EE 2000 SIGNALS AND SYSTEMS

# Ch. 5 Laplace Transform

*(These slides are taken from Dr. Jingxian Wu, University of Arkansas, 2020.)*

# OUTLINE

- **Introduction**
- **Laplace Transform**
- **Properties of Laplace Transform**
- **Inverse Laplace Transform**
- **Applications of Laplace Transform**

# INTRODUCTION

- **Why Laplace transform?**
  - Frequency domain analysis with Fourier transform is extremely useful for the studies of signals and LTI system.
    - Convolution in time domain ➔ Multiplication in frequency domain.
  - Problem: many signals do not have Fourier transform

$$x(t) = \exp(at)u(t), a > 0 \qquad\qquad x(t) = tu(t)$$

  - Laplace transform can solve this problem
    - It exists for most common signals.
    - Follow similar property to Fourier transform
    - It doesn't have any physical meaning; just a mathematical tool to facilitate analysis.
      - Fourier transform gives us the frequency domain representation of signal.

# OUTLINE

- Introduction
- **Laplace Transform**
- Properties of Laplace Transform
- Inverse Lapalace Transform
- Applications of Fourier Transform

# LAPLACE TRANSFORM: BILATERAL LAPLACE TRANSFORM

- **Bilateral Laplace transform (two-sided Laplace transform)**

$$X_B(s) = \int_{-\infty}^{+\infty} x(t)\exp(-st)dt, \qquad s = \sigma + j\omega$$

  - $s = \sigma + j\omega$ is a complex variable
  - $s$ is often called the complex frequency
  - Notations: $X_B(s) = L[x(t)]$

$$x(t) \leftrightarrow X_B(s)$$

- **Time domain v.s. S-domain**
  - $x(t)$ : a function of time t → $x(t)$ is called the time domain signal
  - $X_B(s)$ : a function of s → $X_B(s)$ is called the s-domain signal
    - S-domain is also called as the complex frequency domain

# LAPLACE TRANSFORM

- **Time domain v.s. s-domain**
  - $x(t)$: a function of time t → $x(t)$ is called the time domain signal
  - $X_B(s)$: a function of s → $X_B(s)$ is called the s-domain signal
    - S-domain is also called the complex frequency domain
  - By converting the time domain signal into the *s*-domain, we can usually greatly simplify the analysis of the LTI system.
  - S-domain system analysis:
    - 1. Convert the time domain signals to the s-domain with the Laplace transform
    - 2. Perform system analysis in the s-domain
    - 3. Convert the s-domain results back to the time-domain

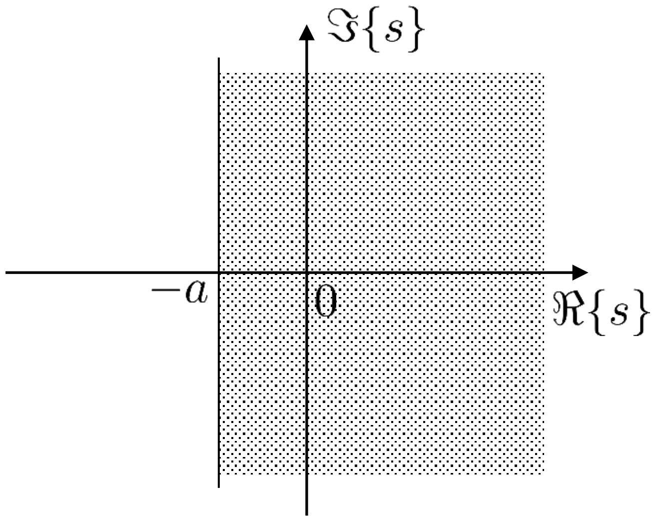
# LAPLACE TRANSFORM: BILATERAL LAPLACE TRANSFORM

- **Example**
  - Find the Bilateral Laplace transform of $x(t) = \exp(-at)u(t)$

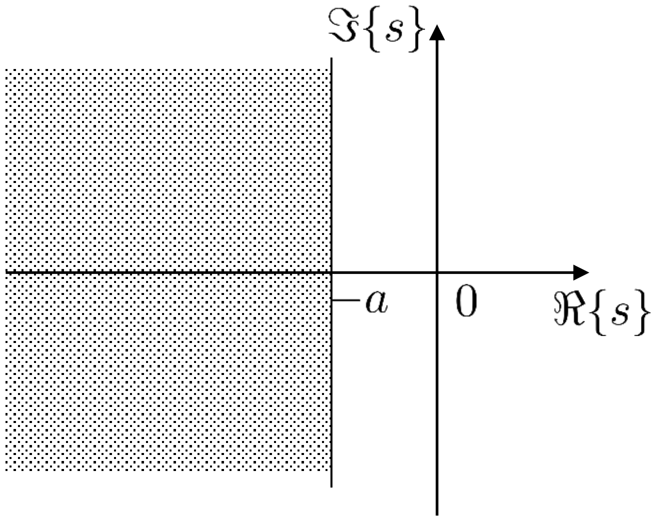
- **Region of Convergence (ROC)**
  - The range of *s* that the Laplace transform of a signal converges.
  - The Laplace transform always contains two components
    - The mathematical expression of Laplace transform
    - ROC.

# LAPLACE TRANSFORM: BILATERAL LAPLACE TRANSFORM

- **Example**
  - Find the Laplace transform of $x(t) = -\exp(-at)u(-t)$




$X_B(s) = \frac{1}{s+a}, \ \Re(s) > -a$




$X_B(s) = \frac{1}{s+a}, \ \Re(s) < -a$

# LAPLACE TRANSFORM: BILATERAL LAPLACE TRANSFORM

- **Example**
  - Find the Laplace transform of $x(t) = 3\exp(-2t)u(t) + 4\exp(t)u(-t)$

# LAPLACE TRANSFORM: UNILATERAL LAPLACE TRANSFORM

- **Unilateral Laplace transform (one-sided Laplace transform)**

$$X(s) = \int_{0^-}^{+\infty} x(t)\exp(-st)dt$$

  - $0^-$ :The value of $x(t)$ at $t = 0$ is considered.
  - Useful when we dealing with causal signals or causal systems.
    - Causal signal: $x(t) = 0$, $t < 0$.
    - Causal system: $h(t) = 0$, $t < 0$.
  - We are going to simply call unilateral Laplace transform as Laplace transform.

# LAPLACE TRANSFORM: UNILATERAL LAPLACE TRANSFORM

- **Example: find the unilateral Laplace transform of the following signals.**
  - 1. $x(t) = A$
  - 2. $x(t) = \delta(t)$

# LAPLACE TRANSFORM: UNILATERAL LAPLACE TRANSFORM

- **Example**
  - 3. $x(t) = \exp(j2t)$
  - 4. $x(t) = \cos(2t)$
  - 5. $x(t) = \sin(2t)$

# LAPLACE TRANSFORM: UNILATERAL LAPLACE TRANSFORM

| Signal | Transform | ROC |
| --- | --- | --- |
| $\delta(t-t_0)$ | $\exp(-st_0)$ | for all $s$ |
| $u(t)$ | $\frac{1}{s}$ | $\Re(s) > 0$ |
| $u(t) - u(t-t_0)$ | $\frac{1}{s}\left[1-\exp(-st_0)\right]$ | $\Re(s) > 0$ |
| $t^n u(t)$ | $\frac{n!}{s^{n+1}},\ n = 1, 2, \cdots$ | $\Re(s) > 0$ |
| $\exp(-at)u(t)$ | $\frac{1}{s+a}$ | $\Re(s) > -a$ |
| $t^n \exp(-at)u(t)$ | $\frac{n!}{(s+a)^{n+1}}$ | $\Re(s) > -a$ |
| $\cos(\omega_0 t)u(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\Re(s) > 0$ |
| $\sin(\omega_0 t)u(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\Re(s) > 0$ |
| $\cos^2(\omega_0 t)u(t)$ | $\frac{s^2+2\omega_0^2}{s(s^2+4\omega_0^2)}$ | $\Re(s) > 0$ |
| $\sin^2(\omega_0 t)u(t)$ | $\frac{2\omega_0^2}{s(s^2+4\omega_0^2)}$ | $\Re(s) > 0$ |
| $\exp(-at)\cos(\omega_0 t)u(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\Re(s) > -a$ |
| $\exp(-at)\sin(\omega_0 t)u(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\Re(s) > -a$ |
| $t\cos(\omega_0 t)u(t)$ | $\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$ | $\Re(s) > 0$ |
| $t\sin(\omega_0 t)u(t)$ | $\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$ | $\Re(s) > 0$ |

# OUTLINE

- Introduction
- Laplace Transform
- **Properties of Laplace Transform**
- Inverse Lapalace Transform
- Applications of Fourier Transform

# PROPERTIES: LINEARITY

- **Linearity**
  - If $x_1(t) \leftrightarrow X_1(s)$ $\quad x_2(t) \leftrightarrow X_2(s)$
  - Then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$

    The ROC is the intersection between the two original signals

- **Example**
  - Find the Laplace transfrom of $\left[A + B\exp(-bt)\right]u(t)$

# PROPERTIES: TIME SHIFTING

- **Time shifting**
  - If $x(t) \leftrightarrow X(s)$ and $t_0 > 0$
  - Then $x(t-t_0)u(t-t_0) \leftrightarrow X(s)\exp(-st_0)$

    The ROC remain unchanged

# PROPERTIES: SHIFTING IN THE *s* DOMAIN

- **Shifting in the s domain**
  - If $x(t) \leftrightarrow X(s)$ $\quad \mathrm{Re}(s) > \sigma$
  - Then $y(t) = x(t)\exp(s_0 t) \leftrightarrow X(s - s_0)$ $\quad \mathrm{Re}(s) > \sigma + \mathrm{Re}(s_0)$

- **Example**
  - Find the Laplace transform of $x(t) = A\exp(-at)\cos(\omega_0 t)u(t)$

# PROPERTIES: TIME SCALING

- **Time scaling**
  - If $x(t) \leftrightarrow X(s)$ $\quad \mathrm{Re}\{s\} > \sigma_1$
  - Then $x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$ $\quad \mathrm{Re}\{s\} > a\sigma_1$

- **Example**
  - Find the Laplace transform of $x(t) = u(at)$

# PROPERTIES: DIFFERENTIATION IN TIME DOMAIN

- **Differentiation in time domain**
  - If $g(t) \leftrightarrow G(s)$
  - Then $\dfrac{dg(t)}{dt} \leftrightarrow sG(s) - g(0^-)$

$$\frac{d^2 g(t)}{dt^2} \leftrightarrow s^2 G(s) - sg(0^-) - g'(0^-)$$

$$\frac{d^n g(t)}{dt^n} \leftrightarrow s^n G(s) - s^{n-1} g(0^-) - \cdots - sg^{(n-2)}(0^-) - g^{(n-1)}(0^-)$$

- **Example**
  - Find the Laplace transform of $g(t) = \sin^2 \omega t \cdot u(t)$, $g(0^-) = 0$

# PROPERTIES: DIFFERENTIATION IN TIME DOMAIN

- **Example**
  - Use Laplace transform to solve the differential equation

$$y''(t)+3y'(t)+2y(t)=0, \qquad y(0^-)=3 \qquad y'(0^-)=1$$

# PROPERTIES: DIFFERENTIATION IN S DOMAIN

- **Differentiation in s domain**
  - If $x(t) \leftrightarrow X(s)$
  - Then $$(-t)^n x(t) \leftrightarrow \frac{d^n X(s)}{ds^n}$$

- **Example**
  - Find the Laplace transform of $t^n u(t)$

# PROPERTIES: CONVOLUTION

- **Convolution**
  - If $x(t) \leftrightarrow X(s)$ $h(t) \leftrightarrow H(s)$
  - Then $x(t) \otimes h(t) \leftrightarrow X(s)H(s)$

    The ROC of $X(s)H(s)$ is the intersection of the ROCs of $X(s)$ and $H(s)$

# PROPERTIES: INTEGRATION IN TIME DOMAIN

- **Integration in time domain**
  - If $x(t) \leftrightarrow X(s)$
  - Then $\int_0^t x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s)$

- **Example**
  - Find the Laplace transform of $r(t) = tu(t)$

# PROPERTIES: CONVOLUTION

- **Example**
  - Find the convolution $$rect\left(\frac{t-a}{2a}\right)\otimes rect\left(\frac{t-a}{2a}\right)$$

# PROPERTIES: CONVOLUTION

- **Example**
  - For a LTI system, the input is $x(t) = \exp(-2t)u(t)$, and the output of the system is

$$y(t) = \left[\exp(-t) + \exp(-2t) - \exp(-3t)\right]u(t)$$

Find the impulse response of the system

# PROPERTIES: CONVOLUTION

- **Example**
  - Find the Laplace transform of the impulse response of the LTI system described by the following differential equation

$$2y''(t) - 3y'(t) + y(t) = 3x'(t) + x(t)$$

assume the system was initially relaxed ( $y^{(n)}(0) = x^{(n)}(0) = 0$ )

# PROPERTIES: MODULATION

- **Modulation**
  - If $x(t) \leftrightarrow X(s)$
  - Then

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(s+j\omega_0)+X(s-j\omega_0)\right]$$

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(s+j\omega_0)-X(s-j\omega_0)\right]$$

# PROPERTIES: MODULATION

- **Example**
  - Find the Laplace transform of $x(t) = \exp(-at)\sin(\omega_0 t)u(t)$

# PROPERTIES: INITIAL VALUE THEOREM

- **Initial value theorem**
  - If the signal $x(t)$ is infinitely differentiable on an interval around $x(0^+)$ then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

$s = \infty$ must be in ROC

  - The behavior of x(t) for small t is determined by the behavior of X(s) for large s.

# PROPERTIES: INITIAL VALUE THEOREM

- **Example**
  - The Laplace transform of x(t) is $$X(s) = \frac{cs+d}{(s-a)(s-b)}$$
    Find the value of $x(0^+)$

# PROPERTIES: FINAL VALUE THEOREM

- **Final value theorem**
  - If $x(t) \leftrightarrow X(s)$
  - Then: $\lim_{t\to\infty} x(t) \leftrightarrow \lim_{s\to 0} sX(s)$

$s = 0$ must be in ROC

- **Example**
  - The input $x(t) = Au(t)$ is applied to a system with transfer function $H(s) = \dfrac{c}{s(s+b)+c}$ , find the value of $\lim_{t\to\infty} y(t)$

# PROPERTIES

| Properties | time-domain | $s$-domain |
|---|---|---|
| Linearity | $\sum_{n=1}^{N} \alpha_n x_n(t)$ | $\sum_{n=1}^{N} \alpha_n X_n(s)$ |
| Time shift | $x(t-t_0)u(t-t_0)$ | $X(s)\exp(-st_0)$ |
| Frequency shift | $\exp(s_0 t)x(t)$ | $X(s-s_0)$ |
| Time scaling | $x(\alpha t), \alpha > 0$ | $X(s/\alpha)/\alpha$ |
| Multiplication by $t$ | $tx(t)$ | $-\frac{dX(s)}{ds}$ |
| Differentiation | $dx(t)/dt$ | $sX(s) - x(0^{-1})$ |
| Integration | $\int_{0^-}^{t} x(\tau)d\tau$ | $X(s)/s$ |
| Modulation | $x(t)\cos(\omega_0 t)$ | $\frac{1}{2}\left[X(s-j\omega_0) + X(s+j\omega_0)\right]$ |
| | $x(t)\sin(\omega_0 t)$ | $\frac{1}{2j}\left[X(s-j\omega_0) - X(s+j\omega_0)\right]$ |
| Convolution | $x(t) \otimes h(t)$ | $X(S)H(S)$ |
| Initial value | $x(0^+)$ | $\lim_{s\to\infty} sX(s)$ |
| Final value | $\lim_{t\to\infty} x(t)$ | $\lim_{s\to 0} sX(s)$ |

# OUTLINE

- Introduction
- Laplace Transform
- Properties of Laplace Transform
- **Inverse Lapalace Transform**
- Applications of Fourier Transform

# INVERSE LAPLACE TRANSFORM

- **Inverse Laplace transform**

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\exp(st)ds$$

  – Evaluation of the above integral requires the use of contour integration in the complex plan ➔ difficult.

- **Inverse Laplace transform: special case**
  – In many cases, the Laplace transform can be expressed as a rational function of s

$$X(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

  – Procedure of Inverse Laplace Transform
    - 1. Partial fraction expansion of $X(s)$
    - 2. Find the inverse Laplace transform through Laplace transform table.

# INVERSE LAPLACE TRANSFORM

- **Review: Partial Fraction Expansion with non-repeated linear factors**

$$X(s) = \frac{A}{s-a_1} + \frac{B}{s-a_2} + \frac{C}{s-a_3}$$

$$A = (s-a_1)X(s)\Big|_{s=a_1} \qquad B = (s-a_2)X(s)\Big|_{s=a_2} \qquad C = (s-a_3)X(s)\Big|_{s=a_3}$$

- **Example**
  - Find the inverse Laplace transform of $X(s) = \dfrac{2s+1}{s^3+3s^2-4s}$

# INVERSE LAPLACE TRANSFORM

- **Example**
  - Find the Inverse Laplace transform of $$X(s) = \frac{2s^2}{s^2 + 3s + 2}$$

- If the numerator polynomial has order higher than or equal to the order of denominator polynomial, we need to rearrange it such that the denominator polynomial has a higher order.

# INVERSE LAPLACE TRANSFORM

- **Partial Fraction Expansion with repeated linear factors**

$$X(s) = \frac{1}{(s-a)^2(s-b)} = \frac{A_2}{(s-a)^2} + \frac{A_1}{s-a} + \frac{B}{s-b}$$

$$A_2 = (s-a)^2 X(s)\Big|_{s=a} \qquad A_1 = \frac{d}{ds}\left[(s-a)^2 X(s)\right]\Big|_{s=a} \qquad B = (s-b)X(s)\Big|_{s=b}$$

# INVERSE LAPLACE TRANSFORM

- **High-order repeated linear factors**

$$X(s)=\frac{1}{(s-a)^N(s-b)}=\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_N}{(s-a)^N}+\frac{B}{s-b}$$

$$A_k=\frac{1}{(N-k)!}\frac{d^{N-k}}{ds^{N-k}}\left[(s-a)^N X(s)\right]\Big|_{s=a} \qquad k=1,\cdots,N$$

$$B=(s-b)X(s)\big|_{s=b}$$

# OUTLINE

- Introduction
- Laplace Transform
- Properties of Laplace Transform
- Inverse Lapalace Transform
- **Applications of Laplace Transform**

## APPLICATION: LTI SYSTEM REPRESENTATION

- **LTI system**
  - System equation: a differential equation describes the input output relationship of the system.

$$y^{(N)}(t)+a_{N-1}y^{(N-1)}(t)+\cdots+a_1y^{(1)}(t)+a_0y(t)=b_Mx^{(M)}(t)+\cdots+b_1x^{(1)}(t)+b_0x(t)$$

$$y^{(N)}(t)+\sum_{n=0}^{N-1}a_n y^{(n)}(t)=\sum_{m=0}^{M}b_m x^{(m)}(t)$$

  - S-domain representation

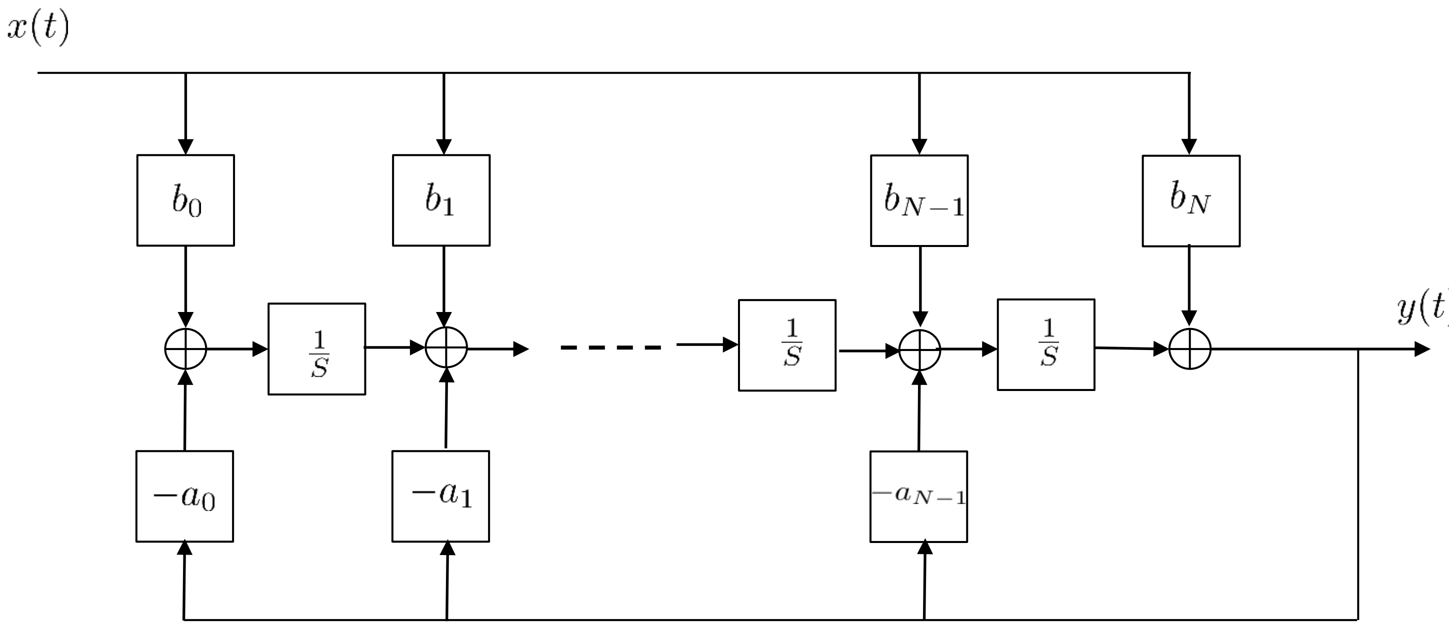
$$\left[s^N+\sum_{n=0}^{N-1}a_n s^n\right]Y(s)=\left[\sum_{m=0}^{M}b_m s^m\right]X(s)$$

  - Transfer function

$$H(s)=\frac{Y(s)}{X(s)}=\frac{\sum_{m=0}^{M}b_m s^m}{s^N+\sum_{n=0}^{N-1}a_n s^n}$$

# APPLICATION: LTI SYSTEM REPRESENTATION

- **Simulation diagram (first canonical form)**




Simulation diagram

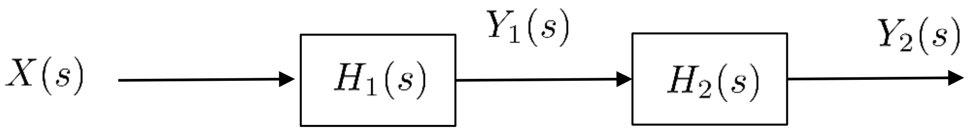
# APPLICATION: LTI SYSTEM REPRESENTATION

- **Example**
  - Show the first canonical realization of the system with transfer function

$$H(S) = \frac{s^2 - 3s + 2}{s^3 + 6s^2 + 11s + 6}$$

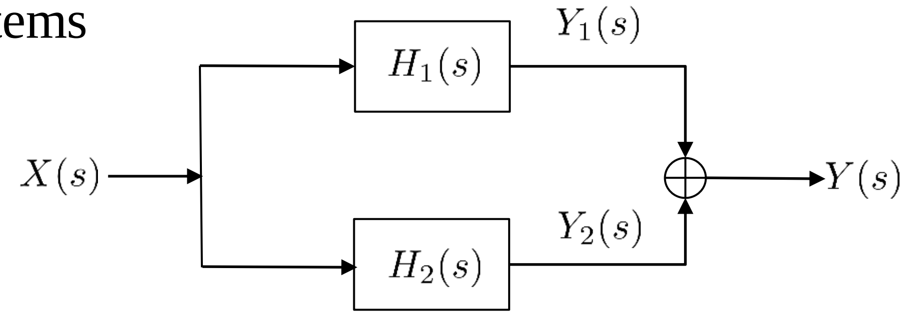
# APPLICATION: COMBINATIONS OF SYSTEMS

- **Combination of systems**
  - Cascade of systems




$$H(S) = H_1(s)H_2(s)$$

  - Parallel systems




$$H(S) = H_1(s) + H_2(s)$$

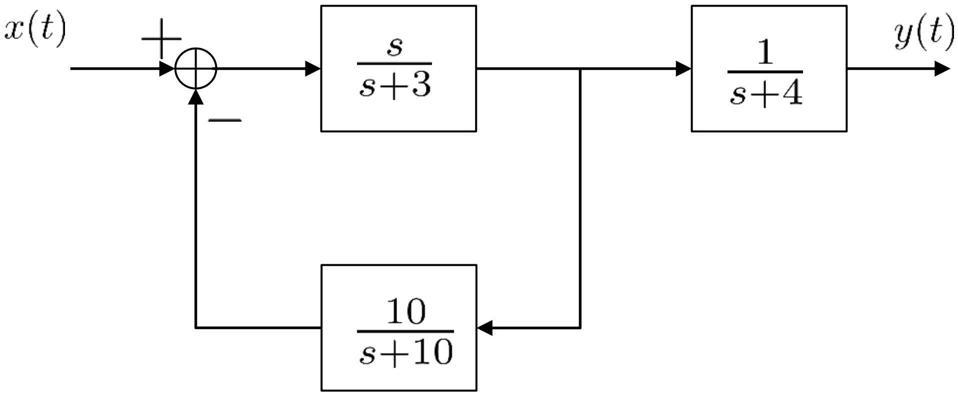
## APPLICATION: LTI SYSTEM REPRESENTATION

- **Example**
  - Represent the system to the cascade of subsystems.

$$H(S) = \frac{s^2 - 3s + 2}{s^3 + 6s^2 + 11s + 6}$$

# APPLICATION: LTI SYSTEM REPRESENTATION

- **Example:**
  - Find the transfer function of the system




LTI system

# APPLICATION: LTI SYSTEM REPRESENTATION

- **Poles and zeros**

$$H(s)=\frac{(s-z_M)(s-z_{M-1})\cdots(s-z_1)}{(s-p_N)(s-p_{N-1})\cdots(s-p_1)}$$

  - Zeros: $z_1, z_2, \cdots, z_M$
  - Poles: $p_1, p_2, \cdots, p_N$

# APPLICATION: STABILITY

- **Review: BIBO Stable**
  - Bounded input always leads to bounded output

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$$

- **The positions of poles of H(s) in the s-domain determine if a system is BIBO stable.**

$$H(s) = \frac{A_1}{s - s_1} + \frac{A_2}{(s - s_2)^m} + \cdots + \frac{A_N}{s - s_N}$$

  - Simple poles: the order of the pole is 1, e.g. $s_1$ $s_N$
  - Multiple-order poles: the poles with higher order. E.g. $s_2$

# APPLICATION: STABILITY

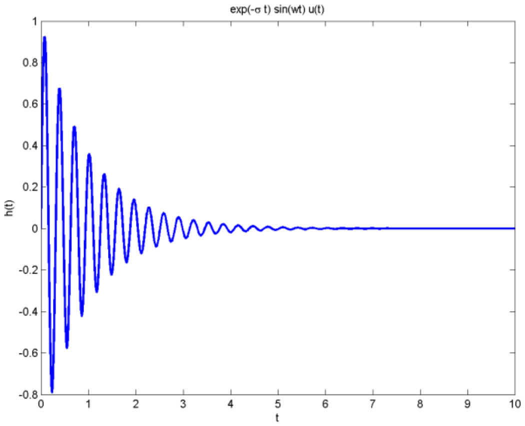
- **Case 1: simple poles in the left half plane**

$$\frac{1}{(s-\sigma_k)^2+\omega_k^2}=\frac{1}{(s-\sigma_k+j\omega_k)(s-\sigma_k-j\omega_k)} \qquad \sigma_k<0$$

$$p_1=\sigma_k-j\omega_k \qquad\qquad p_2=\sigma_k+j\omega_k$$

$$h_k(t)=\frac{1}{\omega_k}\exp(\sigma_k t)\sin(\omega_k t)u(t)$$

$$\int_{-\infty}^{+\infty}\left|h_k(t)\right|dt=$$




Impulse response

- **If all the poles of the system are on the left half plane, then the system is stable.**

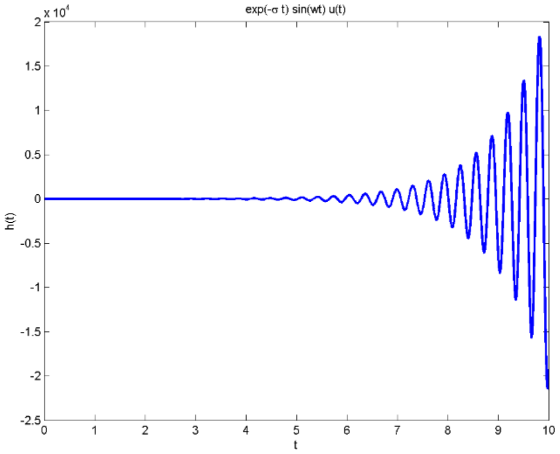
# APPLICATION: STABILITY

- **Case 2: Simple poles on the right half plane**

$$\frac{1}{(s-\sigma_k)^2+\omega_k^2}=\frac{1}{(s-\sigma_k+j\omega_k)(s-\sigma_k-j\omega_k)} \qquad \sigma_k>0$$

$$p_1=\sigma_k+j\omega_k \qquad p_2=\sigma_k-j\omega_k$$

$$h_k(t)=\frac{1}{\omega_k}\exp(\sigma_k t)\sin(\omega_k t)u(t)$$




Impulse response

- **If at least one pole of the system is on the right half plane, then the system is unstable.**

# APPLICATION: STABILITY

- **Case 3: Simple poles on the imaginary axis**

$$\frac{1}{(s-\sigma_k)^2+\omega_k^2}=\frac{1}{(s-\sigma_k+j\omega_k)(s-\sigma_k-j\omega_k)} \qquad \sigma_k=0$$

$$h_k(t)=\frac{1}{\omega_k}\sin(\omega_k t)u(t)$$

- **If the pole of the system is on the imaginary axis, it's unstable.**

# APPLICATION: STABILITY

- **Case 4: multiple-order poles in the left half plane**

$$h_k(t) = \frac{1}{\omega_k} t^m \exp(\sigma_k t)\sin(\omega_k t)u(t) \qquad \sigma_k < 0$$

**stable**

- **Case 5: multiple-order poles in the right half plane**

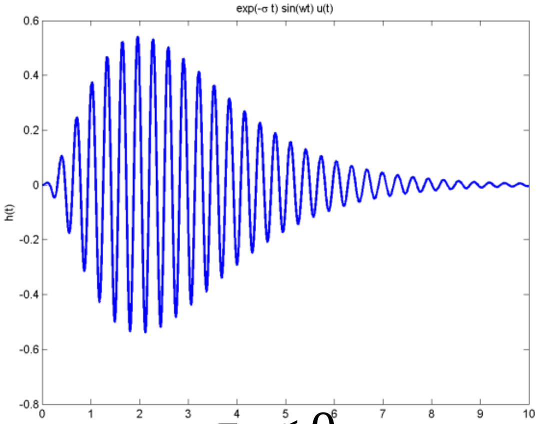
$$h_k(t) = \frac{1}{\omega_k} t^m \exp(\sigma_k t)\sin(\omega_k t)u(t) \qquad \sigma_k > 0$$

**unstable**

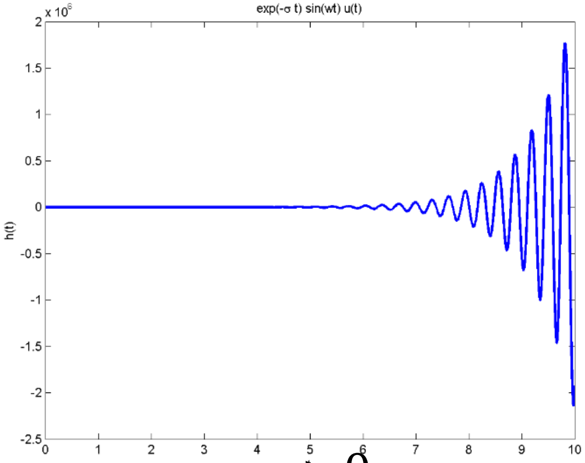
- **Case 6: multiple-order poles on the imaginary axis**

$$h_k(t) = \frac{1}{\omega_k} t^m \sin(\omega_k t)u(t)$$

**unstable**




$\sigma_k < 0$




$\sigma_k > 0$

# APPLICATION: STABILITY

- **Example:**
  - Check the stability of the following system.

$$H(s) = \frac{3s+2}{s^2+6s+13}$$