Lecture 9: Stability and feasibility of MPC

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Outline

- Model predictive control: basic idea, possibilities and challenges
- Recursive feasibility
- Stability of MPC via terminal cost and terminal set
- Example

MPC: basic idea

Subdivide infinite-horizon cost into two

$$
\begin{aligned}
J &= \sum_{t=0}^{\infty} x_t^T Q_1 x_t + u_t^T Q_2 u_t \\
&= \sum_{t=0}^{N-1} x_t^T Q_1 x_t + u_t^T Q_2 u_t + \sum_{t=N}^{\infty} x_t^T Q_1 x_t + u_t^T Q_2 u_t
\end{aligned}
$$

replace tail cost by the cost-to-go from x_N, use finite-horizon formulation

$$
\begin{array}{lll}
\text{mininmize} & \sum_{k=0}^{N-1} x_{t+k|t}^T Q_1 x_{t+k|t} + u_{t+k|t}^T Q_2 u_{t+k|t} + V(x_{t+N|t}) & \\
\text{subject to} & x_{t+k+1|t} = A x_{t+k|t} + B u_{t+k|t} & k = 0, \dots, N-1 \\
& x_{t+k|t} \in X, \quad u_{t+k|t} \in U & k = 0, \dots, N-1 \\
& x_{t|t} = x_t &
\end{array}
$$

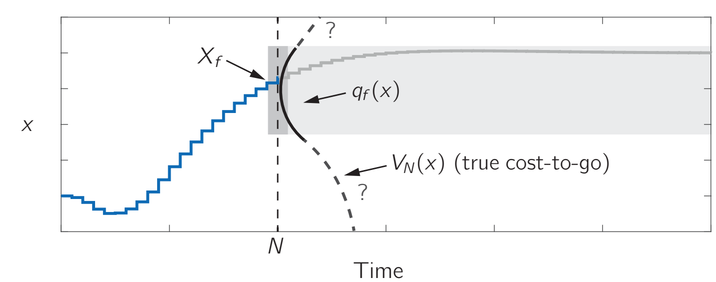
Apply $u_t = u^\star_{t|t}$, repeat in receding-horizon fashion.

MPC: basic idea in pictures

Know that $v(x_N)$ is quadratic if operation is linear from $t+N$ and on.

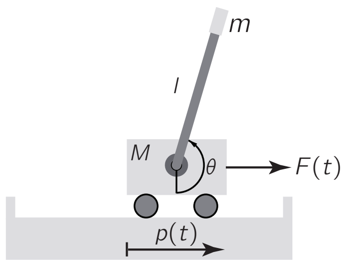
– add terminal set which forces x_N into region of linear operation

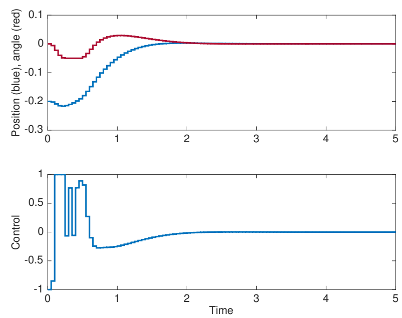
– intuitively, should work fine if N is large



MPC: possibilities

From last lecture: easy to deal with state and control constraints.

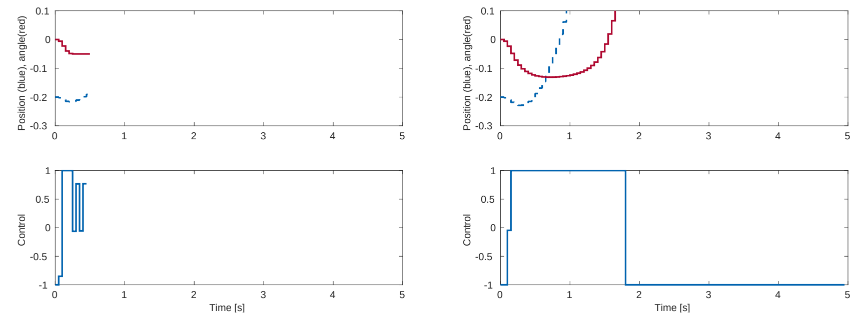




MPC: things that can go wrong

From last lecture: (at least) two important things can go wrong

- we can loose feasibility of the finite-time horizon planning problem
- the MPC controller may fail to make the closed-loop system stable



Problems often due to MPC being too short-sighted

- compensated for by prediction horizon, terminal state and terminal cost

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Recursive feasibility

Definition. The MPC problem is *recursively feasible* from x_0 if the existence of a solution from $x_{0|0} = x_0$ implies that the MPC problem will remain feasible from every state along the closed-loop trajectory.

Definition. $\mathcal{C} \subseteq \mathbb{R}^n$ is *positively control invariant* under the dynamics

$$x_{t+1} = f(x_t, u_t)$$

and control constraint $u_t \in \mathcal{U}$ for $t \geq 0$ if

$$x_t \in \mathcal{C} \Rightarrow \exists \{u_t, u_{t+1}, \dots\} \text{ such that } u_t \in \mathcal{U} \text{ and } x_t \in \mathcal{C}\ \forall t \geq 0$$

Theorem. If $X_f \subseteq X$ and X_f is control invariant, then the MPC problem is recursively feasible from all initial states x_0 which admits a feasible solution.

Recursive feasibility

Proof. By assumption, the MPC problem admits a solution

$$\{u^\star_{0|0}, u^\star_{1|0}, \dots, u^\star_{N-2|0}, u^\star_{N-1|0}\}, \quad \{x^\star_{0|0}, x^\star_{1|0}, \dots, x^\star_{N|0}\}$$

Since $x_1 = Ax_0 + Bu^\star_{0|0} = x^\star_{1|0}$, we can apply the control sequence

$$\{u^\star_{1|0}, u^\star_{2|0}, \dots, u^\star_{N-1|0}, \tilde{u}_N\}$$

from x_1 to obtain the state evolution

$$\{x_1, x^\star_{2|0}, \dots, x^\star_{N|0}, \tilde{x}_{N+1}\}$$

where $\tilde{x}_{N+1} = Ax^\star_{N|0} + B\tilde{u}_N$.

As $x^\star_{N|0} \in X_f$ and X_f is control invariant, $\exists \tilde{u}_N \in U$ such that $\tilde{x}_{N+1} \in X_f$.

Since the same argument applies to the new sequences, proof is complete.

Easily computable control invariant sets

Easily computable control invariant sets include

- $X_f = \{0\}$
- X_f which is is invariant for $x_{t+1} = (A - BL)x_t$ for some fixed $L \in \mathbb{R}^{m\times n}$, and such that $X_f \subseteq X \cup \{x \mid Lx \in U\}$.

Maximal positively control invariant set more complex to calculate.

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Stability of receding-horizon LQR

Consider the discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t$$

under the receding-horizon control

$$u_t = u^\star_{t|t}$$

where $\{u^\star_{t|t}, u^\star_{t+1|t}, \dots, u^\star_{t+N-1|t}\}$ is the optimizer of

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N-1} x^T_{t+k|t} Q_1 x_{t+k|t} + u^T_{t+k|t} Q_2 u_{t+k|t} + x^T_{t+N|t} Q_f x_{t+N|t} \\ \text{subject to} & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \quad x_{t|t} = x_t \end{array} \tag{1}$$

Q: For which N, Q_f is the closed-loop system asymptotically stable?

Recall: stability of receding-horizon LQR

Theorem. Consider the linear system with (A, B) reachable. Let $Q_2 \succ 0$ and $(A, Q_1^{1/2})$ be observable. Then, if $Q_f = P$ where

$$P = A^T PA + Q_1 - (B^T PA)^T (Q_2 + B^T PB)^{-1} B^T PA$$

the receding-horizon LQR results in an asymptotically stable closed-loop system for all values of $N \geq 1$. Moreover, the control is a linear feedback $u_t = -Lx_t$ where L satisfies

$$L = (Q_2 + B^T PB)^{-1} B^T PA$$

Recall: Stability of receding-horizon LQR

Proof. Since $x^T Px$ is the cost-to-go for the infinite-horizon LQR, the finite-horizon LQR problem is equivalent to the infinite-horizon LQR problem, for which the optimal solution is $u_t = -Lx_t$.

From last lecture, we know that under the given conditions, the associated closed-loop system is guaranteed to be asymptotically stable.

Formally, the LQR stability proof uses infinite-horizon cost-to-go function

$$J(x) = x^T Px$$

as Lyapunov function (i.e. P satisfies the ARE).

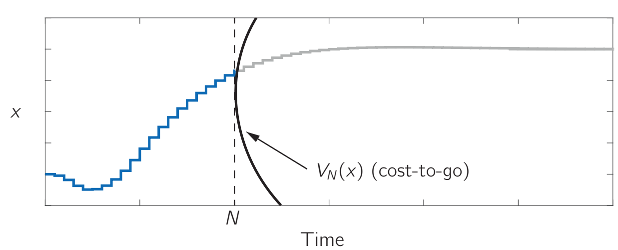
Recall: LQR, dynamic programming and cost-to-go

Recall that the LQR solution is obtained by optimizing over a single stage

$$v_t(z) = \min_w z^T Q_1 z + w^T Q_2 w + V_{t+1}(Az + Bw)$$

if we account for the cost-to-go from where our control actions take us.

The role of $x^T Q_f x$: to account for cost-to-go beyond prediction horizon



Stability of receding-horizon LQR

Closed-loop stability guaranteed if $x^T Q_f x$ is infinite-horizon cost-to-go.

Q: Is this the only choice of Q_f that guarantees stability?

Q: Is infinite-horizon cost the only Lyapunov function that works?
What about the predicted MPC cost:

$$\begin{aligned} J(x_t) \;=\; & \min_{x,u} && \sum_{k=0}^{N-1} x_{t+k|t}^T Q_1 x_{t+k|t} + u_{t+k|t}^T Q_2 u_{t+k|t} + x_{t+N|t}^T Q_f x_{t+N|t} \\ & \text{subject to} && x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} \\ & && x_{t|t} = x_t \end{aligned}$$

(which is equivalent to infinite-horizon cost for appropriate Q_f)

Let's try to find out!

Stability of receding-horizon LQR

Let $U_t^\star = (u^\star_{t|t}, \dots, u^\star_{t+N-1|t})$ be the optimizer of (1) for $x_{t|t} = x_t$. Then

$$J(x_t) = \sum_{k=0}^{N-1} (x^\star_{t+k|t})^T Q_1 x^\star_{t+k|t} + \left(u^\star_{t+k|t}\right)^T Q_2 u^\star_{t+k|t} + (x^\star_{t+N|t})^T Q_f x^\star_{t+N|t}$$

At time $t+1$, consider the control

$$\tilde{U}_{t+1} = (u^\star_{t+1|t}, u^\star_{t+2|t}, \dots, u^\star_{t+N-1|t}, \tilde{u}_N)$$

Since $x_{t+1} = Ax_t + Bu^\star_{t|t} = x^\star_{t+1|t}$, the associated cost is

$$\begin{aligned}\tilde{J}(x_{t+1}) &= \sum_{k=1}^{N} (x^\star_{t+k|t})^T Q_1 x^\star_{t+k|t} + (u^\star_{t+k|t})^T Q_2 u^\star_{t+k|t} + \tilde{x}^T_{t+N+1} Q_f \tilde{x}_{t+N+1} = \\ &= J(x_t) - (x^\star_{t|t})^T Q_1 x^\star_{t|t} - (u^\star_{t|t})^T Q_2 u^\star_{t|t} - (x^\star_{t+N|t})^T Q_f x^\star_{t+N|t} + \\ &\quad + (x^\star_{t+N|t})^T Q_1 x^\star_{t+N|t} + \tilde{u}_N^T Q_2 \tilde{u}_N + \tilde{x}^T_{N+1} Q_f \tilde{x}_{N+1}\end{aligned}$$

with $\tilde{x}_{N+1} = Ax^\star_{t+N|t} + B\tilde{u}_N$.

Stability of receding-horizon LQR

Since $J(x_{t+1}) \leq \tilde{J}(x_{t+1})$, we have

$$\begin{aligned}J(x_{t+1}) \leq J(x_t) &- \left((x^\star_{t|t})^T Q_1 x^\star_{t|t} + (u^\star_{t|t})^T Q_2 u^\star_{t|t}\right) + \\ &+ \tilde{x}^T_{N+1} Q_f \tilde{x}_{N+1} - (x^\star_{t+N|t})^T Q_f x^\star_{t+N|t} + (x^\star_{t+N|t})^T Q_1 x^\star_{t+N|t} + \tilde{u}_N^T Q_2 \tilde{u}_N\end{aligned}$$

Since stage cost is positive if $x_t \neq 0$, it is enough that

$$\tilde{x}^T_{N+1} Q_f \tilde{x}_{N+1} - (x^\star_{t+N|t})^T Q_f x^\star_{t+N|t} \leq -(x^\star_{t+N|t})^T Q_1 x^\star_{t+N|t} + \tilde{u}_N^T Q_2 \tilde{u}_N) \quad (2)$$

Note that $\tilde{x}_{N+1} = Ax^\star_{t+N|t} + B\tilde{u}_N$, so this ressembles a Lyapunov inequality.

If we let $\tilde{u}_N = -\tilde{L}x^\star_{t+N|t}$, then (2) is satisfied for all $\tilde{x}^\star_{t+N|t}$ if

$$(A - B\tilde{L})^T Q_f (A - B\tilde{L}) - Q_f = -Q_1 - \tilde{L}^T Q_2 \tilde{L} \quad (3)$$

Stability of receding-horizon LQR

From the Lyapunov lecture, we know that (3) admits a unique solution $Q_f \succ 0$ if $Q_1 \succeq 0$, $Q_2 \succ 0$, $(A - B\tilde{L})$ is Schur annd $(A, Q_1^{1/2})$ is observable.

Thus, to find a Q_f which ensures an asymptotically stable closed-loop, we can pick *any* stabilizing state feedback gain $\tilde{L}$ and solve (3).

Note. $x_0^T Q_f x_0$ is the infinite-horizon LQR cost for $u_t = -\tilde{L}x_t$.

Stability of receding-horizon LQR

Theorem. Consider the discrete-time linear system with (A, B) reachable. If $(A, Q_1^{1/2})$ is observable and $Q_f = P$ where P satisfies

$$(A - B\tilde{L})^T P (A - B\tilde{L}) - P = -(Q_1 + \tilde{L}^T Q_2 \tilde{L})$$

for some $\tilde{L}$ such that $A - B\tilde{L}$ is Schur, then the receding-horizon LQR results in an asymptotically stable closed-loop for all values of $N \geq 1$.

Model predictive control of constrained linear systems

Consider the discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t \tag{4}$$

under the constraints

$$x_t \in X, \quad u_t \in U, \quad \text{for all } t \geq 0 \tag{5}$$

When will closed-loop system under MPC be stable and satisfy constriants?

Model predictive control of constrained linear systems

We consider the receding-horizon MPC control law

$$u_t = u^\star_{t|t} \tag{6}$$

where $(u^\star_{t|t}, \dots, u^\star_{t+N-1|t})$ is the optimizer of

$$\begin{array}{lll} \text{minimize} & \sum_{k=0}^{N-1} q(x_{t+k|t}, u_{t+k|t}) + q_f(x_{t+N|t}) & \\ \text{subject to} & x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} & \\ & x_{t+k|t} \in X & \\ & u_{t+k|t} \in U & k = 0, \dots, N-1 \\ & x_{t+N|t} \in X_f & \\ & x_{t|t} = x_t & \end{array} \tag{7}$$

We will assume that both stage cost q and terminal cost q_f are quadratic:

$$q(x, u) = x^T Q_1 x + u^T Q_2 u \qquad\qquad q_f(x) = x^T Q_f x$$

Closed-loop stability of MPC

Theorem. Consider the reachable linear system (4) with constraints (5) under the model predictive control (6),(7) . If the following conditions hold:

(a) $Q_1 \succeq 0$ with $(A, Q_1^{1/2})$ observable, $Q_2 \succ 0$, and $Q_f \succ 0$.

(b) X, U and X_f are closed and contain 0 in their interior.

(c) X_f is control invariant and $X_f \subseteq X$.

(d) $\min\limits_{\tilde{u} \in U, Ax+B\tilde{u} \in X_f} q_f(Ax + B\tilde{u}) - q_f(x) + q(x, \tilde{u}) \leq 0$ for all $x \in X_f$.

then x_t converges asymptotically to zero, $\lim_{t\to\infty} x_t = 0$ from all initial values x_0 for which (7) admits a feasible solution.

Closed-loop stability of MPC

Proof sketch. By control invariance of X_f, there exists $\tilde{u}_N$ such that

$$\tilde{U}_{t+1} = \{u^\star_{t+1|t}, u^\star_{t+2|t}, \dots, u^\star_{t+N-1|t}, \tilde{u}_N)\}$$

is feasible. Applying this control sequence, we see that

$$\begin{aligned} \tilde{J}(x_{t+1}) &= \sum_{k=1}^{N-1} q(x_{t+k|t}, u^\star_{t+k|t}) + q(x^\star_{t+N|t}, \tilde{u}_N) + q_f(Ax^\star_{t+N|t} + B\tilde{u}_N) = \\ &= J(x_t) - q(x_t, u_t) - q_f(x^\star_{t+N|t}) + q(x^\star_{t+N|t}, \tilde{u}_N) + q_f(Ax^\star_{t+N|t} + B\tilde{u}_N) \end{aligned}$$

Since this control sequence is suboptimal, we have

$$J(x_{t+1}) \leq J(x_t) - q(x_t, u_t) + q_f(Ax^\star_{t+N|t} + B\tilde{u}_N) - q_f(x^\star_{t+N|t}) + q(x^\star_{t+N|t}, \tilde{u}_N)$$

So, if $q_f(Ax^\star_{t+N|t} + B\tilde{u}_N) - q_f(x^\star_{t+N|t}) + q(x^\star_{t+N|t}, \tilde{u}_N) \leq 0$, it holds that

$$J(x_{t+1}) - J(x_t) \leq -q(x_t, u_t)$$

$J(x)$ is a Lyapunov proving asymptotic stability of the closed loop.

Closed-loop stability of MPC

Comments:

- Proof sketch, since we have not shown that $J(x)$ is continuous.

Conditions of MPC stability theorem are quite complex to apply

- control invariant set may be costly to compute, complex to represent
- condition (d) difficult to verify

Common to simply use LQR-invariant set and cost-to-go.

Closed-loop stability of MPC

Corollary. Consider the reachable linear system with constraints (5) under the model predictive control law (6), (7). If conditions (a) and (b) of earlier theorem hold, $Q_f = P$ where P satisfies the ARE

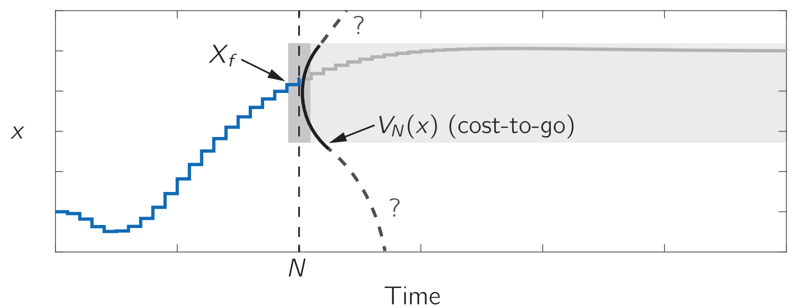
$$P = Q_1 + A^T PA - (B^T PA)^T (Q_2 + B^T PB)^{-1}(B^T PA)$$

and X_f is invariant for $x_{t+1} = (A - BL)x_t$ with

$$L = (Q_2 + B^T PB)^{-1} B^T PA$$

then x_t converges asymptotically to zero, $\lim_{t\to\infty} x_t = 0$ from all initial values x_0 for which (7) admits a feasible solution.

Corollary in pictures



Terminal state X_f:

- is control invariant
- allows us to compute quadratic cost-to-go, use as terminal penalty

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Example

Consider the system

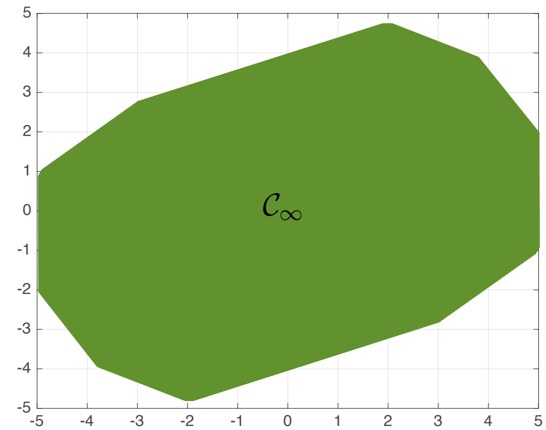
$$x_{t+1} = \begin{bmatrix} 0.5 & -1.25 \\ 1.0 & 0.0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t$$

under constraints $|x_i(t)| \leq 1$, and $|u(t)| \leq 1$.

Let the stage-cost be defined by $Q_1 = I$ and $Q_2 = \rho I$.

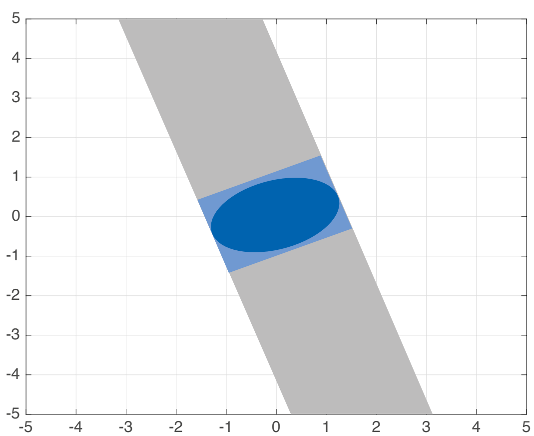
Suggest terminal set, terminal cost, and horizon length for MPC.

Control invariant set



Initial states outside $\mathcal{C}_\infty$ cannot be driven to origin by *any* controller (which satisfies the constraints on states and controls)

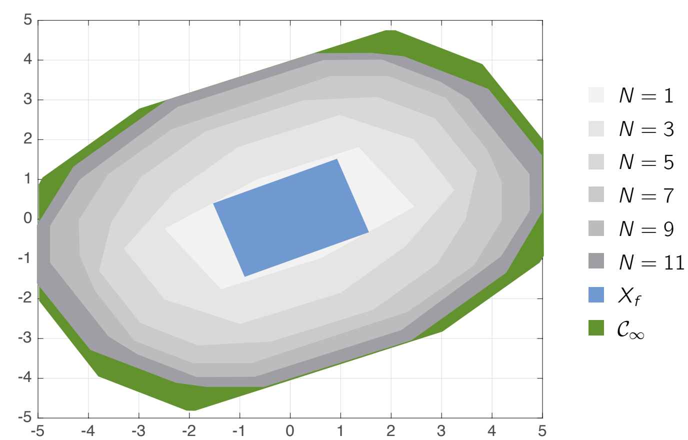
Invariant set for stationary LQR controller



Region of linear operation for $u_t = -L_{\mathrm{LQ}} x_t$ in gray.
Corresponding ellipsoidal (dark blue) and polyhedral (light) invariant sets.

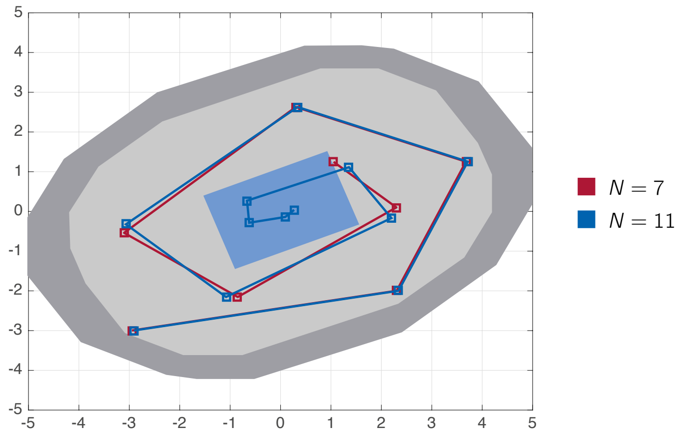
Terminal state constraint and feasible region

Increasing prediction horizon (N) enlarges set of recursively feasible states.



Predicted optimal trajectories

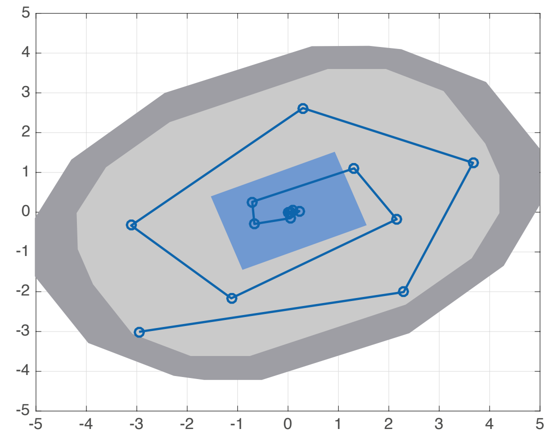
Open-loop predictions from $x(0) = (-3, -3)$.



MPC with $N = 7$ must deviate from optimal trajectory to reach X_f in time.

Closed-loop trajectories

Are in this case identical.



Discussion

Terminal sets not often used in practice:

- not well understood by practitioners
- requires advanced theory, tools to compute
- reduces set of recursively feasible states

In addition, it is not always needed:

- can be implicitly enforced via large horizons
- less critical if open loop system is stable, constraints not hard, . . .
- more next lecture. . .

Summary

- Potential issues:
  - stability and feasibility
- Stability of receding horizon LQR:
  - infinite-horizon cost-to-go, predicted cost
- Recursive feasibility of MPC
  - control invariant terminal sets
- Stability of MPC with terminal sets
  - control invariant terminal sets, cost-to-go as terminal penalty

Reading instructions: Lecture notes §5.1–5.3