Lecture 11: Disturbance models and integral action

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Outline

- Output feedback MPC
- Constant disturbances and integral action
- Compensating for non-constant disturbances

Offset-free MPC

Aim: eliminate stationary errors due to disturbances/reference changes

Two basic ideas:

1. model disturbances and compensate for them
2. re-formulate problem to eliminate influence of stationary disturbances

Offset-free MPC: the disturbance observer approach

Model influence of disturbances

$$x_{t+1} = Ax_t + Bu_t + B_d d_t$$
$$y_t = Cx_t + C_d d_t$$

Include model of disturbance dynamics (compare Ch. 4.5 in notes)

– constant disturbance $d_{t+1} = d_t$
– can also model ramps, sinusoidal/periodic disturbances, etc.

Construct observer that estimates both x_t and d_t.

Offset-free MPC for constant disturbances

Combine system and disturbance dynamics into augmented system

$$\begin{bmatrix} x_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t \tag{1}$$

$$y_t = \begin{bmatrix} C & C_d \end{bmatrix} \begin{bmatrix} x_t \\ d_t \end{bmatrix} \tag{2}$$

Under which conditions can we estimate x and d from observations of y?

Observability of augmented model

Proposition. Assume that (A, C) is observable. Then the augmented system (2) is observable if and only if

$$\begin{bmatrix} (A-I) & B_d \\ C & C_d \end{bmatrix}$$

has rank $n + n_d$. This can only happen if $n_d \leq p$.

Proof. By PBH, there should be no $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \neq 0$ such that

$$\begin{aligned} Av_1 + B_d v_2 &= \lambda v_1 \\ v_2 &= \lambda v_2 \\ Cv_1 + C_d v_2 &= 0 \end{aligned}$$

$v_2 = 0 \Rightarrow v = 0$, since (A, C) observable. Must have $\lambda = 1$.
Thus, there should be no $v \neq 0$ such that

$$\begin{bmatrix} (A-I) & B_d \\ C & C_d \end{bmatrix} v = 0$$

Observability of augmented model: interpretation

In steady-state, we have

$$\begin{bmatrix} (A-I) & B_d \\ C & C_d \end{bmatrix} \begin{bmatrix} x^{\text{eq}} \\ d^{\text{eq}} \end{bmatrix} = \begin{bmatrix} -Bu^{\text{eq}} \\ y^{\text{eq}} \end{bmatrix}$$

Interpretation of full rank condition:

- must be able to deduce unique $(x^{\text{eq}}, d^{\text{eq}})$ from $(u^{\text{eq}}, y^{\text{eq}})$

Combined state and disturbance observer

Estimate state and disturbance using observer

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_t + \begin{bmatrix} K_x \\ K_d \end{bmatrix} (y_t - C\hat{x}_t - C_d \hat{d}_t) \tag{3}$$

Gains can be determined using pole placement, Kalman filter approach, ...

Critical that $(A - KC)$ is Schur (all eigenvalues strictly inside unit circle)

Penalizing deviations from equilibrium

Recall that reference-tracking MPC penalizes deviations from steady-state

$$\sum_{k=0}^{N-1}(x_k - x^{\mathrm{eq}})^T Q_1 (x_k - x^{\mathrm{eq}}) + (u_k - u^{\mathrm{eq}})^T Q_2 (u_k - u^{\mathrm{eq}}) + \dots$$

If the observer is asymptotically stable, then in steady-state

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{x}^{\mathrm{eq}} \\ u^{\mathrm{eq}} \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}^{\mathrm{eq}} \\ y^{\mathrm{eq}} - C_d d^{\mathrm{eq}} \end{bmatrix}$$

Can find $\hat{x}_{ss}$ and u_{ss} for every right-hand side if

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix}$$

has rank $n + p$. Only possible if $m \geq p$.

MPC controller with disturbance compensation

Similar to reference-tracking MPC

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{k=0}^{N-1} q(x_{t+k|t} - x^{\mathrm{eq}}, u_{k+t|t} - u^{\mathrm{eq}}) + q_f(x_{t+N|t} - x^{\mathrm{eq}}) \\ \text{subject to} \quad & x_{k+1} = Ax_k + Bu_k + B_d d \\ & x_{t+k|t} \in X \\ & u_{t+k|t} \in U \\ & x_{t+N|t} - x^{\mathrm{eq}} \in X_f \\ & \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x^{\mathrm{eq}} \\ u^{\mathrm{eq}} \end{bmatrix} = \begin{bmatrix} -B_d d \\ r - C_d d \end{bmatrix} \\ & x_{t|t} = \hat{x}_t \\ & d = \hat{d}_t \end{aligned}$$

where $\hat{x}_t$ and $\hat{d}_t$ are estimated using the observer (3).

Note:

- same issue with terminal set as for reference tracking MPC.
- may also need to soften constraints to cope with estimation errors

Off-set free tracking

Theorem. Let $n_d = p$, assume that the matrices

$$\begin{bmatrix} A - I & B_d \\ C & C_d \end{bmatrix}, \qquad \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix}$$

have full rank, and let the observer gain be chosen to get asymptotically stable estimation error dynamics.

Assume that the MPC problem in the previous slide is feasible for all times, and unconstrained for all $t \geq T$ for some fixed time T.

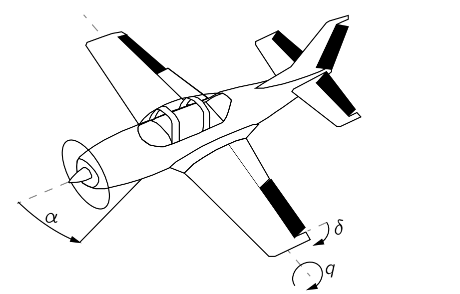
If the closed-loop system reaches an equlibrium $(u^{\mathrm{eq}}, x^{\mathrm{eq}})$ then

$$\lim_{t \to \infty} Cx_t = r$$

Note. May need to add "artifical disturbances" to ensure $n_d = p$.
(has advantage of adding robustness, see Lecture notes for details.)

Aircraft example from Lecture 9

Control elevator surface deflection δ_k to track reference angle α_r.



$$\begin{bmatrix} \alpha_{k+1} \\ q_{k+1} \end{bmatrix} = \begin{bmatrix} 0.9719 & 0.0155 \\ 0.2097 & 0.9705 \end{bmatrix} \begin{bmatrix} \alpha_k \\ q_k \end{bmatrix} + \begin{bmatrix} 0.0071 \\ 0.3263 \end{bmatrix} \delta_k$$

Can only measure α_k. System subject to input disturbance.

Aircraft example

Use observer

$$\begin{bmatrix}\hat{\alpha}_{k+1}\\ \hat{q}_{k+1}\\ \hat{d}_{k+1}\end{bmatrix} = \begin{bmatrix}0.9719 & 0.0155 & 0.0071\\ 0.2097 & 0.9705 & 0.3263\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\alpha}_{k}\\ \hat{q}_{k}\\ \hat{d}_{k}\end{bmatrix} + \begin{bmatrix}0.0071\\ 0.3263\\ 0\end{bmatrix}\delta_k + K(\alpha_k - \hat{\alpha}_k)$$

with

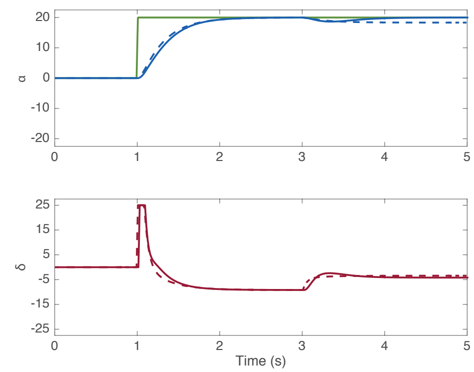
$$K = \begin{bmatrix}0.5424 & 6.0523 & 1.4239\end{bmatrix}^T$$

(places eigenvalues of $A - KC$ in $\begin{bmatrix}0.8 & 0.85 & 0.9\end{bmatrix}$)

Apply pseudo-reference MPC with disturbance compensation.

Aircraft example

Reference change at $t =$, constant input disturbance at $t =$



Simulations with (full) and without (dashed) disturbance compensation.

Extensions

Can also deal with the problem when we want

$$Ey_t \to r$$

for some $E \in \mathbb{R}^{n_r \times p}$ and $r \in \mathbb{R}^{n_r}$.

A few subtleties arise when $p \geq n_r$ (see lecture notes).

Disturbance observers and integral action

Can *prove* that disturbance observer creates integral action in controller.

Basic insight can be obtained from studying scalar system

$$\begin{aligned} x_{t+1} &= ax_t + u_t + d_t \\ y_t &= x_t \end{aligned}$$

under constraints

$$|u_t| \leq 1$$

Aim: regulate x to zero, despite constant and unknown d.

Disturbance observers and integral action

MPC controller with horizon $N = 1$:

$$\begin{aligned} \underset{u_0}{\text{minimize}} \quad & (u_0 - u_0^{\text{eq}})^2 + (x_1 - x_0^{\text{eq}})^2 \\ \text{subject to} \quad & x_1 = a\hat{x}_t + u_0 + \hat{d}_t \\ & -1 \leq u_0 \leq 1 \\ & \begin{bmatrix} a-1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_0^{\text{eq}} \\ u_0^{\text{eq}} \end{bmatrix} = \begin{bmatrix} -\hat{d}_t \\ 0 \end{bmatrix} \end{aligned}$$

Can compute optimal solution explicitly

$$u_0^\star = \begin{cases} 1 & \text{if } -\hat{d}_t - \frac{a}{2}\hat{x}_t > 1 \\ -1 & \text{if } -\hat{d}_t - \frac{a}{2}\hat{x}_t < -1 \\ -\hat{d}_t - a\hat{x}_t/2 & \text{otherwise} \end{cases}$$

Disturbance observers and integral action

Controller dynamics

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} a - K_x & 1 \\ -K_d & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t^{\text{MPC}} + \begin{bmatrix} K_x \\ K_d \end{bmatrix} y_t$$

Inserting the expressions for $u_0^\star$, we find

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} a - K_x & 1 \\ -K_d & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} K_x \\ K_d \end{bmatrix} y_t$$

when $u_0^\star$ is saturated, and otherwise

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} a/2 - K_x & 0 \\ -K_d & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{d}_t \end{bmatrix} + \begin{bmatrix} K_x \\ K_d \end{bmatrix} y_t$$

Disturbance observers and integral action

Readily verified that

$$\begin{bmatrix} a/2 - K_x & 0 \\ -K_d & 1 \end{bmatrix}$$

has eigenvalues at $z = 1$ and $z = a/2 - K_x$.

- in linear operation, the MPC controller implements integral action!

In saturation, the dynamics are simply the stable observer dynamics

- an anti-windup mechanism is incorporated automatically

Can prove similar properties for general set-up.

Alternative formulation: predictions in velocity form

When disturbances are constant, can express predictions in terms of $\Delta x_t = x_t - x_{t-1}$, $\Delta u_t = u_t - u_{t-1}$, $\Delta y_t = y_t - y_{t-1}$:

$$\begin{aligned} \Delta x_{t+1} &= A\Delta x_t + B\Delta u_t \\ \Delta y_t &= C\Delta x_t \end{aligned}$$

$\Rightarrow$ constant disturbances ($d_t \equiv d$) do not affect increments.

Basic idea:

- construct estimator for Δx_t
- use these in MPC scheme to compute Δu_t
- accumulate increments to obtain $u_t = \sum_{k=0}^{t} \Delta u_k$ (assuming $u_{-1} = 0$).

Similar to how we introduced integral action in LQR

Velocity form: augmented system

Form augmented system

$$\begin{bmatrix} \Delta x_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \Delta x_t \\ e_t \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \Delta u_t$$
$$e_t = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \Delta x_t \\ e_t \end{bmatrix}$$

where $e_t = r - y_t$.

Proposition. The augmented system is observable iff (A, C) is observable.

Note: due to output integrators, accurate estimation of Δx_t from e_t slow.

Velocity-form observer

So far, we have used predictive observers (computing $\hat{x}_{t+1|t}$)

$$\hat{x}_{t|t} = x_{t|t-1} + \bar{K}(y_t - C\hat{x}_{t|t-1})$$
$$\hat{x}_{t+1|t} = A\hat{x}_{t|t} + Bu_t$$

In the offset-free MPC literature, one often compute and use $\hat{x}_{t|t}$:

$$\hat{x}_{t|t} = A\hat{x}_{t-1|t-1} + Bu_{t-1} + \bar{K}(y_t - CA\hat{x}_{t-1|t-1} - CBu_{t-1})$$

where $\bar{K}$ is such that $A - \bar{K}CA$ is Schur (e.g $\bar{K}$ from Kalman lecture)

Velocity-form observer

In terms of the augmented system:

$$\begin{bmatrix} \widehat{\Delta x}_t \\ \hat{e}_t \end{bmatrix} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \begin{bmatrix} \widehat{\Delta x}_{t-1} \\ \hat{e}_{t-1} \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} \Delta u_{t-1} +$$
$$+ \begin{bmatrix} \bar{K}_x \\ \bar{K}_e \end{bmatrix} \left(e_t - \begin{bmatrix} CA & I \end{bmatrix} \begin{bmatrix} \widehat{\Delta x}_{t-1} \\ \hat{e}_{t-1} \end{bmatrix} - CB\Delta u_{t-1} \right)$$

Common to use $\bar{K}_e = I$ (so that $\hat{e}_t = e_t$) to make observer faster.

Offset-free MPC in velocity form

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N-1} e_{t+k|t}^T Q_1 e_{t+k|t} + \Delta u_{t+k|t}^T Q_2 \Delta u_{t+k|t} + q_f(\Delta x_{t+N|t}) \\ \text{subject to} & \Delta x_{t+k+1|t} = A\Delta x_{t+k|t} + B\Delta u_{t+k|t} \\ & e_{t+k+1|t} = CA\Delta x_{t+k|t} + e_{t+k|t} + CB\Delta u_{t+k|t} \\ & u + \sum_{i=0}^{k} \Delta u_{t+i|t} \in U \\ & r - e_{t+k|t} \in Y \\ & r - e_{t+N|t} \in Y_f \\ & u = u_{t-1} \\ & e_{t|t} = r - y_t \end{array}$$

Note.

- we do not estimate x, so we only constraint y (which we measure)
- terminal set $y_{t+N|t} \in Y_f$ dealt with using dual mode
- $q_f(\Delta x_{t+N|t})$ chosen as infinite-horizon cost-to-go for cost.

Summary

- Constant disturbances and integral action
  - Disturbance observer
  - Conditions for observability and unique steady-state
  - Convergence theorem and integral action
- Alternative formulation based on velocity form
- Yet another formulation on Friday exercise!

Reading instructions. Lecture notes §5.5