The Cubic Formula Derivation Consider the arbitrary cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

for real numbers a, b, c, d with $a \neq 0$. By the fundamental theorem of algebra this equation has three roots x_1, x_2, x_3, over the complex numbers. Using the factor theorem gives the factorization

$$ax^3 + bx^2 + cx + d = a(x - x_1)(x - x_2)(x - x_3).$$

Expanding out the right-hand side gives

$$ax^3 - a(x_1 + x_2 + x_3)x^2 + a(x_1x_2 + x_1x_3 + x_2x_3)x - ax_1x_2x_3,$$

and equating coefficients with the original expression gives the following system of equations: b = - a(x$_1 + x_2 + x_3)$
$c = a(x_1x_2 + x_1x_3 + x_2x_3)$
$d = -ax_1x_2x_3$ And now it is non-obvious how to proceed and solve for x_1, x_2, x_3.

Introduction of New Variables It turns out that it is fruitful to introduce the new variables y_1, y_2 by the definitions y$_1 = a(x_1 + \omega x_2 + \omega^2 x_3)$
$y_2 = a(x_1 + \omega^2 x_2 + \omega x_3)$ where $\omega = e^{2\pi i/3} = (-1 + \sqrt{-3})/2$ is a cube root of unity. Then the following identities are satisfied: y$_1^3 + y_2^3 = -2b^3 + 9abc - 27a^2d$
$y_1y_2 = b^2 - 3ac$ These can be found using Maple, for example.

Solving for the Variables Note that y_1^3, y_2^3 are the roots of the quadratic polynomial $(y - y_1^3)(y - y_2^3) = y^2 - (y_1^3 + y_2^3)y + (y_1y_2)^3$. In light of the above identities, this polynomial can be written as

$$y^2 - (-2b^3 + 9abc - 27a^2d)y + (b^2 - 3ac)^3.$$

From which the roots y_1^3 and y_2^3 can be found using the quadratic formula. Taking cube roots gives y_1 and y_2, although only 3 of the 9 possible choices for cube roots will work correctly; namely those which satisfy the identity $y_1y_2 = b^2 - 3ac$. This identity will not always be satisfied if one simply uses principal roots, but one can use a modified principal root function.

Once y_1 and y_2 have been determined, one can solve the linear system b = - a(x$_1 + x_2 + x_3)$
$y_1 = a(x_1 + \omega x_2 + \omega^2 x_3)$
$y_2 = a(x_1 + \omega^2 x_2 + \omega x_3)$ for x_1, x_2, and x_3. The solution is given by x$_1 = (-b + y_1 + y_2)/(3a)$
$x_2 = (-b + \omega^2 y_1 + \omega y_2)/(3a)$
$x_3 = (-b + \omega y_1 + \omega^2 y_2)/(3a)$ or more compactly by

$$x = \frac{-b + \omega^n y_1 + \omega^{2n} y_2}{3a}$$

for $n = 0, 1, 2$.