3) a) $\sum_{n=1}^{+\infty} \frac{\sin n}{n^2}$ ; ta có $a_n = \frac{\sin n}{n^2}$

Lại có $|a_n| = \left|\frac{\sin n}{n^2}\right| < \frac{1}{n^2}$ mà $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ hội tụ

(do tiêu chuẩn TP)

$\Rightarrow \sum_{n=1}^{+\infty} \left|\frac{\sin n}{n^2}\right|$ h tụ $\Rightarrow \sum_{n=1}^{+\infty} \frac{\sin n}{n^2}$ hội tụ TĐ.

b) $\sum_{n=2}^{+\infty} \frac{(-1)^n . n}{n^2+1}$ là chuỗi đan dấu với $a_n = \frac{n}{n^2+1}$.

Dễ thấy $a_n > 0 \ \forall n \in [2, +\infty)$; $\lim_{n \to +\infty} a_n = 0$.

Xét hsố $f(x) = \frac{x}{x^2+1}$ trên $[2, +\infty) \Rightarrow f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \ \forall x \in [2, +\infty)$

$\Rightarrow f(x)$ đơn điệu giảm $\Rightarrow a_n$ giảm

$\Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n . n}{n^2+1}$ là chuỗi đan dấu htụ theo tiêu chuẩn Leibnitz.

mà lại có: $\sum_{n=2}^{+\infty} |a_n| = \sum_{n=2}^{+\infty} \frac{n}{n^2+1}$

Khi $n \to +\infty$ $0 < \frac{n}{n^2+1} \sim \frac{1}{n}$ mà $\sum_{n=2}^{+\infty} \frac{1}{n}$ pkỳ (theo tiêu chuẩn TP)

$\Rightarrow \sum_{n=2}^{+\infty} |a_n|$ pkỳ (theo tiêu chuẩn SS tương đương)

$\Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n . n}{n^2+1}$ bán hội tụ.

c) $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$ với $a_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \frac{(-1)^n.(\sqrt{n}-(-1)^n)}{n+1}$

$= (-1)^n . \frac{\sqrt{n}}{n+1} - \frac{1}{n+1}$.

$$\Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \sum_{n=2}^{+\infty} (-1)^n . \frac{\sqrt{n}}{n+1} - \sum_{n=2}^{+\infty} \frac{1}{n+1}.$$

Dễ thấy $\sum_{n=2}^{+\infty} \frac{1}{n+1}$ ~~ht~~ pkỳ (theo tiêu chuẩn tp)

$\sum_{n=2}^{+\infty} (-1)^n . \frac{\sqrt{n}}{n+1}$ chuỗi đan dấu $u_n = \frac{\sqrt{n}}{n+1} > 0 \ \forall n \in [2, +\infty)$

$\lim_{n\to+\infty} u_n = 0$, $u_n$ giảm $\Rightarrow$ HT theo đly Lepnitz

$$\Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} \text{ pkỳ}$$

mà $\sum_{n=2}^{+\infty} |a_n| = \sum_{n=2}^{+\infty} \frac{1}{\sqrt{n}+(-1)^n}$ pkỳ theo tiêu chuẩn So sánh

$$\left( \frac{1}{\sqrt{n}+(-1)^n} \overset{n\to+\infty}{\sim} \frac{1}{\sqrt{n}} \text{ mà } \sum_{n=2}^{+\infty} \frac{1}{\sqrt{n}} \text{ pkỳ} \right)$$

$$\Rightarrow \sum_{n=2}^{+\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} \text{ k° HTTĐ, k° Bán HT.}$$

d) $\sum_{n=1}^{+\infty} \sin(\pi.\sqrt{n^2+1})$, Xét $a_n = \sin(\pi.\sqrt{n^2+1})$

$$= \sin((\pi\sqrt{n^2+1} - \pi n) + \pi n)$$

$$= \sin(\pi\sqrt{n^2+1} - \pi.n).\underbrace{\cos(\pi n)}_{(-1)^n} + \underbrace{\cos(\pi\sqrt{n^2+1} - \pi n) . \sin(n\pi)}_{=0}$$

$$= (-1)^n . \sin(\pi.\sqrt{n^2+1} - \pi n).$$

Xét $\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} (-1)^n . \sin(\pi.(\sqrt{n^2+1} - n))$ với $u_n = \sin(\ldots)$

Ta có: $u_n > 0 \ \forall n \in [1, +\infty)$, $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \sin(0.\pi) = 0.$

Xét $f(x) = \sin(\pi.(\sqrt{x^2+1} - x))$ có $f'(x) = \pi.\cos(\pi(\sqrt{x^2+1} - x)).\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}$

Dễ thấy $f'(x) < 0 \ \forall x \in [1, +\infty) \Rightarrow u_n$ giảm

$$\Rightarrow \sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \sin(\pi.\sqrt{n^2+1})$$ htụ theo tiêu chuẩn leibnitz

Xét $$\sum_{n=1}^{+\infty} |a_n| = \sum_{n=1}^{+\infty} |\sin(\pi.\sqrt{n^2+1})| = \sum_{n=1}^{+\infty} \sin(\pi(\sqrt{n^2+1}-n))$$

Lại có khi $n \to +\infty$ $\sin(\pi.(\sqrt{n^2+1}-n)) \sim \dfrac{\pi}{\sqrt{n^2+1}+n} \sim \dfrac{\pi}{2\sqrt{n^2}} \sim \dfrac{\pi}{2n}$

mà $\sum_{n=1}^{+\infty} \dfrac{\pi}{2n}$ pkỳ (theo TC tích phân)

$\Longrightarrow$ Chuỗi số đã cho bán hội tụ.

e) $$\sum_{n=2}^{+\infty} \frac{1}{\ln^2 n} \cdot \cos\left(\frac{\pi n^2}{n+1}\right)$$ với $u_n = \dfrac{\cos\left(\frac{\pi n^2}{n+1}\right)}{\ln^2(n)}$

Trình bày gợi ý: tự trình bày:

Ta có: $$\cos\left(\frac{\pi n^2}{n+1}\right) = \cos\left(\frac{\pi(n^2-1)+\pi}{n+1}\right) = \cos\left((n-1)\pi + \frac{\pi}{n+1}\right)$$

$$= (-1)^{n-1}.\cos\left(\frac{\pi}{n+1}\right)$$

mà $$\sum_{n=2}^{+\infty} \frac{(-1)^{n-1}}{\ln^2 n}.\cos\left(\frac{\pi}{n+1}\right)$$ với $a_n = \dfrac{\cos\left(\frac{\pi}{n+1}\right)}{\ln^2(n)}$

$a_n > 0$

$\Longrightarrow \lim_{n\to+\infty} a_n = 0$; $a_n \downarrow \Rightarrow$ HT.

mà $$\sum_{n=2}^{+\infty} |u_n| = \sum_{n=2}^{+\infty} \frac{\cos\left(\frac{\pi}{n+1}\right)}{\ln^2 n}$$

Dễ thấy $\dfrac{\cos\left(\frac{\pi}{n+1}\right)}{\ln^2(n)} \sim \dfrac{1}{\ln^2(n)}$

mà $\lim_{n\to+\infty} \dfrac{1}{n} : \dfrac{1}{\ln^2 n} = \lim_{n\to+\infty} \dfrac{\ln^2 n}{n} \overset{L}{=} \lim_{n\to+\infty} \dfrac{2\ln n}{n} \overset{L}{=}$

$= \lim_{n\to+\infty} 2 \cdot \dfrac{1}{n} = 0$

$\Rightarrow \dfrac{1}{n} < \dfrac{1}{\ln^2 n} \Longrightarrow$ Pk $\Rightarrow$ Bán HT.