Lecture 6 Matrix

6.1 What is a Matrix

A matrix is an array of numbers and it is usually in this form:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

This one has 2 rows and 3 columns

6.2 How to multiply Matrix

To multiply a matrix by a single number:

$$2 \times \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix} = \begin{bmatrix} 2\times 6 & 2\times 4 & 2\times 24 \\ 2\times 1 & 2\times -9 & 2\times 8 \end{bmatrix} = \begin{bmatrix} 12 & 8 & 48 \\ 2 & -18 & 16 \end{bmatrix}$$

To multiply a matrix with another matrix:

Step 1: Check if matrix dimension agrees

2x~~2~~ and ~~2~~x3 = 2x3 matrix
4x~~6~~ and ~~6~~x2 = 4x2 matrix

Step 2: Use the dot operator to calculate:

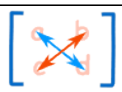
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a,b).(e,g) & (a,b).(f,h) \\ (c,d).(e,g) & (c,d).(f,h) \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Another example:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix} = \begin{bmatrix} (a,b,c).(g,i,k) & (a,b,c).(h,j,l) \\ (d,e,f).(g,i,k) & (d,e,f).(h,j,l) \end{bmatrix}$$
$$= \begin{bmatrix} \quad \end{bmatrix}$$

6.3 Determinant of a Matrix

- The determinant of a matrix is a special number that can be calculated from the matrix. It is useful in performing systems of linear equations, calculus, and others.
- If A is a matrix, then |A| is the determinant of the matrix A
- Matrix determinant can only be derived from Square Matrix such as the 2x2, 3x3 or 4x4 matrix

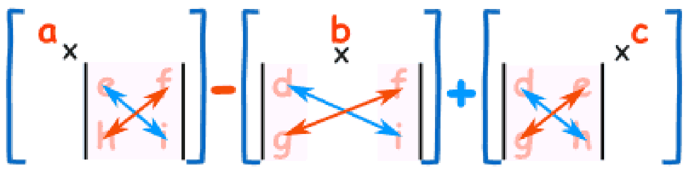
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	
$\lvert A \rvert = ad - bc$	• Blue means positive (+ad) • Red means negative (-bc)

Try the following:

a) B = $\begin{bmatrix} 4 & 1 \\ -9 & 5 \end{bmatrix}$ What is |B|? 29

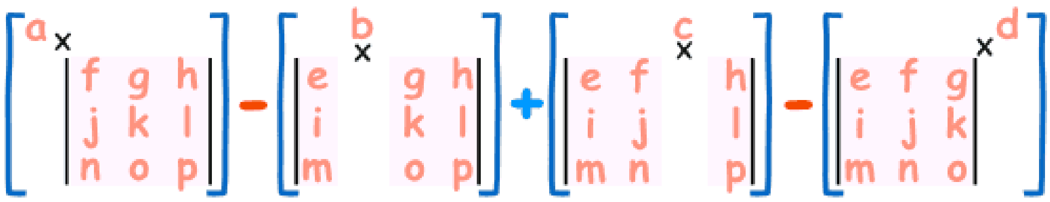
b) S = $\begin{bmatrix} 2 & 3 & 1 \\ 6 & 5 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ What is |S|? -4

For a larger matrix such as the 3x3 here is how you derive the determinant

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	
$\lvert A \rvert = a(ei - fh) - b(di - gf)$ $+c(dh - eg)$	$\lvert A \rvert = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

So for a 4x4 matrix and higher (taken from http://www.mathsisfun.com)

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the + - + - pattern (+a... -b... +c... -d...). This is important to remember.

The pattern continues for 5×5 matrices and higher.

6.4 Solving System of Linear Equations using the Cramer's Method

You've seen how we derive the determinant from a square matrix. You'll be using it to solve linear equations system. Say you've got a 2x2:

$$3x - y = 7$$
$$-5x + 4y = -2$$

You can change this into a "coefficient matrix"

$$\begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

Step 1: Solve the determinant D for the 2x2 matrix

$$D = \begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}$$
$$|D| = ad - bc = 12 - (5) = 7$$

Step2: Solve the determinant for the x-guys

$$D_x = \begin{bmatrix} 7 & -1 \\ -2 & 4 \end{bmatrix}$$

$$|D_x| = 26$$

Step3: Solve the determinant for the y-guys

$$D_y = \begin{bmatrix} 3 & 7 \\ -5 & -2 \end{bmatrix}$$

$$|D_y| = 29$$

Step 4: Get the x and y values

$$x = \frac{D_x}{D} = \frac{26}{7}$$

$$y = \frac{D_y}{D} = \frac{29}{7}$$

Try solving the following using Cramer's Method:

$$x + y - z = 4$$
$$x - 2y + 3z = -6$$
$$2x + 3y + z = 7$$

Step 1: Put in coefficient matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$$

Step 2: Find Determinants (D, Dx, Dy, Dz)

D = $1 \times \begin{bmatrix} -2 & 3 \\ 3 & 1 \end{bmatrix} - 1 \times \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} + (-1) \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

$D = 1 \times |-2(1) - (3)(3)| - 1|1(1) - 2(3)| + (-1)|1(3) - 2(-2)|$

$D = 1(-11) - 1(-5) + (-1)(7)$

$D = -11 + 5 - 7 = -13$

Dx = -13

Dy = -26

Dz = 13

X = 1, y=2, z= -1

6.5 Matrix Transpose

Say you have the following:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The matrix transpose for M is

$$M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

What is the transpose for the following?

$$F = \begin{bmatrix} 2 & 6 & 10 \\ 1 & 3 & 3 \\ 6 & 1 & 0 \end{bmatrix}$$

6.6 Finding Inverse of a Matrix (using Minors, Cofactors and Adjugate)

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!!! (Notes taken from http://www.mathsisfun.com)

Say you are asked to find the inverse to the Matrix A, given as follows:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1: Perform matrix of Minors:

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

$$\cdots$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}}$$

Step 2: Perform Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, you need to change the sign of alternate cells, like this:

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}} \rightsquigarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}}_{\text{Matrix of CoFactors}}$$

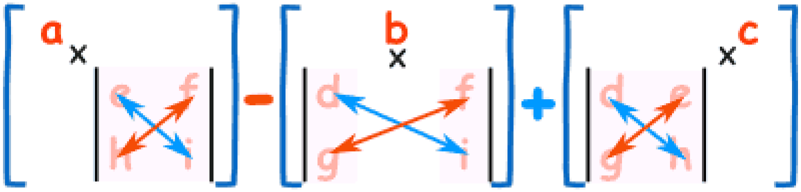
Step 3: Perform Matrix Transpose

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

6.7 Step 4: Multiply by 1/Determinant ($\frac{1}{D}$)

Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".



So: multiply the top row elements by their matching "minor" determinants:

Determinant = 3×2 - 0×2 + 2×2 = **10**

And now multiply the Adjugate by 1/Determinant:

$$A^{-1} = \frac{1}{10}\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{Adjugate}} = \underbrace{\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}}_{\text{Inverse}}$$

And we are done!