CHAPTER 8

8.1a. Find $\mathbf{H}$ in cartesian components at $P(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2-8z+29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2-8z+29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2-8z+29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y \ \mu\text{A/m}}$

b. Repeat if the filament is located at $x=-1$, $y=2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2-8z+26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2-8z+26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2-8z+26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y \ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts a and b, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y \ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

8.2. A current filament of $3\mathbf{a}_x$ A lies along the x axis. Find $\mathbf{H}$ in cartesian components at $P(-1,3,2)$: We use the Biot-Savart law,

$$\mathbf{H} = \int \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2}$$

where $Id\mathbf{L} = 3dx\mathbf{a}_x$, $\mathbf{a}_R = [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]/R$, and $R = \sqrt{x^2+2x+14}$. Thus

$$\mathbf{H}_P = \int_{-\infty}^{\infty} \frac{3dx\mathbf{a}_x \times [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]}{4\pi(x^2+2x+14)^{3/2}} = \int_{-\infty}^{\infty} \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)\,dx}{4\pi(x^2+2x+14)^{3/2}}$$
$$= \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)(x+1)}{4\pi(13)\sqrt{x^2+2x+14}}\Bigg|_{-\infty}^{\infty} = \frac{2(9\mathbf{a}_z - 6\mathbf{a}_y)}{4\pi(13)} = \underline{0.110\mathbf{a}_z - 0.073\mathbf{a}_y \ \text{A/m}}$$

8.3. Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ (note typographical error in problem statement) and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction.

a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$: One way to do this is to use the field from an infinite line and subtract from it that portion of the field that would arise from the current segment at $-a < z < a$, found from the Biot-Savart law. Thus,

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi - \int_{-a}^{a} \frac{I\,dz\,\mathbf{a}_z \times [\rho\,\mathbf{a}_\rho - z\,\mathbf{a}_z]}{4\pi[\rho^2 + z^2]^{3/2}}$$

The integral part simplifies and is evaluated:

$$\int_{-a}^{a} \frac{I\,dz\,\rho\,\mathbf{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} = \frac{I\rho}{4\pi}\,\mathbf{a}_\phi\,\frac{z}{\rho^2\sqrt{\rho^2+z^2}}\Bigg|_{-a}^{a} = \frac{Ia}{2\pi\rho\sqrt{\rho^2+a^2}}\,\mathbf{a}_\phi$$

Finally,

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[1 - \frac{a}{\sqrt{\rho^2+a^2}}\right]\mathbf{a}_\phi \ \text{A/m}$$

b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament? We require

$$\left[1 - \frac{a}{\sqrt{\rho^2+a^2}}\right]_{\rho=1} = \frac{1}{2} \;\Rightarrow\; \frac{a}{\sqrt{1+a^2}} = \frac{1}{2} \;\Rightarrow\; a = \underline{1/\sqrt{3}}$$

8.4a.) A filament is formed into a circle of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Find $\mathbf{H}$ at the origin: We use the Biot-Savart law, which in this case becomes:

$$\mathbf{H} = \int_{loop} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_0^{2\pi} \frac{Ia\,d\phi\,\mathbf{a}_\phi \times (-\mathbf{a}_\rho)}{4\pi a^2} = \underline{0.50\,\frac{I}{a}\,\mathbf{a}_z \ \text{A/m}}$$

b.) A filament of the same length is shaped into a square in the $z = 0$ plane. The sides are parallel to the coordinate axes and a current I flows in the general $\mathbf{a}_\phi$ direction. Again, find $\mathbf{H}$ at the origin: Since the loop is the same length, its perimeter is $2\pi a$, and so each of the four sides is of length $\pi a/2$. Using symmetry, we can find the magnetic field at the origin associated with each of the 8 half-sides (extending from 0 to $\pm\pi a/4$ along each coordinate direction) and multiply the result by 8: Taking one of the segments in the y direction, the Biot-Savart law becomes

$$\mathbf{H} = \int_{loop} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = 8\int_0^{\pi a/4} \frac{I dy\,\mathbf{a}_y \times \left[-(\pi a/4)\,\mathbf{a}_x - y\,\mathbf{a}_y\right]}{4\pi\left[y^2 + (\pi a/4)^2\right]^{3/2}}$$

$$= \frac{aI}{2}\int_0^{\pi a/4} \frac{dy\,\mathbf{a}_z}{\left[y^2 + (\pi a/4)^2\right]^{3/2}} = \frac{aI}{2}\,\frac{y\,\mathbf{a}_z}{(\pi a/4)^2\sqrt{y^2 + (\pi a/4)^2}}\Bigg|_0^{\pi a/4} = \underline{0.57\,\frac{I}{a}\,\mathbf{a}_z \ \text{A/m}}$$

8.5. The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$: We need an expression for $\mathbf{H}$ in cartesian coordinates. We can start with the known $\mathbf{H}$ in cylindrical for an infinite filament along the z axis: $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, which we transform to cartesian to obtain:

$$\mathbf{H} = \frac{-Iy}{2\pi(x^2+y^2)}\,\mathbf{a}_x + \frac{Ix}{2\pi(x^2+y^2)}\,\mathbf{a}_y$$

If we now rotate the filament so that it lies along the x axis, with current flowing in positive x, we obtain the field from the above expression by replacing x with y and y with z:

$$\mathbf{H} = \frac{-Iz}{2\pi(y^2+z^2)}\,\mathbf{a}_y + \frac{Iy}{2\pi(y^2+z^2)}\,\mathbf{a}_z$$

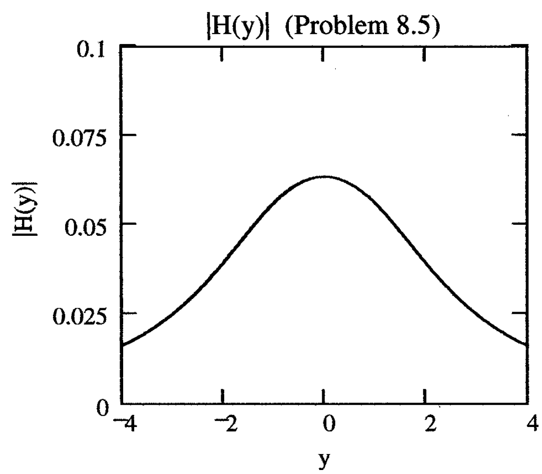
Now, with two filaments, displaced from the x axis to lie at $y = \pm 1$, and with the current directions as shown in the figure, we use the previous expression to write

$$\mathbf{H} = \left[\frac{Iz}{2\pi[(y+1)^2+z^2]} - \frac{Iz}{2\pi[(y-1)^2+z^2]}\right]\mathbf{a}_y + \left[\frac{I(y-1)}{2\pi[(y-1)^2+z^2]} - \frac{I(y+1)}{2\pi[(y+1)^2+z^2]}\right]\mathbf{a}_z$$

We now evaluate this at $z = 2$, and find the magnitude ($\sqrt{\mathbf{H}\cdot\mathbf{H}}$), resulting in

$$|\mathbf{H}| = \frac{I}{2\pi}\left[\left(\frac{2}{y^2+2y+5} - \frac{2}{y^2-2y+5}\right)^2 + \left(\frac{(y-1)}{y^2-2y+5} - \frac{(y+1)}{y^2+2y+5}\right)^2\right]^{1/2}$$

This function is plotted below



8.6a. A current filament I is formed into circle, $\rho = a$, in the $z = z'$ plane. Find H_z at $P(0, 0, z)$ if I flows in the $\mathbf{a}_\phi$ direction: Use the Biot-Savart law,

$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where in this case $Id\mathbf{L} = Id\phi\mathbf{a}_\phi$, $\mathbf{a}_R = [-a\mathbf{a}_\rho + (z-z')\mathbf{a}_z]/R$, and $R = \sqrt{a^2+(z-z')^2}$. The setup becomes

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times [-a\mathbf{a}_\rho + (z-z')\mathbf{a}_z]}{4\pi[a^2+(z-z')^2]^{3/2}} = \int_0^{2\pi} \frac{Ia[a\mathbf{a}_z + (z-z')\mathbf{a}_\rho]\,d\phi}{4\pi[a^2+(z-z')^2]^{3/2}}$$

At this point we need to be especially careful. Note that we are integrating a vector with an $\mathbf{a}_\rho$ component around a complete circle, where the vector has no ϕ dependence. This sum of all $\mathbf{a}_\rho$ components will be zero – even though this doesn't happen when we go ahead with the integration without this knowledge. The problem is that the integral "interprets" $\mathbf{a}_\rho$ as a constant direction, when in fact – as we know – $\mathbf{a}_\rho$ continually changes direction as ϕ varies. We drop the $\mathbf{a}_\rho$ component in the integral to give

$$\mathbf{H} = \int_0^{2\pi} \frac{Ia^2\mathbf{a}_z\, d\phi}{4\pi[a^2+(z-z')^2]^{3/2}} = \frac{\pi a^2 I\mathbf{a}_z}{2\pi[a^2+(z-z')^2]^{3/2}} = \underline{\frac{\mathbf{m}}{2\pi[a^2+(z-z')^2]^{3/2}}\ \text{A/m}}$$

where $\mathbf{m} = \pi a^2 I\mathbf{a}_z$ is the *magnetic moment* of the loop.

b) Find H_z at P caused by a uniform surface current density $\mathbf{K} = K_0\mathbf{a}_\phi$, flowing on the cylindrical surface, $\rho = a$, $0 < z < h$. The results of part a should help: Using part a, we can write down the differential field at P arising from a circular current ribbon of differential height, dz', at location z'. The ribbon is of radius a and carries current $K_0 dz'\mathbf{a}_\phi$ A:

$$d\mathbf{H} = \frac{\pi a^2 K_0 dz'\mathbf{a}_z}{2\pi[a^2+(z-z')^2]^{3/2}}\ \text{A/m}$$

The total magnetic field at P is now the sum of the contributions of all differential rings that comprise the cylinder:

$$\begin{aligned} H_z &= \int_0^h \frac{\pi a^2 K_0 dz'}{2\pi[a^2+(z-z')^2]^{3/2}} = \frac{a^2K_0}{2}\int_0^h \frac{dz'}{[a^2+z^2-2zz'+(z')^2]^{3/2}} \\ &= \frac{a^2K_0}{2}\frac{2(2z'-2z)}{4a^2\sqrt{a^2+z^2-2zz'+(z')^2}}\Bigg|_0^h = \frac{K_0(z'-z)}{2\sqrt{a^2+(z'-z)^2}}\Bigg|_0^h \\ &= \underline{\frac{K_0}{2}\left[\frac{(h-z)}{\sqrt{a^2+(h-z)^2}} + \frac{z}{\sqrt{a^2+z^2}}\right]}\ \text{A/m} \end{aligned}$$

8.7. Given points $C(5,-2,3)$ and $P(4,-1,2)$; a current element $Id\mathbf{L} = 10^{-4}(4,-3,1)$ A $\cdot$ m at C produces a field $d\mathbf{H}$ at P.

a) Specify the direction of $d\mathbf{H}$ by a unit vector $\mathbf{a}_H$: Using the Biot-Savart law, we find

$$d\mathbf{H} = \frac{Id\mathbf{L}\times\mathbf{a}_{CP}}{4\pi R_{CP}^2} = \frac{10^{-4}[4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z]\times[-\mathbf{a}_x+\mathbf{a}_y-\mathbf{a}_z]}{4\pi 3^{3/2}} = \frac{[2\mathbf{a}_x+3\mathbf{a}_y+\mathbf{a}_z]\times 10^{-4}}{65.3}$$

from which

$$\mathbf{a}_H = \frac{2\mathbf{a}_x+3\mathbf{a}_y+\mathbf{a}_z}{\sqrt{14}} = \underline{0.53\mathbf{a}_x + 0.80\mathbf{a}_y + 0.27\mathbf{a}_z}$$

b) Find $|d\mathbf{H}|$.

$$|d\mathbf{H}| = \frac{\sqrt{14}\times 10^{-4}}{65.3} = 5.73\times 10^{-6}\ \text{A/m} \quad = \underline{5.73\ \mu\text{A/m}}$$

c) What direction $\mathbf{a}_l$ should $Id\mathbf{L}$ have at C so that $d\mathbf{H} = 0$? $Id\mathbf{L}$ should be collinear with $\mathbf{a}_{CP}$, thus rendering the cross product in the Biot-Savart law equal to zero. Thus the answer is $\mathbf{a}_l = \underline{\pm(-\mathbf{a}_x+\mathbf{a}_y-\mathbf{a}_z)/\sqrt{3}}$

8.8. For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1: The Biot-Savart law reads:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{\rho \tan\alpha_1}^{\rho \tan\alpha_2} \frac{I dz \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{3/2}} = \int_{\rho \tan\alpha_1}^{\rho \tan\alpha_2} \frac{I\rho \mathbf{a}_\phi \, dz}{4\pi(\rho^2 + z^2)^{3/2}}$$

The integral is evaluated (using tables) and gives the desired result:

$$\mathbf{H} = \frac{I z \mathbf{a}_\phi}{4\pi\rho\sqrt{\rho^2 + z^2}}\Bigg|_{\rho \tan\alpha_1}^{\rho \tan\alpha_2} = \frac{I}{4\pi\rho}\left[\frac{\tan\alpha_2}{\sqrt{1 + \tan^2\alpha_2}} - \frac{\tan\alpha_1}{\sqrt{1 + \tan^2\alpha_1}}\right]\mathbf{a}_\phi = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

8.9. A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$: Using the Biot-Savart law, we write

$$\mathbf{H}_P = \int\int \frac{\mathbf{K} \times \mathbf{a}_R \, dx\, dy}{4\pi R^2} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\mathbf{a}_x \times (-x\mathbf{a}_x - y\mathbf{a}_y + 3\mathbf{a}_z)}{4\pi(x^2 + y^2 + 9)^{3/2}}\, dx\, dy$$

Taking the cross product gives:

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\mathbf{a}_z - 3\mathbf{a}_y)\, dx\, dy}{4\pi(x^2 + y^2 + 9)^{3/2}}$$

We note that the z component is anti-symmetric in y about the origin (odd parity). Since the limits are symmetric, the integral of the z component over y is zero. We are left with

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\mathbf{a}_y\, dx\, dy}{4\pi(x^2 + y^2 + 9)^{3/2}} = -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{x}{(y^2 + 9)\sqrt{x^2 + y^2 + 9}}\Bigg|_{-\infty}^{\infty} dy$$

$$= -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{2}{y^2 + 9}\, dy = -\frac{12}{\pi}\mathbf{a}_y\, \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\Bigg|_{-2}^{2} = -\frac{4}{\pi}(2)(0.59)\,\mathbf{a}_y = \underline{-1.50\,\mathbf{a}_y \text{ A/m}}$$

8.10. Let a filamentary current of 5 mA be directed from infinity to the origin on the positive z axis and then back out to infinity on the positive x axis. Find $\mathbf{H}$ at $P(0, 1, 0)$: The Biot-Savart law is applied to the two wire segments using the following setup:

$$\mathbf{H}_P = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_0^{\infty} \frac{-I dz \mathbf{a}_z \times (-z\mathbf{a}_z + \mathbf{a}_y)}{4\pi(z^2 + 1)^{3/2}} + \int_0^{\infty} \frac{I dx \mathbf{a}_x \times (-x\mathbf{a}_x + \mathbf{a}_y)}{4\pi(x^2 + 1)^{3/2}}$$

$$= \int_0^{\infty} \frac{I dz \mathbf{a}_x}{4\pi(z^2 + 1)^{3/2}} + \int_0^{\infty} \frac{I dx \mathbf{a}_z}{4\pi(x^2 + 1)^{3/2}} = \frac{I}{4\pi}\left[\frac{z\mathbf{a}_x}{\sqrt{z^2 + 1}}\Bigg|_0^{\infty} + \frac{x\mathbf{a}_z}{\sqrt{x^2 + 1}}\Bigg|_0^{\infty}\right]$$

$$= \frac{I}{4\pi}(\mathbf{a}_x + \mathbf{a}_z) = \underline{0.40(\mathbf{a}_x + \mathbf{a}_z) \text{ mA/m}}$$

8.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad \Rightarrow \quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

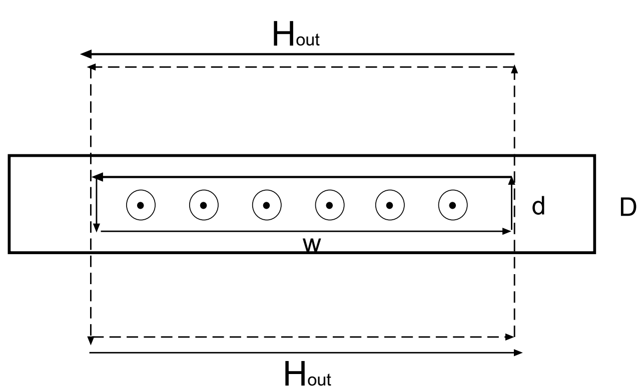
Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

8.12. In Fig. 8.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. The problem asks you to find $\mathbf{H}$ at various positions. Before continuing, we need to know how to find $\mathbf{H}$ for this type of current configuration. The sketch below shows one of the slabs (of thickness D) oriented with the current coming out of the page. The problem statement implies that both slabs are of infinite length and width. To find the magnetic field *inside* a slab, we apply Ampere's circuital law to the rectangular path of height d and width w, as shown, since by symmetry, $\mathbf{H}$ should be oriented horizontally. For example, if the sketch below shows the upper slab in Fig. 8.22, current will be in the positive y direction. Thus $\mathbf{H}$ will be in the positive x direction above the slab midpoint, and will be in the negative x direction below the midpoint.



8.12 (continued) In taking the line integral in Ampere's law, the two vertical path segments will cancel each other. Ampere's circuital law for the interior loop becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2H_{in} \times w = I_{encl} = J \times w \times d \;\; \Rightarrow \;\; H_{in} = \frac{Jd}{2}$$

The field outside the slab is found similarly, but with the enclosed current now bounded by the slab thickness, rather than the integration path height:

$$2H_{out} \times w = J \times w \times D \;\; \Rightarrow \;\; H_{out} = \frac{JD}{2}$$

where H_{out} is directed from right to left below the slab and from left to right above the slab (right hand rule). Reverse the current, and the fields, of course, reverse direction. We are now in a position to solve the problem.

Find **H** at:

a) $z = -0.2$m: Here the fields from the top and bottom slabs (carrying opposite currents) will cancel, and so $\mathbf{H} = \underline{0}$.

b) $z = 0.2$m. This point lies within the lower slab above its midpoint. Thus the field will be oriented in the negative x direction. Referring to Fig. 8.22 and to the sketch on the previous page, we find that $d = 0.1$. The total field will be this field plus the contribution from the upper slab current:

$$\mathbf{H} = \underbrace{\frac{-10(0.1)}{2}\mathbf{a}_x}_{\text{lower slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{upper slab}} = \underline{-2\mathbf{a}_x \text{ A/m}}$$

c) $z = 0.4$m: Here the fields from both slabs will add constructively in the negative x direction:

$$\mathbf{H} = -2\frac{10(0.3)}{2}\mathbf{a}_x = \underline{-3\mathbf{a}_x \text{ A/m}}$$

d) $z = 0.75$m: This is in the interior of the upper slab, whose midpoint lies at $z = 0.85$. Therefore $d = 0.2$. Since 0.75 lies below the midpoint, magnetic field from the upper slab will lie in the negative x direction. The field from the lower slab will be negative x-directed as well, leading to:

$$\mathbf{H} = \underbrace{\frac{-10(0.2)}{2}\mathbf{a}_x}_{\text{upper slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{lower slab}} = \underline{-2.5\mathbf{a}_x \text{ A/m}}$$

e) $z = 1.2$m: This point lies above both slabs, where again fields cancel completely: Thus $\mathbf{H} = \underline{0}$.

8.13. A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a \mathbf{a}_\phi$.

a) Show that H is not a function of ϕ or z: Consider this situation as illustrated in Fig. 8.11. There (sec. 8.2) it was stated that the field will be entirely z-directed. We can see this by applying Ampere's circuital law to a closed loop path whose orientation we choose such that current is enclosed by the path. The only way to enclose current is to set up the loop (which we choose to be rectangular) such that it is oriented with two parallel opposing segments lying in the z direction; one of these lies inside the cylinder, the other outside. The other two parallel segments lie in the ρ direction. The loop is now cut by the current sheet, and if we assume a length of the loop in z of d, then the enclosed current will be given by Kd A. There will be no ϕ variation in the field because where we position the loop around the circumference of the cylinder does not affect the result of Ampere's law. If we assume an infinite cylinder length, there will be no z dependence in the field, since as we lengthen the loop in the z direction, the path length (over which the integral is taken) increases, but then so does the enclosed current – by the same factor. Thus H would not change with z. There would also be no change if the loop was simply moved along the z direction.

b) Show that H_ϕ and H_ρ are everywhere zero. First, if H_ϕ were to exist, then we should be able to find a closed loop path *that encloses current*, in which all or or portion of the path lies in the ϕ direction. This we cannot do, and so H_ϕ must be zero. Another argument is that when applying the Biot-Savart law, there is no current element that would produce a ϕ component. Again, using the Biot-Savart law, we note that radial field components will be produced by individual current elements, but such components will cancel from two elements that lie at symmetric distances in z on either side of the observation point.

c) Show that $H_z = 0$ for $\rho > a$: Suppose the rectangular loop was drawn such that the outside z-directed segment is moved further and further away from the cylinder. We would expect H_z outside to decrease (as the Biot-Savart law would imply) but the same amount of current is always enclosed no matter how far away the outer segment is. We therefore must conclude that the field outside is zero.

d) Show that $H_z = K_a$ for $\rho < a$: With our rectangular path set up as in part a, we have no path integral contributions from the two radial segments, and no contribution from the outside z-directed segment. Therefore, Ampere's circuital law would state that

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z d = I_{encl} = K_a d \quad \Rightarrow \quad H_z = K_a$$

where d is the length of the loop in the z direction.

e) A second shell, $\rho = b$, carries a current $K_b \mathbf{a}_\phi$. Find $\mathbf{H}$ everywhere: For $\rho < a$ we would have both cylinders contributing, or $H_z(\rho < a) = K_a + K_b$. Between the cylinders, we are outside the inner one, so its field will not contribute. Thus $H_z(a < \rho < b) = K_b$. Outside $(\rho > b)$ the field will be zero.

8.14. A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find **H** at the point $P(\rho, \phi, z)$: The construction is similar to that of the toroid of round cross section as done on p.239. Again, magnetic field exists only inside the toroid cross section, and is given by

$$\mathbf{H} = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi \quad (2 < \rho < 3)\text{ cm}, \quad (1 < z < 2.5)\text{ cm}$$

where I_{encl} is found from the given current density: On the outer radius, the current is

$$I_{outer} = -50(2\pi \times 3 \times 10^{-2}) = -3\pi \text{ A}$$

This current is directed along negative z, which means that the current on the *inner* radius ($\rho = 2$) is directed along *positive* z. Inner and outer currents have the same magnitude. It is the inner current that is enclosed by the circular integration path in $\mathbf{a}_\phi$ within the toroid that is used in Ampere's law. So $I_{encl} = +3\pi$ A. We can now proceed with what is requested:

a) $P_A(1.5\text{cm}, 0, 2\text{cm})$: The radius, $\rho = 1.5$ cm, lies outside the cross section, and so $\mathbf{H}_A = \underline{0}$.

b) $P_B(2.1\text{cm}, 0, 2\text{cm})$: This point does lie inside the cross section, and the ϕ and z values do not matter. We find

$$\mathbf{H}_B = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi = \frac{3\mathbf{a}_\phi}{2(2.1 \times 10^{-2})} = \underline{71.4\,\mathbf{a}_\phi \text{ A/m}}$$

c) $P_C(2.7\text{cm}, \pi/2, 2\text{cm})$: again, ϕ and z values make no difference, so

$$\mathbf{H}_C = \frac{3\mathbf{a}_\phi}{2(2.7 \times 10^{-2})} = \underline{55.6\,\mathbf{a}_\phi \text{ A/m}}$$

d) $P_D(3.5\text{cm}, \pi/2, 2\text{cm})$. This point lies outside the cross section, and so $\mathbf{H}_D = \underline{0}$.

8.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\,\mathbf{a}_z$ V/m is present.

a) Find **J**: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z \text{ kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\int \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\rho\, d\rho\, d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\left[-150\rho - 1\right]\Big|_0^{\rho_0} \text{ kA}$$
$$= \underline{12.6\left[1 - (1 + 150\rho_0)e^{-150\rho_0}\right] \text{ A}}$$

c) Make use of Ampere's circuital law to find **H**: Symmetry suggests that **H** will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part b, except with ρ_0 replaced by the variable, ρ. We obtain

$$H_\phi = \underline{\frac{2.00}{\rho}\left[1 - (1 + 150\rho)e^{-150\rho}\right] \text{ A/m}}$$

8.16. The cylindrical shell, 2mm $< \rho <$ 3mm, carries a uniformly-distributed total current of 8A in the $-\mathbf{a}_z$ direction, and a filament on the z axis carries 8A in the $\mathbf{a}_z$ direction. Find $\mathbf{H}$ everywhere: We use Ampere's circuital law, noting that from symmetry, $\mathbf{H}$ will be $\mathbf{a}_\phi$ directed. Inside the shell ($\rho <$ 2mm), A circular integration path centered on the z axis encloses only the filament current along z: Therefore

$$\mathbf{H}(\rho < 2\text{mm}) = \frac{8}{2\pi\rho}\,\mathbf{a}_\phi = \underline{\frac{4}{\pi\rho}\,\mathbf{a}_\phi \text{ A/m}} \;\; (\rho \text{ in m})$$

With the circular integration path within ($2 < \rho <$ 3mm), the enclosed current will consist of the filament plus that portion of the shell current that lies inside ρ. Ampere's circuital law applied to a loop of radius ρ is:

$$\oint \mathbf{H}\cdot d\mathbf{L} = I_{filament} + \int\int_{shell\,area} \mathbf{J}\cdot d\mathbf{S}$$

where the current density is

$$\mathbf{J} = -\frac{8}{\pi(3\times10^{-3})^2 - \pi(2\times10^{-3})^2}\,\mathbf{a}_z = -\frac{8\times10^6}{5\pi}\mathbf{a}_z \text{ A/m}^2$$

So

$$2\pi\rho H_\phi = 8 + \int_0^{2\pi}\int_{2\times10^{-3}}^{\rho}\left(\frac{-8}{5\pi}\times10^6\right)\mathbf{a}_z\cdot\mathbf{a}_z\,\rho'\,d\rho' d\phi = 8 - 1.6\times10^6\,(\rho')^2\Big|_{2\times10^{-3}}^{\rho}$$

Solve for H_ϕ to find:

$$\mathbf{H}(2 < \rho < 3\,\text{mm}) = \underline{\frac{4}{\pi\rho}\left[1-(2\times10^5)(\rho^2 - 4\times10^{-6})\right]\mathbf{a}_\phi \;\; \text{A/m}} \quad (\rho \text{ in m})$$

Outside ($\rho >$ 3mm), the total enclosed current is zero, and so $\mathbf{H}(\rho > 3\text{mm}) = \underline{0}$.

8.17. A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of 0.5 $\mathbf{a}_z$ A/m and $-0.2\,\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at:

a) $\rho = 0.5$ cm: Here, we are either just inside or just outside the first current sheet, so both we will calculate $\mathbf{H}$ for both cases. Just inside, applying Ampere's circuital law to a circular path centered on the z axis produces:

$$2\pi\rho H_\phi = 7 \times 10^{-3} \quad \Rightarrow \quad \mathbf{H}(\text{just inside}) = \frac{7 \times 10^{-3}}{2\pi(0.5 \times 10^{-2}}\mathbf{a}_\phi = \underline{2.2 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

Just outside the current sheet at .5 cm, Ampere's law becomes

$$2\pi\rho H_\phi = 7 \times 10^{-3} - 2\pi(0.5 \times 10^{-2})(0.2)$$
$$\Rightarrow \quad \mathbf{H}(\text{just outside}) = \frac{7.2 \times 10^{-4}}{2\pi(0.5 \times 10^{-2})}\mathbf{a}_\phi = \underline{2.3 \times 10^{-2}\mathbf{a}_\phi \text{ A/m}}$$

b) $\rho = 1.5$ cm: Here, all three currents are enclosed, so Ampere's law becomes

$$2\pi(1.5 \times 10^{-2})H_\phi = 7 \times 10^{-3} - 6.28 \times 10^{-3} + 2\pi(10^{-2})(0.5)$$
$$\Rightarrow \quad \mathbf{H}(\rho = 1.5) = \underline{3.4 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

c) $\rho = 4$ cm: Ampere's law as used in part b applies here, except we replace $\rho = 1.5$ cm with $\rho = 4$ cm on the left hand side. The result is $\mathbf{H}(\rho = 4) = \underline{1.3 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$.

d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm? We require that the total enclosed current be zero, and so the net current in the proposed cylinder at 4 cm must be negative the right hand side of the first equation in part b. This will be -3.2×10^{-2}, so that the surface current density at 4 cm must be

$$\mathbf{K} = \frac{-3.2 \times 10^{-2}}{2\pi(4 \times 10^{-2})}\mathbf{a}_z = \underline{-1.3 \times 10^{-1}\,\mathbf{a}_z \text{ A/m}}$$

8.18. Current density is distributed as follows: $\mathbf{J} = 0$ for $|y| > 2$ m, $\mathbf{J} = 8y\mathbf{a}_z$ A/m^2 for $|y| < 1$ m, $\mathbf{J} = 8(2-y)\,\mathbf{a}_z$ A/m^2 for $1 < y < 2$ m, $\mathbf{J} = -8(2+y)\,\mathbf{a}_z$ A/m^2 for $-2 < y < -1$ m. Use symmetry and Ampere's law to find $\mathbf{H}$ everywhere.

Symmetry does help significantly in this problem. The current densities in the regions $0 < y < 1$ and $-1 < y < 0$ are mirror images of each other across the plane $y = 0$ – this in addition to being of opposite sign. This is also true of the current densities in the regions $1 < y < 2$ and $-2 < y < -1$. As a consequence of this, we find that the net current in region 1, I_1 (see the diagram on the next page), is equal and opposite to the net current in region 4, I_4. Also, I_2 is equal and opposite to I_3. This means that when applying Ampere's law to the path $a - b - c - d - a$, as shown in the figure, zero current is enclosed, so that $\oint \mathbf{H} \cdot d\mathbf{L} = 0$ over the path. In addition, the symmetry of the current configuration implies that $\mathbf{H} = 0$ outside the slabs along the vertical paths $a - b$ and $c - d$. $\mathbf{H}$ from all sources should completely cancel along the two vertical paths, as well as along the two horizontal paths.

8.18. (continued) To find the magnetic field in region 1, we apply Ampere's circuital law to the path $c - d - e - f - c$, again noting that **H** will be zero along the two horizontal segments and along the right vertical segment. This leaves only the left vertical segment, $e - f$, pointing in the $+x$ direction, and along which is field, H_{x1}. The counter-clockwise direction of the path integral is chosen using the right-hand convention, where we take the normal to the path in the $+z$ direction, which is the same as the current direction. Assuming the height of the path is Δx, we find

$$H_{x1}\Delta x = \Delta x \int_{y_1}^{2} 8(2-y)dy = \Delta x \left[16y - 4y^2\right]_{y_1}^{2} = \Delta x \left[16(2-y_1) - 4(4-y_1^2)\right]$$

Replacing y_1 with y, we find

$$H_{x1} = 4[8 - 4y - 4 + y^2] \;\Rightarrow\; \mathbf{H}_1(1 < y < 2) = \underline{4(y-2)^2\,\mathbf{a}_x \text{ A/m}}$$

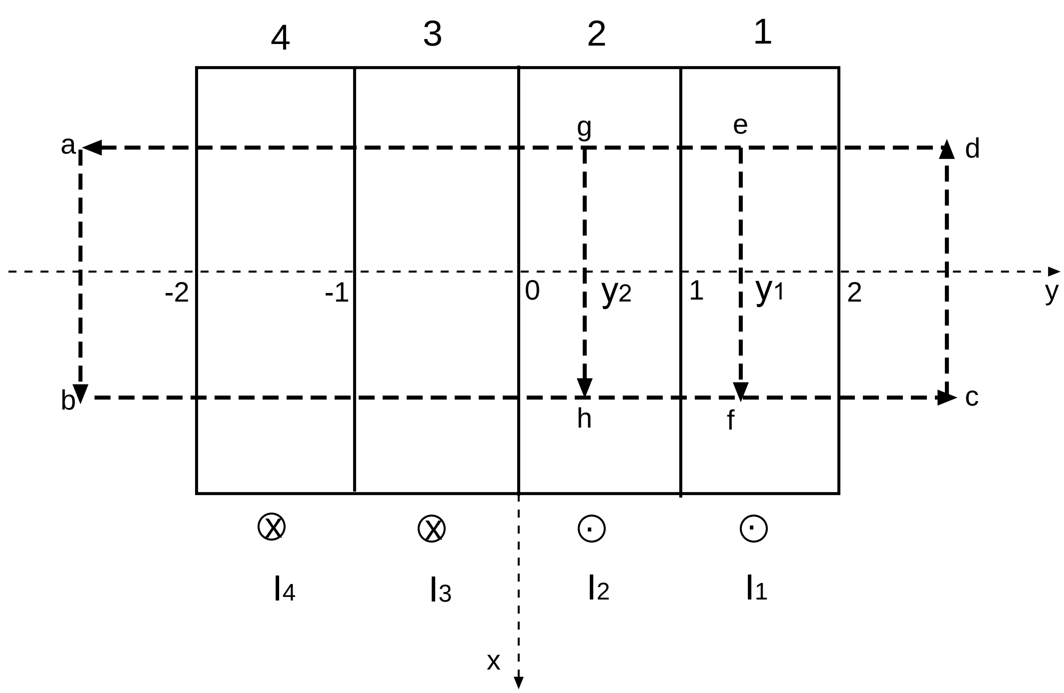
$\mathbf{H}_1$ lies in the positive x direction, since the result of the integration is net positive.

H in region 2 is now found through the line integral over the path $d - g - h - c$, enclosing all of region 1 within Δx and part of region 2 from $y = y_2$ to 1:

$$H_{x2}\Delta x = \Delta x \int_{1}^{2} 8(2-y)\,dy + \Delta x \int_{y_2}^{1} 8y\,dy = \Delta x \left[4(1-2)^2 + 4(1-y_2^2)\right] = 4(2-y_2^2)\Delta x$$

so that in terms of y,

$$\mathbf{H}_2(0 < y < 1) = \underline{4(2-y^2)\mathbf{a}_x \text{ A/m}}$$



8.18. (continued) The procedure is repeated for the remaining two regions, $-2 < y < -1$ and $-1 < y < 0$, by taking the integration path with its right vertical segment within each of these two regions, while the left vertical path is $a - b$. Again the integral is taken counter-clockwise, which means that the right vertical path will be directed along $-x$. But the current is now in the opposite direction of that for $y > 0$, making the enclosed current net negative. Therefore, **H** will be in the opposite direction from that of the right vertical path, which is the positive x direction. The magnetic field will therefore be symmetric about the $y = 0$ plane. We can use the results for regions 1 and 2 to construct the field everywhere:

$$\mathbf{H} = \underline{0 \;\; (y > 2) \;\text{ and }\; (y < -2)}$$

$$\mathbf{H} = \underline{4(2 - |y|^2)\mathbf{a}_x \text{ A/m} \;\; (0 < |y| < 1)}$$

$$\mathbf{H} = \underline{4(|y| - 2)^2\mathbf{a}_x \text{ A/m} \;\; (1 < |y| < 2)}$$

8.19. Calculate $\nabla \times [\nabla(\nabla \cdot \mathbf{G})]$ if $\mathbf{G} = 2x^2yz\,\mathbf{a}_x - 20y\,\mathbf{a}_y + (x^2 - z^2)\,\mathbf{a}_z$: Proceding, we first find $\nabla \cdot \mathbf{G} = 4xyz - 20 - 2z$. Then $\nabla(\nabla \cdot \mathbf{G}) = 4yz\,\mathbf{a}_x + 4xz\,\mathbf{a}_y + (4xy - 2)\,\mathbf{a}_z$. Then

$$\nabla \times [\nabla(\nabla \cdot \mathbf{G})] = (4x - 4x)\,\mathbf{a}_x - (4y - 4y)\,\mathbf{a}_y + (4z - 4z)\,\mathbf{a}_z = \underline{0}$$

8.20. The magnetic field intensity is given in the square region $x = 0$, $0.5 < y < 1$, $1 < z < 1.5$ by $\mathbf{H} = z^2\mathbf{a}_x + x^3\mathbf{a}_y + y^4\mathbf{a}_z$ A/m.

a) evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ about the perimeter of the square region: Using $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and using the given field, we find, in the $x = 0$ plane:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_{.5}^{1} 0\,dy + \int_{1}^{1.5} (1)^4\,dz + \int_{1}^{.5} 0\,dy + \int_{1.5}^{1} (.5)^4\,dz = \underline{0.46875}$$

b) Find $\nabla \times \mathbf{H}$:

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$
$$= \underline{4y^3\mathbf{a}_x + 2z\mathbf{a}_y + 3x^2\mathbf{a}_z}$$

c) Calculate $(\nabla \times \mathbf{H})_x$ at the center of the region: Here, $y = 0.75$ and so $(\nabla \times \mathbf{H})_x = 4(.75)^3 = \underline{1.68750}$.

d) Does $(\nabla \times \mathbf{H})_x = [\oint \mathbf{H} \cdot d\mathbf{L}]/\text{Area Enclosed}$? Using the part a result, $[\oint \mathbf{H} \cdot d\mathbf{L}]/\text{Area Enclosed} = 0.46875/0.25 = \underline{1.8750}$, which is off the value found in part c. Answer: <u>No</u>. Reason: the limit of the area shrinking to zero must be taken before the results will be equal.

8.21. Points A, B, C, D, E, and F are each 2 mm from the origin on the coordinate axes indicated in Fig. 8.23. The value of **H** at each point is given. Calculate an approximate value for $\nabla \times \mathbf{H}$ at the origin: We use the approximation:

$$\text{curl}\,\mathbf{H} \doteq \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta a}$$

where no limit as $\Delta a \to 0$ is taken (hence the approximation), and where $\Delta a = 4\,\text{mm}^2$. Each curl component is found by integrating **H** over a square path that is normal to the component in question.

8.21. (continued) Each of the four segments of the contour passes through one of the given points. Along each segment, the field is assumed constant, and so the integral is evaluated by summing the products of the field and segment length (4 mm) over the four segments. The x component of the curl is thus:

$$(\nabla \times \mathbf{H})_x \doteq \frac{(H_{z,C} - H_{y,E} - H_{z,D} + H_{y,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.69 + 13.88 - 14.35 - 13.10)(250) = 530 \text{ A/m}^2$$

The other components are:

$$(\nabla \times \mathbf{H})_y \doteq \frac{(H_{z,B} + H_{x,E} - H_{z,A} - H_{x,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.82 + 11.11 - 14.21 - 10.88)(250) = 460 \text{ A/m}^2$$

and

$$(\nabla \times \mathbf{H})_z \doteq \frac{(H_{y,A} - H_{x,C} - H_{y,B} H_{x,D})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (-13.78 - 10.49 + 12.19 + 11.49)(250) = -148 \text{ A/m}^2$$

Finally we assemble the results and write:

$$\nabla \times \mathbf{H} \doteq \underline{530\,\mathbf{a}_x + 460\,\mathbf{a}_y - 148\,\mathbf{a}_z}$$

8.22. In the cylindrical region $\rho \leq 0.6$ mm, $H_\phi = (2/\rho) + (\rho/2)$ A/m, while $H_\phi = (3/\rho)$ A/m for $\rho > 0.6$ mm.

a) Determine **J** for $\rho < 0.6$mm: We have only a ϕ component that varies with ρ. Therefore

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left[2 + \frac{\rho^2}{2}\right]\mathbf{a}_z = \mathbf{J} = \underline{1\mathbf{a}_z \text{ A/m}^2}$$

b) Determine **J** for $\rho > 0.6$ mm: In this case

$$\mathbf{J} = \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{3}{\rho}\right]\mathbf{a}_z = \underline{0}$$

c) Is there a filamentary current at $\rho = 0$? If so, what is its value? As $\rho \to 0$, $H_\phi \to \infty$, which implies the existence of a current filament along the z axis: So, <u>YES</u>. The value is found by through Ampere's circuital law, by integrating H_ϕ around a circular path of vanishingly-small radius. The current enclosed is therefore $I = 2\pi\rho(2/\rho) = \underline{4\pi \text{ A}}$.

d) What is **J** at $\rho = 0$? Since a filament current lies along z at $\rho = 0$, this forms a singularity, and so the current density there is <u>infinite</u>.

8.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density **J**: This is found through the curl of **H**, which simplifies to a single term, since **H** varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z \text{ A/m}^2}$$

8.23. (continued)

b) Integrate **J** over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 60\rho\mathbf{a}_z \cdot \rho\, d\rho\, d\phi\mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2\, \mathbf{a}_\phi\big|_{\rho=1} \cdot (1)d\phi\mathbf{a}_\phi = \int_0^{2\pi} 20\, d\phi = \underline{40\pi \text{ A}}$$

8.24. Evaluate both sides of Stokes' theorem for the field $\mathbf{G} = 10\sin\theta\,\mathbf{a}_\phi$ and the surface $r = 3$, $0 \le \theta \le 90°$, $0 \le \phi \le 90°$. Let the surface have the $\mathbf{a}_r$ direction: Stokes' theorem reads:

$$\oint_C \mathbf{G} \cdot d\mathbf{L} = \int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, da$$

Considering the given surface, the contour, C, that forms its perimeter consists of three joined arcs of radius 3 that sweep out 90° in the xy, xz, and zy planes. Their centers are at the origin. Of these three, only the arc in the xy plane (which lies along $\mathbf{a}_\phi$) is in the direction of **G**; the other two (in the $-\mathbf{a}_\theta$ and $\mathbf{a}_\theta$ directions respectively) are perpendicular to it, and so will not contribute to the path integral. The left-hand side therefore consists of only the xy plane portion of the closed path, and evaluates as

$$\oint \mathbf{G} \cdot d\mathbf{L} = \int_0^{\pi/2} 10\sin\theta\big|_{\pi/2}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi\, 3\sin\theta\big|_{\pi/2}\, d\phi = \underline{15\pi}$$

To evaluate the right-hand side, we first find

$$\nabla \times \mathbf{G} = \frac{1}{r\sin\theta}\frac{d}{d\theta}\left[(\sin\theta)10\sin\theta\right]\mathbf{a}_r = \frac{20\cos\theta}{r}\mathbf{a}_r$$

The surface over which we integrate this is the one-eighth spherical shell of radius 3 in the first octant, bounded by the three arcs described earlier. The right-hand side becomes

$$\int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n}\, da = \int_0^{\pi/2}\int_0^{\pi/2} \frac{20\cos\theta}{3}\,\mathbf{a}_r \cdot \mathbf{a}_r\,(3)^2\sin\theta\, d\theta\, d\phi = \underline{15\pi}$$

It would appear that the theorem works.

8.25. (This problem was discovered to be flawed – I will proceed with it and show how). Given the field

$$\mathbf{H} = \frac{1}{2}\cos\frac{\phi}{2}\,\mathbf{a}_\rho - \sin\frac{\phi}{2}\,\mathbf{a}_\phi \text{ A/m}$$

evaluate both sides of Stokes' theorem for the path formed by the intersection of the cylinder $\rho = 3$ and the plane $z = 2$, and for the surface defined by $\rho = 3$, $0 \le \phi \le 2\pi$, and $z = 0$, $0 \le \rho \le 3$: This surface resembles that of an open tin can whose bottom lies in the $z = 0$ plane, and whose open circular edge, at $z = 2$, defines the line integral contour. We first evaluate $\oint \mathbf{H}\cdot d\mathbf{L}$ over the circular contour, where we take the integration direction as clockwise, looking down on the can. We do this because the outward normal from the bottom of the can will be in the $-\mathbf{a}_z$ direction.

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} \mathbf{H}\cdot 3d\phi(-\mathbf{a}_\phi) = \int_0^{2\pi} 3\sin\frac{\phi}{2}d\phi = \underline{12\text{ A}}$$

With our choice of contour direction, this indicates that the current will flow in the negative z direction. Note for future reference that only the ϕ component of the given field contributed here. Next, we evalute $\int\int \nabla\times\mathbf{H}\cdot d\mathbf{S}$, over the surface of the tin can. We find

$$\nabla\times\mathbf{H} = \mathbf{J} = \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\phi}\right)\mathbf{a}_z = \frac{1}{\rho}\left(-\sin\frac{\phi}{2} + \frac{1}{4}\sin\frac{\phi}{2}\right)\mathbf{a}_z = -\frac{3}{4\rho}\sin\frac{\phi}{2}\,\mathbf{a}_z \text{ A/m}$$

Note that *both* field components contribute here. The integral over the tin can is now only over the bottom surface, since $\nabla\times\mathbf{H}$ has only a z component. We use the outward normal, $-\mathbf{a}_z$, and find

$$\int\int \nabla\times\mathbf{H}\cdot d\mathbf{S} = -\frac{3}{4}\int_0^{2\pi}\int_0^3 \frac{1}{\rho}\sin\frac{\phi}{2}\mathbf{a}_z\cdot(-\mathbf{a}_z)\rho\,d\rho\,d\phi = \frac{9}{4}\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = \underline{9\text{ A}}$$

Note that if the radial component of $\mathbf{H}$ were not included in the computation of $\nabla\times\mathbf{H}$, then the factor of 3/4 in front of the above integral would change to a factor of 1, and the result would have been 12 A. What would appear to be a violation of Stokes' theorem is likely the result of a missing term in the ϕ component of $\mathbf{H}$, having zero curl, which would have enabled the original line integral to have a value of 9A. The reader is invited to explore this further.

8.26. Let $\mathbf{G} = 15r\,\mathbf{a}_\phi$.

a) Determine $\oint \mathbf{G}\cdot d\mathbf{L}$ for the circular path $r = 5$, $\theta = 25^\circ$, $0 \le \phi \le 2\pi$:

$$\oint \mathbf{G}\cdot d\mathbf{L} = \int_0^{2\pi} 15(5)\mathbf{a}_\phi\cdot\mathbf{a}_\phi(5)\sin(25^\circ)\,d\phi = 2\pi(375)\sin(25^\circ) = \underline{995.8}$$

b) Evaluate $\int_S(\nabla\times\mathbf{G})\cdot d\mathbf{S}$ over the spherical cap $r = 5$, $0 \le \theta \le 25^\circ$, $0 \le \phi \le 2\pi$: When evaluating the curl of $\mathbf{G}$ using the formula in spherical coordinates, only one of the six terms survives:

$$\nabla\times\mathbf{G} = \frac{1}{r\sin\theta}\frac{\partial(G_\phi\sin\theta)}{\partial\theta}\mathbf{a}_r = \frac{1}{r\sin\theta}15r\cos\theta\,\mathbf{a}_r = 15\cot\theta\,\mathbf{a}_r$$

Then

$$\int_S(\nabla\times\mathbf{G})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{25^\circ} 15\cot\theta\,\mathbf{a}_r\cdot\mathbf{a}_r\,(5)^2\sin\theta\,d\theta\,d\phi$$

$$= 2\pi\int_0^{25^\circ} 15\cos\theta(25)\,d\theta = 2\pi(15)(25)\sin(25^\circ) = \underline{995.8}$$

8.27. The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x + 2y}{z^2}\,\mathbf{a}_y + \frac{2}{z}\,\mathbf{a}_z \text{ A/m}$$

a) Find $\nabla \times \mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x + \frac{\partial H_y}{\partial x}\,\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\,\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \text{ A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla \times \mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 < x < 2$, $3 < y < 5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\int \mathbf{J}\big|_{z=4} \cdot \mathbf{a}_z\, dx\, dy = \int_3^5 \int_1^2 \frac{1}{4^2} dx\, dy = \underline{1/8 \text{ A}}$$

d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint \mathbf{H} \cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint \mathbf{H}\big|_{z=4} \cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}\, dy + \int_5^3 \frac{1+2y}{16}\, dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

8.28. Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi$ A/m in free space:

a) find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like best. I chose the line integral side, where the integration path is the circular path in ϕ around the top edge of the cone, at $r = 5$. The path direction is chosen to be *clockwise* looking down on the xy plane. This, by convention, leads to the normal from the cone surface that points in the positive $\mathbf{a}_\theta$ direction (right hand rule). We find

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} \Big[(3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi\Big]_{r=5,\theta=20} \cdot 5\sin(20^\circ)\, d\phi\, (-\mathbf{a}_\phi)$$
$$= -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ) = \underline{-2.73 \times 10^3 \text{ A}}$$

This result means that there is a component of current that enters the cone surface in the $-\mathbf{a}_\theta$ direction, to which is associated a component of $\mathbf{H}$ in the positive $\mathbf{a}_\phi$ direction.

b) Check the result by using the other side of Stokes' theorem: We first find the current density through the curl of the magnetic field, where three of the six terms in the spherical coordinate formula survive:

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(54r\cos\theta\sin\theta\right)\,\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(54r^2\cos\theta\right)\,\mathbf{a}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3r^3}{\sin\theta}\right)\,\mathbf{a}_\phi = \mathbf{J}$$

Thus

$$\mathbf{J} = 54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi$$

8.28b. (continued)

The calculation of the other side of Stokes' theorem now involves integrating **J** over the surface of the cone, where the outward normal is positive $\mathbf{a}_\theta$, as defined in part *a*:

$$\begin{aligned}\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^5 \left[54 \cot\theta\, \mathbf{a}_r - 108 \cos\theta\, \mathbf{a}_\theta + \frac{9r}{\sin\theta}\, \mathbf{a}_\phi \right]_{\theta=20^\circ} \cdot \mathbf{a}_\theta\, r \sin(20^\circ)\, dr\, d\phi \\ &= -\int_0^{2\pi} \int_0^5 108 \cos(20^\circ) \sin(20^\circ) r\, dr\, d\phi = -2\pi(54)(25) \cos(20^\circ) \sin(20^\circ) \\ &= \underline{-2.73 \times 10^3 \text{ A}}\end{aligned}$$

8.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2 \times 10^{-3})^2} \mathbf{a}_z = \underline{1.59 \times 10^7\, \mathbf{a}_z \text{ A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad \Rightarrow \quad \mathbf{H} = \frac{\rho J}{2} \mathbf{a}_\phi = \underline{7.96 \times 10^6 \rho\, \mathbf{a}_\phi \text{ A/m}}$$

Then $\mathbf{B} = \mu_0 \mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6)\rho \mathbf{a}_\phi = \underline{10\rho\, \mathbf{a}_\phi \text{ Wb/m}^2}$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part *b*, we find,

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi)\, \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1.59 \times 10^7 \rho^2}{2} \right) \mathbf{a}_z = \underline{1.59 \times 10^7\, \mathbf{a}_z \text{ A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi = \underline{\frac{1}{\pi\rho}\, \mathbf{a}_\phi \text{ A/m}}$$

Now $\mathbf{B} = \mu_0 \mathbf{H} = \underline{\mu_0/(\pi\rho)\, \mathbf{a}_\phi \text{ Wb/m}^2}$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi)\, \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{1}{\pi\rho} \right) \mathbf{a}_z = \underline{0} \quad \text{(as expected)}$$

8.30. A solid nonmagnetic conductor of circular cross-section has a radius of 2mm. The conductor is inhomogeneous, with $\sigma = 10^6(1 + 10^6\rho^2)$ S/m. If the conductor is 1m in length and has a voltage of 1mV between its ends, find:

a) **H** inside: With current along the cylinder length (along $\mathbf{a}_z$, and with ϕ symmetry, **H** will be ϕ-directed only. We find $\mathbf{E} = (V_0/d)\mathbf{a}_z = 10^{-3}\mathbf{a}_z$ V/m. Then $\mathbf{J} = \sigma\mathbf{E} = 10^3(1 + 10^6\rho^2)\mathbf{a}_z$ A/m^2. Next we apply Ampere's circuital law to a circular path of radius ρ, centered on the z axis and normal to the axis:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int\int_S \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho} 10^3(1 + 10^6(\rho')^2)\mathbf{a}_z \cdot \mathbf{a}_z \rho' d\rho' d\phi$$

Thus

$$H_\phi = \frac{10^3}{\rho}\int_0^{\rho} \rho' + 10^6(\rho')^3 d\rho' = \frac{10^3}{\rho}\left[\frac{\rho^2}{2} + \frac{10^6}{4}\rho^4\right]$$

Finally, $\mathbf{H} = \underline{500\rho(1 + 5 \times 10^5\rho^3)\mathbf{a}_\phi \text{ A/m } (0 < \rho < 2\text{mm})}$.

b) the total magnetic flux inside the conductor: With field in the ϕ direction, a plane normal to **B** will be that in the region $0 < \rho < 2$ mm, $0 < z < 1$ m. The flux will be

$$\Phi = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_0^1 \int_0^{2\times10^{-3}} \left(500\rho + 2.5 \times 10^8\rho^3\right) d\rho dz = 8\pi \times 10^{-10}\text{ Wb} = \underline{2.5\,\text{nWb}}$$

8.31. The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a non-magnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$:

a) $0 < \rho < 1.2$ cm: We first need to find **J**, **H**, and **B**: The current density will be:

$$\mathbf{J} = \frac{50}{\pi[(1.4 \times 10^{-2})^2 - (1.0 \times 10^{-2})^2]}\mathbf{a}_z = 1.66 \times 10^5\,\mathbf{a}_z \text{ A/m}^2$$

Next we find H_ϕ at radius ρ between 1.0 and 1.4 cm, by applying Ampere's circuital law, and noting that the current density is zero at radii less than 1 cm:

$$2\pi\rho H_\phi = I_{encl} = \int_0^{2\pi}\int_{10^{-2}}^{\rho} 1.66 \times 10^5\rho'\,d\rho'\,d\phi$$

$$\Rightarrow \quad H_\phi = 8.30 \times 10^4\frac{(\rho^2 - 10^{-4})}{\rho} \text{ A/m } \; (10^{-2}\,\text{m} < \rho < 1.4 \times 10^{-2}\,\text{m})$$

Then $\mathbf{B} = \mu_0\mathbf{H}$, or

$$\mathbf{B} = 0.104\frac{(\rho^2 - 10^{-4})}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

Now,

$$\Phi_a = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1\int_{10^{-2}}^{1.2\times10^{-2}} 0.104\left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$

$$= 0.104\left[\frac{(1.2 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4}\ln\left(\frac{1.2}{1.0}\right)\right] = 3.92 \times 10^{-7}\text{ Wb} = \underline{0.392\,\mu\text{Wb}}$$

8.31b) $1.0\,\text{cm} < \rho < 1.4\,\text{cm}$ (note typo in book): This is part *a* over again, except we change the upper limit of the radial integration:

$$\Phi_b = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.4\times10^{-2}} 0.104 \left[\rho - \frac{10^{-4}}{\rho} \right] d\rho\, dz$$
$$= 0.104 \left[\frac{(1.4 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4} \ln\left(\frac{1.4}{1.0}\right) \right] = 1.49 \times 10^{-6}\,\text{Wb} = \underline{1.49\,\mu\text{Wb}}$$

c) $1.4\,\text{cm} < \rho < 20\,\text{cm}$: This is entirely outside the current distribution, so we need **B** there: We modify the Ampere's circuital law result of part *a* to find:

$$\mathbf{B}_{out} = 0.104 \frac{[(1.4 \times 10^{-2})^2 - 10^{-4}]}{\rho} \mathbf{a}_\phi = \frac{10^{-5}}{\rho} \mathbf{a}_\phi \text{ Wb/m}^2$$

We now find

$$\Phi_c = \int_0^1 \int_{1.4\times10^{-2}}^{20\times10^{-2}} \frac{10^{-5}}{\rho} d\rho\, dz = 10^{-5} \ln\left(\frac{20}{1.4}\right) = 2.7 \times 10^{-5}\,\text{Wb} = \underline{27\,\mu\text{Wb}}$$

8.32. The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross-section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m.

a) Specify the current densities on the surfaces at $\rho = 2$ cm, $z = 1$cm, and $z = 4$cm. All surfaces must carry equal currents. With this requirement, we find: $\mathbf{K}(\rho = 2) = \underline{-3\,\mathbf{a}_z \text{ kA/m}}$. Next, the current densities on the $z = 1$ and $z = 4$ surfaces must transistion between the current density values at $\rho = 2$ and $\rho = 3$. Knowing the the radial current density will vary as $1/\rho$, we find $\mathbf{K}(z = 1) = \underline{(60/\rho)\mathbf{a}_\rho \text{ A/m}}$ with ρ in meters. Similarly, $\mathbf{K}(z = 4) = \underline{-(60/\rho)\mathbf{a}_\rho \text{ A/m}}$.

b) Find **H** everywhere: Outside the toroid, $\mathbf{H} = 0$. Inside, we apply Ampere's circuital law in the manner of Problem 8.14:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int_0^{2\pi} \mathbf{K}(\rho = 2) \cdot \mathbf{a}_z\, (2 \times 10^{-2})\, d\phi$$
$$\Rightarrow \quad \mathbf{H} = -\frac{2\pi(3000)(.02)}{\rho} \mathbf{a}_\phi = \underline{-60/\rho\, \mathbf{a}_\phi \text{ A/m (inside)}}$$

c) Calculate the total flux within the toriod: We have $\mathbf{B} = -(60\mu_0/\rho)\mathbf{a}_\phi$ Wb/m^2. Then

$$\Phi = \int_{.01}^{.04} \int_{.02}^{.03} \frac{-60\mu_0}{\rho} \mathbf{a}_\phi \cdot (-\mathbf{a}_\phi)\, d\rho\, dz = (.03)(60)\mu_0 \ln\left(\frac{3}{2}\right) = \underline{0.92\,\mu\text{Wb}}$$

8.33. Use an expansion in cartesian coordinates to show that the curl of the gradient of any scalar field *G* is identically equal to zero. We begin with

$$\nabla G = \frac{\partial G}{\partial x} \mathbf{a}_x + \frac{\partial G}{\partial y} \mathbf{a}_y + \frac{\partial G}{\partial z} \mathbf{a}_z$$

and

$$\nabla \times \nabla G = \left[\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial y}\right) \right] \mathbf{a}_x + \left[\frac{\partial}{\partial z}\left(\frac{\partial G}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial z}\right) \right] \mathbf{a}_y$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right) \right] \mathbf{a}_z = \underline{0} \text{ for any } G$$

8.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

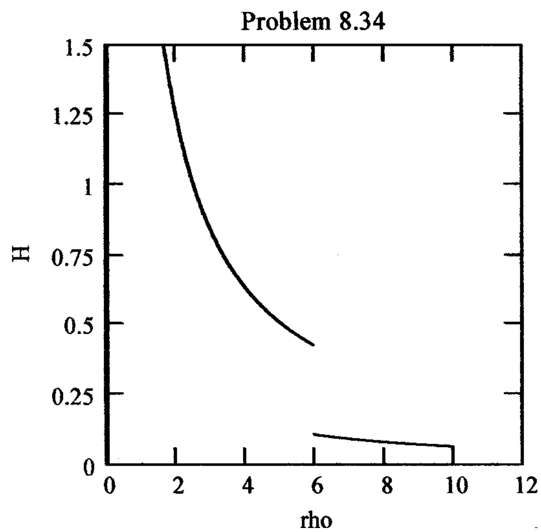
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: This will be

$$\Phi = \int_0^1 \int_1^6 \frac{16\mu_0}{2\pi\rho}\, d\rho\, dz + \int_0^1 \int_6^7 \frac{4\mu_0}{2\pi\rho}\, d\rho\, dz = \frac{2\mu_0}{\pi}\,[4\ln 6 + \ln(7/6)] = \underline{5.9\ \mu\text{Wb}}$$

8.35. A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\,\mathbf{a}_z$ A/m is located at $\rho = 4$.

a.) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$: Since the current is cylindrically-symmetric, we know that $\mathbf{H} = I/(2\pi\rho)\ \mathbf{a}_\phi$, where I is the current enclosed, equal in this case to $2\pi(2)K = 80\pi$ A. Thus, using the result of Section 8.6, we find

$$V_m = -\frac{I}{2\pi}\,\phi = -\frac{80\pi}{2\pi}\,\phi = \underline{-40\phi\ \text{A}}$$

which is valid over the region $2 < \rho < 4$, $-\pi < \phi < \pi$, and $-\infty < z < \infty$. For $\rho > 4$, the outer current contributes, leading to a total enclosed current of

$$I_{net} = 2\pi(2)(20) - 2\pi(4)(10) = 0$$

With zero enclosed current, $H_\phi = 0$, and the magnetic potential is zero as well.

8.35b. Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$: Again, we know that $\mathbf{H} = H_\phi(\rho)$, since the current is cylindrically symmetric. With the current only in the z direction, and again using symmmetry, we expect only a z component of $\mathbf{A}$ which varies only with ρ. We can then write:

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

Thus

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho} \;\; \Rightarrow \;\; A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + C$$

We require that $A_z = 0$ at $\rho = 3$. Therefore $C = [(\mu_0 I)/(2\pi)]\ln(3)$, Then, with $I = 80\pi$, we finally obtain

$$\mathbf{A} = -\frac{\mu_0(80\pi)}{2\pi}\left[\ln(\rho) - \ln(3)\right]\mathbf{a}_z = \underline{40\mu_0 \ln\left(\frac{3}{\rho}\right)\mathbf{a}_z \text{ Wb/m}}$$

8.36. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space.

a) Show that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(3y - z) + \frac{\partial}{\partial y}2xz = \underline{0}$$

b) At $P(2, -1, 3)$, find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$: First $\mathbf{A}_P = \underline{-6\mathbf{a}_x + 12\mathbf{a}_y}$. Then, using the curl formula in cartesian coordinates,

$$\mathbf{B} = \nabla \times \mathbf{A} = -2x\mathbf{a}_x - \mathbf{a}_y + (2z - 3)\mathbf{a}_z \;\; \Rightarrow \;\; \mathbf{B}_P = \underline{-4\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}$$

Now

$$\mathbf{H}_P = (1/\mu_0)\mathbf{B}_P = \underline{-3.2 \times 10^6\mathbf{a}_x - 8.0 \times 10^5\mathbf{a}_y + 2.4 \times 10^6\mathbf{a}_z \text{ A/m}}$$

Then $\mathbf{J} = \nabla \times \mathbf{H} = (1/\mu_0)\nabla \times \mathbf{B} = \underline{0}$, as the curl formula in cartesian coordinates shows.

8.37. Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Fig. 8.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$: Well-within the toroid, we have

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi = -\nabla V_m = -\frac{1}{\rho}\frac{dV_m}{d\phi}\mathbf{a}_\phi$$

Thus

$$V_m = -\frac{NI\phi}{2\pi} + C$$

Then,

$$0 = -\frac{1000(0.8)(0.3\pi)}{2\pi} + C$$

or C = 120. Finally

$$\underline{V_m = \left[120 - \frac{400}{\pi}\phi\right] \text{ A} \quad (0 < \phi < 2\pi)}$$

8.38. The solenoid shown in Fig. 8.11b contains 400 turns, carries a current $I = 5$ A, has a length of 8cm, and a radius $a = 1.2$ cm (hope it doesn't blow up!).

a) Find $\mathbf{H}$ within the solenoid. Assuming the current flows in the $\mathbf{a}_\phi$ direction, $\mathbf{H}$ will then be along the positive z direction, and will be given by

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z = \frac{(400)(5)}{.08}\mathbf{a}_z = \underline{2.5 \times 10^4 \text{ A/m}}$$

b) If $V_m = 0$ at the origin, specify $V_m(\rho, \phi, z)$ inside the solenoid: Since $\mathbf{H}$ is only in the z direction, V_m should vary with z only. Use

$$\mathbf{H} = -\nabla V_m = -\frac{dV_m}{dz}\mathbf{a}_z \quad \Rightarrow \quad V_m = -H_z z + C$$

At $z = 0$, $V_m = 0$, so $C = 0$. Therefore $V_m(z) = \underline{-2.5 \times 10^4 z \text{ A}}$

c) Let $\mathbf{A} = 0$ at the origin, and specify $\mathbf{A}(\rho, \phi, z)$ inside the solenoid if the medium is free space. $\mathbf{A}$ should be in the same direction as the current, and so would have a ϕ component only. Furthermore, since $\nabla \times \mathbf{A} = \mathbf{B}$, the curl will be z-directed only. Therefore

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\mathbf{a}_z = \mu_0 H_z \mathbf{a}_z$$

Then

$$\frac{\partial}{\partial \rho}(\rho A_\phi) = \mu_0 H_z \rho \quad \Rightarrow \quad A_\phi = \frac{\mu_0 H_z \rho}{2} + C$$

$A_\phi = 0$ at the origin, so $C = 0$. Finally,

$$\mathbf{A} = \frac{(4\pi \times 10^{-7})(2.5 \times 10^4)\rho}{2}\mathbf{a}_\phi = \underline{15.7\mathbf{a}_\phi \text{ mWb/m}}$$

8.39. Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$:

a) Find $\mathbf{H}$: Since we have parallel current sheets carrying equal and opposite currents, we use Eq. (12), $\mathbf{H} = \mathbf{K} \times \mathbf{a}_N$, where $\mathbf{a}_N$ is the unit normal directed into the region between currents, and where either one of the two currents are used. Choosing the sheet at $x = 0.2$, we find

$$\mathbf{H} = 30\mathbf{a}_z \times -\mathbf{a}_x = \underline{-30\mathbf{a}_y \text{ A/m}}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: Use

$$\mathbf{H} = -30\mathbf{a}_y = -\nabla V_m = -\frac{dV_m}{dy}\mathbf{a}_y$$

So

$$\frac{dV_m}{dy} = 30 \quad \Rightarrow \quad V_m = 30y + C_1$$

Then

$$0 = 30(0.2) + C_1 \quad \Rightarrow \quad C_1 = -6 \quad \Rightarrow \quad V_m = \underline{30y - 6 \text{ A}}$$

8.39c) Find **B**: $\mathbf{B} = \mu_0 \mathbf{H} = \underline{-30\mu_0 \mathbf{a}_y \text{ Wb/m}^2}$.

d) Obtain an expression for **A** if $\mathbf{A} = 0$ at P: We expect **A** to be z-directed (with the current), and so from $\nabla \times \mathbf{A} = \mathbf{B}$, where **B** is y-directed, we set up

$$-\frac{dA_z}{dx} = -30\mu_0 \quad \Rightarrow \quad A_z = 30\mu_0 x + C_2$$

Then

$$0 = 30\mu_0(0.1) + C_2 \quad \Rightarrow \quad C_2 = -3\mu_0$$

So finally

$$\mathbf{A} = \underline{\mu_0(30x - 3)\mathbf{a}_z \text{ Wb/m}}$$

8.40. Let $\mathbf{A} = (3y^2 - 2z)\mathbf{a}_x - 2x^2 z\mathbf{a}_y + (x + 2y)\mathbf{a}_z$ Wb/m in free space. Find $\nabla \times \nabla \times \mathbf{A}$ at $P(-2, 3, -1)$:

First $\nabla \times \mathbf{A} =$

$$\left(\frac{\partial(x+2y)}{\partial y} - \frac{\partial(-2x^2 z)}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial(3y^2 - 2z)}{\partial z} - \frac{\partial(x+2y)}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial(-2x^2 z)}{\partial x} - \frac{\partial(3y^2 - 2z)}{\partial y}\right)\mathbf{a}_z$$

$$= (2 + 2x^2)\mathbf{a}_x - 3\mathbf{a}_y - (4xz + 6y)\mathbf{a}_z$$

Then

$$\nabla \times \nabla \times \mathbf{A} = \frac{\partial(4xz + 6y)}{\partial x}\mathbf{a}_y - \frac{\partial(4xz + 6y)}{\partial y}\mathbf{a}_x = -6\mathbf{a}_x + 4z\mathbf{a}_y$$

At P this becomes $\nabla \times \nabla \times \mathbf{A}|_P = \underline{-6\mathbf{a}_x - 4\mathbf{a}_y \text{ Wb/m}^3}$.

8.41. Assume that $\mathbf{A} = 50\rho^2 \mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find **H** and **B**: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \underline{-100\rho\, \mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0\, \mathbf{a}_\phi \text{ A/m}}$.

b) Find **J**: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\mathbf{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-100\rho^2}{\mu_0}\right)\mathbf{a}_z = \underline{-\frac{200}{\mu_0}\mathbf{a}_z \text{ A/m}^2}$$

c) Use **J** to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{-200}{\mu_0}\mathbf{a}_z \cdot \mathbf{a}_z\, \rho\, d\rho\, d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0}\mathbf{a}_\phi \cdot \mathbf{a}_\phi\, (1)d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

8.42. Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$. First

$$\nabla_2\left(\frac{1}{R_{12}}\right) = \nabla_2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{-1/2}$$
$$= -\frac{1}{2}\left[\frac{2(x_2-x_1)\mathbf{a}_x + 2(y_2-y_1)\mathbf{a}_y + 2(z_2-z_1)\mathbf{a}_z}{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]^{3/2}}\right] = \frac{-\mathbf{R}_{12}}{R_{12}^3} = \frac{\mathbf{R}_{21}}{R_{12}^3}$$

Also note that $\nabla_1(1/R_{12})$ would give the same result, but of opposite sign.

8.43. Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure: We do this by first finding **B** within the outer conductor and then "uncurling" the result to find **A**. With $-z$-directed current I in the outer conductor, the current density is

$$\mathbf{J}_{out} = -\frac{I}{\pi(7a)^2 - \pi(5a)^2}\mathbf{a}_z = -\frac{I}{24\pi a^2}\mathbf{a}_z$$

Since current I flows in both conductors, but in opposite directions, Ampere's circuital law inside the outer conductor gives:

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_{5a}^{\rho}\frac{I}{24\pi a^2}\rho'\,d\rho'\,d\phi \;\Rightarrow\; H_\phi = \frac{I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Now, with $\mathbf{B} = \mu_0\mathbf{H}$, we note that $\nabla\times\mathbf{A}$ will have a ϕ component only, and from the direction and symmetry of the current, we expect **A** to be z-directed, and to vary only with ρ. Therefore

$$\nabla\times\mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mu_0\mathbf{H}$$

and so

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Then by direct integration,

$$A_z = \int\frac{-\mu_0 I(49)}{48\pi\rho}\,d\rho + \int\frac{\mu_0 I\rho}{48\pi a^2}\,d\rho + C = \frac{\mu_0 I}{96\pi}\left[\frac{\rho^2}{a^2} - 98\ln\rho\right] + C$$

As per Fig. 8.20, we establish a zero reference at $\rho = 5a$, enabling the evaluation of the integration constant:

$$C = -\frac{\mu_0 I}{96\pi}\left[25 - 98\ln(5a)\right]$$

Finally,

$$\underline{A_z = \frac{\mu_0 I}{96\pi}\left[\left(\frac{\rho^2}{a^2}-25\right) + 98\ln\left(\frac{5a}{\rho}\right)\right]\ \text{Wb/m}}$$

A plot of this continues the plot of Fig. 8.20, in which the curve goes negative at $\rho = 5a$, and then approaches a minimum of $-.09\mu_0 I/\pi$ at $\rho = 7a$, at which point the slope becomes zero.

8.44. By expanding Eq.(58), Sec. 8.7 in cartesian coordinates, show that (59) is correct. Eq. (58) can be rewritten as

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

We begin with

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Then the x component of $\nabla(\nabla \cdot \mathbf{A})$ is

$$[\nabla(\nabla \cdot \mathbf{A})]_x = \underline{\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}}$$

Now

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{a}_z$$

and the x component of $\nabla \times \nabla \times \mathbf{A}$ is

$$[\nabla \times \nabla \times \mathbf{A}]_x = \underline{\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial y}}$$

Then, using the underlined results

$$[\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}]_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x$$

Similar results will be found for the other two components, leading to

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \equiv \nabla^2 \mathbf{A} \quad \text{QED}$$