Lecture 7: Lyapunov stability and invariance

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Outline

- Lyapunov stability
- Linear systems and quadratic Lyapunov functions
- Application: closed-loop stability of infinite-horizon LQR
- Positively invariant sets
- Constrained invariant and control invariant sets

Stability of dynamical systems

Consider the nonlinear system

$$x_{t+1} = f(x_t) \tag{1}$$

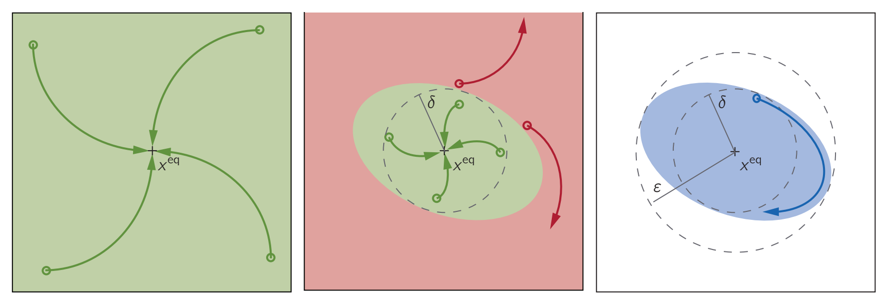
with equilibrium point x^{eq}, *i.e.* $x^{\text{eq}} = f(x^{\text{eq}})$.

Definition. The system is *globally asymptotically stable* if, for every trajectory $\{x_t\}$, we have that $x_t \to x^{\text{eq}}$ as $t \to \infty$.

Definition. The system is *locally asymptotically stable* near x^{eq} if there is a constant $\delta > 0$ such that $\|x_0 - x^{\text{eq}}\| \leq \delta \Rightarrow x_t \to x^{\text{eq}}$ as $t \to \infty$.

Definition. The system is (locally) *stable* if for every (small) $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x_0 - x^{\text{eq}}\| \leq \delta \Rightarrow \|x_t - x^{\text{eq}}\| \leq \varepsilon$ for all $t \geq 0$.

Stability concepts in pictures



Globally asymptotically stable | Locally asymptotically stable | Locally stable

How to ensure stability?

How do we ensure that a system is globally asymptotically stable?

- Easy if system is linear:

$$x_{t+1} = Ax_t$$

  It is necessary and sufficient that all $|\lambda_i(A)| < 1$.
- Much more difficult for nonlinear or constrained systems

Lyapunov theory very powerful for analysis of nonlinear systems

- Basic idea: introduce energy measure, show that does not increase
- Allows to guarantee asymptotic stability, regions of local stability, etc
- Sometimes both necessary and sufficient

A first Lyapunov theorem

Theorem. If there exists a continuous function $V(x)$ whose sublevel sets

$$\mathcal{L}_V(\alpha) = \{x \,|\, V(x) \leq \alpha\}$$

are bounded for every value of α, and

$$\Delta V(x) = V(f(x)) - V(x) \leq 0 \qquad \forall x$$

then all trajectories of (1) are bounded.

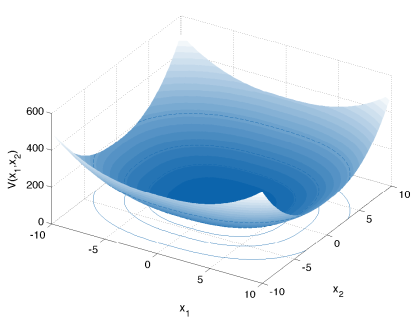
Proof.

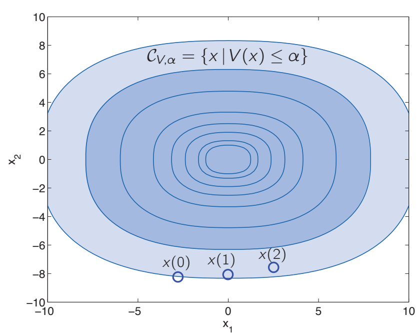
$$V(x_t) = V(x_0) + \sum_{k=0}^{t-1} \Delta V(x_k) \leq V(x_0)$$

so trajectory lies in $\{x \,|\, V(x) \leq V(x_0)\}$ which is bounded by assumption.

Lyapunov theorem in pictures

Lyapunov function (left); level sets and one state trajectory (right)





- once the state enters a level set, it never leaves.
- bounded level sets ensure bounded trajectories (stability).
- a few additional conditions on V will ensure asymptotic stability

Positive definite functions

Definition. A function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is *positive semidefinite* if

$$V(x) \geq 0 \text{ for all } x$$

Definition. A function $V : \mathbb{R}^n \mapsto \mathbb{R}$ is *positive definite* if

(a) $V(x) \geq 0$ for all x

(b) $V(0) = 0$ if and only if $x = 0$

(c) all sublevel sets of V are bounded

Condition (c) is equivalent to $V(x) \to \infty$ as $x \to \infty$

Example. $V(x) = x^T P x$ with $P = P^T$ is positive definite iff $P \succ 0$.

Ensuring asymptotic stability using Lyapunov functions

Theorem. If there exists a continuous function $V : \mathbb{R}^n \mapsto \mathbb{R}$ such that

- $V(x)$ is positive definite
- $V(f(x)) - V(x) \leq -l(x)$

for some positive semidefinite $l(x)$, then $l(x_t) \to 0$ as $t \to \infty$;
If, in addition, $l(x)$ is positive definite, then $x_t \to 0$ as $t \to \infty$.

Lyapunov stability

Proof. The second condition of the theorem implies that

$$l(x_k) \leq V(x_k) - V(x_{k+1})$$

Summing up both sides over $k = 0, 1, \dots$ yields

$$\sum_{k=0}^{\infty} l(x_k) \leq V(x_0) - \lim_{k\to\infty} V(x_k) \leq V(x_0)$$

since $V(x) \geq 0$ for all x. Hence, the infinite sum tends to a finite limit.

By Cauchy's convergence criterion, convergence of the infinite sum implies that $l(x_k) \to 0$ as $k \to \infty$.

When $l(x)$ is positive definite, $l(x) = 0$ implies that $x = 0$.

Lyapunov stability of linear systems

Theorem. The system

$$x_{t+1} = Ax_t$$

is asymptotically stable if and only if, for every $Q \succ 0$ there exists a unique matrix $P \succ 0$ satisfying the Lyapunov equation

$$A^T PA - P + Q = 0. \tag{2}$$

Note. Necessary and sufficient. P is only unique for each given Q.

Lyapunov stability of linear systems

Proof. Consider $V(x) = x^T Px$ where $P \succ 0$ satisfies (2). Then

$$V(x_{t+1}) - V(x_t) = -x_t^T Qx_t \text{ for all } x_t \neq 0$$

Since $V(x) = x^T Px$ and $l(x) = x^T Qx$ are positive definite, $x_t \to 0$. Thus, the Lyapunov equation implies asymptotic stability.

If the system is asymptotically stable, then $|\lambda_i(A)| < 1$ for all i, and

$$P = \sum_{k=0}^{\infty} (A^k)^T QA^k$$

exists and satisfies the Lyapunov equation.

Lyapunov stability of linear systems

To show uniqueness of P, assume that P' also satisfies (2). Then

$$A^T(P-P')A-(P-P')=0$$

Repeated application of this relationship yields

$$P-P'=A^T(P-P')A=\cdots=\lim_{k\to\infty}(A^T)^k(P-P')A^k=0$$

by stability of A. Thus, $P'=P$, i.e. P is unique.

The proof is complete.

Lyapunov stability of linear systems

Theorem. Suppose that $(A, Q^{1/2})$ is observable. Then

$$x_{t+1}=Ax_t$$

is asymptotically stable if and only if there exists a unique matrix $P\succ 0$ which satisfies the Lyapunov equation

$$A^TPA-P+Q=0.$$

Note. $Q=(Q^{1/2})Q^{1/2}$ is only guaranteed to be positive *semidefinite*.

Lyapunov stability of linear systems

Proof. Consider an arbitrary eigenvector $v\neq 0$ of A, *i.e.* $Av=\lambda v$.

Pre- and post-multiplying the Lyapunov equation with v yields

$$(|\lambda|^2-1)v^\star Pv=-\|Q^{1/2}v\|^2$$

By the PBH test, observability implies that $Q^{1/2}v\neq 0$, hence $\|Q^{1/2}v\|>0$.

Now, since $P\succ 0$, this implies that $|\lambda|<1$.

This condition holds for all eigenvalues of A, which is thus stable.

The converse direction relies on establishing that

$$P=\sum_{k=0}^{\infty}(A^k)^T(Q^{1/2})^TQ^{1/2}A^k=\lim_{k\to\infty}\mathcal{O}_k^T\mathcal{O}_k$$

satisfies the Lyapunov equation and is positive definite (due to the observability assumption). Uniqueness follows earlier proof.

Application: closed-loop stability of LQR

Theorem. Let (A,B) be reachable, $(A, Q_1^{1/2})$ observable and $Q_2\succ 0$. Then infinite-horizon LQR control gives asymptotically stable closed-loop.

Recall: the optimal solution to the infinite-horizon LQR problem is

$$u_t=-Lx_t=-(Q_2+B^TPB)^{-1}B^TPAx_t$$

where $P=P^T\succ 0$ satisfies the ARE

$$P=Q_1+A^TPA-A^TPB(Q_2+B^TPB)^{-1}B^TPA$$

Convenient to re-write ARE as

$$P=Q_1+L^TQ_2L+(A-BL)^TP(A-BL)$$

Application: closed-loop stability of LQR

Proof. Use Lyapunov function $V(x) = x^T P x$ where P solves ARE. Then

$$\begin{aligned} V(x_{t+1}) - V(x_t) &= x_t^T (A - BL)^T P (A - BL) x_t - x_t^T P x_t = \\ &= x_t^T ((A - BL)^T P (A - BL) - P) x_t = \\ &= -x_t^T (Q_1 + L^T Q_2 L) x_t \end{aligned}$$

In other words, P satisfies the Lyapunov equation

$$(A - BL)^T P (A - BL) - P = -Q_1 - L^T Q_2 L$$

It may be that the right hand-side is only negative semidefinite.

To guarantee asymptotic stability of the closed-loop, we then have to require that $(A - BL, (Q_1 + L^T Q_2 L)^{1/2})$ is observable. Is it?

Application: closed-loop stability of LQR

Lemma. Let $Q_2 = Q_2^T \in \mathbb{R}^{m \times m}$ be positive definite, and let $L \in \mathbb{R}^{m \times n}$. If $(A, Q_1^{1/2})$ is observable, then so is $(A - BL, (Q_1 + L^T Q_2 L)^{1/2})$.

Proof. If the system is unobservable, then there is $v \neq 0$ such that

$$(A - BL)v = \lambda v, \qquad (Q_1 + L^T Q_2 L)^{1/2} v = 0$$

which implies that

$$v^* (Q_1 + L^T Q_2 L) v = \|Q_1^{1/2} v\|^2 + \|Q_2^{1/2} L v\|^2 = 0$$

i.e. that $Q_1^{1/2} v = 0$ and $Lv = 0$. Thus

$$(A - BL)v = Av = \lambda v \qquad Q_1^{1/2} v = 0$$

which contradicts that $(A, Q_1^{1/2})$ is observable.

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Invariant sets

Definition. The set $\mathcal{I} \subseteq \mathbb{R}^n$ is *(positively) invariant* under $x_{t+1} = f(x_t)$ if

$$x_t \in \mathcal{I} \Rightarrow x_k \in \mathcal{I} \text{ for all } k \geq t \dots$$

Interpretation: if x_t enters $\mathcal{I}$, it will never leave.

Definition. The set $\mathcal{C} \subseteq \mathbb{R}^n$ is *control invariant* under $x_{t+1} = f(x_t, u_t)$ if

$$x_t \in \mathcal{C} \Rightarrow \exists \{u_t, u_{t+1}, \dots\} : x_k \in \mathcal{C} \text{ for all } k \geq t$$

Interpretation: if x_t enters $\mathcal{C}$, there is a control that makes it stay in $\mathcal{C}$.

Invariant sets from Lyapunov functions

Proposition. Let $V(x)$ be a Lyapunov function for $x_t = f(x_t)$. Then

$$\mathcal{L}_V(\alpha) = \{x \,|\, V(x) \leq \alpha\}$$

is invariant under $x_{t+1} = f(x_t)$.

Example. Stable linear systems admit quadratic Lyapunov functions

$$V(x) = x^T P x$$

where $P = P^T > 0$ satisfy $A^T P A - P - Q = 0$ for some $Q = Q^T > 0$. Their level sets

$$\mathcal{L}_V(\alpha) = \{x \,|\, x^T P x \leq \alpha\}$$

define invariant ellipsoids.

Invariant sets from local Lyapunov functions

Theorem. If there exists a continuous function $V(x)$ such that

$$\Delta V(x) = V(f(x)) - V(x) \leq 0 \qquad \forall x \in X$$

then every sub-level set

$$\mathcal{L}_V(\alpha) = \{x \,|\, V(x) \leq \alpha\}$$

which is fully contained in X is invariant under $x_{t+1} = f(x_t)$.

Ellipsoidal invariant sets for constrained linear systems

Example. Consider the autonomous linear system

$$x_{t+1} = \begin{bmatrix} 1.5 & -0.9 \\ 1.0 & 0.0 \end{bmatrix} x_t$$

under state constraints

$$x_t \in X = \{x : \|x\|_\infty \leq 1\} \qquad \text{for all } t = 0, 1, \ldots$$

For which initial values can you guarantee that $x_t \in X$?

Solution. Solve Lyapunov equation

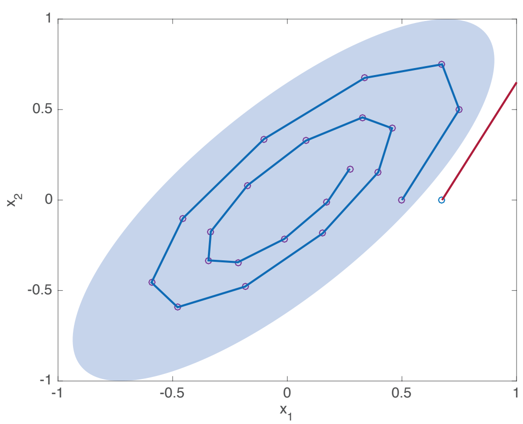
$$A^T P A - P + Q = 0$$

for some $Q > 0$. Find the largest value of α such that

$$\mathcal{L}_V(\alpha) = \left\{x \,|\, x^T P x \leq \alpha\right\} \subset X$$

Ellipsoidal invariant sets for constrained linear systems

Invariant set from Lyapunov function (shaded) and two trajectories.



Reasonably tight estimate (for this system).

Polyhedral invariant sets

Natural to consider polyhedral invariant sets.

Proposition. Assume that a polyhedral constraint set

$$X = \{x \mid Px \le p\}$$

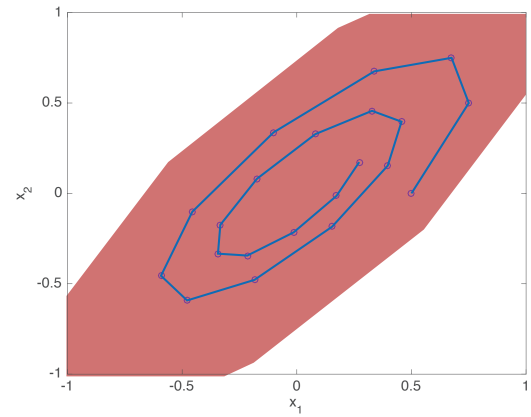
is given. The largest invariant set contained in X under the dynamics $x_{t+1} = Ax_t$ is the polyhedron

$$\begin{bmatrix} P \\ PA \\ PA^2 \\ \vdots \end{bmatrix} x \le \begin{bmatrix} p \\ p \\ p \\ \vdots \end{bmatrix}$$

Once we note that all inequalities in $PA^k \le p$ are redundant, we can stop.

Polyhedral invariant sets for constrained linear systems

Polyhedral invariant set for autonomous linear system from earlier example.



Control invariant sets

Can also construct control invariant sets from the definition.

The control invariant set contained in $X = \{x \mid Px \le p\}$ is

$$\begin{bmatrix} P & 0 & \dots & 0 \\ PA & PB & \ddots & \vdots \\ PA^2 & PAB & PB & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \\ u_3 \\ \vdots \end{bmatrix} \le \begin{bmatrix} p \\ p \\ p \\ p \\ \vdots \end{bmatrix}$$

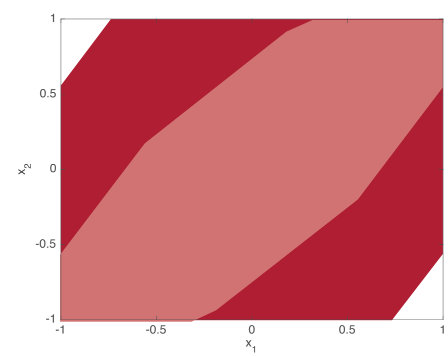
A polyedron in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots$.

We are interested in the projection onto the first n coordinates.

Control invariant set for constrained linear system

Control invariant set (dark) for

$$x_{t+1} = \begin{bmatrix} 1.5 & -0.9 \\ 1.0 & 0.0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t$$



Significantly larger than invariant set for autonomous system (light).

Summary

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Reading instructions: Lecture notes Chapter 2.