REGLERTEKNIK

School of Electrical Engineering, KTH

EL2700 Model predictive control

Exam (tentamen) 2017–10–23, kl 08.00–13.00

Aids: The course notes and slides for EL2700; books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be justified.
Lacking justification will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Only write on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Mikael Johansson 08-7907436

Results: Will be posted no later than November 13, 2017.

Good Luck!

---

1. Consider the infinite-horizon optimal control problem

$$
\begin{array}{lll}
\underset{\{u_0,u_1,\dots\}}{\text{minimize}} & \sum_{k=0}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k & \\
\text{subject to} & x_{k+1} = Ax_k + Bu_k, & k = 1, 2, \dots \\
& |u_k| \leq U_{\max} & k = 0, 1, \dots \\
& |x_k| \leq X_{\max} & k = 1, 2, \dots
\end{array} \tag{1}
$$

where Q_1 and Q_2 are positive definite matrices.

(a) Explain why it is difficult to solve this infinite-horizon control problem directly using dynamic programming or quadratic programming. (2p)

(b) Rather than solving the infinite-horizon problem, we consider the finite-horizon formulation

$$
\begin{array}{lll}
\underset{\{u_0,\dots,u_{N-1}\}}{\text{minimize}} & x_N^T P_f x_N + \sum_{k=0}^{N-1} x_k^T Q_1 x_k + u_k^T Q_2 u_k & \\
\text{subject to} & x_{k+1} = Ax_k + Bu_k, & k = 1, \dots, N-1 \\
& |u_k| \leq U_{\max} & k = 0, \dots, N-1 \\
& |x_k| \leq X_{\max} & k = 1, \dots, N \\
& x_N \in X_f &
\end{array} \tag{2}
$$

We would like this finite-horizon problem to be close to (and ideally equivalent to) the infinite-horizon problem. To this end, it includes a terminal cost and a terminal state constraint, which we need to choose in a particular way. Please answer the following questions:

(i) How do you choose the terminal cost?

(ii) Why do we need terminal state constraints?

(iii) What requirements does the terminal state constraint (i.e., terminal constraint set) need to satisfy? (6p)

(c) Can you guarantee that there will always be a solution to the finite-horizon optimization problem for any initial state? If not, can you suggest a technique that allows you to guarantee that the optimizer will always return a solution (even if this solution does not satisfy all constraints)? (2p)

2. Consider the stirred tank shown in Figure 1. The tank is fed with two inflows with flow rates $F_1(t)$ and $F_2(t)$. Both feeds contain material with constant concentrations c_1 and c_2. It is assumed that the tank is stirred well, so that the concentration of the outgoing flow equals the concentration $c(t)$ in the tank.

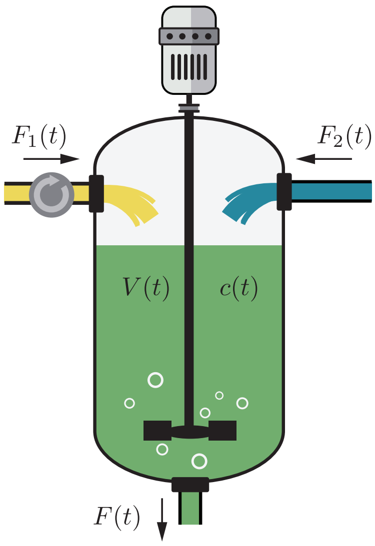

Figure 1: A stirred tank with controlled inflow F_1 and disturbance inflow F_2.

The tank dynamics can be described by the following linear system

$$\frac{dx(t)}{dt} = \begin{bmatrix} -\frac{1}{2\theta} & 0 \\ 0 & -\frac{1}{\theta} \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ \frac{c_1 - c_0}{V_0} \end{bmatrix} F_1(t) + \begin{bmatrix} 1 \\ \frac{c_2 - c_0}{V_0} \end{bmatrix} F_2(t)$$

As usual, all variables and inputs describe deviations from the equilibrium values V_0, c_0, F_{10} and F_{20}. The first state is the deviation in product volume present in the tank and the second state describes the difference between the actual material concentration in the tank and its equilibrium value. We will assume that $F_1(t)$ is under our control but consider $F_2(t)$ as a disturbance.

(a) Derive an exact discrete-time equivalent system under the assumption that the signals are sampled every T seconds and that the two inflows are held constant between sample instances. (4p)

(b) For a certain sampling interval and system parameters, the discrete-time equivalent is

$$x_{t+1} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.90 \end{bmatrix} x_t + \begin{bmatrix} 5 \\ -1 \end{bmatrix} F_t^{(1)} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} F_t^{(2)}$$

Recall that $F_t^{(1)}$ is our control variable, while $F_t^{(2)}$ is a disturbance. Is the discrete-time system with $F^{(1)}$ as input controllable? (1p)

(c) Determine a linear state feedback

$$F_t^{(1)} = -Lx_t$$

which places the closed-loop poles in $z = -0.5$ and $z = -0.6$. (3p)

(d) Your main target is to keep the concentration

$$z_t = \begin{bmatrix} 0 & 1 \end{bmatrix} x_t$$

at its target value 0, despite a constant input disturbance $F_2(t) = 1$. However, when you try your controller from (c), you notice a stationary error.

Show that by including an integral state in the controller, *i.e.* using

$$F_t^{(1)} = -Lx_t - l_i i_t$$

where the integral state satisfies

$$i_{t+1} = i_t + (r_t - y_t) = i_t - y_t$$

guarantees that $\lim_{t\to\infty} z_t = 0$ in steady-state.

You may assume that the feedback gains L and l_i are chosen so that the closed-loop system is asymptotically stable and that the steady-state is attained. (2p)

3. This problem considers constrained infinite-horizon LQR control

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{\infty} y_k^2 + u_k^2 \\ \text{subject to} & x_{k+1} = Ax_k + Bu_k \\ & y_k = Cx_k \\ & -1 \le u_k \le 1 \end{array}$$

We will proceed in steps towards a control design.

(a) First, consider the autonomous linear system

$$x_{t+1} = \begin{bmatrix} 0 & 1 \\ 0 & \alpha \end{bmatrix} x_t, \qquad y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t$$

where α is a constant. Show that

$$-1 \le y_t \le 1 \qquad\qquad \text{for all } t \ge 0$$

if and only if $|\alpha| \le 1$ and

$$\begin{bmatrix} -1 \\ -1 \end{bmatrix} \le x_0 \le \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(2p)

(b) Next, we consider the slightly more complex system

$$x_{t+1} = \begin{bmatrix} \beta & 1 \\ 0 & \alpha \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t, \qquad y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \tag{3}$$

under the control

$$u_t = -\begin{bmatrix} \beta & 0 \end{bmatrix} x_t. \tag{4}$$

We are interested in evaluating the infinite sum

$$\sum_{k=N}^{\infty} y_k^2 + u_k^2$$

where $\{y_k\}$ and $\{u_k\}$ are generated by the closed-loop system (3) and (4). Show that for any finite N, if $|\alpha| < 1$, we have

$$\sum_{k=N}^{\infty} y_k^2 + u_k^2 = x_N^T \begin{bmatrix} \beta^2 + 1 & 0 \\ 0 & (\beta^2+1)/(1-\alpha^2) \end{bmatrix} x_N$$

(4p)

(c) Consider the model-predictive control policy obtained by solving

$$\begin{array}{ll} \underset{\check{u}_0,\ldots,\check{u}_{N-1}}{\text{minimize}} & \sum_{k=0}^{N-1} \check{y}_k^2 + \check{u}_k^2 + \check{x}_N^T P_f \check{x}_N \\ \text{subject to} & \check{x}_{k+1} = A\check{x}_k + B\check{u}_k \\ & \check{y}_k = C\check{x}_k \\ & -1 \leq \check{u}_k \leq 1 \\ & \check{x}_N \in X_f \\ & \check{x}_0 = x_t \end{array}$$

and applying $u_t = \check{u}_0^\star$ in a receding-horizon fashion. What can you say about the stability and optimality of the associated closed-loop control if we let

$$P_f = \begin{bmatrix} \beta^2 + 1 & 0 \\ 0 & (\beta^2+1)/(1-\alpha^2) \end{bmatrix}$$

and set

$$X_f = \left\{ x \; : \; \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq |\beta| x \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(2p)

(d) What would happen if you replace X_f by $\{0\}$ in (c), *i.e.* force the receding-horizon policy to use a control that can drive the state to zero at the end of the prediction horizon? Will the closed-loop system be stable? Are there any disadvantages of this terminal set compared to the one suggested in (c)? (2p)

4. You currently own $x_c > N$ units of company shares, of some company C. You have reason to believe that company C will go bankrupt in N years, at which your shares would be worth nothing. However, up to that time, you receive dividends equal to $\theta \times$ your current share holdings, where $0 < \theta < 1$. In addition, every year you are allowed to buy or sell at most *one* unit of shares without any extra cost. Your intuition tells you that it should be possible to make a profit if you act in a clever way during these N years (assuming your bankruptcy assumption is true). Therefore, you formulate the following discrete optimal control problem:

$$\begin{aligned} x_c + \underset{u_k}{\text{maximize}} \quad & \sum_{k=0}^{N-1} \{\theta x_k\} - 2x_N \\ \text{subject to} \quad & x_{k+1} = x_k + u_k \\ & |u_k| \leq 1 \\ & x_0 = x_c \end{aligned} \tag{5}$$

The intuition of the objective is that you each year receive dividends θx_k, and at the end of the period you lose whatever amount x_N you have left as well as gain/lose $x_c - x_N$ depending on how many shares you sold/bought. Note, that x_c is constant and can be removed from the maximization. Furthermore, $x_c > N$ so it is not possible to sell of all shares and you can assume $x_k > 0$ throughout the period. (It is assumed that unit shares have unit cost, and that you have infinite capital.)

(a) For the specific case $x_c = 3$, $N = 2$, and $\theta = 0.4$, solve the above problem (5) using dynamic programming. Specify the optimal strategy $\hat{u}_0, \hat{u}_1$, and the resulting profit. (2p)

(b) Formulate a value function $V_{k+1}(x_{k+1})$ and show by induction that it is indeed a valid value function for the dynamic programming recursion:

$$V_k(x_k) = \underset{|u_k|\leq 1}{\text{maximize}} \{\theta x_k + V_{k+1}(x_{k+1})\}$$

Hint: Consider using two parameters a_{k+1} and b_{k+1} in the value function. Think carefully about what structure the value function had in (a). (3p)

(c) Use $V_{N-1}(x_{N-1})$ as a starting point of the recursion obtained in (b), and determine an optimal control law $\hat{u}_k$ as a function of θ and N. (3p)

(d) Explain qualitatively how the optimal strategy depends on θ and N. (2p)

5. Lyapunov stability has been a central component of the course. We have shown how quadratic Lyapunov functions are necessary and sufficient for asymptotic stability of discrete-time linear systems. In this problem, we will revisit the theory and show that a small modification of our Lyapunov inequalities allow us to ensure that the poles are located in specific regions of the complex plane.

   (a) We have defined positive definite functions from $\mathbb{R}^n$ to $\mathbb{R}$, but it is sometimes useful to define them for complex arguments. Show that if P is a positive definite matrix, then the function $f : \mathbb{C}^n \mapsto \mathbb{R}$ defined by

   $$f(x) = x^* P x$$

   is positive for all $x \neq 0$. Here x^* denotes complex conjugate transpose (that is, if $x = \alpha + i\beta$, then $x^\star = \alpha^T - i\beta^T$) (1p)

   (b) Now consider the discrete-time autonomous linear system

   $$x_{t+1} = Ax_t$$

   and the associated Lyapunov equation

   $$A^T PA - P + Q = 0.$$

   for some positive definite Q.
   Show by direct calculations that a positive definite solution P to the Lyapunov equation implies that $|\lambda_i(A)| < 1$ for all $i = 1, \ldots, n$. (2p)
   Hint. Remember that we have

   $$\lambda v = Av$$

   for every eigenvalue λ and associated (possibly complex-valued) eigenvector v of the matrix A.

   (c) Prove that a solution $P > 0$ to the modified Lyapunov equation

   $$(A - cI)^T P(A - cI) - r^2 P + Q = 0 \qquad (6)$$

   guarantees that all eigenvalues of A lie in a disk which is centered at c and has radius r. (5p)

   (d) Use the modified Lyapunov equation (6) to show that the poles of the discrete-time linear system

   $$x_{t+1} = \begin{bmatrix} 0.6 & -0.1 \\ 0.2 & 0.5 \end{bmatrix} x_t$$

   are located in a disc centered at 1/2 and with radius 1/5. (2p)