1. (a) Dynamic programming is hard to apply, since the presence of input and state constraints makes the cost-to-go function difficult to compute and represent. Although quadratic programming could be formally applied, the infinite horizon leads to an infinite number of decision variables, which standard solvers are unable to handle.

(b) (i) The terminal cost should be taken as the infinite-horizon cost-to-go function, or an approximation thereof. In the absence of constraints, the infinite-horizon cost is quadratic and has the form $x^T Px$, where P satisfies the algebraic Riccati equation. Therefore, in the equivalent finite horizon optimization problem, we can include a terminal state penalty on the form $x^T Px$ in combination with a terminal state constraints which guarantees that the quadratic cost is a good estimate of the true cost-to-go.

(ii) Terminal constraints ensure that predictions satisfy the system input and state constraints over the infinite horizon at prediction times $k = N, N + 1, \dots$). This provides a recursive guarantee of feasibility, ensuring that, at each sampling instant k, the tail of the predicted input and state sequences that solve the constrained MPC optimization at time k will satisfy the constraints of the MPC optimization at time $k + 1$. This yields a guarantee that the optimal predicted cost decreases with k, and hence a guarantee of closed loop stability. If we, in addition, choose the terminal set as an invariant set of the infinite-horizon LQR-optimal control, then the quadratic cost-to-go will be a good approximation of the true infinite-horizon cost.

(iii) A terminal constraint set S must be:

* invariant under some feedback law $u_k = K(x_k)$ from N to ∞, i.e., $x_k \in S$ implies $x_{k+1} \in S$, for all $k \geq N$.
* feasible, i.e., state constraints are instantaneously satisfied and input constraints are satisfied by the feedback law $u_k = K(x_k)$ from N to ∞, for all $x \in S$.

(c) No, depending on the initial state, model mismatch and disturbances, we may end up in a state from which the optimization problem has no solution. A common remedy is to soften the constraints using slack variables. This can help avoiding infeasibility at the cost of having to solve a slightly larger QP in every iteration (and also accepting that constraints are sometimes violated).

2. (a) We introduce the following notation for the continuous-time system

$$\dot{x}(t) = A_c x(t) + B_{1c} F_1(t) + B_{2c} F_2(t)$$

and

$$x_{t+1} = A x_t + B u_t + B_w w_t$$

where $x_t = x(kh)$, $u_t = F_1(kh)$ and $w_t = F_2(kh)$ for $k = 0, 1, \dots$. The formulas for zero-order hold sampling yield

$$A = e^{A_c h} = \begin{bmatrix} e^{-\frac{h}{2\theta}} & 0 \\ 0 & e^{-\frac{h}{\theta}} \end{bmatrix}$$

while

$$B = \int_{s=0}^{h} e^{A_c s} B_{1c} \, ds = \int_{s=0}^{h} \begin{bmatrix} e^{-\frac{s}{2\theta}} \\ \frac{c_1 - c_0}{V_0} e^{-\frac{s}{\theta}} \end{bmatrix} ds = \begin{bmatrix} 2\theta(1 - e^{-\frac{h}{2\theta}}) \\ \theta \frac{c_1 - c_0}{V_0}(1 - e^{-\frac{h}{\theta}}) \end{bmatrix}$$

and

$$B_w = \int_{s=0}^{h} e^{A_c s} B_{2c} \, ds = \int_{s=0}^{h} \begin{bmatrix} e^{-\frac{s}{2\theta}} \\ \frac{c_2 - c_0}{V_0} e^{-\frac{s}{\theta}} \end{bmatrix} ds = \begin{bmatrix} 2\theta(1 - e^{-\frac{h}{2\theta}}) \\ \theta \frac{c_2 - c_0}{V_0}(1 - e^{-\frac{h}{\theta}}) \end{bmatrix}$$

(b) To check controllability, we form the controllability matrix

$$\mathcal{C}_2 = \begin{bmatrix} B & AB \end{bmatrix} == \begin{bmatrix} 5 & 4.75 \\ -1 & -0.9 \end{bmatrix}$$

Since the matrix is square, we can show that it has full rank by verifying that its determinant is non-zero. In this case

$$\det \mathcal{C}_2 = 0.25$$

so the matrix has full rank and the system is controllable.

(c) The closed loop system matrix is

$$\begin{aligned} A - BL &= \begin{bmatrix} 0.95 & 0 \\ 0 & 0.90 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \\ &= \begin{bmatrix} 0.95 - 5l_1 & -5l_2 \\ l_1 & 0.9 + l_2 \end{bmatrix} \end{aligned}$$

and its eigenvalues are the roots of its characteristic polynomial

$$\begin{aligned} p(\lambda) &= \det(\lambda I - A + BL) = \\ &= \lambda^2 + (5l_1 - l_2 - 1.85)\lambda + 0.8550 - 4.5l_1 + 0.95l_2 \end{aligned}$$

The desired characteristic polynomial is

$$p_{\text{des}}(\lambda) = (\lambda + 0.5)(\lambda + 0.6) = \lambda^2 + 1.1\lambda + 0.3$$

The two coincide if we choose l_1 and l_2 such that

$$\begin{aligned} 5l_1 - l_2 - 1.85 &= 1.1 \\ 0.855 - 4.5l_1 + 0.95l_2 &= 0.3 \end{aligned}$$

This system of equations has solution

$$l_1 = 8.99 \qquad\qquad l_2 = 42$$

(d) With the added integrator state, the closed-loop dynamics is given by

$$\begin{aligned} x_{t+1} &= (A - BL)x_t + Bl_i i_t + B_w w \\ i_{t+1} &= -Cx_t + i_t \end{aligned}$$

At steady-state, $i_{t+1} = i_t$, which implies that

$$\lim_{t\to\infty} y_t = \lim_{t\to\infty} Cx_t = 0$$

3 (a) The dynamics are stable if and only if $|\alpha| < 1$, which is therefore a requirement for $|y(t)| \leq 1$ for all $t \geq 0$. Also note that

$$
\begin{aligned}
y(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(0) \\
y(1) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(1) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} x(0) \\
y(2) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(2) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \alpha \\ 0 & \alpha^2 \end{bmatrix} x(0) \\
&\vdots \\
y(i) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \alpha^{i-1} \\ 0 & \alpha^i \end{bmatrix} x(0) = \begin{bmatrix} 0 & \alpha^{i-1} \end{bmatrix} x(0), \text{ for all } i > 0
\end{aligned}
$$

so that

$$
y(t) = \begin{cases} [\ 1 \quad 0\ ]x(0) & t = 0 \\ [\ 0 \quad \alpha^{t-1}\ ]x(0) & t > 0 \end{cases}
$$

So, we can conclude that $|y(t)| \leq 1$ for all $t \geq 0$ if and only if $\alpha \leq 1$ and each element of $x(0)$ is less than or equal to 1 in absolute value.

(b) With the proposed control law, we have

$$
\sum_{k=N}^{\infty} x_k^T \begin{bmatrix} 1+\beta^2 & 0 \\ 0 & 0 \end{bmatrix} x_k := \sum_{k=N}^{\infty} x_k^T Q x_k
$$

From the lecture notes on Lyapunov stability, we know that we can evaluate this cost as

$$
x_N^T P x_N
$$

where P satisfies the Lyapunov equation

$$
A_{cl}^T P A_{cl} - P + Q = 0
$$

where

$$
A_{cl} = A - BL = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix}
$$

Hence, we need to solve

$$
\begin{bmatrix} 0 & 0 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1+\beta^2 & 0 \\ 0 & 0 \end{bmatrix} = 0
$$

The equation leads to the elementwise equalities

$$
\begin{aligned}
-p_{11} + (1+\beta^2) &= 0 \\
p_{12} &= 0 \\
p_{11} + 2\alpha p_{12} + \alpha^2 p_{22} &= 0
\end{aligned}
$$

with solution

$$\begin{aligned}
p_{11} &= (1+\beta^2) \\
p_{12} &= 0 \\
p_{22} &= p_{11}/(1-\alpha^2) = (1+\beta^2)/(1-\alpha^2)
\end{aligned}$$

which agrees with the postulated values.

(c) One of the most popular strategies for ensuring closed-loop stability is to use the predicted optimization cost as a Lyapunov function. The analysis is simplified if we include both a terminal cost and a terminal constraint set in the optimal control problem. The terminal cost is chosen to be equal to the infinite-horizon value function in a suitable neighborhood of the origin (i.e., within the terminal constraint set). We proved in (b) that using $x^T P_f x$ as a terminal cost respects the ARE, which means that the finite-horizon predicted cost is equivalent to the infinite-horizon one if no constraints are active. Since the problem is constrained, we need to ensure that the terminal constraint is state admissible, input admissible, and control invariant. State and input admissibility are obtained using results from (a) by replacing the constraint $|y_t| \leq 1$ with $|u_t| \leq 1$ and noting that $u_t = -\beta y_t$ for all $t \geq N$. Therefore, the proposed terminal cost and terminal set guarantee closed-loop stability. If, in addition, the horizon is long enough, the policy will also be optimal.

(d) Although the terminal equality constraint $x_N = 0$ would ensure recursive feasibility closed loop stability, it could severely restrict the operating region of the controller. In particular the second element of the state is uncontrollable so this terminal constraint would require the second element of the state to be equal to zero at all points in the operating region.

4 (a) The first step of the dynamic programming recursion is given by

$$\begin{aligned}V_1(x_1) &= \max_{|u_1|\le 1}\{0.4x_1 - 2(x_1+u_1)\}\\ &= \max_{|u_1|\le 1}\{-1.6x_1 - 2u_2\}\\ &= -1.6x_1 + 2\end{aligned}$$

with $\hat{u}_1 = -1$ The next step, which is the last step, is given by

$$\begin{aligned}V_0(x_0) &= \max_{|u_0|\le 1}\{0.4x_0 + V_1(x_0+u_0)\}\\ &= \max_{|u_0|\le 1}\{0.4x_0 - 1.6(x_0+u_0) + 2\}\\ &= \max_{|u_0|\le 1}\{-1.2x_0 - 1.6u_0 + 2\}\\ &= -1.2x_0 + 3.6\end{aligned}$$

with $\hat{u}_0 - 1$. In brief, the optimal strategy is $\hat{u}_0 = -1$, $\hat{u}_1 = -1$ with profit

$$3 + V_0(3) = 3$$

i.e. sell of as much as possible each year.

(b) From the results in (a), a reasonable value function candidate would be

$$V_{k+1}(x_{k+1}) = a_{k+1}x_{k+1} + b_{k+1}$$

Inserting the guess in the dynamic programming recursion gives

$$\begin{aligned}V_k(x_k) &= \max_{|u_k|\le 1}\{\theta x_k + a_{k+1}(x_k+u_k) + b_{k+1}\}\\ &= \max_{|u_k|\le 1}\{(a_{k+1}+\theta)x_k + a_{k+1}u_k + b_{k+1}\end{aligned}$$

The optimal choice of u_k is given by

$$\hat{u}_k = \begin{cases}1, & a_{k+1}\ge 0\\ -1, & a_{k+1} < 0\end{cases} = \mathrm{sgn}(a_{k+1})$$

so that

$$V_k(x_k) = \begin{cases}(a_{k+1}+\theta)x_k + b_{k+1} + a_{k+1}, & a_{k+1}\ge 0\\ (a_{k+1}+\theta)x_k + b_{k+1} - a_{k+1}, & a_{k+1} < 0\end{cases}$$

Hence, $V_k(x_k)$ is on the form $V_k(x_k) = a_k x_k + b_k$, where

$$a_k = a_{k+1} + \theta$$

and

$$b_k = \begin{cases}b_{k+1} + a_{k+1}, & a_{k+1}\ge 0\\ b_{k+1} - a_{k+1}, & a_{k+1} < 0\end{cases} = b_{k+1} + |a_{k+1}|$$

By induction, it follows that the above value function is valid.

(c) At $N-1$, the value function is given by

$$\begin{aligned} V_{N-1}(x_{N-1}) &= \max_{|u_{N-1}|\leq 1} \{\theta x_{N-1} - 2(x_{N-1} + u_{N-1})\} \\ &= (\theta - 2)x_{N-1} + 2 \end{aligned}$$

with $\hat{u}_{N-1} = -1$. Therefore, $a_{N-1} = \theta - 2$ and $b_{N-1} = 2$. Using the recurrence relation obtained in (b), a closed form expression for a_k is given by

$$a_k = (N-k)\theta - 2 \Rightarrow a_{k+1} = (N-k-1)\theta - 2$$

Thus, the optimal input is given by

$$\hat{u}_k(\theta, N) = \begin{cases} 1, & \theta \geq \frac{2}{N-k-1} \\ -1, & \theta < \frac{2}{N-k-1} \end{cases} \quad k = 0, 1, \ldots, N-2$$

and $\hat{u}_{N-1}(\theta, N) = -1$

(d) θ defines a cut-off point, which specifies how many years it is profitable to increase the share holdings. Specifically, it is optimal to buy a share each year up to the cut-off point

$$k = \left\lfloor \frac{\theta(N-1) - 2}{\theta} \right\rfloor$$

and afterwards sell off a share each year. The horizon, N, has to be large enough for it to even be profitable to increase the share holdings. Specifically, for a given θ, if

$$N < \frac{2}{\theta} + \theta$$

then

$$\theta < \frac{2}{N-k-1}, \quad k = 0, 1, \ldots, N-2$$

and it is optimal to sell off a share each year (as in (a)). In brief, large θ and N implies a longer period spent on increasing share holdings, whereas lower values on θ and N implies that most of the time is spent on selling of shares.

5 (a) Let

$$f(\alpha + i\beta) = (\alpha - i\beta)^T P(\alpha + i\beta) = \alpha^T P\alpha + \beta^T P\beta$$

If P is a positive definite matrix, then f is positive unless $\alpha = \beta = 0$.

(b) Assume that the Lyapunov equality

$$A^T PA - P + Q = 0$$

is satisfied, and let λ be an eigenvalue of A with associated eigenvector v. We pre-multiply the equality by v^* and post-multiply it by v:

$$\begin{aligned} v^*(A^T PA - P + Q)v &= \lambda^* v^* P\lambda v - v^* Pv + v^* Qv = \\ &= (|\lambda|^2 - 1)v^* Pv + v^* Qv = 0 \end{aligned}$$

Since Q is assumed to be positive definite, then we have

$$(|\lambda|^2 - 1)v^* Pv < 0$$

Now, if P is also positive definite, then the inequality implies that

$$|\lambda|^2 - 1 < 0$$

(c) Proceeding similarly as in (b), we find

$$v^*((A - cI)^T P(A - cI) - r^2 P + Q)v = (|\lambda - c|^2 - r^2)v^* Pv + v^* Qv = 0$$

Hence, if both Q and P are positive definite, then the equality implies

$$|\lambda - c|^2 < r$$

which is exactly what we were asked to prove.

(d) We first note that

$$A - cI = \begin{bmatrix} 0.6 & -0.1 \\ 0.2 & 0.5 \end{bmatrix} - \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.1 & -0.1 \\ 0.2 & 0 \end{bmatrix}$$

Letting $Q = I$, we thus look for a solution P to

$$\begin{bmatrix} 0.1 & 0.2 \\ -0.1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0.1 & -0.1 \\ 0.2 & 0 \end{bmatrix} - 0.04 \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The elementwise equalities read

$$\begin{aligned} -0.03p_{11} + 0.04p_{12} + 0.04p_{22} &= -1 \\ -0.01p_{11} - 0.06p_{12} &= 0 \\ 0.01p_{11} - 0.04p_{22} &= -1 \end{aligned}$$

which have the solution

$$P = \begin{bmatrix} 75 & -12.5 \\ -12.5 & 43.75 \end{bmatrix}$$

Since P is positive definite, we have verified that the eigenvalues of A are located in the desired region.