REGLERTEKNIK

School of Electrical Engineering and Computer Science, KTH

EL2700 Model predictive control

Exam (tentamen) 2018–10–20, kl 09.00–14.00

Aids: The course notes and slides for EL2700; books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Justify every step of your solutions.
Lacking justification will result in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Write only on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Mikael Johansson 08-7907436

Results: Will be posted no later than November 9, 2018.

Good Luck!

---

1. Consider the following model predictive controller formulation

$$
\begin{aligned}
\min_{\{u_0,\dots,u_{N-1}\}} \quad & \sum_{k=0}^{N-1} x_k^T Q_1 x_k + u_k^T Q_2 u_k + x_N^T Q_f x_N \\
\text{subject to} \quad & x_{k+1} = Ax_k + Bu_k,\ k = 1, \dots, N-1 \\
& u_k \in \mathcal{U},\ k = 0, \dots, N-1 \\
& x_k \in \mathcal{X},\ k = 1, \dots, N-1 \\
& x_N \in \mathcal{X}_f
\end{aligned}
$$

Here, the sets $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{X}_f$ are polytopes enclosing the origin that represent the state, input, and terminal state constraint sets respectively. Q_1, Q_2, and Q_f are positive definite cost matrices.

(a) Explain briefly how a terminal cost $x_N^T Q_f x_N$, which is only dependent on x_N, can be seen as the infinite cost $\sum_{k=N}^{\infty} x_k^T Q_1 x_k + u_k^T Q_2 u_k$ from a static feedback $u_t = Kx_t$. (2p)

(b) State two important properties of a terminal set. (2p)

(c) Show that $\mathcal{X}_f = \{\bar{0}\}$ is a valid terminal set with regards to these properties. (2p)

(d) Why is $\mathcal{X}_f = \{\bar{0}\}$ in general not a good choice for a terminal set? (2p)

(e) What happens with the feasible set when the horizon N increases? Why? (2p)

2. The basic kinematics of lateral control of a vehicle is captured by the so-called bicycle model. When this model is linearized along a straight-line path (modeling, for example, lane following of a car), the lateral deviations from the path can be described by the second-order system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \gamma \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)\end{aligned}$$

where γ is a system parameter. In this description, the first state is normalized deviation of the car's center of mass from the path and the second state is the heading of the car. The control input $u(t)$ is the steering angle (the angle between the front wheels and the axis of the car).

(a) Sample the system using zero-order hold with sampling time h. Give explicit expressions for how the elements of the matrices A, B and C depend on the sampling time and the system parameter γ. (3p)

(b) Show that the discrete-time system is reachable for all values of γ. (1p)

(c) Validate that $h = 1$ and $\gamma = 1/2$ gives the discrete-time system

$$\begin{aligned}x_{t+1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t \\ y_t &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_t\end{aligned}$$

(1p)

(d) Consider the special case $h = 1$ and $\gamma = 1/2$ studied in (c). Design a state feedback law

$$u_t = -Lx_t$$

that places both closed-loop poles at 0.5. (3p)

(e) Augment your control law from (d) with a feed-forward from the reference, *i.e.* consider the controller

$$u_t = -Lx_t + l_r r_t.$$

Design l_r so if r_t is constant and equal to r, then $y_t = r$ in strationarity. (2p)

3. You currently own x_c units of shares, of some company C. You have reason to believe that company C will bankrupt in N years, at which point your shares would be worth nothing. However, up to that time, you receive dividends equal to $\theta \times$ your current share holdings, where $0 < \theta < 1$. In addition, every year you are allowed to buy or sell at most *one* unit of shares without any extra cost. Your intuition tells you that it should be possible to make a profit if you act in a clever way during these N years (assuming your bankruptcy asssumption is true). Therefore, you formulate the following discrete optimal control problem:

$$
\begin{aligned}
x_c + \underset{u_k}{\text{maximize}} \quad & \sum_{k=0}^{N-1} \{\theta x_k\} - 2x_N \\
\text{subject to} \quad & x_{k+1} = x_k + u_k, && k = 0, \dots, N-1 \\
& u_k \in \{-1, 0, 1\}, && k = 0, \dots, N-1 \\
& x_k \geq 0, && k = 0, \dots, N \\
& x_0 = x_c
\end{aligned}
\tag{1}
$$

The intuition of the objective is that you each year receive dividends θx_k, and at the end of the period you lose whatever amount x_N you have left as well as gain/lose $x_c - x_N$ depending on how many shares you sold/bought. Note, that x_c is constant and can be removed from the maximization. Furthermore, you cannot own a negative amount of shares, so $x_k \geq 0$. It is assumed that unit shares have unit cost, and that you have infinite capital.

(a) For the specific case $x_c = 3$, $N = 2$, and $\theta = 0.4$, solve (1) using dynamic programming. Specify the optimal strategy $\hat{u}_0, \hat{u}_1$, and the resulting profit.

Hint: Since $x_c > N$, the constraint $x_k \geq 0$ can be ignored. (2p)

(b) Consider now the case $x_c = 1$, $N = 3$, and $\theta = 0.8$. Solve (1) using dynamic programming. Specify the optimal strategy $\hat{u}_0, \hat{u}_1, \hat{u}_2$, and the resulting profit.

Note: Solution by enumeration will not be an acceptable solution. (8p)

4. A model aircraft has the following discrete dynamics in the vertical direction

$$\begin{bmatrix} h_{t+1} \\ \gamma_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} h_t \\ \gamma_t \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} u_t$$

where h_t is the altitude above ground and γ_t is the flight path angle, see Figure 1. Note that a negative angle means that the aircraft is decreasing its altitude. We have the following two constraints on the system

$$\begin{array}{rcll} -0.25 & \leq \gamma_t \leq & 0.25 & \text{[rad]} \\ 0 & \leq h_t \leq & 100 & \text{[m]}. \end{array} \tag{2}$$

We will design a controller to bring the aircraft to land, that is, we wish to drive the aircraft altitude to $h = 0$.

(a) We first try to solve this problem using LQR, where we minimize

$$J = \sum_{i=0}^{\infty} (x_{t+i}^T Q_1 x_{t+i} + u_{t+i}^T Q_2 u_{t+i}), \qquad Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 35 \end{bmatrix}, \ Q_2 = \begin{bmatrix} 633 \end{bmatrix}$$

The solution to the discrete-time algebraic Riccati equation is given by

$$P = Q_1 + A^T P A - A^T P B (Q_2 + B^T P B)^{-1} B^T P A \implies P \approx \begin{bmatrix} 20 & 240 \\ 240 & 3805 \end{bmatrix}$$

Given this P, what is the optimal LQR feedback solution $u_t = -L x_t$? (1p)

(b) Next we formulate a finite-horizon predictive control law. At each step $u_t = u_t^\star$, where $\{u_t^\star, u_{t+1}^\star, \ldots, u_{t+N-1}^\star\}$ is the minimizing argument of

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N-1} (x_{t+i}^T Q_1 x_{t+i} + u_{t+i}^T Q_2 u_{t+i}) + x_{t+N}^T Q_f x_{t+N} \\ \text{subject to} & x_{t+k+1} = A x_{t+k} + B u_{t+k} \qquad\qquad k = 0, 1, \ldots, N \end{array}$$

where $Q_f = P$ from the previous task. Show that if the system is not subject to any state constraints, then this control law yields a stable closed-loop system. (2p)

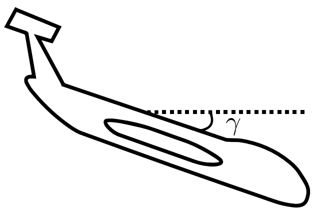

Figure 1: The aircraft vertical state is here described with two variables – altitude [m] and flight path angle [rad].

(c) In practice, it is easy to show that both the LQR controller in subtask (a) and the unconstrained MPC in subtask (b) will break the constraints (2) for many initial conditions. Therefore, we shall now try to construct a constrained finite horizon predictive controller.

In addition to the constraints (2), we also have to limit the vertical velocity at touchdown. The vertical velocity is approximated as

$$\dot{h} = \sin\gamma \cdot 20,$$

where we have assumed that the plane is traveling with a constant velocity $v_0 = 20$ m/s. If we want the vertical velocity at touchdown to be limited to

$$\dot{h} \geq -1 \text{ [m/s]},$$

what is the approximately equivalent constraint

$$\gamma \geq \gamma^{td} \text{ [rad]}$$

which must be enforced at touchdown, assuming that γ is small? (1p)

(d) Next, you will add a constraint on the flight path angle such that there is a continuous transition between the constraint $\gamma \geq \gamma^{min} = -0.25$ at $h = 10$ m and $\gamma \geq \gamma^{td}$ at $h = 0$ m.

Start by formulating the constraint on the form

$$\gamma + k_1 \cdot h \geq k_2.$$

After this, find the matrix H and vector h such that you can write all the state constraints at time t, including (2), on the form

$$Hx_t \leq h. \tag{3}$$

Draw the resulting polygon that represents the constraint set, and indicate the vertex coordinates. (4p)

(e) Assume that the mode two control is given by

$$u_t = \begin{bmatrix} -0.04 & 0.6 \end{bmatrix} x_t.$$

Is the state constraint set (3) which you found in (d), a suitable terminal set with regards to invariance? (2p)

5. In class, we claimed that the cost-to-go function for the infinite-horizon LQR problem is quadratic, but we never actually showed it. This problem gives you the opportunity to formally show this fact. Specifically, we will show that for the infinite-horizon LQR problem, the minimum cost-to-go starting in state x_0 is a quadratic form in x_0.

   Consider a reachable discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t,$$

   let $Q_1 \succeq 0$, $Q_2 \succ 0$ and define the cost

$$J(u, x_0) = \sum_{i=0}^{\infty} (x_t^T Q_1 x_t + u_t^T Q_2 u_t)$$

   where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}^m$. Let u denote an infinite sequence $\{u_0, u_1, \ldots\}$ and

$$V(x_0) = \min_u J(u, x_0)$$

   be the associated value function for initial value x_0.

   (a) Use the fact that

$$x_t = A^t x_0 + \sum_{i=1}^{t} A^{i-1} B u_{t-i}.$$

   to show that for all $\lambda \in \mathbb{R}$, $J(\lambda u, \lambda x_0) = \lambda^2 J(u, x_0)$, and conclude that

$$V(\lambda x_0) = \lambda^2 V(x_0). \tag{4}$$

   (2p)

   (b) Let u and $\tilde{u}$ be two input sequences, and let x_0 and $\tilde{x}_0$ be two initial states. Show that

$$J(u + \tilde{u}, x_0 + \tilde{x}_0) + J(u - \tilde{u}, x_0 - \tilde{x}_0) = 2J(u, x_0) + 2J(\tilde{u}, \tilde{x}_0)$$

   Demonstrate how minimizing the right-hand-side with respect to u and $\tilde{u}$ allows to conclude that

$$V(x_0 + \tilde{x}_0) + V(x_0 - \tilde{x}_0) \leq 2V(x_0) + 2V(\tilde{x}_0)$$

   (2p)

   (c) Let $x_0 = (x_0' + \tilde{x}_0')/2$ and $\tilde{x}_0 = (x_0' - \tilde{x}_0')/2$. Show how the above inequality implies that

$$V(x_0 + \tilde{x}_0) + V(x_0 - \tilde{x}_0) = 2V(x_0) + 2V(\tilde{x}_0) \tag{5}$$

   (2p)

(d) Recall that $V : \mathbb{R}^n \mapsto \mathbb{R}$ and that $\nabla V(z) = \frac{d}{dz}V(z)$. Show how (5) can be used to derive the following relation

$$\nabla V(x_0 + \tilde{x}_0) = \nabla V(x_0) + \nabla V(\tilde{x}_0) \tag{6}$$

Hint: Consider adding the gradients of (5) *with respect to* x_0 *and* $\tilde{x}_0$. (1p)

(e) Demonstrate how (4) implies that

$$\nabla V(\lambda x_0) = \lambda \nabla V(x_0) \tag{7}$$

(1p)

(f) The results (6) and (7) show that $\nabla V(x_0)$ is linear in x_0. Therefore, $\exists M \in \mathbb{R}^{n\times n}$ such that $\nabla V(x_0) = Mx_0$.

Next, to conclude the proof and show that the cost-to-go is quadratic in x_0, show that $V(x_0) = x_0^T P x_0$, where $P = \frac{1}{4}(M + M^T)$. Show that $P \succ 0$.

Hint:

$$V(x_0) = V(0) + \int_0^1 \nabla V(\theta x_0)^T x_0 \, d\theta$$

(2p)