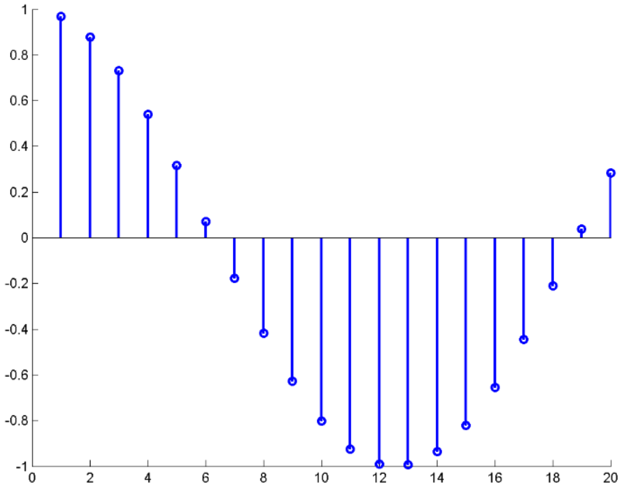
EE 2000 SIGNALS AND SYSTEMS

Ch. 6 Discrete-Time System

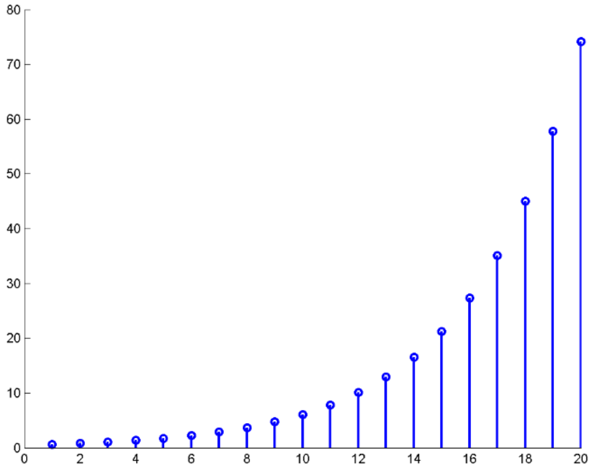
(These slides are taken from Dr. Jingxian Wu, University of Arkansas, 2020.)

SIGNAL

- **Discrete-time signal**
  - The time takes discrete values



$$x(n) = \cos\left(\frac{n}{4}\right)$$



$$x(n) = \frac{1}{2}\exp\left(\frac{n}{4}\right)$$

SIGNAL: CLASSIFICATION

- **Energy signal v.s. Power signal**
  - Energy:

$$E = \lim_{N\to\infty} \sum_{n=-N}^{N} |x(n)|^2$$

  - Power:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

  - Energy signal: $E < \infty$
  - Power signal: $P < \infty$

SIGNAL: CLASSIFICATION

- **Periodic signal v.s. aperiodic signal**
  - Periodic signal $x(n) = x(n+N)$
    - The smallest value of N that satisfies this relation is the fundamental periods.
  - Is $\cos(\omega n)$ periodic?

$\cos(\omega n)$ is periodic if $\dfrac{2k\pi}{\omega}$ is integer for integer k.

- Example: $\cos(3n)$

  $\cos(\pi n)$

  $\cos(\frac{3}{4}\pi n)$

SIGNAL: ELEMENTARY SIGNAL

- **Unit impulse function**

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

- **Unit step function**

$$u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

- **Relation between unit impulse function and unit step function**

$$\delta(n) = u(n) - u(n-1)$$

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

SIGNAL: ELEMENTARY SIGNAL

- **Exponential function**

$$x(n) = \exp(\alpha n)$$

- **Complex exponential function**

$$x(n) = \exp(j\omega_0 n) = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

OUTLINE

- **Discrete-time signals**
- **Discrete-time systems**
- **Z-transform**

SYSTEM: IMPULSE RESPONSE

- **Impulse response of LTI system**
  - The response of the system when the input is $\delta(n)$

$$x(n)=\delta(n) \longrightarrow \boxed{\text{System}} \longrightarrow y(n)=h(n)$$

LTI system

- **System response for arbitrary input**
  - Any signal can be decomposed as the sum of time-shifted impulses

$$x(n)=\sum_{k=-\infty}^{+\infty} x(k)\delta(n-k)$$

  - Time invariant

$$\delta(n-k) \longrightarrow \boxed{\text{System}} \longrightarrow h(n-k)$$

LTI system

  - Linear

$$\sum_{k=-\infty}^{+\infty} x(k)\delta(n-k) \longrightarrow \boxed{\text{System}} \longrightarrow \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

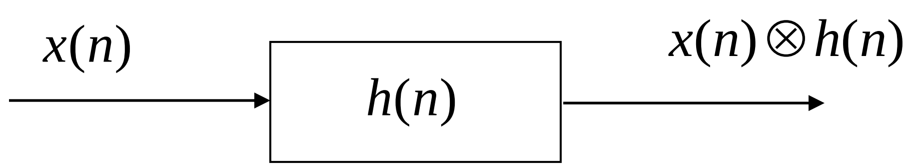
LTI system

SYSTEM: CONVOLUTION SUM

- **Convolution sum**
  - The convolution sum of two signals $x(n)$ and $h(n)$ is

$$x(n) \otimes h(n) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

- **Response of LTI system**
  - The output of a LTI system is the convolution sum of the input and the impulse response of the system.



LTI system

SYSTEM: CONVOLUTION SUM

- **Example**
  - 1. $x(n)\otimes\delta(n-m)$
  - 2. $x(n)=\alpha^{n}u(n),\qquad h(n)=\beta^{n}u(n)$

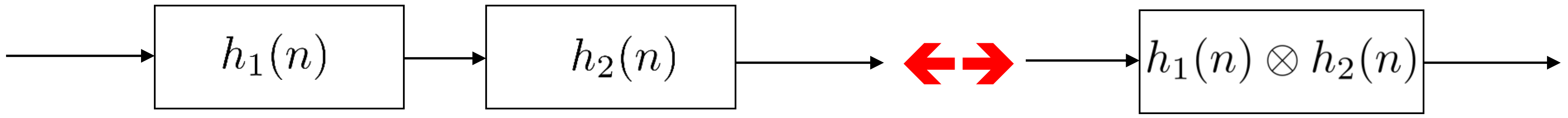
    $x(n)\otimes h(n)=$

SYSTEM: CONVOLUTION SUM

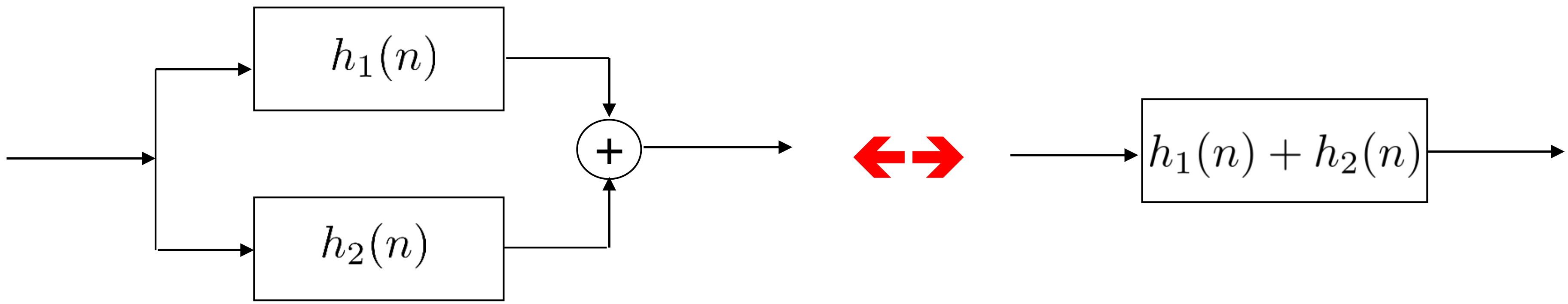
- **Example:**
  - Let $x(n) = [\underset{\uparrow}{1}, 3, -1, -2]$ $\quad h(n) = [1, \underset{\uparrow}{2}, 0, -1, 1]$, be two sequences, find $x(n) \otimes h(n)$

STSTEM: COMBINATION OF SYSTEMS

- **Combination of systems**



Two systems in series



Two systems in parallel

SYSTEM: DIFFERENCE EQUATION REPRESENTATION

- **Difference equation representation of system**

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

OUTLINE

- **Discrete-time signals**
- **Discrete-time systems**
- **Z-transform**

Z-TRANSFORM

- **Bilateral Z-transform**

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

- **Unilateral Z-transform**

$$X(z) = \sum_{n=0}^{+\infty} x(n) z^{-n}$$

- **Z-transform:**
  - Ease of analysis
  - Doesn’t have any physical meaning (the frequency domain representation of discrete-time signal can be obtained through discrete-time Fourier transform)
  - Counterpart for continuous-time systems: Laplace transform.

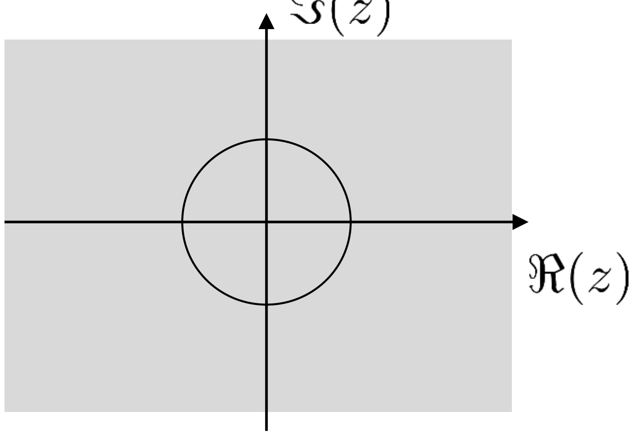
Z-TRANSFORM

- **Example: find Z-transforms**
  - 1. $x(n) = \delta(n)$
  - 2. $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Z-TRANSFORM

- **Example**
  - 3. $$x(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$$

- **Region of convergence (ROC)**



Region of convergence

Z-TRANSFORM: CONVERGENCE

- **Convergence of causal signal**

$$x(n) = \alpha^n u(n)$$

- **Convergence of anti-causal signal**

$$x(n) = \beta^n u(-n-1)$$

Z-TRANSFORM: TIME SHIFTING PROPERTY

- **Time Shifting**
  - Let $x(n)$ be a causal sequence with the Z-transform $X(z)$
  - Then

$$Z\left[x(n+n_0)\right] = z^{n_0} X(z) - z^{n_0} \sum_{m=0}^{n_0-1} x(m) z^{-m}$$

$$Z\left[x(n-n_0)\right] = z^{-n_0} X(z) + z^{-n_0} \sum_{m=-n_0}^{-1} x(m) z^{-m}$$

Z-TRANSFORM: LTI SYSTEM

- **LTI System**
  - Difference equation representation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

  - Z-domain representation

$$\left[\sum_{k=0}^{N} a_k z^{-k}\right] Y(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] X(z)$$

  - Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left[\sum_{k=0}^{M} b_k z^{-k}\right]}{\left[\sum_{k=0}^{N} a_k z^{-k}\right]}$$

Z-TRANSFORM: LTI SYSTEM

- **Example**
  - Find the transfer function of the system described by the following difference equation

$$y(n) - 2y(n-1) + 2y(n-2) = x(n) + \frac{1}{2}x(n-1)$$

Z-TRANSFORM: STABILITY

- **Stability**

$$H(z) = \frac{z}{z-a} \qquad\qquad h(n) = a^n u(n)$$

  - A LTI system is BIBO stable is all the poles are within the unit circle (|a| < 1)
  - A LTI system is unstable is at least one pole is on or outside of the unit circult ($|a| \geq 1$)