Karnaugh Maps

Objectives

1. Given a function (completely or in completely specified) of three to five variable, plot it on a Karnaugh map.
   The function may be given in minterm, maxterm, or algebraic form.

2. Determine the essential prime implicants of a function from a map.

3. Obtain the minimum sum-of-products or minimum product-of-sums form of a function from the map.

4. Determine all of the prime implicants of a function from a map.

5. Understand the relation between operations performed using the map and the corresponding algebraic operation.

Circuit Synthesis

- there are 5 ways to describe a Logic Expression

1) Truth Table
2) Minterm List
3) Canonical Sum
4) Maxterm List
5) Canonical Product

- we can directly synthesis circuits from SOP and POS expressions

SOP = AND-OR structure

POS = OR-AND structure

Circuit Synthesis

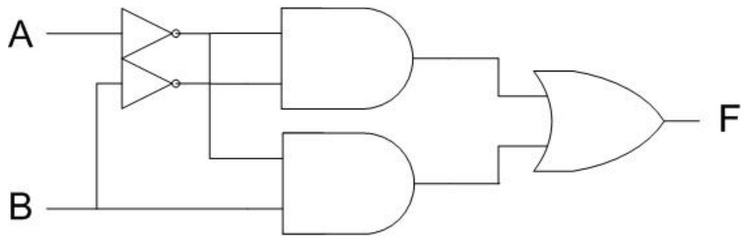
- ## Circuit Synthesis

- For the given Truth Table, synthesize the SOP and POS Logic Diagrams

Row	A B	Minterm	Maxterm	F
0	0 0	A'·B'	A+B	1
1	0 1	A'·B	A+B'	1
2	1 0	A·B'	A'+B	0
3	1 1	A·B	A'+B'	0

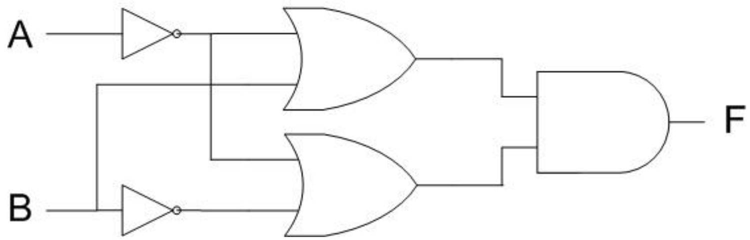
Minterm List & SOP

$F = \Sigma_{A,B}\,(0,1) = A' \cdot B' + A' \cdot B$



Maxterm List & POS

$F = \Pi_{A,B}\,(2,3) = (A'+B) \cdot (A'+B')$



Logic Minimization

Logic Minimization

- We've seen that we can directly translate a Truth Table into a SOP/POS and in turn a Logic

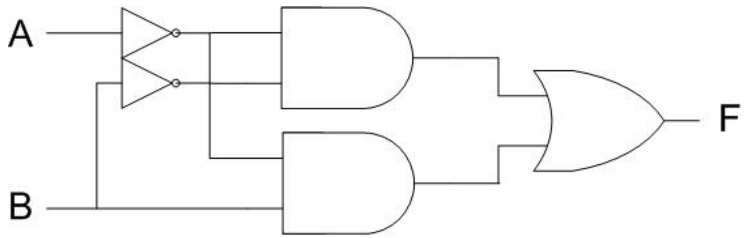
- However, this type of expression is NOT minimized

ex)

Row	A B	Minterm	Maxterm	F
0	0 0	A'·B'	A+B	1
1	0 1	A'·B	A+B'	1
2	1 0	A·B'	A'+B	0
3	1 1	A·B	A'+B'	0

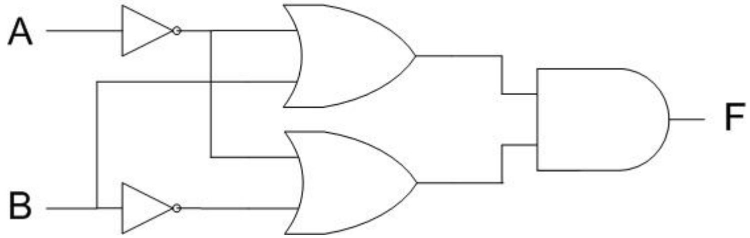
Minterm List & SOP

$F = \Sigma_{A,B}(0,1) = A' \cdot B' + A' \cdot B$



Maxterm List & POS

$F = \Pi_{A,B}(2,3) = (A'+B) \cdot (A'+B')$



- # Logic Minimization

  - using our Axioms and Theorems, we can manually minimize the expressions…

Minterm List & SOP	Maxterm List & POS
$F = A' \cdot B' + A' \cdot B$	$F = (A'+B) \cdot (A'+B')$
$F = A' \cdot (B'+B) = A'$	$F = A' + (B' \cdot B) = A'$

  - doing this by hand can be difficult and requires that we recognize patterns associated with our 5 Axioms and our 15+ Theorems

- # Karnaugh Maps

  - a graphical technique to minimize a logic expression

5.1 Minimum Forms of Switching Functions

1. Combine terms by using $XY' + XY = X$

Do this repeatedly to eliminates as many literals as possible.

A given term may be used more than once because $X + X = X$

2. Eliminate redundant terms by using the consensus theorems.

5.1 Minimum Forms of Switching Functions

Example: Find a minimum sum-of-products

$$F(a,b,c)=\sum m(0,1,2,5,6,7)$$

$$F=a'b'c'+a'b'c+a'bc'+ab'c+abc'+abc$$

$$= a'b' + b'c + bc' + ab$$

$$F=a'b'c'+a'b'c+a'bc'+ab'c+abc'+abc$$

$$= a'b' + bc' + ac$$

5.1 Minimum Forms of Switching Functions

Example: Find a minimum product-of-sums

$$(A+B'+C+D')(A+B'+C'+D')(A+B'+C'+D)(A'+B'+C'+D)(A+B+C'+D)(A'+B+C'+D)$$

$$= (A+B'+D') \quad (A+B'+C') \quad (B'+C'+D) \quad (B+C'+D)$$

$$= (A+B'+D') \quad (A+B'+C') \quad (C'+D)$$

$$= (A+B'+D')(C'+D)$$

Eliminate by consensus

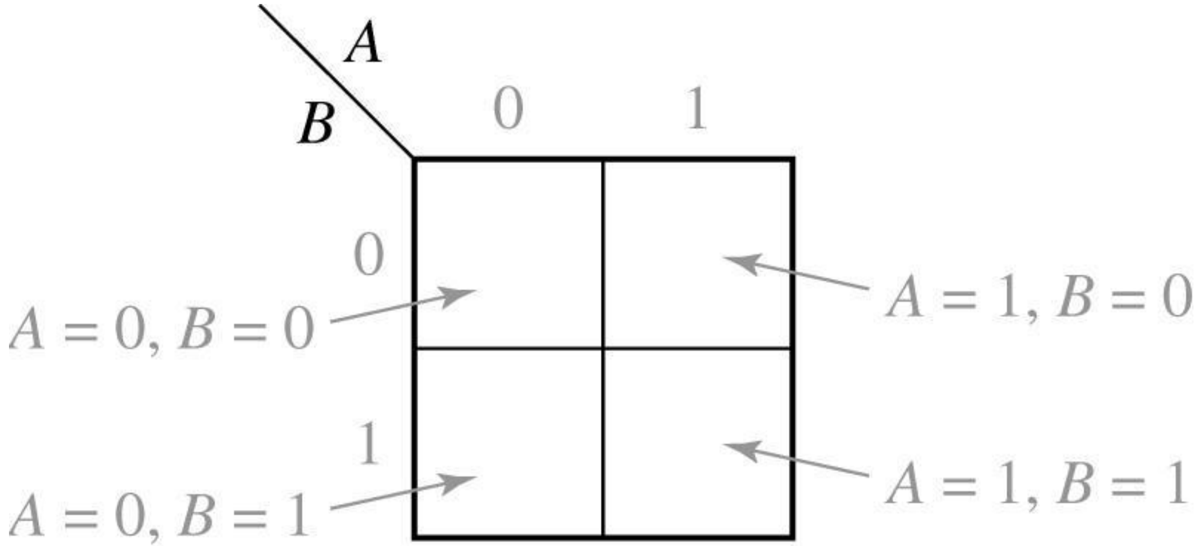
Problems in Algebraic Simplification

1) The procedures are difficult to apply in a systematic way.

2) It is difficult to tell when you have arrived at a minimum solution. (minimum SOP, POS) => Karnaugh map (K-map) is the solution.

5.2 Two- and Three-Variable Karnaugh Maps

A 2-variable Karnaugh Map

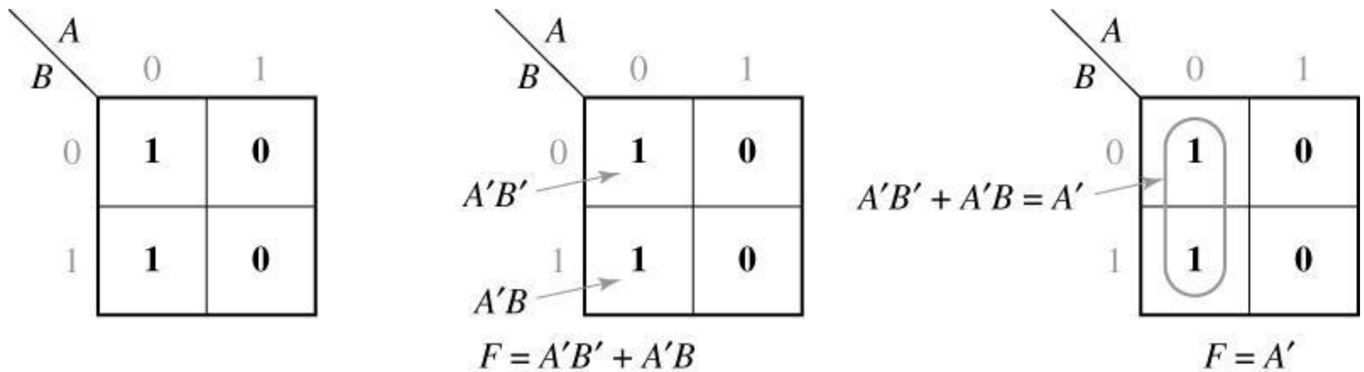


5.2 Two- and Three-Variable Karnaugh Maps

Truth Table for a function F

A B	F
0 0	1
0 1	1
1 0	0
1 1	0

(a)



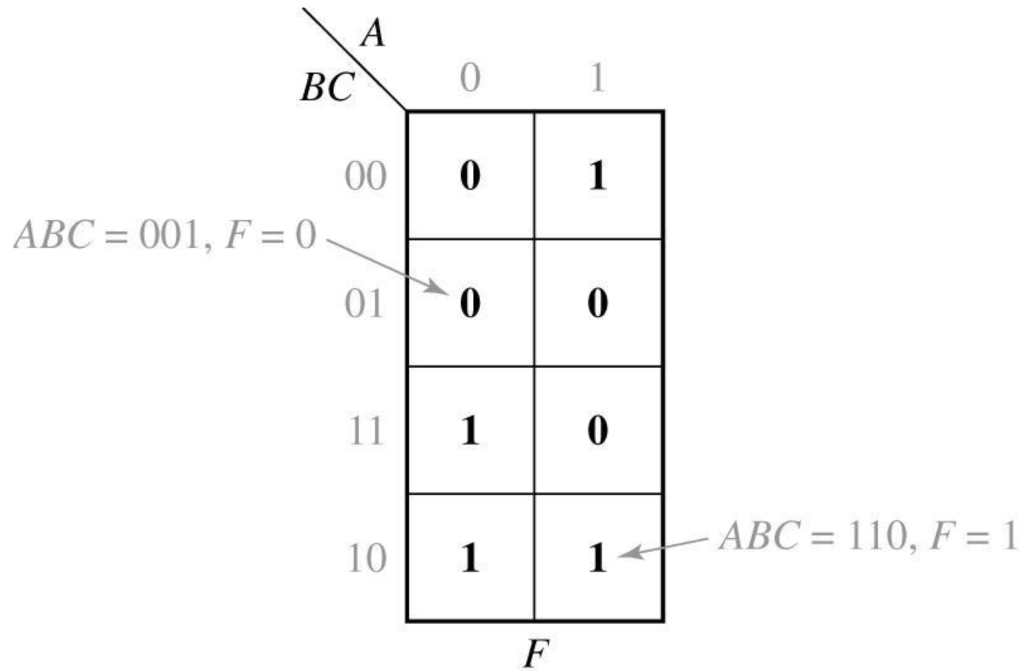
(b) (c) (d)

5.2 Two- and Three-Variable Karnaugh Maps

Truth Table and Karnaugh Map for Three-Variable Function

A B C	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

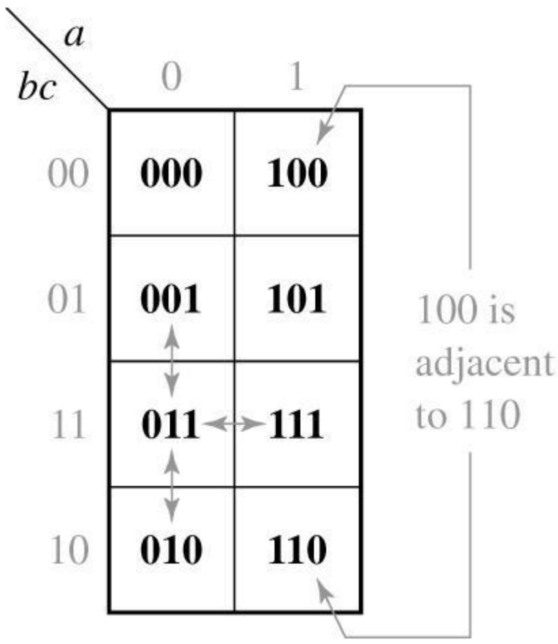
(a)



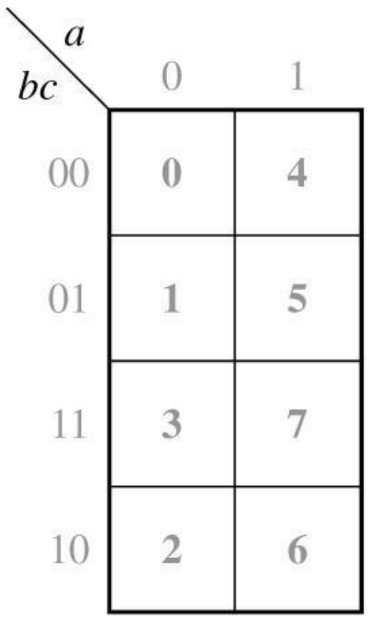
(b)

5.2 Two- and Three-Variable Karnaugh Maps

Location of Minterms on a Three-Variable Karnaugh Map



(a) Binary notation



(b) Decimal notation

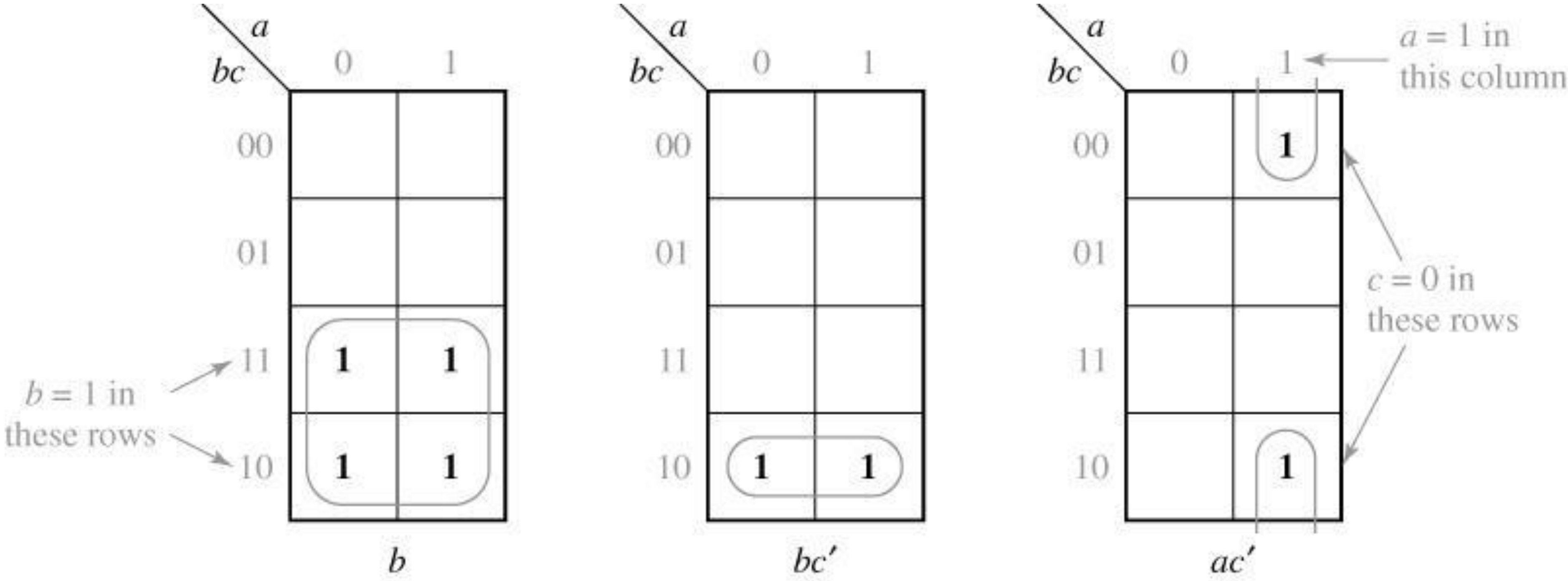
5.2 Two- and Three-Variable Karnaugh Maps

Karnaugh Map of F(*a*, *b*, *c*) = ∑ m(1, 3, 5) = ∏(0, 2, 4, 6, 7)

bc \ *a*	0	1
00	**0** $_{0}$	**0** $_{4}$
01	**1** $_{1}$	**1** $_{5}$
11	**1** $_{3}$	**0** $_{7}$
10	**0** $_{2}$	**0** $_{6}$

5.2 Two- and Three-Variable Karnaugh Maps

Karnaugh Maps for Product Terms

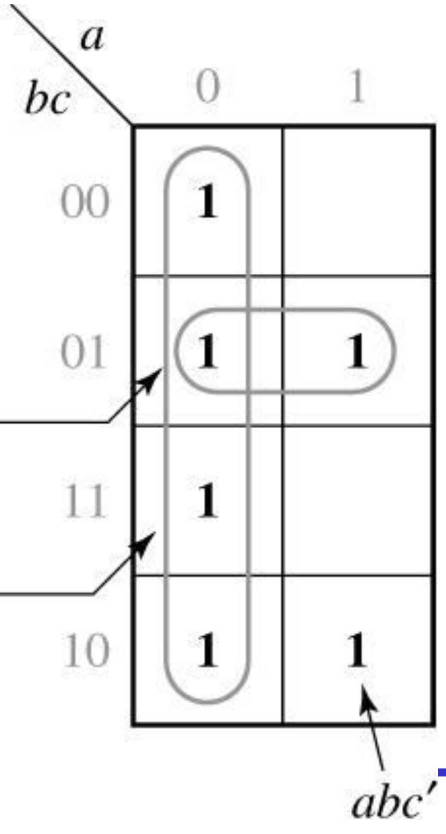


5.2 Two- and Three-Variable Karnaugh Maps

Given Function

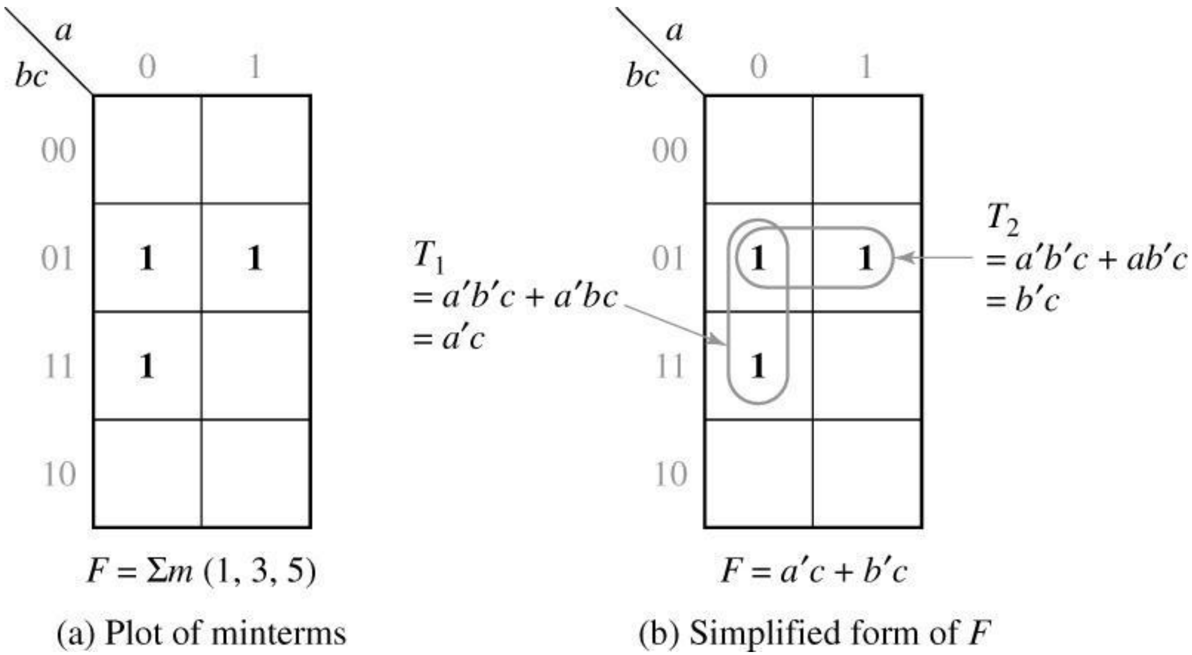
$$f(a,b,c) = abc' + b'c + a'$$

1. The term abc' is 1 when $a = 1$ and $bc = 10$, so we place a 1 in the square which corresponds to the $a = 1$ column and the $bc = 10$ row of the map.
2. The term $b'c$ is 1 when $bc = 01$, so we place 1's in both squares of the $bc = 01$ row of the map.
3. The term a' is 1 when $a = 0$, so we place 1's in all the squares of the $a = 0$ column of the map. (Note: Since there already is a 1 in the $abc = 001$ square, we do not have to place a second 1 there because $x + x = x$.)



5.2 Two- and Three-Variable Karnaugh Maps

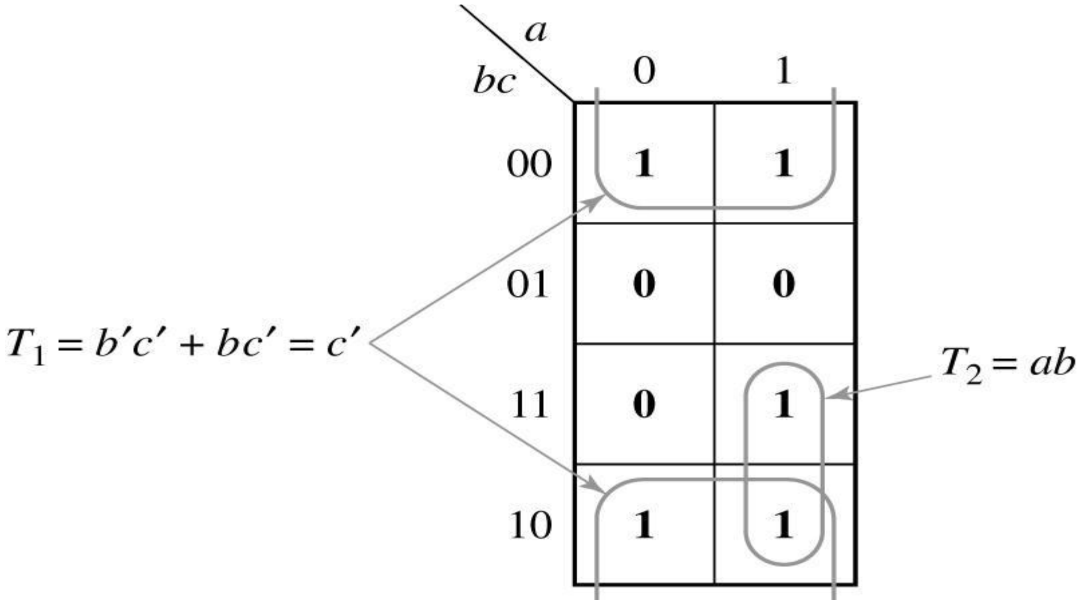
Simplification of a Three-Variable Function



$$F = T_1 + T_2 = a'c + b'c$$

5.2 Two- and Three-Variable Karnaugh Maps

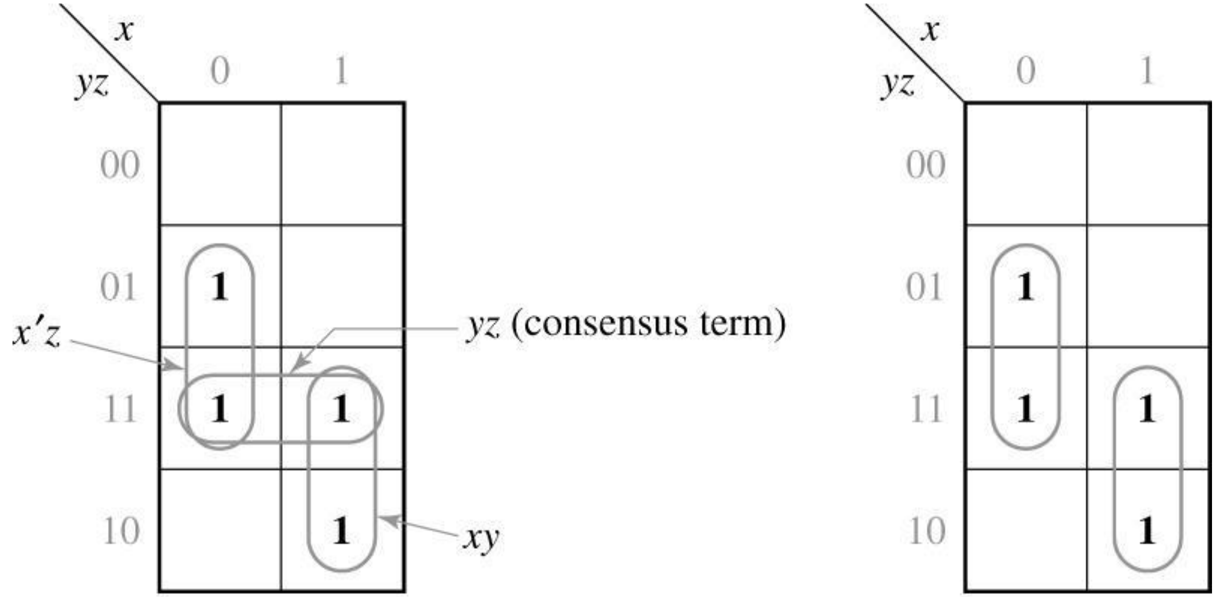
Complement of Map in Figure 5-6(a)



$$F = T_1 + T_2 = c' + ab$$

5.2 Two- and Three-Variable Karnaugh Maps

Karnaugh Maps Which Illustrate the Consensus Theorem



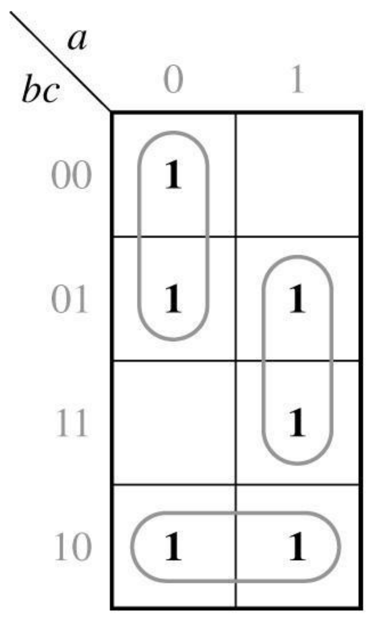
$$xy + x'z + yz = xy + x'z$$

Consensus term is redundant

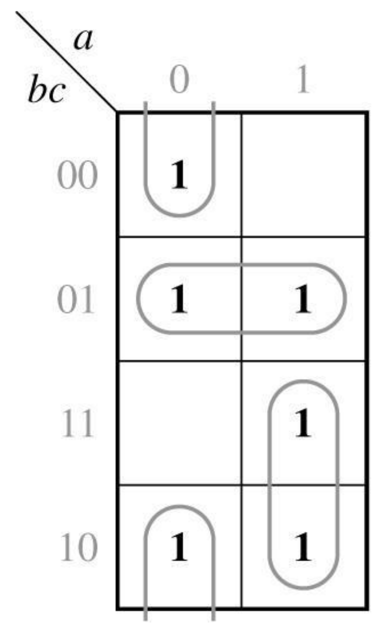
5.2 Two- and Three-Variable Karnaugh Maps

Function with Two Minimal Forms

$$F = \sum m(0,1,2,5,6,7)$$



$F = a'b' + bc' + ac$



$F = a'c' + b'c + ab$

5.3 Four-Variable Karnaugh Maps

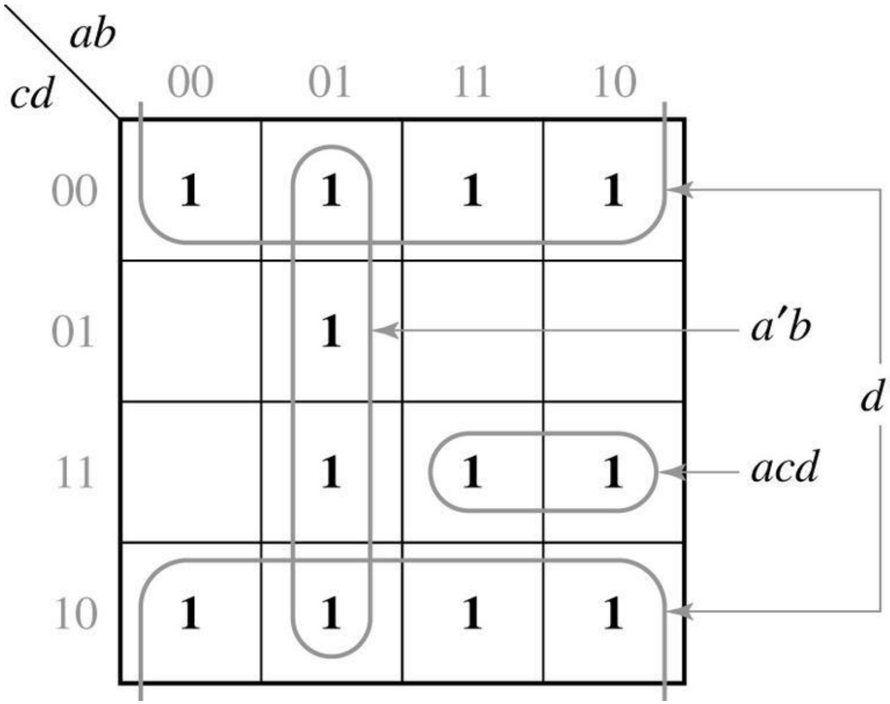
Location of Minterms on Four-Variable Karnaugh Map

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

5.3 Four-Variable Karnaugh Maps

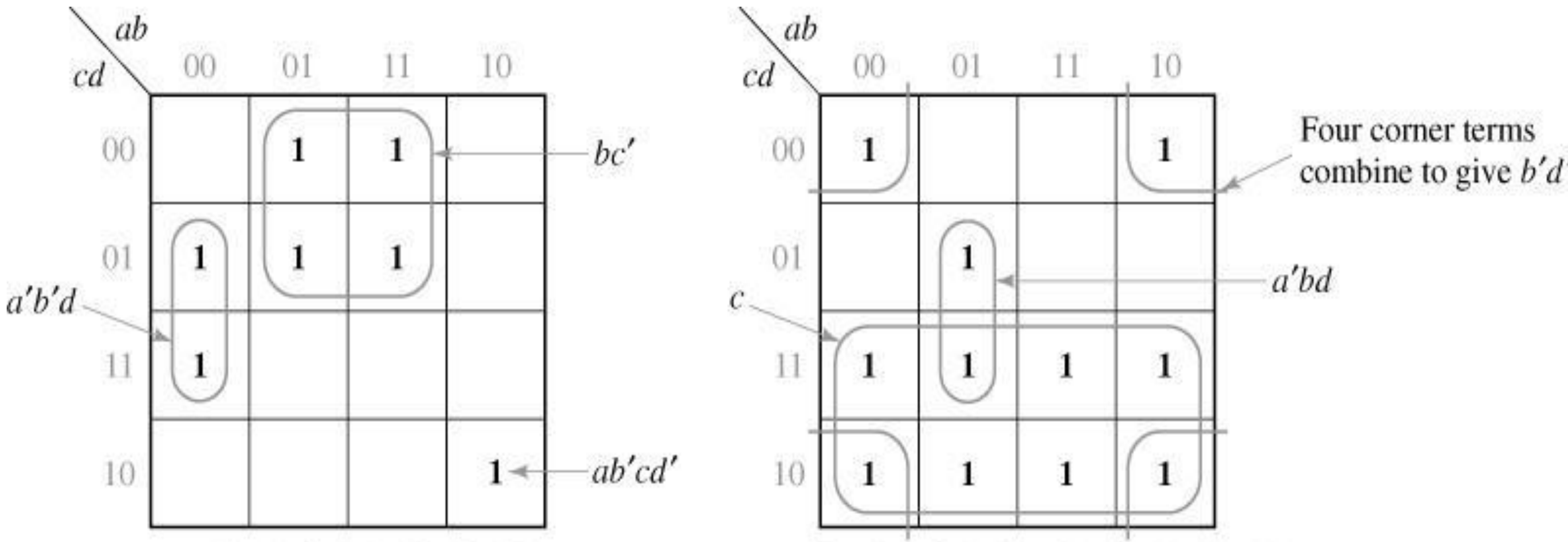
Plot of ***acd + a'b + d'***

$$f(a,b,c,d) = acd + a'b + d'$$



5.3 Four-Variable Karnaugh Maps

Simplification of Four-Variable Functions



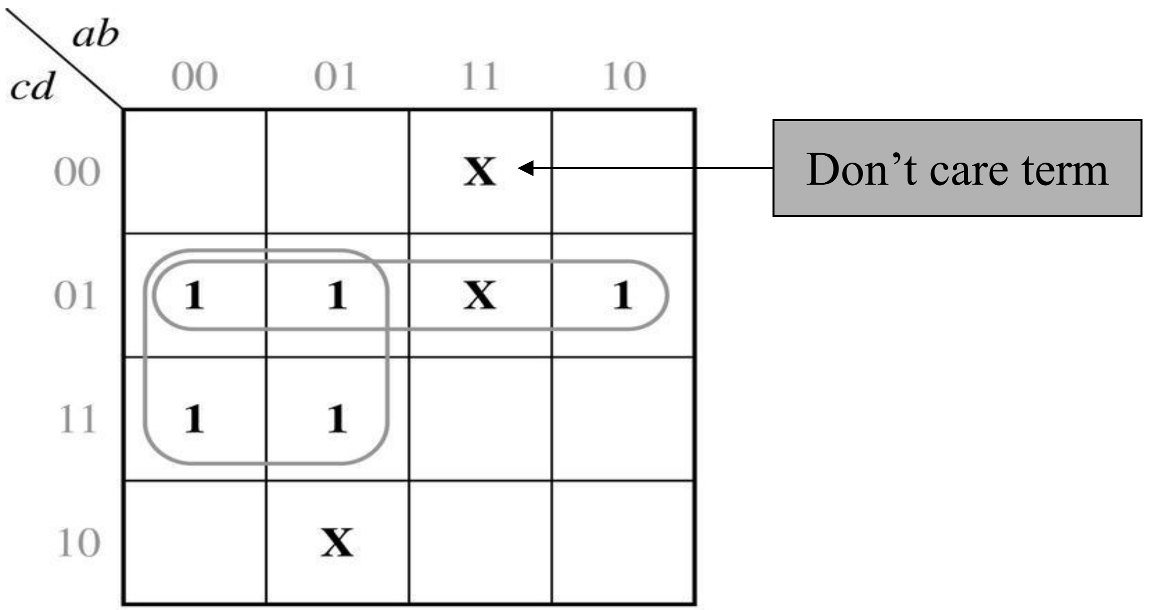
$f_1 = \Sigma m(1, 3, 4, 5, 10, 12, 13)$
$= bc' + a'b'd + ab'cd'$

$f_2 = \Sigma m(0, 2, 3, 5, 6, 7, 8, 10, 11, 14, 15)$
$= c + b'd' + a'bd$

(a) (b)

5.3 Four-Variable Karnaugh Maps

Simplification of an Incompletely Specified Function



$$f = \Sigma m(1, 3, 5, 7, 9) + \Sigma d(6, 12, 13)$$
$$= a'd + c'd$$

5.3 Four-Variable Karnaugh Maps

Figure 5-14

1's of f

$$f = x'z' + wyz + w'y'z' + x'y$$

0's of f

$$f' = y'z + wxz' + w'xy$$

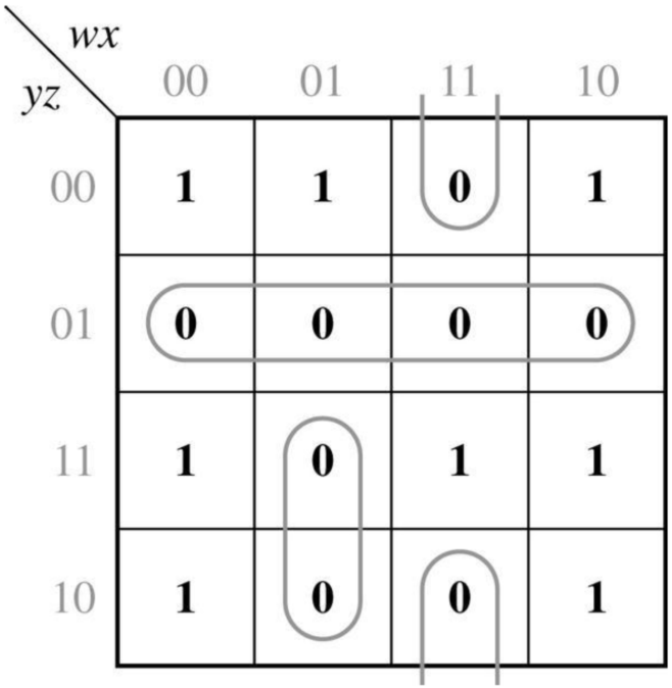
yz \ wx	00	01	11	10
00	1	1	0	1
01	0	0	0	0
11	1	0	1	1
10	1	0	0	1

$$f = (y + z')(w' + x'z)(w + x' + y')$$

minimum product of sum for f

5.3 Four-Variable Karnaugh Maps

Figure 5-14

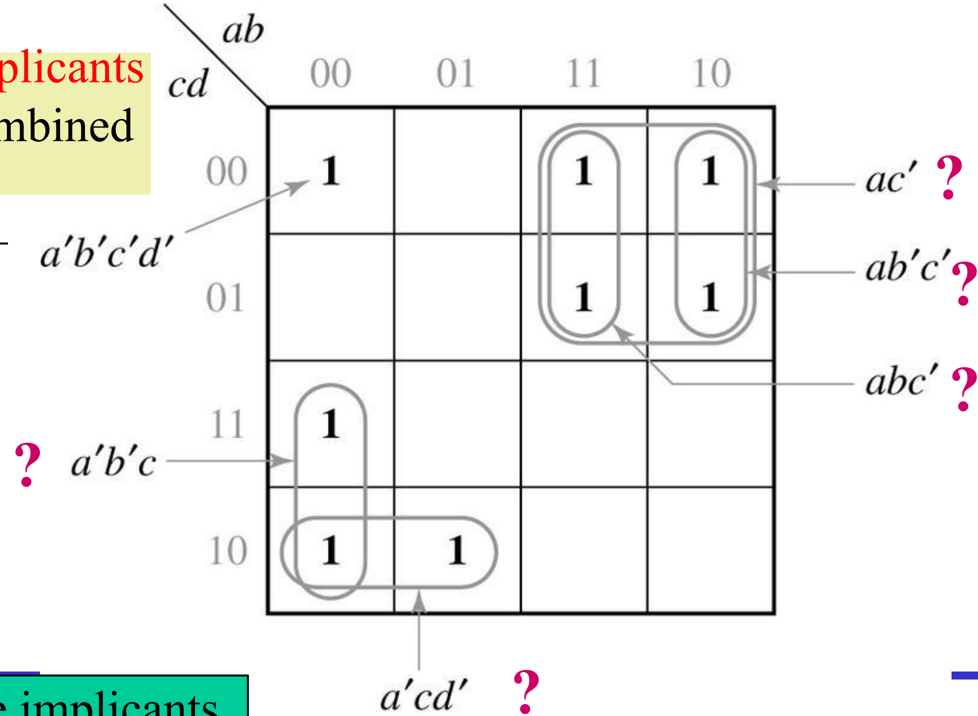


5.4 Determination of Minimum Expressions Using Essential Prime Implicants

- ***Implicants of F*** : Any single '1' or any group of "1's which can be combined together on a Map

- ***prime Implicants of F*** : A product term if it can not be combined with other terms to eliminate variable

It is not Prime implicants since it can be combined with other terms

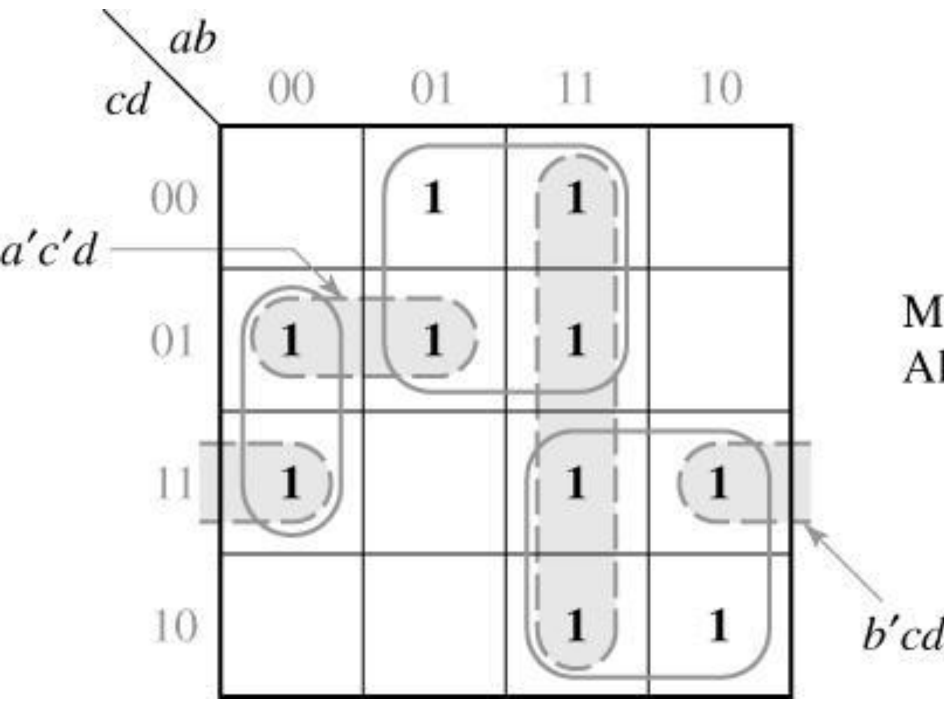


Prime implicants

Prime implicants

5.4 Determination of Minimum Expressions Using Essential Prime Implicants

Determination of All Prime Implicants

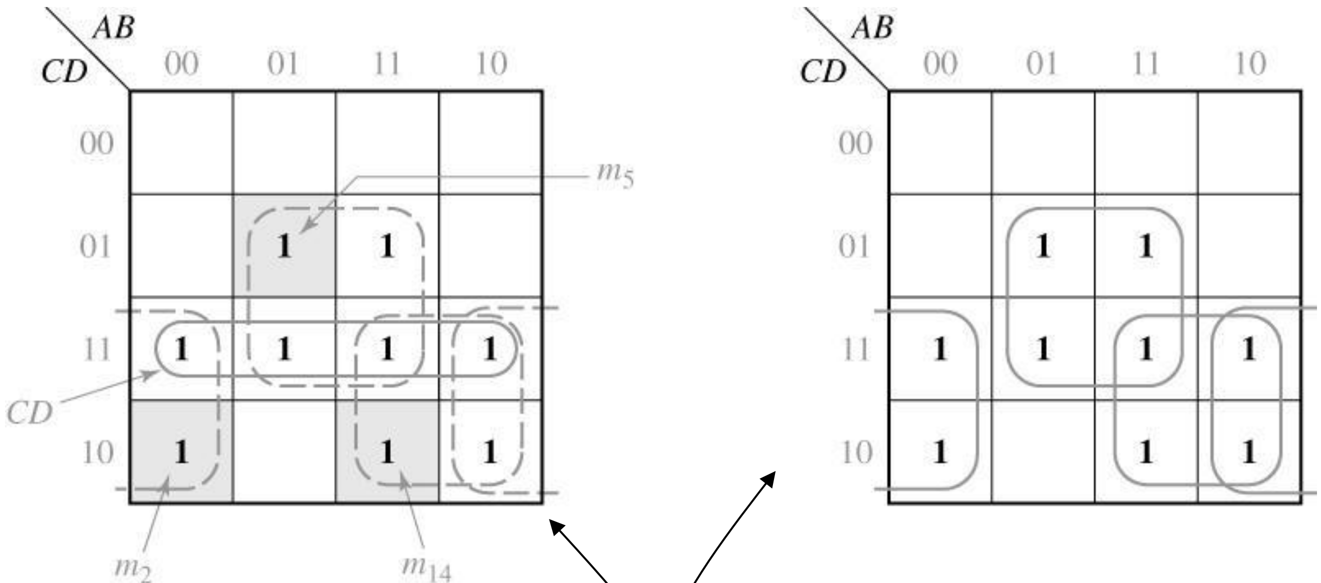


Minimum solution: $F = a'b'd + bc' + ac$

All prime implicants: $a'b'd$, bc', ac, $a'c'd$, ab, $b'cd$

5.4 Determination of Minimum Expressions Using Essential Prime Implicants

Because all of the prime implicants of a function are generally not needed in forming the minimum sum of products, selecting prime implicants is needed.



$f = CD + BD + B'C + AC$

(a)

$f = BD + B'C + AC$

(b)

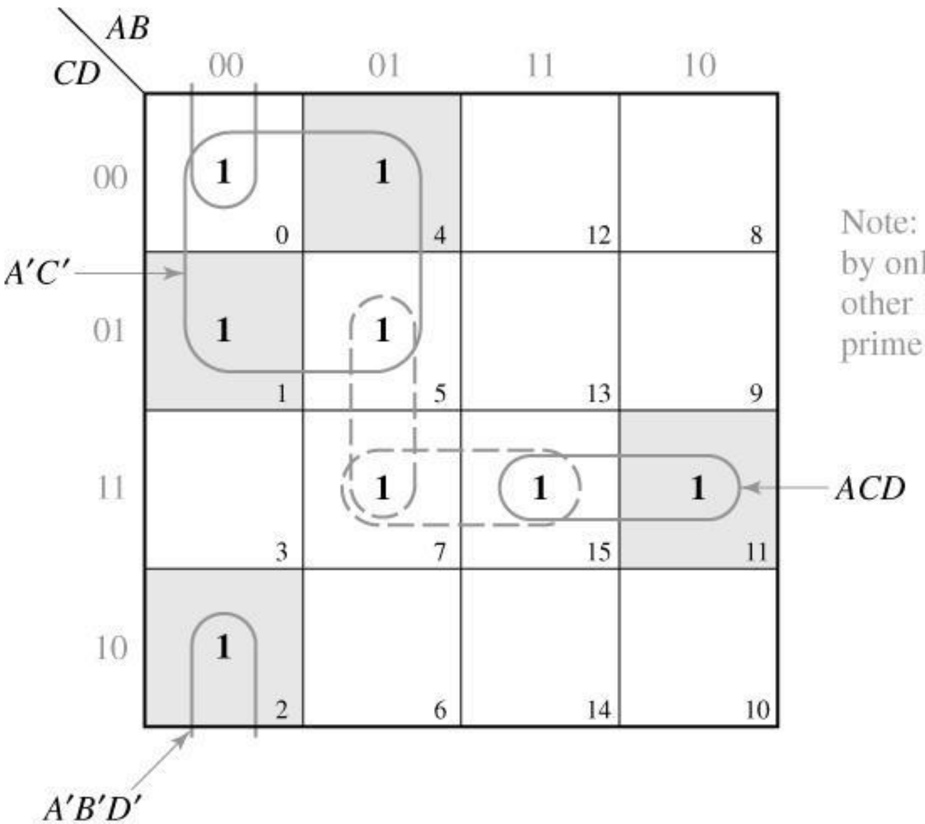
- ***CD*** **is not needed to cover for minimum expression**

-*B'C* , *AC* , *BD* are "essential" prime implicants

- ***CD*** **is not an "essential " prime implicants**

5.4 Determination of Minimum Expressions Using Essential Prime Implicants

1. **First, find essential prime implicants**
2. **If minterms are not covered by essential prime implicants only, more prime implicants must be added to form minimum expression.**

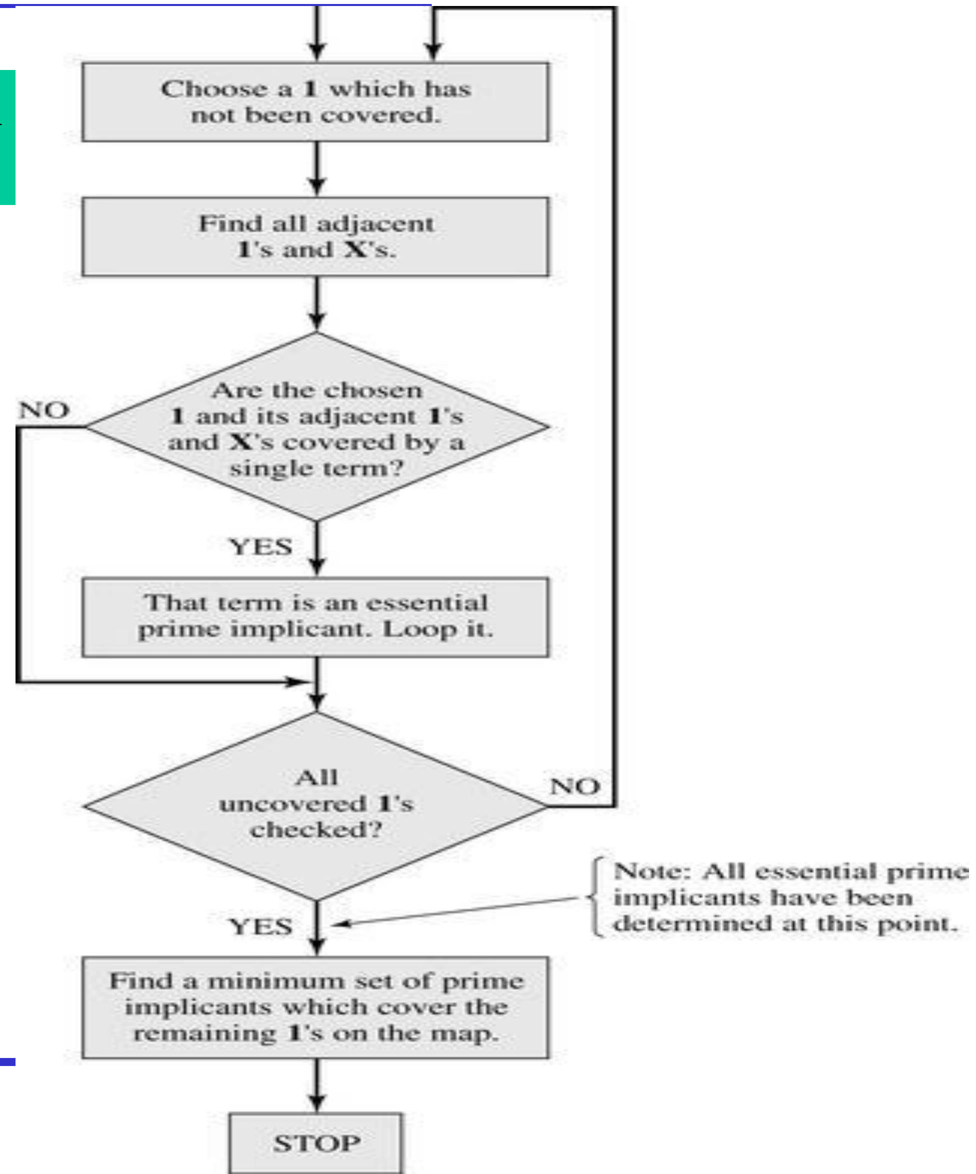


Note: 1's shaded in blue are covered by only one prime implicant. All other 1's are covered by at least two prime implicants.

$$A'C + A'B'D' + ACD + \begin{cases} A'BD \\ \text{or} \\ BCD \end{cases}$$

5.4 Determination of Minimum Expressions Using Essential Prime Implicants

Flowchart for Determining a Minimum Sum of Products Using a Karnaugh Map

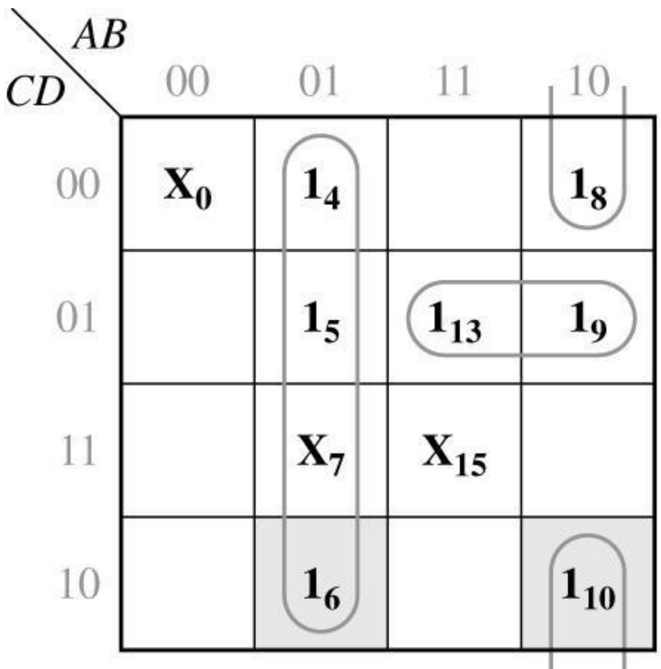


5.4 Determination of Minimum Expressions Using Essential Prime Implicants

1) ***A'B*** **covers I_6 and its adjacent → essential PI**

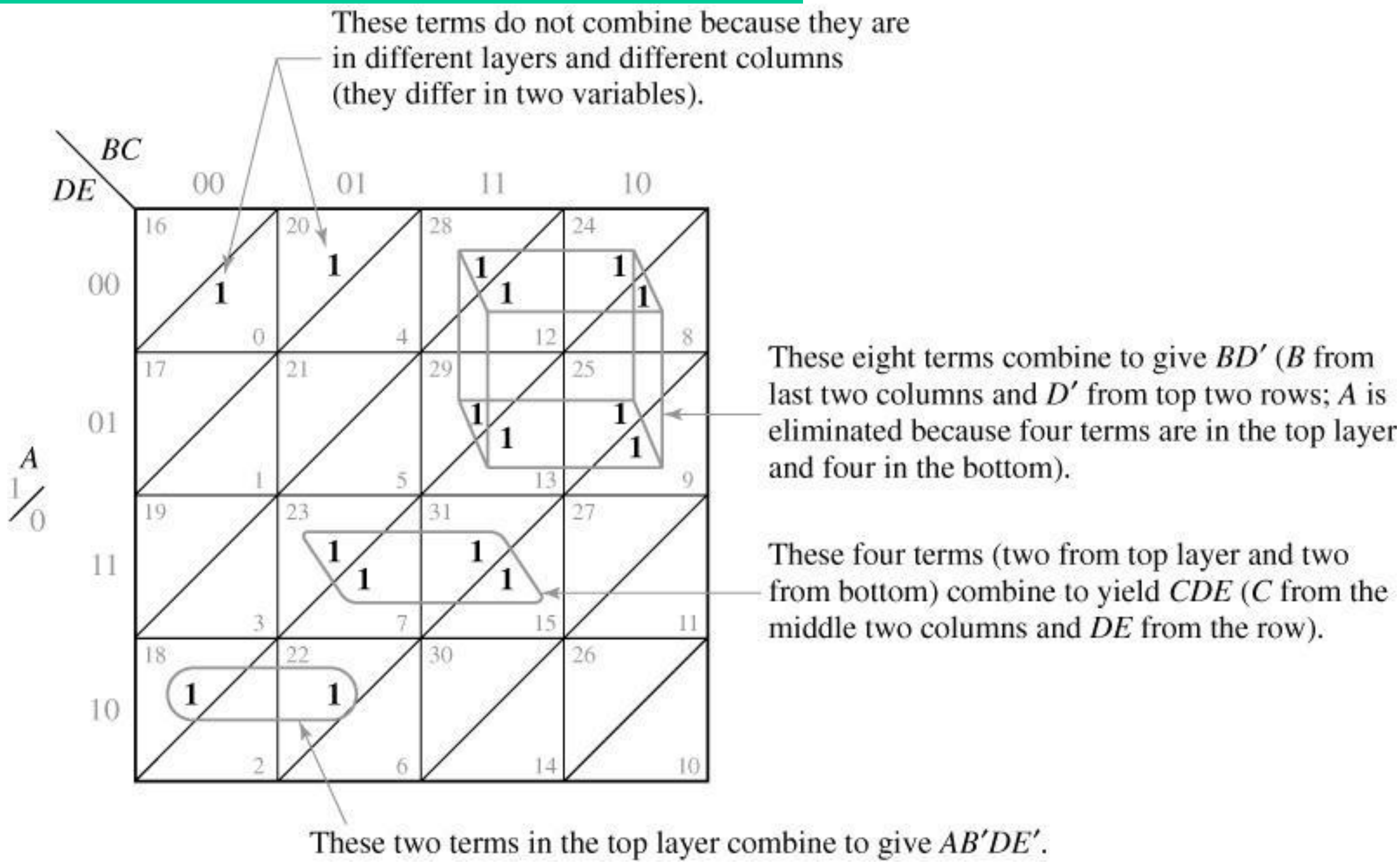
2) ***AB'D'*** **covers I_{10} and its adjacent → essential PI**

3) ***AC'D*** **is chosen for minimal cover →** ***AC'D*** **is not an essential PI**



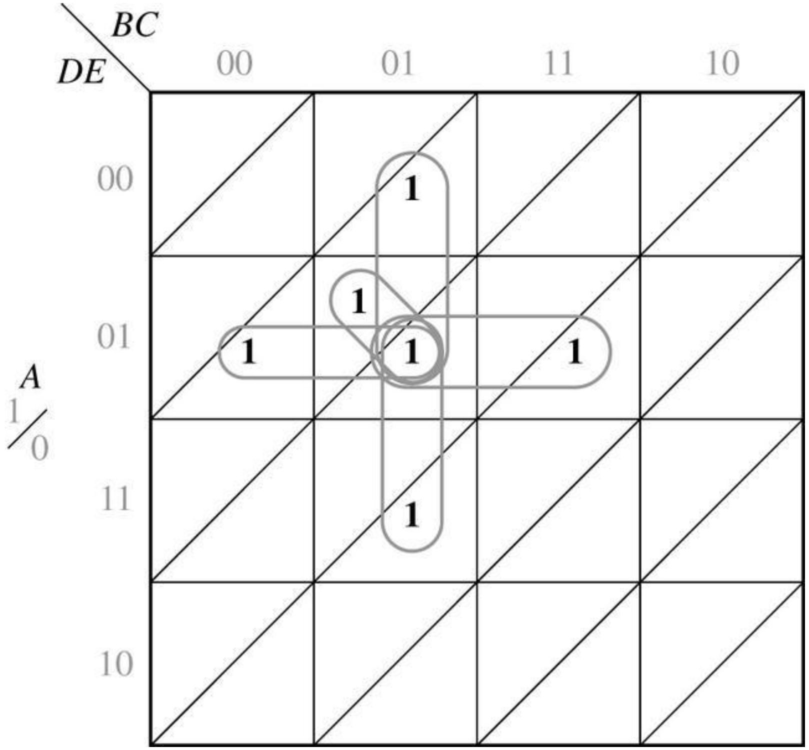
Shaded 1's are covered by only one prime implicant.

Five-Variable Karnaugh Map



5.5 Five-Variable Karnaugh Maps

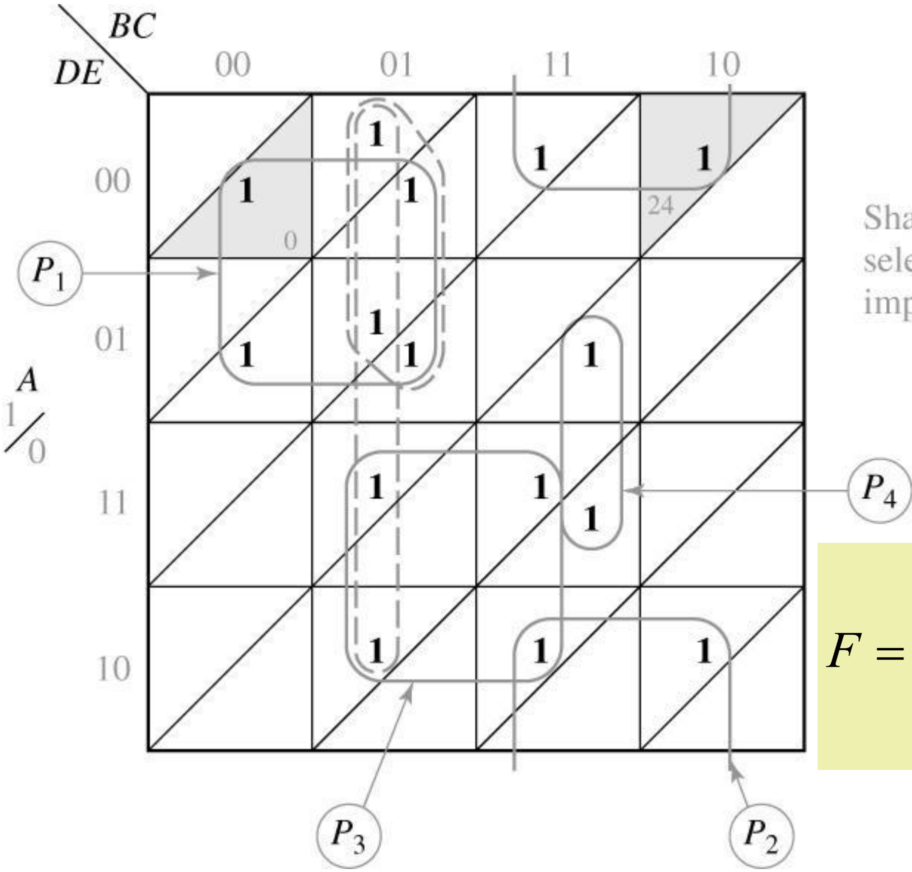
Figure 5-22



5.5 Five-Variable Karnaugh Maps

Figure 5-23

$$F(A,B,C,D,E)=\sum m(0,1,4,5,13,15,20,21,22,23,24,26,28,30,31)$$



Shaded 1's are used to select essential prime implicants.

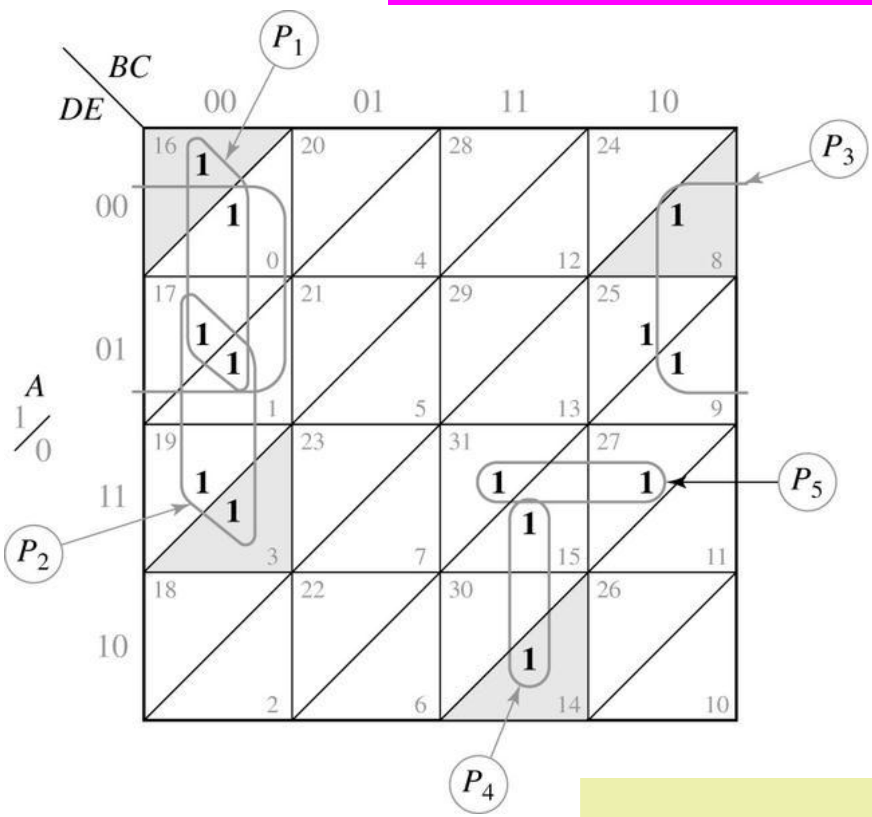
Resulting minimum solution

$$F=\underbrace{A'B'D'}_{P_1}+\underbrace{ABE'}_{P_2}+\underbrace{ACD}_{P_3}+\underbrace{A'BCE}_{P_4}+\left\{\begin{matrix} AB'C \\ \text{or} \\ B'CD' \end{matrix}\right\}$$

5.5 Five-Variable Karnaugh Maps

Figure 5-24

$$F(A,B,C,D,E)=\sum m(0,1,3,8,9,14,15,16,17,19,25,27,31)$$



Final solution

$$F=\underset{P_1}{B'C'D'}+\underset{P_2}{B'C'E}+\underset{P_3}{A'C'D'}+\underset{P_4}{A'BCD}+\underset{P_5}{ABDE}+\begin{matrix}C'D'E\\ \text{or}\\ A'C'E\end{matrix}$$

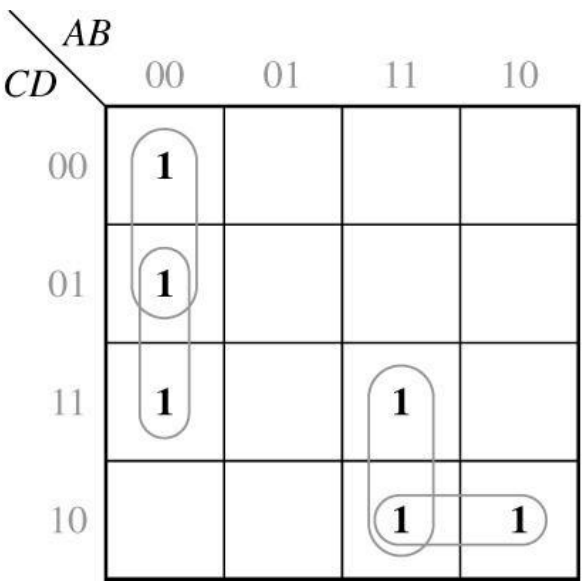
5.6 Other Uses of Karnaugh Maps

minturm expansion of f is $\quad f = \sum m(0,2,3,4,8,10,11,15)$

maxterm expansion of f is $\quad f = \prod M(1,5,6,7,9,12,16,14)$

same

Figure 5-25



$$F = A'B'(C' + D) + AC(B + D')$$

5.6 Other Uses of Karnaugh Maps

Figure 5-26

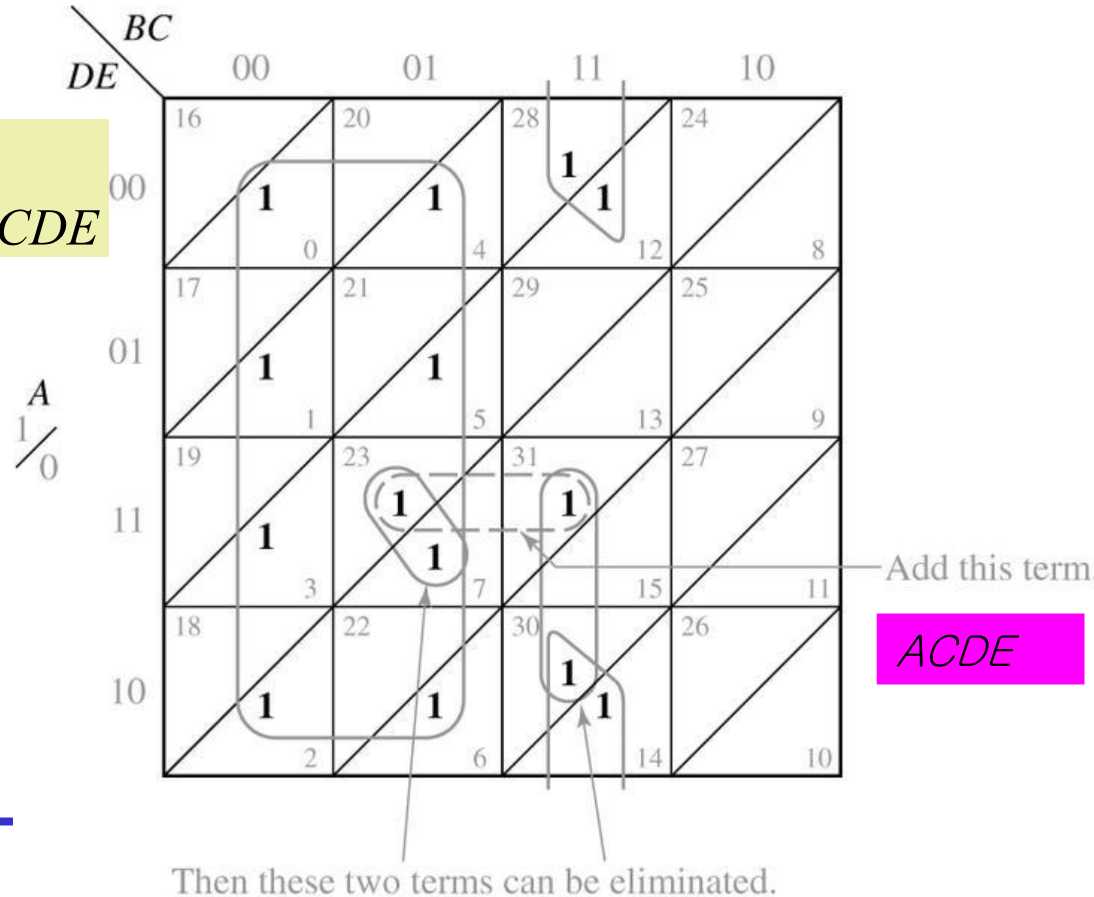
$$F = ABCD + B'CDE + A'B' + BCE'$$

Using the consensus theorem:

$$F = ABCD + B'CDE + A'B' + BCE' + ACDE$$

minimum solution:

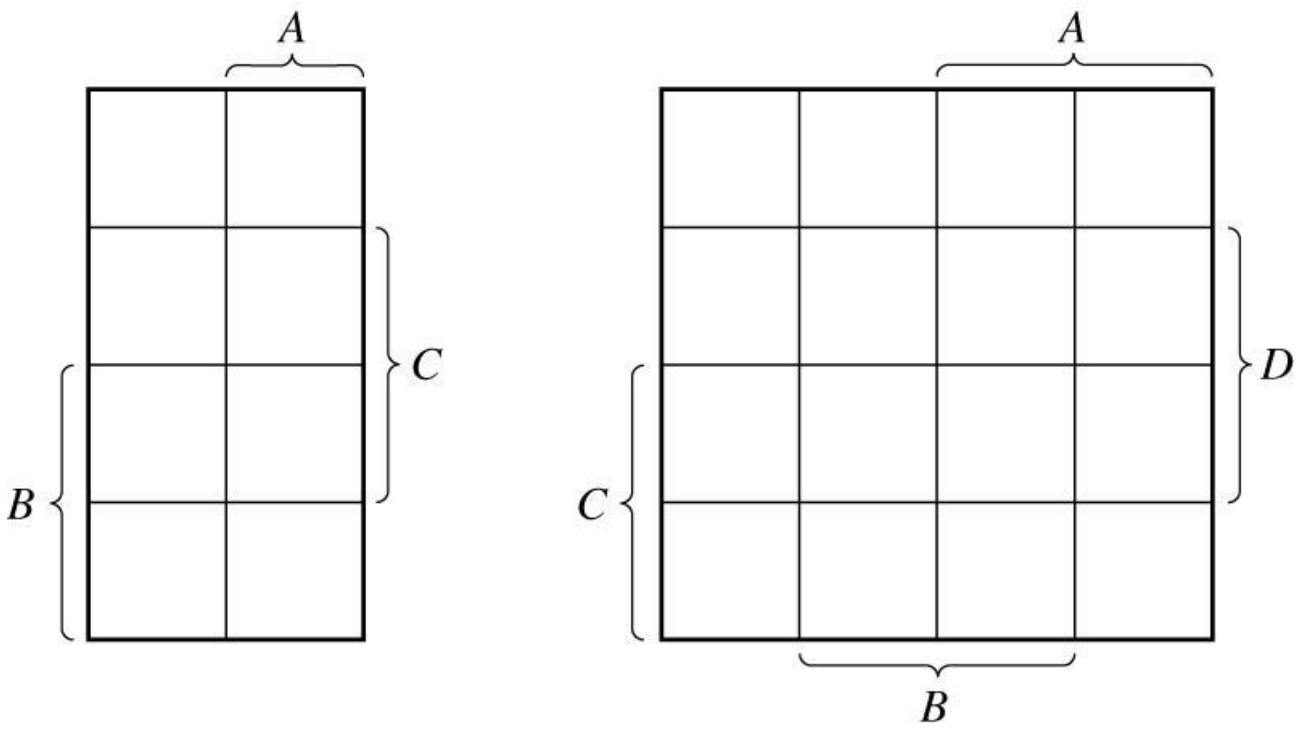
$$F = A'B' + BCE' + ACDE$$



ACDE

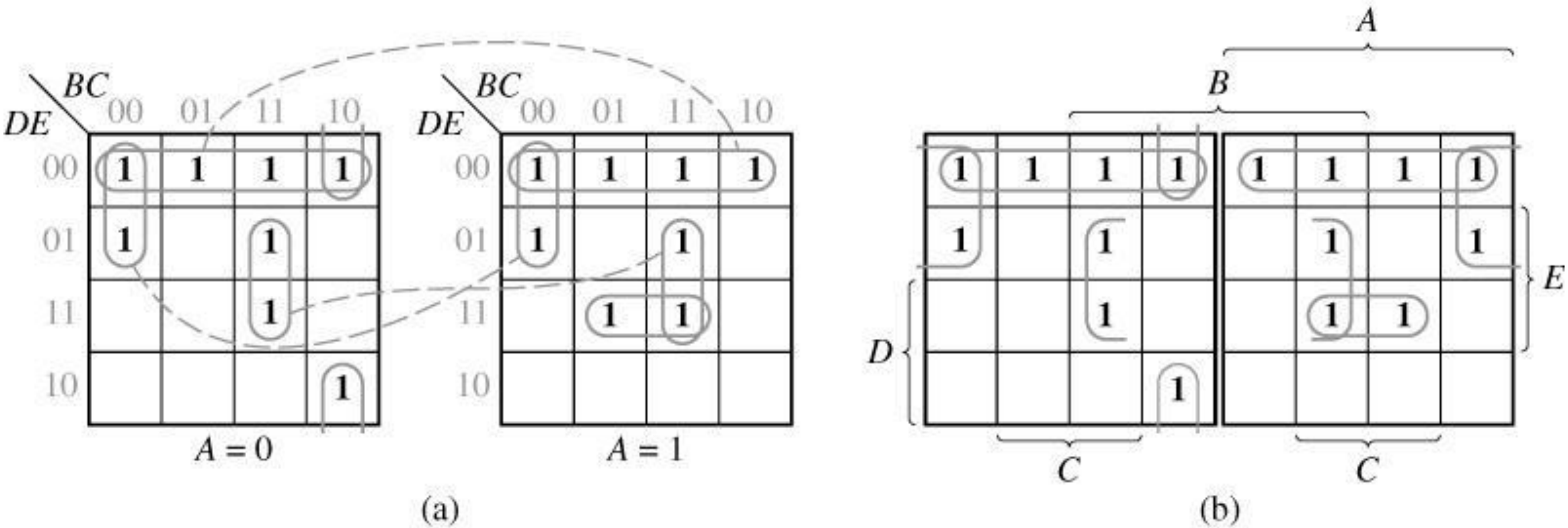
5.7 Other Forms of Karnaugh Maps

Figure 5-27. Veitch Diagrams



5.7 Other Forms of Karnaugh Maps

Figure 5-28. Other Forms of Five-Variable Karnaugh Maps



$$F = D'E' + B'C'D' + BCE + A'BC'E' + ACDE$$

Quine-McKlusky Method

Objectives

1. Find the prime implicants of a function by using the Quine-McCluskey method.

2. Define prime implicants and essential prime implicants

3. Given the prime implicants, find the essential prime implicants and a minimum sum-of-products expression for a function, using a prime implicant chart and using Petrick method

4. Minimize an incompletely specified function, using the Quine-McCluskey method

5. Find a minimum sum-of-products expression for a function, using the method of map-entered variables

6.1 Determination of Prime Implicants

- The minterms are represented in binary notation and combined using

$$XY + XY' = X$$

- The binary notation and its algebraic equivalent

$$AB'CD' + AB'CD = AB'C$$

$$\underbrace{1\ 0\ 1}_{X}\ \underbrace{0}_{Y} + \underbrace{1\ 0\ 1}_{X}\ \underbrace{1}_{Y'} = \underbrace{1\ 0\ 1}_{X}\ -$$ -- (the dash indicates a missing variable)

$$AB'CD' + AB'CD$$ (will not combine)

$$1\ 0\ 1\ 0 + 1\ 0\ 1\ 1$$ (will not combine)

6.1 Determination of Prime Implicants

- the binary minterms are sorted into groups

$$f(a,b,c,d)=\sum m(0,1,2,5,6,7,8,9,10,14)$$

Is repersented by the following list of minterms:

group 0	0	0000
	1	0001
group 1	2	0010
	8	1000
	5	0101
group 2	6	0110
	9	1001
	10	1110
group 3	7	0111
	14	1110

6.1 Determination of Prime Implicants

- Determination of Prime Implicants (Table 6-1)

	Column I		Column II		Column III	
group 0	0	0000 ✓	0, 1	000- ✓	0, 1, 8, 9	-00-
group 1	1	0001 ✓	0, 1	00-0 ✓	0, 2, 8, 10	-0-0
	2	0010 ✓	0, 8	-000 ✓	~~0, 8, 1, 9~~	~~-00-~~
	8	1000 ✓	1, 5	0-01	~~0, 2, 8, 10~~	~~-0-0~~
group 2	5	0101 ✓	1, 9	-001 ✓	2, 6, 10, 14	--10
	6	0110 ✓	2, 6	0-10 ✓	~~2, 10, 6, 14~~	~~--10~~
	9	1001 ✓	2, 10	-010 ✓		
	10	1010 ✓	8, 9	10-0 ✓		
group 3	7	0111 ✓	8, 10	10-0 ✓		
	14	1110 ✓	5, 7	01-1		
			6, 7	011-		
			6, 14	-110 ✓		
			10, 14	1-10 ✓		

6.1 Determination of Prime Implicants

The function is equal to the sum of its prime implicants

$$f = \underset{(1,5)}{a'c'd} + \underset{(5,7)}{a'bd} + \underset{(6,7)}{a'bc} + \underset{(0,1,8,9)}{b'c'} + \underset{(0,2,8,10)}{b'd'} + \underset{(2,7,10,14)}{cd'}$$

Using the consensus theorem to eliminate redundant terms yields

$$f = a'bd + b'c' + cd'$$

Definition: Given a function F of n variables, a product term P is an implicants of F iff for every combination of values of the n variables for which P=1, F is also equal to 1.

Definition: A Prime implicants of a function F is a product term implicant which is no longer an implicant if any literal is deleted from it.

6.2 The Prime Implicant Chart

Prime Implicant Chart (Table 6-2)

Essential Prime Implicant : ⊗

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	x	x					x	⊗		
(0, 2, 8, 10)	$b'd'$	x		x				x		x	
(2, 6, 10, 14)	cd'			x		x				x	⊗
(1, 5)	$a'c'd$		x		x						
(5, 7)	$a'bd$				x		x				
(6, 7)	$a'bc$					x	x				

Remaining cover

The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

6.2 The Prime Implicant Chart

The resulting chart (Table 6-3)

		0	1	2	5	6	7	8	9	10	14
(0, 1, 8, 9)	$b'c'$	x	x					x	x		
(0, 2, 8, 10)	$b'd'$	x		x				x		x	
(2, 6, 10, 14)	cd'			x		x				x	x
(1, 5)	$a'c'd$		x		x						
(5, 7)	$a'bd$				x		x				
(6, 7)	$a'bc$					x	x				

The resulting minimum sum of products is

$$f = b'c' + cd' + a'bd$$

6.2 The Prime Implicant Chart

Example: **cyclic prime implicants**(two more X's in every column in chart)

$$F = \sum m\,(0, 1, 2, 5, 6, 7)$$

Derivation of prime implicants

0	000	✓	0,1	00-
1	001	✓	0,2	0-0
2	010	✓	1,5	-01
5	101	✓	2,6	-10
6	110	✓	5,7	1-1
7	111	✓	6,7	11-

6.2 The Prime Implicant Chart

The resulting prime implicant chart (Table 6-4)

				0	1	2	5	6	7
①	→	(0,1)	$a'b'$	x	x				
		(0,2)	$a'c'$	x		x			
		(1,5)	$b'c$		x		x		
②	→	(2,6)	bc'			x		x	
③	→	(5,7)	ac				x		x
		(6,7)	ad					x	x

One solution:

$$F = a'b' + bc' + ac$$

6.2 The Prime Implicant Chart

Again starting with the other prime implicant that covers column 0.

The resulting table (Table6-5)

			0	1	2	5	6	7
P_1	(0,1)	$a'b'$	x	x				
P_2	(0,2)	$a'c'$	x		x			
P_3	(1,5)	$b'c$		x		x		
P_4	(2,6)	bc'			x		x	
P_5	(5,7)	ac				x		x
P_6	(6,7)	ad					x	x

Finish the solution and show that $F = a'c' + b'c + ab.$

6.3 Petrick's Method

– A technique for determining all minimum SOP solution from a PI chart

			0	1	2	5	6	7
P_1	(0,1)	$a'b'$	x	x				
P_2	(0,2)	$a'c'$	x		x			
P_3	(1,5)	$b'c$		x		x		
P_4	(2,6)	bc'			x		x	
P_5	(5,7)	ac				x		x
P_6	(6,7)	ad					x	x

Because we must cover all of the minterms, the following function must be true:

$$P = (P_1 + P_2)(P_1 + P_3)(P_2 + P_4)(P_3 + P_5)(P_4 + P_6)(P_5 + P_6) = 1$$

minterm0 minterm1 ....

6.3 Petrick's Method

– Reduce P to a minimum SOP

First, we multiply out, using (X+Y)(X+Z) = X+YZ and the ordinary Distributive law

$$
\begin{aligned}
P &= (P_1 + P_1P_1)(P_1 + P_1P_1)(P_1 + P_1P_1) \\
&= (P_1P_4 + P_1P_2P_6 + P_2P_3P_4 + P_2P_3P_6)(P_5 + P_3P_6) \\
&= P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_2P_3P_5P_6 + P_1P_3P_4P_6 \\
&\quad + P_1P_2P_3P_6 + P_2P_3P_4P_6 + P_2P_3P_6
\end{aligned}
$$

Use X+XY=X to eliminate redundant terms from P

$$P = P_1P_4P_5 + P_1P_2P_5P_6 + P_2P_3P_4P_5 + P_1P_3P_4P_6 + P_2P_3P_6$$

– Choose *P1,P4,P5* or *P2,P3,P6* for minimum solution

$F = a'b' + bc' + ac$ or $F = a'c' + b'c + ab.$

Example: $F(A,B,C,D)=\sum m(2,3,7,9,11,13)+\sum d(1,10,15)$

Don't care terms are treated like required minterms…

1	0001	✓	(1, 3)	00-1	✓	(1, 3, 9, 11)	-0-1
2	0010	✓	(1, 9)	-001	✓	(2, 3, 10, 11)	-01-
3	0011	✓	(2, 3)	001-	✓	(3, 7, 11, 15)	--11
9	1001	✓	(2, 10)	-010	✓	(9, 11, 13, 15)	1--1
10	1010	✓	(3, 7)	0-11	✓		
7	0111	✓	(3, 11)	-011	✓		
11	1011	✓	(9, 11)	10-1	✓		
13	1101	✓	(9, 13)	1-01	✓		
15	1111	✓	(10, 11)	101-	✓		
			(7, 15)	-111	✓		
			(11, 15)	1-11	✓		
			(13, 15)	11-1	✓		

Don’t care columns are omitted when forming the PI chart...

	2	3	7	9	11	13
$(1,3,9,11)$		X		X	X	
* $(2,3,10,11)$	X	X			X	
* $(3,7,11,15)$		X	X		X	
* $(9,11,13,15)$				X	X	X

$$F = B'C + CD + AD$$

Replace each term in the final expression for F

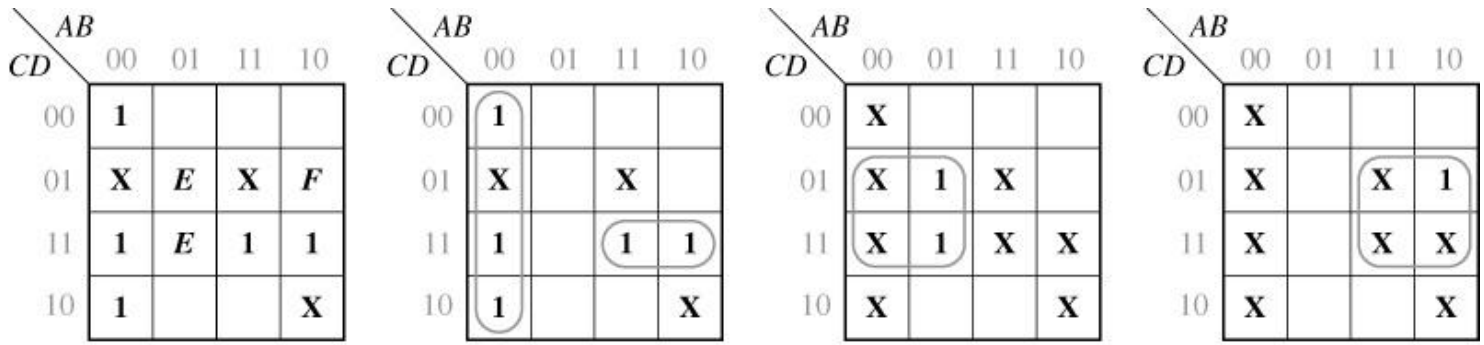
$$F = (m_2 + m_3 + m_{10} + m_{11}) + (\cancel{m_3} + m_7 + \cancel{m_{11}} + m_{15}) + (m_9 + \cancel{m_{11}} + m_{13} + \cancel{m_{15}})$$

The don’t care terms in the original truth table for F

$$for \quad ABCD = 0001,\ F = 0; \qquad for\ 1010, F = 1; \qquad for\ 1111,\ F = 1$$

6.5 Simplification Using Map-Entered Variables

Using of Map-Entered Variables (Figure 6-1)



G (a)

$E = F = 0$, $MS_0 = A'B' + ACD$ (b)

$E = 1, F = 0$, $MS_1 = A'D$ (c)

$E = 0, F = 1$, $MS_2 = AD$ (d)

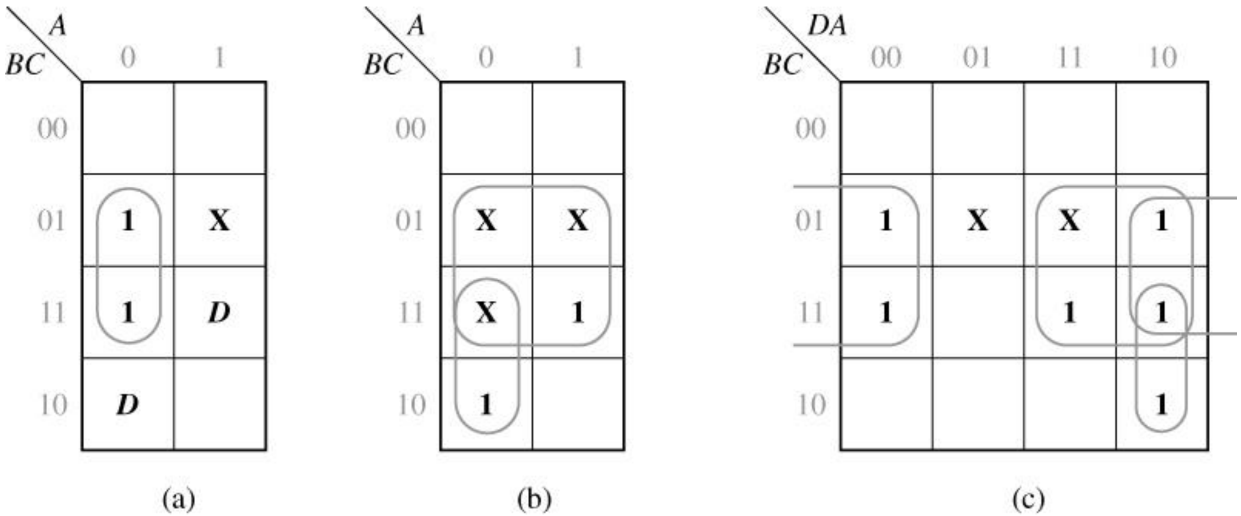
The map represents the 6-variable function

$$G(A,B,C,D,E,F) = m_0 + m_2 + m_3 + Em_5 + Em_7 + Fm_9 + m_{11} + m_{15}$$
$$(\text{+don't care terms})$$

Use a 3-variable map to simplify the function:

$$F(A,B,C,D) = A'B'C + A'BC + A'BC'D + ABCD + (AB'C)$$

Simplification Using a Map-Entered Variable (Figure 6-2)



From Figure 6-2(b),

$$F = A'C + D(C + A'B) = A'C + CD + A'BD$$

Find a sum-of-products expression for F of the form

$$F = MS_0 + + P_1MS_1 + P_2MS_2 + \ldots$$

MS_0 : minimum sum obtained by $P_1=P_2=\ldots=0$

MS_1 : minimum sum obtained by $P_1=1$, $P_j=0(j\neq 1)$ and replacing all '1"s on the map with 'don't cares(X)'

MS_2 :....

The resulting expression is a minimum sum of products for G(Fig. 6-1) :

$$G = A'B' + ACD + EA'D + FAD$$

Design Flow

Combinational Logic Design Flow

- We now have all the pieces for a complete design process

1) Design Specifications : description of what we want to do

2) Truth Table : listing the logical operation of the system

3) Describe using SOP/POS/Minterm/Maxterm : creating the logic expression

4) Logic Minimization : K-maps

5) Logic Manipulation : Convert to desired technology (NAND/NAND, …)

6) Hazard Prevention