Lecture 3 Relations

3.1 Definition

A **relation** is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the **domain** corresponds to **at least one** member of the **range**.

Relations is an extended study of Sets (from Lecture 2). Remember that given a Set A with **elements** {1, 2, 3} and Set B with **elements** {4, 5, 6}, we can write them down as follows:

$A = \{1, 2, 3\}\ and\ B = \{4, 5, 6\}$

In this module we learn about the Relations (**R**) of the elements inside multiple sets

So say we refer to elements inside Set **A** as $\boldsymbol{a}$ and elements inside Set B as $\boldsymbol{b}$, then the following two statements can be written:

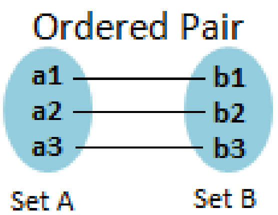
(1) $\boldsymbol{aRb}\ \ or\ (\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ which means a is related to b by R
(2) $\boldsymbol{aRb}\ \ or\ (\boldsymbol{a}, \boldsymbol{b}) \notin \boldsymbol{R}$ which means a is not related to b by R

When a is related to b by R, then R can be defined as follows:
$R = \{(a1, b1), (a2, b2), (a3, b3), \ldots\}$
Can you see the notation of R is similar to a normal set?

Q: What is R if elements in Set A and Set B (defined at the beginning) is related by R?
A: $R = \{(1, 4), (2, 5), (3, 6)\}$

3.2 Ordered Pair

The elements inside R are also called an ordered pair. As the name suggests the order matters which means (a, b) is not the same as (b, a) unless there's a condition/rule that says a = b.

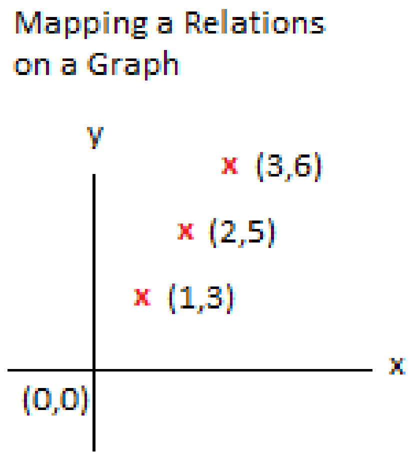


first element taken from A and
the second element taken from B

Note:

a) If A and B are sets, and A≠B, then the relations $A\ x\ B = \{(a, b) : a \in A\ and\ b \in B\}$
b) If A and B are sets, and A = B, then the relations $A\ x\ B\ equals\ to\ A\ x\ A = A^2$
c) Instead of the notation (a, b) we can also use the notation $a \sim b$
Example: $R = \{(1, 4), (2, 5), (3, 6)\}$ is the same as $R :\ 1{\sim}4, 2{\sim}5, 3{\sim}6$

R can be treated as sets of coordinates and can be mapped as follows:



3.3 Types of Relations

There are 4 types of Relations:

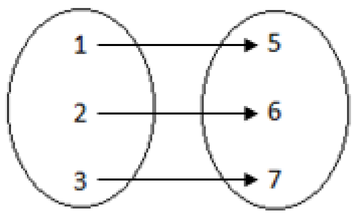
a) One – to – One
A relation A→B is **one-to-one relation** if no two elements of A is paired with the same element in B

b) One – to – Many
A relation A→B is **one-to-many relation** if an element of A is related to 2 or more elements of B

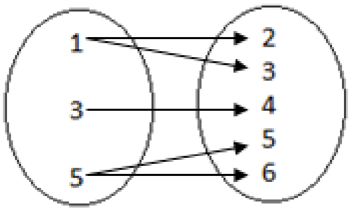
c) Many – to – One
A relation A→B is a **many-to one relation** if 2 or more elements of A are related to 1 element of B.

d) Many – to – Many
A relation A→B is **many-to-many relation** if 2 or more elements of A are related to 2 or more elements of B

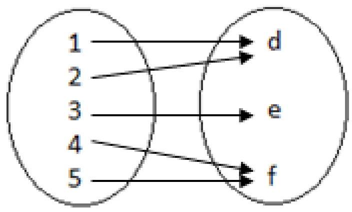
Can you guess which is which?



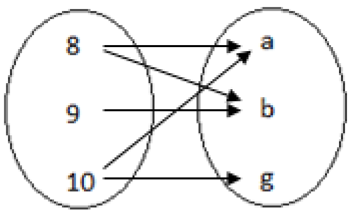
One to one



One to many



Many to One



Many to many

3.4 Relations on a Set (Composition)

Relations are set, so we can apply the usual set operations to them

A relation on the set A is a relation from A to A.
In other words, a relation on the set A is a subset of $A \times A$.

Example: Let A = {1, 2, 3, 4}.
If no condition, then R = A x A = {(a1,a1), (a2,a2), (a3,a3)…}
So, R = {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4) … }

But say A is given a condition. The condition is $\{(a, b) \mid a < b\}$? Which ordered pairs are in the relation?

$$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$$

R	1	2	3	4
1		*	*	*
2			*	*
3				*
4				

If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R1 \cup R2$, $R1 \cap R2$, or $R1 - R2$.
In each case, the result will be **another relation from A to B**.

When we combine two relations together, we call it **composite of relations**

Say R be the relation from set A to set B, $(a, b) \in R$ where $a \in A$ and $b \in B$
Say S be the relation from set B to set C, $(b, c) \in S$ where $b \in B$ and $c \in C$

If we are going to combine the relations R and S together, we use the symbol $\boldsymbol{S \circ R}$ to denote their **composite**

The ***composite*** of R and S is the relation consisting of ordered pairs (a, c)

Example 1:
Given $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d\}$ $C = \{x, y, z)$
And:
$R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ (A has been mapped to B)

$S = \{(b,x),(b,z),(c,y),(d,z)\}$ (B has been mapped to C)

(b, x) o (3, b) = (3, x)
(b, z) o (3, b) = (3, z)
Answer:
$S \circ R = \{(2,z),(3,x),(3,z)\}$

Remember, ordered pair, S→R and R→S is the same or <u>not</u>?

Find the composite of S → R?

Solution:
a) (b, x) in S maps to element (3, b) in R, which means (3, x) is in $S \circ R$
b) (b, z) in S maps to element (3, b) in R, which means (3, z) is in $S \circ R$
c) (c, y) in S maps to no element in R (ignore)
d) (d, z) in S maps to element (2, d) and (3, d) in R meaning (2, z) and (3, z) is in $S \circ R$

<u>Example 2:</u>
Let D and S be relations on $A = \{1,2,3,4\}$
Given:
$D = \{(a,b) \mid b = 5 - a\}$ *"b equals (5 – a)"*
$S = \{(a,b) \mid a < b\}$ *"a is smaller than b"*
Find $S \circ D$?

Solution:
a) From notes, A relation on the set A is a relation from A *to* A, so list down all elements of the relation A x A:
A x A = {(1, 1), (1, 2), (1, 3), (1, 4),
(2, 1), (2, 2), (2, 3), (2, 4),
(3, 1), (3, 2), (3, 3), (3, 4),
(4, 1), (4, 2), (4, 3), (4, 4)}

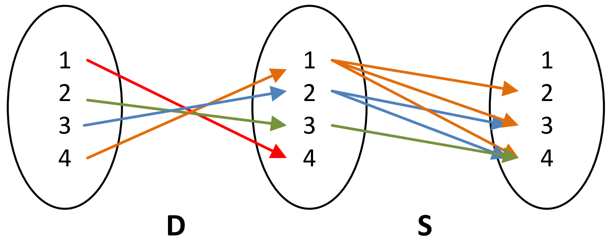
b) Find elements in A x A that satisfies D conditions:
D = {(a, 5-a)}
$D = \{(1,4),(2,3),(3,2),(4,1)\}$

c) Find elements in A x A that satisfies S conditions:
S = {a, a<b}
$S = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

d) Find $S \circ D$: (give me your answer)

Given
S = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}
D = {(1,4),(2,3),(3,2),(4,1)}

Draw the relation of S o D



<u>Answer</u>

SoD = {(2, 4),(3, 3),(3, 4),(4, 2),(4, 3),(4, 4)}

3.5 Inverse Relations

The inverse of a relation R from A to B is denoted R^{-1}, and defined from B to A as $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

<u>Example 1</u>:

What is R^{-1} ?
Given $R = \{(2,3), (2,5), (3,4), (3,6), (6,6)\}$

Solution:

$$R^{-1} = \{(3,2), (5,2), (4,3), (6,3), (6,6)\}$$

<u>Example 2:</u>

Given $R = \{(1,3), (2,1), (4,5), (6,6)\}$

Solutions $R^{-1} = \{(3,1), (1,2), (5,4), (6,6)\}$

3.6 Representing Relations using Matrix

Another way to represent relations is by using the **Zero – One Matrix**

If R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

m_{ij} = 1, if $(a_i, b_j) \in R$, and
m_{ij} = 0, if $(a_i, b_j) \notin R$

Note that for creating this matrix we first need to list the elements in A and B in an order

Example:

Suppose that A = {1, 2, 3} and B = { 1, 2}. Let R be the relation from *A* to *B* containing (*a*, *b*) if $a \in A,\ b \in B$, and a > b. How can we represent the relation R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

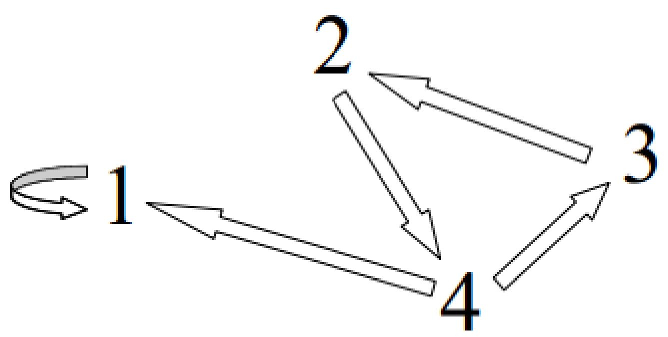
Solution:

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Note:
The matrices representing a **relation on a set** (a relation from A to A) are called **square** matrices.

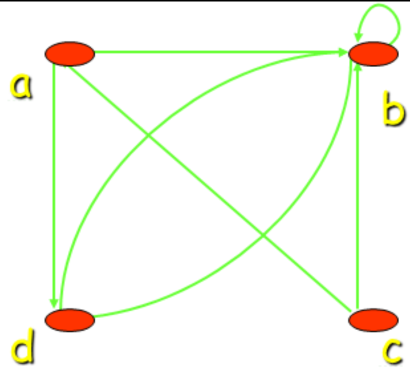
3.7 Representing Relations using Digraphs

When R is a relation on a set A, we can draw it using a **directed graph**. For example, if R = {(1,1), (2,4), (3,2), (4,1), (4,3)}, then its directed graph is:



A directed graph, or digraph, consists of a set *V* of vertices (or nodes) together with a set *E* of ordered pairs of elements of *V* called edges (or arcs).The vertex *a* is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge. We can use arrows to display graphs.

<table>
<tr>
<td>Example:
Display the digraph with

V = {a, b, c, d},
E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.</td>
<td>

An edge of the form (b, b) is called a loop.</td>
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3.8 Properties of Relations

3.8.1 Reflexive Relations

A Relation R on a set A is said to be ***reflexive*** if $(a, a) \in R$ where $a \in A$

(a, a) ∈ R

Examples:
The relation R on {1,2,3} given by R = {(1,1), (2,2), (2,3), (3,3)} is reflexive.
(All loops are present)

Why is R = {(1,1), (2,2), (3,3)} not reflexive on {1,2,3,4}?

Solution: Because (4,4) is missing

3.8.2 Symmetric Relations

A relation R on a set A is said to be ***symmetric*** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$

(a, b) ∈ R implies (b, a) ∈ R

Examples:
The relation R on {1, 2, 3} given by R = {(1,1), (1,2), (2,1), (1,3), (3,1)} is symmetric.
(All paths are 2-way)

Why is R = {(1,2), (2,1), (3,1)} not symmetric?

Solution: Because (1,3) is missing

3.8.3 Transitive Relations

A relation R on a set A is said to be ***transitive*** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for every $a, b, c \in A$

If (a, b) $\in R$ and (b, c) $\in R$, this implies (a, c) $\in R$

Examples:
The relation R on {1,2,3} given by R = {(1,1), (1,2), (2,1), (2,2), (2,3), (1,3)} is transitive.
(If I can get from one point to another in 2 steps, then I can get there in 1 step)

Why is R = {(1,2), (2,3), (1,3), (2,1)} not transitive?

Because (1,1) and (2,2) are missing

3.8.4 Equivalence Relations

If a relation is (i) transitive, (ii) symmetric, (iii) transitive, then it is called an equivalence relation

Is {(1,1), (2,2), (3,3)} reflexive? symmetric? transitive?

Yes! Yes! Yes!