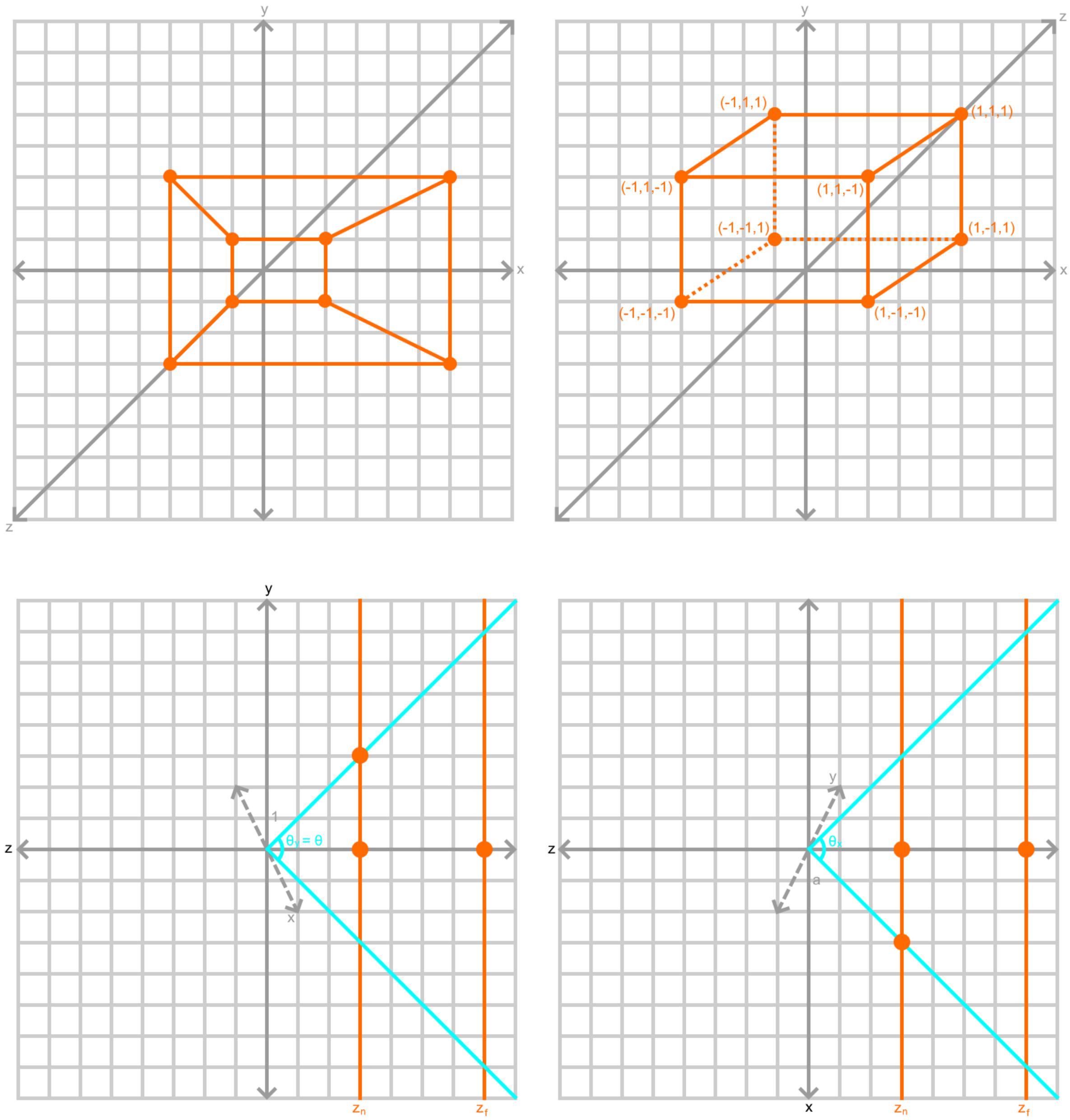
Perspective Projection



Frustum	Cube		Frustum	Cube
$(-a\tan(\frac{\theta}{2})z_n, \tan(\frac{\theta}{2})z_n, -z_n)$	$(-1, 1, -1)$		$(0, \tan(\frac{\theta}{2})z_n, -z_n)$	$(0, 1, -1)$
$(-a\tan(\frac{\theta}{2})z_n, -\tan(\frac{\theta}{2})z_n, -z_n)$	$(-1, -1, -1)$		$(0, -\tan(\frac{\theta}{2})z_n, -z_n)$	$(0, -1, -1)$
$(a\tan(\frac{\theta}{2})z_n, \tan(\frac{\theta}{2})z_n, -z_n)$	$(1, 1, -1)$		$(a\tan(\frac{\theta}{2})z_n, 0, -z_n)$	$(1, 0, -1)$
$(a\tan(\frac{\theta}{2})z_n, -\tan(\frac{\theta}{2})z_n, -z_n)$	$(1, -1, -1)$		$(-a\tan(\frac{\theta}{2})z_n, 0, -z_n)$	$(-1, 0, -1)$
$(-a\tan(\frac{\theta}{2})z_f, \tan(\frac{\theta}{2})z_f, -z_f)$	$(-1, 1, 1)$		$(0, \tan(\frac{\theta}{2})z_f, -z_f)$	$(0, 1, 1)$
$(-a\tan(\frac{\theta}{2})z_f, -\tan(\frac{\theta}{2})z_f, -z_f)$	$(-1, -1, 1)$		$(0, -\tan(\frac{\theta}{2})z_f, -z_f)$	$(0, -1, 1)$
$(a\tan(\frac{\theta}{2})z_f, \tan(\frac{\theta}{2})z_f, -z_f)$	$(1, 1, 1)$		$(a\tan(\frac{\theta}{2})z_f, 0, -z_f)$	$(1, 0, 1)$
$(a\tan(\frac{\theta}{2})z_f, -\tan(\frac{\theta}{2})z_f, -z_f)$	$(1, -1, 1)$		$(-a\tan(\frac{\theta}{2})z_f, 0, -z_f)$	$(-1, 0, 1)$
$(0, 0, -z_n)$	$(0, 0, -1)$		$(0, 0, -z_f)$	$(0, 0, 1)$

Note that the transformation from coordinates in the frustum to the unit cube includes a division by the z-value. Division is NOT a linear transformation and thus, matrix transformations cannot perform this operation. Also, note that since the z-value is also divided by the z-value, the z-value will be lost. Thus, in order to perform the perspective projection matrix, there needs to be a 4-th coordinate w which stores the information from the z. Let w = 1. Then:

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w=1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ -z \end{bmatrix}$$

Now in order to get the 3D coordinates, after applying the perspective projection matrix, the resulting (x,y,z) needs to be divided by -z. Now, using $(0, 0, -z_n)$ and $(0, 0, -z_f)$:

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -z_n \\ w=1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -z_n \\ z_n \end{bmatrix} \qquad \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -z_f \\ w=1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_f \\ z_f \end{bmatrix}$$

The reason for $[0\ 0\ A\ B]$ is the final z is only dependent on the initial z (A) and an offest (B).

$$-Az_n + B = -z_n \qquad -Az_f + B = z_f$$
$$-Az_f + Az_n = z_f + z_n \qquad \Rightarrow \qquad \text{A} = \frac{z_n+z_f}{z_n-z_f}$$
$$-\frac{z_n+z_f}{z_n-z_f} z_f + B = z_f \qquad \Rightarrow \qquad \text{B} = \frac{z_n z_f - z_f z_f}{z_n-z_f} + \frac{z_n z_f + z_f z_f}{z_n-z_f} = \frac{2 z_n z_f}{z_n-z_f}$$

$$\begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & \frac{z_n+z_f}{z_n-z_f} & \frac{2z_n z_f}{z_n-z_f} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w=1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ -z \end{bmatrix}$$

Now, using $(0, \tan(\frac{\theta}{2})z_n, -z_n)$. Thus:

$$\begin{bmatrix} ? & 0 & 0 & 0 \\ ? & \frac{1}{\tan(\frac{\theta}{2})} & 0 & 0 \\ 0 & 0 & \frac{z_n+z_f}{z_n-z_f} & \frac{2z_n z_f}{z_n-z_f} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \tan(\frac{\theta}{2})z_n \\ -z_n \\ w=1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ z_n \\ -z_n \\ z_n \end{bmatrix}$$

Now, using $(a\tan(\frac{\theta}{2})z_n, 0, -z_n)$. Thus:

$$\begin{bmatrix} \frac{1}{a\tan(\frac{\theta}{2})} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan(\frac{\theta}{2})} & 0 & 0 \\ 0 & 0 & \frac{z_n+z_f}{z_n-z_f} & \frac{2z_n z_f}{z_n-z_f} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a\tan(\frac{\theta}{2})z_n \\ 0 \\ -z_n \\ w=1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} z_n \\ 0 \\ -z_n \\ z_n \end{bmatrix}$$

Thus, the perspective projection matrix from (0,0,0) pointing to the -z-axis:

$$\text{P} = \begin{bmatrix} \frac{1}{a\tan(\frac{\theta}{2})} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan(\frac{\theta}{2})} & 0 & 0 \\ 0 & 0 & \frac{z_n+z_f}{z_n-z_f} & \frac{2z_n z_f}{z_n-z_f} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

The above matrix assumes the camera is located at the origin (0,0,0). For a general camera position (x_c, y_c, z_c), first transform the camera's new position to the original position and then apply the perspective projection matrix. This transformation is simply a translation of $(-x_c, -y_c, -z_c)$ so any point $(x, y, z, w = 1)$ becomes $(x - x_c, y - y_c, z - z_c, w = 1)$.

$$\begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w=1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x - x_c \\ y - y_c \\ z - z_c \\ 1 \end{bmatrix} \qquad \text{so} \qquad \text{M} = \begin{bmatrix} 1 & 0 & 0 & -x_c \\ 0 & 1 & 0 & -y_c \\ 0 & 0 & 1 & -z_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the perspective projection matrix at a camera position (x_c, y_c, z_c) is PM.

The above matrix also assumes that the camera's view is pointed towards the -z axis. Suppose the camera turned an angle of ϕ_h in the xz-plane where $\phi_h > 0$ is to the right and $\phi_h < 0$ is to the left. Also, suppose the camera turned an angle of ϕ_v in the yz-plane where $\phi_v > 0$ is up and $\phi_h < 0$ is down. To transform the new camera's view back to the original -z-xis view, rotate any point $(x, y, z, w = 1)$ in the xz-plane an angle of $-\theta_h$ and yz-plane an angle of $-\theta_v$.

$$R_h = \begin{bmatrix} \cos(\phi_h) & 0 & \sin(\phi_h) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi_h) & 0 & \cos(\phi_h) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi_v) & \sin(\phi_v) & 0 \\ 0 & -\sin(\phi_v) 0 & \cos(\phi_v) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The camera can also be tilted ϕ_t in the xy-plane where $\phi_t > 0$ is to right and $\phi_t < 0$ to left.

$$R_t = \begin{bmatrix} \cos(\phi_t) & -\sin(\phi_t) & 0 & 0 \\ \sin(\phi_t) & \cos(\phi_t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining everything gives the perspective projection matrix at a camera position (x_c, y_c, z_c) turned at an angle of ϕ_h horizontally and ϕ_v vertically with a tilt of ϕ_t is $PR_tR_vR_hM$.

Overall, there are quite a few parameters to define.

Aspect ratio	a $= \frac{\text{width}}{\text{height}}$
Closest viewing distance	$z_n > 0$
Furthest viewing distance	$z_f > z_n$
Camera's vertical field of view	θ
Camera's position	(x_c, y_c, z_c)
Camera's direction horizontal tilt (Positive = Right)	ϕ_h
Camera's direction vertical tilt (Positive = Up)	ϕ_v
Camera's direction diagonal tilt (Positive = Right)	ϕ_t