$

This indicates that we can get rid of the cross-terms by considering $\tilde{u}_t$ as the input to our linear system and modifying the matrices defining the cost function. Specifically, if we consider

$$\tilde{u}_t = u_t + Q_2^{-1} Q_{12}^T x_t$$

as an input to our system, *i.e.*, study the linear system

$$x_{t+1} = \underbrace{(A - BQ_2^{-1}Q_{12}^T)}_{\tilde{A}} x_t + B\tilde{u}_t := \tilde{A}x_t + B\tilde{u}_t \tag{4.8}$$

then the corresponding cost function

$$\tilde{J} = \sum_{t=0}^{N} x_t^T \tilde{Q}_1 x_t + \tilde{u}_t^T Q_2 \tilde{u}_t \tag{4.9}$$

does not have any cross-term and can be dealt with using the results that we have derived earlier. Clearly, the optimal values of J and $\tilde{J}$ are the same, and if the optimal control under modified dynamics (4.8) and modified cost (4.9) is

$$\tilde{u}_t^\star = -\tilde{L}x_t$$

then the optimal control for the problem with cross-terms is

$$u_t^\star = -(\tilde{L} + Q_2^{-1}Q_{12}^T)x_t.$$

This observation holds for both the finite and infinite horizon problems derived above, and allow us to use all the preceding theory also for systems with cross-terms.

4.1.3 Properties of the LQ-optimal control law

The LQ cost function attempts to strike a balance between the transient response and control effort. We will now explore this trade-off in some more detail, in order to gain insight into how the LQ-optimal controller depends on the weight matrices Q_1 and Q_2. We begin with a numerical example to develop a basic intuition.

■ **Example 4.1** Consider the mechanical system model from Example 3.2

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0.95 & 0.94 \\ -0.09 & 0.86 \end{bmatrix} x_t + \begin{bmatrix} 0.48 \\ 0.94 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \end{aligned} \tag{4.10}$$

We study the LQ-optimal controllers under the criterion given by

$$Q_1 = C^T C, \qquad Q_2 = \rho I, \qquad Q_{12} = 0$$

for different values of the parameter ρ. Figure 4.1 shows the output and control signals from the same initial value but for three different values of ρ. As expected, a larger value of ρ results in restrictive use of input energy at the expense of a slower output response, while a low value of ρ gives a fast response but large control actions. ■

It turns out that we can be much more precise about the trade-off between transient response and control effort. For simplicity of exposition, let us consider a scalar system

$$x_{t+1} = ax_t + u_t$$

and the infinite-horizon criterion defined by $Q_1 = 1$ and $Q_2 = \rho$, *i.e.*

$$J = \sum_{k=0}^{N} x_k^2 + \rho u_k^2.$$

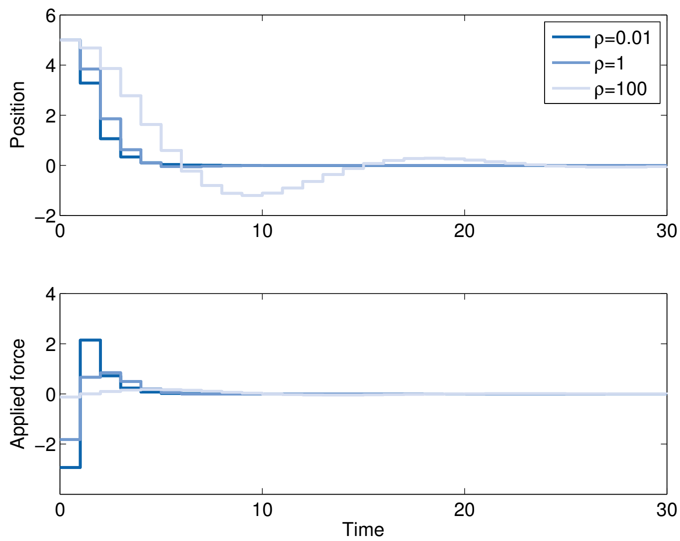

Figure 4.1: Initial value responses of LQR-controller for three different values of the ρ parameter.

To find the optimal control, we form the associated algebraic Riccati equation

$$P = 1 + a^2 P - \frac{a^2 P^2}{\rho + P}.$$

By re-writing the ARE as

$$P^2 - \underbrace{(1 - \rho(a^2 - 1))}_{\phi(\rho)} P - \rho = 0$$

we find the positive solution

$$P^\star = \frac{1}{2}\phi(\rho) + \frac{1}{2}\sqrt{\phi^2(\rho) + 4\rho}$$

and the optimal feedback gain

$$L^\star = \frac{aP^\star}{\rho + P^\star}.$$

Notice that already for the scalar case, the optimal feedback gain has a complicated analytical expression. However, we can understand its properties at the extremes when $\rho \to 0$ and $\rho \to \infty$.

When $\rho \to 0$ ("control is cheap"), $\phi(\rho) \to 1$, $P^\star \to 1$ and $L^\star \to a$. Thus, the closed loop dynamics becomes $x_{t+1} = (a - L^\star)x_t \to 0$, *i.e.* the closed-loop pole tends to the origin and the state converges to zero in a single time step.

As $\rho \to \infty$ ("control is expensive"), on the other hand,

$$\frac{P^\star}{\rho} \to \frac{a^2 - 1}{2} + \left|\frac{a^2 - 1}{2}\right|$$

and the closed-loop dynamics tends to

$$x_{t+1} = \begin{cases} ax_t & \text{if } |a| \le 1 \\ a^{-1}x_t & \text{otherwise .} \end{cases}$$

Thus, if the open-loop is stable it is optimal to not apply any control. If the system is unstable, on the other hand, we need to stabilize it and it is optimal to place the closed-loop pole in $1/a$.

It turns out that the properties for the cheap and expensive control scenarios that we have discovered in the scalar case also hold for linear systems with arbitrary state, input and output dimension. The result for multiple-input multiple-output systems require some concepts that are outside the scope of these notes, but we can state the results for single-input single-output systems.

Theorem 4.1.4 Consider the linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned}$$

with $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}$ and $y_t \in \mathbb{R}$. Let (A,B) be reachable, (A,C) be observable and consider the cost criterion

$$J = \sum_{t=0}^{\infty} y_t^2 + \rho u_t^2$$

Assume that the open loop system from u_t to y_t has q zeros located at $z_1, \dots, z_q$, p poles at the origin and $n-p$ poles located at p_i. Then,

(a) as $\rho \to \infty$, $n-p$ closed-loop poles remain at zero and the others tend to

$$\pi = \begin{cases} p_i & \text{if } |p_i| \leq 1 \\ p_i^{-1} & \text{if } |p_1| \geq 1 \end{cases}$$

(b) as $\rho \to 0$, $n-q$ closed-loop poles tend to zero, and the remaining ones to

$$\pi = \begin{cases} z_i & \text{if } |z_i| \leq 1 \\ z_i^{-1} & \text{if } |z_i| \geq 1 \end{cases}$$

The proof of this theorem, along with generalizations to the multiple-input multiple-output case can be found in Kwakernaak and Sivan [**KwS:72**]. The next example illustrates the use of the theorem on several representative systems.

■ **Example 4.2** First, consider the system

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \end{aligned}$$

with $a_{21} = -0.16$ and $a_{22} = 1$. This system has two stable poles in 0.2 and 0.8, and no zeros. Figure 4.2(left) shows how the poles are left unaltered when ρ is large, but are then gradually moved towards the origin as the control becomes increasingly cheap. Next, we consider the system given by $a_{21} = -0.248$ and $a_{22} = 1.45$. This system has one stable pole in 0.2 and one unstable pole in 1.25. As seen in Figure 4.2 (right), the LQ-optimal controller places the poles in 0.2 and $1/1.25 = 0.8$ when control is expensive, and then moves the closed loop poles increasingly close to the origin as the control becomes cheaper.

To show the influence of zeros, we change the B matrix to be on the form

$$B = \begin{bmatrix} 1 \\ a_{22} - \tilde{b} \end{bmatrix}$$

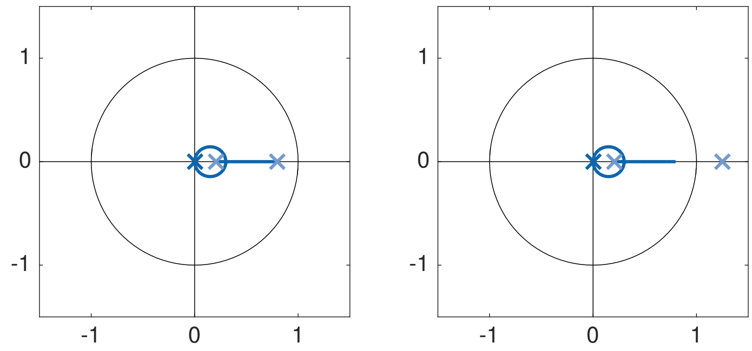

Figure 4.2: Open-loop pole loections (light crosses) and closed-loop poles for cheap control (dark crosses). The dark line represent the pole locations when the control goes from expensive to cheap. Left figure is for a system with two open-loop stable poles, while the right figure is for a system with one (open-loop) stable and one unstable pole.

which introduces a zero at $\tilde{b}$. We consider the stable A-matrix above, and use two values of $\tilde{b}$. First, we let $\tilde{b} = 0.5$. Figure 4.3 (left) shows how the closed-loop poles start at the open loop locations, and then move to the origin and to the stable zero location as control becomes increasingly cheap. We then let $\tilde{b} = 4/3$. Again, as seen in Figure 4.3 (right), the closed-loop poles move form their open-loop location to the origin and $1/\tilde{b}$, respectively. ■

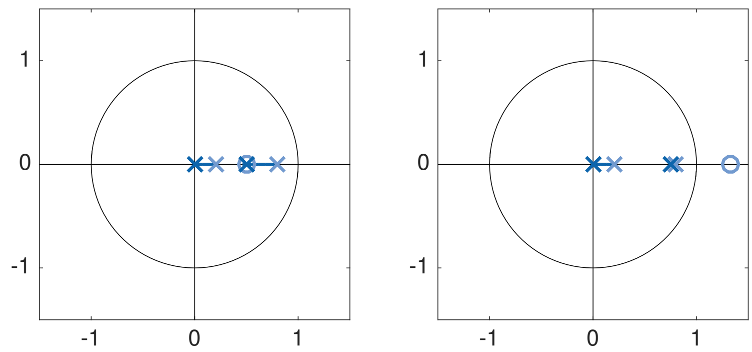

Figure 4.3: Open-loop poles (light crosses) and zeros (light rings). Dark crosses represent closed-loop poles under cheap control, and the dark line shows how the closed-loop poles move when control goes from expensive to cheap. Left figure is for a system with a zero inside the unit disc, while the right figure is for a system with a zero outside the unit disc.

As illustrated in the previous example, the closed-loop poles move smoothly from one extreme to the other as we change the weight parameter ρ. The next example validates that this effect.

An optimal trade-off*

The trade-off that we have just explored is, in a certain sense, optimal. Specifally, consider the following energy-constrained optimal control problem

$$\begin{array}{ll} \text{minimize} & \sum_t y_t^2 \\ \text{subject to} & \sum_t u_t^2 \leq e_{\max} \\ & x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t \end{array}$$

By introducing a Lagrange multiplier λ for the energy constraint (cf. Appendix C), we can form an associated dual problem whose dual function is

$$g(\lambda) = \inf_{\{u_t\}} \left\{ \sum_t x_t^T C^T C x_t + \lambda \sum_t u_t^2 \mid x_{t+1} = Ax_t + Bu_t \right\} - \lambda e_{\max}$$

We recognize the first term as the optimal LQ cost for $Q_1 = C^T C$ and $Q_2 = \lambda I$. If strong duality holds (which it does, for example, when the open loop system is stable, or when the system is controllable and the initial value is such that it is possible to drive the system to rest using less input energy than $e_{\max}$) then there is a $\lambda^\star$ (which depends on $e_{\max}$ and the system parameters) such that the optimal solution to the energy-constrained problem is given by the LQ-optimal controller for the criterion (4.3) with $\rho = \lambda^\star$. Thus, by varying ρ in the LQ criterion, we are able to trace the optimal trade-off surface between $\sum_k y_k^2$ and $\sum_k u_k^2$.

■ **Example 4.3** Figure 4.4 shows the trade-off surface between control energy and output energy for the mechanical system studied in Example 4.1. There is no controller which can do better in terms of these criteria, as long as it has access to the same information as the LQ-optimal controller. Therefore, the performance of every other controller will lie in the shaded area. ■

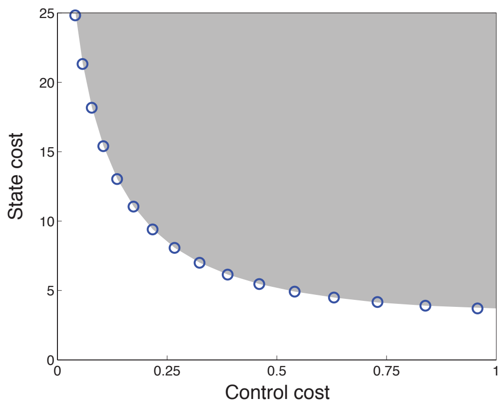

Figure 4.4: Trade-off between input and output energy for mechanical system.

4.2 Error-free reference tracking

In many cases, the aim of the control system is not to drive the system state to zero, but rather to make the output follow a given reference sequence $\{r_t\}$. It turns out that such servo problems can also be dealt with using linear-quadratic regulator theory.

A feed forward solution

Let us first assume that the reference is constant, $r_t = r$ for all t. Then, error-free tracking is possible, if there is an equilibrium state x^{eq} and corresponding constant input u^{eq} such that

$$\begin{cases} x^{\text{eq}} &= Ax^{\text{eq}} + Bu^{\text{eq}} \\ r &= Cx^{\text{eq}} \end{cases} \tag{4.11}$$

If we introduce $\Delta x_t = x_t - x^{\text{eq}}$ and $\Delta u_t = u_t - u^{\text{eq}}$, we notice that

$$\Delta x_{t+1} = A\Delta x_t + B\Delta u_t. \tag{4.12}$$

Hence, the servo problem of minimizing the criterion

$$\sum_{t=0}^{\infty} \Delta x_t^T Q_1 \Delta x_t + \Delta u_t^T Q_2 \Delta u_t \tag{4.13}$$

can be solved with standard LQR theory applied to (4.12) and yields an optimal control on the form

$$\Delta u_t = -L\Delta x_t.$$

Using the definitions of Δu_t and Δx_t, this control can be written on the form $u_t = -Lx_t + u^{\text{eq}} + Lx^{\text{eq}}$, where both u^{eq} and x^{eq} depend on r. We will now show that we can find a u^{eq} on the form

$$u^{\text{eq}} = -Lx^{\text{eq}} + l_r r \tag{4.14}$$

so that the optimal control is on the form

$$u_t = -Lx_t + u^{\text{eq}} + Lx^{\text{eq}} = -Lx_t + l_r r$$

i.e. a linear combination of feedback from states and feed-forward from the reference. Specifically, inserting this expression into (4.11), and noting that asymptotic stability of the closed-loop implies that $(I-(A-BL))$ is invertible, the equilibrium conditions are satisfied if we choose l_r such that

$$C(I-(A-BL))^{-1}Bl_r = I_{p\times p}. \tag{4.15}$$

Hence, by combining the LQR-optimal control for (A,B) with a feed-forward term $l_r r$ results in a control law which achieves error-free tracking minimizing the criterion (4.13).

A feedback solution

A drawback with feed-forward compensation is that it tends to be sensitive to modeling errors. In the solution above, l_r is computed based on perfect knowlege of A,B and C. It is often a more reliable solution to use feedback, and in particular integral action, to ensure error-free tracking. The simplest way to include integral action is to introduce a controller state which accumulates the errors between the reference and the output:

$$i_{t+1} = i_t + r_t - y_t = -Cx_t + i_t + r_t$$

Then, any equilibrium point $(x^{\text{eq}}, i^{\text{eq}}, u^{\text{eq}})$ of the augmented system

$$\begin{bmatrix} x_{t+1} \\ i_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x_t \\ i_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ I \end{bmatrix} r_t$$

satisfies $Cx^{\text{eq}} = r$. Proceeding as in the feed-forward case, we let $\Delta x_t = x_t - x^{\text{eq}}$, $\Delta i_t = i_t - i^{\text{eq}}$ and $\Delta u_t = u_t - u^{\text{eq}}$. In these coordinates, the system dynamics is given by

$$\begin{bmatrix} \Delta x_{t+1} \\ \Delta i_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} \Delta x_t \\ \Delta i_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \Delta u_t \tag{4.16}$$

so the optimal controller for the criterion

$$J = \sum_{t=0}^{\infty} \begin{bmatrix} \Delta x_t \\ \Delta i_t \end{bmatrix}^T \bar{Q}_1 \begin{bmatrix} \Delta x_t \\ \Delta i_t \end{bmatrix} + \Delta u_t^T \bar{Q}_2 \Delta u_t \tag{4.17}$$

can be computed using standard LQR theory for the augmented system (4.16) and cost (4.17), provided that the reachability and observability criteria hold for the augmented system matrices.

Proposition 4.2.1 Assume that the nominal system (A,B) is reachable. Then the augmented system is reachable if and only if the matrix

$$\begin{bmatrix} A-I & B \\ -C & 0 \end{bmatrix} \tag{4.18}$$

has full rank.

Proof. By the PBH test, the augmented system is reachable if and only if there is no λ and no $w=(w_1,w_2)\neq 0$ such that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}^T \begin{bmatrix} A & 0 & B \\ -C & I & 0 \end{bmatrix} = \begin{bmatrix} \lambda w_1^T & \lambda w_2^T & 0 \end{bmatrix}$$

First note that if $w_2=0$, then w_1 must also be a zero vector, due to the assumption that the nominal system is reachable. On the other hand, if $w_2\neq 0$, then any solution must have $\lambda=1$ and satisfy

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}^T \begin{bmatrix} A-I & B \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Due to the rank assumption, the only solution to this system of equations is $w=0$. Hence, by the PBH test, the augmented system is reachable. ■

The rank condition requires that $m\geq p$ and that both B and C have rank of at least p. Thus, for error-free tracking, one need at least as many inputs as the number of outputs one wants to track. The observability condition on the augmented system and $\bar{Q}_1$ will be ensured if we let

$$\begin{bmatrix} x_t \\ i_t \end{bmatrix}^T \bar{Q}_1 \begin{bmatrix} x_t \\ i_t \end{bmatrix} = x_t^T Q_1 x_t + i_t^T \operatorname{diag}(q_1,\dots,q_p) i_t$$

with $q_1,\dots q_p$ positive.

It is useful to reflect a little bit on the difference between the feedforward and the feedback approaches to error-free reference following. If the actual system differs from the model which we used to calculate the optimal feedforward compensation, then there is in general no guarantee that the output will be able to follow the reference. With integral action, on the other hand, we only need to require that the designed feedback law stabilizes the real system to guarantee that the output will converge to the desired reference value. The next example illustrates the use of integral action in linear-quadratic regulators, and also demonstrates how LQR design extends seamlessly from scalar systems to systems with multiple inputs and outputs.

■ **Example 4.4** The quadruple-tank apparatus shown in Figure 4.5 is a common laboratory process used to demonstrate various aspect of control of multiple-input/multiple-output systems. It consists of two double-tank systems, where a fraction of the inflow generated by the pump to each upper tank is fed into the lower tank of the other systems.

This cross-coupling of the inflows allows to generate a wide range of challenging dynamics. We will consider a relatively benign configuration, described by the discrete-time linear system

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 0.9921 & 0 & 0.0206 & 0 \\ 0 & 0.9945 & 0 & 0.0165 \\ 0 & 0 & 0.9793 & 0 \\ 0 & 0 & 0 & 0.9835 \end{bmatrix} x_t + \begin{bmatrix} 0.0415 & 0.0002 \\ 0.0001 & 0.0313 \\ 0 & 0.0237 \\ 0.0155 & 0 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix} x_t \end{aligned}$$

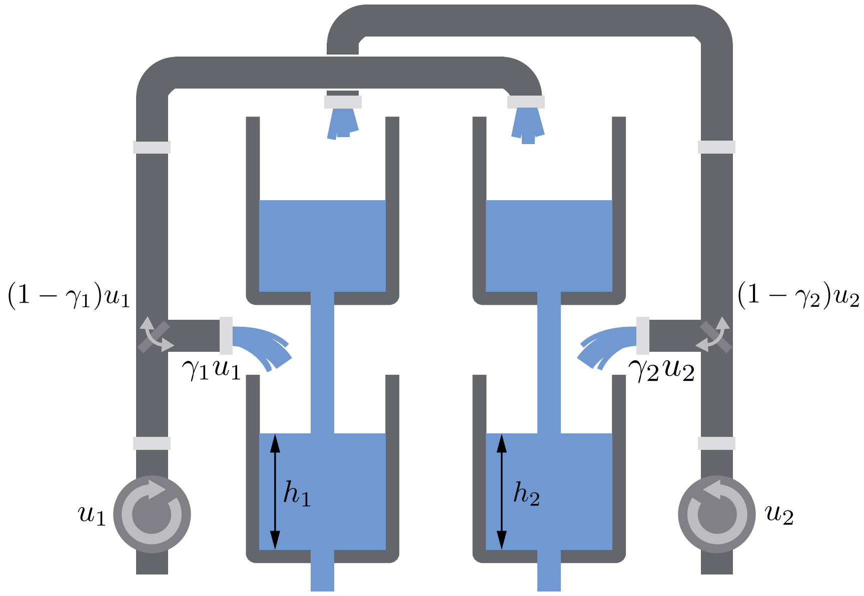

Figure 4.5: The quadruple tank apparatus.

The system has an equal number of inputs and outputs, and one can readily verify that the rank condition on (4.18) is satisfied. We perform a LQR design with

$$Q_1 = \begin{bmatrix} C^T C & 0 \\ 0 & I \end{bmatrix}, \qquad Q_2 = I.$$

Figure 4.6 shows a simulation of reference changes in the two lower tank levels, followed by added disturances (constant additional inflows) in the lower tanks. Notice how integral action allows the controller to perform error-free tracking and suppress the constant disturbances in stationarity. ■

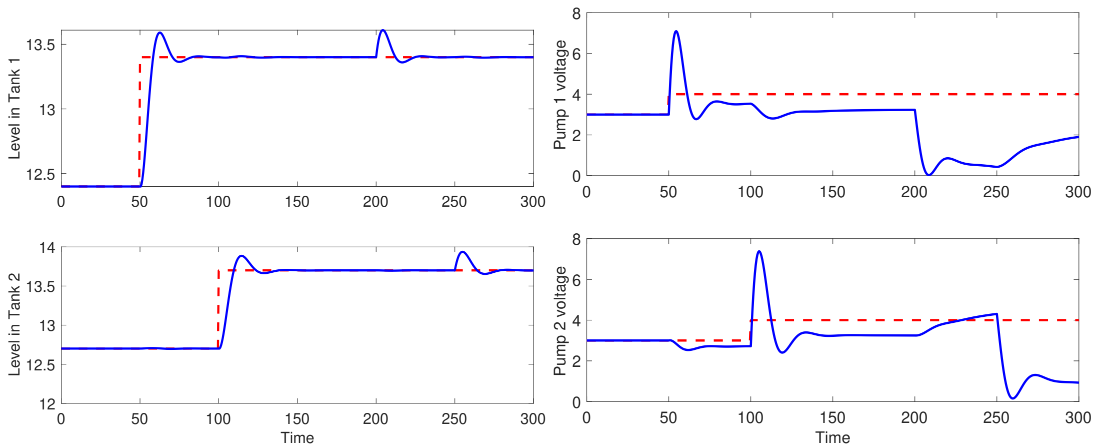

Figure 4.6: By incorporating integral action, the LQR controller allows to perform error free tracking of both outputs (left). In addition, constant disurbances are eliminated in stationarity. The associated control signals in the right figure demonstrate the multi-variable nature of the controller. For example, the reference change in the level of the first tank is dealt with by simultaneously increasing the voltage to the first pump and decreasing the voltage to the second.

4.3 Least-squares state estimation

The linear-quadratic regulator has been derived under the assumption of perfect knowledge of the full state vector. In practice, it is often impractical or even impossible to measure all the states. Instead, one attempts to estimate (or reconstruct) the state vector from the available measurements.

In the next few pages, we will show how one can construct an optimal state estimate recursively. We will also show how this estimator is intimately related to the celebrated Kalman filter for linear systems subject to stochastic disturbances on the output and the state vector.

4.3.1 A least-squares filtering principle

Assume that we are given observations $\{a_i, b_i\}_{i=1}^m$ with $a_i \in \mathbb{R}^n$ and b_i in R. We believe that the observations are related by a linear model

$$a_i^T x = b_i \qquad\qquad i = 1, \dots, m \tag{4.19}$$

for some parameter vector $x \in \mathbb{R}^n$. To compute the x which describes our observations, we organize the relations in matrix form

$$Ax - b = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} x - \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0$$

If $m = n$ and the a_i-vectors are linearly independent, then one could simply compute the unique solution $x^\star = A^{-1}b$. However, in most practical cases, there is no x that describes all observations perfectly (for example, since the linear model is wrong or the observations are subject to noise). It is then natural to introduce a residual vector $r \in \mathbb{R}^m$, write the observation equations as

$$Ax - b = r.$$

We then try to compute the parameter vector x which minimizes the size of the mismatch between the best possible model on the form (4.19) and our observations,

$$\begin{aligned} &\underset{x,r}{\text{minimize}} && r^T r \\ &\text{subject to} && Ax - b = r \end{aligned}$$

We can eliminate r from this problem and find the optimal x by minimizing

$$J = (Ax - b)^T (Ax - b).$$

The optimal solution $x^\star$ is called the *least-squares estimate*, which we have shown satisfies

$$A^T A x^\star = A^T b.$$

Even if A is not square, if A has full rank then $A^T A$ is invertible and the least-squares estimate can be written as $x^\star = A^+ b$ where $A^+ = (A^T A)^{-1} A^T$ is the Moore-Penrose pseudoinverse of A.

4.3.2 Recursive least-squares estimation

Assume that the sequences $\{x_t\}$ and $\{y_t\}$ are generated by the linear system

$$\begin{aligned} x_{t+1} &= Ax_t \\ y_t &= Cx_t \end{aligned}$$

from some initial state x_0. The state estimation problem amounts to estimating (or reconstructing) the state trajectory $\{x_t\}$ from measurements of the associated output sequence $\{y_t\}$.

In absence of disturbances, it is enough to estimate x_0, since we can then use the model to generate the future states. Earlier in these notes, we have shown that it is possible to compute a unique x_0 which matches the given output sequence if and only if the system is observable.

In practice, however, the system model will not be a perfect description of the true system dynamics and the output measurements will subject to errors. Even if the model is observable, it may then be impossible to find any initial state such that the output sequence of the model matches the observations. To reconcile these inconsistencies, we can introduce state and output disturbance sequences, $\{w_t\}$ and $\{v_t\}$, and consider the model

$$
\begin{aligned}
x_{t+1} &= Ax_t + w_t \\
y_t &= Cx_t + v_t
\end{aligned}
$$

By the least-squares principle, it is natural to try to minimize the size of the disturbance sequences which allow us to match the predicted and measured output sequences. Specifically, we will compute an estimate of the state sequence via the quadratic program

$$
\begin{array}{ll}
\underset{x_0,\{w_t\},\{v_t\}}{\text{minmize}} & (x_0-\bar{x}_0)^T R_0 (x_0-\bar{x}_0) + \sum_{k=0}^{t} w_t^T R_1 w_t + v_t^T R_2 v_t \\
\text{subject to} & x_{t+1} = Ax_t + w_t \\
& y_t = Cx_t + v_t
\end{array}
\tag{4.20}
$$

Here $\bar{x}_0$ is a prior guess of the initial state vector while R_0, R_1 and R_2 are positive semi-definite matrices. These matrices are tuning parameters for our estimate and play a similar role as the weight matrices in the linear-quadratic regulator problem. If we have high faith in the state update $x_{t+1} = Ax_t$, then we should choose a large R_1; and if we believe that our output measurements are error-free we should let R_2 to be large (relative to the other weight matrices).

The least-squares estimation problem (4.20) can be solved using dynamic programming, resulting in a the following recursive estimator of the state sequence $\{\hat{x}_{t|t}\}$

Theorem 4.3.1 The least-squares state estimation problem (4.20) can be solved recursively by repeated application of the following updates

Measurement update:

$$
\begin{aligned}
\bar{K}_t &= (S_t + C^T R_2 C)^{-1} C^T R_2 \\
\hat{x}_{t|t} &= \hat{x}_{t|t-1} + \bar{K}_t (y_t - C\hat{x}_{t|t-1}) \\
\bar{S}_t &= S_t + C^T R_2 C
\end{aligned}
$$

Prediction step:

$$
\begin{aligned}
\hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\
S_{t+1} &= R_1 - R_1 A (\bar{S}_t + A^T R_1 A)^{-1} A^T R_1
\end{aligned}
$$

from initial values $\hat{x}_{0|-1} = \bar{x}_0$ and $S_0 = R_0$.

Proof. The proof is given in Appendix D.1. ■

Note that the one-step predictions have the classical observer-structure

$$
\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + K_t (y_t - C\hat{x}_{t|t-1})
$$

where the observer gain $K_t = A\bar{K}_t$ is a time-varying and computed recursively via Theorem 4.3.1. If the system is also driven by a known control signal, *i.e.*

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

then the optimal estimator is simply

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + Bu_t + K_t(y_t - C\hat{x}_{t|t-1})$$

4.3.3 The Kalman filter

The least-squares estimator that we have derived above is intimately related to the celebrated Kalman filter. Although the Kalman filter is derived from a stochastic perspective, it can be seen as a least-squares filter which maintains

$$P_t = S_t^{-1}$$

and parameterizes the criterion as

$$J = (x_0 - \bar{x}_0)^T \Sigma_0 (x_0 - \bar{x}_0) + \sum_{t=0}^{N} w_t^T \Sigma_1 w_t + v_t^T \Sigma_2 v_t$$

where $\Sigma_0 = R_0^{-1}$, $\Sigma_1 = R_1^{-1}$ and $\Sigma_2 = R_2^{-1}$. The corresponding filter updates can be derived by applying matrix inversion identities to the updates in Theorem 4.3.1.

Theorem 4.3.2 The estimation problem (4.20) with $R_0 = \Sigma_0^{-1}$, $R_1 = \Sigma_w^{-1}$ and $R_2 = \Sigma_v^{-1}$ can be solved recursively by repeated application of the updates

Measurement update:

$$\begin{aligned}
\bar{K}_t &= P_t C^T (\Sigma_v + C^T P_t C)^{-1} \\
\hat{x}_{t|t} &= \hat{x}_{t|t-1} + \bar{K}_t (y_t - C\hat{x}_{t|t-1}) \\
\bar{P}_t &= P_t - P_t C^T (\Sigma_v + C P_t C^T)^{-1} C P_t
\end{aligned}$$

Prediction step:

$$\begin{aligned}
\hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\
P_{t+1} &= \Sigma_w + A\bar{P}_t A^T
\end{aligned}$$

from initial values $\hat{x}_{0|-1} = \bar{x}_0$ and $P_0 = \Sigma_0$.

Proof. The proof is given in Appendix D.2. ■

Analogously to the least-squares filter, a high confidence in the model translates to a small value of Σ_w, while error-free measurements translates into a small value of Σ_v. The following example works out the Kalman filter equation for a simple scalar system and verifies that the optimal estimator behaves as expected.

■ **Example 4.5** Consider the scalar system

$$\begin{aligned}
x_{t+1} &= ax_t + w_t \\
y_t &= x_t + v_t
\end{aligned}$$

with $\Sigma_w = 1$ and $\Sigma_v = r$. The stationary Kalman filter gain is

$$K = \frac{aP}{r+P}$$

where P satisfies the Riccati equation

$$P = 1 + a^2P - \frac{a^2P^2}{r+P}$$

To gain insight into the optimal estimator, we consider the dynamics of the estimation error

$$e_{t+1} = (a-K)e_t$$

When $\Sigma_v = r \to 0$ (which corresponds to error-free measurements), $K \to a$ and

$$\hat{x}_{t+1} = ay_t = ax_t$$

Thus, the observer disregards any previous information and constructs the state estimate using the last measurement only. When $r \to \infty$ (very corrupted measurements), on the other hand, we have

$$a - K \to \begin{cases} a & \text{if } |a| \leq 1 \\ a^{-1} & \text{otherwise} \end{cases}$$

Thus, if the process is open-loop stable, the estimator disregards the measurements and uses its model to predict the state estimate. If the system is unstable, on the optimal gain is such that the error dynamics are stable and its pole is the inverse of the open-loop system pole. ■

If we are only interested in the one-step ahead prediction $\hat{x}_{t|t-1}$, then it is natural to eliminate the intermediate variables $\bar{P}_t$ and write updates for P_t as

$$P_{t+1} = \Sigma_w + AP_tA^T - AP_tC^T(\Sigma_v + CP_tC^T)^{-1}CP_tA^T. \tag{4.21}$$

Many of the results that we have derived for the linear-quadratic regulator applies also to this Riccati equation. For example, any stationary solution must satisfy the algebraic Riccati equation

$$P = \Sigma_w + APA^T - APC^T(\Sigma_v + CPC^T)^{-1}CPA^T = \tag{4.22}$$

$$= (A-KC)P(A-KC)^T + \Sigma_w + K\Sigma_vK^T \tag{4.23}$$

where $K = APC^T(\Sigma_v + C^TPC)^{-1}$. If (A,C) is observable and $(A, \Sigma_v^{1/2})$ is reachable, then this equation admits a unique positive definite solution P and the estimation error dynamics $e_{t+1} = (A-KC)e_t$ are asymptotically stable. We refer to [**AnM:71**] for complete details.

A note on duality between estimation and control

The infinite-horizon control and estimation problems bear some striking ressemblances: the optimal control is a linear state feedback $u_t = -Lx_t$ that drives the state (error) dynamics $x_{t+1} = (A-BL)x_t$ to zero, while the optimal estimator adjusts the state estimates by a factor K_tCe_t such that the estimation error dynamics $e_{t+1} = (A-KC)e_t$ tends to zero asymptotically. In both cases, the optimal gains are given by the solutions to similar AREs: (4.4), (4.5) and (4.22), (4.23).

It turns out that this is no coincidence: estimation and control are dual in a precise mathematical sense. To the discrete-time linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned}$$

we can associate a dual system

$$\tilde{x}_{t+1} = A^T \tilde{x}_t + C^T \tilde{u}_t$$
$$\tilde{y}_t = B^T \tilde{x}_t$$

By applying the reachability and observability tests in Chapter 1, we notice that the original system is reachable if and only if its dual is observable (and vice versa). In addition, we notice that the optimal estimator for the primal system with LQR cost parameterized by Σ_w and Σ_v coincides with the optimal LQR controller for the dual system with $Q_1 = \Sigma_w$ and $Q_2 = \Sigma_v$ (and vice versa).

One consequence of this duality is that it is enough that numerical control design packages provide support for solving either the LQR or the Kalman filter ARE: the other one can be solved by transforming the input data as revealed by the duality argument.

We will explore another insight of the duality argument, namely how to select weight matrices in the Kalman filter to ensure fast error dynamics. Note that if we apply $\tilde{u}_t = -K^T \tilde{x}_t$ in the dual system, we get the closed-loop dynamics $\tilde{x}_{t+1} = (A^T - C^T K^T)\tilde{x}_t = (A - KC)^T \tilde{x}_t$. By the cheap control argument, we know that this closed-loop dynamics will become increasingly fast if we use a criterion that puts an increasingly large penalty on $\tilde{y}_t^T \tilde{y}_t = \tilde{x}_t^T BB^T \tilde{x}_t$, for example, letting $Q_1 = \sigma BB^T$ and $Q_2 = I$ and increasing σ. By the duality argument, we should be able to compute this K by solving the estimation problem for the original system with $\Sigma_v = \sigma BB^T$ and $\Sigma_w = I$. The next example illustrates that this intution is indeed correct.

■ **Example 4.6** Consider the mechanical system (4.10). The dual system has a two complex conjugate poles near the unit circle, and one zero at -1. Computing the optimal Kalman filter with $\Sigma_w = \sigma BB^T$ and $\Sigma_v = I$, and letting σ vary from 1 to 10000 yields the eigenvalues of $A - KC$ shown in Figure 4.7. As can be expected from the cheap control analogy, one pole tends to the origin, while the other tends to the system zero. ■

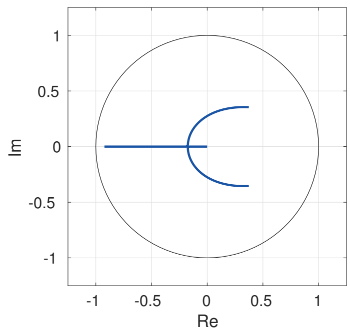

Figure 4.7: The poles of the estimation error dynamics in dark blue: one pole tends to the origin while the other one tends to the system zero.

4.4 Output feedback control

The LQR-optimal controller can be combined with the least-squares estimator to form an output feedback controller. This controller measures the system output, reconstructs the state vector using the observer, and computes the control action as a linear feedback from the estimated states; see Figure 4.8. The controller is described by the equations

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + K(y_t - \hat{y}_t) = (A - BL - KC)\hat{x}_t + Ky_t$$
$$u_t = -L\hat{x}_t$$

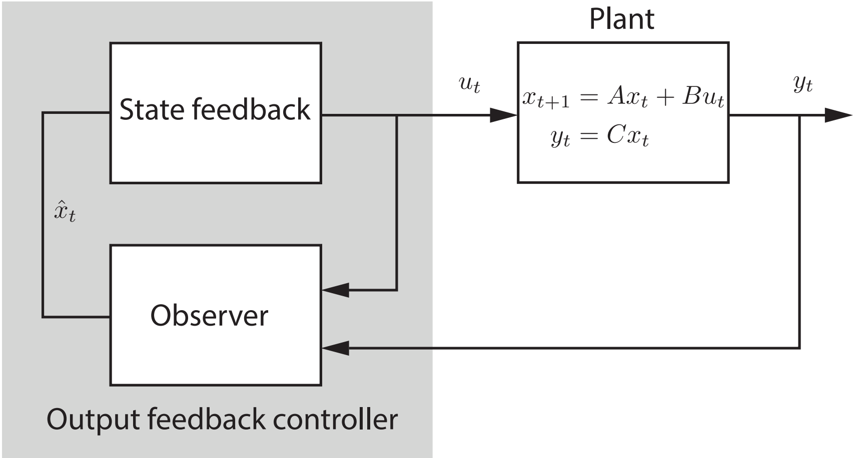

Figure 4.8: Output feedback control: combining a state estimator and linear feedback from the estimated states.

If the observer and state feedback gains are computed independently of each other, it is in general difficult to establish any optimality properties of this output feedback strategy. A notable exception is the linear-quadratic Gaussian control problem, for which the combination of a Kalman filter and LQR optimal controller indeed constitutes the optimal output feedback policy [**AnM:71**].

However, even though the LQG controller is optimal in a precise way, it turns out that it can be very sensitive to modelling errors. In practice, the combination of estimator and feedback from estimated states can often be made to work well, provided that the observer dynamics is significantly faster than the closed-loop dynamics. For the Kalman filter, we can increase the bandwidth by adding an artifical noise to the state equations, *e.g.* letting

$$\Sigma_w = \Sigma_w^{\text{nom}} + \sigma_w BB^T$$

and adjust the parameter σ_w. This procedure is known as *loop transfer recovery*, and has been analyzed thoroughly in the literature. However, the procedure is not guaranteed to give good robustness properties and the closed-loop system should always be analyzed for robustness and influence of unmodeled disturbances.

4.5 Disturbance modeling and compensation

We have discussed earlier how known exogenous signals, such as references or measurable disturbances, can be compensated for using feed-forward. When we cannot measure the disturbance signals, it may still be possible to estimate them using an observer. If we can model a disturbance signal as the output of a linear system, we can combine the disturbance model with the system dynamics into an extended system. Estimating the state of the extended system and performing feedback from the estimated state vector of the extended system gives a structured approach for disturbance compensation. To describe this approach, consider the system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + B_d d_t \\ y_t &= Cx_t + D_d d_t \end{aligned}$$

where d_t is a vector of disturbance signals which we do not measure. Rather than considering the disturbances as arbitrary, we assume that we can model them as the output of a linear system

$$\begin{aligned} \chi_{t+1} &= A_\chi \chi_t \\ d_t &= C_\chi \chi_t \end{aligned} \tag{4.24}$$

The full system dynamics is then described by an *extended system* on the form

$$\begin{aligned}\begin{bmatrix} x_{t+1} \\ \chi_{t+1} \end{bmatrix} &= \begin{bmatrix} A & B_d C_\chi \\ 0 & A_\chi \end{bmatrix} \begin{bmatrix} x_t \\ \chi_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} C & D_d C_\chi \end{bmatrix} \begin{bmatrix} x_t \\ \chi_t \end{bmatrix}\end{aligned} \tag{4.25}$$

We will now design an output feedback controller for the extended system based on a Kalman filter and linear feedback from estimated states. We thus create an observer that produces estimates $\hat{x}_t$ and $\hat{\chi}_t$ of the system states and the disturbance signal generator states, respectively. We then design a linear-quadratic regulator for the extended system, and use the estimated states as if they were the true states. This leads to a control on the form

$$u_t = -L\hat{x}_t - L_d \hat{\chi}_t.$$

We can view this as a combination of feedback from the estimated process states and compensation for the estimated disturbances; see Figure 4.9.

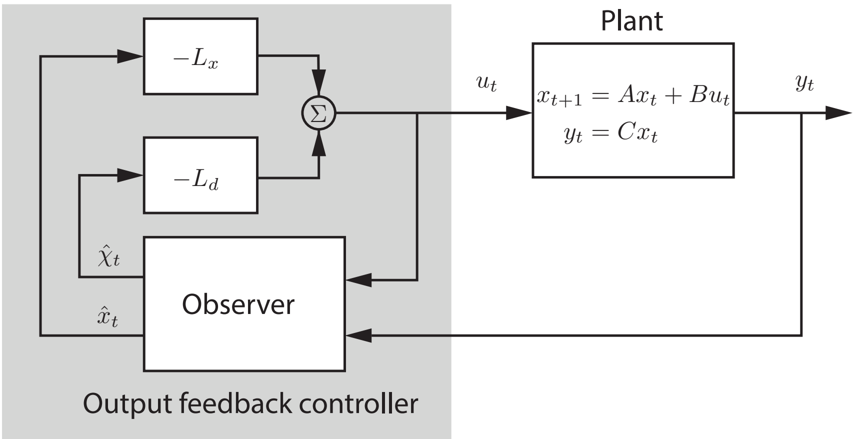

Figure 4.9: Output feedback controller with combined state and disturbance estimator.

Many common disturbances in control systems can be modeled as outputs of autonomous linear systems. For example, a constant disturbance of unknown amplitude can be modeled by

$$\begin{aligned}\chi_{t+1} &= \chi_t \\ d_t &= \chi_t\end{aligned}$$

The unknown initial value represents the constant level of the disturbance. A sinusoidal disturbance with natural frequency ω_0 rad/sec can be modelled as the output of a harmonic oscillator, *i.e.*

$$\begin{aligned}\chi_{t+1} &= \begin{bmatrix} \cos(\omega_0 h) & \sin(\omega_0 h) \\ -\sin(\omega_0 h) & \cos(\omega_0 h) \end{bmatrix} \chi_t \\ d_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} \chi_t\end{aligned}$$

where h is the sampling time of the discrete system (in seconds). The first state of this model is the amplitude of the disturbance, while the second one describes the rate-of-change. Finally, a periodic disturbance of unknown shape but with known period of n samples can be described by

$$\begin{aligned}\chi_{t+1} &= \begin{bmatrix} 0_{(n-1)\times 1} & I_{(n-1)\times(n-1)} \\ 1 & 0_{1\times(n-1)} \end{bmatrix} \chi_t \\ d_t &= \begin{bmatrix} 1 & 0_{(1\times(n-1))} \end{bmatrix} \chi_t\end{aligned}$$

A problem with the two disturbance models described above is that their dynamics is only stable, and not asymptotically stable. Since the control signal cannot affect the disturbance state vector in (4.25), the system extended to include the disturbance model is not stabilizable. The traditional engineering solution to this problem is to perturb the nominal disturbance dynamics slightly so that it is asymptotically stable.

We demonstrate the disturbance modeling and compensation ideas on a simple example.

■ **Example 4.7** DVD is an optical storage format where digital data is read from (and written to) compact discs using laser. The discs are organized in tracks, $0.076\mu m$ apart, which the laser head needs to find and track. Since all discs asymmetric, the tracks oscillate relative to a fixed laser position when the disc spins. The magnitude of these oscillations is dramatic, about 5000 times larger than the acceptable tracking error. DVD drives therefore rely on a high-performance controller for positioning the laser head to track the tracks; see 4.10.

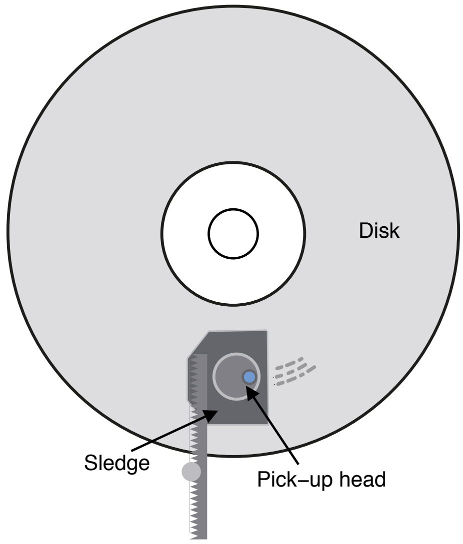

Figure 4.10: A typical DVD design: a pickup head with laser, lens and light detector is mounted on a sledge which can move radially across the disc. To achieve the desired position accuracy, the laser is mounted onto the sledge using springs and can be moved relative to the sledge by electromagnets. In this way, it can also be moved small amounts quickly and with high accuracy.

The following linear system describes the radial dynamics of a DVD servo

$$x_{t+1} = \begin{bmatrix} 0.968 & -0.545 \\ 0.143 & 0.759 \end{bmatrix} x_t + \begin{bmatrix} 0.288 \\ -0.924 \end{bmatrix} u_t$$
$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t$$

To design an output feedback controller, we first start designing an infinite-horizon LQR-optimal state feedback controller for the criterion given by $Q_1 = C^T C$ and $Q_2 = 1$. The optimal state feedback controller

$$u_t = -0.412x_1 + 0.740x_2$$

gives a fast and well-damped response with closed-loop poles in $0.462 \pm 0.155i$.

Next, we design an observer using the Kalman filter parameterization with $\Sigma_w = I + B^T B$ and $\Sigma_v = 0.1$. The optimal observer gain is

$$K = \begin{bmatrix} 1.264 \\ -0.342 \end{bmatrix}$$

which places the observer poles in 0.350 and 0.113.

In our simulations, we include a reference scaling l_r to ensure that the static gain from reference input to system output is one. The resulting controller is

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t) = (A - BL - KC)\hat{x}_t + Ky_t + Kl_r r_t$$
$$u_t = -L\hat{x}_t + l_r r_t$$

The real system is subject to a 20 Hz sinusoidal disturbance of amplitude 0.1 acting at the output. As shown in Figure 4.11 (top), the effect of the disturbance is not well compensated for by the nominal controller. To enhance the disturbance suppression, we model the disturbance signal as the output of a harmonic oscillator (slightly perturbed to have asymptotically stable dynamics) which acts on the system output. For the extended system, we then design a Kalman filter and an LQR controller with weights $Q_1 = C_e^T C_e$ and $Q_2 = 0.1$. The optimal controller is

$$u = -\begin{bmatrix}1.233 & -1.084\end{bmatrix}\hat{x}_t - \begin{bmatrix}0.035 & -0.019\end{bmatrix}\hat{\chi}_t.$$

Clearly, the controller uses both state and disturbance estimates. As shown in Figure 4.11 (bottom), the controller yields a near-perfect disturbance suppression. ■

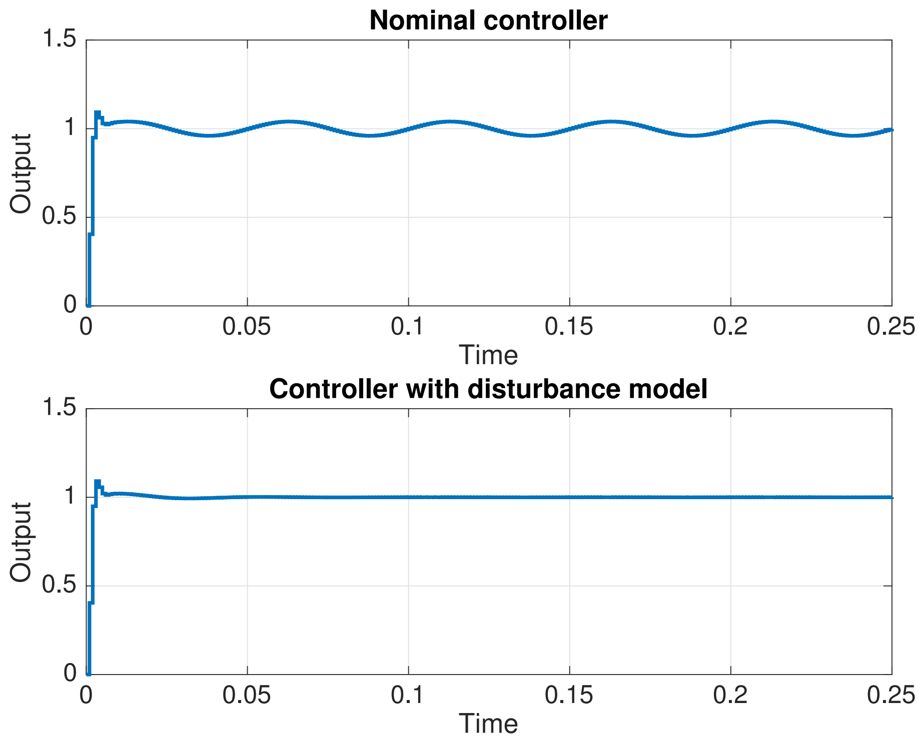

Figure 4.11: Effect of 20Hz output disturbance on nominal closed-loop system (top) and closed-loop system under the output feedback which uses an internal disturbance model (bottom).

Disturbance observers and integral action

From the basic control course, we know that integral action in a controller allows to compensate for constant disturbances. It is therefore natural to ask if the combination of a disturbance observer and feedback from estimated states gives a controller with integral action when the disturbance is assumed to be constant. It turns out that this is indeed true. We will demonstrate it on a system with a scalar input disturbance, which can be described by the extended system (4.25) with matrices

$$B_d = B, \qquad C_\chi = 1, \qquad A_\chi = 1, \qquad D_d = 0$$

The combination of Kalman filter and state feedback from the estimated states of the extended system gives an output feedback controller on the form

$$
\begin{aligned}
\hat{x}_{t+1} &= A\hat{x}_t + B\hat{\chi}_t + Bu_t + K_x(y_t - \hat{y}_t) \\
\chi_{t+1} &= \chi_t + K_\chi(y_t - \hat{y}_t) \\
u_t &= -L_x\hat{x}_t - L_\chi\hat{\chi}_t
\end{aligned}
$$

which we can write as

$$
\begin{aligned}
\begin{bmatrix} \hat{x}_{t+1} \\ \hat{\chi}_{t+1} \end{bmatrix} &= \begin{bmatrix} A - BL_x & B(1 - L_\chi) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{\chi}_t \end{bmatrix} + \begin{bmatrix} K_x \\ K_\chi \end{bmatrix} (y_t - \hat{y}_t) \\
u_t &= -\begin{bmatrix} L_x & L_\chi \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{\chi}_t \end{bmatrix}
\end{aligned}
$$

Due to the block diagonal structure of the system matrix, the controller will have n poles at the eigenvalues of $A - BL_x$ and one pole at $+1$. Thus, the controller will have an integral state, and the control law can be seen as a feedback from the estimated plant state $L_x\hat{x}_t$ plus an integral term $L_\chi\hat{\chi}_t$.

4.6 LQR controller tuning and design

A well-designed control system should satisfy many diverse requirements: it should yield a stable closed-loop system with good transient response; it should suppresses external disturbances and measurement noise; it should be robust to model uncertainties and process variations; and it should make sure that the system satisfies safety constraints. In addition, it should sometimes be able to follow a given reference trajectory with small or vanishing errors. All these requirements should be met without the use excessive control actions, be it in terms of energy, magnitude or variation.

Many of the requirements on a typical control system are in conflict: for example, a faster transient response almost always requires us to use more control energy; faster load disturbance suppression tends to result in higher sensitivity to measurement noise; etc. Although the LQR controller allows us to make an optimal trade-off between transient response and control effort, it does not directly address the other requirements. This limitation is not unique for LQR. In fact, most modern control design techniques are good at addressing *some* of the typical control system requirements, but less effective when it comes to others. So, one typically needs perform an initial design, evaluate its performance by simulations or analysis, and then adjust the "tuning knobs" of the design framework to address any shortcomings that have been detected. In the case of linear quadratic control, it is the weight matrices that influence the state feedback gains and, hence, the closed-loop properties. In the next few pages we will try to develop some insight into how the cost matrices affect the closed-loop system and present some useful tricks for control tuning.

4.6.1 Tuning rules

The linear-quadratic framework has many design parameters. Even in the absence of cross-terms, there are $n(n+1)/2 + m(m+1)/2$ free parameters in the cost function, and it can be difficult to obtain good results without a structured approach to designing the weight matrices. Below, we outline a tuning procedure where the degrees of freedom are kept low and each tuning parameter which is introduced has a clear objective.

1. Tune the closed-loop bandwidth of the LQR controller

It is usually a good idea to begin by addressing the basic trade-off between output and control energy. We thus let

$$
Q_1 = C^T C, \quad Q_2 = \rho I
$$

and adjust ρ to get the desired closed-loop bandwidth.

2. Adjust for heterogeneous requirements by Bryson's rule

The basic idea of trading off the energy in the output signal against the required control energy has a straight-forward extension to systems with many input and outputs. In this case, it is natural to consider the quadratic cost

$$J_N = \sum_{i=1}^{p} \lambda_i \sum_{t=1}^{N} (y_t^{(i)})^2 + \sum_{i=1}^{m} \rho_i \sum_{t=1}^{N} (u_t^{(i)})^2 \tag{4.26}$$

This cost has $m+p$ parameters, and is considerably more complex to tune than the simple cost involving only a single input and a single output.

A good initial guess for the weight parameters is obtained by *Bryson's rule*. It suggests that one should pick the weights λ_i inversely proportional to the square of the maximum allowed magnitude of the corresponding output or input:

$$\lambda_i \sim \frac{1}{\bar{y}_i^2}, \qquad \rho_i \sim \frac{1}{\bar{u}_i^2}$$

In these expressions $\bar{z}$ is the maximum allowed magnitude of signal z.

3. Fine-tune performance by introducing additional performance outputs

If the closed-loop behaviour is still not as desired, one can introduce additional performance outputs y_i in (4.26). These performance outputs are then typically key variables which need to obey certain magnitude limitations of have responses with desired bandwidths. In other cases, these outputs are chosen artificially to craft the system dynamics. For example, if we would like the i^{th} system state to evolve as

$$x_{t+1}^{(i)} = a_i x_t^{(i)}$$

we can add the performance output

$$y^{(i)} = x_{t+1}^{(i)} - a_i x_t^{(i)}$$

and penalize its square deviation from zero. We illustrate this idea on the mechanical system used in earlier examples.

■ **Example 4.8** Consider the mechanical system (4.10). Assume that we would like the second state to evolve according to

$$x_{t+1}^{(2)} = 0.5 x_t^{(2)}$$

It would then be natural to include the performance output

$$y_t = x_{t+1}^{(2)} - 0.5 x_t^{(2)} = -0.09 x_t^{(1)} + 0.36 x_t^{(2)} + 0.94 u_t$$

and to minimize the criterion determined by cost matrices Q_1, Q_{12} and Q_2 such that

$$y_t^2 = \begin{bmatrix} x_t \\ u_t \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

i.e. by

$$Q_1 = \begin{bmatrix} 0.0081 & -0.0324 \\ -0.0324 & 0.1296 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} 0.0054 \\ -0.0216 \end{bmatrix}, \qquad Q_2 = 0.0036$$

Solving the ARE (4.4), we find

$$L = \begin{bmatrix} -0.0957 & 0.3830 \end{bmatrix}$$

which results in the closed-loop matrix

$$A - BL = \begin{bmatrix} 0.9960 & 0.7562 \\ 0.0000 & 0.5000 \end{bmatrix}$$

We have thus accomplished our goal. ■

We would like to emphasize that although the trick worked out fine in this example, we will not be able to assign arbitrary closed-loop matrices in general.

4. Tune Kalman filter to achieve desired closed-loop properties

Once the state feedback controller behaves as desired, we add the Kalman filter. Unless the disturbances have a specific correlation structure, we propose to use a simple criterion such as

$$\Sigma_w = I \qquad\qquad \Sigma_v = \sigma_v I + \sigma_{\mathrm{LTR}} BB^T$$

and increase the weights of σ_v and σ_{LTR} until the closed-loop system has the desired properties.

At this stage, it is essential to study the transfer functions from all inputs and possible disturbances or uncertainties, to the control signal and system outputs. It is not uncommon to detect that the design choices made earlier have to be revised, and one may need several iterations before finding good tuning parameters.

4.6.2 Inverse optimality

An alternative approach for defining the cost criterion is to reverse engineer it from an existing feedback law

$$u_t = -L_{\mathrm{tgt}} x_t$$

designed using some alternative design technique. To derive the corresponding cost function, we notice that $u(t) = -L_{\mathrm{tgt}} x(t)$ minimizes

$$\|Ax_t + Bu_t - (Ax_t - BL_{\mathrm{tgt}} x_t)\|_2^2 = \|B(u_t + L_{\mathrm{tgt}} x_t)\|_2^2 =$$
$$= \begin{bmatrix} x_t \\ u_t \end{bmatrix}^T \begin{bmatrix} L_{\mathrm{tgt}}^T B^T B L_{\mathrm{tgt}} & L_{\mathrm{tgt}}^T B^T B \\ B^T B L_{\mathrm{tgt}} & B^T B \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

Under some technical conditions (for example, $A - BL_{\mathrm{tgt}}$ must be asymptotically stable), one can indeed verify that the state feedback gains L_{tgt} are optimal for the infinite-horizon LQ problem with weight matrices

$$Q_1 = \begin{bmatrix} L_{\mathrm{tgt}}^T B^T B L_{\mathrm{tgt}} \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} B^T B \end{bmatrix}, \qquad Q_1 2 = \begin{bmatrix} L_{\mathrm{tgt}} B^T B \end{bmatrix}$$