REGLERTEKNIK

School of Electrical Engineering, KTH

EL2700 Model predictive control

Exam (tentamen) 2016–10–22, kl 09.00–14.00

Aids: The course notes for EL2700 and books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Each step in your solutions must be justified.
Lacking motivation will results in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Only write on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

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Resultat: Will be posted no later than November 12, 2016.

Good Luck!

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1. Consider the discrete-time linear time-invariant system defined by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k). \end{aligned} \tag{1}$$

The system is controlled by a predictive controller that minimizes the predicted cost

$$J(k) = \sum_{i=0}^{N-1} \left[x(k+i|k)^T Q_1 x(k+i|k) + u(k+i|k)^T Q_2 u(k+i|k) \right] \tag{2}$$

at each time-step k. Here, $u(k+i|k)$ and $x(k+i|k)$ denote predicted input and state vectors at time $k+i$ based on the information available at time k.

(a) Explain the receding horizon principle used in MPC. (2p)

(b) For a linear system without constraints, we can consider the infinite-horizon cost (letting $N \to \infty$ in the formulation above) and compute the optimal feedback law as a linear state feedback $u(k) = Lx(k)$. In the MPC framework, on the other hand, we always minimize a finite-horizon cost function. Why? (2p)

(c) Is there any way that we can modify the criterion (2) so that the receding-horizon control for the unconstrained MPC always agrees with the infinite-horizon optimal control? Please, write down the new cost function. (3p)

(d) Show how the optimization problem of minimizing (2) subject to the linear dynamics (1) can be written as a quadratic optimization problem on the form

$$\underset{U}{\text{minimize}} \quad U^T P U + q^T U,$$

where $U = [u(k|k), ..., u(k+N-1|k)]^T$. Express P and q in terms of A, B, Q_1, and Q_2. (3p)

2. In this problem, we will perform a simple pole-placement design of an observer-based controller for a system described by the continuous-time dynamics

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

   (a) Sample the system with sampling time h to obtain an equivalent discrete-time state-space model (equivalent here means that the discrete-time model should exactly predict the state of the continuous system at sampling intervals, provided that the input is held constant between samples).

(4p)

   (b) You decide to sample your system with $h = 2$, which gives the discrete-time model

$$x(t+1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

   Verify that the system is reachable, and determine a discrete-time state feedback control law

$$u(t) = -Lx(t)$$

   that places the closed-loop poles in $z = 1/2$. (3p)

   (c) Your controller cannot measure the full state vector, but only the output $y(t)$. You will therefore have to construct an observer which estimates the full state vector based on knowledge of $u(t)$ and $y(t)$. Compute the observer gains which place the poles of the observer error dynamics in $z = 1/4$. (3p)

3. Consider the following sequential decision-making problem

$$\begin{array}{lll} \underset{\{u_k\}}{\text{maximize}} & x_N + \sum_{k=0}^{N-1}(1-u_k)x_k & \\ \text{subject to} & x_{k+1} = x_k + 0.5x_k u_k & k = 0, 1, \ldots, N-1 \\ & 0 \leq u_k \leq 1 & k = 0, 1, \ldots, N-1 \\ & x_0 \geq 0 & \end{array}$$

(a) Show that $x_k \geq 0$ for all $k = 0, 1, \ldots, N$. (1p)

(b) Write down the dynamic programming recursion for this problem, i.e. write the Bellman equation which characterizes $V_t(x)$ in terms of $V_{t+1}(x)$. Make sure to specify how to initialize the recursion (how to select V_N). (3p)

(c) Solve the allocation problem using dynamic programming for $N = 3$. Determine the optimal controls $u_0^\star$, $u_1^\star$, $u_2^\star$ and the optimal objective (as a function of x_0) (6p)

Remark. If you would like a motivation for this decision problem, it comes from investment planning where one have to decide if asset returns should be cashed in or re-invested to obtain higher returns in the future.

In this setting, x_k is return from your investment at time k, and u_k is the percentage of your returns in period k that you re-invest. If you do not re-invest ($u_k = 0$) the return on your investment will stay be the same in the next period (since you did not do any further investments), but you will be able to cash in x_k (increasing your objective). If you decide to re-invest your full return x_k, on the other hand, you will not get any cash in that period (so the objective function stays the same) but your reinvestment will earn interest which allow you to obtain higher returns in the future. The control u_k allow you to take any action in between these two extremes.

4. We are interested in designing an MPC controller for the system

$$x(t+1) = 3x(t) + u(t),$$

where the input and the state are constrained according to

$$\begin{aligned} -20 \leq & u(t) \leq 25, \\ & |x(t)| \leq 8. \end{aligned}$$

(a) Let us first disregard the input and state constraints and formulate the standard (unconstrained) LQR problem. The cost function we wish to minimize is

$$\sum_{k=0}^{\infty} 17x_k^2 + 2u_k^2.$$

Compute the optimal feedback control law $u(k) = -Lx(k)$ for this problem.

(2p)

(b) Second, we convert the LQR problem to an equivalent *unconstrained* MPC problem with control hoizon N_u and prediction horizon N. Write down the complete formulation of the equivalent *unconstrained* MPC (in the form of "minimize ..., subject to ..."). Justify why the two formulation are equivalent. Finally, determine the control law $u(t) = -Kx(t)$ for the MPC mode 2 (i.e, the control law to use from the end of the control horizon until the end of the prediction horizon) such that the predicted trajectories in the LQR and this *unconstrained* MPC are equivalent.

(4p)

(c) We are now ready to introduce the input and state constraints in our MPC formulation. Compute the final state constraints necessary to guarantee the stability of the MPC formulation. Assume that the MPC mode 2 responds to the control law $u(t) = -Kx(t)$, where K has been defined in (b). Write down the resulting MPC formulation.

Note. To get full point on (c), the terminal set should be as large as possible.

(4p)

5. In class, we have learned how to compute the optimal control law for the criterion

$$J_{N,t}(x_t) = \sum_{k=t}^{N} x_k^T Q_1 x_k + u_k^T Q_2 u_k$$

when the underlying system dynamics is linear and on the form

$$x_{k+1} = Ax_k + Bu_k$$

When N tends to infinity, the optimal control law is a linear state-feedback.

In this problem, we are interested in estimating the quadratic cost for an arbitrary *given* (not necessarily optimal) linear control law

$$u_k = -Lx_k$$

Of course, if N is small and we are only interested in one particular initial value x_0, we could simply simulate the control law and evaluate the cost. But as N grows large, this is increasingly inefficient. We will therefore derive an alternative approach.

(a) Use (backward) induction to show that $J_{N,0}(x_0)$ is a quadratic function of the initial state, *i.e.* $J_{N,0}(x_0) = x_0^T V_{N,0} x_0$ for some matrix $V_{N,0}$. Derive a recursion for how $V_{N,k}$ depends on $V_{N,k+1}$. Make sure to specify how the recursion should be initialized (how $V_{N,N}$ should be selected). (6p)

(b) Assume that the iteration converges as $N \to \infty$, *i.e.* $V_{N,k} = V_{N,k+1} = V_\infty$ for all finite values of k. Derive a matrix equation that determines V_∞ in terms of A, B and L. Explain how you can solve the equation that you derived using a numerical routine for solving Lyapunov equations (i.e. a numerical routine for computing P such that $\Phi^T P \Phi - P + Q = 0$ for given Φ and Q). (4p)