Exercise 3

Problem 1. Consider the general scalar linear-quadratic problem

$$\begin{aligned} \underset{u_0,\dots,u_{N-1}}{\text{minimize}} \quad & \sum_{t=0}^{N-1} q_1 x_t^2 + q_2 u_t^2 + q_3 x_N^2 \\ \text{subject to} \quad & x_{t+1} = a x_t + b u_t, \\ & x_0 \text{ given,} \end{aligned}$$

where $x_{t+1} = a x_t + b u_t$ is controllable and $q_1 > 0$, $q_2 > 0$, $q_3 > 0$.

(a) Using dynamic programming, derive an optimal control law as a closed form recursion $u_t = -K_t x_t$.

(b) Under the given assumptions, there exists an optimal state feedback to the infinite-horizon problem

$$\begin{aligned} \underset{u_0,u_1,\dots}{\text{minimize}} \quad & \sum_{t=0}^{+\infty} q x_t^2 + r u_t^2 \\ \text{subject to} \quad & x_{t+1} = a x_t + b u_t, \\ & x_0 \text{ given,} \end{aligned}$$

Derive the optimal feedback law in this case.

Problem 2. Consider the system

$$x_{t+1} = 0.9 x_t + u_t,$$

and the infinite–horizon problem

$$\begin{aligned} \underset{u_0,u_1,\dots}{\text{minimize}} \quad & \sum_{t=0}^{+\infty} q x_t^2 + r u_t^2 \\ \text{subject to} \quad & x_{t+1} = 0.9 x_t + u_t, \\ & x_0 \text{ given,} \end{aligned}$$

where $q = 1$, while r can take three different values: $r = 2$, $r = 10$, and $r = 100$. Based on these values, three optimal controllers were designed. The initial condition was set to $x(0) = 10$, and performances of the controllers were tested. Figure 2 shows the system trajectory, and Figure 4 the control efforts. Pair the responses and control efforts with different values of r.

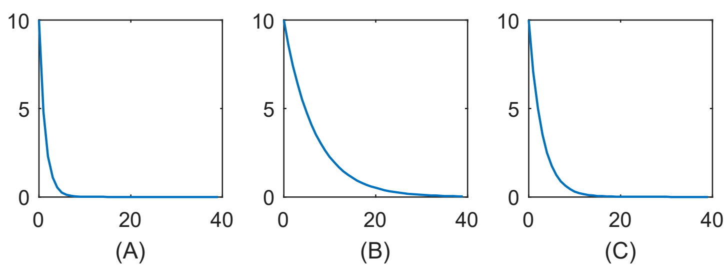

Fig. 1: Evolution of the state x (Problem 2).

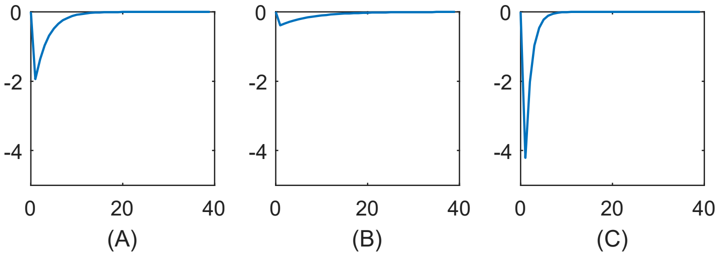

Fig. 3: Evolution of the control input u (Problem 2).

Problem 3. Consider the discrete-time LTI system defined by

$$x_{t+1} = \begin{bmatrix} 4/3 & -2/3 \\ 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t, \quad y_t = \begin{bmatrix} -2/3 & 1 \end{bmatrix} x_t.$$

(a) Let $Q = C^T C + 0.0001 I_2$, $R = 0.001$, $Q_N = Q$. Use the discrete-time Riccati recursion to compute the optimal control law that minimizes the following cost:

$$V = \sum_{i=0}^{N-1} (x_t^T Q x_t + u_t^T R u_t) + x_N^T Q_N x_N.$$

(b) Compute LQR controller.

(c) Compute the closed-loop state trajectory in a receding horizon fashion from state $x = [10\ 10]^T$. Find the minimum horizon length $N^\star$ that stabilizes the system. Motivate why increasing the horizon stabilizes the closed loop system.

(d) Is it possible to avoid the problem of short horizon by choosing suitable value for Q_N?

Problem 4. We consider the system

$$x_{t+1} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_t, \qquad y_t = \begin{bmatrix} 0 & 1 \end{bmatrix} x_t,$$

(a) Design LQR controller of the form $u_t = -Lx_t - l_r r_t$ using Matlab. The performance weights $Q = I_2$, $R = 1$ should be used, and static gain from r to y should equal 1.

(b) Assume now that the there exists disturbance acting on the input, so $u_t' = u_t + d$. Assume $r_t = 1$ and $d_t = 1$. Do you expect the static error $\lim_{t\to\infty} e_t = \lim_{t\to\infty} y_t - r_t$ to be present? Simulate the response of y_t to verify or disprove intuition.

(c) Introduce additional integral state i_t in the system and design LQR controller $i_t = -Lx_t - l_i i_t$ using Matlab. Use the performance weights $Q = I_3$, $R = 1$.

(d) Assume again $d_t = 1$ and $r_t = 1$. Do you expect the static error to be present now? Simulate the response of y_t to verify or disprove intuition.