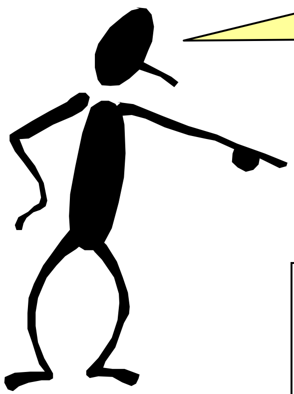
CHAPTER 4 : MODELLING & ANALYSIS FOR PROCESS CONTROL



When I complete this chapter, I want to be able to do the following.

- **Analytically solve linear dynamic models of first and second order**
- **Express dynamic models as transfer functions**
- **Predict important features of dynamic behavior from model <u>without solving</u>**

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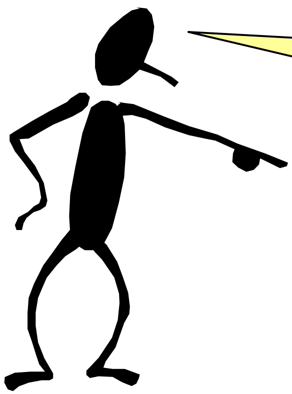
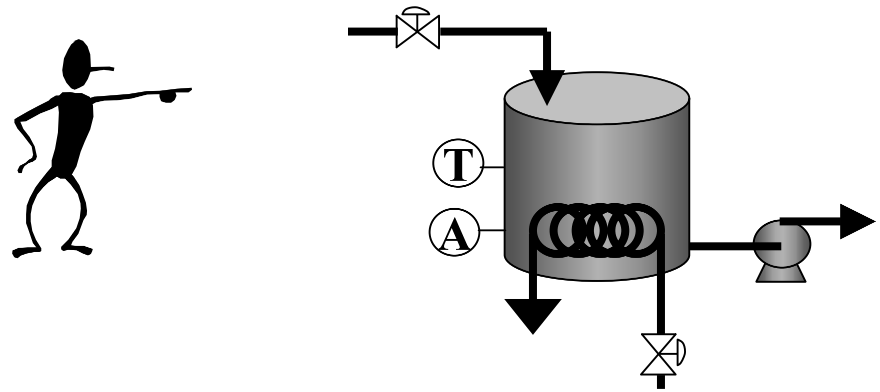


Outline of the lesson.

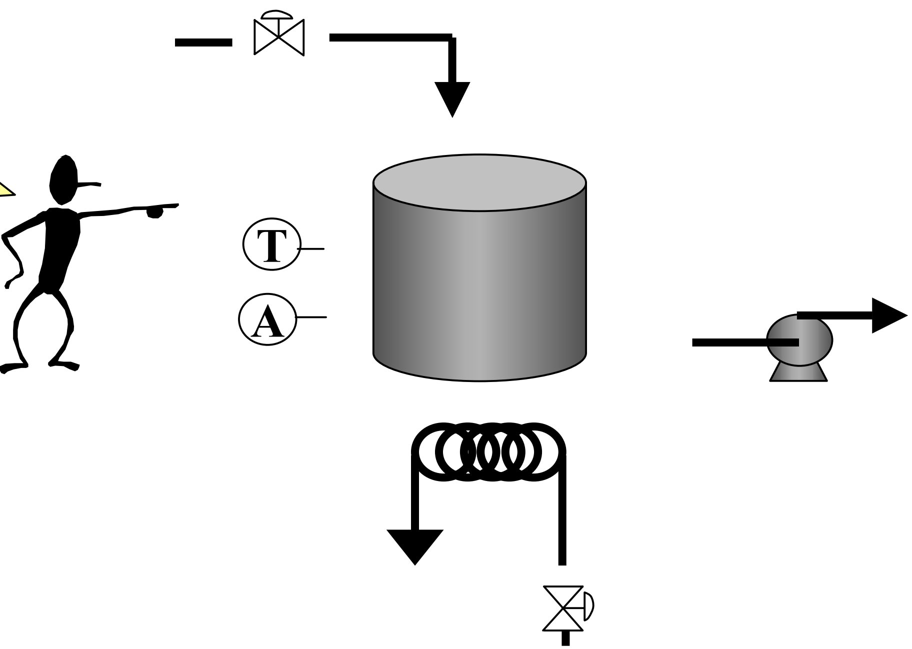
- **Laplace transform**
- **Solve linear dynamic models**
- **Transfer function model structure**
- **Qualitative features directly from model**
- **Frequency response**
- **Workshop**

WHY WE NEED MORE DYNAMIC MODELLING

I can model this;
what more do
I need?

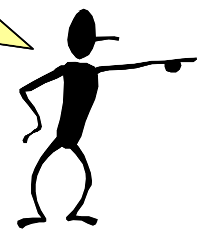


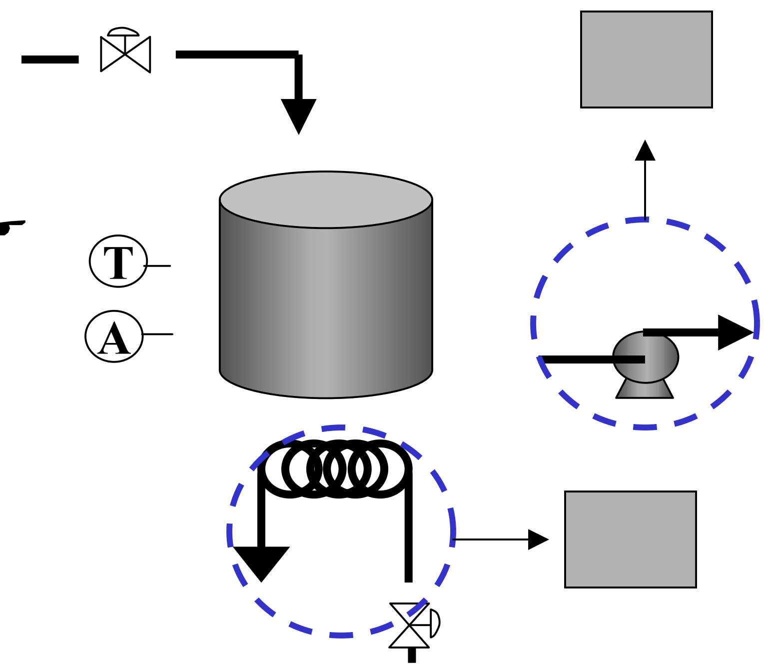
I would like to
- model elements individually
- combine as needed
- determine key dynamic features w/o solving



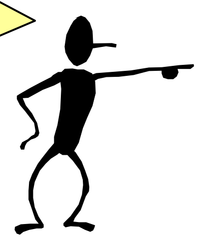
WHY WE NEED MORE DYNAMIC MODELLING

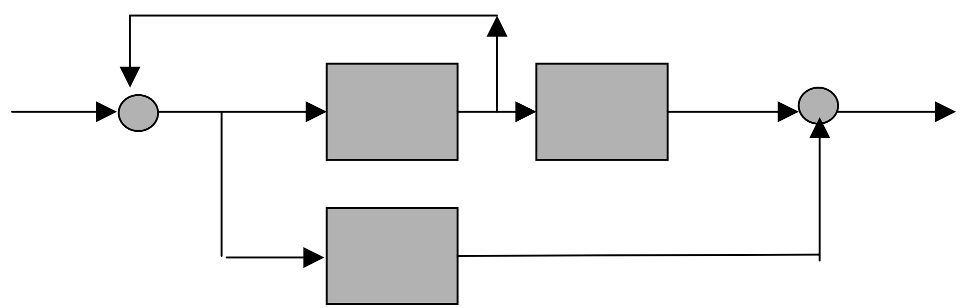
I would like to
- model elements individually
- This will be a "transfer function"





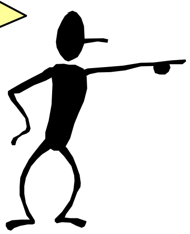
Now, I can combine elements to model many process structures

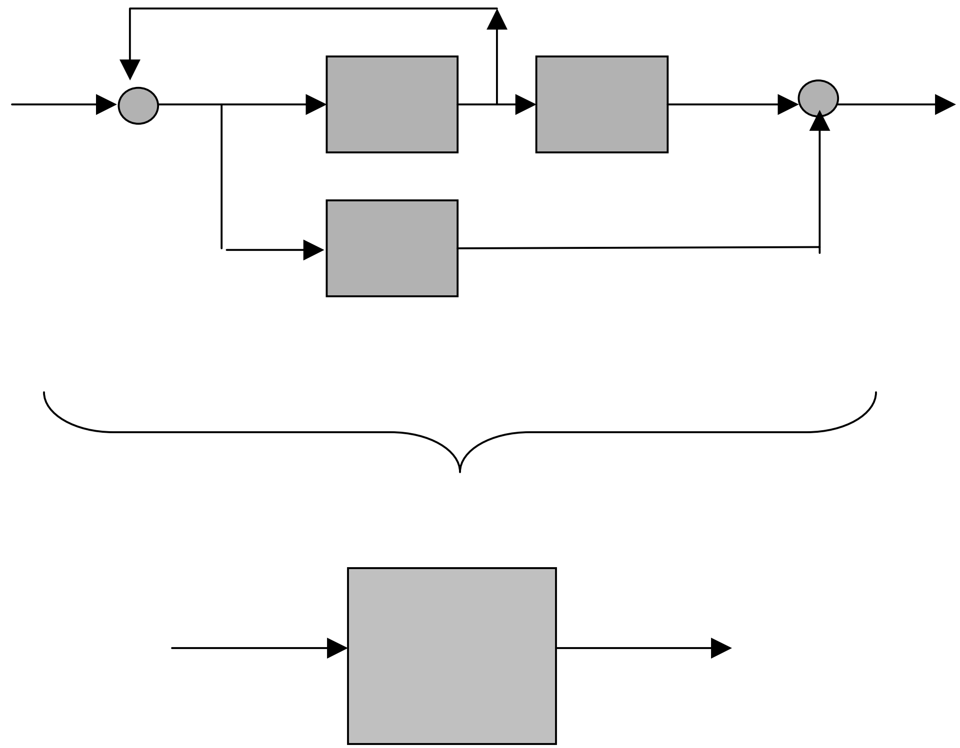




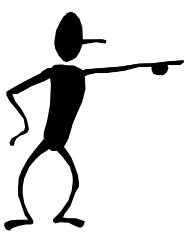
WHY WE NEED MORE DYNAMIC MODELLING

Now, I can combine elements to model many process structures





Even more amazing, I can combine to derive a simplified model!

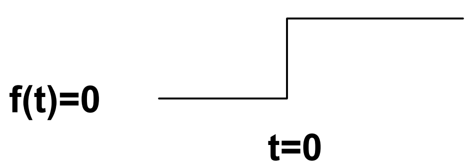


THE FIRST STEP: LAPLACE TRANSFORM

$$L(f(t)) = f(s) = \int_0^{\infty} f(t) e^{-st} dt$$

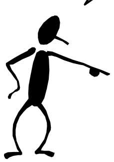
<u>Constant</u> : $L(C) = \int_0^{\infty} C e^{-st} dt = -\frac{C}{s} e^{-st} \Big|_{t=0}^{t=\infty} = \frac{C}{s}$

<u>Step Change</u> at t=0: Same as constant for t=0 to t=∞



THE FIRST STEP: LAPLACE TRANSFORM

We have seen this term often! It's the step response to a first order dynamic system.



$$L(f(t)) = f(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$\mathbf{L}\left(\boxed{(1-\mathbf{e}^{-\mathbf{t}/\tau})}\right) = \int_0^\infty (1-\mathbf{e}^{-\mathbf{t}/\tau})\mathbf{e}^{-\mathbf{st}}\mathbf{dt} = \underbrace{\int_0^\infty \mathbf{e}^{-\mathbf{st}}\mathbf{dt}}_{=1/\mathbf{s}} + \int_0^\infty -\mathbf{e}^{-\mathbf{t}/\tau}\mathbf{e}^{-\mathbf{st}}\mathbf{dt}$$

$$= -\int_0^\infty \mathbf{e}^{-(1/\tau+\mathbf{s})\mathbf{t}}\mathbf{dt} = \frac{1}{\mathbf{s}+1/\tau}\mathbf{e}^{-(1/\tau+\mathbf{s})\mathbf{t}}\Big|_0^\infty = -\frac{1}{\mathbf{s}+1/\tau}$$

$$= \frac{1}{\mathbf{s}} - \frac{1}{\mathbf{s}+1/\tau} = \frac{1}{\mathbf{s}} - \frac{\tau}{\tau\,\mathbf{s}+1} = \boxed{\frac{1}{\mathbf{s}(\tau\,\mathbf{s}+1)}}$$

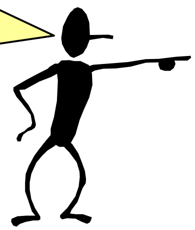
THE FIRST STEP: LAPLACE TRANSFORM

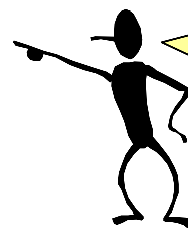
Let's consider plug flow through a pipe. Plug flow has no backmixing; we can think of this a a hockey puck traveling in a pipe.

What is the dynamic response of the outlet fluid property (e.g., concentration) to a step change in the inlet fluid property?

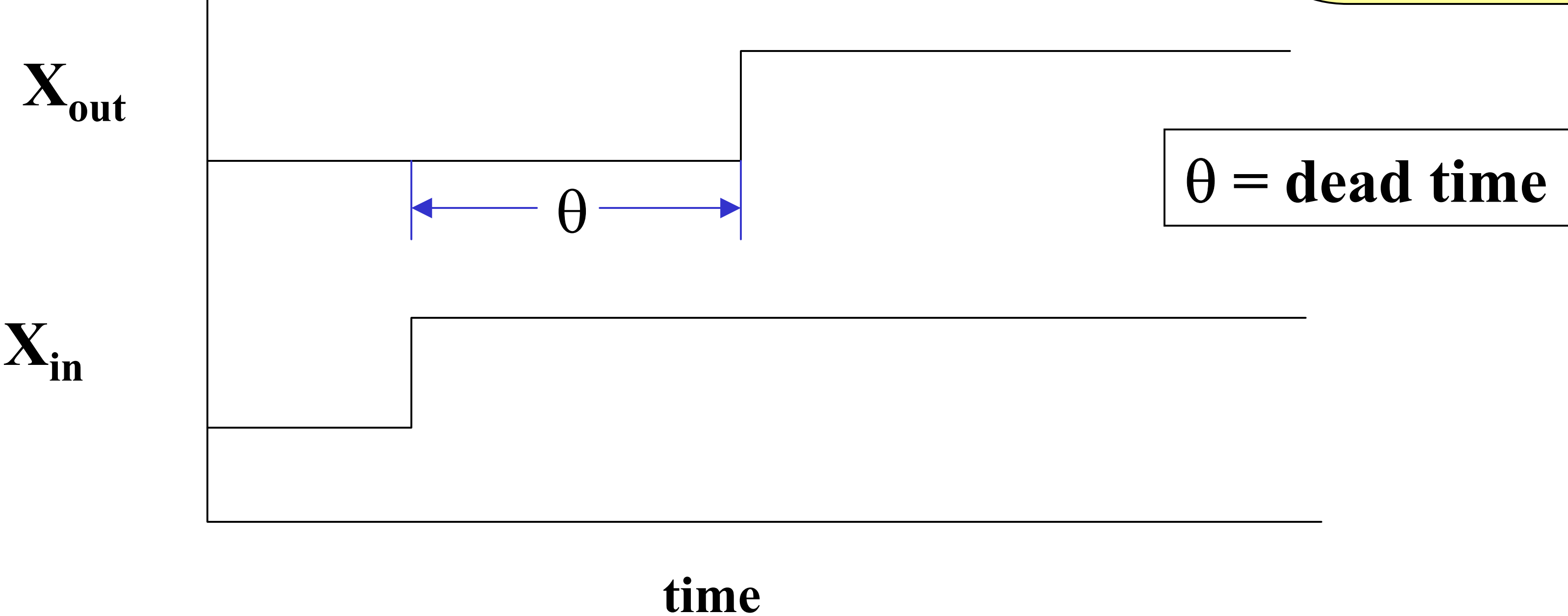
THE FIRST STEP: LAPLACE TRANSFORM

Let's learn a new dynamic response & its Laplace Transform



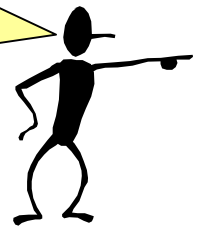


What is the value of dead time for plug flow?

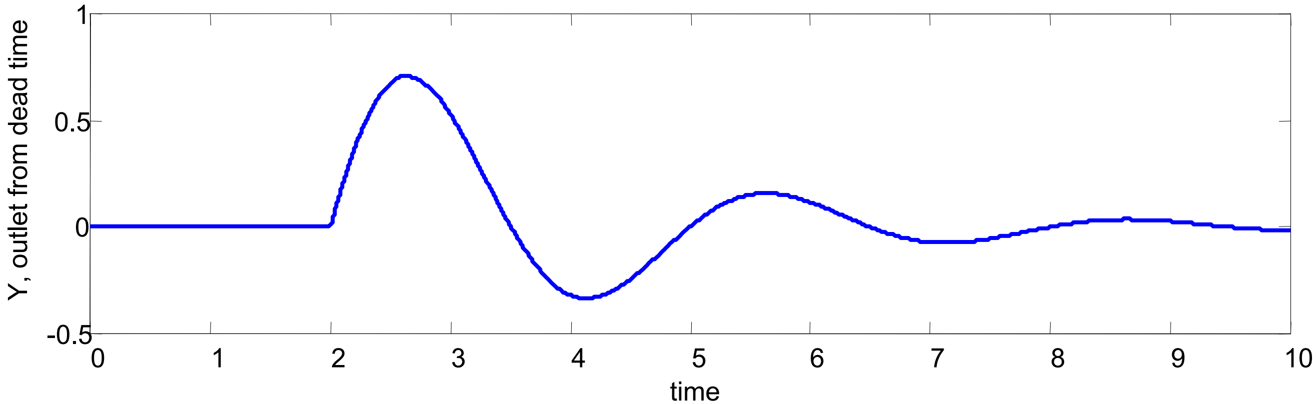


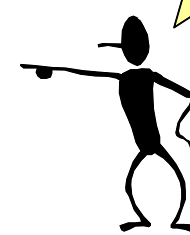
THE FIRST STEP: LAPLACE TRANSFORM

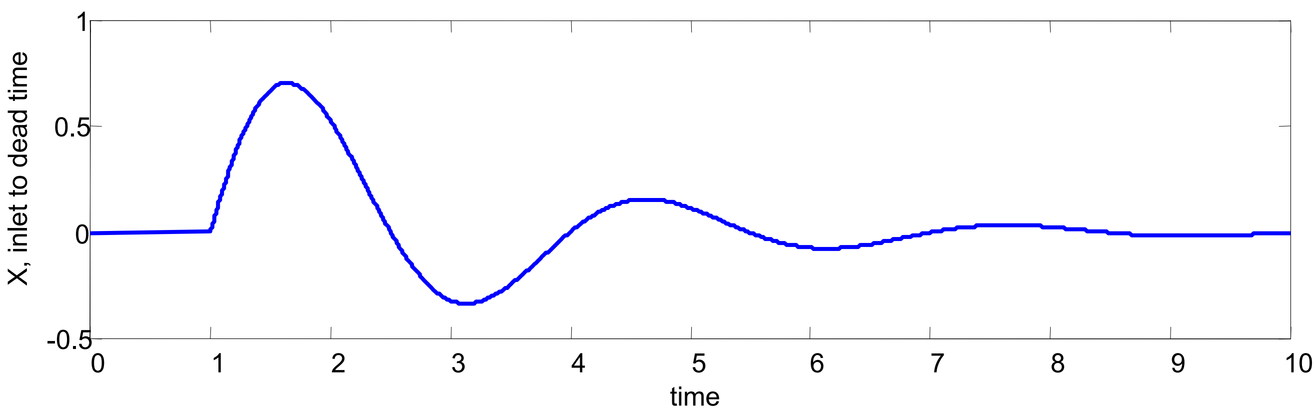
Let’s learn a new dynamic response & its Laplace Transform



- Is this a dead time?
- What is the value?

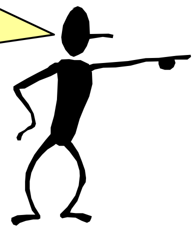




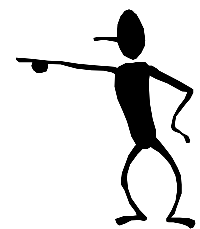


THE FIRST STEP: LAPLACE TRANSFORM

Let's learn a new dynamic response & its Laplace Transform



Our plants have pipes. We will use this <u>a lot!</u>



The dynamic model for dead time is

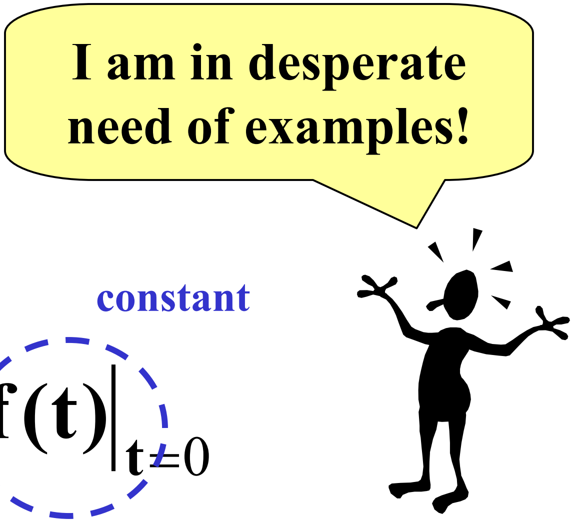
$$\mathbf{X_{out}(t) = X_{in}(t-\theta)}$$

The Laplace transform for a variable after dead time is

$$L(X_{out}(t)) = L(X_{in}(t-\theta)) = e^{-\theta s} X_{in}(s)$$

THE FIRST STEP: LAPLACE TRANSFORM

We need the Laplace transform of derivatives for solving dynamic models.



<u>First derivative</u>:

$$L\left[\frac{df(t)}{dt}\right] = sf(s) - \overbrace{f(t)\Big|_{t=0}}^{\text{constant}}$$

<u>General</u>:

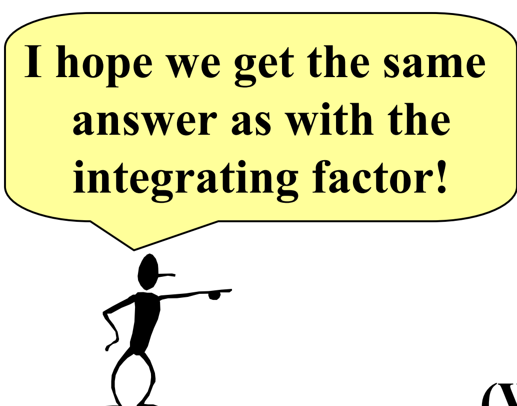
$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n f(s) - \overbrace{\left(s^{n-1} f(t)\Big|_{t=0} + s^{n-1}\frac{df(t)}{dt}\Big|_{t=0} + \ldots. + \frac{d^{n-1}f(t)}{dt^{n-1}}\Big|_{t=0} \right)}^{\text{constant}}$$

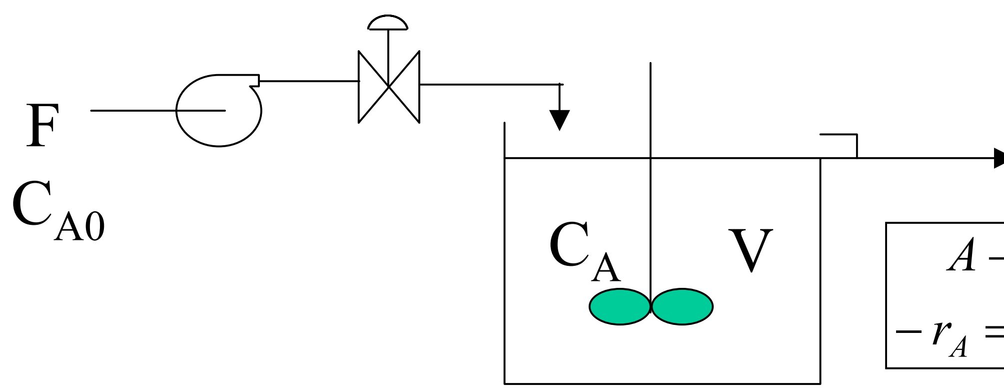
SOLVING MODELS USING THE LAPLACE TRANSFORM

<u>Textbook Example 3.1</u>: The CSTR (or mixing tank) experiences a step in feed composition with all other variables are constant. Determine the dynamic response.

$$V\frac{dC'_A}{dt} = F(C'_{A0} - C'_A) - VkC'_A$$

$$\tau\frac{dC'_A}{dt} + C'_A = KC'_{A0} \quad \textbf{with } \tau = \frac{\mathbf{V}}{\mathbf{F+kV}} \quad \textbf{and K} = \frac{\mathbf{F}}{\mathbf{F+kV}}$$





$$A \rightarrow B$$
$$-r_A = kC_A$$

(We'll solve this in class.)

SOLVING MODELS USING THE LAPLACE TRANSFORM

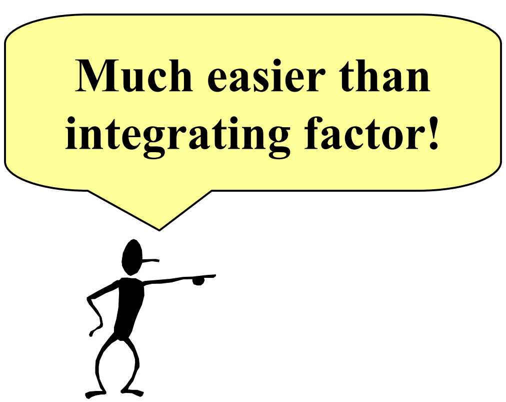
Two isothermal CSTRs are initially at steady state and experience a step change to the feed composition to the first tank. Formulate the model for C_{A2}.

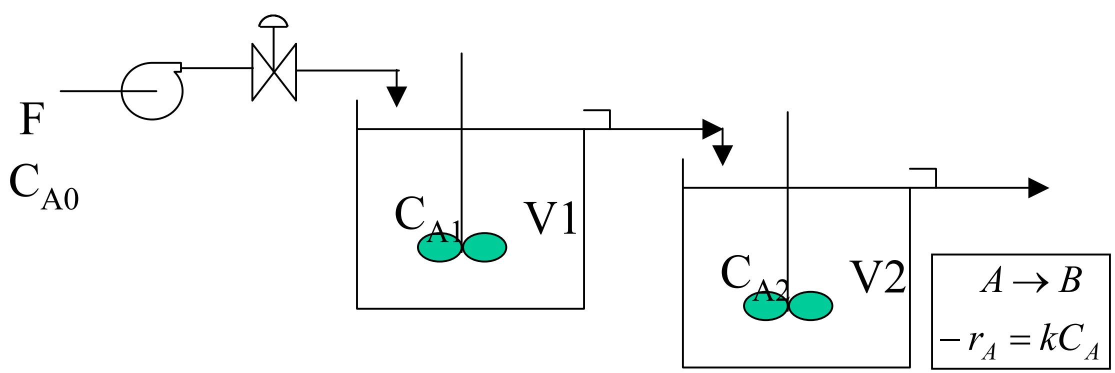
$$V_1 \frac{dC'_{A1}}{dt} = F(C'_{A0} - C'_{A1}) - V_1 k_1 C'_{A1}$$

$$V_2 \frac{dC'_{A2}}{dt} = F(C'_{A1} - C'_{A2}) - V_2 k_2 C'_{A2}$$

$$\tau_1 \frac{dC'_{A1}}{dt} + C'_{A1} = K_1 C'_{A0}$$

$$\tau_2 \frac{dC'_{A2}}{dt} + C'_{A2} = K_2 C'_{A1}$$

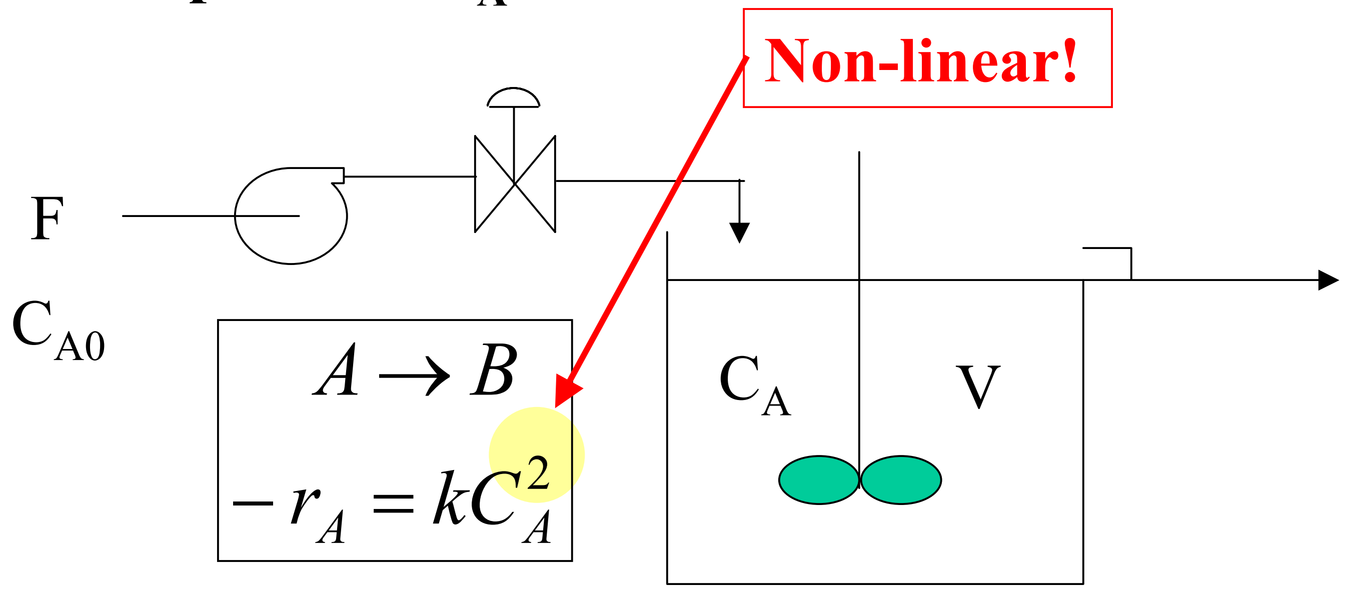




(We'll solve this in class.)

SOLVING MODELS USING THE LAPLACE TRANSFORM

Textbook Example 3.5: The feed composition experiences a step. All other variables are constant. Determine the dynamic response of C_A.



(We'll solve this in class.)

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

Let's rearrange the Laplace transform of a dynamic model

$$Y(s) = G(s)\,X(s) \qquad\qquad X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

A <u>TRANSFER FUNCTION</u> is the output variable, Y(s), divided by the input variable, X(s), with all initial conditions zero.

$$G(s) = Y(s)/X(s)$$

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

$$G(s) = Y(s) / X(s)$$

$$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

- **How do we achieve zero initial conditions for every model?**
- **We don’t have “primes” on the variables; why?**
- **Is this restricted to a step input?**
- **What about non-linear models?**
- **How many inputs and outputs?**

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

$$G(s) = Y(s)/X(s)$$

$$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

Some examples:

Mixing tank : $$\frac{C_A(s)}{C_{A0}(s)} = G(s) = ?$$

Two CSTRs : $$\frac{C_{A2}(s)}{C_{A0}(s)} = G(s) = ?$$

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

$$G(s) = Y(s) / X(s)$$

$$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

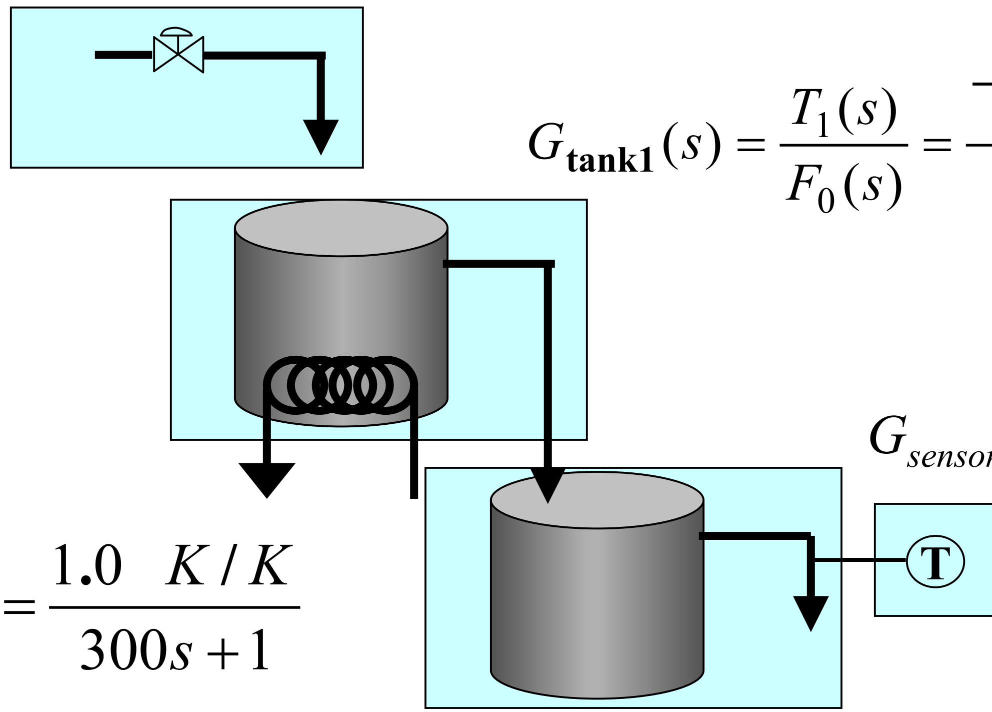
Why are we doing this?

- **To torture students.**
- **We have individual models that we can combine easily - <u>algebraically</u>.**
- **We can determine lots of information about the system <u>without solving</u> the dynamic model.**

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

$$G_{valve}(s) = \frac{F_0(s)}{v(s)} = .10 \ \ m^3/s \Big/ \% \ open$$

$$G_{\mathbf{tank1}}(s) = \frac{T_1(s)}{F_0(s)} = \frac{-1.2 \ \ \mathbf{K} \big/ \mathbf{m^3/s}}{250s+1}$$



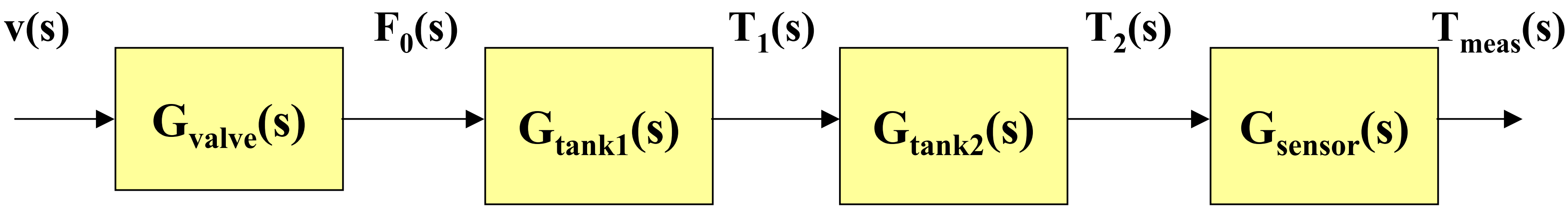
$$G_{sensor}(s) = \frac{T_{measured}(s)}{T_2(s)} = \frac{1.0 \ \ K/K}{10s+1}$$

$$G_{\mathbf{tank2}}(s) = \frac{T_2(s)}{T_1(s)} = \frac{1.0 \ \ K/K}{300s+1}$$

(Time in seconds)

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

The BLOCK DIAGRAM

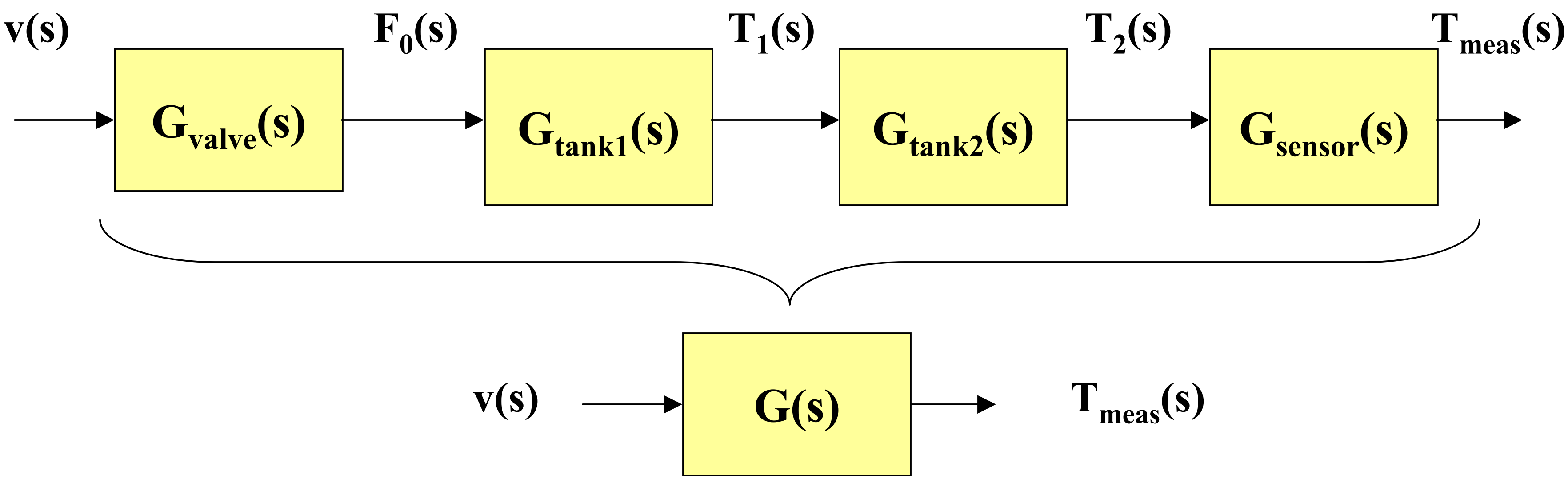


It's a picture of the model equations!

- Individual models can be replaced easily
- Helpful visualization
- Cause-effect by arrows

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

Combine using BLOCK DIAGRAM ALGEBRA



$$\frac{T_{meas}(s)}{v(s)} = G(s) = \left[\frac{T_{meas}(s)}{T_2(s)}\right]\left[\frac{T_2(s)}{T_1(s)}\right]\left[\frac{T_1(s)}{F_0(s)}\right]\left[\frac{F_0(s)}{v(s)}\right]$$

$$= G_s(s)G_{T2}(s)G_{T1}(s)G_v(s)$$

TRANSFER FUNCTIONS: MODELS VALID FOR ANY INPUT FUNCTION

Key rules for BLOCK DIAGRAM ALGEBRA

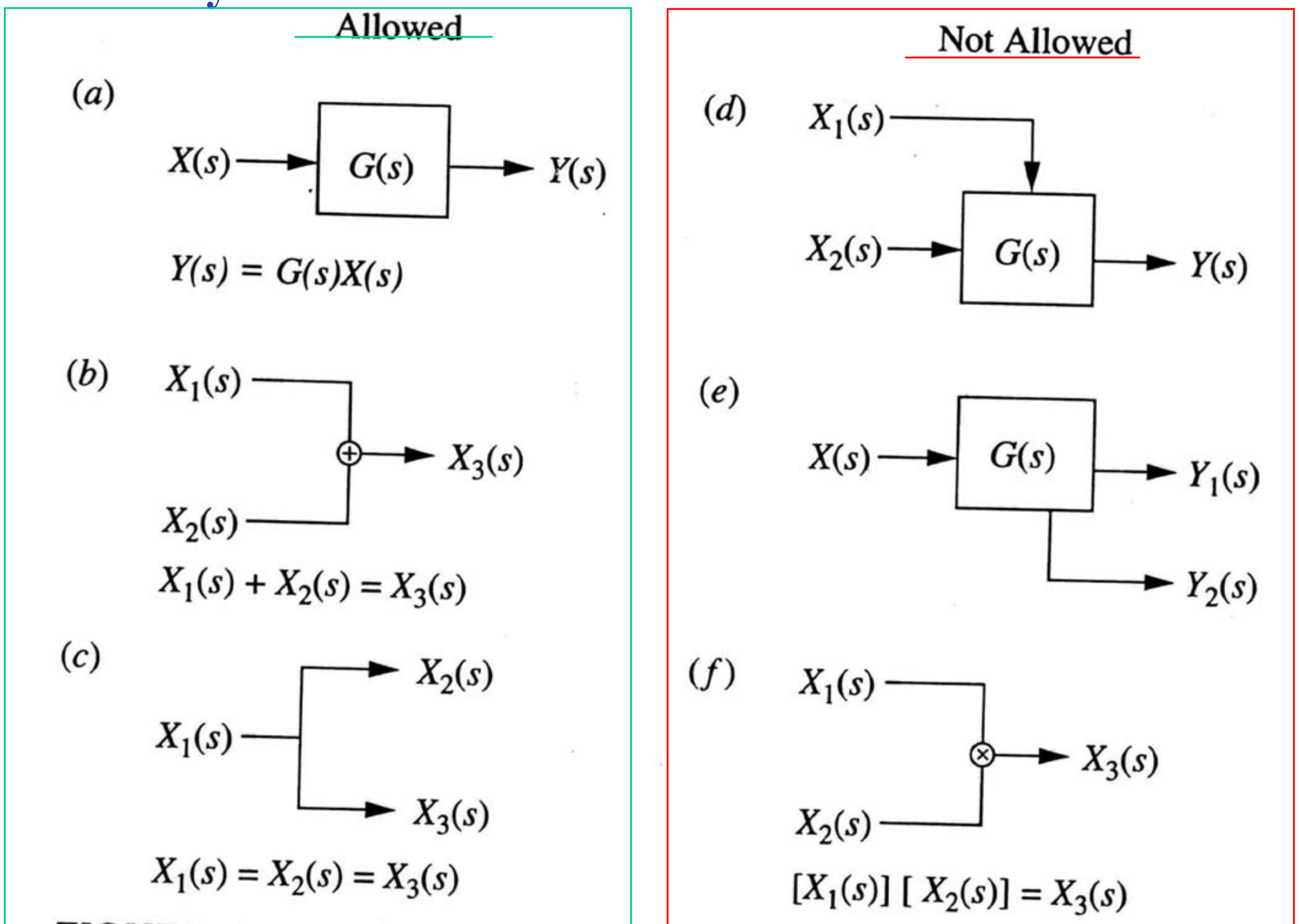


FIGURE 4.5

QUALITATIVE FEATURES W/O SOLVING

FINAL VALUE THEOREM: Evaluate the final valve of the output of a dynamic model without solving for the entire transient response.

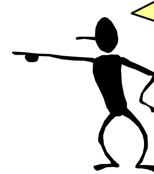
$$Y(t)\Big|_{t\to\infty} = \lim_{s\to\infty} \mathbf{sY(s)}$$

Example for first order system

$$C_A(t)\,|_{t\to\infty} = \lim_{s\to 0} s \frac{\Delta C_{A0} K_p}{s(\tau s + 1)} = \Delta C_{A0} K_p$$

QUALITATIVE FEATURES W/O SOLVING

We can use partial fraction expansion to prove the following key result.



What about dynamics can we determine without solving?

$$Y(s) = G(s)X(s) = [N(s)/D(s)]X(s) = C_1/(s-\alpha_1) + C_2/(s-\alpha_2) + \ldots$$

With α_i the solution to the denominator of the transfer function being zero, D(s) = 0.

$$Y(t) = A_0 + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \ldots + (B_0 + B_1 t + B_2 t^2 + \ldots) e^{\alpha_p t} + \ldots + [C_1 \cos(\omega t) + C_2 \sin(\omega t)] e^{\alpha_q t} + \ldots$$

Real, distinct α_i

Complex α_i
α_q is Re(α_i)

Real, repeated α_i

QUALITATIVE FEATURES W/O SOLVING

With α_i the solutions to D(s) = 0, which is a polynomial.

$$Y(t) = A_0 + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \ldots + (B_0 + B_1 t + B_2 t^2 + ..)e^{\alpha_p t} +$$

$$\ldots + [C_1 \cos(\omega t) + C_2 \sin(\omega t)]e^{\alpha_q t} + \ldots$$

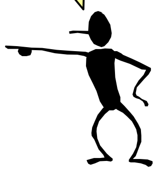
1. **If all α_i are ???, Y(t) is stable**

   If any one α_i is ???, Y(t) is unstable

2. **If all α_i are ???, Y(t) is overdamped (does not oscillate)**

   If one pair of α_i are ???, Y(t) is underdamped

Complete statements based on equation.



QUALITATIVE FEATURES W/O SOLVING

With α_i the solutions to D(s) = 0, which is a polynomial.

$$Y(t) = A_0 + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \ldots + (B_0 + B_1 t + B_2 t^2 + ..)e^{\alpha_p t} +$$

$$\ldots + [C_1 \cos(\omega t) + C_2 \sin(\omega t)]e^{\alpha_q t} + \ldots$$

1. **If all real [α_i] are < 0, Y(t) is stable**

   If any one real [α_i] is $\geq$ 0, Y(t) is unstable

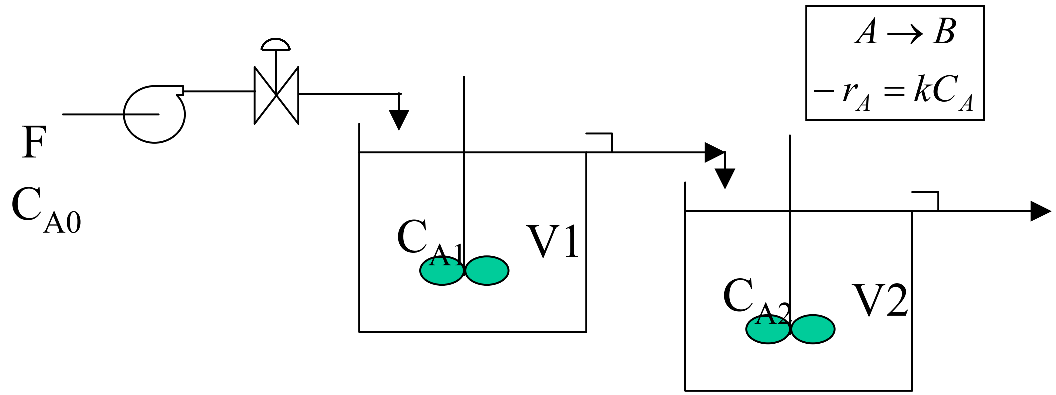
2. **If all α_i are real, Y(t) is overdamped (does not oscillate)**

   If one pair of α_i are complex, Y(t) is underdamped

QUALITATIVE FEATURES W/O SOLVING

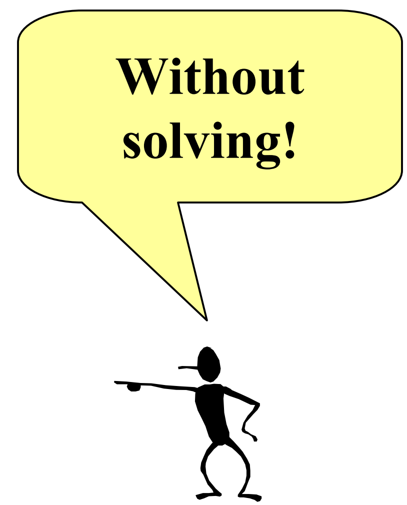
$$\tau_1 \frac{dC'_{A1}}{dt} + C'_{A1} = K_1 C'_{A0}$$

$$\tau_2 \frac{dC'_{A2}}{dt} + C'_{A2} = K_2 C'_{A1}$$



1. **Is this system stable?**
2. **Is this system over- or underdamped?**
3. **What is the order of the system?**

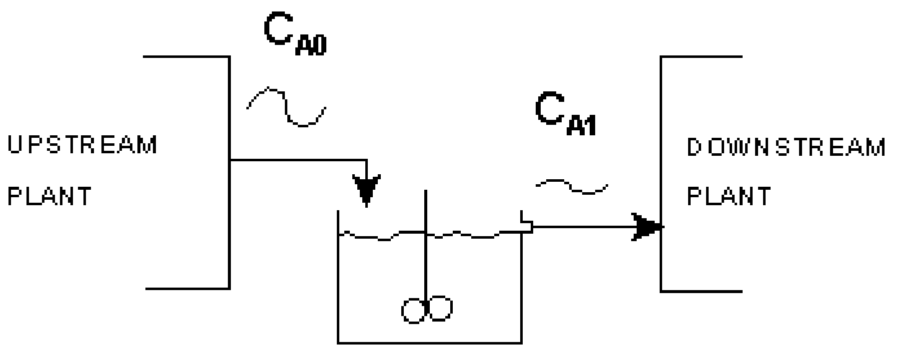
   (Order = the number of derivatives between the input and output variables)
4. **What is the steady-state gain?**



(We'll solve this in class.)

QUALITATIVE FEATURES W/O SOLVING

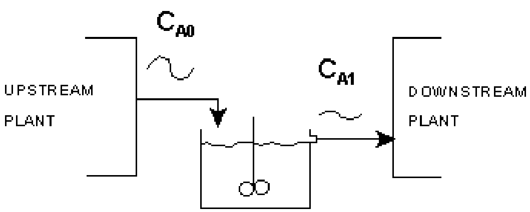
FREQUENCY RESPONSE:The response to a sine input of the output variable is of great practical importance. Why?



Sine inputs almost never occur. However, many periodic disturbances occur and other inputs can be represented by a combination of sines.

For a process without control, we want a sine input to have a small effect on the output.

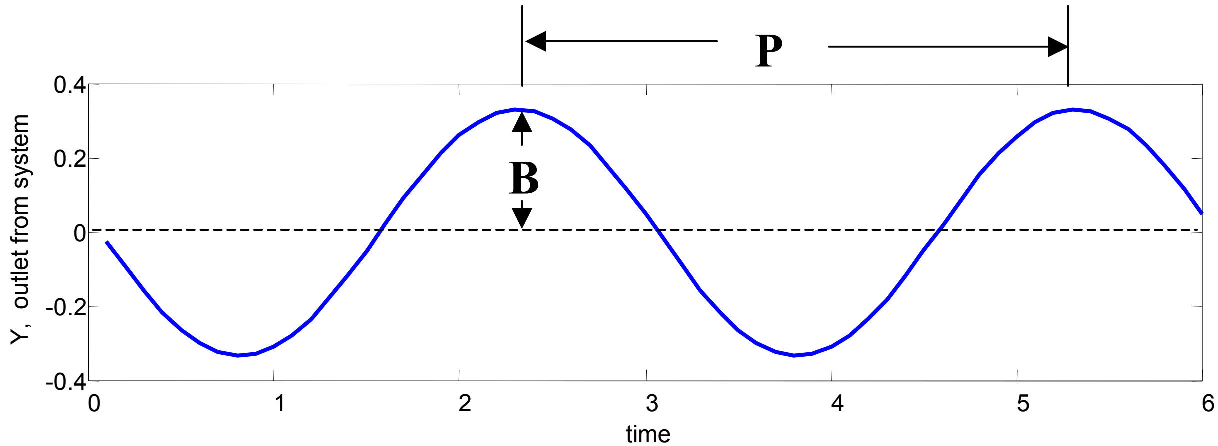
QUALITATIVE FEATURES W/O SOLVING



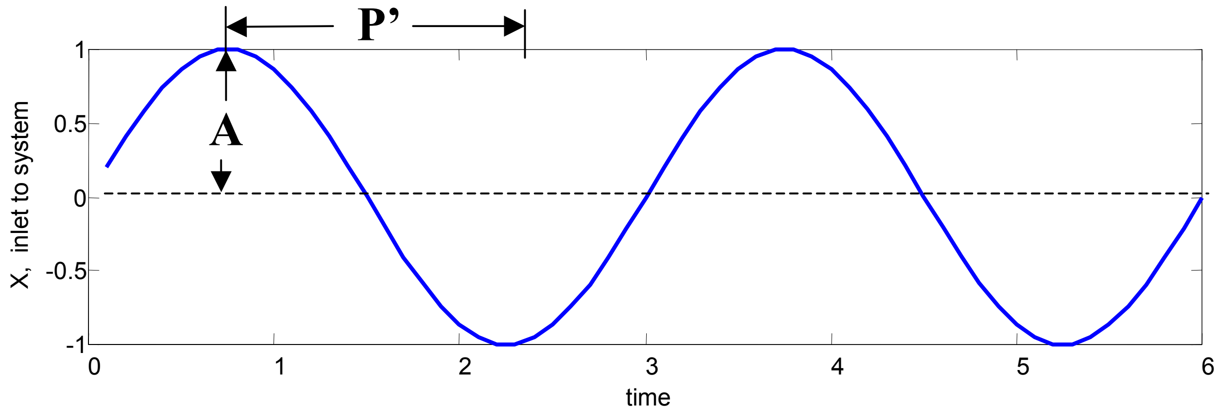
Amplitude ratio = $|Y'(t)|_{max} \,/\, |X'(t)|_{max}$

Phase angle = phase difference between input and output

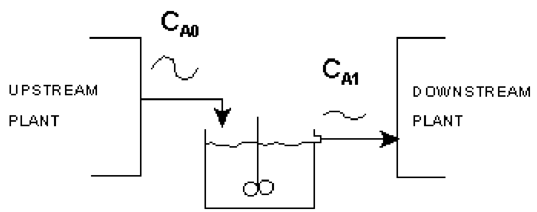
output



input



QUALITATIVE FEATURES W/O SOLVING



Amplitude ratio = $|Y'(t)|_{max} / |X'(t)|_{max}$

Phase angle = phase difference between input and output

For linear systems, we can evaluate directly using transfer function!
Set s = jω, with ω= frequency and j = complex variable.

$$\textbf{Amp. Ratio} = AR = \left|G(j\omega)\right| = \sqrt{\mathbf{Re}(G(j\omega))^2 + \mathbf{Im}(G(j\omega))^2}$$

$$\textbf{Phase angle} = \varphi = \angle G(j\omega) = \tan^{-1}\left(\frac{\mathbf{Im}(G(j\omega))}{\mathbf{Re}(G(j\omega))}\right)$$

These calculations are tedious by hand but easily performed in standard programming languages.

QUALITATIVE FEATURES W/O SOLVING

Example 4.15 Frequency response of mixing tank.

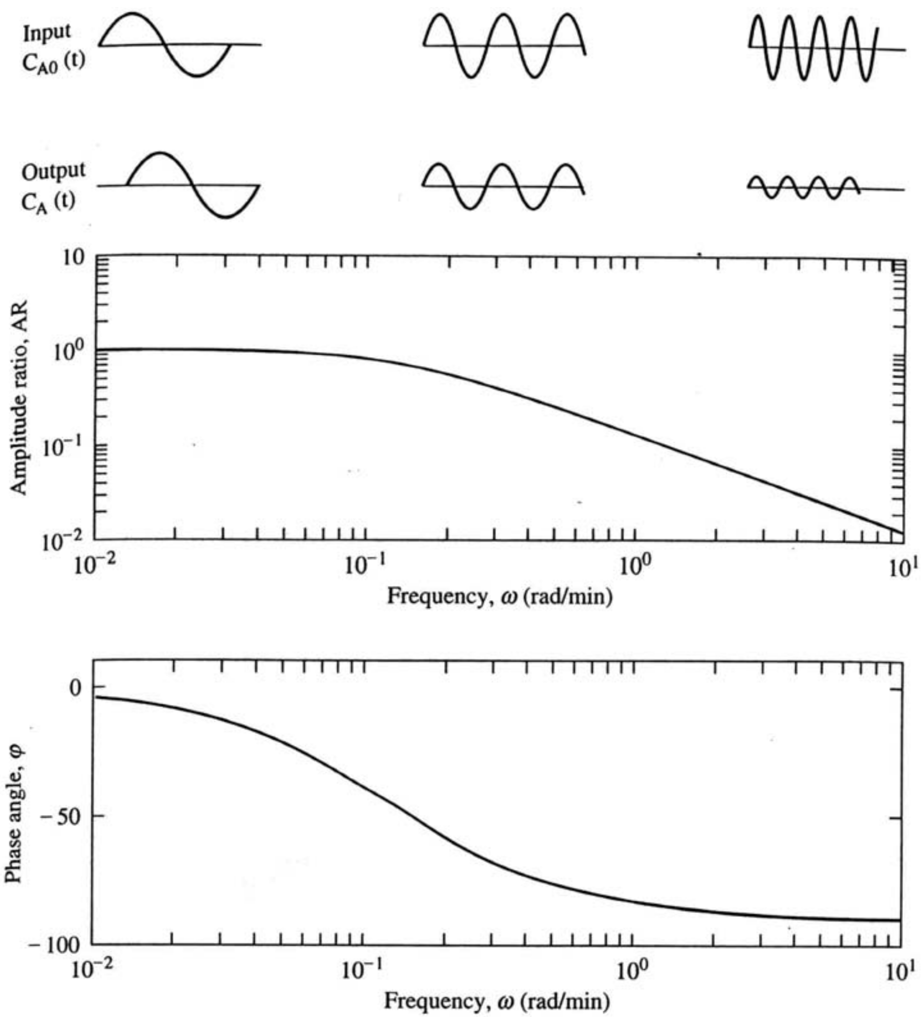


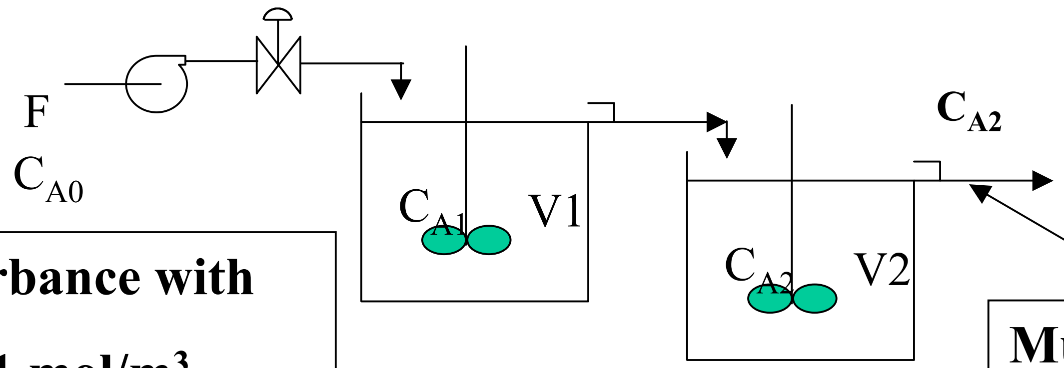
FIGURE 4.10

Time-domain behavior.

Bode Plot - Shows frequency response for a range of frequencies

- **Log (AR) vs log(ω)**
- **Phase angle vs log(ω)**

QUALITATIVE FEATURES W/O SOLVING



Sine disturbance with

amplitude = 1 mol/m³

frequency = 0.20 rad/min

Data from 2 CSTRs

Must have fluctuations

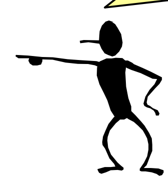
< 0.050 mol/m³

Using equations for the frequency response amplitude ratio

$$\frac{|C_{A2}|}{|C_{A0}|} = |G(j\omega)| = \frac{K_p}{(1+\omega^2\tau^2)}$$

$$|C_{A2}| = |C_{A0}| \frac{K_p}{(1+\omega^2\tau^2)}$$

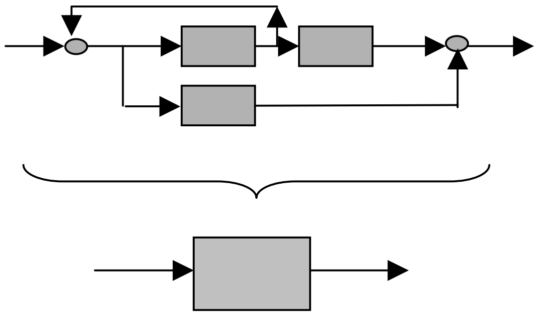
$$|C_{A2}| = (1.0)(0.12) \boxed{= 0.12 > 0.050}$$

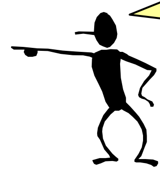


Not acceptable. We need to reduce the variability. How about feedback control?

OVERVIEW OF ANALYSIS METHODS

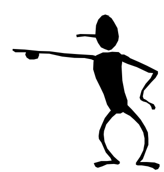
Transfer function and block diagram





We can determine individual models and combine

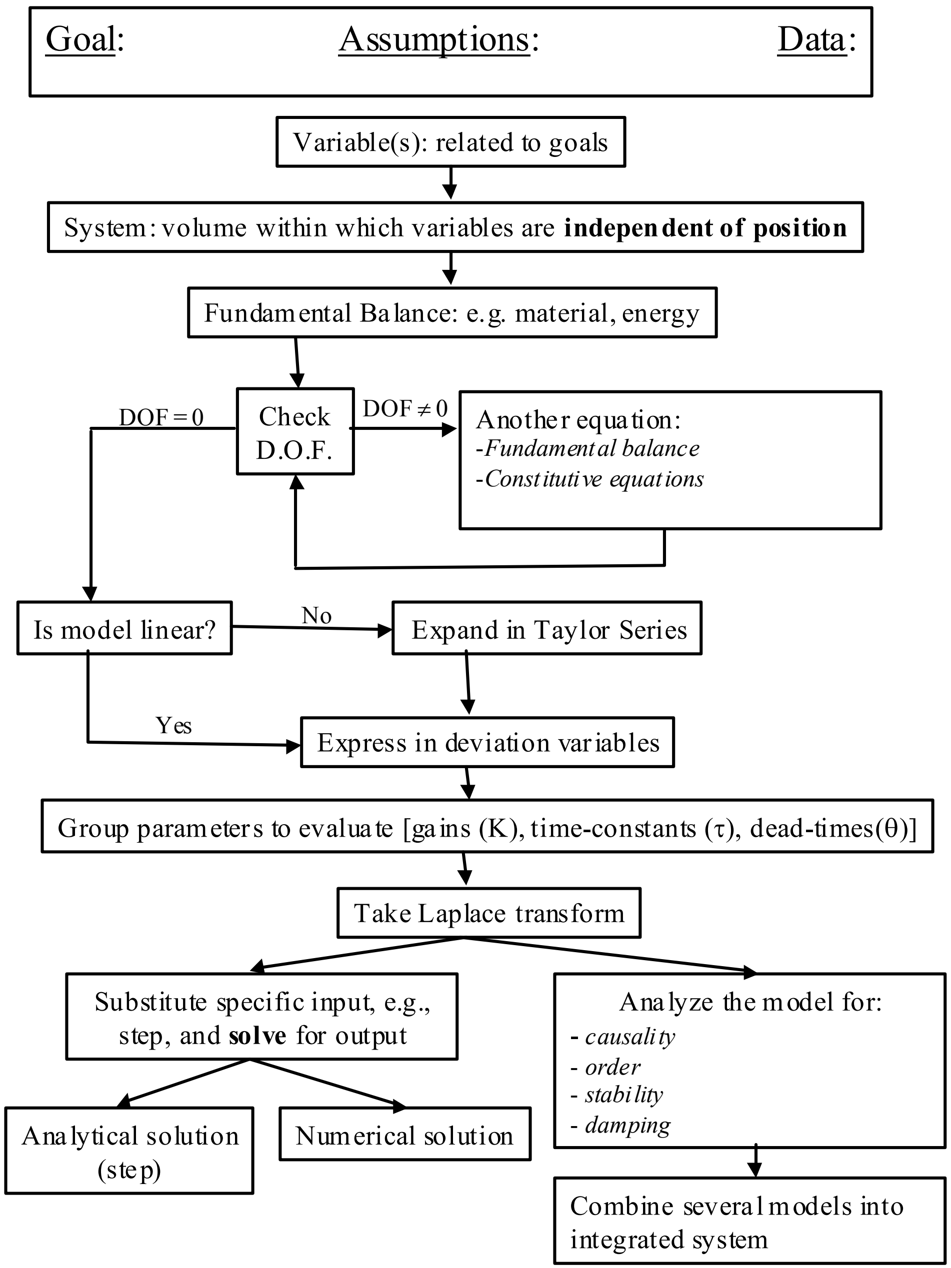
1. System order
2. Final Value
3. Stability
4. Damping
5. Frequency response



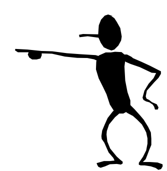
We can determine these features without solving for the entire transient!!

Combining Chapters 3 and 4

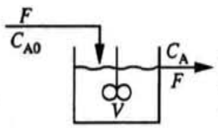
Flowchart of Modeling Method



We can use a standard modelling procedure to focus our creativity!



Too small to read here - check it out in the textbook!



$A \rightarrow B$

$r_A = -kC_A^2$

$V, F = \text{constant}$

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - VkC_A^2$$

$$V\frac{dC_A}{dt} = F(C_{A0} - C_A) - [VkC_{As}^2 + 2VkC_{As}(C_A - C_{As})]$$

Formulate Model Based on Conservation Balances and Constitutive Relationships

- "Exact" dynamic behavior described by model

Linearize Nonlinear Terms

- Easier to solve analytically
- Useful for determining some properties, e.g., stability

Numerical Simulation

- Determine the complete transient response

```
% Pseudocode for Euler's integration
T(1)=0              % Initialize
CA(1)=CAINIT
FOR N=2:  NMAX
  IF N>NSTEP, CAO=STEP, END
  DER=(F/V)*(CAO-CA(N-1))
    -K*(CA(N-1))^2
  CA(N) = CA(N-1)+DELTAT*DER
  T(N)=T(N-1)+DELTAT
END
```

$$\frac{dC'_A}{dt} + \frac{1}{\tau}C'_A = \frac{K_p}{\tau}C'_{A0}$$

$$\tau = \frac{V}{F + 2VkC_{As}} \qquad K_p = \frac{F}{F + 2VkC_{As}}$$

Express in Deviation Variables

- Required so that transfer functions are linear operators

$$sC'_A(s) - C'_A(t)\big|_{t=0} + \frac{C'_A(s)}{\tau} = \frac{K_p}{\tau}C'_{A0}(s)$$

Take the Laplace Transform

Transfer Function: $\dfrac{C_A(s)}{C_{A0}(s)} = \dfrac{K_p}{(\tau s + 1)} = G(s)$

Final Value: $\lim_{t\to\infty} C'_A(t) = \lim_{s\to 0} sC_A(s)$

$= \lim_{s\to 0} s\,\dfrac{\Delta C_{A0}}{s}\,\dfrac{K_p}{(\tau s+1)}$

$= K_p\Delta C_{A0}$

For $C'_A(t)\big|_{t=0} = 0 \qquad C'_{A0}(s) = \dfrac{\Delta C'_{A0}}{s}$

$$C'_A(s) = \frac{K_p \Delta C_{A0}}{s(\tau s + 1)}$$

Table 4.1, entry 5

$$C'_A(t) = K_p\Delta C_{A0}(1 - e^{-t/\tau})$$

Solve Analytically
(Invert to time domain)

- Use Table 4.1
- Expand using partial fractions
- General initial conditions and input forcing

Results: Complete transient of the linearized system

Formulate Transfer Function
(Do not solve for entire dynamic response)

- Set all initial conditions to zero
- Draw block diagram of system
- Derive overall transfer function using block diagram algebra

Results: Final value, stability, and frequency response

$$C_{A0}(s) \rightarrow \boxed{\frac{K_p}{(\tau s+1)}} \rightarrow C_A(s)$$

Shows cause-effect direction

Stability: Pole $s = \dfrac{-1}{\tau} < 0$

$\therefore$ stable

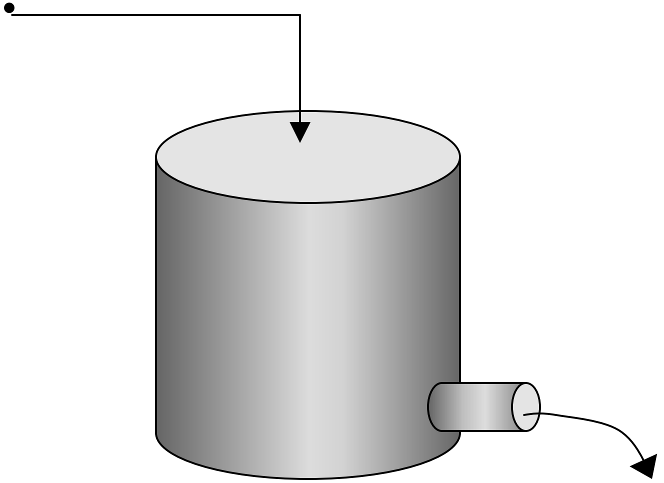
Frequency Response:

$$\text{AR} = |G(j\omega)| = \frac{K_p}{\sqrt{1 + \omega^2\tau^2}}$$

$$\Phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$$

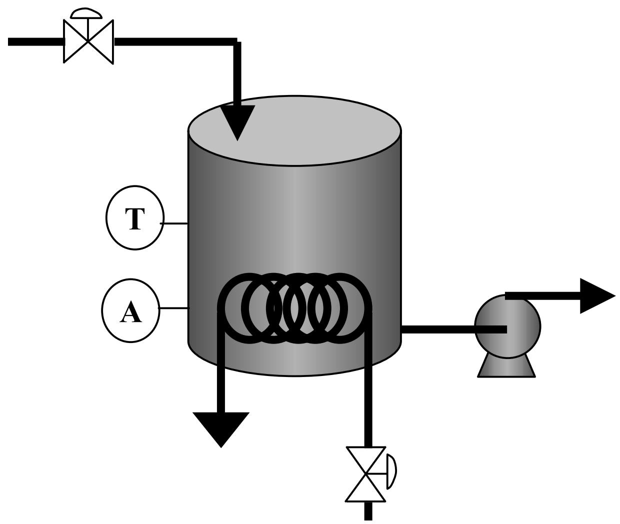
CHAPTER 4: MODELLING & ANALYSIS WORKSHOP 1

Example 3.6 The tank with a drain has a continuous flow in and out. It has achieved initial steady state when a step decrease occurs to the flow in. Determine the level as a function of time.



Solve the linearized model using Laplace transforms

CHAPTER 4: MODELLING & ANALYSIS WORKSHOP 2



The dynamic model for a non-isothermal CSTR is derived in Appendix C. A specific example has the following transfer function.

$$\frac{T(s)}{F_c(s)} = \frac{(-6.07s - 45.83)}{(s^2 + 1.79s + 35.80)}$$

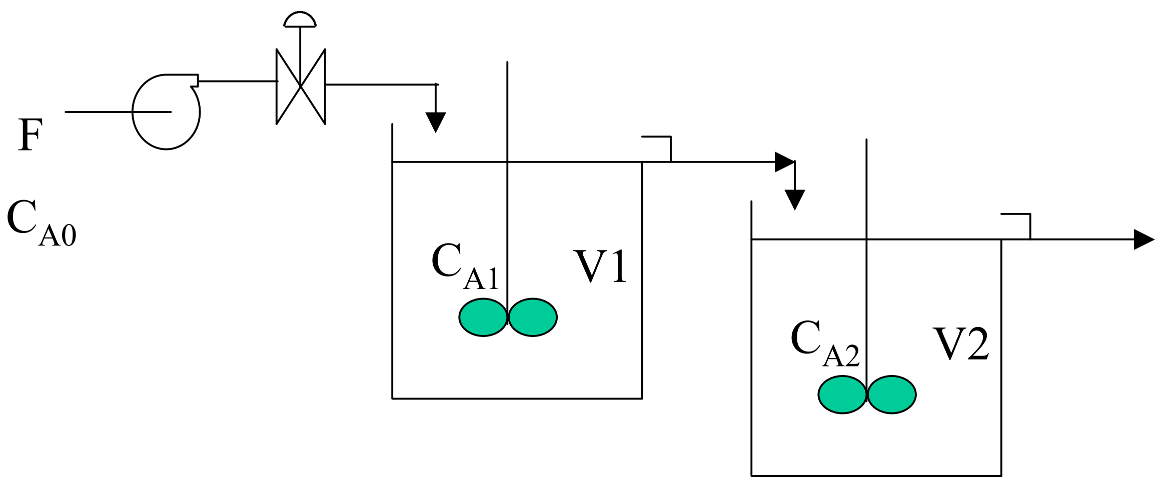
Determine the features in the table for this system.

1. **System order**
2. **Final Value**
3. **Stability**
4. **Damping**
5. **Frequency response**

CHAPTER 4: MODELLING & ANALYSIS WORKSHOP 3

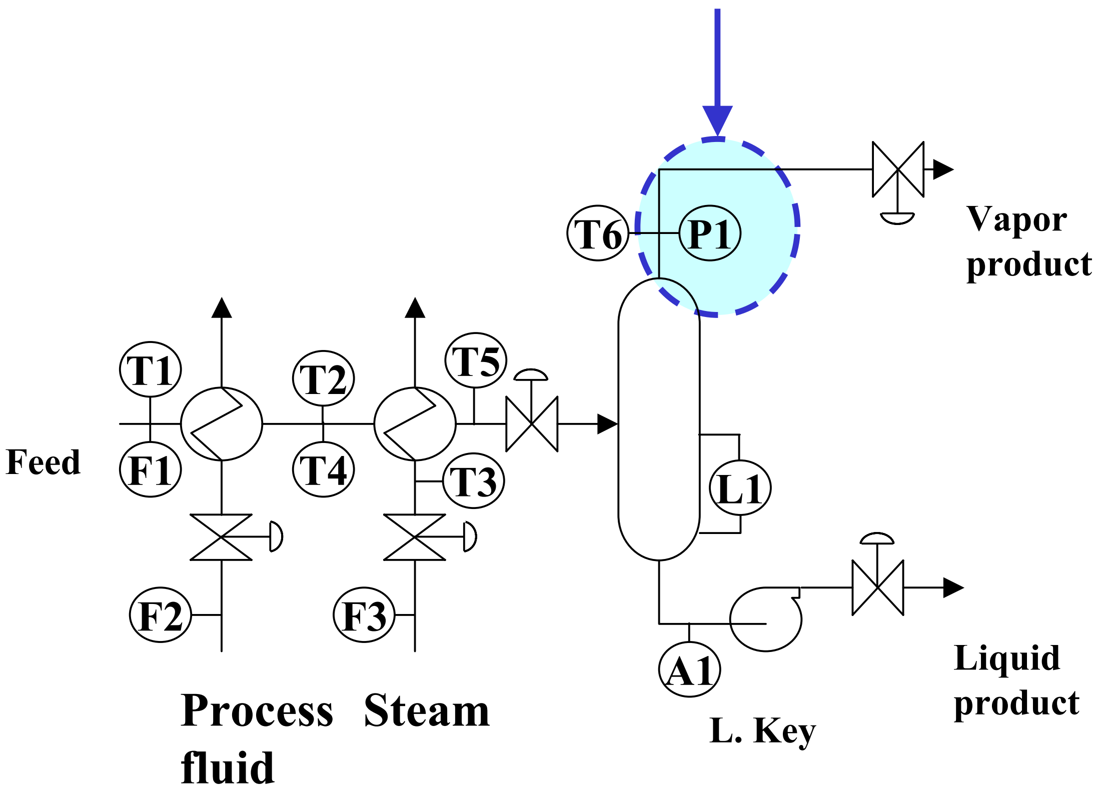
Answer the following using the MATLAB program S_LOOP.

Using the transfer function derived in Example 4.9, determine the frequency response for $C_{A0} \rightarrow C_{A2}$. Check one point on the plot by hand calculation.

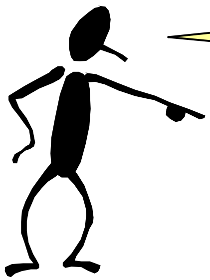


CHAPTER 4: MODELLING & ANALYSIS WORKSHOP 4

We often measure pressure for process monitoring and control. Explain three principles for pressure sensors, select one for P1 and explain your choice.

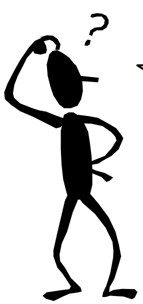


CHAPTER 4 : MODELLING & ANALYSIS FOR PROCESS CONTROL



When I complete this chapter, I want to be able to do the following.

- **Analytically solve linear dynamic models of first and second order**
- **Express dynamic models as transfer functions**
- **Predict important features of dynamic behavior from model <u>without solving</u>**



Lot's of improvement, but we need some more study!

- **Read the textbook**
- **Review the notes, especially learning goals and workshop**
- **Try out the self-study suggestions**
- **Naturally, we'll have an assignment!**

LEARNING RESOURCES

- **<u>SITE PC-EDUCATION WEB</u>**

  - Instrumentation Notes

  - Interactive Learning Module (Chapter 4)

  - Tutorials (Chapter 14)

- **Software Laboratory**

  - S_LOOP program

- **Other textbooks on Process Control (see course outline)**

SUGGESTIONS FOR SELF-STUDY

1. Why are variables expressed as deviation variables when we develop transfer functions?

2. Discuss the difference between a second order reaction and a second order dynamic model.

3. For a sine input to a process, is the output a sine for a

   a. Linear plant?
   b. Non-linear plant?

4. Is the amplitude ratio of a plant always equal to or greater than the steady-state gain?

SUGGESTIONS FOR SELF-STUDY

5. Calculate the frequency response for the model in Workshop 2 using S_LOOP. Discuss the results.

6. Decide whether a linearized model should be used for the fired heater for

   a. A 3% increase in the fuel flow rate.
   b. A 2% change in the feed flow rate.
   c. Start up from ambient temperature.
   d. Emergency stoppage of fuel flow to 0.0.