are very powerful. They are useful in all coordinate systems, and they can be used in problems in which the potential varies with all three coordinates.

We have merely introduced the subject here, and more information can be obtained from the references below, several of which devote hundreds of pages to the solution of Laplace's equation.

SUGGESTED REFERENCES

1. Dekker, A. J.: (see Suggested References for Chap. 5).
2. Hayt, W. H., Jr., and J. E. Kemmerly: "Engineering Circuit Analysis," 5th ed., McGraw-Hill Book Company, New York, 1993.
3. Push, E. M., and E. W. Pugh: "Principles of Electricity and Magnetism," 2d ed., Addison-Wesley Publishing Co., Reading, Mass., 1970. This text provides the physicist's view of electricity and magnetism, but electrical engineering students should find it easy to read. The solution to Laplace's equation by a number of methods is discussed in chap. 4.
4. Ramo, S., J. R. Whinnery, and T. Van Duzer: (see Suggested References for Chap. 6). A more complete and advanced discussion of methods of solving Laplace's equation is given in chap. 7.
5. Seeley, S., and A. D. Poularikas: "Electromagnetics: Classical and Modern Theory and Applications," Marcel Dekker, Inc., New York, 1979. Several examples of the solution of Laplace's equation by separation of variables appear in chap. 4.
6. Smythe, W. R.: "Static and Dynamic Electricity," 3d ed., McGraw-Hill Book Company, New York, 1968. An advanced treatment of potential theory is given in chap. 4.
7. Weber, E.: (see Suggested References for Chap. 6). There are a tremendous number of potential solutions given with the original references.

PROBLEMS

7.1 Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find: (*a*) V at P; (*b*) $\mathbf{E}$ at P; (*c*) ρ_v at P; (*d*) the equation of the equipotential surface passing through P; (*e*) the equation of the streamline passing through P. (*f*) Does V satisfy Laplace's equation?

7.2 A potential field V exists in a region where $\epsilon = f(x)$. Find $\nabla^2 V$ if $\rho_v = 0$.

7.3 Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$.

7.4 Given the potential field $V = A \ln\left(\tan^2 \dfrac{\theta}{2}\right) + B$: (*a*) show that $\nabla^2 V = 0$; (*b*) select A and B so that $V = 100\,\text{V}$ and $E_\theta = 500\,\text{V/m}$ at $P(r = 5, \theta = 60^\circ, \phi = 45^\circ)$.

7.5 Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$: (*a*) show that $\nabla^2 V = 0$; (*b*) select A and B so that $V = 100\,\text{V}$ and $|\mathbf{E}| = 500\,\text{V/m}$ at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$.

7.6 If $V = \dfrac{20\sin\theta}{r^3}$ V in free space, find: (*a*) ρ_v at $P(r = 2, \theta = 30^\circ, \phi = 0)$; (*b*) the total charge within the spherical shell $1 < r < 2\,\text{m}$.

7.7 Let $V = \dfrac{\cos 2\phi}{\rho}$ V in free space. (*a*) Find the volume charge density at point $A(\frac{1}{2}, 60^\circ, 1)$. (*b*) Find the surface charge density on a conductor surface passing through the point $B(2, 30^\circ, 1)$.

7.8 Let $V_1(r, \theta, \phi) = \dfrac{20}{r}$ and $V_2(r, \theta, \phi) = \dfrac{4}{r} + 4$. (*a*) State whether V_1 and V_2 satisfy Laplace's equation. (*b*) Evaluate V_1 and V_2 on the closed surface $r = 4$. (*c*) Conciliate your results with the uniqueness theorem.

7.9 The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \le \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$. (*a*) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*b*) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? (*c*) Are the functions V_2, $V_1 + V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1?

7.10 Conducting planes at $z = 2\,\text{cm}$ and $z = 8\,\text{cm}$ are held at potentials of -3 V and 9 V, respectively. The region between the plates is filled with a perfect dielectric with $\epsilon = 5\epsilon_0$. Find and sketch: (*a*) $V(z)$; (*b*) $E_z(z)$; (*c*) $D_z(z)$.

7.11 The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find: (*a*) V at $P(5, 2, 6)$; (*b*) **E** at P.

7.12 Conducting cylinders at $\rho = 2\,\text{cm}$ and $\rho = 8\,\text{cm}$ in free space are held at potentials of 60 mV and -30 mV, respectively. (*a*) Find $V(\rho)$. (*b*) Find $E_\rho(\rho)$. (*c*) Find the surface on which $V = 30\,\text{mV}$.

7.13 Coaxial conducting cylinders are located at $\rho = 0.5\,\text{cm}$ and $\rho = 1.2\,\text{cm}$. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100 V and the outer at 0 V, find: (*a*) the location of the 20-V equipotential surface; (*b*) $E_{\rho\,\max}$; (*c*) ϵ_R if the charge per meter length on the inner cylinder is 20 nC/m.

7.14 Two semi-infinite planes are located at $\phi = -\alpha$ and $\phi = \alpha$, where $\alpha < \pi/2$. A narrow insulating strip separates them along the z axis. The potential at $\phi = -\alpha$ is V_0, while $V = 0$ at $\phi = \alpha$. (*a*) Find $V(\phi)$ in terms of α and V_0. (*b*) Find E_ϕ at $\phi = 20^\circ$, $\rho = 2\,\text{cm}$, if $V_0 = 100\,V$ and $\alpha = 30^\circ$.

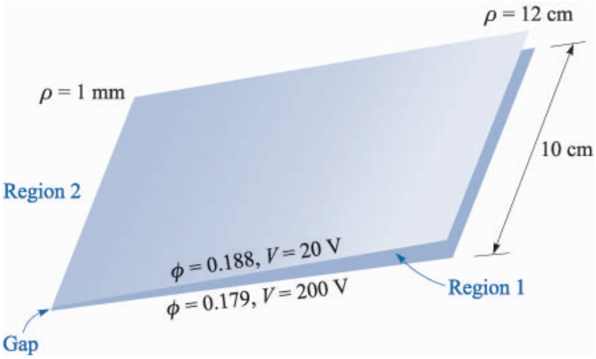

FIGURE 7.8
See Prob. 15.

7.15 The two conducting planes illustrated in Fig. 7.8 are defined by $0.001 < \rho < 0.120\,\text{m}$, $0 < z < 0.1\,\text{m}$, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find: (*a*) $V(\phi)$; (*b*) $\mathbf{E}(\rho)$; (*c*) $\mathbf{D}(\rho)$; (*d*) ρ_S on the upper surface of the lower plane; (*e*) Q on the upper surface of the lower plane. (*f*) Repeat (*a*) to (*c*) for region 2 by letting the location of the upper plane be $\phi = 0.188 - 2\pi$, and then find ρ_S and Q on the lower surface of the lower plane. (*g*) Find the total charge on the lower plane and the capacitance between the planes.

7.16 (*a*) Solve Laplace's equation for the potential field in the homogeneous region between two concentric conducting spheres with radii a and b, $b > a$, if $V = 0$ at $r = b$, and $V = V_0$ at $r = a$. (*b*) Find the capacitance between them.

7.17 Concentric conducting spheres are located at $r = 5\,\text{mm}$ and $r = 20\,\text{mm}$. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer at 0 V: (*a*) find the location of the 20-V equipotential surface; (*b*) find $E_{r,\max}$; (*c*) find ϵ_R if the surface charge density on the inner sphere is $100\,\mu\text{C/m}^2$.

7.18 Concentric conducting spheres have radii of 1 and 5 cm. There is a perfect dielectric for which $\epsilon_R = 3$ between them. The potential of the inner sphere is 2 V and that of the outer is −2 V. Find: (*a*) $V(r)$; (*b*) $\mathbf{E}(r)$; (*c*) V at $r = 3\,\text{cm}$; (*d*) the location of the 0-V equipotential surface; (*e*) the capacitance between the spheres.

7.19 Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100\,\text{V}$ and $V_B = 20\,\text{V}$. Find: (*a*) α for each cone; (*b*) V at $P(1, 1, 1)$.

7.20 A potential field in free space is given as $V = 100\ln[\tan(\theta/2)] + 50\,\text{V}$. (*a*) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8\,\text{m}$, $60^\circ < \phi < 90^\circ$. (*b*) Describe the surface $V = 80\,\text{V}$.

7.21 In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$. (*a*) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$. (*b*) Now find $V(r)$ by using Gauss's law and a line integral.

7.22 Let the volume charge density in Fig. 7.3*a* be given by $\rho_v = \rho_{v0}(x/a)e^{-|x|a}$. (*a*) Determine $\rho_{v,\max}$ and $\rho_{v,\min}$ and their locations. (*b*) Find E_x and $V(x)$ if $V(0) = 0$ and $E_x \to 0$ as $x \to \infty$. (*c*) Use a development similar to that of Sec. 7.4 to show that $C = dQ/dV_0 = \epsilon_0 S/(4\sqrt{2}a)$.

7.23 A rectangular trough is formed by four conducting planes located at $x = 0$ and 8 cm and $y = 0$ and 5 cm in air. The surface at $y = 5$ cm is at a potential of 100 V, the other three are at zero potential, and the necessary gaps are placed at two corners. Find the potential at $x = 3$ cm, $y = 4$ cm.

7.24 The four sides of a square trough are held at potentials of 0, 20, −30, and 60 V; the highest and lowest potentials are on opposite sides. Find the potential at the center of the trough.

7.25 In Fig. 7.7 change the right side so that the potential varies linearly from 0 at the bottom of that side to 100 V at the top. Solve for the potential at the center of the trough.

7.26 If X is a function of x and $X'' + (x-1)X' - 2X = 0$, assume a solution in the form of an infinite power series and determine numerical values for a_2 to a_8 if $a_0 = 1$ and $a_1 = -1$.

7.27 It is known that $V = XY$ is a solution of Laplace's equation, where X is a function of x alone and Y is a function of y alone. Determine which of the following potential functions are also solutions of Laplace's equation: (*a*) $V = 100X$; (*b*) $V = 50XY$; (*c*) $V = 2XY + x - 3y$; (*d*) $V = xXY$; (*e*) $V = X^2 Y$.

7.28 Assume a product solution of Laplace's equation in cylindrical coordinates, $V = PF$, where V is not a function of z, P is a function only of ρ, and F is a function only of ϕ. (*a*) Obtain the two separated equations if the separation constant is n^2. Select the sign of n^2 so that the solution of the ϕ equation leads to trigonometric functions. (*b*) Show that $P = A\rho^n + B\rho^{-n}$ satisfies the ρ equation. (*c*) Construct the solution $V(\rho, \phi)$. Functions of this form are called *circular harmonics*.