REGLERTEKNIK

School of Electrical Engineering, KTH

# EL2700 Model predictive control

Exam (tentamen) 2017–12–18, kl 14.00–19.00

**Aids:** The course notes and slides for EL2700; books from other control courses; mathematical tables and pocket calculator. Note that exercise materials are NOT allowed. You may add hand-written notes to the material that you bring, as long as these notes are not exercises or solutions.

**Observe:** Do not treat more than one problem on each page.
Write only on one side of each sheet.
Each step in your solutions must be justified.
Lacking justification will result in point deductions.
Write a clear answer to each question
Write name and personal number on each page.
Mark the total number of pages on the cover

The exam consists of five problems of which each can give up to 10 points. The points for subproblems have marked.

**Grading:** Grade A: $\geq 43$, Grade B: $\geq 38$
Grade C: $\geq 33$, Grade D: $\geq 28$
Grade E: $\geq 23$, Grade Fx: $\geq 21$

**Responsible:** Mikael Johansson 08-7907436

**Results:** Will be posted no later than January 15, 2018.

*Good Luck!*

1. Consider the following model predictive controller formulation for tracking a constant reference $(\bar{x}, \bar{u})$

$$
\begin{aligned}
\min_{\{u_0,\ldots,u_{N-1}\}} \quad & \sum_{k=0}^{N-1} (x_k - \bar{x})^T Q_1 (x_k - \bar{x}) + (u_k - \bar{u})^T Q_2 (u_k - \bar{u}) + (x_N - \bar{x})^T Q_f (x_N - \bar{x}) \\
\text{subject to} \quad & x_{k+1} = Ax_k + Bu_k, \;\; k = 1, ..., N-1 \\
& u_k \in \mathcal{U}, \;\; k = 0, ..., N-1 \\
& x_k \in \mathcal{X}, \;\; k = 1, ..., N-1 \\
& x_N - \bar{x} \in \mathcal{X}_f
\end{aligned}
$$

Here, the sets $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{X}_f$ and are polytopes representing the state, input, and terminal state constraint sets, respectively. Also, $Q_1$, $Q_2$, and $Q_f$ are positive definite.

(a) What relationship needs to hold between $\bar{x}$ and $\bar{u}$ for the controller to be able to guarantee error-free tracking? (1p)

(b) In contrast to other control methods, MPC has the ability to include reference preview. Explain what reference preview is and how you would modify the MPC formulation above to support reference preview. (2p)

(c) The terminal state $\mathcal{X}_f$ is typically designed to ensure recursive feasibility. Explain what recursive feasibility is. (2p)

(d) Suppose that you want to account for disturbances in the prediction model, i.e.

$$x_{k+1} = Ax_k + Bu_k + B_d d$$

where $d$ is a constant disturbance which we cannot measure directly. Explain how the state space model can be augmented to include a representation of this disturbance. Given that an MPC controller has been designed for the augmented model, what additional system component is required for the controller to work? (3p)

(e) Explain what modifications you need to the controller above in order to guarantee offset-free tracking in the presence of a constant disturbance. (2p)

2. A process is described by the discrete-time linear system

$$
\begin{aligned}
x_{t+1} &= \begin{bmatrix} 0.4 & 0.8 \\ 0.4 & 0 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t \\
y_t &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_t
\end{aligned}
$$

(a) Prove that the open-loop system is asymptotically stable. (2p)

(b) Is the system controllable? (1p)

(c) Design a state feedback controller (2p)

$$u_t = -Lx_t$$

which places both closed-loop poles in $z = 0.5$.

(d) Is the system observable? (1p)

(e) Design an observer (2p)

$$
\begin{aligned}
\hat{x}_{t+1} &= A\hat{x}_t + Bu_t + K(y_t - \hat{y}_t) \\
\hat{y}_t &= C\hat{x}_t
\end{aligned}
$$

which places one observer pole at $z = 0$ and the other at $z = -0.4$.

(f) Can you find an observer gain which places both observer poles at the origin? Explain! (2p)

3. Consider the discrete time system

$$x_{t+1} = Ax_t + Bu_t,$$
$$y_t = Cx_t$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix}.$$

We want to design a model predictive controller that minimizes the cost function

$$J = \sum_{k=0}^{\infty} \frac{1}{2}(y_k^2 + u_k^2).$$

(a) Show that if $u_k = \frac{1}{\sqrt{2}} y_k$, then

$$\sum_{k=0}^{\infty} \frac{1}{2}(y_k^2 + u_k^2) = \|x_0\|^2.$$

(3p)

(b) A predictive control law is defined at each time step $t$ by

$$u_t = \check{u}_t$$

where $\{\check{u}_t, \check{u}_{t+1}, \ldots, \check{u}_{t+N-1}\}$ is the minimizing argument of

$$\sum_{k=0}^{N-1} \frac{1}{2}(\check{y}_{t+k}^2 + \check{u}_{t+k}^2) + \|\check{x}_{t+N}\|^2$$

and $\{\check{y}_{t+k}\}$ and $\check{x}_{t+N}$ are predicted values of the corresponding system signals given state measurements at time $t$. Show that the control law yields a stable closed-loop system. (2p)

(c) The system is now subject to the constraint $-1 \le y_t \le 1$, for all $t$.
Show that if $u_t = \frac{1}{\sqrt{2}} y_t$, then if $|y_t| \le 1$ and $|y_{t+1}| \le 1$, it will hold that

$$-1 \le y_{t+k} \le 1 \qquad \forall k \ge 0$$

(3p)

(d) Now, consider the model predictive control law $u_t = \check{u}_t$ where $\{\check{u}_t, , \ldots, \check{u}_{t+N}\}$ is the minimizing argument of

$$\begin{array}{lll} \text{minimize} & \sum_{k=0}^{N-1} \frac{1}{2}(\check{y}_{t+k}^2 + \check{u}_{t+k}^2) + \|\check{x}_{t+N}\|^2 & \\ \text{subject to} & \check{x}_{t+k+1} = A\check{x}_{t+k} + B\check{u}_{t+k} & k = 0, 1, \ldots, N \\ & -1 \le \check{y}_{t+k} \le 1, & k = 0, 1, \ldots, N+1 \end{array}$$

Will this MPC controller guarantee a stable closed-loop system? (2p)

4. A public company has profit $x_k$ at year $k$. This profit is distributed partly to the shareholders as dividends and partly as reinvestment in the company itself. Reinvesting increases the company profit by $\theta\times$ the invested capital. To boost reputation, the company decides to maximize the amount distributed to the shareholdes over an $N$ year period. Therefore, the following discrete optimal control problem is formulated:

$$\begin{aligned} \underset{u_k}{\text{maximize}} \quad & \sum_{k=0}^{N-1}(1-u_k)x_k \\ \text{subject to} \quad & x_{k+1} = x_k + \theta u_k x_k \\ & 0 \leq u_k \leq 1 \\ & x_0 = x_c \end{aligned} \tag{1}$$

The intuition of the objective is that each year the profit $x_k$ is divided into $u_k x_k$ which is reinvested and $(1-u_k)x_k$ which is given to the shareholders. $x_c > 0$ is the current profit which can be distributed the first year.

(a) For the specific case $x_c = 3$, $N = 2$, and $\theta = 0.4$, solve the above problem (1) using dynamic programming. Specify the optimal strategy $\hat{u}_0, \hat{u}_1$, and the total distributed profit. (3p)

(b) Formulate a value function $V_{k+1}(x_{k+1})$ and show by induction that it is indeed a valid value function for the dynamic programming recursion:

$$V_k(x_k) = \underset{0\leq u_k \leq 1}{\text{maximize}} \{(1-u_k)x_k + V_{k+1}(x_{k+1})\}$$

Next, determine an optimal control law $\hat{u}_k$ as a function of $\theta$ and $N$.
*Hint: Think carefully about what structure the value function had in (a).* (7p)

5. Consider the constrained optimal control problem

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & \sum_{k=0}^{\infty} x_k^2 + \sum_{k=0}^{\infty} u_k^2 \\ \text{subject to} \quad & x_{k+1} = x_k + \frac{3}{2} u_k \\ & x_0 \text{ given} \\ & |u_k| \leq 1 \end{aligned} \tag{2}$$

over an infinite horizon.

(a) Determine an optimal feedback law to the corresponding unconstrained problem

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & \sum_{k=0}^{\infty} x_k^2 + \sum_{k=0}^{\infty} u_k^2 \\ \text{subject to} \quad & x_{k+1} = x_k + \frac{3}{2} u_k \\ & x_0 \text{ given} \end{aligned}$$

(2p)

(b) Determine an approximate optimal feedback law to (2) by solving

$$\begin{aligned} \underset{u_k}{\text{minimize}} \quad & \sum_{k=0}^{N} x_k^2 + \sum_{k=0}^{N} u_k^2 \\ \text{subject to} \quad & x_{k+1} = x_k + \frac{3}{2} u_k \\ & x_0 \text{ given} \\ & |u_k| \leq 1 \end{aligned}$$

using dynamic programming, with a quadratic value function

$$V_{k+1}(x_{k+1}) = p_{k+1} x_{k+1}^2$$

as a starting guess. Next, determine the optimal feedback law to (2) by letting $N \to \infty$.
*Hint: The optimal feedback $\hat{u}$ for the unconstrained problem is still optimal if $|\hat{u}| \leq 1$.*

(5p)

(c) Show that the controller you computed in (b) is stabilizing, regardless of the initial state $x_0$. Note that, if necessary, you can use LQR stability without proof.

(3p)

Note: If you did not solve (b), show instead that the controller

$$u = -\text{sat}(Kx)$$

is stabilizing for a certain value of $K$. Here, $\text{sat}(\cdot)$ denotes the standard saturation function

$$\text{sat}(z) = \begin{cases} +1 & \text{if } z \geq 1 \\ -1 & \text{if } z \leq -1 \\ z & \text{otherwise} \end{cases} .$$

Specify at least one stabilizing gain $K$, if such a gain exists.