REGLERTEKNIK
School of Electrical Engineering, KTH

EL2700 Model predictive control

Exam (tentamen) 2017–12–18, kl 14.00–19.00

Aids: The course notes and slides for EL2700; books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Write only on one side of each sheet.
Each step in your solutions must be justified.
Lacking justification will result in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

Responsible: Mikael Johansson 08-7907436

Results: Will be posted no later than January 15, 2018.

Good Luck!

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1. (a) For an offset free tracking, $\bar{x}$ and $\bar{u}$ must hold the following relashionship $\bar{x} = A\bar{x} + B\bar{u}$.

   (b) If the future reference and constraints are known, they can be included in the optimization problem formulation over a finite prediction horizon. This gives the advantage of knowing a priori how does the reference change in future and optimize the input sequence accordingly. This is extremely advantageous when the reference is time-varying.

   (c) Recursive feasibility is the property that solving the MPC optimization problem for an initial feasible state results in the next state being feasible as well. This results in a sequence of feasible (solvable) optimal control problems.

   (d) Since the disturbance is constant we have $d_{k+1} = d_k$. To include the effects of the disturbance in the process model we can extend the state vector:

$$z_k = \begin{bmatrix} x_k \\ d_k \end{bmatrix},$$

   which allows an augmented model to be constructed:

$$\begin{bmatrix} x_{k+1} \\ d_{k+1} \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ d_k \end{bmatrix}$$

   Since the extended state vector contains the unknown disturbance, a state observer will be needed. For example, a linear observer or a Kalman filter can be used. A stable observer will converge to the correct value of d_k.

   (e) We would need to find the steady-state reference, and use a pseudo-reference MPC with disturbance compensation, i.e., penalize deviations from $\bar{x}_{ss}$ and $\bar{u}_{ss}$, which can be calculated by (after the observer converged)

$$\begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{ss} \\ \bar{u}_{ss} \end{bmatrix} = \begin{bmatrix} -B_d \hat{d}_{ss} \\ \bar{r} \end{bmatrix}$$

2. (a) The open-loop system is asymptotically stable if all the eigenvalues of the system matrix A has magnitude strictly less than one. Since

$$p(z) = \det zI - A = \det \begin{bmatrix} z - 0.4 & -0.8 \\ -0.4 & z \end{bmatrix} = z^2 - 0.4z - 0.32 = (z - 0.8)(z + 0.4)$$

The open loop system has poles in $z = 0.8$ and $z = -0.4$, and is thus stable.

(b) The controllability matrix

$$\begin{bmatrix} B \\ AB \end{bmatrix} = \begin{bmatrix} 0 & 0.8 \\ 1 & 0 \end{bmatrix}$$

has full rank. Hence, the system is reachable, and therefore also controllable.

(c) The closed-loop system matrix $A - BL$ has characteristic polynomial

$$p(z) = \det zI - (A - BL) = \det \begin{bmatrix} z - 0.4 & -0.8 \\ -0.4 + l_1 & z + l_2 \end{bmatrix} =$$
$$= z^2 + (l_2 - 0.4)z - 0.32 + 0.8l_1 - 0.4l_2$$

The desired closed-loop polynomial is

$$p_{\text{des}}(z) = (z - 0.5)^2 = z^2 - z + 0.25$$

Identifying the coefficients, we get

$$\begin{aligned} l_2 - 0.4 &= -1 & &\Rightarrow l_2 = -0.6 \\ -0.32 - 0.4l_2 + 0.8l_1 &= 0.25 & &\Rightarrow l_1 = 33/80 \end{aligned}$$

(d) The observability matrix

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.8 & 0.8 \end{bmatrix}$$

does not have full rank (its rows are linearly dependent). Hence, the system is not observable.

(e) The error dynamics of the observer is governed by the matrix

$$A - KC = \begin{bmatrix} 0.4 - k_1 & 0.8 - k_1 \\ 0.4 - k_2 & -k_2 \end{bmatrix}$$

whose characteristic polynomial is

$$p(z) = \det zI - (A - KC) = z^2 + (k_1 + k_2 - 0.4)z - 0.4(k_1 + k_2 - 0.8).$$

With desired poles in $z = 0$ and $z = -0.4$ we have

$$p_{\text{des}}(z) = z(z + 0.4) = z^2 + 0.4z$$

which is satisfied for any gains for which $k_1 + k_2 = 0.8$.

(f) There are no values for k_1 and k_2 which gives the desired characteristic polynomial is $p_{\text{des}}(z) = z^2$. This is to be expected. Since the system is not observable, we are not guaranteed to be able to place the poles in arbitrary locations.

3. (a) The infinite cost is given by

$$\sum_{k=0}^{\infty} \frac{1}{2}(y_k^2 + u_k^2) = x_0^T P x_0,$$

where P respects the ARE

$$(A+BK)^T P(A+BK) - P = -Q - K^T RK,$$

where $Q = \frac{1}{2}C^T C$, $R = \frac{1}{2}$, $K = \frac{1}{\sqrt{2}}C$. Therefore, if we note that $P = I$ and substitute it in the ARE, then we get

$$(A+BK)^T P(A+BK) - P = -Q - K^T RK = \begin{bmatrix} -\frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{3}{8} \end{bmatrix}$$

Hence the solution of the ARE is $P = I$, which implies $\sum_{k=0}^{\infty} \frac{1}{2}(y_k^2+u_k^2) = ||x_0||^2$.

(b) From part (a), the finite-horizon cost function is equal to infinite-horizon cost function and the control input is $u_k = \frac{1}{\sqrt{2}}y_k$ for all $k \geq N$. Let $J^\star(x(t))$ be the minimum value of this cost over $\{u^\star_{0|t}, ..., u^\star_{N-1|t}\}$. Then, at time $t+1$, the predicted input sequence is $\{u^\star_{1|t}, ..., u^\star_{N-1|t}, Ky_N\}$, which gives

$$J(x(t+1)) = \sum_{k=1}^{\infty} \frac{1}{2}(y_{k|t}^2 + u_{k|t}^2) = J^\star(x(t)) - \frac{1}{2}(y_k^2 + u_k^2)$$

and since the optimal cost at time $t+1$ satisfies

$$J^\star(x(t+1)) \leq J(x(t+1)),$$

we can conclude that

$$J^\star(x(t+1)) \leq J^\star(x(t)) - \frac{1}{2}(y_k^2 + u_k^2).$$

This implies closed loop stability because $J^\star(x(t))$ is positive definite in $x(t)$ since (A, C) is observable (the observability matrix is full rank) and (A, B) is reachable (the controllability matrix is full rank).

(c) At time t, if the constraint is

$$-1 \leq y_t = Cx_t = \frac{1}{\sqrt{2}}[1\ 1]x_t \leq -1$$

and at time $t+1$ if the constraint is

$$-1 \leq y_{t+1} = C(A+BCK)x_t = \frac{1}{\sqrt{2}}[-1\ 1]x_t \leq -1$$

that implies that at $t+k$

$$-1 \leq y_{t+k} = C(A+BCK)^{t+k}x_t \leq -1.$$

(d) The closed loop system will be stable if the predicted trajectories satisfies $-1 \leq y_t \leq 1$ for all $t \geq 0$. Imposing constraints in the MPC formulation, ensures constraint satisfaction $-1 \leq y_t \leq 1$ for $t = 1, 2, ..., N$. Also, by imposing the constraint on $N + 1$ we ensure constraint satisfaction to all $t > 0$, as proven in (c). Moreover, we have proven in (a) and (b) that the unconstrained controller is stable and that the finite cost is equivalent to the infinite cost, under the control law $\check{u}_t$. Therefore, the closed loop system will be necessarily stable.

4. (a) The first step of the dynamic programming recursion is given by

$$\begin{aligned} V_1(x_1) &= \max_{0 \le u_1 \le 1} \{(1 - u_1)x_1\} \\ &= x_1 \end{aligned}$$

with $\hat{u}_1 = 0$ The next step, which is the last step, is given by

$$\begin{aligned} V_0(x_0) &= \max_{0 \le u_0 \le 1} \{(1 - u_0)x_0 + V_1(x_0 + 0.4u_0x_0)\} \\ &= \max_{0 \le u_0 \le 1} \{(1 - u_0)x_0 + x_0 + 0.4u_0x_0\} \\ &= \max_{0 \le u_0 \le 1} \{(2 - 0.6u_0)x_0\} \\ &= 2x_0 \end{aligned}$$

with $\hat{u}_0 = 0$. In brief, the optimal strategy is $\hat{u}_0 = 0$, $\hat{u}_1 = 0$ with profit

$$V_0(3) = 6$$

being distributed to the shareholders.

(b) From the results in (a), a reasonable value function candidate would be

$$V_{k+1}(x_{k+1}) = c_{k+1}x_{k+1}$$

Inserting the guess in the dynamic programming recursion gives

$$\begin{aligned} V_k(x_k) &= \max_{0 \le u_k \le 1} \{\theta(1 - u_k)x_k + c_{k+1}(x_k + \theta u_k x_k)\} \\ &= \max_{0 \le u_k \le 1} \{((1 + c_{k+1}) + (c_{k+1}\theta - 1)u_k)x_k \\ &= \max \{1 + c_{k+1}, (1 + \theta)c_{k+1}\}x_k \end{aligned}$$

Now, introduce the recursion

$$c_k = c_{k+1} + \max[1, \theta c_{k+1}]$$

so that

$$V_k(x_k) = c_k x_k$$

By induction, it follows that the above value function is valid. The optimal choice of u_k is determined by the largest expression in the max term above. In brief, $u_k = 0$ when

$$\theta \le \frac{1}{c_{k+1}}$$

and 1 otherwise. Since $V_N(x_N) = 0$, it holds that

$$c_k = c_{k+1} + 1 = N - k$$

until the cut-off point $\tilde{k}$ when

$$\theta > \frac{1}{c_{\tilde{k}+1}} = \frac{1}{N-k-1}$$

Therefore, the optimal control law is given by

$$\hat{u}_k = \begin{cases} 1, & \theta > \frac{1}{N-k-1} \\ 0, & \theta \leq \frac{1}{N-k-1} \end{cases}$$

In other words, invest all profits in the early years to build capital, and distribute everything during the last years of the period.

5. (a) This is a scalar LQR problem, where the optimal feedback is given by

$$\hat{u}_k = -\frac{3/2p}{1+(3/2)^2p}x_k = -\frac{6p}{4+9p}x_k$$

where p solves the scalar riccati equation

$$p = 1 + p - \frac{(3/2)^2p^2}{1+(3/2)^2p} = 1 + p - \frac{9p^2}{4+9p}$$

$$\Leftrightarrow 9p^2 - 9p - 4 = 0$$

with solutions

$$p = \frac{1}{2} \overset{+}{(-)} \frac{5}{6} = \frac{4}{3}$$

so that

$$\hat{u}_k = -\frac{1}{2}x_k$$

(b) The dynamic programming recursion is given by

$$\begin{aligned} V_k(x_k) &= \min_{|u_k|\leq 1} \left\{x_k^2 + u_k^2 + p_{k+1}x_{k+1}^2\right\} \\ &= \min_{|u_k|\leq 1} \left\{x_k^2 + u_k^2 + p_{k+1}\left(x_k + \frac{3}{2}u_k\right)^2\right\} \end{aligned}$$

Without constraints, the optimal solution is given by

$$\hat{u}_k = -\frac{6p_k}{4+9p_k}x_k$$

where p_k satisfies the recursion

$$p_k = 1 + p_{k+1} - \frac{9p_{k+1}^2}{4+9p_{k+1}}$$

If the optimal feedback is attainable, i.e. within the constraints on u_k, then

$$\left|\frac{6p_k}{4+9p_k}x_k\right| \leq 1$$

so that

$$|x_k| \leq \frac{4+9p_k}{6p_k} = \frac{3}{2} + \frac{2}{3p_k}$$

When x_k is outside this region, the optimum is achieved at either constraint bound. Due to the quadratic expressions it is straightforward to see that

$$x_k + \frac{3}{2}u_k$$

governs the optimal choice of u_k. If $x_k > \frac{3}{2} + \frac{2}{3p_k}$ then the optimal decision will be to counteract x_k by choosing $u_k = -1$. Likewise, if $x_k < -\frac{3}{2} - \frac{2}{3p_k}$ then $u_k = 1$ is the best choice. Therefore, the optimal feedback law at stage k is given by

$$\hat{u}_k = \begin{cases} 1, \quad x_k < -\frac{3}{2} - \frac{2}{3p_k} \\ -\frac{6p_k}{4+9p_k} x_k, \quad -\frac{3}{2} - \frac{2}{3p_k} \le x_k \le \frac{3}{2} + \frac{2}{3p_k} \\ -1, \quad x_k > \frac{3}{2} + \frac{2}{3p_k} \end{cases}$$

where p_k satisfies

$$p_k = 1 + p_{k+1} - \frac{9p_{k+1}^2}{4 + 9p_{k+1}}$$

and $p_N = 1$. Moreover, when $N \to \infty$, p_k converges to the stationary solution of the riccati recursion. This solution was found in (a) as $p_k = \frac{4}{3}$. In conclusion, the optimal feedback law to the constrained LQR problem is

$$\hat{u}_k = \begin{cases} 1, \quad x_k < -2 \\ -\frac{1}{2} x_k, \quad -2 \le x_k \le 2 \\ -1, \quad x_k > 2 \end{cases}$$

(c) Three cases have to be covered:

i) $-2 \le x_0 \le 2$

ii) $x_0 < -2$

iii) $x_0 > 2$

i) First, if $-2 \le x_0 \le 2$, then the dynamics under the optimal feedback law from (b) are given by

$$x_{k+1} = x_k + \frac{3}{2}\left(-\frac{1}{2}x_k\right) = \frac{1}{4}x_k$$

which is a stable closed loop system (alternatively, argue that LQR feedback laws yield a stable system under the given conditions). Note, that the subset $\mathcal{X} = \{x_k \mid -2 \le x_k \le 2\}$ is invariant under the feedback law, so that $x_k \in \mathcal{X} \Rightarrow x_{k+1} \in \mathcal{X}$, and hence any starting point $x_0 \in \mathcal{X}$ will evolve to zero without ever leaving $\mathcal{X}$.

ii) The dynamics are now given by

$$x_{k+1} = x_k + \frac{3}{2}$$

so that x_k is increasing. Note, that $x_{k+1} - x_k = \frac{3}{2} < 4$, so that if $x_k < -2$ then $x_{k+1} < 2$. Hence, for any $x_0 < -2$ there is some k such that $-2 \le x_k \le 2$, which is the situation of case i). Thus, any $x_0 < -2$ will also evolve to zero.

iii) Likewise, if $x_0 > 2$ then

$$x_{k+1} = x_k - \frac{3}{2}$$

so that x_k is always decreasing and there is some k such that $-2 \leq x_k \leq 2$, which again yields a stable system. In other words, any $x_0 > 2$ will evolve to zero.

In conclusion, the optimal feedback law determined in (b) is stabilizing.

$\square$