REGLERTEKNIK

School of Electrical Engineering, KTH

EL2700 Model predictive control

Exam (tentamen) 2016–12–19, kl 14.00–19.00

Aids: The course notes for EL2700 and books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be justified.
Lacking motivation will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Only write on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Mikael Johansson 08-7907436

Resultat: Will be posted no later than January 9, 2017.

Good Luck!

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1. Consider the discrete-time linear time-invariant system defined by

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k).$$

To regulate the system, we design the following MPC with a prediction horizon N

$$\begin{array}{ll} \underset{u(k|k),\ldots,u(k+N-1|k)}{\text{minimize}} & \displaystyle\sum_{i=0}^{N-1} \left[x(k+i|k)^T Q_1 x(k+i|k) + u^T Q_2 u(k+i|k)\right] + x_N^T Q_f x_N + \lambda \Delta^T \Delta \\ \text{subject to} & x(k+1) = Ax(k) + Bu(k), \quad \text{for all } k = 0,\, 1,\, \ldots,\, N-1, \\ & |x(k)| < \mathbf{1}h + \Delta \end{array}$$

where $\mathbf{1}$ is a N-dimensional vector and h is a user-defined parameter that defines the state constraint.

(a) In the MPC framework, what are the main considerations that govern the choice of the prediction horizon N?

(2p)

(b) In the inequality constraint, we included a new variable Δ. How is that variable called and in what circumstances is it used? What are the advantages and disadvantages of such approach?

(3p)

(c) Qualitatively explain how does Q_1 and Q_2 affect the closed-loop behaviour of the system.

(2p)

(d) In the light of MPC stability theory, explain the function of terminal constraints in an MPC for a system with input or state constraints. Define two principal properties that must be satisfied by a terminal constraint set.

(3p)

2. Consider the linear system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t \end{aligned}$$

with matrices

$$A = \begin{bmatrix} 0 & -1/2 \\ 3/2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

(a) For which values of the parameter b is the system open-loop stable? (2p)

(b) For which values of the parameter b is the system controllable? (2p)

(c) Let $b = 0$. Compute a state-feedback law $u_t = -Lx_t$ which places all system poles at the origin. After how many steps does the state, at most, arrive at the origin with this controller? (3p)

(d) Consider the observer

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + K(y_t - C\hat{x}_t)$$

with

$$K = \begin{bmatrix} -1/4 \\ -1 \end{bmatrix}$$

Find a Lyapunov function $V(x) = x^T P x$ which proves that the error dynamics

$$e_{t+1} = (A - KC)e_t$$

is asymptotically stable (here, $e_t = x_t - \hat{x}_t$). (3p)

3. A first-order system with discrete-time model

$$x_{t+1} = 1.5x_t + u_t$$

is to be controlled using a model predictive controller that minimizes the criterion

$$J = \sum_{k=0}^{1}(x_{t+k}^2 + 10u_{t+k}^2) + qx_{t+2}^2$$

(a) Show that when $q = 1$, the MPC controller has the form

$$u_t = -0.35x_t$$

(2p)

(b) Determine the optimal control law with respect to the infinite-horizon cost

$$J_\infty = \sum_{k=0}^{\infty} x_k^2 + 10u_k^2$$

Determine a value of q such that the finite-horizon cost in (a) and the infinite horizon cost above yield the same feedback control law. (4p)

(c) The predicted cost is to be minimized subject to the input constraint

$$-0.5 \leq u_{t+k} \leq 1$$

Show that if we use the q value computed in (b), *i.e.* if the predicted control is $u_{t+k} = -0.88x_{t+k}$, then the MPC optimization problem is feasible for all future times if u_{t+k} satisfies the constraints for $k = 0, 1$ and 2. (4p)

4. We want to solve the following optimization problem

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & x_N^2 + \sum_{k=0}^{N-1} u_k^2 \\ \text{subject to} \quad & x_{k+1} = x_k + u_k, \quad \text{for all } k = 0,\, 1,\, \ldots,\, N-1, \\ & x_0 > 0. \end{aligned}$$

using dynamic programming.

(a) Write down the dynamic programming recursion for this problem, i.e., write the Bellman equation which gives $V_t(x)$ recursively in terms of $V_{t+1}(x)$.

(2p)

(b) Solve the optimization problem using dynamic programming when $N = 3$ and $x_0 = 1$. Determine the optimal cost, optimal controls $\{u_0^\star, u_1^\star, u_2^\star\}$ and the optimal total allocation value $V_0(x_0)$. The variables u_k are real numbers.

(4p)

(c) Extend the result to solve the general optimization problem for a fixed given $N = n$.

Hint: Use an initial guess of the optimal cost-to-go function $V_t(x)$ based on the results in part 2. This function depends on t and n. Then, compute $V_{t-1}(x)$ and the corresponding $u_{t-1}^\star$ and verify that the initial guess is valid.

(4p)

5. Observability characterizes our ability to estimate the initial state from measurements of the system output. A disadvantage of the observability tests that we have discussed in class is that they are binary: systems are either observable or not. In practice, however, systems can be observable to varying degree, and different initial states can be more or less hard to estimate. In this problem, we will investigate one observability measure that can make these distinctions.

   Consider an autonomous discrete-time system

$$\begin{aligned} x_{t+1} &= Ax_t \\ y_t &= Cx_t \end{aligned}$$

   If the system is not observable, then there will be initial states x_0 which do not generate any output energy, i.e. for which

$$V_t = \sum_{k=0}^{t} y_k^T y_k \tag{3}$$

   is zero. Initial states x_0 which generate little output energy can be assumed hard to estimate, while initial states which generate a lot of output energy are easier to estimate.

   Of course, for a given initial value, one can estimate the quantity (3) using simulation. But such a simulation can take long time if t is large, and one may need to redo the simulation for many different initial values to gain an understanding of the system.

   (a) Use forward induction to show that V_t is a quadratic function of x_0, i.e.

$$V_t = x_0^T S_t x_0$$

   for some positive (semi-)definite matrix S_t. Derive a recursion for how S_{t+1} depends on the system matrices A and C, and S_t. (4p)

   (b) Assume that the iteration converges, in the sense that $S_t \to S$ as $t \to \infty$. Then

$$V_\infty = \sum_{k=0}^{\infty} y_k^T y_k = x_0^T S x_0$$

   so the infinite sum is readily evaluated, and we can also see which initial values generate a lot (and a little) output energy. Derive a matrix equation for computing S. (2p)

   (c) Consider the system given by the matrices

$$A = \begin{bmatrix} 0.8 & a \\ 0 & 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.6 & 0 \end{bmatrix}.$$

   Without carrying out any numerical computations, what structure would you expect S to have when $a = 0$? Justify your answer! (1p)

(d) Let instead $a = 0.115$. Compute the matrix S and determine the ratio between the output energy generated from initial value $x_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and initial value $x_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}$. (3p)