Lecture 5: Infinite-horizon LQR

Mikael Johansson
KTH - Royal Institute of Technology

Recap: finite-horizon linear-quadratic regulation

Given the *linear* system

$$x_{t+1} = Ax_t + Bu_t$$

with initial state x_0, find control sequence

$$U_N = \{u_0, u_1, \cdots, u_{N-1}\}$$

that minimizes the *quadratic* cost function

$$J(U_N) = \sum_{k=0}^{N-1} (x_k^T Q_1 x_k + u_k^T Q_2 u_k) + x_N^T Q_f x_N := J'(U_N) + x_0^T Q_1 x_0$$

for given state cost, control cost, and final cost matrices

$$Q_1 \succeq 0, \quad Q_2 \succ 0, \quad Q_f \succeq 0$$

N is called the *horizon* of the problem. Note the final state cost.

Recap: finite-horizon LQR solution

Elegant solution via dynamic programming.

First, solve Riccati recursion backward in time from $P_N = Q_f$

$$P_{t-1} := Q_1 + A^T P_t A - A^T P_t B(Q_2 + B^T P_t B)^{-1} B^T P_t A$$

Then compute time-varying feedback gains

$$L_t = (Q_2 + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

Optimal control is the time-varying state feedback law

$$u_t = -L_t x_t$$

(can also solve directly using quadratic programming)

Outline

- Infinite-horizon optimal LQR
- Weight selection for trading conflicting objectives
- Understanding LQR at the extremes: cheap and expensive control
- A design example
- Reference feedforward and integral action

Infinite-horizon LQR

Given linear sytem

$$x_{t+1} = Ax_t + Bu_t$$

find feedback policy $u_t = \mu_t(x_t)$ which minimizes infinite-horizon cost

$$J = \sum_{k=0}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k$$

with $Q_1 \succeq 0$ and $Q_2 \succ 0$.

Note: no terminal penalty (since it has no impact on the optimal solution).

Reachability ensures bounded cost

Infinite-horizon cost can become unbounded, *e.g.* for

$$x_{t+1} = 2x_t + 0u_t$$

However, if (A, B) is reachable, then there exists a control sequence

$$\{u_0, u_1, \ldots, u_{n-1}\}$$

which drives any initial state x_0 to $x_n = 0$.

This control has a finite LQ-cost, so the *optimal* cost must also be finite.

Infinite-horizon DP and the Bellman equation

Value function does not depend on time (since remaining horizon is infinite)

$$v(x) = \min_{\{u_0, u_1, \ldots\}} \left\{ \sum_{k=0}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k \mid x_{k+1} = Ax_k + Bu_k,\ x_0 = x \right\}$$

and the DP recursion is replaced by the Bellman equation

$$v(x) = \min_u \left\{ x^T Q_1 x + u^T Q_2 u + v(Ax + Bu) \right\}$$

Infinite-horizon optimal LQR

For LQR, it turns out that $v(x)$ is quadratic, $v(x) = x^T P x$ with $P \succeq 0$.

Thus, we need to find $u = \mu(x)$ such that

$$x^T P x = \min_u \left\{ x^T Q_1 x + u^T Q_2 u + (Ax + Bu)^T P (Ax + Bu) \right\} \qquad \forall x$$

By completion-of-squares, optimal solution is time-invariant state feedback

$$u(t) = -Lx(t) \qquad L = (Q_2 + B^T P B)^{-1} B^T P A$$

where P is the unique solution to the algebraic Riccati equation (ARE):

$$P = Q_1 + A^T P A - A^T P B (Q_2 + B^T P B)^{-1} B^T P A$$

If $(Q_1^{1/2}, A)$ is observable, closed-loop is stable and $P \succ 0$ (more later...)

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Trading conflicting critera

Example. an imaginary gadget of size x costs $c(x) = c_0 + x^2$ dollars and generates emissions of $e(x) = (x - a)^2$ kilograms over its life time.

Impossible to minimize $c(x)$ and $e(x)$ at the same time.

Natural to try to minimize weighted cost

$$J(x) = e(x) + \rho c(x)$$

with optimal solution $x^\star = a/(1+\rho)$.

Interpretation

- when ρ is large, focus in on minimizing costs ($x^\star$ tends to zero);
- when ρ tends to zero, focus is on minimizing emissions ($x^\star$ tends to a)

Note that it is the *relative* weight between the two criteria that matters! ($J'(x) = 10e(x) + 10\rho c(x)$ has same optimizer as $J(x)$)

LQ weight selection to trade off state errors and controls

The LQR criterion

$$J(x,u) = \sum_t x_t^T Q_1 x_t + u_t^T Q_2 u_t$$

makes a trade-off between the cost of large states and large controls.

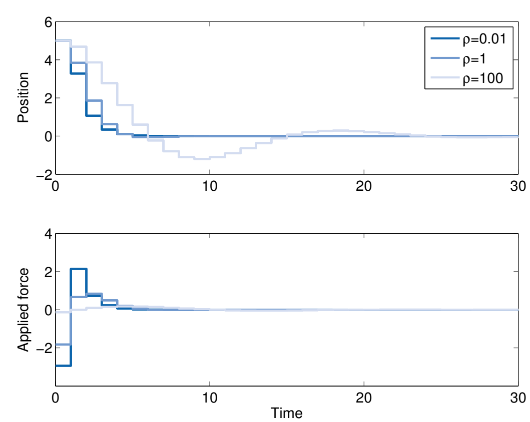
Easiest to see when $Q_1 = I$ and $Q_2 = \rho I$, *i.e.*

$$J(x,u) = \sum_t x_t^T x_t + \rho u_t^T u_t \tag{1}$$

The higher the value of ρ, the more expensive it is to use the controls.

LQ weight selection to trade off state errors and controls

Initial responses and controls for mechanical system from Lecture 3.



Increasing ρ gives gentler control, but also slower response.

Intuition from the scalar case

Consider the scalar LQR problem

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{\infty} x_t^2 + \rho u_t^2 \\ \text{subject to} & x_{t+1} = a x_t + u_t \end{array}$$

Need to find $P > 0$ satisfying the ARE with $Q_1 = 1$ and $Q_2 = \rho$

$$P = 1 + a^2 P - \frac{a^2 P^2}{\rho + P}$$

or, equivalently

$$P^2 - \underbrace{\left(1 + \rho(a^2 - 1)\right)}_{\phi(\rho)} P - \rho = 0$$

Solution $P^\star$ and optimal feedback gain $L^\star$

$$P^\star = \frac{1}{2}\phi(\rho) + \frac{1}{2}\sqrt{\phi^2(\rho) + 4\rho} \qquad L^\star = aP^\star/(\rho + P^\star)$$

Complicated expressions, but what about the extremes $\rho \to 0$ and $\rho \to \infty$?

Intuition from the scalar case

When $\rho \to 0$ (control is cheap), $L^\star \to a$, so the closed-loop dynamics

$$x_{t+1} = (a - L^\star)x_t \to 0$$

i.e. the system converges to zero in one step.

When $\rho \neq 0$, the closed-loop system is

$$x_{t+1} = \frac{a}{1 + P^\star/\rho} x_t$$

As $\rho \to \infty$, $P^\star/\rho \to (a^2 - 1)/2 + |(a^2 - 1)|/2$, with closed-loop dynamics

$$x_{t+1} = \begin{cases} a x_t & \text{if } |a| \le 1 \\ a^{-1} x_t & \text{otherwise} \end{cases}$$

Do nothing if open-loop stable; otherwise cheapest to place pole in $1/a$.

Understanding the LQR criterion

Can prove the following properties of LQR control with criterion

$$J = \sum_{t=0}^{\infty} y_t^2 + \rho u_t^2$$

if $y_t = Cx_t \in \mathbb{R}$ and $u_t \in \mathbb{R}$, and system is reachable and observable.

Assume that the open-loop from u to y has q zeros at z_i, p poles at the origin and $n - p$ poles p_i. Then,

(a) as $\rho \to \infty$, p closed-loop poles remain at origin, the others tend to

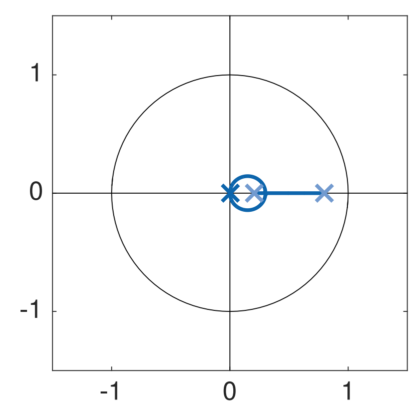
$$\pi_i = \begin{cases} p_i & \text{if } |p_i| < 1 \\ 1/p_i & \text{if } |p_i| > 1 \end{cases}$$

(b) as $\rho \to 0$, $n - q$ closed-loop poles tend to origin, the remaining ones to

$$\pi_i = \begin{cases} z_i & \text{if } |z_i| < 1 \\ 1/z_i & \text{if } |z_i| \ge 1 \end{cases}$$

Example: stable system

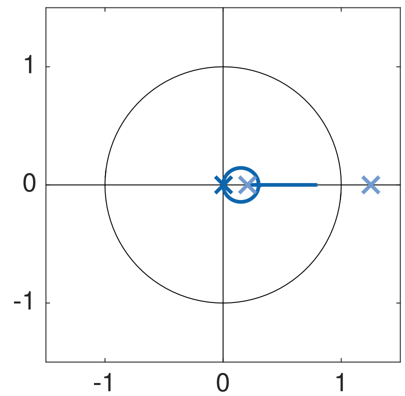
Open-loop poles at $z = 0.2$ and $z = 0.8$ (hence stable)



Closed-loop poles tend to zero as $\rho \to 0$.

Example: unstable system

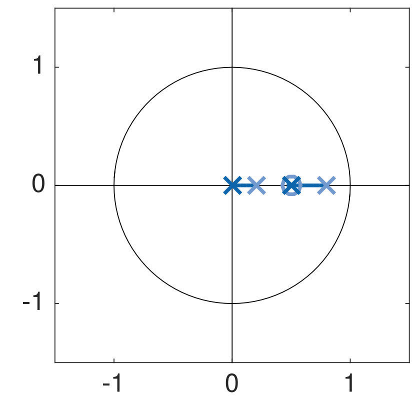
Open-loop poles at $z = 0.2$ and $z = 1.25$ (hence unstable)



Closed-loop poles begin in $z = 0.2$ and $z/1.25 = 0.8$, tend $\to 0$ as $\rho \to 0$.

Example: system with stable zero

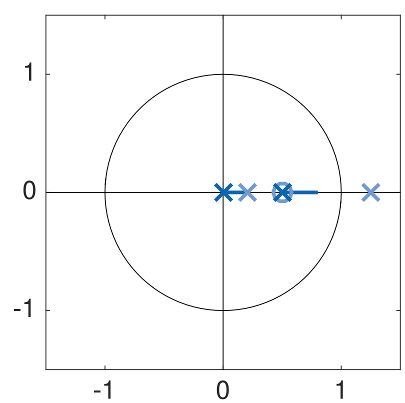
Open-loop poles at $z = 0.2$ and $z = 0.8$, zero for $z = 0.5$



As $\rho \to 0$, one closed-loop pole tends to origin, the other to the zero.

Example: unstable system with zero

Open-loop poles at $z = 0.2$ and $z = 1.25$; zero at $z = 0.5$.



Closed-loop poles begin at $0.2, 1/2.5$ tend to origin and zero location.

Cheap and expensive control: general case

Statements only hold when there are as many inputs as outputs. The cost

$$J = \sum_t x_t^T x_t + \rho u_t^T u_t$$

corresponds to $C = I$ (that is, n outputs).

Can derive similar results by properties of the transmission zeros of the *multi-variable system* from u to $y = x$ (not covered in this course!)

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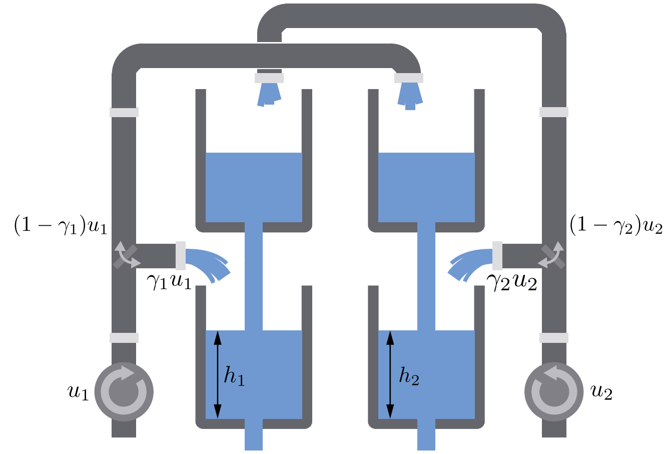
Example: LQR control of quadruple tank

Two cross-connected double-tank systems.



Common teaching equipment for multivariable control.

Example: LQR control of quadruple tank

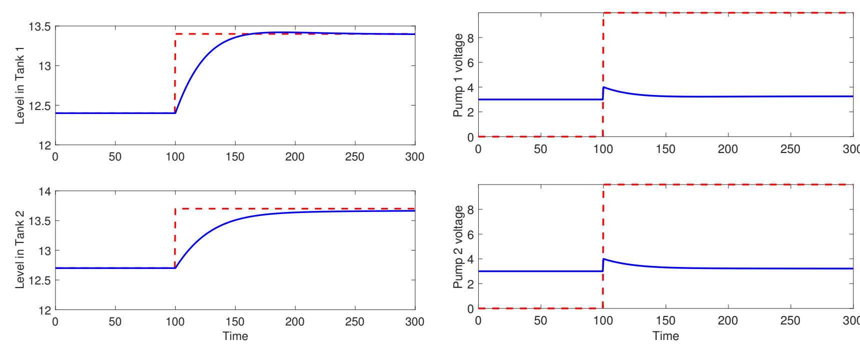
Multiple-input, multiple-output system:

- four states (the levels in the tanks)
- two inputs (voltage to the two pumps)
- two outputs (the two lower tank levels)

Detailed discrete-time model in the lecture notes.

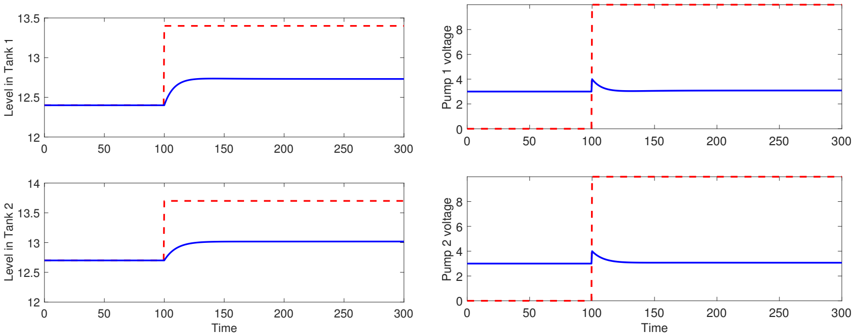
A first design

Let us start with $Q_1 = I$ and $Q_2 = I$.



Adjusting the bandwidth

Increase bandwidth by increasing $Q1$ to $10I$ (leaving $Q_2 = I$).



Large stationary error! (in fact, earlier design also had stationary error)

Reference feedforward

Stationary output when $u_t = -Lx_t + r$

$$x^{\mathrm{eq}} = (A - BL)x^{\mathrm{eq}} + Br$$
$$y^{\mathrm{eq}} = Cx^{\mathrm{eq}}$$

If $I - (A - BL)$ is invertible (it is if closed-loop is asymptotically stable),

$$y^{\mathrm{eq}} = C(I - (A - BL))^{-1}Br$$

Stationary output changes as we change the feedback gain!

Simple solution: use

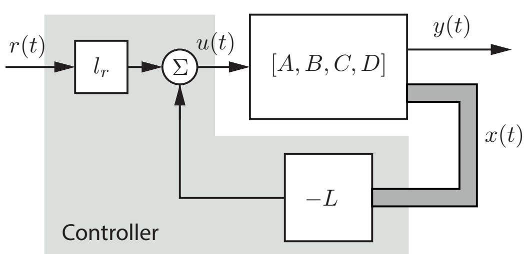
$$u_t = -Lx_t + l_r r$$

where l_r is adjusted so that $y_t = r$ in steady-state, i.e.

$$l_r = 1/(C(I - (A - BL))^{-1}B)$$

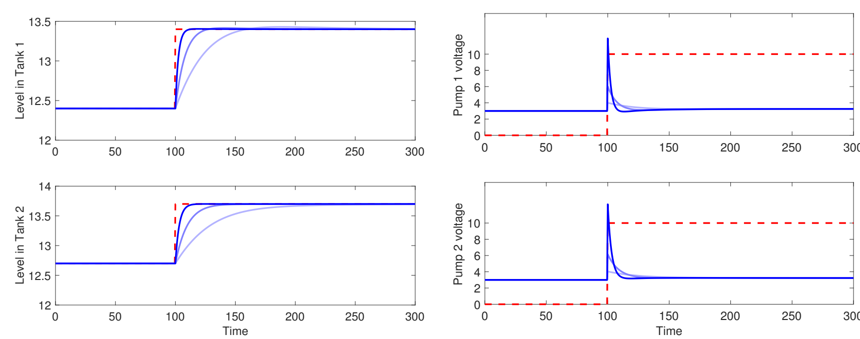
Reference feedforward

Standard state feedback extended with feedforward from reference



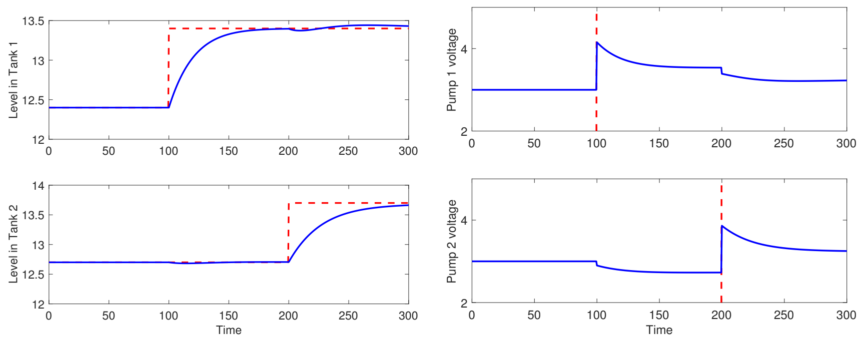
Quadruple tank with reference feedforward

$Q_1 = I, 10I$ and $100I$ (leaving $Q_2 = I$), all with feedforward.



The multivariable nature of the controller

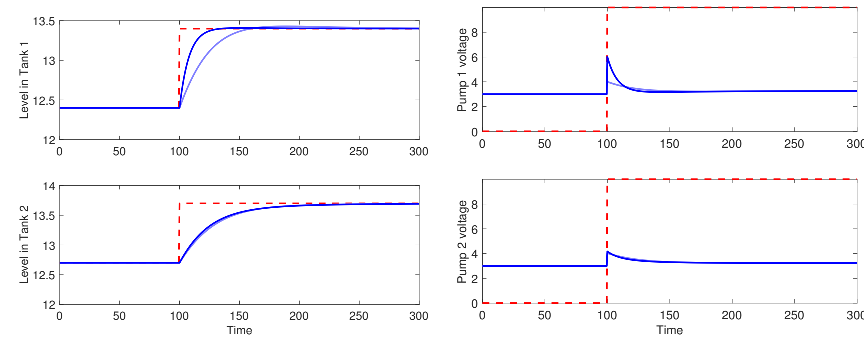
Altering reference change times reveals the MIMO nature of controller



Different costs on different outputs

Increasing the weight on first output by using Q_1 and Q_2 such that

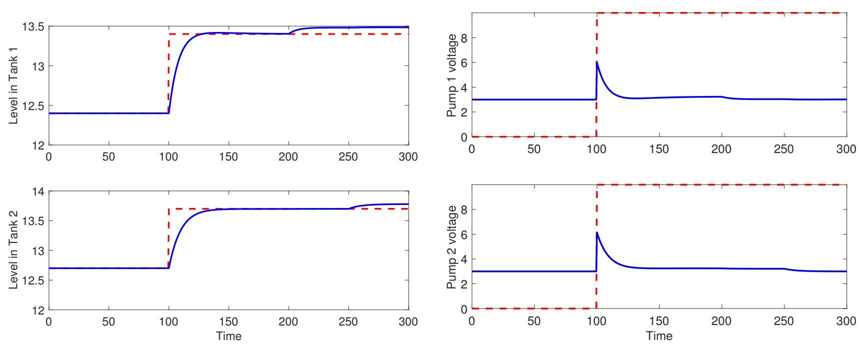
$$J = \sum_t 10(y_t^{(1)})^2 + (y_t^{(2)})^2 + (u_t^{(1)})^2 + (u_t^{(2)})^2$$



Faster response in first output, same as before in second.

The response to input disturbances

Feed-forward from reference does not help to counter-act disturbances



From basic course: need integral action.

Introducing integral action

Control structure

$$u_t = -Lx_t - l_i i_t$$

where i_t is the state of the discrete-time integrator

$$i_{t+1} = i_t + r_t - y_t$$

How can we include the tuning of l_i in the LQR framework?

Introducing integral action

Simplest approach (single input): augment model with an integral state

$$\begin{bmatrix} x_{t+1} \\ i_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 1 \end{bmatrix} \begin{bmatrix} x_t \\ i_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r_t$$

Design LQR-optimal controller

$$u_t = -\bar{L}\bar{x}_t = -Lx_t - l_i i_t$$

minimizing the criterion

$$J = \sum_{t=0}^{\infty} \bar{x}_t^T \bar{Q} \bar{x}_t + u_t^T R u_t$$

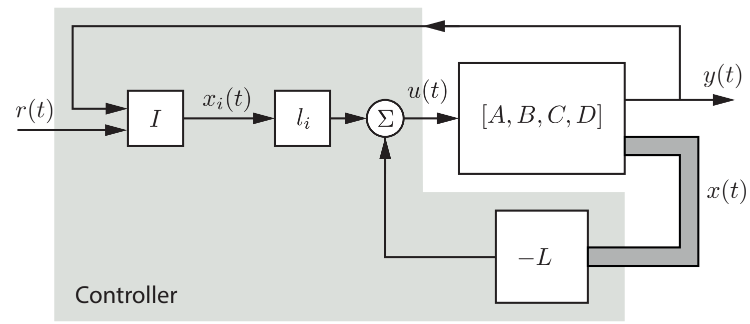
Implement dynamic controller (which measures x_t and r_t):

$$\begin{aligned} i_{t+1} &= i_t - Cx_t + r_t) \\ u_t &= -Lx_t - l_i i_t \end{aligned}$$

(we will discuss many more options later. . .)

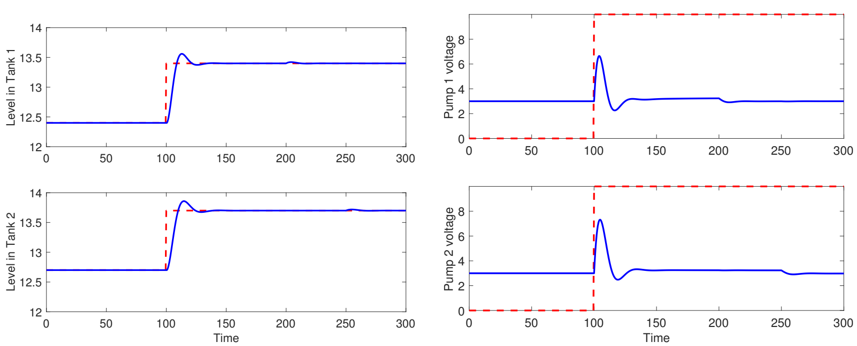
LQR with integral action

LQR controller including internal integral state



IQuadruple tank with integral action

Integral action design: $Q_1 = \text{diag}(I, I)$ and $Q_2 = I$.



Essentially same response as with reference scaling (slight overshoot, why?)

Summary

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Reading instructions: lecture notes Chapter 4.1–4.2.