EE 2000 SIGNALS AND SYSTEMS

Ch. 2 Continuous-Time Systems

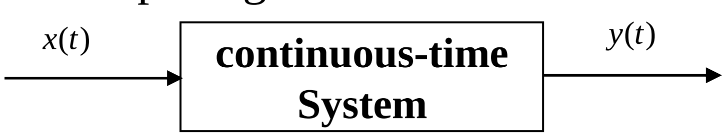
(These slides are taken from Dr. Jingxian Wu, University of Arkansas, 2020.)

OUTLINE

- **Classifications of continuous-time system**
- **Linear time-invariant system (LTI)**
- **Properties of LTI system**
- **System described by differential equations**

CLASSIFICATIONS: SYSTEM DEFINITION

- **What is system?**
  - A system is a process that transforms input signals into output signals
    - Accept an input
    - Process the input
    - Send an output (also called: the response of the system to input)
  - System examples:
    - Radio: input: electrical signals from air, output: music
    - Robot: input: electrical control signals, output: motion or action
- **Continuous-time system**
  - A system in which continuous-time input signals are transformed to continuous-time output signals
- **Discrete-time system**
  - A system in which discrete-time input signals are transformed to discrete-time output signals.



Continuous-time system

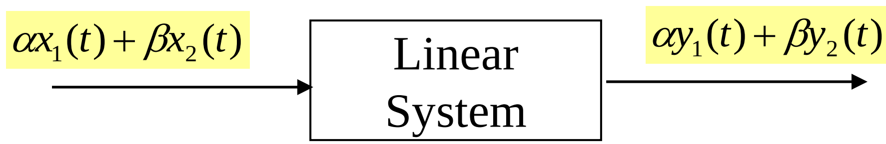
discrete-time system

CLASSIFICATIONS: SYSTEM DEFINITION

- **Classifications**
  - Linear v.s. non-linear
  - Time-invariant v.s. time-varying
  - Dynamic v.s. static (memory v.s. memoryless)
  - Causal v.s. non-causal
  - Invertible v.s. non-invertible
  - Stable v.s. non-stable

CLASSIFICATIONS: LINEAR AND NON-LINEAR

- **Linear system**
  - Let $y_1(t)$ be the response of a system to an input $x_1(t)$
  - Let $y_2(t)$ be the response of a system to an input $x_2(t)$
  - The system is linear if the superposition principle is satisfied:
    - 1. the response to $x_1(t)+x_2(t)$ is $y_1(t)+y_2(t)$
    - 2. the response to $\alpha x_1(t)$ is $\alpha y_1(t)$



Linear system

- **Non-linear system**
  - If the superposition principle is not satisfied, then the system is a non-linear system

CLASSIFICATIONS: LINEAR AND NON-LINEAR

- **Example: check if the following systems are linear**
  - System 1: $y(t) = \exp[x(t)]$
  - System 2: charge a capacitor. Input: $i(t)$, output $v(t)$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

  - System 3: inductor. Input: $i(t)$, output $v(t)$

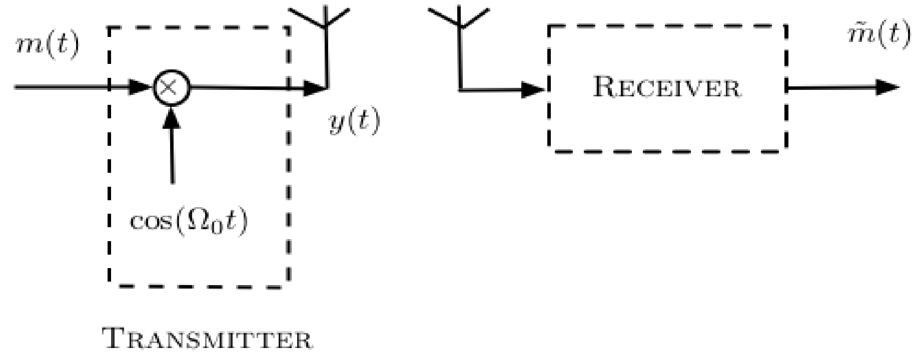
$$v(t) = L\frac{di(t)}{dt}$$

CLASSIFICATIONS: LINEAR AND NON-LINEAR

- **Example**
  - System 4: $$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)d\tau + B$$
  - System 5: $$y(t) = |x(t)|$$
  - System 6: $$y(t) = x^2(t)$$

CLASSIFICATIONS: LINEAR V.S. NON-LINEAR

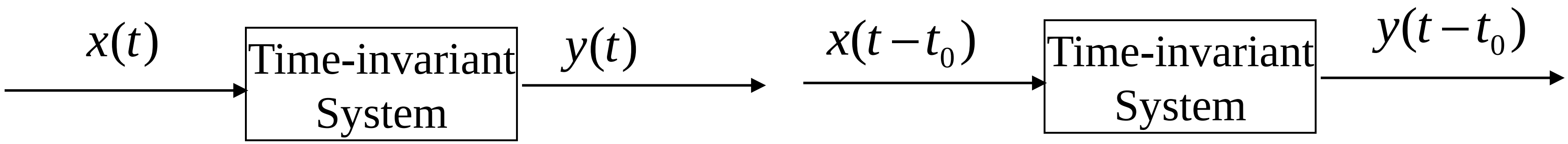
- **Example:**
  - Amplitude Modulation:
    - Is it linear?



Amplitude modulation

CLASSIFICATIONS: TIME-VARYING V.S. TIME-INVARIANT

- **Time-invariant**
  - A system is time-invariant if a time shift in the input signal causes an identical time shift in the output signal



Time-invariant system

- **Examples**
  - $y(t) = \cos(x(t))$
  - $y(t) = \int_0^t x(v)dv$

CLASSIFICATIONS: MEMORY V.S. MEMORYLESS

- **Memoryless system**
  - If the present value of the output depends only on the present value of input, then the system is said to be memoryless (or instantaneous).
  - Example: input x(t): the current passing through a resistor
    output y(t): the voltage across the resistor

$$y(t) = Rx(t)$$

  - The output value at time *t* depends only on input value at time *t*.
- **System with memory**
  - If the present value of the output depends on not only present value of input, but also previous input values, then the system has memory.
  - Example: capacitor, current: x(t), output voltage: y(t)

$$y(t) = \frac{1}{C}\int_0^t x(\tau)d\tau$$

  - the output value at *t* depends on all input values before *t*

CLASSIFICATIONS: MEMORY V.S. MEMORYLESS

- **Examples: determine if the systems has memory or not**
  - $y(t) = \sum_{i=0}^{N} a_i x(t - T_i)$
  - $y(t) = \sin(2x^2(t) + \theta)x(t)$

CLASSIFICATIONS: CAUSAL V.S. NON-CAUSAL

- **Causal system**
  - A system is causal if the output $y(t_0)$ depends only on values of input for $t \le t_0$
    - The output depends on only input from the past and present
  - Example

$$y(t) = \frac{1}{C}\int_0^t x(\tau)d\tau$$

- **Non-causal system**
  - A system is non-causal if the output depends on the input from the future (prediction).
  - Examples:

$$y(t) = x(t+a) \qquad a > 0$$

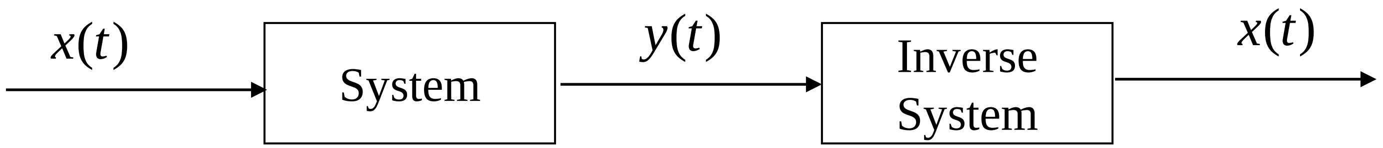
$$y(t) = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)d\tau$$

  - The output value at *t* depends on the input value at *t* + *a* (from future)

  - All practical systems are causal.

CLASSIFICATION: INVERTIBILITY

- **Invertible**
  - A system is invertible if
    - by observing the output, we can determine its input.



invertible system

  - Question: for a system, if two different inputs result in the same output, is this system invertible?

- **Example**

$$y(t) = 2x(t)$$

$$y(t) = \cos[x(t)]$$

- If two different inputs result in the same output, the system is non-invertible

CLASSIFICATION: STABILITY

- **Bounded signal**
  - Definition: a signal $x(t)$ is said to be bounded if

$$|x(t)| < B < \infty \qquad \forall t$$

- **Bounded-input bounded-output (BIBO) stable system**
  - Definition: a system is BIBO stable if, for any bounded input $x(t)$, the response $y(t)$ is also bounded.

$$|x(t)| < B_1 < \infty \Rightarrow |\mathrm{y(t)}| < B_2 < \infty \qquad \forall t$$

- **Example: determine if the systems are BIBO stable**

$$y(t) = \exp\left[x(t)\right]$$

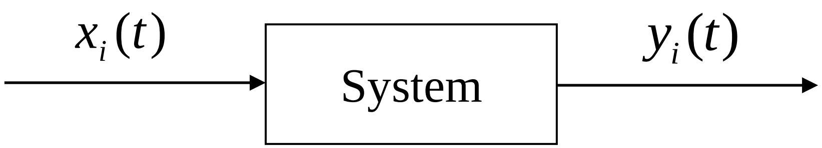
$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

OUTLINE

- Classifications of continuous-time system
- **Linear time-invariant system (LTI)**
- Properties of LTI system
- System described by differential equations

LTI: DEFINTION

- **Linear time-invariant (LTI) system**
  - Definition: a system is said to be LTI if it's linear and time-invariant



system

- Linear

Input: $x(t) = a_1 x_1(t) + a_2 x_2(t) + \cdots + a_N x_N(t) = \sum_{i=1}^{N} a_i x_i(t)$

Output: $y(t) = a_1 y_1(t) + a_2 y_2(t) + \cdots + a_N y_N(t) = \sum_{i=1}^{N} a_i y_i(t)$

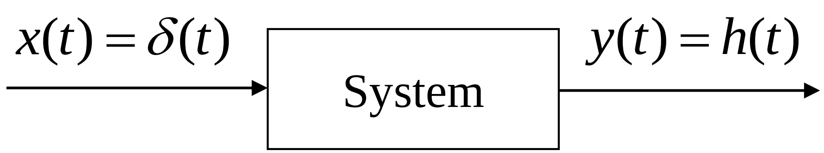
- Time-invariant

Input: $x(t) = x_i(t - t_0)$

Output: $y(t) = y_i(t - t_0)$

LTI: IMPULSE RESPONSE

- **Impulse response of LTI system**
  - Def: the output (response) of a system when the input is a unit impulse function (delta function).
    - Usually denoted as $h(t)$



LTI system

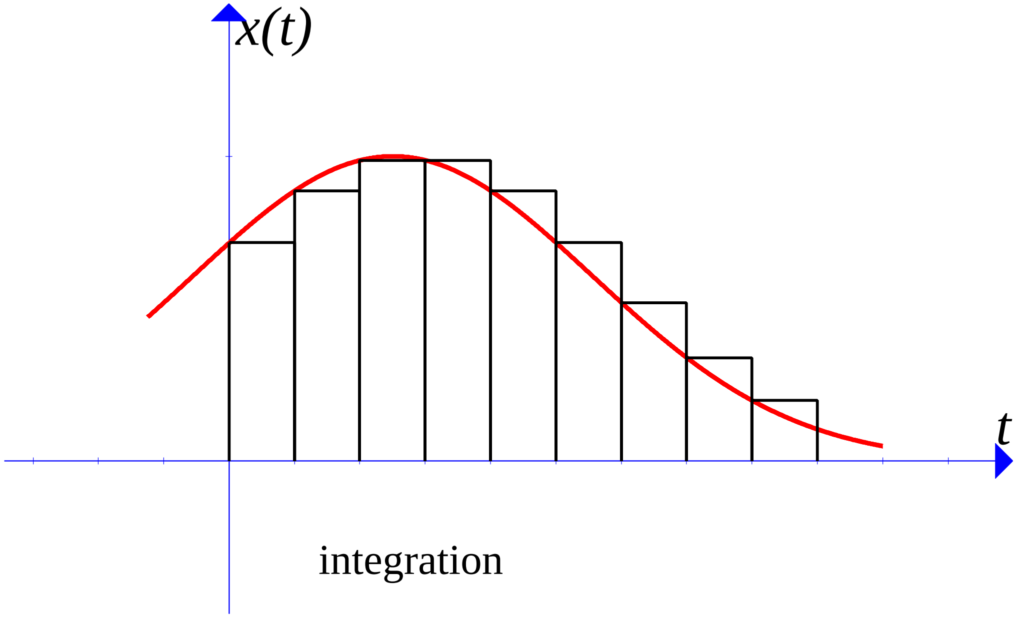
- **For system with an arbitrary input $x(t)$, we want to find out the output $y(t)$.**
  - Method 1: differential equations
  - Methods 2: convolution integral
  - Methods 3: Laplace transform, Fourier transform,

LTI: CONVOLUTION

- **Derivation**
  - Any signal can be approximated by the sum of a sequence of delta functions

$$\int_{-\infty}^{+\infty} z(\tau)d\tau = \lim_{\Delta \to 0} \sum_{n=-\infty}^{+\infty} z(n\Delta)\Delta$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau = \lim_{\Delta \to 0} \sum_{n=-\infty}^{+\infty} x(n\Delta)\delta(t-n\Delta)\Delta$$



integration

LTI: CONVOLUTION

- **Derivation**
  - Any signal can be approximated by the sum of a sequence of delta functions

$$x(t)=\int_{-\infty}^{+\infty}x(\tau)\delta(t-\tau)d\tau=\lim_{\Delta\to 0}\sum_{n=-\infty}^{+\infty}x(n\Delta)\delta(t-n\Delta)\Delta$$

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$$

  - Time invariant

$$\delta(t-n\Delta) \longrightarrow \boxed{\text{System}} \longrightarrow h(t-n\Delta)$$

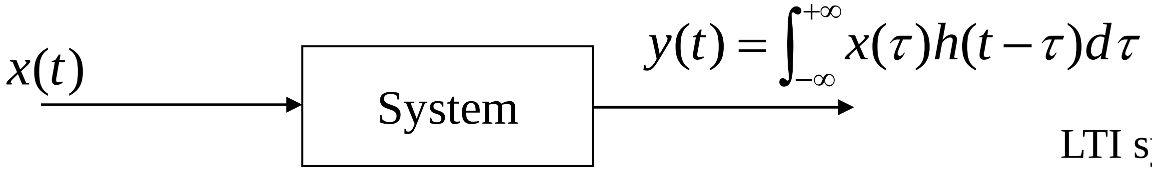
  - Linear

$$\sum_{n=-\infty}^{+\infty}x(n\Delta)\delta(t-n\Delta)\Delta \longrightarrow \boxed{\text{System}} \longrightarrow \sum_{n=-\infty}^{+\infty}x(n\Delta)h(t-n\Delta)\Delta$$

LTI system

LTI: CONVOLUTION

- **Convolution**



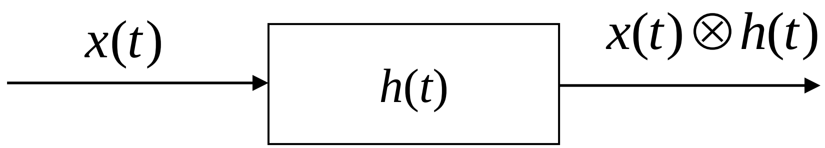
LTI system

– Definition: the convolution of two signals $x(t)$ and $h(t)$ is defined as

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

– The operation of convolution is usually denoted with the symbol $\otimes$

$$y(t)=x(t)\otimes h(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$



LTI system

For LTI system, if we know input x(t) and impulse response h(t), Then the output is $x(t)\otimes h(t)$

LTI: CONVOLUTION

- **Examples**

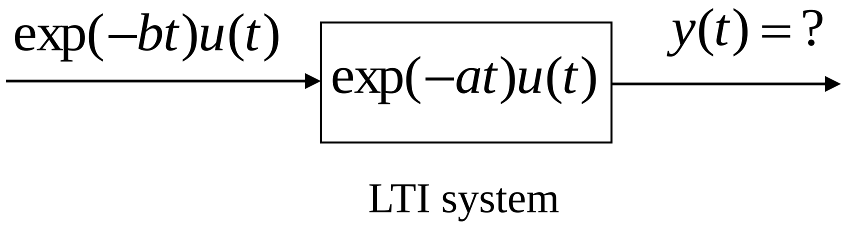
$$x(t) \otimes \delta(t)$$

$$x(t) \otimes \delta(t - t_0)$$

$$x(t) \otimes u(t)$$

LTI: CONVOLUTION

- **Examples**



LTI: CONVOLUTION

- **Example**
  - Obtain the impulse response of a capacitor and use it to find the unit-step response by using convolution. Assume the input is the current, and the output is the voltage. Let C = 1F.

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

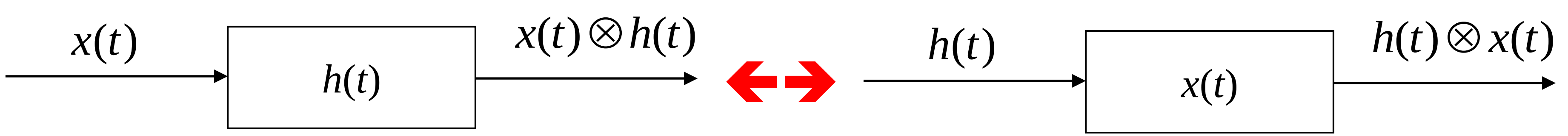
LTI: CONVOLUTION PROPERTIES

- **Commutativity**

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

  - Proof:

$$x(t) \otimes y(t) = \int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau$$



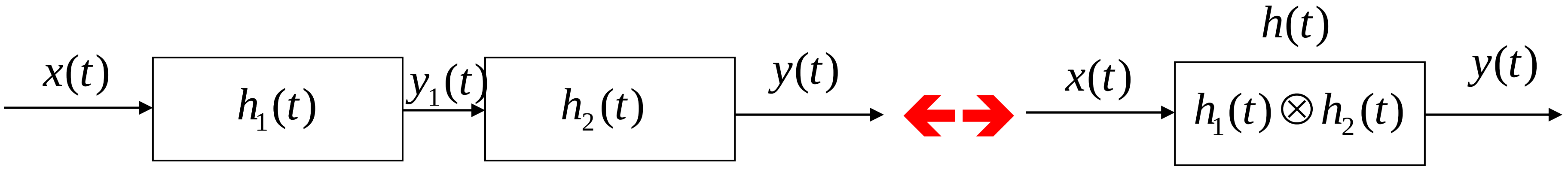
commutativity

LTI: CONVOLUTION PROPERTIES

- **Associativity**

$$x(t)\otimes h_1(t)\otimes h_2(t)=\left[x(t)\otimes h_1(t)\right]\otimes h_2(t)=x(t)\otimes\left[h_1(t)\otimes h_2(t)\right]$$

  - proof



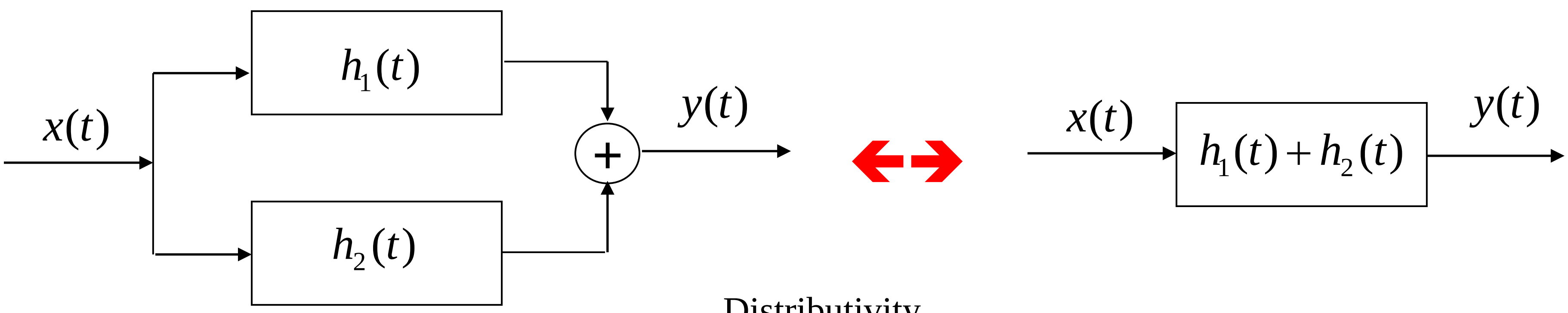
Associativity

LTI: CONVOLUTION PROPERTIES

- **Distributivity**

$$x(t)\otimes\left[h_1(t)+h_2(t)\right]=\left[x(t)\otimes h_1(t)\right]+\left[x(t)\otimes h_1(t)\right]$$

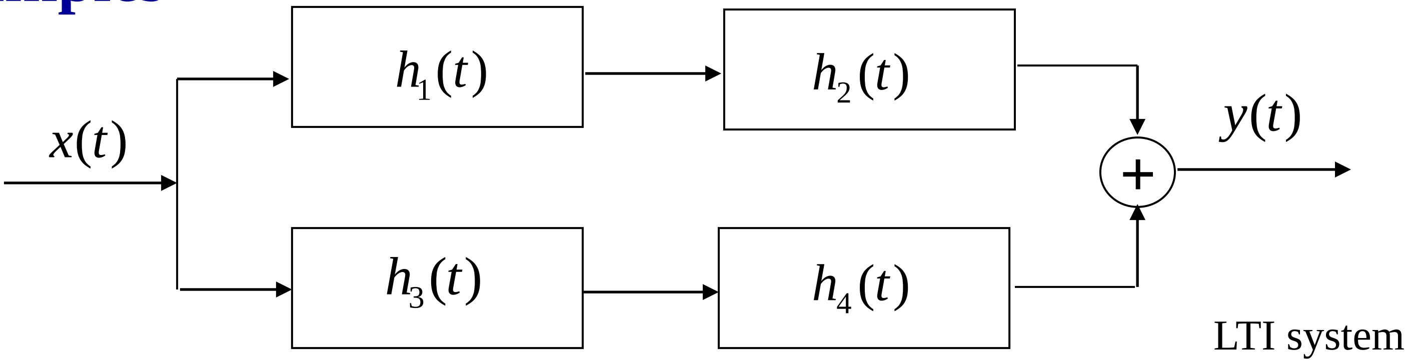
  – proof



Distributivity

LTI: CONVOLUTION PROPERTIES

- **Examples**



$$h_1(t) = \exp(-2t)u(t) \qquad h_2(t) = 2\exp(-t)u(t)$$

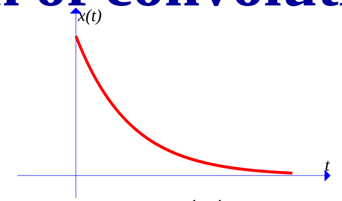
$$h_3(t) = \exp(-3t)u(t) \qquad h_4(t) = 4\delta(t)$$

$$h(t) = ?$$

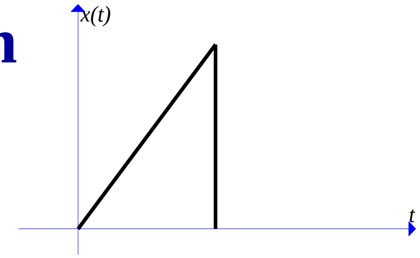
LTI: GRAPHICAL CONVOLUTION

- **Graphical interpretation of convolution**

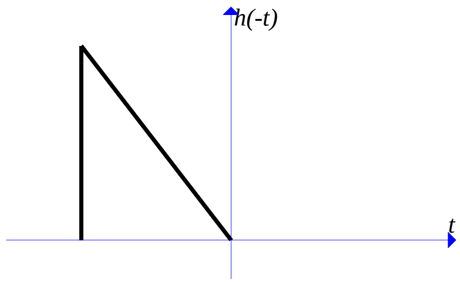
$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$



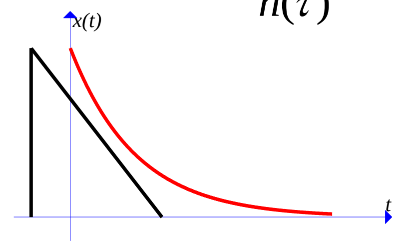
$x(\tau)$



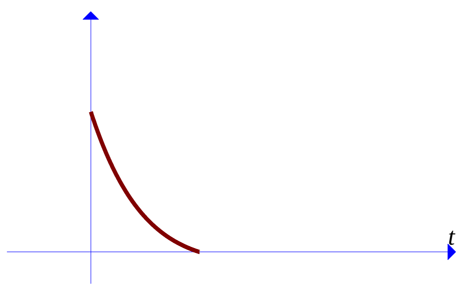
$h(\tau)$



– 1. Reflection $g(\tau) = h(-\tau)$



– 2. Shift $g(\tau - t_0) = h(-(\tau - t_0)) = h(t_0 - \tau)$



– 3. Multiplication $x(\tau) h(t_0 - \tau)$

– 4. Integration $y(t_0) = \int_{-\infty}^{+\infty} x(\tau) h(t_0 - \tau) d\tau$

LTI: GRAPHICAL CONVOLUTION

- **Example**

$$y(t) = [2a \cdot p_{2a}(t)] \otimes [2a \cdot p_{2a}(t-a)]$$

OUTLINE

- **Classifications of continuous-time system**
- **Linear time-invariant system (LTI)**
- **Properties of LTI system**
- **System described by differential equations**

LTI PROPERTIES

- **Memoryless LTI system**
  - Review: present output only depends on present input

$$y(t) = Kx(t)$$

  - The impulse response of Memoryless LTI system is

$$h(t) = K\delta(t)$$

- **Causal LTI system**
  - Review: output depends on only current input and past input.
  - The impulse response of causal LTI system must satisfy:

$$h(t) = 0 \qquad \text{for } \ t < 0$$

  - Why?

LTI PROPERTIES

- **Invertible LTI Systems**
  - Review: a system is invertible iff (if and only if) there is an inverse system that, when connected in cascade with the original system, yields an output equal to original system input

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \boxed{g(t)} \longrightarrow x(t)$$

$$x(t) \otimes h(t) \otimes g(t) = x(t)$$

  - For invertible LTI systems with IR (impulse response) $h(t)$, there exists inverse system $g(t)$ such that

$$g(t) \otimes h(t) = \delta(t)$$

  - Example: find the inverse system of LTI system $h(t) = \delta(t - t_0)$

LTI PROPERTIES

- **BIBO Stable LTI state**
  - Review: a system is BIBO stable iff every bounded input produces a bounded output.
  - LTI system: an LTI system is BIBO stable iff

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty$$

    - Proof:

LTI PROPERTIES

- **Examples**
  - Determine: causal or non-causal, memory or memoryless, stable or unstable
  - 1. $h_1(t) = t\exp(-2t)u(t) + \exp(3t)u(-t) + \delta(t-1)$
  - 2. $h_2(t) = -3\exp(2t)u(t)$
  - 3. $h_3(t) = 5\delta(t+5)$

OUTLINE

- Classifications of continuous-time system
- Linear time-invariant system (LTI)
- Properties of LTI system
- **System described by differential equations**

DIFFERENTIAL EQUATIONS

- **LTI system can be represented by differential equations**

$$a_0 y(t) + a_1 y'(t) + \cdots + a_N y^{(N)}(t) = b_0 x(t) + b_1 x'(t) + \cdots + b_M x^{(M)}(t)$$

  - Initial conditions:

$$\left.\frac{d^k y(t)}{dt^k}\right|_{t=0} \qquad k = 0, \cdots, N-1$$

  - Notation: n-th derivative:

$$y^{(n)}(t) = \frac{d^n y(t)}{dt^n}$$

DIFFERENTIAL EQUATION

- **Example:**
  - Consider a circuit with a resistor R = 1 Ohm and an inductor L = 1H, with a voltage source $v(t) = Bu(t)$, and I_o is the initial current in the inductor. The output of the system is the current across the inductor.
    - Represent the system with a differential equation.
    - Find the output of the system with $I_o = 0$ and $I_o = 1$

DIFFERENTIAL EQUATION

$$a_0 y(t) + a_1 y'(t) + \cdots + a_N y^{(N)}(t) = b_0 x(t) + b_1 x'(t) + \cdots + b_M x^{(M)}(t)$$

$$\left.\frac{d^k y(t)}{dt^k}\right|_{t=0} \qquad k = 0, \cdots, N-1$$

- **Zero-state response**
  - The output of the system when the initial conditions are zero
  - Denoted as $y_{zs}(t)$
- **Zero-input response**
  - The output of the system when the input is zero
  - Denoted as $y_{zi}(t)$
- **The actual output of the system**

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

DIFFERENTIAL EQUATION

- **Example**
  - Find the zero-state output and zero-input response of the RL circuit in the previous example.