Lecture 3: Finite-time control via optimization

Mikael Johansson
KTH - Royal Institute of Technology

Outline

- The finite-time optimal control problem
- Mathematical programming: convexity, LPs and QPs.
- A few quadratic programs with analytical solutions
- Application: energy-optimal state transfer

Finite-time optimal control

Given a dynamical system

$$x_{t+1} = f_t(x_t, u_t)$$

with initial state x_0, find input sequence $\{u_0, \dots, u_{N-1}\}$ which minimizes

$$\sum_{t=0}^{N-1} g_t(x_t, u_t) + g_N(x_N)$$

while satisfying state and control constraints $x_t \in X_t$, $u_t \in U_t$ for all t.

Here, N is the *horizon* of the planning problem.

Finite-time optimal control on standard form

Convenient to represent optimal control problems on standard form

$$\begin{array}{lll} \underset{\{u_0,\dots,u_{N-1}\}}{\text{minimize}} & \sum_{t=0}^{N-1} g_t(x_t, u_t) + g_N(x_N) & \\ \text{subject to} & x_{t+1} = f_t(x_t, u_t) & t = 0, \dots, N-1 \\ & x_t \in X_t & t = 0, \dots, N \\ & u_t \in U_t & t = 0, \dots, N-1 \end{array}$$

Note: optimal solution may be either

- an open-loop sequence $\{u_0, \dots, u_{N-1}\}$, or
- a feedback policy $u_t = \varphi_t(x_t)$ for some functions $\varphi_t : \mathbb{R}^n \mapsto \mathbb{R}^m$

Example: energy-optimal state transfer

Example. Find minimum-energy input which drives linear system

$$x_{t+1} = Ax_t + Bu_t$$

from $x_0 = 0$ to $x_N = x_{\text{tgt}}$.

Finite-time optimal control formulation:

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{N-1} u_t^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t \\ & x_0 = 0 \\ & x_N = x_{\text{tgt}} \end{array}$$

Last lecture: solution always exists if system is reachable and $N \geq n$.

Today: how to find *optimal* solutions via mathematical programming.

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Mathematical programming: standard form and notation

Standard form for constrained optimization problems

$$\begin{array}{lll} \underset{z \in \mathbb{R}^n}{\text{minimize}} & f_0(z) & \\ \text{subject to} & f_i(z) \leq 0, & i = 1, \dots, m \\ & g_i(z) = 0, & i = 1, \dots, p \end{array} \tag{1}$$

Notation:

- $z \in \mathbb{R}^n$ is the *decision vector*, representing the free variables
- $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$ is the *objective function*, representing the operating cost
- $f_i(z) \leq 0, \ i = 1, \dots, m$ and $g_i(z) = 0, \ i = 1, \dots, p$ are *constraints*

Furthermore,

- z is *feasible*, if it satisfies all constraints.
- the optimization problem is feasible, if it admits at least one feasible z
- $z^\star$ is *optimal*, if it attains the smallest value of f_0 among all feasible z
- $p^\star = f_0(z^\star)$ is the *optimal value* of the optimization problem

Hard and easy optimization problems

Sometimes convenient to write optimization problem as

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & z \in Z \end{array} \tag{2}$$

where we have introduced the *feasible set*

$$Z = \{z \,|\, f_i(z) \leq 0, \ i = 1, \dots, m \wedge g_i(z) = 0, \ i = 1, \dots, p\}.$$

Without further assumptions, (2) may be easy or *very difficult* to solve.

We focus on *convex optimization problems*, where f_0 and Z are convex

- powerful and useful theory, efficient numerical solvers

Unconstrained optimization: optimality conditions

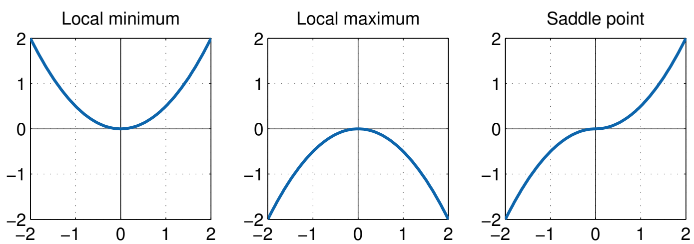
Consider the unconstrained minimization problem

$$\underset{z}{\text{minimize}} \quad f_0(z)$$

with $f_0 : \mathbb{R}^n \mapsto \mathbb{R}$. If f is differentiable, any minimizer $z^\star$ must satisfy

$$\nabla f_0(z^\star) = 0$$

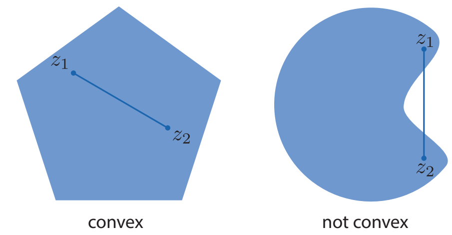
Condition not sufficient: $z^\star$ could be minimum, maximum or saddle point.



Can say more if f_0 is a convex function.

Convex sets

Definition. The set $Z \subseteq \mathbb{R}^n$ is *convex* if for any $z_1, z_2 \in Z$, and any $\theta \in [0, 1]$ we have $\theta z_1 + (1 - \theta) z_2 \in Z$.

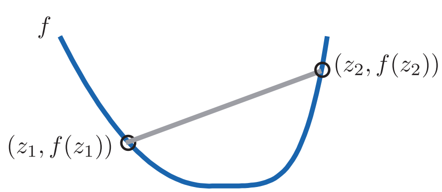


"line segment between any two points in Z also in Z"

Convex functions

Definition. A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is *convex* if its domain is a convex set and if for all $z_1, z_2 \in \operatorname{dom} f$ and $\theta \in [0, 1]$, we have

$$f(\theta z_1 + (1 - \theta) z_2) \leq \theta f(z_1) + (1 - \theta) f(z_2)$$



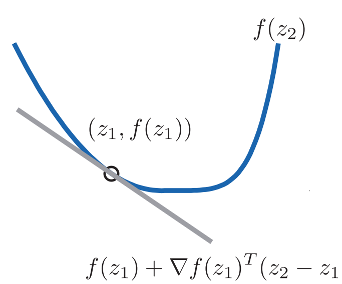
"line segment between $(z_1, f(z_1))$ and $(z_2, f(z_2))$ always above graph of f"

The role of convexity

A continuously differentiable function f is convex if and only if

$$f(z_2) \geq f(z_1) + \nabla f(z_1)^T (z_2 - z_1) \qquad \forall z_1, z_2 \in \operatorname{dom} f$$

"Every linearization is a global lower bound"



Consequences:

- first-order optimality conditions necessary and sufficient
- stationary points of convex functions are *global minima!*

Some convex functions

Claim. The following functions are convex:

(a) affine functions

$$f(z) = a^T z + b$$

(b) quadratic functions

$$f(x) = z^T P z$$

where P is positive semindefinite ($P \succeq 0$)

(c) The sum of two convex functions

$$f(z) = f_1(z) + f_2(z)$$

Consequence: $f(z) = z^T P z + 2q^T z + r$ is convex if (and only if) $P \succeq 0$.

Convex sets induced by constraints

Claim. The following sets are convex

(a) $Z = \{z \mid f(z) \leq 0\}$ where f is a convex function.

(b) $Z = \{z \mid g^T z = h\}$

(c) $Z = Z_1 \cap Z_2$ where Z_1 and Z_2 are convex.

Note. $Z = \{z \mid f(z) = 0\}$ is not necessarily convex, even if f is.

Convex optimization problems on standard form

Standard form for constrained optimization problems

$$\begin{array}{lll} \underset{z \in \mathbb{R}^n}{\text{minimize}} & f_0(z) & \\ \text{subject to} & f_i(z) \leq 0, & i = 1, \dots, m \\ & g_i^T z = h_i, & i = 1, \dots, p \end{array} \tag{3}$$

where $f_0, f_1, \dots, f_m$ are convex functions.

Note. equality constraints must be linear.

Example: linear program (LP)

Minimize a linear function subject to linear constraints:

$$\begin{array}{lll} \text{minimize} & c^T z & \\ \text{subject to} & a_i^T z \leq b_i, & i = 1, \dots, m \\ & g_i^T z = h_i, & i = 1, \dots, p \end{array}$$

- Strong theory with insightful geometrical interpretations.
- Very efficient solvers (100 millions of constraints, billions of variables)
- A mature technology

Example: quadratic program (QP)

Minimize a convex quadratic function subject to linear constraints

$$\begin{array}{lll} \text{minimize} & z^T P z + 2q^T z + r & \\ \text{subject to} & a_i^T z \le b_i, & i = 1, \dots, m \\ & g_i^T z = h_i, & i = 1, \dots, p \end{array}$$

with $P \succeq 0$.

Similarly to LP: strong and useful theory, efficient numerical solvers.

Note. Easy to solve numerically, but can only rarely find analytical solution.

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Completion of squares lemma

Lemma. All minimizers of the quadratic function

$$f(z) = z^T P z + 2q^T z + r$$

with $P \succeq 0$ satisfy the normal equations

$$Pz + q = 0.$$

If $P \succ 0$, then the minimizer is unique and given by

$$z^\star = -P^{-1} q$$

with corresponding minimal value

$$f^\star = r - q^T P^{-1} q = r - (z^\star)^T P z^\star.$$

Moreover, f can be written as a completion-of-squares

$$f(z) = (z - z^\star)^T P (z - z^\star) + r - (z^\star)^T P z^\star.$$

Least-norm solution to linear equations

Proposition. Let $z \in \mathbb{R}^n$, $d \in \mathbb{R}^m$ and $C \in \mathbb{R}^{m \times n}$ with $m < n$, and consider

$$\begin{array}{ll} \text{minimize} & z^T z \\ \text{subject to} & Cz = d \end{array}$$

If $\text{rank}(C) = m$, then the optimal solution is

$$z^\star = C^T (CC^T)^{-1} d.$$

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Energy-optimal state transfer

Find minimum-energy input which drives linear system

$$x_{t+1} = Ax_t + Bu_t$$

from $x_0 = 0$ to $x_N = x_{\text{tgt}}$.

Finite-time optimal control formulation:

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{N-1} u_t^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t \\ & x_0 = 0 \\ & x_N = x_{\text{tgt}} \end{array} \tag{4}$$

This is a quadratic program in $z = (u_0, \dots, u_{N-1}, x_0, \dots, x_N)$.

Energy-optimal state transfer: compact formulation

Can eliminate x from the decision vector.

For the constraints, the the prediction equations (and $x_0 = 0$) yields

$$x_N = \sum_{k=0}^{N-1} A^k B u_{N-1-k} := \mathcal{C}_N \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix} = \mathcal{C}_N U_N$$

For the objective, $\sum_{k=0}^{N-1} u_k^2 = U_N^T U_N$, so (4) is equivalent to

$$\begin{array}{ll} \text{minimize} & U_N^T U_N \\ \text{subject to} & \mathcal{C}_N U_N = x_{\text{tgt}} \end{array}$$

Least-norm state transfer

Least-energy state transfer can be found by solving the quadratic program

$$\begin{array}{ll} \text{minimize} & U_N^T U_N \\ \text{subject to} & \mathcal{C}_N U_N = x_{\text{tgt}} \end{array}$$

We have shown that the optimal solution is

$$U_N^\star = \mathcal{C}_N^T (\mathcal{C}_N \mathcal{C}_N^T)^{-1} x_{\text{tgt}}$$

with associated optimal value (minimum energy cost)

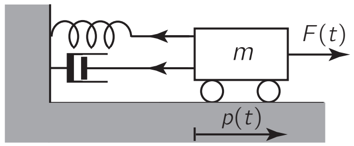
$$\mathcal{E}(x_{\text{tgt}}, N) = (U_N^\star)^T U_N^\star = x_{\text{tgt}}^T (\mathcal{C}_N \mathcal{C}_N^T)^{-1} x_{\text{tgt}} = x_{\text{tgt}}^T (\sum_{k=0}^{N-1} A^k B B^T (A^T)^k)^{-1} x_{\text{tgt}}$$

Note:

- $\mathcal{E}(x_{\text{tgt,N}})$ measures the energy is needed to reach x_{tgt} in N steps.
- $\{x_{\text{tgt}} \,|\, \mathcal{E}(x_{\text{tgt}}, N) \leq 1\}$ is an ellipsoid, whose size grows as N increases.

Least-norm state transfer

Example. Reachable sets with unit energy for mechanical system



Continuous-time model

$$m\ddot{p}(t) = F(t) - kp(t) - d\dot{p}(t)$$

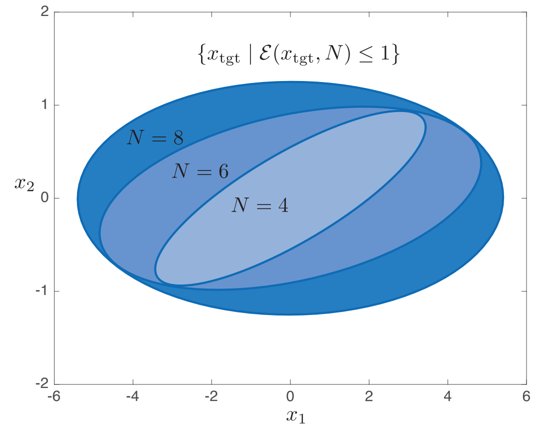
$m = 1$, $k = d = 0.1$, and $h = 1$ gives the discrete-time model

$$x(t+1) = \begin{bmatrix} 0.95 & 0.94 \\ -0.09 & 0.86 \end{bmatrix} x(t) + \begin{bmatrix} 0.48 \\ 0.94 \end{bmatrix} u(t)$$

(first state is position, second is velocity; control signal is applied force)

Least-norm state transfer

Reachable sets for different horizon length N



Correspond well with physical intuition.

Finite-time state transfer with bounded controls

Consider the minimum energy state transfer with the additional constraint

$$u_{\min} \leq u_t \leq u_{\max}, \qquad t = 0, 1, \ldots, N-1$$

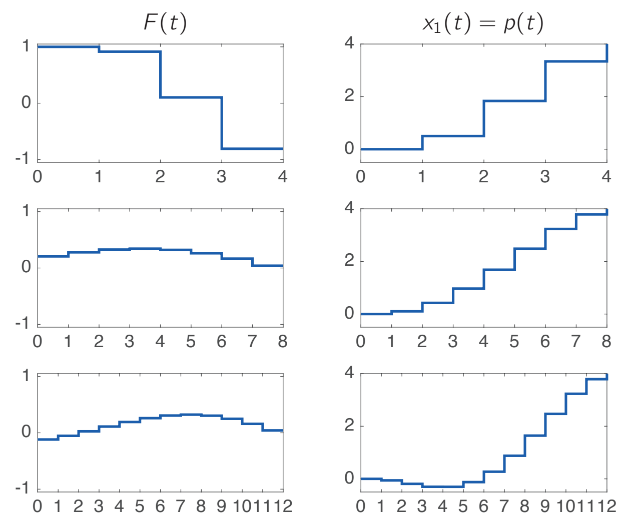
Optimal control solves the quadratic program

$$\begin{array}{ll} \text{minimize} & U_T^T U_T \\ \text{subject to} & \mathcal{C}_T U_T = x_{\text{tgt}} \\ & U_T \leq u_{\max}\mathbf{1} \\ & -U_T \leq -u_{\min}\mathbf{1} \end{array}$$

Not easy to find explicit solution, but can solve numerically.

Finite-time state transfer with bounded controls

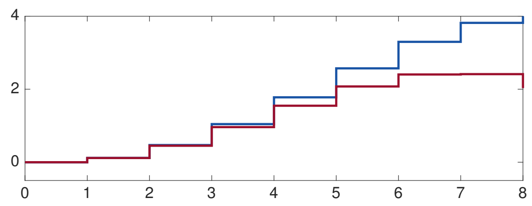
Optimal controls and trajectories for $x_{\text{tgt}} = (4, 0)$, $u_{\max} = 1$, $u_{\min} = -1$.



Warning: open-loop control is fragile

Open loop control is sensitive to modeling errors and disturbances.

Example. Open-loop optimal input on system with larger spring constant



Nominal response (blue) and actual (red). Target state no longer reached!

- similar problems when disturbances act on system
- need to introduce feedback to compensate for uncertainties

Summary and reading instructions

Summary:

- The finite-time optimal control problem
- Mathematical programming: convexity, LP and QP
- A few quadratic programs with analytical solutions
- Application: minimum energy state transfer

Reading instructions: lecture notes Chapter 3.1-3.2 + Appendices A and B.

Extra: energy-optimal state transfer as QP

With $z = (u_0, u_1, \dots, u_{N-1}, x_0, x_1, \dots, x_N)$, objective is

$$f_0(z) = z^T P z = z^T \begin{bmatrix} I_{N\times N} & 0 \\ 0 & 0_{(N+1)\times(N+1)} \end{bmatrix} z$$

The linear dynamics induces the constraints

$$\begin{bmatrix} B & 0 & \dots & 0 & A & -I & 0 & 0 & \dots & 0 \\ 0 & B & \dots & 0 & 0 & A & -I & 0 & \dots & 0 \\ \vdots & & & & & & & & & \\ 0 & 0 & \dots & B & 0 & \dots & 0 & \dots & A & -I \end{bmatrix} z = 0$$

while initial and target constraints read

$$\begin{bmatrix} 0_{n\times Nm} & I_{n\times n} & 0_{n\times Nn} \end{bmatrix} z = 0_{n\times 1}$$
$$\begin{bmatrix} 0_{n\times Nm} & 0_{n\times Nn} & I_{n\times n} \end{bmatrix} z = x_{\text{tgt}}$$

QP, since P is positive semidefinite (f_0 is convex) and constraints are linear.