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EL2700 Model predictive control

Exam solution
2016–10–22, kl 9.00–14.00

1. (a) The receding horizon principal involves finding an open-loop control sequence that minimizes a certain cost function over a finite horizon. This procedure is performed each time new measurements are available. The first step in the computed control sequence is used as the control signal. Although the computed control sequences are open-loop sequences, the calculation of a new sequence at each sample can be thought of as providing feedback.

   (b) One of the main strengths of MPC is the possibility of including input and state constraints. Therefore, we need to solve an optimization problem over a finite number of free variables.

   (c) We know that the infinite cost is quadratic and has the form $x^T P x$, where P is positive semi-definite and is the solution to the algebraic Riccati equation. Therefore, in the equivalent finite horizon optimization problem, we can include a terminal state penalty weighted by P. The new cost function is then

$$J(k) = \sum_{i=0}^{N-1} \left[x(k+i|k)^T Q_1 x(k+i|k) + u^T Q_2 u(k+i|k) \right] + x_N^T P x_N$$

   (d) We start by defining the matrices $\bar{Q}_1 = \text{diag}(Q_1, Q_1, ..., Q_f)$ and $\bar{Q}_2 = \text{diag}(Q_2, Q_2, ..., Q_2)$. Then, we can rewrite the cost function in matricial form as

$$X^T \bar{Q}_1 X + U^T \bar{Q}_2 U,$$

   where $X = [x(1), ..., x(N)]^T$ By replacing X by $\bar{A}x_0 + \bar{B}U$ we get

$$(\bar{A}x_0 + \bar{B}U)^T \bar{Q}_1 (\bar{A}x_0 + \bar{B}U) + U^T \bar{Q}_2 U,$$

   which gives

$$x_0^T \bar{A}^T \bar{Q}_1 \bar{A} x_0 + 2x_0^T \bar{A}^T \bar{Q}_1 \bar{B} U + U^T \bar{B}^T \bar{Q}_1 \bar{B} U + U^T \bar{Q}_2 U,$$

   which yields

$$U^T P U + q^T U,$$

   where $P = \bar{B}^T \bar{Q}_1 \bar{B} + \bar{Q}_2$ and $q^T = 2x_0^T \bar{A}^T \bar{Q}_1 \bar{B}$.

2. (a) The discrete state matrix A can be computed as $A = e^{A_c h}$. In order to compute the matrix exponential we use the following formula that can be obtained using the Taylor theorem,

$$e^{Mh} = I + Mh + \frac{Mh^2}{2!}h^2 + \frac{Mh^3}{3!}h^3 + \ldots.$$

We can now compute A as

$$A = e^{A_c h} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} h^2 = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}.$$

The discrete input matrix B can be computed as

$$B = \int_0^h e^{A_c s} B ds = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix}$$

(b) The controllability matrix is given by

$$W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 2 & 2 \end{bmatrix}$$

which has full rank. Hence, the system is reachable.

The closed-loop matrix is $A - BL$, which evaluates to

$$\begin{bmatrix} 1 - 2l_1 & 2 - 2l_2 \\ -2l_1 & 1 - 2l_2 \end{bmatrix}$$

Its characteristic polynomial is

$$\begin{aligned} \det(zI - (A - BL)) &= (z - 1 + 2l_1)(z - 1 + 2l_2) + 4l_1 - 4l_1 l_2 = \\ &= z^2 + z(-2 + 2l_1 + 2l_2) + (1 - 2l_1)(1 - 2l_2) + 4l_1 - 4l_1 l_2 = \\ &= z^2 + z(-2 + 2l_1 + 2l_2) + 1 - 2l_1 - 2l_2 + 4l_1 \end{aligned}$$

The desired characteristic polynomial is

$$p(z) = (z - \frac{1}{2})^2 = z^2 - z + \frac{1}{4}$$

which implies that

$$\begin{aligned} -2 + 2l_1 + 2l_2 &= -1 \\ 1 + 2l_1 - 2l_2 &= 1/4 \end{aligned}$$

Adding the two equations yields $4l_1 = 1/4$, *i.e.* $l_1 = 1/16$; The first equation then gives that $l_2 = 7/16$.

(c) The error dynamics is governed by the matrix $A - KC$

$$A - KC = \begin{bmatrix} 1 - k_1 & 2 \\ -k_2 & 1 \end{bmatrix}$$

whose characteristic polynomial is

$$\det(zI - (A - KC)) = (z - 1 + k_1)(z - 1) + 2k_2 = z^2 + z(k_1 - 2) + 1 - k_1 + 2k_2$$

The desired characteristic polynomial is

$$p(z) = \left(z - \frac{1}{4}\right)^2 = z^2 - \frac{1}{2}z + \frac{1}{16}$$

which implies that we must have

$$\begin{aligned} k_1 - 2 &= -\frac{1}{2} \\ 1 - k_1 + 2k_2 &= \frac{1}{16} \end{aligned}$$

In other words,

$$k_1 = \frac{3}{2}, \qquad k_2 = \frac{9}{32}$$

3. (a) Lets consider the first state update where $x_1 = x_0 + 0.5x_0u_0$. Since $x_0 \geq 0$ and $0 \leq u_0 \leq 1$ then $x_1 \geq 0$. Since the system is time-invariant,

$$x_k = x_k + 0.5x_ku_k \geq 0 \text{ for all } k.$$

(b) The dynamic programming recursion is given by

$$\begin{aligned} V_N(x) &= x_N \\ V_t(x) &= \max_{0\leq u\leq 1}\{(1-u_t)x_t + V_{t+1}(x_t + 0.5x_tu_t)\} \end{aligned}$$

(c) We have

$$\begin{aligned} V_3(x) &= x_3 \\ V_2(x) &= \max_{0\leq u\leq 1}\{(1-u_2)x_2 + (x_2 + 0.5x_2u_2)\} = 2x_2 \text{ with } u_2^\star = 0 \\ V_1(x) &= \max_{0\leq u\leq 1}\{(1-u_1)x_1 + 2(x_1 + 0.5x_1u_1)\} = 3x_1 \text{ with } u_1^\star = 1 \\ V_0(x) &= \max_{0\leq u\leq 1}\{(1-u_0)x_1 + 3(x_0 + 0.5x_0u_0)\} = 4.5x_0 \text{ with } u_0^\star = 1. \end{aligned}$$

Hence $V_0(x_0) = 4.5x_0$ and $u^\star = \{1, 1, 0\}$

4. (a) In order to compute L, we solve the Riccati algebraic equation

$$P = 17 + 3^2 P - \frac{3^2 P^2}{2 + P}$$

and the corresponding optimal control law

$$L = \frac{3P}{2 + P}$$

Solving the two equations we obtain $P = 34$ and $L = 51/18$. The LQR feedback control law is therefore $u(t) = -(51/18)x(t)$.

(b) In order to move from the infinite horizon to finite horizon summation, we add the final cost $x_N^{\mathrm{T}} Q_f x_N$, where $Q_f = P = 34$ is the solution to the algebraic Riccati equation. The resulting MPC formulation is

$$\begin{aligned} \underset{\{u_0, \dots, u_{N-1}\}}{\text{minimize}} \quad & 34x_N^2 + \sum_{k=0}^{N-1} 17x_k^2 + 2u_k^2, \\ \text{subject to} \quad & x_{k+1} = 3x_k + u_k, \\ & x_0 = x(t), \end{aligned}$$

The LQR and unconstrained MPC formulations are equivalent because, by definition of the algebraic Riccati equation,

$$x_0^{\mathrm{T}} P x_0 = \min_{\{u_0, u_1, \dots\}} \sum_{i=0}^{\infty} x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i,$$

subject to the system dynamics. In order to have equivalent predicted trajectories in the LQR and in the unconstrained MPC cases, the MPC mode 2 control law should be chosen according the LQR feedback law, i.e., $u_k = -(51/18)x_k$, for $k = N,\ N+1,\ \dots$.

(c) In order to guarantee the stability of the MPC, the final state constraint should be state-feasible (i.e., satisfy the state constraint), input-feasible (i.e., satisfy the input constraint), and invariant while the mode 2 control law is used. These conditions are needed for proving that the cost function is a valid Lyapunov function for the MPC controller.

Let us define the final state constraint as $x_l \leq x_N \leq x_u$.

- To be state-admissible, $x_l \geq -8$ and $x_u \leq 8$.
- To be input-admissible, $-20 \leq -Kx_N \leq 25$, for all $x_l \leq x_N \leq x_u$. Since the constraint is linear we only need to check that it is satisfied for the extreme value of x_N. Therefore, (i) $-25 \leq -Kx_l \leq 20$ and (ii) $-25 \leq -Kx_u \leq 20$, where $K = (51/18)$. We can rewrite the constraints as

$$-\frac{450}{51} \leq x_l \leq \frac{360}{51} \quad \text{and} \quad -\frac{450}{51} \leq x_u \leq \frac{360}{51}$$

- finally, in order to be invariant, the final constraint should satisfy $x_l \leq (A - BK)x_N \leq x_u$, for all $x_l \leq x_N \leq x_u$, where $A - BK = 1/6$. Because of the linearity of the constraint, we need to only check again that it is satisfied for the extreme value of x_N. We therefore need to guarantee that $x_l \leq x_l/6 \leq x_u$ and $x_l \leq x_u/6 \leq x_u$. Therefore, $x_l \leq 0$ and $x_u \geq 0$.

Combining the three conditions and looking for the less restrictive constraint on x_N, we obtain $x_l = -8$ and $x_u = 360/51$.

Note that also the final constraint $x_N = 0$ would be enough to prove the MPC stability. However this answer will not entitle you to get full points.

5. (a) First, we note that

$$J_{N,N}(N) = x_N^T(Q_1 + L^T Q_2 L)x_N := x_N^T V_{N,N} x_N$$

Now, assume that $J_{N,t}(x_t) = x_t^T V_{N,t} x_t$. Then,

$$\begin{aligned} J_{N,t-1}(x_{t-1}) = \sum_{k=t-1}^{N} x_k^T Q_1 x_k + u_k^T Q_2 u_k &= x_{t-1}^T Q_1 x_{t-1} + u_{t-1}^T Q_2 u_{t-1} + x_t^T V_{N,t} x_t = \\ &= x_{t-1}^T \left(Q_1 + L^T Q_2 L + (A - BL)^T V_{N,t}(A - BL)\right) x_{t-1} := x_{t-1}^T V_{N,t-1} x_{t-1} \end{aligned}$$

We have thus derived the recursion

$$V_{N,t-1} = Q_1 + L^T Q_2 L + (A - BL)^T V_{N,t}(A - BL)$$

(b) When the iterations converge, we get the equality

$$V_\infty = Q_1 + L^T Q_2 L + (A - BL)^T V_\infty (A - BL)$$

which we can re-write as

$$(A - BL)^T V_\infty (A - BL) - V_\infty + (Q_1 + L^T Q_2 L) = 0$$

We notice that this equation has the structure of a Lyapunov equation

$$\Phi^T P \Phi - P + Q = 0 \tag{1}$$

with $\Phi = A - BL$ and $Q = Q_1 + L^T Q_2 L$. So, we can compute V_∞ by forming Φ and Q and calling the routine for solving (1).