CHAPTER 5

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]$ kA/m^2:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1\left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right]dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1-e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part b, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.2. Let the current density be $\mathbf{J} = 2\phi\cos^2\phi\mathbf{a}_\rho - \rho\sin 2\phi\mathbf{a}_\phi$ A/m^2 within the region $2.1 < \rho < 2.5$, $0 < \phi < 0.1$ rad, $6 < z < 6.1$. Find the total current I crossing the surface:

a) $\rho = 2.2$, $0 < \phi < 0.1$, $6 < z < 6.1$ in the $\mathbf{a}_\rho$ direction: This is a surface of constant ρ, so only the radial component of $\mathbf{J}$ will contribute: At $\rho = 2.2$ we write:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_0^{0.1} 2(2)\cos^2\phi\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,2d\phi dz = 2(2.2)^2(0.1)\int_0^{0.1}\frac{1}{2}(1+\cos 2\phi)\,d\phi$$
$$= 0.2(2.2)^2\left[\frac{1}{2}(0.1) + \frac{1}{4}\sin 2\phi\Big|_0^{0.1}\right] = \underline{97\,\text{mA}}$$

b) $\phi = 0.05$, $2.2 < \rho < 2.5$, $6 < z < 6.1$ in the $\mathbf{a}_\phi$ direction: In this case only the ϕ component of $\mathbf{J}$ will contribute:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_{2.2}^{2.5} -\rho\sin 2\phi\big|_{\phi=0.05}\,\mathbf{a}_\phi\cdot\mathbf{a}_\phi\,d\rho\,dz = -(0.1)^2\frac{\rho^2}{2}\Big|_{2.2}^{2.5} = \underline{-7\,\text{mA}}$$

5.2c. Evaluate $\nabla \cdot \mathbf{J}$ at $P(\rho = 2.4, \phi = 0.08, z = 6.05)$:

$$\nabla \cdot \mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho J_\rho) + \frac{1}{\rho}\frac{\partial J_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 \cos^2 \phi) - \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho \sin 2\phi) = 4\cos^2\phi - 2\cos 2\phi \Big|_{0.08}$$
$$= \underline{2.0\,\mathrm{A/m^3}}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2 + 4}\,\mathbf{a}_r \ \mathrm{A/m^2}$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J}\cdot\mathbf{n}\Big|_S\, da = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} \frac{400\sin\theta}{(.8)^2 + 4}(.8)^2 \sin\theta\, d\theta\, d\phi = \frac{400(.8)^2 2\pi}{4.64}\int_{.1\pi}^{.3\pi} \sin^2\, d\theta$$
$$= 346.5\int_{.1\pi}^{.3\pi} \frac{1}{2}[1 - \cos(2\theta)]\, d\theta = \underline{77.4\,\mathrm{A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta\, d\theta\, d\phi = 1.46\,\mathrm{m^2}$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\,\mathbf{a}_r = \underline{53.0\,\mathbf{a}_r\ \mathrm{A/m^2}}$.

5.4. The cathode of a planar vacuum tube is at $z = 0$. Let $\mathbf{E} = -4 \times 10^6\,\mathbf{a}_z$ V/m for $z > 0$. An electron ($e = 1.602 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg) is emitted from the cathode with zero initial velocity at $t = 0$.

a) Find $v(t)$: Using Newton's second law, we write:

$$\mathbf{F} = m\mathbf{a} = q\mathbf{E} \ \Rightarrow\ \mathbf{a} = \frac{(-1.602 \times 10^{-19})(-4 \times 10^6)\mathbf{a}_z}{(9.11 \times 10^{-31})} = 7.0 \times 10^{17}\mathbf{a}_z\ \mathrm{m/s^2}$$

Then $v(t) = at = \underline{7.0 \times 10^{17} t\ \mathrm{m/s}}$.

b) Find $z(t)$, the electron location as a function of time: Use

$$z(t) = \int_0^t v(t')dt' = \frac{1}{2}(7.0 \times 10^{17})t^2 = \underline{3.5 \times 10^{17} t^2\ \mathrm{m}}$$

c) Determine $v(z)$: Solve the result of part b for t, obtaining

$$t = \frac{\sqrt{z}}{\sqrt{3.5 \times 10^{17}}} = 1.7 \times 10^9 \sqrt{z}$$

Substitute into the result of part a to find $v(z) = 7.0 \times 10^{17}(1.7 \times 10^{-9})\sqrt{z} = \underline{1.2 \times 10^9 \sqrt{z}\ \mathrm{m/s}}$.

5.4d. Make the assumption that the electrons are emitted continuously as a beam with a 0.25 mm radius and a total current of 60 μA. Find $\mathbf{J}(z)$ and $\rho(z)$:

$$\mathbf{J}(z) = \frac{-60 \times 10^{-6}}{\pi(0.25)^2(10^{-6})}\mathbf{a}_z = \underline{-3.1 \times 10^2\, \mathbf{a}_z \text{ A/m}^2}$$

(negative since we have electrons flowing in the positive z direction) Next we use $\mathbf{J}(z) = \rho_v(z)\mathbf{v}(z)$, or

$$\rho_v(z) = \frac{J}{v} = \frac{-3.1 \times 10^2}{1.2 \times 10^9\sqrt{z}} = -\frac{2.6 \times 10^{-7}}{\sqrt{z}} \text{ C/m}^3 = \underline{\frac{-26}{\sqrt{z}}\ \mu\text{C/m}^3}$$

5.5. Let

$$\mathbf{J} = \frac{25}{\rho}\mathbf{a}_\rho - \frac{20}{\rho^2 + 0.01}\mathbf{a}_z \text{ A/m}^2$$

a) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$: Use

$$I = \int\int_S \mathbf{J} \cdot \mathbf{n}\Big|_{z=.2} da = \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\rho\, d\rho\, d\phi$$
$$= -\left(\frac{1}{2}\right) 20 \ln(.01 + \rho^2)\Big|_0^{.4}(2\pi) = -20\pi \ln(17) = \underline{-178.0 \text{ A}}$$

b) Calculate $\partial\rho_v/\partial t$: This is found using the equation of continuity:

$$\frac{\partial\rho_v}{\partial t} = -\nabla \cdot \mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2 + .01}\right) = \underline{0}$$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$: This will be

$$I = \int_0^{.2}\int_0^{2\pi} \frac{25}{.01}\mathbf{a}_\rho \cdot (-\mathbf{a}_\rho)(.01)\, d\phi\, dz + \int_0^{.2}\int_0^{2\pi} \frac{25}{.4}\mathbf{a}_\rho \cdot (\mathbf{a}_\rho)(.4)\, d\phi\, dz$$
$$+ \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z \cdot (-\mathbf{a}_z)\,\rho\, d\rho\, d\phi + \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z \cdot (\mathbf{a}_z)\,\rho\, d\rho\, d\phi = \underline{0}$$

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for $\mathbf{J}$ and the surface specified in part b. In part c, the net outward flux was found to be zero, and in part b, the divergence of $\mathbf{J}$ was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.

5.6. Let $\epsilon = \epsilon_0$ and $V = 90z^{4/3}$ in the region $z = 0$.

a) Obtain expressions for $\mathbf{E}$, $\mathbf{D}$, and ρ_v as functions of z: First,

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{4}{3}(90)z^{1/3}\mathbf{a}_z = \underline{-120z^{1/3}\mathbf{a}_z \text{ V/m}}$$

5.6a. (continued)

Next, $\mathbf{D} = \epsilon_0 \mathbf{E} = \underline{1.06 z^{1/3} \mathbf{a}_z \text{ nC/m}^2}$. Then

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{dD_z}{dz} = -\frac{1}{3}(120)\epsilon_0 z^{-2/3} = \underline{-354 z^{-2/3} \text{ pC/m}^3}$$

b) If the velocity of the charge density is given as $v_z = 5 \times 10^6 z^{2/3}$ m/s, find J_z at $z = 0$ and $z = 0.1$ m (note that v_z is written as v_x through a missprint): Use $J_z = \rho_v v_z = (-354 \times 10^{-12}) z^{-2/3} (5 \times 10^6) z^{2/3} = \underline{-1.8 \text{ mA/m}^2}$ at any z.

5.7. Assuming that there is no transformation of mass to energy or vice-versa, it is possible to write a continuity equation for mass.

a) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? These would be, respectively, <u>mass flux density</u> in (kg/m^2 − s) and <u>mass density</u> in (kg/m^3).

b) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, -9.85, 1.75, -2.00, -4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center. We may write the continuity equation for mass as follows, also invoking the divergence theorem:

$$\int_v \frac{\partial \rho_m}{\partial t} dv = -\int_v \nabla \cdot \mathbf{J}_m \, dv = -\oint_s \mathbf{J}_m \cdot \mathbf{dS}$$

where

$$\oint_s \mathbf{J}_m \cdot \mathbf{dS} = 10.25 - 9.85 + 1.75 - 2.00 - 4.05 + 4.45 = 0.550 \text{ mg/s}$$

Treating our 1 cm^3 volume as differential, we find

$$\frac{\partial \rho_m}{\partial t} \doteq -\frac{0.550 \times 10^{-3} \text{ g/s}}{10^{-6} \text{ m}^3} = \underline{-550 \text{ g/m}^3 - \text{s}}$$

5.8. The continuity equation for mass equates the divergence of the mass rate of flow (mass per second per square meter) to the negative of the density (mass per cubic meter). After setting up a cartesian coordinate system inside a star, Captain Kirk and his intrepid crew make measurements over the faces of a cube centered at the origin with edges 40 km long and parallel to the coordinate axes. They find the mass rate of flow of material outward across the six faces to be -1112, 1183, 201, -196, 1989, and -1920 kg/m$^2 \cdot$ s.

a) Estimate the divergence of the mass rate of flow at the origin: We make the estimate using the definition of divergence, but without taking the limit as the volume shrinks to zero:

$$\text{Div}\, \mathbf{J}_m \doteq \frac{\oint \mathbf{J}_m \cdot d\mathbf{S}}{\Delta v} = \frac{(-1112 + 1183 + 201 - 196 + 1989 - 1920)(40)^2}{(40)^3} = \underline{3.63 \text{ kg/km}^3 \cdot \text{s}}$$

b) Estimate the rate of change of the density at the origin: The continuity equation for mass reads: Div $\mathbf{J}_m = -\partial \rho_m / \partial t$. Therefore, the rate of change of density at the origin will be just the negative of the part *a* result, or $\partial \rho_m / \partial t \doteq \underline{-3.63 \text{ kg/km}^3 \cdot \text{s}}$.

5.9a. Using data tabulated in Appendix C, calculate the required diameter for a 2-m long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it:

The required resistance will be

$$R = \frac{V^2}{P} = \frac{l}{\sigma(\pi a^2)}$$

Thus the diameter will be

$$d = 2a = 2\sqrt{\frac{lP}{\sigma\pi V^2}} = 2\sqrt{\frac{2(450)}{(10^6)\pi(120)^2}} = 2.8 \times 10^{-4}\,\text{m} = \underline{0.28\,\text{mm}}$$

b) Calculate the rms current density in the wire: The rms current will be $I = 450/120 = 3.75\,\text{A}$. Thus

$$J = \frac{3.75}{\pi\left(2.8 \times 10^{-4}/2\right)^2} = \underline{6.0 \times 10^7\ \text{A/m}^2}$$

5.10. A steel wire has a radius of 2 mm and a conductivity of 2×10^6 S/m. The steel wire has an aluminum ($\sigma = 3.8 \times 10^7$ S/m) coating of 2 mm thickness. Let the total current carried by this hybrid conductor be 80 A dc. Find:

a) J_{st}. We begin with the fact that electric field must be the same in the aluminum and steel regions. This comes from the requirement that **E** tangent to the boundary between two media must be continuous, and from the fact that when integrating E over the wire length, the applied voltage value must be obtained, regardless of the medium within which this integral is evaluated. We can therefore write

$$E_{Al} = E_{st} = \frac{J_{Al}}{\sigma_{Al}} = \frac{J_{st}}{\sigma_{st}} \quad \Rightarrow \quad J_{Al} = \frac{\sigma_{Al}}{\sigma_{st}} J_{st}$$

The net current is now expressed as the sum of the currents in each region, written as the sum of the products of the current densities in each region times the appropriate cross-sectional area:

$$I = \pi(2 \times 10^{-3})^2 J_{st} + \pi[(4 \times 10^{-3})^2 - (2 \times 10^{-3})^2] J_{Al} = 80\,\text{A}$$

Using the above relation between J_{st} and J_{Al}, we find

$$80 = \pi\left[(2 \times 10^{-3})^2\left[1 - \left(\frac{3.8 \times 10^7}{6 \times 10^6}\right)\right] + (4 \times 10^{-3})^2\left(\frac{3.8 \times 10^7}{6 \times 10^6}\right)\right] J_{st}$$

Solve for J_{st} to find $J_{st} = \underline{3.2 \times 10^5\ \text{A/m}^2}$.

b)

$$J_{Al} = \frac{3.8 \times 10^7}{6 \times 10^6}(3.2 \times 10^5) = \underline{2.0 \times 10^6\ \text{A/m}^2}$$

c,d) $E_{st} = E_{Al} = J_{st}/\sigma_{st} = J_{Al}/\sigma_{Al} = \underline{5.3 \times 10^{-2}\ \text{V/m}}$.

e) the voltage between the ends of the conductor if it is 1 mi long: Using the fact that 1 mi $= 1.61 \times 10^3$ m, we have $V = El = (5.3 \times 10^{-2})(1.61 \times 10^3) = \underline{85.4\ \text{V}}$.

5.11. Two perfectly-conducting cylindrical surfaces are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. *Assume the cylinders are both of length l.*

a) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05\,\text{S/m}$ is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho \text{ A/m}^2$$

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot\mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}\text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\ \Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E}\cdot\mathbf{J}\,dv = \int_0^l\int_0^{2\pi}\int_{.03}^{.05}\frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\,\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l}\ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\text{ W}$$

which is in agreement with the power density integration.

5.12. The spherical surfaces $r = 3$ and $r = 5$ cm are perfectly conducting, and the total current passing radially outward through the medium between the surfaces is 3 A dc.

a) Find the voltage and resistance between the spheres, and **E** in the region between them, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < r < 5$ cm. We first find **J** as a function of radius by dividing the current by the area of a sphere of radius r:

$$\mathbf{J} = \frac{I}{4\pi r^2}\,\mathbf{a}_r = \frac{3}{4\pi r^2}\,\mathbf{a}_r \text{ A/m}^2$$

Then

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{3}{4\pi r^2(0.05)}\,\mathbf{a}_r = \underline{\frac{4.77}{r^2}\,\mathbf{a}_r \text{ V/m}}$$

5.12a. (continued)

$$V = -\int_{r_2}^{r_1} \mathbf{E}\cdot d\mathbf{L} = -\int_{.05}^{.03} \frac{4.77}{r^2}\,dr = 4.77\left[\frac{1}{.03} - \frac{1}{.05}\right] = \underline{63.7\text{ V}}$$

Finally, $R = V/I = 63.7/3 = \underline{21.2\ \Omega}$.

b) Repeat if $\sigma = 0.0005/r$ for $3 < r < 5$ cm: First, $\mathbf{J} = 3\mathbf{a}_r/(4\pi r^2)$ as before. The electric field is now

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{3r\mathbf{a}_r}{4\pi(.0005)r^2} = \underline{\frac{477}{r}\,\mathbf{a}_r\text{ V/m}}$$

Now

$$V = -\int_{r_2}^{r_1} \mathbf{E}\cdot d\mathbf{L} = -\int_{.05}^{.03} \frac{477}{r}\,dr = -477\ln\left(\frac{.03}{.05}\right) = \underline{244\text{ V}}$$

Finally, $R = V/I = 244/3 = \underline{81.3\ \Omega}$.

c) Show that integrating the power dissipated per unit volume in part b over the volume gives the total dissipated power: The dissipated power density is

$$p_d = \mathbf{E}\cdot\mathbf{J} = \left(\frac{3}{4\pi(.0005)r}\right)\left(\frac{3}{4\pi r^2}\right) = \frac{114}{r^3}\text{ W/m}^2$$

We integrate this over the volume between spheres:

$$P_d = \int_0^{2\pi}\int_0^{\pi}\int_{.03}^{.05} \frac{114}{r^3}\,r^2\sin\theta\,dr\,d\theta\,d\phi = 4\pi(114)\ln\left(\frac{5}{3}\right) = \underline{732\text{ W}}$$

The dissipated power should be just $I^2R = (3)^2(81.3) = 732$ W. So it works.

5.13. A hollow cylindrical tube with a rectangular cross-section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5\times 10^7$ S/m. A current of 200 A dc is flowing down the tube.

a) What voltage drop is present across a 1m length of the tube? Converting all measurements to meters, the tube resistance over a 1 m length will be:

$$R_1 = \frac{1}{(1.5\times 10^7)\left[(2.54)(2.54/2)\times 10^{-4} - 2.54(1-.1)(2.54/2)(1-.2)\times 10^{-4}\right]}$$
$$= 7.38\times 10^{-4}\ \Omega$$

The voltage drop is now $V = IR_1 = 200(7.38\times 10^{-4} = \underline{0.147\text{ V}}$.

b) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5\times 10^5$ S/m: The resistance of the filling will be:

$$R_2 = \frac{1}{(1.5\times 10^5)(1/2)(2.54)^2\times 10^{-4}(.9)(.8)} = 2.87\times 10^{-2}\ \Omega$$

The total resistance is now the parallel combination of R_1 and R_2:
$R_T = R_1R_2/(R_1+R_2) = 7.19\times 10^{-4}\ \Omega$, and the voltage drop is now $V = 200R_T = \underline{.144\text{ V}}$.

5.14. Find the magnitude of the electric field intensity in a conductor if:

a) the current density is $5\,\text{MA/m}^2$, the electron mobility is $3 \times 10^{-3}\ \text{m}^2/\text{V}\cdot\text{s}$, and the volume charge density is $-2.4 \times 10^{10}\ \text{C/m}^3$: In magnitude, we have

$$E = \frac{J}{\mu_e \rho_v} = \frac{5 \times 10^6}{(2.4 \times 10^{10})(3 \times 10^{-3})} = \underline{6.9 \times 10^{-2}\ \text{V/m}}$$

b) $J = 3\ \text{MA/m}^2$ and the resistivity is $3 \times 10^{-8}\ \Omega \cdot \text{m}$: $E = J\rho = (3 \times 10^6)(3 \times 10^{-8}) = \underline{9 \times 10^{-2}\ \text{V/m}}$.

5.15. Let $V = 10(\rho + 1)z^2 \cos\phi$ V in free space.

a) Let the equipotential surface $V = 20$ V define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\underline{(\rho + 1)z^2 \cos\phi = 2}$$

b) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$: At the given values of ϕ and z, we solve the equation of the surface found in part a for ρ, obtaining $\rho = \underline{.10}$. Then

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi - \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= -10z^2 \cos\phi\, \mathbf{a}_\rho + 10\,\frac{\rho + 1}{\rho} z^2 \sin\phi\, \mathbf{a}_\phi - 20(\rho + 1)z \cos\phi\, \mathbf{a}_z$$

Then

$$\mathbf{E}(.10, .2\pi, 1.5) = \underline{-18.2\,\mathbf{a}_\rho + 145\,\mathbf{a}_\phi - 26.7\,\mathbf{a}_z\ \text{V/m}}$$

c) Find $|\rho_s|$ at that point: Since $\mathbf{E}$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s = \epsilon_0 \mathbf{E} \cdot \mathbf{n}\Big|_{\text{surface}} = \epsilon_0 \frac{\mathbf{E} \cdot \mathbf{E}}{|\mathbf{E}|} = \epsilon_0 \sqrt{\mathbf{E} \cdot \mathbf{E}} = \epsilon_0 \sqrt{(18.2)^2 + (145)^2 + (26.7)^2} = \underline{1.32\ \text{nC/m}^2}$$

5.16. A potential field in free space is given as $V = (80 \cos\theta \sin\phi)/r^3$ V. Point $P(r = 2, \theta = \pi/3, \phi = \pi/2)$ lies on a conducting surface.

a) Write the equation of the conducting surface: The surface will be an equipotential, where the value of the potential is V_P:

$$V_P = \frac{80 \cos(\pi/3) \sin(\pi/2)}{(2)^3} = 5$$

So the equation of the surface is

$$\frac{80 \cos\theta \sin\phi}{r^3} = 5 \quad \text{or} \quad \underline{16 \cos\theta \sin\phi = r^3}$$

5.16c. (I will work parts b and c in reverse order)

Find $\mathbf{E}$ at P:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi$$
$$= \frac{80(3)\cos\theta\sin\phi}{r^4}\,\mathbf{a}_r + \frac{80\sin\theta\sin\phi}{r^4}\,\mathbf{a}_\theta - \frac{80\cos\theta\cos\phi}{r^4\sin\theta}\,\mathbf{a}_\phi$$

Now

$$\mathbf{E}_P = \frac{80(1/2)(1)(3)}{16}\,\mathbf{a}_r + \frac{80(\sqrt{3}/2)(1)}{16}\,\mathbf{a}_\theta - 0\,\mathbf{a}_\phi = \underline{7.5\,\mathbf{a}_r + 4.3\,\mathbf{a}_\theta \text{ V/m}}$$

b) Find a unit vector directed outward to the surface, assuming the origin is inside the surface: Such a unit normal can be construced from the result of part c:

$$\mathbf{a}_N = \frac{7.5\,\mathbf{a}_r + 4.3\,\mathbf{a}_\theta}{4.33} = \underline{0.87\,\mathbf{a}_r + 0.50\,\mathbf{a}_\theta}$$

5.17. Given the potential field

$$V = \frac{100xz}{x^2+4}\text{ V}$$

in free space:

a) Find $\mathbf{D}$ at the surface $z = 0$: Use

$$\mathbf{E} = -\nabla V = -100z\frac{\partial}{\partial x}\left(\frac{x}{x^2+4}\right)\mathbf{a}_x - 0\,\mathbf{a}_y - \frac{100x}{x^2+4}\,\mathbf{a}_z \text{ V/m}$$

At $z = 0$, we use this to find

$$\mathbf{D}(z=0) = \epsilon_0\mathbf{E}(z=0) = \underline{-\frac{100\epsilon_0 x}{x^2+4}\,\mathbf{a}_z \text{ C/m}^2}$$

b) Show that the $z = 0$ surface is an equipotential surface: There are two reasons for this: 1) $\mathbf{E}$ at $z = 0$ is everywhere z-directed, and so moving a charge around on the surface involves doing no work; 2) When evaluating the given potential function at $z = 0$, the result is 0 for all x and y.

c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$: We have

$$\rho_s = \mathbf{D}\cdot\mathbf{a}_z\Big|_{z=0} = -\frac{100\epsilon_0 x}{x^2+4}\text{ C/m}^2$$

So

$$Q = \int_{-3}^{0}\int_{0}^{2} -\frac{100\epsilon_0 x}{x^2+4}\,dx\,dy = -(3)(100)\epsilon_0\left(\frac{1}{2}\right)\ln(x^2+4)\Big|_0^2 = -150\epsilon_0\ln 2 = \underline{-0.92\,\text{nC}}$$

5.18. Let us assume a field $\mathbf{E} = 3y^2z^3\,\mathbf{a}_x + 6xyz^3\,\mathbf{a}_y + 9xy^2z^2\,\mathbf{a}_z$ V/m in free space, and also assume that point $P(2, 1, 0)$ lies on a conducting surface.

a) Find ρ_v just adjacent to the surface at P:

$$\rho_v = \nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = 6xz^3 + 18xy^2z \text{ C/m}^3$$

Then at P, $\rho_v = \underline{0}$, since $z = 0$.

b) Find ρ_s at P:

$$\rho_s = \mathbf{D} \cdot \mathbf{n}\Big|_P = \epsilon_0 \mathbf{E}\dot{\mathbf{n}}\Big|_P$$

Note however, that this computation involves evaluating $\mathbf{E}$ at the surface, yielding a value of 0. Therefore the surface charge density at P is $\underline{0}$.

c) Show that $V = -3xy^2z^3$ V: The simplest way to show this is just to take $-\nabla V$, which yields the given field: A more general method involves deriving the potential from the given field: We write

$$E_x = -\frac{\partial V}{\partial x} = 3y^2z^3 \;\Rightarrow\; V = -3xy^2z^3 + f(y, z)$$

$$E_y = -\frac{\partial V}{\partial y} = 6xyz^3 \;\Rightarrow\; V = -3xy^2z^3 + f(x, z)$$

$$E_z = -\frac{\partial V}{\partial z} = 9xy^2z^2 \;\Rightarrow\; V = -3xy^2z^3 + f(x, y)$$

where the integration "constants" are functions of all variables other than the integration variable. The general procedure is to adjust the functions, f, such that the result for V is the same in all three integrations. In this case we see that $f(x, y) = f(x, z) = f(y, z) = 0$ accomplishes this, and the potential function is $V = -3xy^2z^3$ as given.

d) Determine V_{PQ}, given $Q(1, 1, 1)$: Using the potential function of part c, we have

$$V_{PQ} = V_P - V_Q = 0 - (-3) = \underline{3 \text{ V}}$$

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At 0 V}: \;\; 2x^2y - z = 0$$

$$\text{At 60 V}: \;\; 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

5.19b. (continued) Now

$$\mathbf{E} = -\nabla V = -40xyz\,\mathbf{a}_x - 20x^2z\,\mathbf{a}_y - [20xy - 20z]\,\mathbf{a}_z$$

Then, at the given point, we have

$$\mathbf{D}(2, 7/8, 1) = \epsilon_0\mathbf{E}(2, 7/8, 1) = -\epsilon_0[70\,\mathbf{a}_x + 80\,\mathbf{a}_y + 50\,\mathbf{a}_z]\,\mathrm{C/m^2}$$

We know that since this is the higher potential surface, **D** must be directed away from it, and so the charge density would be positive. Thus

$$\rho_s = \sqrt{\mathbf{D}\cdot\mathbf{D}} = 10\epsilon_0\sqrt{7^2+8^2+5^2} = \underline{1.04\,\mathrm{nC/m^2}}$$

c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface: This will be in the direction of **E** and **D** as found in part *b*, or

$$\mathbf{a}_n = -\left[\frac{7\mathbf{a}_x + 8\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{7^2+8^2+5^2}}\right] = \underline{-[0.60\mathbf{a}_x + 0.68\mathbf{a}_y + 0.43\mathbf{a}_z]}$$

5.20. A conducting plane is located at $z = 0$ in free space, and a 20 nC point charge is present at $Q(2, 4, 6)$.

a) If $V = 0$ at $z = 0$, find V at $P(5, 3, 1)$: The plane can be replaced by an image charge of -20 nC at $Q'(2, 4, -6)$. Vectors **R** and **R**$'$ directed from Q and Q' to P are $\mathbf{R} = (5, 3, 1) - (2, 4, 6) = (3, -1, -5)$ and $\mathbf{R}' = (5, 3, 1) - (2, 4, -6) = (3, -1, 7)$. Their magnitudes are $R = \sqrt{35}$ and $R' = \sqrt{59}$. The potential at P is given by

$$V_P = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R'} = \frac{20\times10^{-9}}{4\pi\epsilon_0\sqrt{35}} - \frac{20\times10^{-9}}{4\pi\epsilon_0\sqrt{59}} = \underline{7.0\ \mathrm{V}}$$

b) Find **E** at P:

$$\begin{aligned}\mathbf{E}_P &= \frac{q\mathbf{R}}{4\pi\epsilon_0 R^3} - \frac{q\mathbf{R}'}{4\pi\epsilon_0 (R')^3} = \frac{(20\times10^{-9})(3,-1,-5)}{4\pi\epsilon_0(35)^{3/2}} - \frac{(20\times10^{-9})(3,-1,7)}{4\pi\epsilon_0(59)^{3/2}}\\ &= \frac{20\times10^{-9}}{4\pi\epsilon_0}\left[(3\mathbf{a}_x - \mathbf{a}_y)\left(\frac{1}{(35)^{3/2}} - \frac{1}{(59)^{3/2}}\right) - \left(\frac{7}{(59)^{3/2}} + \frac{5}{(35)^{3/2}}\right)\mathbf{a}_z\right]\\ &= \underline{1.4\mathbf{a}_x - 0.47\mathbf{a}_y - 7.1\mathbf{a}_z\ \mathrm{V/m}}\end{aligned}$$

c) Find ρ_s at $A(5, 3, 0)$: First, find the electric field there:

$$\mathbf{E}_A = \frac{20\times10^{-9}}{4\pi\epsilon_0}\left[\frac{(5,3,0)-(2,4,6)}{(46)^{3/2}} - \frac{(5,3,0)-(2,4,-6)}{(46)^{3/2}}\right] = -6.9\mathbf{a}_z\ \mathrm{V/m}$$

Then $\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_{surface} = -6.9\epsilon_0\mathbf{a}_z\cdot\mathbf{a}_z = \underline{-61\ \mathrm{pC/m^2}}$.

5.21. Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$.

a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$: The line charges will image across the plane, producing image line charges of -30 nC/m each at $x = 0$, $y = -1$, and $x = 0$, $y = -2$. We find the potential at P by evaluating the work done in moving a unit positive charge from the $y = 0$ plane (we choose the origin) to P: For each line charge, this will be:

$$V_P - V_{0,0,0} = -\frac{\rho_l}{2\pi\epsilon_0}\ln\left[\frac{\text{final distance from charge}}{\text{initial distance from charge}}\right]$$

where $V_{0,0,0} = 0$. Considering the four charges, we thus have

$$\begin{aligned} V_P &= -\frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{2}}{1}\right) - \ln\left(\frac{\sqrt{10}}{1}\right) - \ln\left(\frac{\sqrt{17}}{2}\right)\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\ln(2) + \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{10}\right) + \ln\left(\frac{\sqrt{17}}{2}\right)\right] = \frac{30\times 10^{-9}}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{10}\sqrt{17}}{\sqrt{2}}\right] \\ &= \underline{1.20\,\text{kV}} \end{aligned}$$

b) Find $\mathbf{E}$ at P: Use

$$\begin{aligned} \mathbf{E}_P &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,2,0)-(0,1,0)}{|(1,1,0)|^2} + \frac{(1,2,0)-(0,2,0)}{|(1,0,0)|^2}\right. \\ &\quad \left. - \frac{(1,2,0)-(0,-1,0)}{|(1,3,0)|^2} - \frac{(1,2,0)-(0,-2,0)}{|(1,4,0)|^2}\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,1,0)}{2} + \frac{(1,0,0)}{1} - \frac{(1,3,0)}{10} - \frac{(1,4,0)}{17}\right] = \underline{723\,\mathbf{a}_x - 18.9\,\mathbf{a}_y \text{ V/m}} \end{aligned}$$

5.22. Let the plane $x = 0$ be a perfect conductor in free space. Locate a point charge of 4nC at $P_1(7, 1, -2)$ and a point charge of $-3nC$ at $P_2(4, 2, 1)$.

a) Find $\mathbf{E}$ at $A(5, 0, 0)$: Image charges will be located at $P_1'(-7, 1, -2)$ (-4nC) and at $P_2'(-4, 2, 1)$ (3nC). Vectors from all four charges to point A are:

$$\mathbf{R}_1 = (5,0,0) - (7,1,-2) = (-2,-1,2)$$

$$\mathbf{R}_1' = (5,0,0) - (-7,1,-2) = (12,-1,2)$$

$$\mathbf{R}_2 = (5,0,0) - (4,2,1) = (1,-2,-1)$$

and

$$\mathbf{R}_2' = (5,0,0) - (-4,2,1) = (9,-2,-1)$$

Replacing the plane by the image charges enables the field at A to be calculated through:

$$\begin{aligned} \mathbf{E}_A &= \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{(4)(-2,-1,2)}{9^{3/2}} - \frac{(3)(1,-2,-1)}{6^{3/2}} - \frac{(4)(12,-1,2)}{(149)^{3/2}} + \frac{(3)(9,-2,-1)}{(86)^{3/2}}\right] \\ &= \underline{-4.43\mathbf{a}_x + 2.23\mathbf{a}_y + 4.42\mathbf{a}_z \text{ V/m}} \end{aligned}$$

5.22b. Find $|\rho_s|$ at $B(0,0,0)$ (note error in problem statement): First, $\mathbf{E}$ at the origin is done as per the setup in part a, except the vectors are directed from the charges to the origin:

$$\mathbf{E}_B = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{(4)(-7,-1,2)}{(54)^{3/2}} - \frac{(3)(-4,-2,-1)}{(21)^{3/2}} - \frac{(4)(7,-1,2)}{(54)^{3/2}} + \frac{(3)(4,-2,-1)}{(21)^{3/2}}\right]$$

Now $\rho_s = \mathbf{D}\cdot\mathbf{n}|_{surface} = \mathbf{D}\cdot\mathbf{a}_x$ in our case (note the other components cancel anyway as they must, but we still need to express ρ_s as a scalar):

$$\rho_{sB} = \epsilon_0 \mathbf{E}_B \cdot \mathbf{a}_x = \frac{10^{-9}}{4\pi}\left[\frac{(4)(-7)}{(54)^{3/2}} - \frac{(3)(-4)}{(21)^{3/2}} - \frac{(4)(7)}{(54)^{3/2}} + \frac{(3)(4)}{(21)^{3/2}}\right] = \underline{8.62\,\mathrm{pC/m^2}}$$

5.23. A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\mathrm{C}\cdot\mathrm{m}$ is located at $A(1,0,0)$ in free space, and the $x = 0$ plane is perfectly-conducting.

a) Find V at $P(2,0,1)$. We use the far-field potential for a z-directed dipole:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0}\frac{z}{[x^2+y^2+z^2]^{1.5}}$$

The dipole at $x = 1$ will image in the plane to produce a second dipole of the opposite orientation at $x = -1$. The potential at any point is now:

$$V = \frac{p}{4\pi\epsilon_0}\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right]$$

Substituting $P(2,0,1)$, we find

$$V = \frac{.1\times 10^6}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right] = \underline{289.5\ \mathrm{V}}$$

b) Find the equation of the 200-V equipotential surface in cartesian coordinates: We just set the potential exression of part a equal to 200 V to obtain:

$$\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right] = 0.222$$

5.24. The mobilities for intrinsic silicon at a certain temperature are $\mu_e = 0.14\ \mathrm{m^2/V\cdot s}$ and $\mu_h = 0.035\ \mathrm{m^2/V\cdot s}$. The concentration of both holes and electrons is $2.2\times 10^{16}\,\mathrm{m^{-3}}$. Determine both the conductivity and the resistivity of this silicon sample: Use

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = (1.6\times 10^{-19}\mathrm{C})(2.2\times 10^{16}\,\mathrm{m^{-3}})(0.14\,\mathrm{m^2/V\cdot s} + 0.035\,\mathrm{m^2/V\cdot s})$$
$$= \underline{6.2\times 10^{-4}\ \mathrm{S/m}}$$

Conductivity is $\rho = 1/\sigma = \underline{1.6\times 10^3\ \Omega\cdot\mathrm{m}}$.

5.25. Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ C/m^3. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ m^2/V · s and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ m^2/V · s, where the temperature, T, *is in degrees Kelvin*. The conductivity will thus be

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = 6200T^{1.5}e^{-7000/T}\left[2.1 \times 10^5 T^{-2.5} + 2.3 \times 10^5 T^{-2.7}\right]$$
$$= \frac{1.30 \times 10^9}{T}e^{-7000/T}\left[1 + 1.095T^{-.2}\right] \text{ S/m}$$

Find σ at:

a) 0° C: With $T = 273$°K, the expression evaluates as $\sigma(0) = \underline{4.7 \times 10^{-5} \text{ S/m}}$.

b) 40° C: With $T = 273 + 40 = 313$, we obtain $\sigma(40) = \underline{1.1 \times 10^{-3} \text{ S/m}}$.

c) 80° C: With $T = 273 + 80 = 353$, we obtain $\sigma(80) = \underline{1.2 \times 10^{-2} \text{ S/m}}$.

5.26. A little donor impurity, such as arsenic, is added to pure silicon so that the electron concentration is 2×10^{17} conduction electrons per cubic meter while the number of holes per cubic meter is only 1.1×10^{15}. If $\mu_e = 0.15$ m^2/V · s for this sample, and $\mu_h = 0.045$ m^2/V · s, determine the conductivity and resistivity:

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = (1.6 \times 10^{-19})\left[(2 \times 10^{17})(0.15) + (1.1 \times 10^{15})(0.045)\right] = \underline{4.8 \times 10^{-3} \text{ S/m}}$$

Then $\rho = 1/\sigma = \underline{2.1 \times 10^2\ \Omega \cdot \text{m}}$.

5.27. Atomic hydrogen contains 5.5×10^25 atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m.

a) Find P: With all identical dipoles, we have

$$P = Nqd = (5.5 \times 10^{25})(1.602 \times 10^{-19})(7.1 \times 10^{-19}) = 6.26 \times 10^{-12} \text{ C/m}^2 = \underline{6.26 \text{ pC/m}^2}$$

b) Find ϵ_R: We use $P = \epsilon_0\chi_e E$, and so

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

Then $\epsilon_R = 1 + \chi_e = \underline{1.000176}$.

5.28. In a certain region where the relative permittivity is 2.4, $\mathbf{D} = 2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z$ nC/m^2. Find:

$$a) \qquad \mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{(2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z) \times 10^{-9}}{(2.4)(8.85 \times 10^{-12})} = \underline{94\mathbf{a}_x - 188\mathbf{a}_y + 235\mathbf{a}_z \text{ V/m}}$$

$$b) \qquad \mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}(\epsilon_R - 1) = \frac{(2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z) \times 10^{-9}}{2.4}(2.4 - 1)$$
$$= \underline{1.2\mathbf{a}_x - 2.3\mathbf{a}_y + 2.9\mathbf{a}_z \text{ nC/m}^2}$$

$$c) \qquad |\nabla V| = |\mathbf{E}| = [(94.1)^2 + (188)^2 + (235)^2]^{1/2} = \underline{315 \text{ V/m}}$$

5.29. A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_R = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho$ nC/m^2 in the dielectric, find:

a) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ: Use

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0(\epsilon_R - 1)} = \frac{(2/\rho) \times 10^{-9}\mathbf{a}_\rho}{(8.85 \times 10^{-12})(1.56)} = \underline{\frac{144.9}{\rho}\mathbf{a}_\rho \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \frac{2 \times 10^{-9}\mathbf{a}_\rho}{\rho}\left[\frac{1}{1.56} + 1\right] = \frac{3.28 \times 10^{-9}\mathbf{a}_\rho}{\rho} \text{ C/m}^2 = \underline{\frac{3.28\mathbf{a}_\rho}{\rho} \text{ nC/m}^2}$$

b) Find V_{ab} and χ_e: Use

$$V_{ab} = -\int_3^{0.8} \frac{144.9}{\rho}\, d\rho = 144.9 \ln\left(\frac{3}{0.8}\right) = \underline{192 \text{ V}}$$

$\chi_e = \epsilon_r - 1 = \underline{1.56}$, as found in part a.

c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$: Use

$$\mathbf{p} = \frac{\mathbf{P}}{N} = \frac{(2 \times 10^{-9}/\rho)}{4 \times 10^{19}}\,\mathbf{a}_\rho = \underline{\frac{5.0 \times 10^{-29}}{\rho}\,\mathbf{a}_\rho \text{ C} \cdot \text{m}}$$

5.30. Given the potential field $V = 200 - 50x + 20y$ V in a dielectric material for which $\epsilon_R = 2.1$, find:

a) $\mathbf{E} = -\nabla V = \underline{50\mathbf{a}_x - 20\mathbf{a}_y \text{ V/m}}$.

b) $\mathbf{D} = \epsilon\mathbf{E} = (2.1)(8.85 \times 10^{-12})(50\mathbf{a}_x - 20\mathbf{a}_y) = \underline{930\mathbf{a}_x - 372\mathbf{a}_y \text{ pC/m}^2}$.

c) $\mathbf{P} = \epsilon_0\mathbf{E}(\epsilon_R - 1) = (8.85 \times 10^{-12})(50\mathbf{a}_x - 20\mathbf{a}_y)(1.1) = \underline{487\mathbf{a}_x - 195\mathbf{a}_y \text{ pC/m}^2}$.

d) $\rho_v = \nabla \cdot \mathbf{D} = \underline{0}$.

e) $\rho_b = -\nabla \cdot \mathbf{P} = \underline{0}$

f) $\rho_T = \nabla \cdot \epsilon_0\mathbf{E} = \underline{0}$

5.31. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_R = \epsilon_{R1} = 3$, while $\epsilon_{R2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find:

a) E_{N1}: This will be $\mathbf{E}_1 \cdot \mathbf{a}_x = \underline{80 \text{ V/m}}$.

b) $\mathbf{E}_{T1}$. This consists of components of $\mathbf{E}_1$ *not* normal to the surface, or $\mathbf{E}_{T1} = \underline{-60\mathbf{a}_y - 30\mathbf{a}_z \text{ V/m}}$.

c) $E_{T1} = \sqrt{(60)^2 + (30)^2} = \underline{67.1 \text{ V/m}}$.

d) $E_1 = \sqrt{(80)^2 + (60)^2 + (30)^2} = \underline{104.4 \text{ V/m}}$.

e) The angle θ_1 between $\mathbf{E}_1$ and a normal to the surface: Use

$$\cos\theta_1 = \frac{\mathbf{E}_1 \cdot \mathbf{a}_x}{E_1} = \frac{80}{104.4} \;\Rightarrow\; \theta_1 = \underline{40.0^\circ}$$

5.31 (continued)

f) $D_{N2} = D_{N1} = \epsilon_{R1}\epsilon_0 E_{N1} = 3(8.85 \times 10^{-12})(80) = \underline{2.12\,\text{nC/m}^2}$.

g) $D_{T2} = \epsilon_{R2}\epsilon_0 E_{T1} = 5(8.85 \times 10^{-12})(67.1) = \underline{2.97\,\text{nC/m}^2}$.

h) $\mathbf{D}_2 = \epsilon_{R1}\epsilon_0 E_{N1}\mathbf{a}_x + \epsilon_{R2}\epsilon_0 \mathbf{E}_{T1} = \underline{2.12\mathbf{a}_x - 2.66\mathbf{a}_y - 1.33\mathbf{a}_z \text{ nC/m}^2}$.

i) $\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0\mathbf{E}_2 = \mathbf{D}_2\,[1 - (1/\epsilon_{R2})] = (4/5)\mathbf{D}_2 = \underline{1.70\mathbf{a}_x - 2.13\mathbf{a}_y - 1.06\mathbf{a}_z \text{ nC/m}^2}$.

j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface: Use

$$\cos\theta_2 = \frac{\mathbf{E}_2 \cdot \mathbf{a}_x}{E_2} = \frac{\mathbf{D}_2 \cdot \mathbf{a}_x}{D_2} = \frac{2.12}{\sqrt{(2.12)^2 = (2.66)^2 + (1.33)^2}} = .581$$

Thus $\theta_2 = \cos^{-1}(.581) = \underline{54.5^\circ}$.

5.32. In Fig. 5.18, let $\mathbf{D} = 3\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z$ nC/m^2 and find:

a) $\mathbf{D}_2$: First, the electric field in region 1 is

$$\mathbf{E}_1 = \left[\frac{3}{2\epsilon_0}\mathbf{a}_x - \frac{4}{2\epsilon_0}\mathbf{a}_y + \frac{5}{2\epsilon_0}\mathbf{a}_z\right] \times 10^{-9} \text{ V/m}$$

Since, at the dielectric interface, tangential electric field and normal electric flux density are continuous, we may write

$$\mathbf{D}_2 = \epsilon_{R2}\epsilon_0\mathbf{E}_{T1} + \mathbf{D}_{N1} = \left(\frac{5}{2}\right)3\mathbf{a}_x - \left(\frac{5}{2}\right)4\mathbf{a}_y + 5\mathbf{a}_z = \underline{7.5\mathbf{a}_x - 10\mathbf{a}_y + 5\mathbf{a}_z \text{ nC/m}^2}$$

b) $\mathbf{D}_{N2} = \underline{5\mathbf{a}_z}$, as explained above.

c) $\mathbf{D}_{T2} = \epsilon_{R2}\epsilon_0\mathbf{E}_{T2} = \epsilon_{R2}\epsilon_0\mathbf{E}_{T1} = \underline{7.5\mathbf{a}_x - 10\mathbf{a}_y \text{ nC/m}^2}$.

d) the energy density in each region:

$$w_{e1} = \frac{1}{2}\epsilon_{R1}\epsilon_0\mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{1}{2}(2)\epsilon_0\left[\left(\frac{3}{2\epsilon_0}\right)^2 + \left(\frac{4}{2\epsilon_0}\right)^2 + \left(\frac{5}{2\epsilon_0}\right)^2\right] \times 10^{-18} = \underline{1.41\ \mu\text{J/m}^3}$$

$$w_{e2} = \frac{1}{2}\epsilon_{R2}\epsilon_0\mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{1}{2}(5)\epsilon_0\left[\left(\frac{3}{2\epsilon_0}\right)^2 + \left(\frac{4}{2\epsilon_0}\right)^2 + \left(\frac{5}{5\epsilon_0}\right)^2\right] \times 10^{-18} = \underline{2.04\ \mu\text{J/m}^3}$$

e) the angle that $\mathbf{D}_2$ makes with $\mathbf{a}_z$: Use $\mathbf{D}_2 \cdot \mathbf{a}_z = |\mathbf{D}_2|\cos\theta = D_z = 5$. where $|\mathbf{D}_2| = \left[(7.5)^2 + (10)^2 + (5)^2\right]^{1/2} = 13.5$. So $\theta = \cos^{-1}(5/13.5) = \underline{68^\circ}$.

f) $D_2/D_1 = \left[(7.5)^2 + (10)^2 + (5)^2\right]^{1/2}/\left[(3)^2 + (4)^2 + (5)^2\right]^{1/2} = \underline{1.91}$.

g) P_2/P_1: First $\mathbf{P}_1 = \epsilon_0\mathbf{E}_1(\epsilon_{R1} - 1) = 1.5\mathbf{a}_x - 2\mathbf{a}_y + 2.5\mathbf{a}_z$ nC/m^2.
Then $\mathbf{P}_2 = \epsilon_0\mathbf{E}_2(\epsilon_{R2} - 1) = 6\mathbf{a}_x - 8\mathbf{a}_y + 4\mathbf{a}_z$ nC/m^2. So

$$\frac{P_2}{P_1} = \frac{[(6)^2 + (8)^2 + (4)^2]^{1/2}}{[(1.5)^2 + (2)^2 + (2.5)^2]^{1/2}} = \underline{3.04}$$

5.33. Two perfect dielectrics have relative permittivities $\epsilon_{R1} = 2$ and $\epsilon_{R2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$: We need to find the components of $\mathbf{E}_1$ that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = \mathbf{E}_1 \cdot \mathbf{n}$. Taking $f = x - y + 2z$, the unit vector that is normal to the surface is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}} \left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right]$$

This normal will point in the direction of increasing f, which will be away from the origin, or into region 2 (you can visualize a portion of the surface as a triangle whose vertices are on the three coordinate axes at $x = 5$, $y = -5$, and $z = 2.5$). So $E_{N1} = (1/\sqrt{6})[100 - 200 - 100] = -81.7$ V/m. Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$\mathbf{E}_{N1} = -81.65 \left(\frac{1}{\sqrt{6}}\right) [\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z] = -33.33\mathbf{a}_x + 33.33\mathbf{a}_y - 66.67\mathbf{a}_z \text{ V/m}$$

Now, the tangential component will be

$$\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z$$

Our boundary conditions state that $\mathbf{E}_{T2} = \mathbf{E}_{T1}$ and $\mathbf{E}_{N2} = (\epsilon_{R1}/\epsilon_{R2})\mathbf{E}_{N1} = (1/4)\mathbf{E}_{N1}$. Thus

$$\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = \mathbf{E}_{T1} + \frac{1}{4}\mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z - 8.3\mathbf{a}_x + 8.3\mathbf{a}_y - 16.67\mathbf{a}_z$$
$$= \underline{125\mathbf{a}_x + 175\mathbf{a}_y \text{ V/m}}$$

5.34. Let the spherical surfaces $r = 4$ cm and $r = 9$ cm be separated by two perfect dielectric shells, $\epsilon_{R1} = 2$ for $4 < r < 6$ cm and $\epsilon_{R2} = 5$ for $6 < r < 9$ cm. If $\mathbf{E}_1 = (2000/r^2)\mathbf{a}_r$ V/m, find:

a) $\mathbf{E}_2$: Since $\mathbf{E}$ is normal to the interface between ϵ_{R1} and ϵ_{R2}, $\mathbf{D}$ will be continuous across the boundary, and so

$$\mathbf{D}_1 = \frac{2\epsilon_0(2000)}{r^2}\mathbf{a}_r = \mathbf{D}_2$$

Then

$$\mathbf{E}_2 = \frac{\mathbf{D}_2}{5\epsilon_0} = \left(\frac{2}{5}\right)\frac{2000}{r^2}\mathbf{a}_r = \underline{\frac{800}{r^2}\mathbf{a}_r \text{ V/m}}$$

b) the total electrostatic energy stored in each region: In region 1, the energy density is

$$w_{e1} = \frac{1}{2}\epsilon_{R1}\epsilon_0|\mathbf{E}_1|^2 = \frac{1}{2}(2)\epsilon_0\frac{(2000)^2}{r^4} \text{ J/m}^3$$

In region 2:

$$w_{e2} = \frac{1}{2}\epsilon_{R2}\epsilon_0|\mathbf{E}_2|^2 = \frac{1}{2}(5)\epsilon_0\frac{(800)^2}{r^4} \text{ J/m}^3$$

5.34. (continued)

The energies in each region are then

$$\text{Region 1}: \quad W_{e1} = (2000)^2 \epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_{.04}^{.06} \frac{1}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 4\pi\epsilon_0 (2000)^2 \left[\frac{1}{.04} - \frac{1}{.06} \right] = \underline{3.7 \text{ mJ}}$$

$$\text{Region 2}: \quad W_{e2} = (800)^2 \left(\frac{5}{2}\right) \epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_{.06}^{.09} \frac{1}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 4\pi\epsilon_0 (800)^2 \left(\frac{5}{2}\right) \left[\frac{1}{.06} - \frac{1}{.09} \right] = \underline{0.99 \text{ mJ}}$$

5.35. Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{R1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{R2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{E}_2$: The interfaces between the two media will lie on planes of constant ϕ, to which $\mathbf{E}_1$ is parallel. Thus the field is the same on either side of the boundaries, and so $\underline{\mathbf{E}_2 = \mathbf{E}_1}$.

b) the total electrostatic energy stored in a 1m length of each region: In general we have $w_E = (1/2)\epsilon_R \epsilon_0 E^2$. So in region 1:

$$W_{E1} = \int_0^1 \int_0^{\pi/2} \int_4^9 \frac{1}{2}(2)\epsilon_0 \frac{(2000)^2}{\rho_2} \rho \, d\rho \, d\phi \, dz = \frac{\pi}{2}\epsilon_0 (2000)^2 \ln\left(\frac{9}{4}\right) = \underline{45.1\,\mu\text{J}}$$

In region 2, we have

$$W_{E2} = \int_0^1 \int_{\pi/2}^{2\pi} \int_4^9 \frac{1}{2}(5)\epsilon_0 \frac{(2000)^2}{\rho_2} \rho \, d\rho \, d\phi \, dz = \frac{15\pi}{4}\epsilon_0 (2000)^2 \ln\left(\frac{9}{4}\right) = \underline{338\,\mu\text{J}}$$

5.36. Let $S = 120\,\text{cm}^2$, $d = 4$ mm, and $\epsilon_R = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:
$C = \epsilon_R \epsilon_0 S/d = [12\epsilon_0(120 \times 10^{-4})]/[4 \times 10^{-3}] = 3.19 \times 10^{-10} = \underline{319\,\text{pF}}$.

b) After connecting a 40 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: $E = V/d = 40/(4 \times 10^{-3}) = \underline{10^4\,\text{V/m}}$. $D = \epsilon_R \epsilon_0 E = 12\epsilon_0 \times 10^4 = \underline{1.06\,\mu\text{C/m}^2}$. Then $Q = \mathbf{D} \cdot \mathbf{n}|_{surface} \times S = 1.06 \times 10^{-6} \times (120 \times 10^{-4}) = 1.27 \times 10^{-8}\text{C} = \underline{12.7\,\text{nC}}$. Finally $W_e = (1/2)CV_0^2 = (1/2)(319 \times 10^{-12})(40)^2 = \underline{255\,\text{nJ}}$.

c) The source is now removed and the dielectric is carefully withdrawn from between the plates. Again calculate E, D, Q, and the energy: With the source disconnected, the charge is constant, and thus so is D: Therefore, $Q = \underline{12.7\,\text{nC}}$, $D = \underline{1.06\,\mu\text{C/m}^2}$, and $E = D/\epsilon_0 = 10^4/8.85 \times 10^{-12} = \underline{1.2 \times 10^5\,\text{V/m}}$. The energy is then

$$W_e = \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \times S = \frac{1}{2}(1.06 \times 10^{-6})(1.2 \times 10^5)(120 \times 10^{-4})(4 \times 10^{-3}) = \underline{3.05\,\mu\text{J}}$$

d) What is the voltage between the plates? $V = E \times d = (1.2 \times 10^5)(4 \times 10^{-3}) = \underline{480\,\text{V}}$.

5.37. Capacitors tend to be more expensive as their capacitance and maximum voltage, V_{max}, increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: $\epsilon_R = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_R = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_R = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_R = 2.26$, $E_{BD} = 4.7$ MV/m? Note that $V_{max} = E_{BD}d$, where d is the plate separation. Also, $C = \epsilon_R\epsilon_0 A/d$, and so $V_{max}C = \epsilon_R\epsilon_0 A E_{BD}$, where A is the plate area. The maximum CV_{max} product is found through the maximum $\epsilon_R E_{BD}$ product. Trying this with the given materials yields the winner, which is <u>barium titanate</u>.

5.38. A dielectric circular cylinder used between the plates of a capacitor has a thickness of 0.2 mm and a radius of 1.4 cm. The dielectric properties are $\epsilon_R = 400$ and $\sigma = 10^{-5}$ S/m.

a) Calculate C:

$$C = \frac{\epsilon_R\epsilon_0 S}{d} = \frac{(400)(8.854\times10^{-12})\pi(1.4\times10^{-2})^2}{2\times10^{-4}} = 1.09\times10^{-8} = \underline{10.9 \text{ nF}}$$

b) Find the quality factor Q_{QF} ($Q_QF = \omega RC$) of the capacitor at $f = 10$ kHz: Use the relation $RC = \epsilon/\sigma$ to write

$$Q_{QF} = \omega RC = \frac{2\pi f\epsilon}{\sigma} = \frac{(2\pi\times10^4)(400)(8.854\times10^{-12})}{10^{-5}} = \underline{22.3}$$

c) If the maximum field strength permitted in the dielectric is 2 MV/m, what is the maximum permissible voltage across the capacitor? $V_{max} = E_{BD}d = (2\times10^6)(2\times10^{-4}) = \underline{400 \text{ V}}$.

d) What energy is stored when this voltage is applied?

$$W_{e,max} = \frac{1}{2}CV_{max}^2 = \frac{1}{2}(10.9\times10^{-9})(400)^2 = 8.7\times10^{-4} = \underline{0.87 \text{ mJ}}$$

5.39. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_R = 2 + 2\times10^6x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. **D** will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that **D** *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that **D**, which is normal to the layers, will be continuous across each boundary, and so **D** is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0\epsilon_R} = \frac{\rho_s}{\epsilon_0(2+2\times10^6x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\,dx}{\epsilon_0(2+2\times10^6x^2)} = \frac{\rho_s}{\epsilon_0}\frac{1}{\sqrt{4\times10^6}}\tan^{-1}\left(\frac{x\sqrt{4\times10^6}}{2}\right)\Big|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0}\frac{1}{2\times10^3}\left(\frac{\pi}{4}\right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s(.02)\epsilon_0(2\times10^3)(4)}{\rho_s\pi} = 4.51\times10^{-10}\,\text{F} = \underline{451\,\text{pF}}$$

5.40a. The width of the region containing ϵ_{R1} in Fig. 5.19 is 1.2 m. Find ϵ_{R1} if $\epsilon_{R2} = 2.5$ and the total capacitance is 60 nF: The plate areas associated with each capacitor are $A_1 = 1.2(2) = 2.4$ m^2 and $A_2 = 0.8(2) = 1.6$ m^2. Having parallel capacitors, the capacitances will add, so

$$C = C_1 + C_2 \quad \Rightarrow \quad 60 \times 10^{-9} = \frac{\epsilon_{R1}\epsilon_0(2.4)}{2 \times 10^{-3}} + \frac{2.5\epsilon_0(1.6)}{2 \times 10^{-3}}$$

Solve this to obtain $\epsilon_{R1} = \underline{4.0}$.

b) Find the width of each region (containing ϵ_{R1} and ϵ_{R2}) if $C_{total} = 80$ nF, $\epsilon_{R2} = 3\epsilon_{R1}$, and $C_1 = 2C_2$: Let w_1 be the width of region 1. The above conditions enable us to write:

$$\left[\frac{\epsilon_{R1}\epsilon_0 w_1(2)}{2 \times 10^{-3}}\right] = 2\left[\frac{3\epsilon_{R1}\epsilon_0(2 - w_1)(2)}{2 \times 10^{-3}}\right] \quad \Rightarrow \quad w_1 = 6(2 - w_1)$$

So that $w_1 = 12/7 = \underline{1.7\,\text{m}}$ and $w_2 = \underline{0.3\,\text{m}}$.

5.41. Let $\epsilon_{R1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{R2} = 4$ for $1 < y < 3$ mm, and ϵ_{R3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: a) ϵ_{R3} is that of air; b) $\epsilon_{R3} = \epsilon_{R1}$; c) $\epsilon_{R3} = \epsilon_{R2}$; d) ϵ_{R3} is silver: The combination will be three capacitors in series, for which

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{d_1}{\epsilon_{R1}\epsilon_0(1)} + \frac{d_2}{\epsilon_{R2}\epsilon_0(1)} + \frac{d_3}{\epsilon_{R3}\epsilon_0(1)} = \frac{10^{-3}}{\epsilon_0}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\epsilon_{R3}}\right]$$

So that

$$C = \frac{(5 \times 10^{-3})\epsilon_0\epsilon_{R3}}{10 + 4.5\epsilon_{R3}}$$

Evaluating this for the four cases, we find a) $C = \underline{3.05\,\text{nF}}$ for $\epsilon_{R3} = 1$, b) $C = \underline{5.21\,\text{nF}}$ for $\epsilon_{R3} = 2.5$, c) $C = \underline{6.32\,\text{nF}}$ for $\epsilon_{R3} = 4$, and d) $C = \underline{9.83\,\text{nF}}$ if silver (taken as a perfect conductor) forms region 3; this has the effect of removing the term involving ϵ_{R3} from the original formula (first equation line), or equivalently, allowing ϵ_{R3} to approach infinity.

5.42. Cylindrical conducting surfaces are located at $\rho = 0.8$ cm and 3.6 cm. The region $0.8 < \rho < a$ contains a dielectric for which $\epsilon_R = 4$, while $\epsilon_R = 2$ for $a < \rho < 3.6$.

a) Find a so that the voltage across each dielectric layer is the same: Assuming charge density ρ_s on the inner cylinder, we have $\mathbf{D} = \rho_s(0.8)/\rho\,\mathbf{a}_\rho$, which gives $\mathbf{E}(0.8 < \rho < a) = (0.8\rho_s)/(4\epsilon_0\rho)\mathbf{a}_\rho$ and $\mathbf{E}(a < \rho < 3.6) = (0.8\rho_s)/(2\epsilon_0\rho)\mathbf{a}_\rho$. The voltage between conductors is now

$$V_0 = -\int_{3.6}^{a} \frac{0.8\rho_s}{2\epsilon_0\rho}\,d\rho - \int_{a}^{0.8} \frac{0.8\rho_s}{4\epsilon_0\rho}\,d\rho = \frac{0.8\rho_s}{2\epsilon_0}\left[\ln\left(\frac{3.6}{a}\right) + \frac{1}{2}\ln\left(\frac{a}{0.8}\right)\right]$$

We require

$$\ln\left(\frac{3.6}{a}\right) = \frac{1}{2}\ln\left(\frac{a}{0.8}\right) \quad \Rightarrow \quad \frac{3.6}{a} = \sqrt{\frac{a}{0.8}} \quad \Rightarrow \quad a = \underline{2.2\text{ cm}}$$

b) Find the total capacitance per meter: Using the part a result, have

$$V_0 = \frac{0.8\rho_s}{2\epsilon_0}\left[\ln\left(\frac{3.6}{2.2}\right) + \frac{1}{2}\ln\left(\frac{2.2}{0.8}\right)\right] = \frac{0.4\rho_s}{\epsilon_0}$$

5.42b. (continued) The charge on a unit length of the inner conductor is $Q = 2\pi(0.8)(1)\rho_s$. The capacitance is now

$$C = \frac{Q}{V_0} = \frac{2\pi(0.8)(1)\rho_s}{0.4\rho_s/\epsilon_0} = 4\pi\epsilon_0 = \underline{111\ \text{pF/m}}$$

Note that throughout this problem, I left all dimensions in cm, knowing that all cm units would cancel, leaving the units of capacitance to be those used for ϵ_0.

5.43. Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_R = 4$. Find the capacitance if

a) $c = 2$ cm, $d = 3$ cm: This is two capacitors in series, and so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\frac{1}{4}\ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\right] \Rightarrow C = \underline{143\,\text{pF}}$$

b) $d = 4$ cm, and the volume of the dielectric is the same as in part a: Having equal volumes requires that $3^2 - 2^2 = 4^2 - c^2$, from which $c = 3.32$ cm. Now

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\ln\left(\frac{3.32}{2}\right) + \frac{1}{4}\ln\left(\frac{4}{3.32}\right)\right] \Rightarrow C = \underline{101\,\text{pF}}$$

5.44. Conducting cylinders lie at $\rho = 3$ and $\rho = 12$ mm; both extend from $z = 0$ to $z = 1$ m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho < 6$ mm, $\epsilon_R = 4$ for $6 < \rho < 9$ mm, and $\epsilon_R = 8$ for $9 < \rho < 12$ mm.

a) Calculate C: First we know that $\mathbf{D} = (3\rho_s/\rho)\mathbf{a}_\rho$ C/m^2, with ρ expressed in mm. Then, with ρ in mm,

$$\mathbf{E}_1 = \frac{3\rho_s}{\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (3 < \rho < 6)$$

$$\mathbf{E}_2 = \frac{3\rho_s}{4\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (6 < \rho < 9)$$

and

$$\mathbf{E}_3 = \frac{3\rho_s}{8\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (9 < \rho < 12)$$

The voltage between conductors will be:

$$V_0 = \left[-\int_{12}^{9}\frac{3\rho_s}{8\epsilon_0\rho}\,d\rho - \int_{9}^{6}\frac{3\rho_s}{4\epsilon_0\rho}\,d\rho - \int_{6}^{3}\frac{3\rho_s}{\epsilon_0\rho}\,d\rho\right] \times 10^{-3}(\text{m/mm})$$
$$= \frac{.003\rho_s}{\epsilon_0}\left[\frac{1}{8}\ln\left(\frac{12}{9}\right) + \frac{1}{4}\ln\left(\frac{9}{6}\right) + \ln\left(\frac{6}{3}\right)\right] = \frac{.003\rho_s}{\epsilon_0}(0.830)\text{ V}$$

Now, the charge on the 1 m length of the inner conductor is $Q = 2\pi(.003)(1)\rho_s$. The capacitance is then

$$C = \frac{Q}{V_0} = \frac{2\pi(.003)(1)\rho_s}{(.003)\rho_s(.830)/\epsilon_0} = \frac{2\pi\epsilon_0}{.830} = \underline{67\ \text{pF}}$$

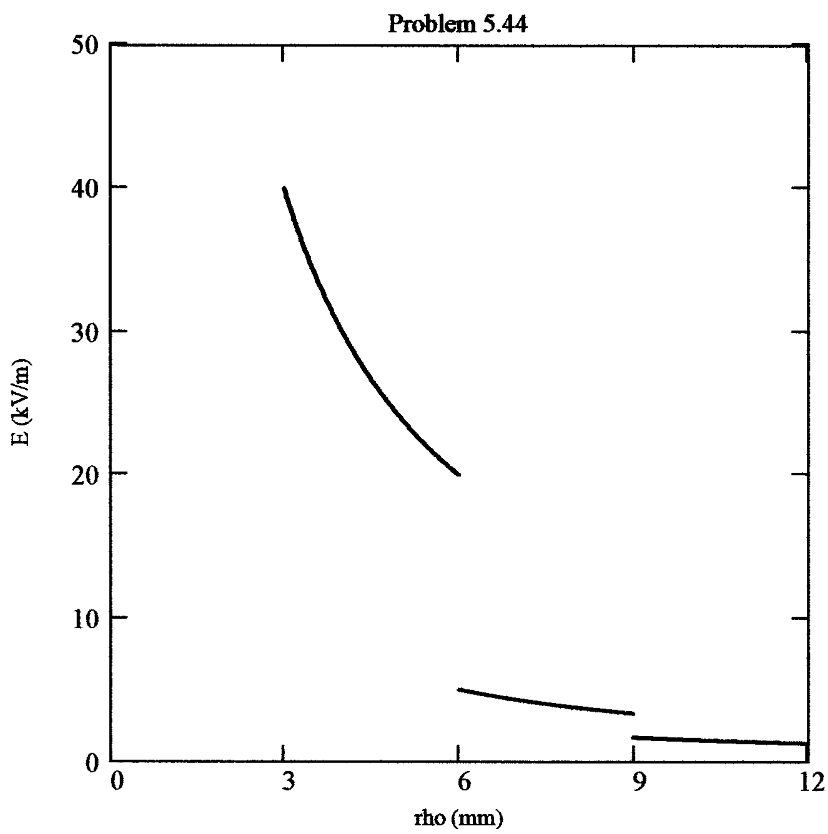
5.44b. If the voltage between the cylinders is 100 V, plot $|E_\rho|$ vs. ρ:
Have $Q = CV_0 = (67 \times 10^{-12})(100) = 6.7\text{nC}$. Then

$$\rho_s = \frac{6.7 \times 10^{-9}}{2\pi(.003)(1)} = 355 \text{ nC/m}^2$$

Then, using the electric field expressions from part a, we find

$$E_1 = \left(\frac{3}{\rho}\right)\frac{355 \times 10^{-9}}{8.854 \times 10^{-12}} = \frac{12 \times 10^4}{\rho} \text{ V/m} = \frac{120}{\rho} \text{ kV/m } (3 < \rho < 6)$$

where ρ is expressed in mm. Similarly, we find $E_2 = E_1/4 = 30/\rho$ kV/m $(6 < \rho < 9)$ and $E_3 = E_1/8 = 15$ kV/m $(9 < \rho < 12)$. These fields are plotted below.



5.45. Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_R = 8$.

a) Find C: For a spherical capacitor, we know that:

$$C = \frac{4\pi\epsilon_R\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi(8)\epsilon_0}{\left(\frac{1}{3} - \frac{1}{6}\right)(100)} = 1.92\pi\epsilon_0 = \underline{53.3\,\text{pF}}$$

b) A portion of the dielectric is now removed so that $\epsilon_R = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_R = 8$, $\pi/2 < \phi < 2\pi$. Again, find C: We recognize here that removing that portion leaves us with two capacitors in parallel (whose C's will add). We use the fact that with the dielectric *completely* removed, the capacitance would be $C(\epsilon_R = 1) = 53.3/8 = 6.67\,\text{pF}$. With one-fourth the dielectric removed, the total capacitance will be

$$C = \frac{1}{4}(6.67) + \frac{3}{4}(53.4) = \underline{41.7\,\text{pF}}$$

5.46. (see Problem 5.44).

5.47. With reference to Fig. 5.17, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C: We have

$$K_1 = \left[\frac{h + \sqrt{h^2 + b^2}}{b}\right]^2 = \left[\frac{15 + \sqrt{(15)^2 + (6)^2}}{6}\right]^2 = \underline{23.0}$$

We then have

$$\rho_L = \frac{4\pi\epsilon_0 V_0}{\ln K_1} = \frac{4\pi\epsilon_0(250)}{\ln(23)} = \underline{8.87\,\text{nC/m}}$$

Next, $a = \sqrt{h^2 - b^2} = \sqrt{(15)^2 - (6)^2} = \underline{13.8\,\text{m}}$. Finally,

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi\epsilon_0}{\cosh^{-1}(15/6)} = \underline{35.5\,\text{pF}}$$

5.48. A potential function in free space is given by

$$V = -20 + 10 \ln \left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right]$$

a) Describe the 0-V equipotential surface: Setting the given expression equal to zero, we find

$$\left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right] = e^2 = 7.39$$

So $6.39x^2 + 6.39y^2 - 83.9y + 160 = 0$. Completing the square in the y trinomial leads to $x^2 + (y - 6.56)^2 = 18.1 = (4.25)^2$, which we recognize as a right circular cylinder whose axis is located at $x = 0$, $y = 6.56$, and whose radius is 4.25.

b) Describe the 10-V equipotential surface: In this case, the given expression is set equal to ten, leading to

$$\left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right] = e^3 = 20.1$$

So $19.1x^2 + 19.1y^2 - 211y + 477 = 0$. Following the same procedure as in part a, this becomes $x^2 + (y - 5.52)^2 = 5.51 = (2.35)^2$, which we recognize again as a right circular cylinder with axis at $x = 0$, $y = 5.52$, and of radius 2.35.

5.49. A 2 cm diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the:

a) cylinder at a point nearest the plane: The cylinder will image across the plane, producing an equivalent two-cylinder problem, with the second one at location 5 cm below the plane. We will take the plane as the zy plane, with the cylinder positions at $x = \pm 5$. Now $b = 1$ cm, $h = 5$ cm, and $V_0 = 100$ V. Thus $a = \sqrt{h^2 - b^2} = 4.90$ cm. Then $K_1 = [(h+a)/b]^2 = 98.0$, and $\rho_L = (4\pi\epsilon_0 V_0)/\ln K_1 = 2.43\,\text{nC/m}$. Now

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -\frac{\rho_L}{2\pi} \left[\frac{(x+a)\mathbf{a}_x + y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\mathbf{a}_x + y\mathbf{a}_y}{(x-a)^2 + y^2} \right]$$

and

$$\rho_{s,\,max} = \mathbf{D} \cdot (-\mathbf{a}_x)\Big|_{x=h-b,\,y=0} = \frac{\rho_L}{2\pi} \left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2} \right] = \underline{473\,\text{nC/m}^2}$$

b) plane at a point nearest the cylinder: At $x = y = 0$,

$$\mathbf{D}(0,0) = -\frac{\rho_L}{2\pi} \left[\frac{a\mathbf{a}_x}{a^2} - \frac{-a\mathbf{a}_x}{a^2} \right] = -\frac{\rho_L}{2\pi} \frac{2}{a} \mathbf{a}_x$$

from which

$$\rho_s = \mathbf{D}(0,0) \cdot \mathbf{a}_x = -\frac{\rho_L}{\pi a} = \underline{-15.8\,\text{nC/m}^2}$$