CHAPTER 7

7.1. Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find:

a) V at P: Substituting the coordinates into V, find $V_P = \underline{-8\,\text{V}}$.

b) $\mathbf{E}$ at P: We use $\mathbf{E} = -\nabla V = -2y^2z^3\mathbf{a}_x - 4xyz^3\mathbf{a}_y - 6xy^2z^2\mathbf{a}_z$, which, when evaluated at P, becomes $\mathbf{E}_P = \underline{8\mathbf{a}_x + 8\mathbf{a}_y - 24\mathbf{a}_z \text{ V/m}}$

c) ρ_v at P: This is $\rho_v = \nabla \cdot \mathbf{D} = -\epsilon_0 \nabla^2 V = \underline{-4xz(z^2 + 3y^2)\ \text{C/m}^3}$

d) the equation of the equipotential surface passing through P: At P, we know $V = -8\,\text{V}$, so the equation will be $\underline{xy^2z^3 = -4}$.

e) the equation of the streamline passing through P: First,

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus

$$ydy = 2xdx, \text{ and so } \frac{1}{2}y^2 = x^2 + C_1$$

Evaluating at P, we find $C_1 = 1$. Next,

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus

$$3xdx = zdz, \text{ and so } \frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations:

$$\underline{y^2 - 2x^2 = 2} \text{ and } \underline{3x^2 - z^2 = 2}$$

f) Does V satisfy Laplace's equation? No, since the charge density is not zero.

7.2. A potential field V exists in a region where $\epsilon = f(x)$. Find $\nabla^2 V$ if $\rho_v = 0$.
First, $\mathbf{D} = \epsilon(x)\mathbf{E} = -f(x)\nabla V$. Then $\nabla \cdot \mathbf{D} = \rho_v = 0 = \nabla \cdot (-f(x)\nabla V)$.

So

$$0 = \nabla \cdot (-f(x)\nabla V) = -\left[\left(\frac{df}{dx}\frac{\partial V}{\partial x} + f(x)\frac{\partial^2 V}{\partial x^2}\right) f(x)\frac{\partial^2 V}{\partial y^2} + f(x)\frac{\partial^2 V}{\partial z^2}\right]$$
$$= -\left[\frac{df}{dx}\frac{\partial V}{\partial x} + f(x)\nabla^2 V\right]$$

Therefore,

$$\nabla^2 V = \underline{-\frac{1}{f(x)}\frac{df}{dx}\frac{\partial V}{\partial x}}$$

7.3. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \;\Rightarrow\; \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \left.\frac{df}{dx}\right|_{x=0} = 0 \text{(as given)}$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

7.4. Given the potential field $V = A \ln \tan^2(\theta/2) + B$:

a) Show that $\nabla^2 V = 0$: Since V is a function only of θ,

$$\nabla^2 V = \frac{1}{r^2 \sin\theta)} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right)$$

where

$$\frac{dV}{d\theta} = \frac{d}{d\theta}\left(A \ln \tan^2(\theta/2) + B\right) = \frac{d}{d\theta}\left(2A \ln \tan(\theta/2)\right) = \frac{A}{\sin(\theta/2)\cos(\theta/2)} = \frac{2A}{\sin\theta}$$

Then

$$\nabla^2 V = \frac{1}{r^2 \sin\theta)} \frac{d}{d\theta}\left(\sin\theta \frac{2A}{\sin\theta}\right) = 0$$

b) Select A and B so that $V = 100$ V and $E_\theta = 500$ V/m at $P(r = 5, \theta = 60^\circ, \phi = 45^\circ)$:

First,

$$E_\theta = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta} = -\frac{2A}{r\sin\theta} = -\frac{2A}{5\sin 60} = -0.462A = 500$$

So $A = \underline{-1082.5}$ V. Then

$$V_P = -(1082.5) \ln \tan^2(30^\circ) + B = 100 \;\Rightarrow\; B = \underline{-1089.3 \text{ V}}$$

Summarizing, $V(\theta) = -1082.5 \ln \tan^2(\theta/2) - 1089.3$.

7.5. Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$:

a) Show that $\nabla^2 V = 0$: In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho(4A\rho^3 - 4B\rho^{-5})\right)\sin 4\phi - \frac{1}{\rho^2}16(A\rho^4 + B\rho^{-4})\sin 4\phi$$
$$= \frac{16}{\rho}(A\rho^3 + B\rho^{-5})\sin 4\phi - \frac{16}{\rho^2}(A\rho^4 + B\rho^{-4})\sin 4\phi = 0$$

b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$: First,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi$$
$$= -4\left[(A\rho^3 - B\rho^{-5})\sin 4\phi\,\mathbf{a}_\rho + (A\rho^3 + B\rho^{-5})\cos 4\phi\,\mathbf{a}_\phi\right]$$

and at P, $\mathbf{E}_P = -4(A - B)\,\mathbf{a}_\rho$. Thus $|\mathbf{E}_P| = \pm 4(A - B)$. Also, $V_P = A + B$. Our two equations are:

$$4(A - B) = \pm 500$$

and

$$A + B = 100$$

We thus have two pairs of values for A and B:

$$\underline{A = 112.5,\ B = -12.5} \text{ or } \underline{A = -12.5,\ B = 112.5}$$

7.6. If $V = 20\sin\theta/r^3$ V in free space, find:

a) ρ_v at $P(r = 2, \theta = 30^\circ, \phi = 0)$: We use Poisson's equation in free space, $\nabla^2 V = -\rho_v/\epsilon_0$, where, with no ϕ variation:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)$$

Substituting:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{60\sin\theta}{r^4}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{20\cos\theta}{r^3}\right)$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-60\sin\theta}{r^2}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{10\sin 2\theta}{r^3}\right)$$
$$= \frac{120\sin\theta}{r^5} + \frac{20\cos 2\theta}{r^5\sin\theta} = \frac{20(4\sin^2\theta + 1)}{r^5\sin\theta} = -\frac{\rho_v}{\epsilon_0}$$

So

$$\rho_{vP} = -\epsilon_0\left[\frac{20(4\sin^2\theta + 1)}{r^5\sin\theta}\right]_{r=2,\theta=30} = -2.5\epsilon_0 = \underline{-22.1\ \text{pC/m}^3}$$

7.6b. the total charge within the spherical shell $1 < r < 2$ m: We integrate the charge density found in part a over the specified volume:

$$Q = -\epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_1^2 \frac{20(4\sin^2\theta + 1)}{r^5 \sin\theta} r^2 \sin\theta \, dr \, d\theta \, d\phi$$
$$= -2\pi(20)\epsilon_0 \int_0^{\pi} \int_1^2 \frac{(4\sin^2\theta + 1)}{r^3} \, dr \, d\theta = -40\pi\epsilon_0 \int_1^2 \frac{3\pi}{r^3} dr = 60\pi^2 \epsilon_0 \frac{1}{r^2}\Big|_1^2 = -45\pi^2 \epsilon_0$$
$$= \underline{-3.9 \text{ nC}}$$

7.7. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$: Use Poisson's equation:

$$\rho_v = -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} \right)$$
$$= -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{-\cos 2\phi}{\rho} \right) - \frac{4}{\rho^2} \frac{\cos 2\phi}{\rho} \right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120^\circ)}{0.5^3} = -12\epsilon_0 = \underline{-106 \, \text{pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30^\circ, 1)$: First, we find $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi$$
$$= \frac{\cos 2\phi}{\rho^2} \mathbf{a}_\rho + \frac{2 \sin 2\phi}{\rho^2} \mathbf{a}_\phi$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60^\circ}{4} \mathbf{a}_\rho + \frac{2 \sin 60^\circ}{4} \mathbf{a}_\phi = 0.125 \, \mathbf{a}_\rho + 0.433 \, \mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm |\mathbf{D}_B| = \pm \epsilon_0 |\mathbf{E}_B| = \pm 0.451 \epsilon_0 = \underline{\pm 0.399 \, \text{pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

7.8. Let $V_1(r, \theta, \phi) = 20/r$ and $V_2(r, \theta, \phi) = (4/r) + 4$.

a) State whether V_1 and V_2 satisfy Laplace's equation:

$$\nabla^2 V_1 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV_1}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{-20}{r^2} \right) \right] = 0$$

$$\nabla^2 V_2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV_2}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{-4}{r^2} \right) \right] = 0$$

7.8b. Evaluate V_1 and V_2 on the closed surface $r = 4$:

$$V_1(r = 4) = \frac{20}{4} = 5 \quad V_2(r = 4) = \frac{4}{4} + 4 = 5$$

c) Conciliate your results with the uniqueness theorem: Uniqueness specifies that there is only one potential that will satisfy all the given boundary conditions. While both potentials have the same value at $r = 4$, they do not as $r \to \infty$. So they apply to different situations.

7.9. The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \leq \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$.

a) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? Yes for the first three, since Laplace's equation is linear. No for $V_1 V_2$.

b) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? At the 100 V surface ($\rho = a$), No for all. At the 0 V surfaces, yes, except for $V_1 + 3$.

c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1? Only V_2 is, since it is given as satisfying all the boundary conditions that V_1 does. Therefore, by the uniqueness theorem, $V_2 = V_1$. The others, not satisfying the boundary conditions, are not the same as V_1.

7.10. Conducting planes at $z = 2$cm and $z = 8$cm are held at potentials of -3V and 9V, respectively. The region between the plates is filled with a perfect dielectric with $\epsilon = 5\epsilon_0$. Find and sketch:

a) $V(z)$: We begin with the general solution of the one-dimensional Laplace equation in rectangular coordinates: $V(z) = Az + B$. Applying the boundary conditions, we write $-3 = A(2) + B$ and $9 = A(8) + B$. Subtracting the former equation from the latter, we find $12 = 6A$ or $A = 2$ V/cm. Using this we find $B = -7$ V. Finally, $V(z) = 2z - 7$ V (z in cm) or $V(z) = \underline{200z - 7\text{ V}}$ (z in m).

b) $E_z(z)$: We use $\mathbf{E} = -\nabla V = -(dV/dz)\mathbf{a}_z = -2\text{ V/cm} = \underline{-200\text{ V/m}}$.

c) $D_z(z)$: Working in meters, have $D_z = \epsilon E_z = -200\epsilon = \underline{-1000\epsilon_0\text{ C/m}^2}$

7.11. The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find:

a) V at $P(5, 2, 6)$: The planes are parallel, and so we expect variation in potential in the direction normal to them. Using the two boundary condtions, our general potential function can be written:

$$V(x, y) = A(2x + 3y - 12) + 100 = A(2x + 3y - 18) + 0$$

and so $A = -100/6$. We then write

$$V(x, y) = -\frac{100}{6}(2x + 3y - 18) = -\frac{100}{3}x - 50y + 300$$

and $V_P = -\frac{100}{3}(5) - 100 + 300 = \underline{33.33\text{ V}}$.

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E} = -\nabla V = \underline{\frac{100}{3}\,\mathbf{a}_x + 50\,\mathbf{a}_y\text{ V/m}}$$

7.12. Conducting cylinders at $\rho = 2$ cm and $\rho = 8$ cm in free space are held at potentials of 60mV and -30mV, respectively.

a) Find $V(\rho)$: Working in volts and meters, we write the general one-dimensional solution to the Laplace equation in cylindrical coordinates, assuming radial variation: $V(\rho) = A\ln(\rho) + B$. Applying the given boundary conditions, this becomes $V(2\text{cm}) = .060 = A\ln(.02) + B$ and $V(8\text{cm}) = -.030 = A\ln(.08) + B$. Subtracting the former equation from the latter, we find $-.090 = A\ln(.08/.02) = A\ln 4 \;\;\Rightarrow\;\; A = -.0649$. B is then found through either equation; e.g., $B = .060 + .0649\ln(.02) = -.1940$. Finally, $\underline{V(\rho) = -.0649\ln\rho - .1940}$.

b) Find $E_\rho(\rho)$: $\mathbf{E} = -\nabla V = -(dV/d\rho)\mathbf{a}_\rho = \underline{(.0649/\rho)\mathbf{a}_\rho \text{ V/m}}$.

c) Find the surface on which $V = 30$ mV:
Use $.03 = -.0649\ln\rho - .1940 \;\;\Rightarrow\;\; \rho = .0317\,\text{m} = \underline{3.17\,\text{cm}}$.

7.13. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (16) we have

$$V(\rho) = 100\frac{\ln(.012/\rho)}{\ln(.012/.005)} \text{ V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100\frac{\ln(.012/\rho)}{\ln(2.4)} \;\;\Rightarrow\;\; \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01 \text{ cm}}$$

b) $E_{\rho\, max}$: We have

$$E_\rho = -\frac{\partial V}{\partial \rho} = -\frac{dV}{d\rho} = \frac{100}{\rho\ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\, max} = \frac{100}{.005\ln(2.4)} = 2.28 \times 10^4 \text{ V/m} = \underline{22.8 \text{ kV/m}}$$

c) ϵ_R if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_R}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_R:

$$\epsilon_R = \frac{(20\times 10^{-9})\ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

7.14. Two semi-infinite planes are located at $\phi = -\alpha$ and $\phi = \alpha$, where $\alpha < \pi/2$. A narrow insulating strip separates them along the z axis. The potential at $\phi = -\alpha$ is V_0, while $V = 0$ at $\phi = \alpha$.

a) Find $V(\phi)$ in terms of α and V_0: We use the one-dimensional solution form for Laplace's equation assuming variation along ϕ: $V(\phi) = A\phi + B$. The boundary conditions are then substituted: $V_0 = -A\alpha + B$ and $0 = A\alpha + B$. Subtract the latter equation from the former to obtain: $V_0 = -2A\alpha \;\Rightarrow\; A = -V_0/(2\alpha)$. Then $0 = -V_0/(2\alpha)\alpha + B \;\Rightarrow\; B = V_0/2$. Finally

$$V(\phi) = \underline{\frac{V_0}{2}\left(1 - \frac{\phi}{\alpha}\right)\ \text{V}}$$

b) Find E_ϕ at $\phi = 20^\circ$, $\rho = 2$ cm, if $V_0 = 100$ V and $\alpha = 30^\circ$:

$$E_\phi = -\frac{1}{\rho}\frac{dV}{d\rho} = \frac{V_0}{2\alpha\rho}\ \text{V/m} \quad \text{Then} \quad E(2\text{cm}, 20^\circ) = \frac{100}{2(30 \times 2\pi/360)(.02)} = \underline{4.8\ \text{kV/m}}$$

7.15. The two conducting planes illustrated in Fig. 7.8 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find:

a) $V(\phi)$: The general solution to Laplace's equation will be $V = C_1\phi + C_2$, and so

$$20 = C_1(.188) + C_2 \ \text{ and } \ 200 = C_1(.179) + C_2$$

Subtracting one equation from the other, we find

$$-180 = C_1(.188 - .179) \;\Rightarrow\; C_1 = -2.00 \times 10^4$$

Then

$$20 = -2.00 \times 10^4(.188) + C_2 \;\Rightarrow\; C_2 = 3.78 \times 10^3$$

Finally, $V(\phi) = \underline{(-2.00 \times 10^4)\phi + 3.78 \times 10^3\ \text{V}}$.

b) $\mathbf{E}(\rho)$: Use

$$\mathbf{E}(\rho) = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi} = \underline{\frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi\ \text{V/m}}$$

c) $\mathbf{D}(\rho) = \epsilon_0\mathbf{E}(\rho) = \underline{(2.00 \times 10^4\epsilon_0/\rho)\,\mathbf{a}_\phi\ \text{C/m}^2}$.

d) ρ_s on the upper surface of the lower plane: We use

$$\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_{surface} = \frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi = \underline{\frac{2.00 \times 10^4}{\rho}\ \text{C/m}^2}$$

e) Q on the upper surface of the lower plane: This will be

$$Q_t = \int_0^{.1}\int_{.001}^{.120} \frac{2.00 \times 10^4\epsilon_0}{\rho}\,d\rho\,dz = 2.00 \times 10^4\epsilon_0(.1)\ln(120) = 8.47 \times 10^{-8}\ \text{C} = \underline{84.7\ \text{nC}}$$

f) Repeat $a)$ to $c)$ for region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. Back to the beginning, we use

$$20 = C_1'(.188 - 2\pi) + C_2' \ \text{ and } \ 200 = C_1'(.179) + C_2'$$

7.15f (continued) Subtracting one from the other, we find

$$-180 = C_1'(.009 - 2\pi) \;\Rightarrow\; C_1' = 28.7$$

Then $200 = 28.7(.179) + C_2' \Rightarrow C_2' = 194.9$. Thus $V(\phi) = \underline{28.7\phi + 194.9}$ in region 2. Then

$$\underline{\mathbf{E} = -\frac{28.7}{\rho}\,\mathbf{a}_\phi \text{ V/m}} \text{ and } \underline{\mathbf{D} = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \text{ C/m}^2}$$

ρ_s on the lower surface of the lower plane will now be

$$\rho_s = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) = \underline{\frac{28.7\epsilon_0}{\rho} \text{ C/m}^2}$$

The charge on that surface will then be $Q_b = 28.7\epsilon_0(.1)\ln(120) = \underline{122 \text{ pC}}$.

g) Find the total charge on the lower plane and the capacitance between the planes: Total charge will be $Q_{net} = Q_t + Q_b = 84.7 \text{ nC} + 0.122 \text{ nC} = \underline{84.8 \text{ nC}}$. The capacitance will be

$$C = \frac{Q_{net}}{\Delta V} = \frac{84.8}{200 - 20} = 0.471 \text{ nF} = \underline{471 \text{ pF}}$$

7.16. a) Solve Laplace's equation for the potential field in the homogeneous region between two concentric conducting spheres with radii a and b, $b > a$, if $V = 0$ at $r = b$ and $V = V_0$ at $r = a$. With radial variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

Multiply by r^2:

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \text{ or } r^2 \frac{dV}{dr} = A$$

Divide by r^2:

$$\frac{dV}{dr} = \frac{A}{r^2} \;\Rightarrow\; V = \frac{A}{r} + B$$

Note that in the last integration step, I dropped the minus sign that would have otherwise occurred in front of A, since we can choose A as we wish. Next, apply the boundary conditions:

$$0 = \frac{A}{b} + B \;\Rightarrow\; B = -\frac{A}{b}$$

$$V_0 = \frac{A}{a} - \frac{A}{b} \;\Rightarrow\; A = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Finally,

$$V(r) = \frac{V_0}{r\left(\frac{1}{a} - \frac{1}{b}\right)} - \frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)} = \underline{V_0 \frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

7.16b. Find the capacitance between them: Assume permittivity ϵ. First, the electric field will be

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}\mathbf{a}_r \text{ V/m}$$

Next, on the inner sphere, the charge density will be

$$\rho_s = \mathbf{D}\Big|_{r=a} \cdot \mathbf{a}_r = \frac{\epsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \text{ C/m}^2$$

The capacitance is now

$$C = \frac{Q}{V_0} = \frac{4\pi a^2 \rho_s}{V_0} = \underline{\frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \text{ F}}$$

7.17. Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere at 0 V:

a) Find the location of the 20 V equipotential surface: Solving Laplace's equation gives us

$$V(r) = V_0 \frac{\frac{1}{r}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$$

where $V_0 = 100$, $a = 5$ and $b = 20$. Setting $V(r) = 20$, and solving for r produces $r = \underline{12.5\text{ mm}}$.

b) Find $E_{r,max}$: Use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0\,\mathbf{a}_r}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

Then

$$E_{r,max} = E(r=a) = \frac{V_0}{a(1-(a/b))} = \frac{100}{5(1-(5/20))} = 26.7 \text{ V/mm} = \underline{26.7 \text{ kV/m}}$$

c) Find ϵ_R if the surface charge density on the inner sphere is $100\,\mu\text{C/m}^2$: ρ_s will be equal in magnitude to the electric flux density at $r = a$. So $\rho_s = (2.67 \times 10^4 \text{ V/m})\epsilon_R\epsilon_0 = 10^{-4}$ C/m^2. Thus $\epsilon_R = \underline{423\ !}$ (obviously a bad choice of numbers here – possibly a misprint. A more reasonable charge on the inner sphere would have been 1 μC/m^2, leading to $\epsilon_R = 4.23$).

7.18. Concentric conducting spheres have radii of 1 and 5 cm. There is a perfect dielectric for which $\epsilon_R = 3$ between them. The potential of the inner sphere is 2V and that of the outer is -2V. Find:

a) $V(r)$: We use the general expression derived in Problem 7.16: $V(r) = (A/r) + B$. At the inner sphere, $2 = (A/.01) + B$, and at the outer sphere, $-2 = (A/.05) + B$. Subtracting the latter equation from the former gives

$$4 = A\left(\frac{1}{.01} - \frac{1}{.05}\right) = 80A$$

so $A = .05$. Substitute A into either of the two potential equations at the boundaries to find $B = -3$. Finally, $V(r) = \underline{(.05/r) - 3}$.

7.18b. $\mathbf{E}(r) = -(dV/dr)\mathbf{a}_r = \underline{(.05/r^2)\mathbf{a}_r \text{ V/m}}$.

c) V at $r = 3$ cm: $V(.03) = (.05/.03) - 3 = \underline{-1.33 \text{ V}}$.

d) the location of the 0-V equipotential surface: Use

$$0 = (.05/r_0) - 3 \quad \Rightarrow \quad r_0 = (.05/3) = .0167\,\text{m} = \underline{1.67\,\text{cm}}$$

e) the capacitance between the spheres:

$$C = \frac{4\pi\epsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi(3)\epsilon_0}{\left(\frac{1}{.01} - \frac{1}{.05}\right)} = \frac{12\pi\epsilon_0}{80} = \underline{4.2\,\text{pF}}$$

7.19. Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find:

a) α for each cone: Have $\alpha_A = \tan^{-1}(1/2) = \underline{26.57^\circ}$ and $\alpha_B = \tan^{-1}(3/2) = \underline{56.31^\circ}$.

b) V at $P(1, 1, 1)$: The potential function between cones can be written as

$$V(\theta) = C_1 \ln\tan(\theta/2) + C_2$$

Then

$$20 = C_1 \ln\tan(56.31/2) + C_2 \;\text{ and }\; 100 = C_1 \ln\tan(26.57/2) + C_2$$

Solving these two equations, we find $C_1 = -97.7$ and $C_2 = -41.1$. Now at P, $\theta = \tan^{-1}(\sqrt{2}) = 54.7^\circ$. Thus

$$V_P = -97.7 \ln\tan(54.7/2) - 41.1 = \underline{23.3\,\text{V}}$$

7.20. A potential field in free space is given as $V = 100 \ln\tan(\theta/2) + 50$ V.

a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. First

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{100}{2r\tan(\theta/2)\cos^2(\theta/2)}\mathbf{a}_\theta = -\frac{100}{2r\sin(\theta/2)\cos(\theta/2)}\mathbf{a}_\theta = -\frac{100}{r\sin\theta}\mathbf{a}_\theta$$

This will maximize at the smallest value of r, or 0.1:

$$\mathbf{E}_{max}(\theta = 40^\circ) = \mathbf{E}(r = 0.1, \theta = 40^\circ) = -\frac{100}{0.1\sin(40)}\mathbf{a}_\theta = \underline{1.56\,\mathbf{a}_\theta \text{ kV/m}}$$

b) Describe the surface $V = 80$ V: Set $100\ln\tan\theta/2 + 50 = 80$ and solve for θ: Obtain $\ln\tan\theta/2 = 0.3 \;\Rightarrow\; \tan\theta/2 = e^{.3} = 1.35 \;\Rightarrow\; \theta = \underline{107^\circ}$ (the cone surface at $\theta = 107$ degrees).

7.21. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \rightarrow 0$ when $r \rightarrow 0$, and also that $V \rightarrow 0$ as $r \rightarrow \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2 \frac{dV}{dr}\right) = -\frac{200}{.6}r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2 E_r \rightarrow 0$ when $r \rightarrow 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4}r^{-.4} + C_2$$

From our second boundary condition, $V \rightarrow 0$ as $r \rightarrow \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4} \text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi \int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0 \frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3r^{-1.4} \text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4} dr' = \underline{833.3r^{-.4} \text{ V}}$$

7.22. Let the volume charge density in Fig. 7.3a be given by $\rho_v = \rho_{v0}(x/a)e^{-|x|/a}$ (note error in the exponent in the formula stated in the book).

a) Determine $\rho_{v,max}$ and $\rho_{v,min}$ and their locations: Let $x' = x/a$. Then $\rho_v = x'e^{-|x'|}$. Differentiate with respect to x' to obtain:

$$\frac{d\rho_v}{dx'} = \rho_{v0}e^{-|x'|}(1 - |x'|)$$

This derivative is zero at $x' = \pm 1$, or the minimum and maximum occur at $x = \underline{\pm a}$ respectively. The values of ρ_v at these points will be $\rho_{v,max} = \rho_{v0}e^{-1} = \underline{0.368\rho_{v0}}$, occurring at $x = a$. $\rho_{v,min} = -\rho_{v0}e^{-1} = \underline{-0.368\rho_{v0}}$, occurring at $x = -a$.

7.22b. Find E_x and $V(x)$ if $V(0) = 0$ and $E_x \to 0$ as $x \to \infty$: We use Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \Rightarrow \quad \frac{d^2V}{dx^2} = -\frac{\rho_{v0}}{\epsilon}\left(\frac{x}{a}\right)e^{-|x|/a}$$

For $x > 0$, this becomes

$$\frac{d^2V}{dx^2} = -\frac{\rho_{v0}}{\epsilon}\left(\frac{x}{a}\right)e^{-x/a}$$

Integrate once over x:

$$\frac{dV}{dx}(x > 0) = -\frac{\rho_{v0}}{\epsilon}\int\left(\frac{x}{a}\right)e^{-x/a}\,dx + C_1 = \frac{a\rho_{v0}}{\epsilon}e^{-x/a}\left(\frac{x}{a}+1\right) + C_1$$

Noting that $E_x = -dV/dx$, we use the first boundary condition, $E_x \to 0$ as $x \to \infty$, to establish that $C_1 = 0$. Over the range $x < 0$, we have

$$\frac{dV}{dx}(x < 0) = -\frac{\rho_{v0}}{\epsilon}\int\left(\frac{x}{a}\right)e^{x/a}\,dx + C_1' = \frac{a\rho_{v0}}{\epsilon}e^{x/a}\left(\frac{-x}{a}+1\right) + C_1'$$

where $C_1' = 0$, since, by symmetry, $E_x \to 0$ as $x \to -\infty$. These two equations can be unified to cover the entire range of x; the final expression for the electric field becomes:

$$E_x = -\frac{dV}{dx} = \underline{-\frac{a\rho_{v0}}{\epsilon}\left(\frac{|x|}{a}+1\right)e^{-|x|/a}\ \text{V/m}}$$

The potential function is now found by a second integration. For $x > 0$, this is

$$V(x) = \frac{a\rho_{v0}}{\epsilon}\int\left[\left(\frac{x}{a}\right)e^{-x/a} + e^{-x/a}\right]dx + C_2 = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{-x/a} - 2e^{-x/a}\right] + C_2$$

We use the second boundary condition, $V(0) = 0$, from which $C_2 = 2a^2\rho_{v0}/\epsilon$. Substituting this yields

$$V(x)\ (x > 0) = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{-x/a} + 2(1 - e^{-x/a})\right]$$

We repeat the procedure for $x < 0$ to obtain

$$V(x) = \frac{a\rho_{v0}}{\epsilon}\int\left[\left(\frac{x}{a}\right)e^{x/a} + e^{x/a}\right]dx + C_2' = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{x/a} - 2e^{x/a}\right] + C_2'$$

Again, with the $V(0) = 0$ boundary condition, we find $C_2' = -2a^2\rho_{v0}/\epsilon$, which when substituted leads to

$$V(x)\ (x < 0) = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{x/a} - 2(1 - e^{x/a})\right]$$

Combining the results for both ranges of x, we write

$$V(x) = \underline{\frac{-a^2\rho_{v0}}{\epsilon}\left[\left(\frac{x}{a}\right)e^{-|x|/a} - \frac{2x}{|x|}\left(1 - e^{-|x|/a}\right)\right]}$$

7.22c. Use a development similar to that of Sec. 7.4 to show that $C = dQ/dV_0 = \epsilon S/8a$ (note error in problem statement): First, the overall potential difference is

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = 2 \times \frac{2a^2\rho_{v0}}{\epsilon} = \frac{4a^2\rho_{v0}}{\epsilon}$$

From this we find $a = \sqrt{(\epsilon V_0)/(4\rho_{v0})}$. Then the total charge on one side will be

$$Q = S\int_0^\infty \rho_{v0}\left(\frac{x}{a}\right)e^{-x/a}\,dx = S\rho_{v0}\,ae^{-x/a}\left[\frac{-x}{a} - 1\right]\Big|_0^\infty = S\rho_{v0}\,a = \frac{1}{2}S\sqrt{\epsilon V_0\rho_{v0}}$$

Now

$$C = \frac{dQ}{dV_0} = \frac{d}{dV_0}\left(\frac{1}{2}S\sqrt{\epsilon V_0\rho_{v0}}\right) = \frac{S}{4}\sqrt{\frac{\rho_{v0}\,\epsilon}{V_0}}$$

But $a = \sqrt{(\epsilon V_0)/(4\rho_{v0})}$, from which $(\rho_{v0}/V_0) = \epsilon/(4a^2)$. Substituting this into the capacitance expression gives

$$C = \frac{S}{4}\sqrt{\frac{\epsilon^2}{4a^2}} = \underline{\frac{\epsilon S}{8a}}$$

7.23. A rectangular trough is formed by four conducting planes located at $x = 0$ and 8 cm and $y = 0$ and 5 cm in air. The surface at $y = 5$ cm is at a potential of 100 V, the other three are at zero potential, and the necessary gaps are placed at two corners. Find the potential at $x = 3$ cm, $y = 4$ cm: This situation is the same as that of Fig. 7.6, except the non-zero boundary potential appears on the top surface, rather than the right side. The solution is found from Eq. (39) by simply interchanging x and y, and b and d, obtaining:

$$V(x, y) = \frac{4V_0}{\pi}\sum_{1,odd}^{\infty}\frac{1}{m}\frac{\sinh(m\pi y/d)}{\sinh(m\pi b/d)}\sin\frac{m\pi x}{d}$$

where $V_0 = 100$ V, $d = 8$ cm, and $b = 5$ cm. We will use the first three terms to evaluate the potential at (3,4):

$$\begin{aligned} V(3,4) &\doteq \frac{400}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(5\pi/8)}\sin(3\pi/8) + \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(15\pi/8)}\sin(9\pi/8) + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(25\pi/8)}\sin(15\pi/8)\right] \\ &= \frac{400}{\pi}[.609 - .040 - .011] = 71.1 \text{ V} \end{aligned}$$

Additional accuracy is found by including more terms in the expansion. Using thirteen terms, and using six significant figure accuracy, the result becomes $V(3,4) \doteq \underline{71.9173 \text{ V}}$. The series converges rapidly enough so that terms after the sixth one produce no change in the third digit. Thus, quoting three significant figures, 71.9 V requires six terms, with subsequent terms having no effect.

7.24. The four sides of a square trough are held at potentials of 0, 20, -30, and 60 V; the highest and lowest potentials are on opposite sides. Find the potential at the center of the trough: Here we can make good use of symmetry. The solution for a single potential on the right side, for example, with all other sides at 0V is given by Eq. (39):

$$V(x,y) = \frac{4V_0}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin\left(\frac{m\pi y}{b}\right)$$

In the current problem, we can account for the three voltages by superposing three solutions of the above form, suitably modified to account for the different locations of the boundary potentials. Since we want V at the center of a square trough, it no longer matters on what boundary each of the given potentials is, and we can simply write:

$$V(\text{center}) = \frac{4(0+20-30+60)}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi/2)}{\sinh(m\pi)} \sin(m\pi/2) = \underline{12.5\,\text{V}}$$

The series converges to this value in three terms.

7.25. In Fig. 7.7, change the right side so that the potential varies linearly from 0 at the bottom of that side to 100 V at the top. Solve for the potential at the center of the trough: Since the potential reaches zero periodically in y and also is zero at $x = 0$, we use the form:

$$V(x,y) = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi x}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

Now, at $x = d$, $V = 100(y/b)$. Thus

$$100\frac{y}{b} = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

We then multiply by $\sin(n\pi y/b)$, where n is a fixed integer, and integrate over y from 0 to b:

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \underbrace{\int_0^b \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) dy}_{=b/2 \text{ if } m=n,\ \text{zero if } m\neq n}$$

The integral on the right hand side picks the nth term out of the series, enabling the coefficients, V_n, to be solved for individually as we vary n. We find in general,

$$V_m = \frac{2}{b\sinh(m\pi/d)} \int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy$$

The integral evaluates as

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \left\{ \begin{array}{c} -100/m\pi \ \text{(m even)} \\ 100/m\pi \ \text{(m odd)} \end{array} \right\} = (-1)^{m+1}\frac{100}{m\pi}$$

7.25 (continued) Thus

$$V_m = \frac{200(-1)^{m+1}}{m\pi b\ \sinh(m\pi d/b)}$$

So that finally,

$$V(x, y) = \frac{200}{\pi b} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{\sinh\,(m\pi x/b)}{\sinh\,(m\pi d/b)} \sin\left(\frac{m\pi y}{b}\right)$$

Now, with a square trough, set $b = d = 1$, and so $0 < x < 1$ and $0 < y < 1$. The potential becomes

$$V(x, y) = \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{\sinh\,(m\pi x)}{\sinh\,(m\pi)} \sin\,(m\pi y)$$

Now at the center of the trough, $x = y = 0.5$, and, using four terms, we have

$$V(.5, .5) \doteq \frac{200}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(\pi)} - \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(3\pi)} + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(5\pi)} - \frac{1}{7}\frac{\sinh(7\pi/2)}{\sinh(7\pi)}\right] = \underline{12.5\text{ V}}$$

where additional terms do not affect the three-significant-figure answer.

7.26. If X is a function of x and $X'' + (x-1)X - 2X = 0$, assume a solution in the form of an infinite power series and determine numerical values for a_2 to a_8 if $a_0 = 1$ and $a_1 = -1$: The series solution will be of the form:

$$X = \sum_{m=0}^{\infty} a_m x^m$$

The first 8 terms of this are substituted into the given equation to give:

$$\begin{aligned}
&(2a_2 - a_1 - 2a_0) + (6a_3 + a_1 - 2a_2 - 2a_1)x + (12a_4 + 2a_2 - 3a_3 - 2a_2)x^2 \\
&+ (3a_3 - 4a_4 - 2a_3 + 20a_5)x^3 + (30a_6 + 4a_4 - 5a_5 - 2a_4)x^4 + (42a_7 + 5a_5 - 6a_6 - 2a_5)x^5 \\
&+ (56a_8 + 6a_6 - 7a_7 - 2a_6)x^6 + (7a_7 - 8a_8 - 2a_7)x^7 + (8a_8 - 2a_8)x^8 = 0
\end{aligned}$$

For this equation to be zero, each coefficient term (in parenthesis) must be zero. The first of these is

$$2a_2 - a_1 - 2a_0 = 2a_2 + 1 - 2 = 0 \quad\Rightarrow\quad a_2 = \underline{1/2}$$

The second coefficient is

$$6a_3 + a_1 - 2a_2 - 2a_1 = 6a_3 - 1 - 1 + 2 = 0 \quad\Rightarrow\quad a_3 = \underline{0}$$

Third coefficient:

$$12a_4 + 2a_2 - 3a_3 - 2a_2 = 12a_4 + 1 - 0 - 1 = 0 \quad\Rightarrow\quad a_4 = \underline{0}$$

Fourth coefficient:

$$3a_3 - 4a_4 - 2a_3 + 20a_5 = 0 - 0 - 0 + 20a_5 = 0 \quad\Rightarrow\quad a_5 = \underline{0}$$

In a similar manner, we find $a_6 = a_7 = a_8 = \underline{0}$.

7.27. It is known that $V = XY$ is a solution of Laplace's equation, where X is a function of x alone, and Y is a function of y alone. Determine which of the following potential function are also solutions of Laplace's equation:

a) $V = 100X$: We know that $\nabla^2 XY = 0$, or

$$\frac{\partial^2}{\partial x^2}XY + \frac{\partial^2}{\partial y^2}XY = 0 \;\Rightarrow\; YX'' + XY'' = 0 \;\Rightarrow\; \frac{X''}{X} = -\frac{Y''}{Y} = \alpha^2$$

Therefore, $\nabla^2 X = 100X'' \neq 0$ – No.

b) $V = 50XY$: Would have $\nabla^2 V = 50\nabla^2 XY = 0$ – Yes.

c) $V = 2XY + x - 3y$: $\nabla^2 V = 2\nabla^2 XY + 0 - 0 = 0$ – Yes.

d) $V = xXY$: $\nabla^2 V = x\nabla^2 XY + XY\nabla^2 x = 0$ – Yes.

e) $V = X^2Y$: $\nabla^2 V = X\nabla^2 XY + XY\nabla^2 X = 0 + XY\nabla^2 X$ – No.

7.28. Assume a product solution of Laplace's equation in cylindrical coordinates, $V = PF$, where V is not a function of z, P is a function only of ρ, and F is a function only of ϕ.

a) Obtain the two separated equations if the separation constant is n^2. Select the sign of n^2 so that the solution of the ϕ equation leads to trigonometric functions: Begin with Laplace's equation in cylindrical coordinates, in which there is no z variation:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

We substitute the product solution $V = PF$ to obtain:

$$\frac{F}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = \frac{F}{\rho}\frac{dP}{d\rho} + F\frac{d^2P}{d\rho^2} + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = 0$$

Next, multiply by ρ^2 and divide by FP to obtain

$$\underbrace{\frac{\rho}{P}\frac{dP}{d\rho} + \frac{\rho^2}{P}\frac{d^2P}{d\rho^2}}_{n^2} + \underbrace{\frac{1}{F}\frac{d^2F}{d\phi^2}}_{-n^2} = 0$$

The equation is now grouped into two parts as shown, each a function of only one of the two variables; each is set equal to plus or minus n^2, as indicated. The ϕ equation now becomes

$$\frac{d^2F}{d\phi^2} + n^2F = 0 \;\Rightarrow\; F = C_n\cos(n\phi) + D_n\sin(n\phi) \;\; (n \geq 1)$$

Note that n is required to be an integer, since physically, the solution must repeat itself every 2π radians in ϕ. If $n = 0$, then

$$\frac{d^2F}{d\phi^2} = 0 \;\Rightarrow\; F = C_0\phi + D_0$$

7.28b. Show that $P = A\rho^n + B\rho^{-n}$ satisfies the ρ equation: From part a, the radial equation is:

$$\rho^2 \frac{d^2 P}{d\rho^2} + \rho \frac{dP}{d\rho} - n^2 P = 0$$

Substituting $A\rho^n$, we find

$$\rho^2 n(n-1)\rho^{n-2} + \rho n \rho^{n-1} - n^2\rho^n = n^2\rho^n - n\rho^n + n\rho^n - n^2\rho^n = 0$$

Substituting $B\rho^{-n}$:

$$\rho^2 n(n+1)\rho^{-(n+2)} - \rho n \rho^{-(n+1)} - n^2\rho^{-n} = n^2\rho^{-n} + n\rho^{-n} - n\rho^{-n} - n^2\rho^{-n} = 0$$

So it works.

c) Construct the solution $V(\rho, \phi)$. Functions of this form are called *circular harmonics*. To assemble the complete solution, we need the radial solution for the case in which $n = 0$. The equation to solve is

$$\rho \frac{d^2 P}{d\rho^2} + \frac{dP}{d\rho} = 0$$

Let $S = dP/d\rho$, and so $dS/d\rho = d^2P/d\rho^2$. The equation becomes

$$\rho \frac{dS}{d\rho} + S = 0 \quad \Rightarrow \quad -\frac{d\rho}{\rho} = \frac{dS}{S}$$

Integrating, find

$$-\ln\rho + \ln A_0 = \ln S \quad \Rightarrow \quad \ln S = \ln\left(\frac{A_0}{\rho}\right) \quad \Rightarrow \quad S = \frac{A_0}{\rho} = \frac{dP}{d\rho}$$

where A_0 is a constant. So now

$$\frac{d\rho}{\rho} = \frac{dP}{A_0} \quad \Rightarrow \quad P_{n=0} = A_0 \ln\rho + B_0$$

We may now construct the solution in its complete form, encompassing $n \geq 0$:

$$V(\rho, \phi) = \underbrace{(A_0 \ln\rho + B_0)(C_0\phi + D_0)}_{n=0 \text{ solution}} + \sum_{n=1}^{\infty} [A_n\rho^n + B_n\rho^{-n}][C_n \cos(n\phi) + D_n \sin(n\phi)]$$