Observable Degree Analysis Using Unscented Information Filter for Nonlinear Estimation Systems

Peng Zhuo
Hangzhou Dianzi University
School of Automaiton
Hangzhou, China 310018
Email: paradoxing@foxmail.com

Quanbo Ge, Teng Shao
Hangzhou Dianzi University
Hangzhou, China 310018
Telephone: (86)571-86915193
Email: qbge@hdu.edu.cn, shaoking@yahoo.com

Yongting Wang, Liang Bai
Science and Technology
on Electro-optic Control Laboratory
Luoyang, Henan Province,china

***Abstract*—Observability and observable degree, which are derived from modern control theory and relative to control and estimation performances, are both important concepts for state estimation. For linear time-invariant systems, judgement of observability and evaluation of observable degree are irrelevant to filtering process and parameters. Thereby, it has limited application ability. Different from linear systems, observability and observable degree becomes more complex and are seldom studied deeply and integrally for nonlinear systems. Due to the complexity of nonlinear filtering, the observable degree depends strongly on filtering process, filter class and associated parameters. Thereby, we study the observable degree analysis of nonlinear estimation systems by using unscented information filter (UIF). A way to evaluate observable degree using singular value decomposition (SVD) of observability matrix is presented based on the UIF and the influencing factors are deeply analyzed. Moreover, differences of observable degree analysis are distinctly discussed between linear and nonlinear systems. The results reveal that evaluation of the observable degree for nonlinear systems can be affected by filtering estimate at last time, initial estimate and filter's parameters such as covariances of process and measurement noises. Some simulations are demonstrated to validate the results.**

***Index Terms*—Nonlinear systems; observable degree analysis; unscented information filter; singular value decomposition; pseudo measurement matrix**

I. INTRODUCTION

Observability is an important concept to explain possibility to determine system state by using outputs sequence in modern control theory. In the original work presented by R.E Kalman [1], the controllability and observability construct the basis of modern control theory or state estimation based on state space method. In particular, the estimation ability depends closely on the observability for control and estimation systems [2]. Unfortunately, the conventional observability study only focuses on distinguishing whether the system is observable for linear time-invariant dynamic systems. Thereby, it has limited help for state estimation or filtering due to its basic definition [3]. In order to enhance application ability of the observability theory, the concept of observable degree, which provides a quantitative measure on observability, was established and the observability theory has been deepened [4]. Accordingly, the observable degree has been a potential way to extend and perfect the conventional Kalman filtering estimation frame based on minimum mean square error (MMSE) criteria from the current preliminary research work. Especially, it is greatly possible to improve estimation performance by using the observable degree analysis (ODA) method to some extent.

Actually, the observable degree analysis is getting more and more attention from engineers, but we should follow with more interests in theoretical research field because it is challenging to improve and perfect the current state estimation theory under the MMSE rule. After the observability method was established, the subsequent study was very slow and no more distinct progress was carried forward until the last century [5-8]. Recently, because the study on the conventional state estimation based on Kalman filtering frame suffer a bottleneck for both theoretical and applied fields, we have to look for new breakthroughs to promote progress of state estimation theory. The ODA method should be one of effective ways. Introducing the ODA into the state estimators or filters, many new problems will appear. Meanwhile, the estimator design process should be modified and many aspects should be changed on state estimation theory. Thereby, it is very significative to study the observable degree analysis theory in order to promote state estimation based on models for practical applications.

The observability study and the associated results both focus on linear time-invariant (LTI) systems for a long while because it is influenced by the original work in modern control theory. Due to the LTI characteristic, the conventional observability judgement method is very simple and judgement conclusions are the same for all of times. Naturally, the associated observable degree analysis process and the results are coincident at every time [9-14]. Additionally, for the LTI system, both the observability and the ODA depend on the analytical and known state transfer matrix and measurement matrix which are not influenced by the estimation process. Unfortunately, for nonlinear systems, all of methods and results taken from the LTI system are not directly applicable. This is because the complectly of the nonlinear system and the associated filtering. Most important of all, there are not analytical and known state transfer matrix and measurement matrix, which are dynamical time variant and usually influenced by online filtering process. Meanwhile, the properties are not the same for different nonlinear filters, and the observability and observable degree analysis depend closely on the detailed nonlinear filters [15-17].

Aiming at the problems mentioned above, we study observable degree analysis for a kind of nonlinear system in this paper and the unscented information filter, which is a information filtering form of unscented Kalman filter (UKF), is considered because of superiority and representativeness compared with the extended Kalman filter (EKF) [18-19].Although unscented Kalman filter (UKF) is based on statistical method and with no strict demonstration, it's a good way for nonlinear system ODA. Moreover, considering two purpose of this work, which are to show the differences between LTI systems and nonlinear systems on the observable degree analysis and discuss factors to influence the ODA of nonlinear systems, the ODA method based on singular value decomposition (SVD) is used.

The rest of this paper is organized as follows. Section II presents an information form of unscented Kalman filter for a kind nonlinear system which has linear state. In Section III, the observable degree analysis method is introduced by using the SVD theory. In Section IV, we explain the differences on the ODA between linear and nonlinear systems. The key factors to influence the ODA process are discussed in details based on the unscented information filter in Section V. Some simulation examples are demonstrated to validate the presented analysis results in Section VI. Finally, we conclude this paper.

II. Unscented Information Filter

A. System Formulation

The following nonlinear dynamical system is considered

$$\begin{aligned} \mathbf{x}_k &= \mathbf{A}\mathbf{x}_{k-1} + \mathbf{w}_{k,k-1} \\ \mathbf{z}_k &= h(\mathbf{x}_k) + \mathbf{v}_k \end{aligned} \tag{1}$$

where, the vector $\mathbf{x}_k \in R^n$ and $\mathbf{z}_k \in R^m$ represent the system state and measurement at time k, respectively.$\mathbf{A} \in R^n, h(\cdot) : R^n \rightarrow R^m$ are known vector functions, and $\mathbf{w}_{k,k-1} \in R^n$ and $\mathbf{v}_k \in R^m$ are independent process and measurement white noises, which are both supposed to be Gaussian with zero means and known covariance matrices $\mathbf{Q}_{k,k-1}$ and $\mathbf{R}_k$. It should be noted that the state model is linear and measurement model is nonlinear. Thereby, the estimation system is specially nonlinear.

B. Unscented Kalman Filter

In the UKF, the state prediction and measurement prediction are both computed by the unscented transformation[18][21]. State estimate $\hat{\mathbf{x}}_{k|k}$ at time k and the associated estimation error covariance $\mathbf{P}_{k|k}$ are evaluated by

$$\left\{ \begin{array}{l} \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}) \\ \mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{P}_{\tilde{z}_k} \mathbf{K}_k^T \\ \mathbf{K}_k = \mathbf{P}_{\tilde{x}_k \tilde{z}_k} (\mathbf{P}_{\tilde{z}_k})^{-1} \end{array} \right. \tag{2}$$

where, one step measurement prediction is denoted as $\hat{\mathbf{z}}_{k|k-1}$. Measurement prediction error covariance $\mathbf{P}_{\tilde{z}_k}$ and cross-covariance $\mathbf{P}_{\tilde{x}_k \tilde{z}_k}$ are computed by

$$\left\{ \begin{array}{l} \mathbf{P}_{\tilde{z}_k} = \sum\limits_{i=0}^{2n} W_i^c (\eta_{k|k-1}^{(i)} - \hat{\mathbf{z}}_{k|k-1})(\eta_{k|k-1}^{(i)} - \hat{\mathbf{z}}_{k|k-1})^T \\ \quad +\mathbf{R}_k \\ \mathbf{P}_{\tilde{x}_k \tilde{z}_k} = \sum\limits_{i=0}^{2n} W_i^c (\varsigma_k^{(i)} - \hat{\mathbf{x}}_{k|k-1})(\eta_{k|k-1}^{(i)} - \hat{\mathbf{z}}_{k|k-1})^T \end{array} \right. \tag{3}$$

The $\hat{\mathbf{z}}_{k|k-1}$ is computed according to

$$\left\{ \begin{array}{l} \hat{\mathbf{z}}_{k|k-1} = \sum\limits_{i=0}^{2n} W_i^m \eta_{k|k-1}^{(i)} \\ \eta_{k|k-1}^{(i)} = h(\varsigma_k^{(i)}) \quad i = 0, 1, \cdots 2n \end{array} \right. \tag{4}$$

where $\varsigma_k^{(i)}$ and the corresponding weights W_i^m and W_i^c $(i = 0, 1, 2 \cdots 2n)$ are computed by

$$\left\{ \begin{array}{l} \varsigma_k^{(0)} = \hat{\mathbf{x}}_{k|k-1} \\ \varsigma_k^{(i)} = \hat{\mathbf{x}}_{k|k-1} + (\sqrt{(n+\lambda)\mathbf{P}_{k|k-1}})i \\ \quad i = 1, 2, \cdots n \\ \varsigma_k^{(i)} = \hat{\mathbf{x}}_{k|k-1} - (\sqrt{(n+\lambda)\mathbf{P}_{k|k-1}})i \\ \quad i = n+1, n+2, \cdots 2n \end{array} \right. \tag{5}$$

and

$$\left\{ \begin{array}{l} W_0^m = \frac{\lambda}{n+\lambda} \\ W_0^c = \frac{\lambda}{n+\lambda} + (1 - \alpha^2 + \beta) \\ W_i^m = W_i^c = \frac{1}{2(n+\lambda)}, i = 1, 2, \cdots 2n \end{array} \right. \tag{6}$$

where $\lambda = \alpha^2(n+K) - n$ is proportionality coefficient and $0 \leq \alpha \leq 1$. In most cases, we normally choose $\alpha = 0.01$ and $K = 0$. β describes the distribution information of state variables and usually $\beta = 2$.

One step state prediction $\hat{\mathbf{x}}_{k|k-1}$ and its error covariance $\mathbf{P}_{k|k-1}$ are given by

$$\left\{ \begin{array}{l} \hat{\mathbf{x}}_{k|k-1} = \mathbf{A}\hat{\mathbf{x}}_{k-1|k-1} \\ \mathbf{P}_{k|k-1} = \mathbf{A}\mathbf{P}_{k-1|k-1}\mathbf{A}^T + \mathbf{Q}_{k,k-1} \end{array} \right. \tag{7}$$

C. Unscented Information Filter (UIF)

Information form of the state filter is a kind of effective way to compute state estimate and attracts many attention from researches. Different from linear Kalman filter, in the computation processes of nonlinear filters besides the unscented information filter, there is not measurement matrix which is available to take the information form of the UKF called unscented information filter (UIF). However, we can construct a pseudo observation matrix according to the following way, namely[20]

$$\mathbf{H}_k^* = (\mathbf{P}_{k|k-1}^{-1} \mathbf{P}_{\tilde{x}_k \tilde{z}_k})^T \tag{8}$$

Let denote

$$\left\{ \begin{array}{l} \mathbf{Y}_{k|k-1} = \mathbf{P}_{k|k-1}^{-1} \quad \mathbf{Y}_{k|k} = \mathbf{P}_{k|k}^{-1} \\ \mathbf{y}_{k|k-1} = \mathbf{P}_{k|k-1}^{-1}\hat{\mathbf{x}}_{k|k-1} = \mathbf{Y}_{k|k-1}\hat{\mathbf{x}}_{k|k-1} \\ \hat{\mathbf{y}}_{k|k} = \mathbf{P}_{k|k}^{-1}\hat{\mathbf{x}}_{k|k} = \mathbf{Y}_{k|k}\hat{\mathbf{x}}_{k|k} \end{array} \right. \tag{9}$$

where

$$\begin{aligned} \mathbf{Y}_{k|k} = \mathbf{P}_{k|k}^{-1} &= \mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^{*T}\mathbf{R}_k^{-1}\mathbf{H}_k^* \\ &= \mathbf{Y}_{k|k-1} + \mathbf{H}_k^{*T}\mathbf{R}_k^{-1}\mathbf{H}_k^* \end{aligned} \tag{10}$$

The information expression of state estimate is

$$\begin{aligned}\hat{\mathbf{y}}_{k|k} &= \mathbf{Y}_{k|k}(\hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1})) \\ &= \hat{\mathbf{y}}_{k|k-1} + \mathbf{H}_k^{*T}\mathbf{R}_k^{-1}(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1} + \mathbf{H}_k^{*}\hat{\mathbf{x}}_{k|k-1})\end{aligned} \tag{11}$$

Accordingly, the state estimate can be taken by

$$\hat{\mathbf{x}}_{k|k} = \mathbf{Y}_{k|k}^{-1} \times \hat{\mathbf{y}}_{k|k} \tag{12}$$

III. Observable Degree Analysis Based SVD Theory

The observability degree analysis mainly includes two parts. One is to judge the system observability and the other is to compute observable degree of estimation system [9-11]. According to the basic theory of system observability, we can get the observability matrix based on the UIF at k as follows

$$\mathbf{G}_k = [\ \mathbf{H}_k^* \quad \mathbf{H}_k^*\mathbf{A} \quad \cdots \quad \mathbf{H}_k^*\ \mathbf{A}^{n-1}]^T \tag{13}$$

Here, when $rank(\mathbf{G}_k) = n$, we think that the system is fully observable.

In linear systems, the observable degree analysis is regard as uniqueness of solutions of linear consistent equations with the system observability matrix. In nonlinear system, we can obtain pseudo observability matrix $\mathbf{G}_k$ by using $\mathbf{H}_k^*$ at time k. Then, the observable degree can be analyzed by SVD decomposition of $\mathbf{G}_k$[12,13].

$$\mathbf{G}_k = \mathbf{U}_k \Sigma {\mathbf{V}_k}^T \tag{14}$$

where $\Sigma = diag(S, 0)$ and $S = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$ are both diagonal matrices with the singular values of $\mathbf{G}_k$ which are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. $\mathbf{U}_k$ and $\mathbf{V}_k$ are orthogonal matrix. Denote $\mathbf{U}_k = [u_1, u_2, \cdots, u_n]$ and $\mathbf{V}_k = [v_1, v_2, \cdots v_n]$, we have [5]

$$\mathbf{Z} = \mathbf{G}_k\mathbf{x}_0 = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \mathbf{x}_0 = \sum_{i=1}^{r} \sigma_i (\mathbf{v}_i^T \mathbf{x}_0)\mathbf{u}_i \tag{15}$$

where $\mathbf{Z}$ is a set which includes all measurements from initial time to time k and $\mathbf{x_0}$ means the initial state. When $\mathbf{Z}$ is a constant norm, $\mathbf{x_0}$ comes into being an ellipsoid, namely

$$|\mathbf{Z}|^2 = \sum_{i=1}^{r} \left(\sigma_i \mathbf{v}_i^T \mathbf{x}_0 \mathbf{u}_i\right)^2 = \sum_{i=1}^{r} \left(\frac{\mathbf{v}_i^T \mathbf{x}_0 \mathbf{u}_i}{\frac{1}{\sigma_i}}\right)^2 \tag{16}$$

where σ_i^{-1} means the axial length of the ellipsoid.

The upper bound of $|\mathbf{x}_0|$ is:

$$|\mathbf{x}_0| \leq |\mathbf{Z}| / \sigma_r, \sigma_r = \min, i = 1, 2, \cdots, n \tag{17}$$

So, the observable degree is defined as follows

$$\eta_j = \sigma_i, i = 1, 2, \cdots, n \tag{18}$$

$$\sigma_i \sim \max\left(\frac{u_i^T \mathbf{Z} v_i}{\sigma_i}\right) \tag{19}$$

It should be noted that singular values correspond to observable degrees of a linear combination of state components. The larger singular value is, the better observable performance is.

It simplifies the definition of the state component observable degree in [20]. Namely, the singular value σ_i can be regarded as the observable degree of corresponding state components, in which the corresponding state component of the largest absolute value number in $\mathbf{v}_i$ has the largest projection in the direction of $\mathbf{v}_i$.

IV. Discussion On The ODA For Nonlinear Systems

A. The ODAs Between Linear and Nonlinear systems

In linear systems, the state transition matrix $\mathbf{A}$ and measurement matrix, which are used to evaluate the observable degree and are independent from the filtering process and estimation result, are both known and analytic. An expression on the ODA sees Fig.1 for linear systems. However, for nonlinear systems, the observable degree analysis is related to the filtering result at each time and is influenced by the filtering process. This is because that computation of pseudo-system parameters such as pseudo-measurement matrix $\mathbf{H}_k^*$, which can be used to compute the observable degree for nonlinear systems, depends closely state estimate at last time. It means there is a feedback on state estimate from current time to next time for the ODA. The details see to Fig.2. In this figure, it's implied several factors, which have relations with the filtering result and can affect the computation of observable degree.

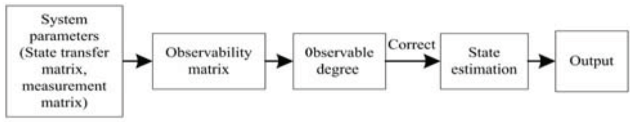

Fig. 1. Observable degree analysis of linear systems

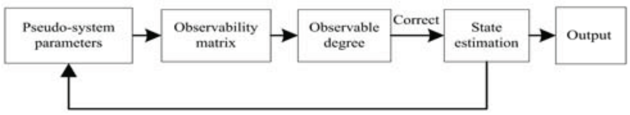

Fig. 2. Observable degree analysis of nonlinear systems

TABLE I
Comparison between linear and nonlinear systems

System class	Observability matrix	Filtering difference	Characteristic of observable degree
LTI system	$\begin{bmatrix}\mathbf{H}\\ \mathbf{HA}\\ \vdots\\ \mathbf{HA}^{n-1}\end{bmatrix}$	optimal estimate	•time-invariant •independent on filtering process •Influenced by only two system parameters
nonlinear system	$\begin{bmatrix}\mathbf{H}_k^*\\ \mathbf{H}_k^*\mathbf{A}\\ \vdots\\ \mathbf{H}_k^*\mathbf{A}^{n-1}\end{bmatrix}$	using the nonlinear approximation to get second-best estimate	•time-varying •depend on filtering result •Infuluenced by many factors

Under general cases, there are nonlinear state equation and nonlinear measurement for nonlinear systems. Certainly, we can obtain pseudo-system parameters $\mathbf{A}_k$ and $\mathbf{H}_k^*$ through state transition function $\phi(\cdot)$ and measurement function $h(\cdot)$, respectively. In this paper, a kind of special nonlinear system is considered, namely, the state equation is linear and the measurement equation is nonlinear.

There are some comparisons on the ODA in details between linear and nonlinear systems in Table I. Observability matrix of linear system is made up by the state transition matrix and measurement matrix. For this reason, the observable degree can be computed before the filtering. Clearly, it's consistent and independent of the filtering result. In nonlinear systems, which is formulated by linear state equation and nonlinear measurement equation, the accurate measurement matrix $\mathbf{H}_k^*$ is not gotten and only the pseudo-measurement matrix at every moment is available for us. For the pseudo-measurement matrix, it is necessary to be time-varying and relying on the filtering result. As a result, the observable degree analysis for nonlinear systems is time-varying and dependent on the real-time filtering process. And then, there are many factors such as initial estimate and covariances of process and measurement noises, which can influence the filtering result and the computation of the observable degree of nonlinear systems.

B. Differences on ODAs Based on EKF and UKF

The ODA depends on the detailed nonlinear filtering methods for nonlinear systems. Here, let's discuss the differences on the ODA between using the EKF and the UKF. The difference is how to obtain pseudo-measurement matrix. In observable degree analysis using EKF, we obtain the pseudo-measure matrix at k by using Jacobian matrix of $h(\cdot)$ and one step state prediction $\hat{\mathbf{x}}_{k|k-1}$ as follows

$$\mathbf{H}_k^* = \frac{\partial h}{\partial \mathbf{x}} \bigg|_{\hat{\mathbf{x}}_{k|k-1}} \tag{20}$$

For the ODA using UKF, we obtain pseudo-measure matrix $\mathbf{H}_k^*$ by using Eq.(8).

V. Key Factors To Influence The ODA of Nonlinear Systems

For the observable degree analysis based on the UKF, the observable degree is computed by using the singular value of $\mathbf{G}_k$. At this time, the observable degree can be related to the pseudo-observability matrix. Each singular value σ_i of $\mathbf{G}_k$ corresponds to the observable degree of a state component. For the nonlinear system which expressed by linear state equation and nonlinear measurement equation, the computation of the observable degree can be affected by state transition matrix $\mathbf{A}$ and Pseudo-measurement matrix $\mathbf{H}_k^*$. Clearly, from Eq.(8) we can know that the computation of $\mathbf{H}_k^*$ is related to the real-time filtering process and influenced by some factors such as covariances $\mathbf{Q}_{k,k-1}$ and $\mathbf{R}_k$ of process and measurement noises, initial state estimate $\mathbf{x}_0$ and the associate estimate error covariance $\mathbf{P}_0$.

A. Process Noise Covariance $\boldsymbol{Q}_{k,k-1}$

According to Eq.(7), we have

$$\mathbf{P}_{k|k-1} = \mathbf{A}\mathbf{P}_{k-1|k-1}\mathbf{A}^T + \mathbf{Q}_{k,k-1} \tag{21}$$

By jointly considering Eq.(8) and Eq.(21), we have the relation expressed by Fig.3. Clearly, the ODA can be affected by the process noise covariance $\mathbf{Q}_{k,k-1}$.

Fig. 3. Factor $\mathbf{Q}_{k,k-1}$

B. Initial State Estimate $\boldsymbol{x}_0$

According to Eq.(3)-(5) clearly, we know that $\mathbf{P}_{\tilde{x}_k\tilde{z}_k}$ is influenced by $\hat{\mathbf{x}}_{k-1|k-1}$ and $\hat{\mathbf{x}}_{k-1|k-1}$ is taken based on $\hat{\mathbf{x}}_{k-2|k-2}$, at last $\mathbf{P}_{\tilde{x}_k\tilde{z}_k}$ is affected by $\mathbf{x}_0$ according to Fig.4. That is to say $\mathbf{x}_0$ can affect the computation of observable degree. For linear systems, the initial state estimate is unrelated to the ODA.

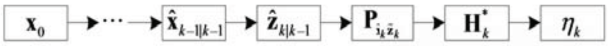

Fig. 4. Factor $\mathbf{x}_0$

C. Initial State Estimate Error Covariance $\boldsymbol{P}_0$

In Eq.(21), it's shown that $\mathbf{P}_{k|k-1}$ is influenced by $\mathbf{P}_{k-1|k-1}$, which is obtained based on $\mathbf{P}_{k-2|k-2}$. So, $\mathbf{P}_{k|k-1}$ is affected by the initial state estimate error covariance $\mathbf{P}_0$ at last. In the other words, $\mathbf{P}_0$ can influence the computation of the observable degree from Fig.5.

Fig. 5. Factor $\mathbf{P}_0$

D. Measurement Noise Covariance $\boldsymbol{R}_k$

In terms of the recursive computation of the UIF, in equation (2),(3), that $\mathbf{P}_{k-1|k-1}$ is under the effect of $\mathbf{R}_k$ and we can get the relation figure shown by Fig.6. Because the observable degree is under influence of $\mathbf{P}_{k-1|k-1}$,it is also affected by $\mathbf{R}_k$.

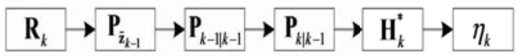

Fig. 6. Factor $\mathbf{R}_k$

From what discussed above, different from observability of linear systems, not only the state transition matrix $\mathbf{A}_k$ and $h(\cdot)$ can affect the observable degree analysis, but other factors like $\mathbf{x_0}$, $\mathbf{P_0}$, $\mathbf{Q}_{k,k-1}$ and $\mathbf{R}_k$ can directly and indirectly influence on the computation of the observable degree.

VI. Simulation Examples

A. System Setup

In the simulation, the nonlinear system with a linear state equation is considered and the time-varying pseudo-measurement matrix $\mathbf{H}_k^*$ is obtained from the unscented information filter. So, it is different from the ODA of linear systems, the observable degree of state component is also time varying.

In this simulation, the state equation is considered as:

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{1,k-1} \\ x_{2,k-1} \\ x_{3,k-1} \end{bmatrix} + \begin{bmatrix} w_{1,k,k-1} \\ w_{2,k,k-1} \\ w_{3,k,k-1} \end{bmatrix}$$

where $\mathbf{A}$ is an 3×3 identity matrix. $\mathbf{w}_{k,k-1}$ is measurement Gaussian noise which covariance $\mathbf{Q}_{k,k-1}$ is given as follows

$$\mathbf{Q}_{k,k-1} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

The measurement equation is given by

$$\begin{bmatrix} z_{1,k} \\ z_{2,k} \\ z_{3,k} \end{bmatrix} = \begin{bmatrix} \sqrt{x_{1,k}^2 + 16} \\ \sqrt{x_{2,k}^2 + 9} \\ \sqrt{x_{3,k}^2 + 1} \end{bmatrix} + \begin{bmatrix} v_{1,k} \\ v_{2,k} \\ v_{3,k} \end{bmatrix}$$

where $\mathbf{v}_{k,k-1}$ is measurement Gaussian noise which covariance

$$\mathbf{R}_k = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}$$

$\mathbf{P}_0 \in 3 \times 3$ is an identity matrix and $\mathbf{x}_0 = [\ 1 \quad 1 \quad 1\ ]^T$. From the measurement function $h(\cdot)$, one can see that z_1, z_2, z_3 contain the information of x_1, x_2, x_3. Through using unscented information filter, we get pseudo-measurement matrix $\mathbf{H}_k^*$ and the observable degree at every moment. The matlab simulation results of the observable degree are shown by Fig.7.

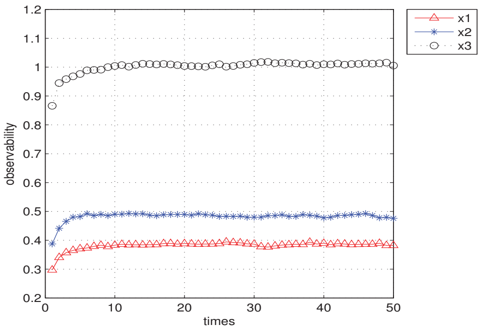

Fig. 7. Observable degree of x_1,x_2,x_3

We can see from the Fig.7 that the observable degree is time varying. For x_1, x_2, x_3, the value of observable degree is stable and close to 0.95, 0.48 and 0.38, respectively.

B. Simulation Examples

1) The Influence of $\boldsymbol{P}_0$: Let

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

where a grows from 1.00 to 8.00 and every interval is 1.00.

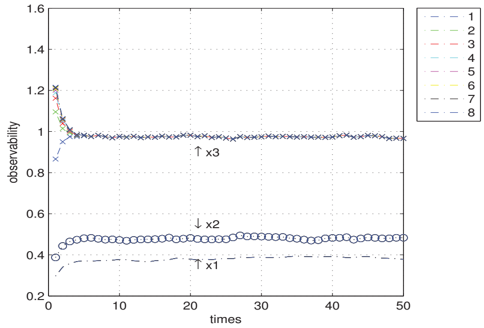

Fig. 8. Observable degree of x_1,x_2,x_3

The results are given from Fig.8, respectively show the observable degree change of x_1, x_2, x_3 through the varying of a. For x_3, at the beginning, the observable degree of x_3 monotonously increases accompanying with the growing a. The observable degrees of x_1 and x_2 are not affected by the growth of a.

2) The influence of $\boldsymbol{x}_0$: Let

$$\mathbf{x}_0 = [\ 1 \quad 1 \quad b\ ]^T$$

where b grows from 1.00 to 8.00 and every interval is 1.00.

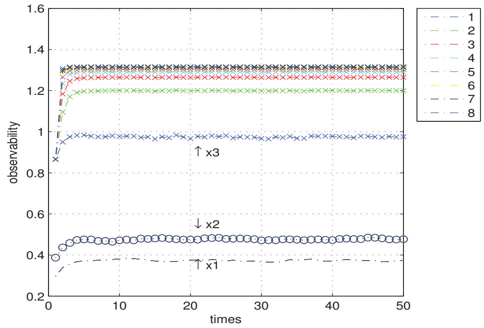

Fig. 9. Observable degree of x_1,x_2,x_3

The results are given from Fig.9, respectively show the observable degree change of x_1, x_2, x_3. Along with the changing of b, there is no change for the observable degree of x_1, x_2. For x_3, observable degree change in a great range and is monotone increasing.

3) The influence of $\boldsymbol{R}_k$: Let

$$\mathbf{R}_k = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \times c^2 \end{bmatrix}$$

where c grows from 1.00 to 8.00 and every interval is 1.00
We can see from the Fig.10. For the observable degree of x_1, x_2 are still unchanged through the grows of c, and the effects of x_3 is relatively obvious. With the grows of c the observable degree of x_3 is increasing.

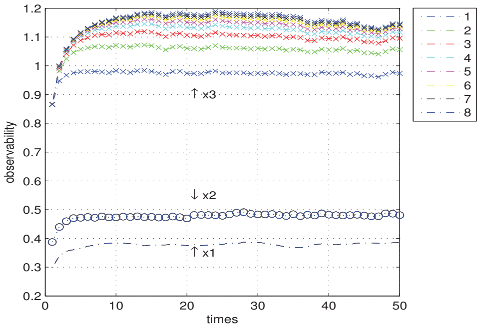

Fig. 10. Observable degree of x_1,x_2,x_3

4) The influence of $\boldsymbol{Q}_{k,k-1}$: Let

$$\mathbf{Q}_{k,k-1} = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \times d^2 \end{bmatrix}$$

where d grows from 1.00 to 8.00 and every interval add 1.00.

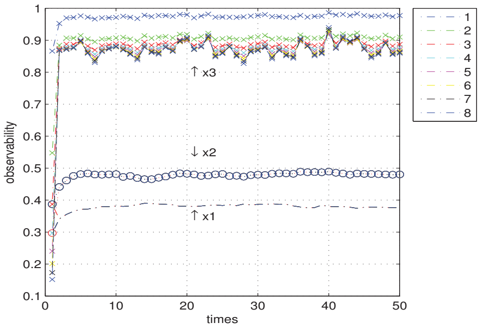

Fig. 11. Observable degree of x_1,x_2,x_3

We can see from the Fig.10 that the observable degree of x_1, x_2 are immune to the change of d. And for x_3, the observable degree is decreasing accompany with the augment of d.

VII. Conclusion

This paper mainly addresses the observable degree analysis problem for the nonlinear system with linear state equation and nonlinear measurement equation. It is different from the ODA of linear systems due to the observable degree of nonlinear system are affected by some filtering factors besides the system structure of nonlinear systems. What's more important is that the observable degree of nonlinear systems is time varying and influenced by the real-time filtering process and many filtering parameters. The conclusion shows that there is a clear difference on the ODAs for linear and nonlinear systems and it clearly indicates that it is challenging to further study the observable degree analysis to perfect the theory of the ODA.

Acknowledgment

We would like to show my deepest gratitude to Dr. Feng Xiaoliang, a respectable, responsible and resourceful scholar,from Henan University of Technology, who has provided us with valuable guidance in every stage of the writing of this thesis. This work was partially supported by Aeronautical Science Foundation of China Grant #201451T002 and the National Nature Science Fund of China (NSFC) Grant #61172133, #61304258 and #61273075).

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