# Problem 1

(a) $x(t+1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$ $\quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u(t) = 0$

$y(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(t)$

$x_2(t+1) = 0\, x_1(t) + 1 \cdot x_2(t) + 0 \cdot u(t)$

$= x_2(t)$

$\Rightarrow x_2(1) = x_2(0) \qquad x_2(2) = x_2(1) = x_2(0)$

$\Rightarrow \boxed{x_2(t) = x(0) = 1} \quad \forall t$

(b) $y(t) = C\,x(t)$ $\longrightarrow$ we need to find $x(t)$ first

Recall

$$x(t) = \underbrace{A^t x(0)}_{\text{Free response}} + \underbrace{\sum_{k=0}^{t-1} A^k B\, u(t-1-k)}_{\text{Driven response}} \qquad (*)$$

We need to find $A^t$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 2\times 0 & 1\times 2 + 2\times 1 \\ 0\times 1 + 1\times 0 & 0\times 2 + 1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2+2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times 1 + (2+2)\times 0 & 2+2+2 \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+2+2 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A^t = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix}}$$

$$\boxed{A^t B} = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 2t\times 0 \\ 0\times 1 + 1\times 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

Free response $\quad A^t x(0) = \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ 1 \end{bmatrix}$

Driven response $\quad \sum_{k=0}^{t-1} \underbrace{A^k B}_{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underbrace{u(t-1-k)}_{5} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 5 = \begin{bmatrix} 5t \\ 0 \end{bmatrix}$

Total $\quad x(t) = \begin{bmatrix} 2t+1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5t \\ 0 \end{bmatrix} = \begin{bmatrix} 7t+1 \\ 1 \end{bmatrix}$

$$\Rightarrow \quad \boxed{y(t)} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 7t+1 \\ 1 \end{bmatrix} = \boxed{7t+3}$$

# Problem 2

Recall $\dot{x}(t) = A_c x(t) + B_c u(t) \xrightarrow{h} x(k+1) = A x(k) + B u(k)$

$$\boxed{A = e^{A_c h}} \qquad \boxed{B = \int_0^h e^{A_c s} B_c \, ds}$$

(a) $\dot{x}(t) = 3u(t) = 0 \cdot x(t) + 3 \cdot u(t)$

$$\left.\begin{aligned} A_c &= 0 \\ B_c &= 3 \end{aligned}\right\} \quad A_c = e^{A_c h} = e^0 = 1$$

$$B_c = \int_0^h e^{0 s} \cdot 3 \, ds = \int_0^h 3 \, ds = 3s \Big|_0^h = 3h$$

(b) $z_1(t) = x(t)$

$z_2(t) = \dot{x}(t)$

$$\Rightarrow \dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = e^{A_c h} = I + \frac{1}{1!} A_c^1 h^1 + \frac{1}{2!} A_c^2 h^2 + \dots$$

$$A_c^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{e^{A_c h}} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h = \boxed{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}} = A$$

$$B = \int_0^h e^{A_c s} B_c \, ds = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds$$

$$= \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{s^2}{2} \\ s \end{bmatrix} \Bigg|_0^h$$

$$= \boxed{\begin{bmatrix} h^2/2 \\ h \end{bmatrix}} = B$$

# Problem 3

The discrete time system $x_{t+1} = A x_t$ is asymptotically stable

$$\Updownarrow$$

$$|\lambda_i(A)| < 1 \quad \forall\, i = 1, \dots, n$$

$\lambda_i$ - eigenvalue of $A$

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(a) $\boxed{\text{Characteristic polynomial} \quad \det(\lambda I - A) = 0}$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -6 & 4 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda-3 & 1 \\ 6 & \lambda-4 \end{bmatrix}\right)$$

$$= (\lambda-3)(\lambda-4) - 6$$

$$= \lambda^2 - 3\lambda - 4\lambda + 12 - 6$$

$$= \lambda^2 - 7\lambda + 6$$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49 - 4 \cdot 6}}{2} = \frac{7 \pm 5}{2} \begin{matrix} \nearrow 6 \\ \searrow 1 \end{matrix}$$

The System is unstable

(b) $\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} \lambda - 1/4 & 1/4 \\ 1/4 & \lambda - 1/4 \end{bmatrix}\right)$$

$$= \left(\lambda - \frac{1}{4}\right)\left(\lambda - \frac{1}{4}\right) - \frac{1}{4} \cdot \frac{1}{4}$$

$$= \lambda^2 - \frac{1}{4}\lambda - \frac{1}{4}\lambda + \cancel{\frac{1}{16}} - \cancel{\frac{1}{16}}$$

$$= \lambda\left(\lambda - \frac{1}{2}\right)$$

$\Rightarrow \quad \lambda_1 = 0 \qquad \lambda_2 = \frac{1}{2}$

The system is stable

# Problem 4

(a) $\circ\ C_n = [A^{n-1}B \dots B]$ - reachability matrix

$\circ$ The system $x_{t+1} = Ax_t + Bu_t$ is reachable

$\Updownarrow$

rank $(C_n) = n$

$n = 2$ $\quad C_2 = [AB \mid B]$

$$= \left[ \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Bigg| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ rank$(C_2) = 1$ $(\det(C_2) = 0)$ $\Rightarrow$ system is not reachable

give intuition

(b) The system $\begin{matrix} x_{t+1} = Ax_t + Bu_t \\ y_t = Cx_t \end{matrix}$ is unobservable

$\Updownarrow$

there exists $v \neq 0, v \in R^n, \lambda \in C$ such that

$$\boxed{\underset{(1)}{Av = \lambda v} \qquad \underset{(2)}{Cv = 0}}$$

From (2) $\Rightarrow [1 \quad 1] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow$ $\boxed{v_2 = -v_1 \neq 0}$

From (1) $\Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(1)+(2) $\Rightarrow$ $2v_1 = \lambda v_1 \Rightarrow \lambda = 2$

$-v_1 = -\lambda v_1 \Rightarrow \lambda = 1$ $\lightning$

$$\boxed{\Rightarrow \text{The system is observable}}$$

## Problem 5 → Explain the feedback first

Recall: The state feedback law $u(t)=-Lx(t)$ allows us to assign the poles of $A-BL$ to arbitrary locations

$\Updownarrow$

The open loop system is reachable

(a) Investigate reachability

$$C = [AB \quad B] = \boxed{\begin{bmatrix} 0 & 0 \\ 0.2 & 1 \end{bmatrix}}$$

$\Rightarrow$ rank$(C)=1$ $\Rightarrow$ the system is not reachable

$\Rightarrow$ we cannot assign the poles arbitrarily

b) $L=[l_1 \quad l_2]$ 1 input $\Downarrow$ L has 1 row 2 states $\Downarrow$ L has 2 columns

$$\boxed{A-BL} = \begin{bmatrix} 0.5 & 0 \\ 0.8 & 0.2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ l_1 & l_2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0.5 & 0 \\ 0.8-l_1 & 0.2-l_2 \end{bmatrix}}$$

$$\det(\lambda I-(A-BL)) = \det\left(\begin{bmatrix} \lambda-0.5 & 0 \\ -0.8+l_1 & \lambda-0.2+l_2 \end{bmatrix}\right)$$

$$= (\lambda-0.5)(\lambda-0.2+l_2)-0$$

$\lambda_1 = 0.5$ $\quad \lambda_2 = -0.2+l_2 \overset{l_2=0.2}{=} 0$

$\Rightarrow$ We can place the poles to $(0.5, 0)$