Một số VD

1. Tìm phép biến đổi Laplace ngược của các hàm sau:

a) $F(s) = \frac{s}{(s^2+4s+13)^2}$, $\quad$ *b)* $F(s) = ln\frac{s^2+1}{(s+2)(s-3)}$

2. Giải các phương trình vi phân sau

a) $tx'' + (4t-2)x' + (13t-4)x = 0$ thỏa mãn $x(0) = 0$

b) $\begin{cases} y'' + 3y' + 2y = f(t) \\ y(0) = 0 \\ y'(0) = -2 \end{cases}$ trong đó $f(t) = \begin{cases} 2 & khi\ 0 \le t < 6 \\ t & khi\ \ 6 \le t < 10 \\ 4 & khi\ \ t \ge 10 \end{cases}$

1.a Tìm phép biến đổi Laplace ngược

$$F(s) = \frac{s}{(s^2+4s+13)^2}$$

Ta có $F(s) = \frac{s}{(s^2+4s+13)^2} = \frac{s+2}{[(s+2)^2+9]^2} - \frac{2}{[(s+2)^2+9]^2}$

$$L^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \frac{1}{3}L^{-1}\left\{\frac{s}{s^2+9}.\frac{3}{s^2+9}\right\} = \frac{1}{3}(cos3t * sin3t)$$

$$= \frac{1}{3}\int_0^t \cos(3t-3\tau)\, sin(3\tau)d\tau = \frac{1}{6}\int_0^t [\sin(3t) - \sin(3t-6\tau)]d\tau$$

$$= \frac{1}{6}\left[\tau\sin(3t) - \frac{1}{6}\cos(3t-6\tau)\right]\Big|_0^t = \frac{1}{6}tsin3t$$

Vậy

$$L^{-1}\left\{\frac{s+2}{[(s+2)^2+9]^2}\right\} = \frac{1}{6}e^{-2t}tsin3t$$

Câu 1.a (tiếp)

Tương tự $L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \frac{1}{9}L^{-1}\left\{\frac{3}{s^2+9}.\frac{3}{s^2+9}\right\} = \frac{1}{9}(sin3t * sin3t)$

$$= \frac{1}{9}\int_0^t \sin(3t-3\tau)\, sin(3\tau)d\tau = -\frac{1}{18}\int_0^t [cos(3t-6\tau) - \mathrm{cos}3t]d\tau$$

$$= -\frac{1}{18}\left[-\frac{1}{6}\sin(3t-6\tau) - \tau\mathrm{cos3t}\right]\Big|_0^t = \frac{1}{18}\left[tcos3t - \frac{1}{3}sin3t\right]$$

Vậy $L^{-1}\left\{\frac{2}{[(s+2)^2+9]^2}\right\} = \frac{1}{9}e^{-2t}\left[tcos3t - \frac{1}{3}sin3t\right]$

Đo đó

$$L^{-1}\left\{\frac{s}{(s^2+4s+13)^2}\right\} = \frac{1}{6}e^{-2t}\ tsin3t - \frac{1}{9}e^{-2t}\left[tcos3t - \frac{1}{3}sin3t\right]$$

Câu 1.b Tìm phép biến đổi Laplace ngược của

$$F(s) = ln\frac{s^2+1}{(s+2)(s-3)}$$

Ta có $F(s) = ln\frac{s^2+1}{(s+2)(s-3)} = ln(s^2+1) - ln(s^2-s-6)$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2s-1}{(s+2)(s-3)} = \frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}$$

$$L^{-1}\left\{ln\frac{s^2+1}{(s+2)(s-3)}\right\} = -\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s+2} - \frac{1}{s-3}\right\}$$

$$= -\frac{1}{t}[2cost - e^{-2t} + e^{3t}]$$

$$2.a \qquad tx'' + (4t-2)x' + (13t-4)x = 0$$

thỏa mãn $x(0) = 0$

Đặt $L\{x(t)\} = X$.

Ta có $L\{x''\} = s^2X - sx(0) - x'(0) = s^2X - x'(0)$

$$L\{tx''\} = -\frac{d}{ds}(L\{x''\}) = -(2sX + s^2X')$$

$$L\{x'\} = sX - x(0) = sX \rightarrow L\{tx'\} = -\frac{d}{ds}(L\{x'\}) = -X - sX'$$

$$L\{tx\} = -\frac{d}{ds}(L\{x\}) = -X'$$

Vậy $L\{tx'' + (4t-2)x' + (13t-4)x\}$

$$= L\{tx''\} + 4L\{tx'\} - 2L\{x'\} + 13L\{tx\} - 4L\{x\} = 0$$

$$-s^2X' - 2sX + 4(-X - sX') - 2[sX - x(0)] - 13X' - 4X = 0$$

$$(s^2 + 4s + 13)X' + (4s + 8)X = 0$$

$$X' = -\frac{4s+8}{s^2+4s+13}X \rightarrow \frac{dX}{X} = -2\frac{2s+4}{s^2+4s+13}ds$$

$$\rightarrow ln|X| = -2ln|s^2 + 4s + 13| + ln\,|C| \rightarrow X = \frac{C}{(s^2+4s+13)^2}$$

Bài 2.a (tiếp)

$$X = \frac{C}{(s^2+4s+13)^2} = \frac{C}{[(s+2)^2+9]^2}$$

Ta có $L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+9}\right\} * L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{9}(sin3t * sin3t)$

$$= \frac{1}{9}\int_0^t \sin 3(t-\tau)\, sin3\tau d\tau = \frac{1}{18}\int_0^t [cos3t - \cos(3t-6\tau)]d\tau$$

$$= \frac{1}{18}\left[\tau cos3t + \frac{1}{6}\sin(3t-6\tau)\right]\Bigg|_0^t = \frac{1}{18}\left[tcos3t - \frac{1}{3}sin3t\right]$$

Vậy

$x(t) = L^{-1}\left\{\frac{C}{[(s+2)^2+9]^2}\right\} = \frac{C}{18}e^{-2t}\left[tcos3t - \frac{1}{3}sin3t\right]$

Bài 2.b

$$f(t) = \begin{cases} 2 \quad\quad khi\ 0 \le t < 6 \\ t \quad khi \quad 6 \le t < 10 \\ 4 \quad\quad\quad khi \quad t \ge 10 \end{cases}$$

$$f(t) = 2[u(t) - u(t-6)] + t[u(t-6) - u(t-10)] + 4[u(t-10)]$$
$$= 2u(t) + (t-2)u(t-6) - (t-4)u(t-10)$$
$$L\{f(t)\} = 2L\{u(t)\} + L\{(t-2)u(t-6)\} - L\{(t-4)u(t-10)\}$$

Ta có $L\{(t-2)u(t-6)\} = L\{(t-6)u(t-6)\} + 4L\{u(t-6)\}$

Tương tự $L\{(t-4)u(t-10)\} = L\{(t-10)u(t-10)\} + 6L\{u(t-10)\}$

Vậy

$$L\{f(t)\} = 2L\{u(t)\} + 4L\{u(t-6)\} - 6L\{u(t-10)\}$$
$$+ L\{(t-6)u(t-6)\} - L\{(t-10)u(t-10)\}$$

Vậy

$$F(s) = L\{f(t)\} = \frac{2 + 4e^{-6s} - 6e^{10s}}{s} + \frac{e^{-6s} - e^{10s}}{s^2}$$

Bài 2b. (tiếp)

$$\begin{cases} y'' + 3y' + 2y = f(t) \\ y(0) = 0 \\ y'(0) = -2 \end{cases} \rightarrow s^2Y + 2 + 3sY + 2Y = F(s)$$

$$(s^2 + 3s + 2)Y + 2 = F(s) \rightarrow Y = \frac{F(s) - 2}{(s+1)(s+2)}$$

$$Y = -\frac{2}{(s+1)(s+2)} + \frac{2 + 4e^{-6s} - 6e^{10s}}{s(s+1)(s+2)} + \frac{e^{-6s} - e^{10s}}{s^2(s+1)(s+2)}$$

Đặt $G(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \rightarrow L^{-1}\{G(s)\} = e^{-t} - e^{-2t} = g(t)$

$$H(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2}.\frac{1}{s+2}$$

$$\rightarrow L^{-1}\{H(s)\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = h(t)$$

$$K(s) = \frac{1}{s^2(s+1)(s+2)} = -\frac{3}{4s} + \frac{1}{2s^2} + \frac{1}{s+1} - \frac{1}{4}.\frac{1}{s+2}$$

$$\rightarrow L^{-1}\{K(s)\} = -\frac{3}{4} + \frac{1}{2}t + e^{-t} - \frac{1}{4}e^{-2t} = k(t)$$

Bài 2b. (tiếp)

Vậy $y(t) = -2g(t) + 2h(t) + 4h(t-6)u(t-6)$
$-6h(t-10)u(t-10) + k(t-6)u(t-6) - k(t-10)u(t-10)$
$= -2g(t) + 2h(t) + [4h(t-6) + k(t-6)]u(t-6)$
$-[6h(t-10) + k9t - 10)]u(t-10).$