REGLERTEKNIK

School of Electrical Engineering and Computer Science, KTH

EL2700 Model predictive control

Exam (tentamen) 2018–10–20, kl 09.00–14.00

Aids: The course notes and slides for EL2700; books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

Observe: Do not treat more than one problem on each page.
Justify every step of your solutions.
Lacking justification will result in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Write only on one side of each sheet.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

Grading: Grade A: ≥ 43, Grade B: ≥ 38
Grade C: ≥ 33, Grade D: ≥ 28
Grade E: ≥ 23, Grade Fx: ≥ 21

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Results: Will be posted no later than November 9, 2018.

Good Luck!

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1. (a) If $u_t = Kx_t$, then all future states and inputs after x_N are dependent on only x_N

$$x_{N+i} = (A+BK)^i x_N$$
$$u_{N+i} = K(A+BK)^i x_N, \qquad \text{for } i = 0, 1 \ldots$$

Therefore, the infinite cost will also be dependent only on x_N.

In fact, we can be even more precise than this, since we have shown that the infinite-horizon cost for a linear state feedback is a quadratic form $x_N^T P x_N$ where P is computed from a specific Lyapunov equation (see Lecture notes).

(b) A terminal constraint set $\mathcal{X}_f$ must be:

- Control invariant
- Recursively feasible

(c) Yes, $\{\bar{0}\}$ id control invariant, because when $x = \bar{0}$,

$$x_{k+1} = Bu_k = \bar{0} \quad \text{if } u = \bar{0}.$$

The set is feasible because $\mathcal{X}$ and $\mathcal{U}$ both contain the origin.

(d) If $\mathcal{X}_f = \{\bar{0}\}$, then the MPC must reach the origin in N steps. This might be a very conservative constraint which gives a small feasible set.

(e) As N increases, the set of initial states x_0 from which the terminal set $\mathcal{X}_N$ can be reached becomes larger. Thus, the feasible set becomes larger.

2. (a) We start by determining

$$
\begin{aligned}
&A = \exp(A_c h) = \\
&I + A_c h + A_c^2 \frac{h^2}{2} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

We can then proceed to compute

$$
\begin{aligned}
B &= \int_{s=0}^{h} \exp(As) B \, ds = \\
&= \int_{s=0}^{h} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} ds = \int_{s=0}^{h} \begin{bmatrix} \gamma + s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \gamma h + h^2/2 \\ h \end{bmatrix}
\end{aligned}
$$

Using $h = 1$, we find the model given by

$$
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} \gamma + 1/2 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
$$

(b) The controllability matrix is given by

$$
W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \gamma + 1 & \gamma + 2 \\ 1 & 1 \end{bmatrix}
$$

Its deterinant is $\det(W_c) = \gamma + 1 - (\gamma + 2) = -1 \neq 0$. Hence, the system is reachable for all values of γ.

(c) Inserting $h = 1$ and $\gamma = 1/2$ in the expressions derived in (a) validates the suggested expressions for A, B and C. 6

(d) The feedback law $u_t = -Lx_t$ gives the closed loop

$$
x_{t+1} = (A - BL)x_t
$$

where

$$
A - BL = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 1 - l_1 & 1 - l_2 \\ -l_1 & 1 - l_2 \end{bmatrix}
$$

whose characteristic polynomial is

$$
\begin{aligned}
p(\lambda) &= \det(\lambda I - (A - BL)) = \det \begin{bmatrix} \lambda - 1 + l_1 & l_2 - 1 \\ l_1 & \lambda + l_2 - 1 \end{bmatrix} \\
&= (\lambda - 1 + l_1)(\lambda + l_2 - 1) - (l_2 - 1)l_1 = \\
&= \lambda^2 + \lambda(l_1 + l_2 - 2) + (l_2 - 1)(l_1 - 1) - l_1(l_2 - 1) = \\
&= \lambda^2 + \lambda(l_1 + l_2 - 2) + 1 - l_2
\end{aligned}
$$

The desired characteristic polynomial is $(\lambda - 0.5)^2 = \lambda^2 - \lambda + 0.25$. Hence

$$\begin{aligned} 1 - l_2 &= 0.25 \\ l_1 + l_2 &= 1 \end{aligned}$$

with solution

$$l_1 = 0.25, \quad l_2 = 0.75$$

(d) The feed-forward gain should satisfy

$$C(I - (A - BL))^{-1} B l_r = 1$$

We have

$$I - (A - BL) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.25 & -0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

with inverse

$$(I - (A - BL))^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

Hence, we should have

$$C(I - (A - BL))^{-1} B l_r = 4 l_r = 1 \Rightarrow l_r = \frac{1}{4}$$

3. (a) The first step of the dynamic programming recursion is given by

$$\begin{aligned}
V_1(x_1) &= \max_{u_1\in\{-1,0,1\}} \{0.4x_1 - 2(x_1+u_1)\} \\
&= \max_{u_1\in\{-1,0,1\}} \{-1.6x_1 - 2u_2\} \\
&= -1.6x_1 + 2
\end{aligned}$$

with $\hat{u}_1 = -1$. The next step, which is the last step, is given by

$$\begin{aligned}
V_0(x_0) &= \max_{u_0\in\{-1,0,1\}} \{0.4x_0 + V_1(x_0+u_0)\} \\
&= \max_{u_0\in\{-1,0,1\}} \{0.4x_0 - 1.6(x_0+u_0) + 2\} \\
&= \max_{u_0\in\{-1,0,1\}} \{-1.2x_0 - 1.6u_0 + 2\} \\
&= -1.2x_0 + 3.6
\end{aligned}$$

with $\hat{u}_0 - 1$. In brief, the optimal strategy is $\hat{u}_0 = -1$, $\hat{u}_1 = -1$ with profit

$$3 + V_0(3) = 3$$

i.e. sell of as much as possible each year.

(b) The first step of the dynamic programming recursion is given by

$$\begin{aligned}
V_2(x_2) &= \max_{u_2\in\{-1,0,1\},x_3\geq 0} \{0.8x_2 - 2(x_2+u_2)\} \\
&= \max_{u_2\in\{-1,0,1\},x_3\geq 0} \{-1.2x_2 - 2u_2\} \\
&= \begin{cases} -1.2x_2 + 2, & x_2 \geq 1 \\ 0, & x_2 = 0 \end{cases}
\end{aligned}$$

with

$$\hat{u}_2 = \begin{cases} -1, & x_2 \geq 1 \\ 0, & x_2 = 0 \end{cases}$$

The next step is given by

$$\begin{aligned}
V_1(x_1) &= \max_{u_1\in\{-1,0,1\},x_2\geq 0} \{0.8x_1 + V_2(x_1+u_1)\} \\
&= \max_{u_1\in\{-1,0,1\},x_2\geq 0} \left\{ 0.8x_1 + \begin{cases} -1.2(x_1+u_1) + 2, & x_1+u_1 \geq 1 \\ 0, & x_1+u_1 = 0 \end{cases} \right\}
\end{aligned}$$

Assume first that $x_1 \geq 2$. It follows that $x_1 + u_1 \geq 1$ and

$$\begin{aligned}
V_1(x_1) &= \max_{u_1\in\{-1,0,1\},x_2\geq 0} \{0.8x_1 - 1.2(x_1+u_1) + 2\} \\
&= \max_{u_1\in\{-1,0,1\},x_2\geq 0} \{-0.4x_1 - 1.2u_1 + 2\} \\
&= -0.4x_1 + 3.2
\end{aligned}$$

with $\hat{u}_1 = -1$. The control is necessarily admissible since $x_1 + u_1 \geq 1 > 0$ was assumed. Next, assume instead that $x_1 = 1$. It follows that

$$\begin{aligned} V_1(x_1) &= \max_{u_1 \in \{-1,0,1\}, x_2 \geq 0} \{0.8 + V_2(1 + u_1)\} \\ &= \max_{u_1 \in \{-1,0,1\}, x_2 \geq 0} \left\{ 0.8 + \begin{cases} -0.4, & u_1 = 1 \\ 0.8, & u_1 = 0 \\ 0, & u_1 = -1 \end{cases} \right\} \\ &= 1.6 \end{aligned}$$

with

$$\hat{u}_1 = 0$$

Finally, assume that $x_1 = 0$. It follows that

$$\begin{aligned} V_1(x_1) &= \max_{u_1 \in \{-1,0,1\}, x_2 \geq 0} \{V_2(u_1)\} \\ &= \max_{u_1 \in \{-1,0,1\}, x_2 \geq 0} \left\{ \begin{cases} -0.4, & u_1 = 1 \\ 0, & u_1 = 0 \end{cases} \right\} \\ &= 0 \end{aligned}$$

with

$$\hat{u}_1 = 0$$

Therefore, it can be concluded that

$$V_1(x_1) = \begin{cases} -0.4x_1 + 3.2, & x_1 \geq 2 \\ 1.6, & x_1 = 1 \\ 0, & x_1 = 0 \end{cases}$$

with

$$\hat{u}_1 = \begin{cases} -1, & x_1 \geq 2 \\ 0, & x_1 < 2 \end{cases}$$

The final step is given by

$$\begin{aligned} V_0(x_0) &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{0.8x_0 + V_1(x_0 + u_0)\} \\ &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \left\{ 0.8x_0 + \begin{cases} -0.4(x_0 + u_0) + 3.2, & x_0 + u_0 \geq 2 \\ 1.6, & x_0 + u_0 = 1 \\ 0, & x_0 + u_0 = 0 \end{cases} \right\} \end{aligned}$$

Assume first that $x_0 \geq 3$. It follows that $x_0 + u_0 \geq 2$ and

$$\begin{aligned}
V_0(x_0) &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{0.8x_0 - 0.4(x_0 + u_0) + 3.2\} \\
&= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{0.4x_0 - 0.4u_0 + 3.2\} \\
&= 0.4x_0 + 3.6
\end{aligned}$$

with $\hat{u_0} = -1$. The control is necessarily admissible since $x_0 + u_0 \geq 2 > 0$ was assumed. Next, assume instead that $x_0 = 2$. It follows that

$$\begin{aligned}
V_0(x_0) &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{1.6 + V_1(2 + u_0)\} \\
&= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \left\{ 1.6 + \begin{cases} 2, & u_0 = 1 \\ 2.4, & u_0 = 0 \\ 1.6, & u_0 = -1 \end{cases} \right\} \\
&= 4
\end{aligned}$$

with

$$\hat{u}_0 = 0$$

Next, assume that $x_0 = 1$. It follows that

$$\begin{aligned}
V_0(x_0) &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{0.8 + V_1(1 + u_0)\} \\
&= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \left\{ 0.8 + \begin{cases} 2.4, & u_0 = 1 \\ 1.6, & u_0 = 0 \\ 0, & u_0 = -1 \end{cases} \right\} \\
&= 3.2
\end{aligned}$$

with

$$\hat{u}_0 = 1$$

Finally, assume that $x_0 = 0$. It follows that

$$\begin{aligned}
V_0(x_0) &= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \{V_1(u_0)\} \\
&= \max_{u_0 \in \{-1,0,1\}, x_1 \geq 0} \left\{ \begin{cases} 1.6, & u_0 = 1 \\ 0, & u_0 = 0 \end{cases} \right\} \\
&= 1.6
\end{aligned}$$

with

$$\hat{u}_0 = 1$$

Therefore, it can be concluded that

$$V_0(x_0) = \begin{cases} 0.4x_0 + 3.6, & x_0 \geq 3 \\ 4, & x_0 = 2 \\ 3.2, & x_0 = 1 \\ 1.6, & x_0 = 0 \end{cases}$$

with

$$\hat{u}_0 = \begin{cases} -1, & x_0 \geq 3 \\ 0, & x_0 = 2 \\ 1, & x_0 < 2 \end{cases}$$

In brief, the optimal strategy is $\hat{u}_0 = 1$, $\hat{u}_1 = -1$, $\hat{u}_2 = -1$ with profit

$$1 + V_0(1) = 4.2$$

4. (a) The optimal feedback is given by

$$L = (Q_2 + B^T P B)^{-1} B^T P A = \frac{1}{633 + 43.05} \begin{bmatrix} 26 & 416.05 \end{bmatrix} = \begin{bmatrix} 0.0385 & 0.6154 \end{bmatrix}$$

this means that the closed loop system will be given by

$$A - BL = \begin{bmatrix} 0.99615 & 1.9385 \\ -0.00385 & 0.8385 \end{bmatrix}$$

In practice, this controller will break the state constraints for many initial values, and additional methods would be needed to ensure constraint satisfaction by limiting the input.

(b) From part (a), the finite-horizon cost function is equal to the infinite-horizon cost function and the control input is $u_k = -Lx_k$ for all $k \geq N$.

This implies closed loop stability if (A, B) is reachable and $(A, Q^{1/2})$ is observable, where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{35} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{35} \end{bmatrix} \implies Q^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{35} \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.09 \end{bmatrix}, \qquad \mathcal{O} = \begin{bmatrix} Q^{1/2} \\ Q^{1/2} A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{35} \\ 1 & 2 \\ 0 & 0.9\sqrt{35} \end{bmatrix}$$

(c) At touchdown

$$\gamma \geq \gamma^{td} = -\frac{1}{20} = -0.05$$

(d) There are two lower constraints on γ

$\gamma \geq \gamma^{td} = -0.05$ at touchdown ($h_t = 0$)

$\gamma \geq \gamma^{min} = -0.25$ at all times

The constraint can be written as

$$\gamma(t) \geq \frac{\gamma_{min} - \gamma^{td}}{h^{final}} h(t) + \gamma^{td} = -0.02h(t) - 0.05$$

The complete state constraint at time t can be written as

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -0.02 & -1 \end{bmatrix} \begin{bmatrix} h(t) \\ \gamma(t) \end{bmatrix} \leq \begin{bmatrix} 100 \\ 0 \\ 0.25 \\ 0.25 \\ 0.05 \end{bmatrix}$$

This is a polyhedron as is illustrated in Figure 1

(e) This is not an invariant set. The closed loop dynamics is with the given feedback

$$A - BL = \begin{bmatrix} 0.996 & 2.06 \\ -0.004 & 0.96 \end{bmatrix}$$

Now, consider e.g. the following vertex of the state constraint set

$$\begin{bmatrix} 0.996 & 2.06 \\ -0.004 & 0.96 \end{bmatrix} \begin{bmatrix} 100 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 99.085 \\ -0.64 \end{bmatrix}$$

which means that the constraint is broken, and the set is not invariant. To find the largest possible invariant set, one would have to iterative methods.

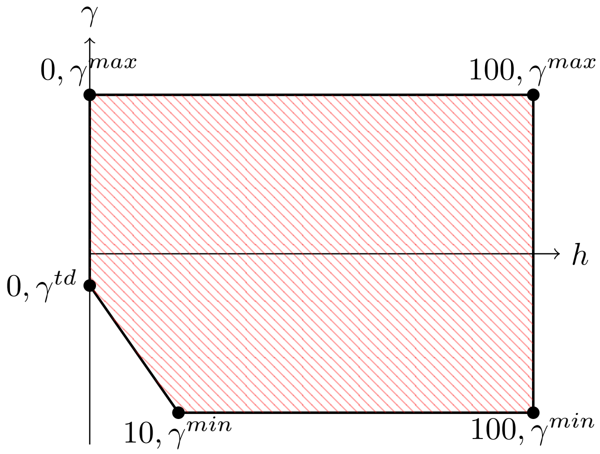

Figure 1: A polyhedron representing the state constraints in question (4)

5. (a) Let x_t be the state that corresponds to an input u_t and an initial condition x_0. The state at time t x_t has a linear dependence on $\left[x_0, u_0, u_1, \ldots, u_{t-1}\right]^T$.

$$x_t = A^t x_0 + \sum_{i=1}^{t} A^{i-1} B u_{t-i}.$$

If we now scale the input u_t and initial condition x_0 with an constant λ, then the state at time t will be scaled accordingly

$$A^t \lambda x_0 + \sum_{i=1}^{t} A^{i-1} B(\lambda u_{t-i}) = \lambda x_t.$$

This leads to

$$J(\lambda u, \lambda x_0) = \sum_{t=0}^{\infty} (\lambda^2 x_t^T Q_1 x_t + \lambda^2 u_t^T Q_2 u_t) = \lambda^2 J(u, x_0).$$

(b) Suppose

$$\begin{aligned} x_{t+1} &= A x_t + B u_t \\ \tilde{x}_{t+1} &= A \tilde{x}_t + B \tilde{u}_t \end{aligned}$$

Adding or subtracting the above equations yields

$$x_{t+1} \pm \tilde{x}_{t+1} = A(x_t \pm \tilde{x}_t) + B(u_t \pm \tilde{u}_t)$$

Therefore, $x_t \pm \tilde{x}_t$ is the state that corresponds to an input $u_t \pm \tilde{u}_t$ and initial condition $x_t \pm \tilde{x}_t$ respectively. Now

$$\begin{aligned} &J(u+\tilde{u}, x_0 + \tilde{x}_0) + J(u - \tilde{u}, x_0 - \tilde{x}_0) \\ &= \sum_{i=0}^{\infty} ((x_t + \tilde{x}_t)^T Q_1 (x_t + \tilde{x}_t) + (x_t - \tilde{x}_t)^T Q_1 (x_t - \tilde{x}_t) \\ &\quad + (u_t + \tilde{u}_t)^T Q_2 (u_t + \tilde{u}_t) + (u_t - \tilde{u}_t)^T Q_2 (u_t - \tilde{u}_t)) \\ &= \sum_{t=0}^{\infty} 2x_t^T Q_1 x_t + 2\tilde{x}_t^T Q_1 \tilde{x}_t + 2u_t^T Q_2 u_t + 2\tilde{u}_t^T Q_2 \tilde{u}_t \\ &= 2J(u, x_0) + 2J(\tilde{u}, \tilde{x}_0) \end{aligned}$$

By minimizing both sides with respect to u and $\tilde{u}$ we obtain

$$\min_{u,\tilde{u}} \{J(u + \tilde{u}, x_0 + \tilde{x}_0) + J(u - \tilde{u}, x_0 - \tilde{x}_0)\} = \min_{u} 2J(u, x_0) + \min_{\tilde{u}} 2J(\tilde{u}, \tilde{x}_0)$$

The right-hand side of this equation is obviously $2V(x_0) + 2V(\tilde{x_0})$. Examining the left-hand side,

$$\min_{u,\tilde{u}}\{J(u+\tilde{u}, x_0+\tilde{x}_0)+J(u-\tilde{u}, x_0-\tilde{x}_0)\} \geq \min_{u,\tilde{u}} J(u+\tilde{u}, x_0+\tilde{x}_0)+\min_{u,\tilde{u}} J(u-\tilde{u}, x_0-\tilde{x}_0)$$

Leads to

$$V(x_0 + \tilde{x_0}) + V(x_0 - \tilde{x_0}) \leq 2V(x_0) + 2V(\tilde{x_0})$$

(c) Performing this substitution yields

$$V(x_0) + V(\tilde{x}_0) \leq 2V(\frac{x_0 + \tilde{x}_0}{2}) + 2V(\frac{x_0 - \tilde{x}_0}{2}) = \frac{1}{2}V(x_0 + \tilde{x}_0) + \frac{1}{2}V(x_0 - \tilde{x}_0)$$

where the result from (a) was used in the last step. The conclusion must be that

$$2V(x_0) + 2V(\tilde{x}_0) = V(x_0 + \tilde{x}_0) + V(x_0 - \tilde{x}_0)$$

(d) Taking derivatives w.r.t. the inner variables result in

$$\begin{aligned}
\nabla_{x_0} V(x_0 + \tilde{x}_0) &= \nabla V(x_0 + \tilde{x}_0) \\
\nabla_{x_0} V(x_0 - \tilde{x}_0) &= \nabla V(x_0 - \tilde{x}_0) \\
\nabla_{\tilde{x}_0} V(x_0 + \tilde{x}_0) &= \nabla V(x_0 + \tilde{x}_0) \\
\nabla_{\tilde{x}_0} V(x_0 - \tilde{x}_0) &= -\nabla V(x_0 - \tilde{x}_0)
\end{aligned}$$

Adding the derivatives as suggested in the hint now results in

$$\begin{aligned}
\nabla_{x_0} V(x_0 + \tilde{x}_0) + \nabla_{x_0} V(x_0 - \tilde{x}_0) + \nabla_{\tilde{x}_0} V(x_0 + \tilde{x}_0) + \nabla_{\tilde{x}_0} V(x_0 - \tilde{x}_0) \\
= 2\nabla V(x_0 + \tilde{x}_0)
\end{aligned}$$

(e) Taking gradients of both sides of equation (**??**), we obtain

$$\lambda \nabla V(\lambda x_0) = \lambda^2 \nabla V(x_0) \implies \nabla V(\lambda x_0) = \lambda \nabla V(x_0)$$

therefore, $\nabla V(x_0)$ is linear in x_0, and as such $\exists M \in \mathbb{R}^{n \times n}$ such that $\nabla V(x_0) = Mx_0$

(f) Substituting λ in (a), we get

$$V(0) = V(0 \cdot x_0) = 0V(x_0) = 0$$

And thus

$$V(x_0) = \int_0^1 \nabla V(\theta x_0)^T x_0 \, d(\theta) = \int \nabla V(x_0)^T x_0 \theta \, d\theta = \frac{1}{2} x_0^T M^T x_0$$

Therefore we can write $V(x_0)$ as

$$V(x_0) = \frac{x_0^T M^T x_0}{2} = \frac{x_0^T M x_0}{2} = \frac{1}{4}\left(x_0^T M x_0 + x_0^T M^T x_0\right) = \frac{1}{4} x_0^T \left(M + M^T\right) x_0$$

Therefore P must be symmetric. Also,

$$\min_u J(u, x_0) \geq 0 \implies V(x_0) = x_0^T P x_0 \geq 0 \implies P \succ 0$$

means that P is also positive definite. This concludes our proof.