Lecture 6: Least-squares and Kalman filtering

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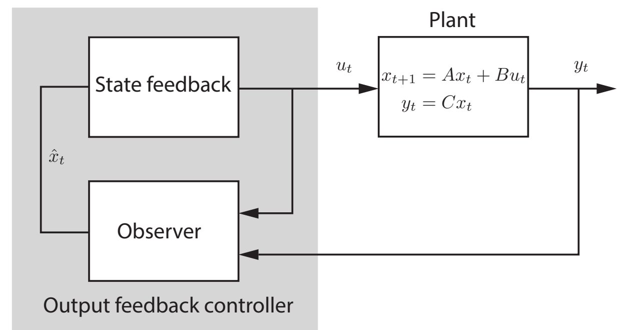
Output feedback linear-quadratic control

Infinite-horizon optimal LQR controller is a linear state feedback

- in practice, may only be able to measures a few noisy output signals
- natural to consider output-feedback control



Q: How can we estimate state from knowledge of $\{u_t\}$ and $\{y_t\}$?

State estimation without disturbances

From Lecture 2, can reconstruct state using observer

$$\begin{aligned} \hat{x}_{t+1} &= A\hat{x}_t + Bu_t + K(y_t - \hat{y}_t) \\ \hat{y}_t &= C\hat{x}_t \end{aligned}$$

Estimation errors $x_t - \hat{x}_t$ converge to zero if $A - KC$ stable.

Can place poles of $A - KC$ arbitrarily if (A, C) is observable.

State estimation with disturbances

Linear system subject to state and measurement disturbances

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned}$$

Process disturbance $w_t \in \mathbb{R}^n$, measurement disturbance $v_t \in \mathbb{R}^p$.

Q: how to construct *optimal* state estimate $\hat{x}_{t|t-1}$ given $\{y_0, \dots, y_{t-1}\}$?

Key results:

- can be done recursively using the Kalman filter
- a linear observer with time-varying gain, computed via Riccati recursion

Outline

- A least-squares filtering principle
- Recursive solution to the least-squares filtering problem
- Relationship to the Kalman filter
- Properties of the optimal filter
- A few words about the stochastic case

A least-squares estimation principle

Consider the problem of finding x from inconsistent system of equations

$$y = Ax$$

Natural to introduce residual $\varepsilon = y - Ax$ and minimize its norm

$$\begin{array}{ll} \underset{x,\varepsilon}{\text{minimize}} & \varepsilon^T \varepsilon \\ \text{subjec to} & y = Ax + \varepsilon \end{array}$$

Interpretation: smallest uncertainty ε that explains observations.

Can eliminate ε and find optimal solution via least-squares

$$\begin{aligned} x^\star &= \underset{x}{\text{argmin}}\ (y - Ax)^T (y - Ax) = \\ &= (A^T A)^{-1} A^T y \end{aligned}$$

Least-squares state estimation

Linear system with state and measurement disturbances

$$\begin{aligned} x_{t+1} &= Ax_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{1}$$

Denote our best prior guess of x_0 by $\bar{x}_0$.

Uncertainty: $\{w_t\}$, $\{v_t\}$, and $x_0 - \bar{x}_0$.

Least-squres estimation principle: find estimate $\{x_t\}$ which minimizes

$$J_N = (x_0 - \bar{x}_0)^T R_0 (x_0 - \bar{x}_0) + \sum_{t=0}^{N-1} w_t^T R_1 w_t + v_t^T R_2 v_t \tag{2}$$

Least-squares state estimation

Use system equations to eliminate $\{w_t\}$ and $\{v_t\}$:

$$\begin{aligned} J_N = &\ (x_0 - \bar{x}_0)^T R_0 (x_0 - \bar{x}_0) + \\ + &\ \textstyle\sum_{t=0}^{N-1} (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t) + \\ + &\ \textstyle\sum_{t=0}^{N-1} (y_t - Cx_t)^T R_2 (y_t - Cx_t) \end{aligned} \tag{3}$$

A function of $x_0, x_1, \ldots, x_N$. Can we minimize it recursively?

Dynamic programming of multi-stage functions

To minimize a multi-stage function on the form

$$J_2(x_0, x_1, x_2) = g_0(x_0) + g_1(x_1, x_0) + g_2(x_1, x_2)$$

we can proceed recursively

$$\begin{aligned} J_2^\star &= \inf_{x_0, x_1, x_2} \; g_0(x_0) + g_1(x_1, x_0) + g_2(x_1, x_2) = \\ &= \inf_{x_1, x_2} \nu_1(x_1) + g_2(x_1, x_2) \end{aligned}$$

where $\nu_1(x_1) = \inf_{x_0} g_0(x_0) + g_1(x_1, x_0)$. Continuing in the same way,

$$J_2^\star = \inf_{x_2} \; \nu_2(x_2)$$

where $\nu_2(x_2) = \inf_{x_1} \nu_1(x_1) + g_2(x_1, x_2)$. We can finally minimize ν_2 over x_2.

Dynamic programming of multi-stage functions

More generally, we can optimize

$$J_N(x_0, \dots, x_N) = g_0(x_0) + \sum_{t=0}^{N-1} g_t(x_t, x_{t+1})$$

by *forward induction* (or forward dynamic programming)

$$\begin{aligned} \nu_0(x_0) &= g_0(x_0) \\ \nu_{t+1}(x_{t+1}) &= \inf_{x_t} \{\nu_t(x_t) + g_t(x_t, x_{t+1})\} \qquad t = 0, \dots, N-1 \end{aligned}$$

Arrival cost $\nu_t(x_t)$ of stage t measures minimal accumulated cost.

Least-squares estimation via dynamic programming

Least-squares estimation problem

$$\begin{aligned} J_N &= (x_0 - \bar{x}_0)^T R_0 (x_0 - \bar{x}_0) \\ &\quad + \sum_{t=0}^{N-1} (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t) + (y_t - Cx_t)^T R_2 (y_t - Cx_t) \end{aligned}$$

has the desired form with

$$\begin{aligned} g_0(x_0) &= (x_0 - \bar{x}_0)^T R_0 (x_0 - \bar{x}_0) \\ g_k(x_t, x_{t+1}) &= (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t) + (y_t - Cx_t)^T R_2 (y_t - Cx_t) \end{aligned}$$

Can thus attempt to solve it using forward induction.

Recursive least-squares estimation

Will show that arrival cost has the form

$$\nu_t(x_t) = (x_t - \hat{x}_{t|t-1})^T S_t (x_t - \hat{x}_{t|t-1}) + c_t$$

and derive recursive updates of $\hat{x}_{t|t-1}$, S_t and c_t.

Note that hypothesis holds for $t = 0$, with $\hat{x}_{0|-1} = \bar{x}_0$, $S_t = R_0$ and $c_t = 0$.

Recursive least-squares estimation

At an arbitrary stage t, we have

$$\begin{aligned}\nu_{t+1}(x_{t+1}) &= \inf_{x_t} \ \nu_t(x_t) + (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t) + \\ &\qquad + (y_t - Cx_t)^T R_2 (y_t - Cx_t) = \\ &= \inf_{x_t} \nu_t^+(x_t) + (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t)\end{aligned}$$

where

$$\begin{aligned}\nu_t^+ &= x_t^T (S_t + C^T R_2 C) x_t - 2(S_t \hat{x}_{t|t-1} + C^T R_2 y_t) x_t \\ &\quad + c_t + \hat{x}_{t|t-1}^T S_t \hat{x}_{t|t-1} + y_t^T R_2 y_t\end{aligned}$$

Can re-write ν_t^+ using completion-of-squares.

Recall: completion-of-squares lemma

Lemma. The quadratic function

$$f(x) = x^T P x + 2q^T x + r$$

with $P \succ 0$ has minimizer

$$x^\star = -P^{-1} q$$

and minimal value

$$f^\star = r - q^T P^{-1} q = r - (x^\star)^T P x^\star.$$

Moreover, $f(x)$ can be re-written as a completion-of-squares

$$f(x) = (x - x^\star)^T P (x - x^\star) + r - (x^\star)^T P x^\star$$

Recursive least-squares estimation cont'd

By the completion-of-squares lemma,

$$\nu_t^+ = (x_t - \hat{x}_{t|t})^T S_t^+ (x_t - \hat{x}_{t|t}) + c_t^+$$

where

$$\begin{aligned}\hat{x}_{t|t} &= \hat{x}_{t|t-1} + \bar{K}_t (y_t - C\hat{x}_{t|t-1}) \\ \bar{K}_t &= (S_t + C^T R_2 C)^{-1} C^T R_2 \\ S_t^+ &= S_t + C^T R_2 C \\ c_t^+ &= c_t + \hat{x}_{t|t-1}^T S_t \hat{x}_{t|t-1} + y_t^T R_2 y_t - \hat{x}_{t|t}^T (S_t + C^T R_2 C) \hat{x}_{t|t}\end{aligned}$$

Recursive least-squares estimation cont'd

We can now form

$$\nu_{t+1}(x_{t+1}) = \inf_{x_t} \ \{\nu_t^+(x_t) + (x_{t+1} - Ax_t)^T R_1 (x_{t+1} - Ax_t)\}$$

By inserting the expression for ν_t^+ and applying completion of squares

$$\nu_{t+1}(x_{t+1}) = \hat{x}_{t|t}^T S_t^+ \hat{x}_{t|t} + x_{t+1}^T R_1 x_{t+1} + c_t^+ - (x_t^\star)^T (S_t^+ + A^T R_1 A) x_t^\star$$

where

$$x_t^\star = \hat{x}_{t|t} + (S_t^+ + A^T R_1 A)^{-1} A^T R_1 (x_{t+1} - A\hat{x}_{t|t})$$

It remains to show that ν_{t+1} admits the desired parameterization.

Recursive least-squares estimation cont'd

A short calculation shows that, indeed,

$$\nu_{t+1}(x_{t+1}) = (x_{t+1} - \hat{x}_{t+1|t})^T S_{t+1}(x_{t+1} - \hat{x}_{t+1|t}) + c_{t+1}$$

where

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\ c_{t+1} &= c_t^+ \\ S_{t+1} &= R_1 - R_1 A(S_t^+ + A^T R_1 A)A^T R_1 \end{aligned}$$

The induction is complete.

Least-squares estimation

Theorem. The least-squares state estimation problem (1), (2) can be solved recursively by repeated application of the following updates

Measurement update:

$$\begin{aligned} \bar{K}_t &= (S_t + C^T R_2 C)^{-1} C^T R_2 \\ \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \bar{K}_t(y_t - C\hat{x}_{t|t-1}) \\ S_t^+ &= S_t + C^T R_2 C \end{aligned}$$

Prediction step:

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\ S_{t+1} &= R_1 - R_1 A(S_t^+ + A^T R_1 A)^{-1} A^T R_1 \end{aligned}$$

from initial values $\hat{x}_{0|-1} = \bar{x}_0$ and $S_0 = R_0$.

Structure of recursive lest-squares estimator

Optimal solution has observer structure with time-varying gain

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + K_t(y_t - C\hat{x}_{t|t-1})$$

where $K_t = A\bar{K}_t$, and $\bar{K}_t$ is computed recursively as shown.

If we instead consider

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned}$$

the optimal observer becomes

$$\hat{x}_{t+1|t} = A\hat{x}_{t|t-1} + Bu_t + K_t(y_t - C\hat{x}_{t|t-1})$$

where K_t is the same as above.

Relationship to Kalman filter

The classical Kalman filter results from the criterion

$$J_N = (x_0 - \bar{x}_0)^T \Sigma_0^{-1}(x_0 - \bar{x}_0) + \sum_{t=0}^{N-1} w_t^T \Sigma_w^{-1} w_t + v_t^T \Sigma_v^{-1} v_t$$

i.e. it can be seen as a least-squares filter with

$$R_0 = \Sigma_0^{-1}, \quad R_1 = \Sigma_w^{-1} \quad R_2 = \Sigma_v^{-1}$$

Kalman filter typically maintains $P_t = S_t^{-1}$

- updates for P_t recovered via matrix inversion identities (see notes)

Kalman filter recursions

Theorem. The estimation problem (1), (2) with $R_0 = \Sigma_0^{-1}$, $R_1 = \Sigma_w^{-1}$ and $R_2 = \Sigma_v^{-1}$ can be solved recursively by repeated application of the updates

Measurement update:

$$\begin{aligned} \bar{K}_t &= P_t C^T (\Sigma_v + C^T P_t C) \\ \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \bar{K}_t (y_t - C\hat{x}_{t|t-1}) \\ P_t^+ &= P_t - P_t C^T (\Sigma_v + C P_t C^T)^{-1} C P_t \end{aligned}$$

Prediction step:

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t} \\ P_{t+1} &= \Sigma_w + A P_t^+ A^T \end{aligned}$$

from initial values $\hat{x}_{0|-1} = \bar{x}_0$ and $P_0 = \Sigma_0$.

A Riccati recursion and the optimal observer

Can eliminate the intermediate variables in the measurement update:

$$P_{t+1} = \Sigma_w + A P_t A^T - A P_t C^T (\Sigma_v + C P_t C^T)^{-1} C P_t A^T$$

Note that $\{P_t\}$ does not depend on observations, could be precomputed.

Riccati recursion evolves forward in time (in contrast to LQR Riccati)

The optimal observer is

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t-1} + A\bar{K}_t (y_t - C\hat{x}_{t|t-1}) \\ &= A\hat{x}_{t|t-1} + K_t (y_t - C\hat{x}_{t|t-1}) \end{aligned}$$

Stationary solution

Similarly to the LQR, this Riccati recursion can be shown to converge.

The stationary solution satisfies the ARE

$$P = APA^T + \Sigma_w - APC^T (\Sigma_v + CPC^T)^{-1} CPA^T$$

and the corresponding time-invariant observer is

$$\begin{aligned} \hat{x}_{t+1|t} &= A\hat{x}_{t|t-1} + K(y_t - \hat{y}_{t|t-1}) \\ \hat{y}_{t|t-1} &= C\hat{x}_{t|t-1} \end{aligned}$$

with $K = APC^T (\Sigma_v + CPC^T)^{-1}$.

Stationary solution: error dynamics

The observer has error dynamics

$$\begin{aligned} e_{t+1} &= x_{t+1} - \hat{x}_{t+1|t} = \\ &= Ax_t + w_t - (A\hat{x}_{t|t-1} + K(Cx_t + v_t - C\hat{x}_{t|t-1})) = \\ &= (A - KC)e_t + w_t - Kv_t \end{aligned}$$

A linear system driven by disturbance $w_t - Kv_t$.

If (A, C) is observable and (A, Σ_w) is controllable, then $A - KC$ is stable.

Example

Consider the scalar system

$$\begin{aligned} x_{t+1} &= ax_t + w_t \\ y_t &= x_t + v_t \end{aligned}$$

with $\Sigma_w = 1$ and $\Sigma_v = r$. Stationary Kalman filter gain is

$$K = \frac{aP}{r+P}$$

where P satisfies

$$P = 1 + a^2 P - \frac{a^2 P^2}{r+P}$$

Example continued

The error dynamics

$$e_{t+1} = (a-K)e_t$$

In the extreme when $r \to 0$ (error-free measurements)

$$K \to a, \qquad \hat{x}_{t+1} = ay_t = ax_t$$

Disregard previous information, use measurements only.

When $r \to \infty$ (very corrupted measurements) on the other hand, we have

$$a - K = \begin{cases} a & \text{if } |a| \leq 1 \\ a^{-1} & \text{otherwise} \end{cases}$$

Disregard y_t if open loop stable, else mirror pole in stability boundary.

Duality between estimation and control

The linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned}$$

admits a dual system

$$\begin{aligned} \tilde{x}_{t+1} &= A^T \tilde{x}_t + C^T \tilde{u}_t \\ \tilde{y}_t &= B^T \tilde{x}_t \end{aligned}$$

- System is reachable if and only if its dual is observable (and vice versa)
- Optimal estimator gains can be found by solving LQR problem for dual (with $Q_1 = \Sigma_w$ and $Q_2 = \Sigma_v$)
- Observer bandwidth increased by $\Sigma_v = I$, $\Sigma_w = \sigma BB^T$ and increasing σ (compare cheap control, with $\sigma = 1/\rho$)

A few words about the stochastic case

Linear system with process noise $\{w_t\}$ and measurement noise $\{v_t\}$:

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned}$$

$x_0 \sim \mathcal{N}(0, \Sigma_0)$, $w_t \sim \mathcal{N}(0, \Sigma_w)$ and $v_t \sim \mathcal{N}(0, \Sigma_v)$; all independent.

Aim: find control $\{u_t\}$ (relying only on $\{y_0, \ldots, y_t\}$) which minimizes

$$\mathbf{E}\left\{ \sum_{t=0}^{N-1} (x_t^T Q_1 x_1 + u_t^T Q_2 u_t + x_N Q_f x_N \right\}$$

A few words about the stochastic case

Optimal solution is on the form

$$u_t = -L_t \mathbf{E}\{x_t | y_0, \ldots y_t\}$$

where L_t is the optimal feedback gain from the associated LQR

Conditional mean $\mathbf{E}\{x_t | y_0, \ldots, y_t\}$ coincides with Kalman estimate.

Called the *separation principle*. The optimal policy

- estimates state via Kalman filter (independently of control cost)
- uses estimated state as if it were the actual (for control)

Warning. LQG controller may be sensitive to modeling errors.

Comments

Weights R_0, R_1 and R_2 in least-squares criterion are tuning parameters

- use lower weight if disturbance is known to be large

Kalman filter computes conditional mean

- if Σ_0, Σ_w and Σ_v are covariance matrices of corresponding signals
- High variance of v gives large Σ_v, but small weight Σ_v^{-1}, etc.

Can be difficult to tune, no matter which perspective one adopts.

Summary

- A least-squares filtering principle
- Recursive solution to the least-squares filtering problem
- Relationship to the Kalman filter
- Properties of the optimal filter
- A few words about the stochastic case