Linear Algebra

Azure

2022

Contents

1   Vectors & Matrices

1.1   Vectors

Definition 1.1.1: Vector

A vector $x \in \mathbb{R}^n$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = (x_1, x_2, ..., x_n) \qquad \text{where each } x_i \in \mathbb{R}$$

Let $0 \in \mathbb{R}^n$ be $0 = \underbrace{(0, ..., 0)}_{n}$. Then for any $x, y \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$, define:

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \qquad cx = (cx_1, cx_2, ..., cx_n)$$

Also, define the length of x:

$$||x|| = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

Theorem 1.1.2: Properties of Vectors

For any $x, y, z \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$:

(a) Commutativity

$$x + y = y + x$$

Proof

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) = (y_1 + x_1, y_2 + x_2, ..., y_n + x_n) = y + x$$

(b) Addititive Associativity

$$(x + y) + z = x + (y + z)$$

Proof

$$(x + y) + z = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) + z$$
$$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, ..., x_n + y_n + z_n)$$
$$= x + (y_1 + z_1, y_2 + z_2, ..., y_n + z_n) = x + (y + z)$$

(c) Additive Identity

There exists a unique $0_v \in \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$:

$$0_v + x = x$$

Moreover, $0_v = 0$. This only holds for $\mathbb{R}^n$ and not all vector spaces.

Proof

Suppose there are $0_{v_1}, 0_{v_2}$ where $0_{v_1} + x = x$ and $0_{v_2} + x = x$ for all x. Then:
$$0_{v_1} = 0_{v_2} + 0_{v_1} = 0_{v_1} + 0_{v_2} = 0_{v_2}$$
Thus, 0_v must be unique.
Since $0 + x = (0 + x_1, ..., 0 + x_n) = (x_1, ..., x_n) = x$, then $0_v = 0$.

(d) Additive Inverse

For any x, there exists a unique -x $\in \mathbb{R}^n$ such that:

$$x + (\text{-x}) = 0_v$$

Moreover, $-x = (-1)x$. This only holds for $\mathbb{R}^n$ and not all vector spaces.

Proof

Suppose there are $(-x)_a, (-x)_b$ where $x + (-x)_a = 0_v$ and $x + (-x)_b = 0_v$. Then:
$$(-x)_a = (-x)_a + 0_v = (-x)_a + x + (-x)_b = 0_v + (-x)_b = (-x)_b$$
Thus, $-x$ must be unique.
Since $(-1)x + x = (-x_1 + x_1, ..., -x_n + x_n) = 0$, then $-x = (-1)x$.

(e) Distributivity
$$c_1(x+y) = c_1x + c_1y \qquad (c_1 + c_2)x = c_1x + c_2x$$
Proof

$$c_1(x+y) = (c_1x_1+c_1y_1, ..., c_1x_n+c_1y_n) = (c_1x_1, ..., c_1x_n) + (c_1y_1, ..., c_1y_n) = c_1x+c_1y$$
$$(c_1 + c_2)x = (c_1x_1 + c_2x_1, ..., c_1x_n + c_2x_n)$$
$$= (c_1x_1, ..., c_1x_n) + (c_2x_1, ..., c_2x_n) = c_1x + c_2x$$

(f) Multiplicative Associativity
$$c_1(c_2)x = (c_1c_2)x$$
Proof

$$c_1(c_2)x = (c_1c_2x_1, ..., c_1c_2x_n) = (c_1c_2)x$$

(g) Multiplicative Identity
$$1\text{x} = \text{x}$$
Proof

$$1\text{x} = (1x_1, ..., 1x_n) = (x_1, ..., x_n) = \text{x}$$

1.2   Matrices

Definition 1.2.1: Matrix
A $m \times n$ matrix $A \in M_{m \times n}(\mathbb{R})$:
$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad \text{where each } a_{ij} \in \mathbb{R}$$

Let $0 \in M_{m \times n}(\mathbb{R})$: be a m × n matrix where the value of all entries are 0.
For A,B $\in M_{m \times n}(\mathbb{R})$ and scalar c $\in \mathbb{R}$, define:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \ldots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \ldots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \ldots & a_{mn} + b_{mn} \end{bmatrix}$$

$$cA = \begin{bmatrix} ca_{11} & ca_{12} & \ldots & ca_{1n} \\ ca_{21} & ca_{22} & \ldots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \ldots & ca_{mn} \end{bmatrix}$$

Theorem 1.2.2: Properties of Matrices
For any A,B,C $\in M_{m \times n}(\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$:
(a) Commutativity
$$A + B = B + A$$
Proof

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \ldots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \ldots & a_{mn} + b_{mn} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & \ldots & b_{1n} + a_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} + a_{m1} & \ldots & b_{mn} + a_{mn} \end{bmatrix} = B + A$$

(b) Addititive Associativity
$$(A + B) + C = A + (B + C)$$
Proof

$$(A + B) + C = \begin{bmatrix} a_{11} + b_{11} + c_{11} & \cdots & a_{1n} + b_{1n} + c_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} + c_{m1} & \cdots & a_{mn} + b_{mn} + c_{mn} \end{bmatrix} = A + (B + C)$$

(c) Additive Identity
There exists a unique $0_M \in M_{m \times n}(\mathbb{R})$ such that for all $A \in M_{m \times n}(\mathbb{R})$:
$$0_M + A = A$$
Moreover, $0_M = 0$. This only holds for $M_{m \times n}(\mathbb{R})$ and not all vector spaces.
Proof

Suppose there are $0_{M_1}, 0_{M_2}$ where $0_{M_1} + A = A$ and $0_{M_2} + A = A$ for all A. Then:
$$0_{M_1} = 0_{M_2} + 0_{M_1} = 0_{M_1} + 0_{M_2} = 0_{M_2}$$
Thus, 0_M must be unique.

$$\text{Since } 0 + A = \begin{bmatrix} 0 + a_{11} & \cdots & 0 + a_{1n} \\ \vdots & \ddots & \vdots \\ 0 + a_{m1} & \cdots & 0 + a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = A, \text{ then } 0_M = 0.$$

(d) Additive Inverse
For any A, there exists a unique -A $\in M_{m \times n}(\mathbb{R})$ such that:
$$A + (\text{-}A) = 0_M$$
Moreover, $-A = (-1)A$. This only holds for $M_{m \times n}(\mathbb{R})$ and not all vector spaces.
Proof

Suppose there are $(-A)_a, (-A)_b$ where $A + (-A)_a = 0_M$ and $A + (-A)_b = 0_M$.
$$(-A)_a = (-A)_a + 0_M = (-A)_a + A + (-A)_b = 0_M + (-A)_b = (-A)_b$$
Thus, $-M$ must be unique.

$$\text{Since } (\text{-1})A + A = \begin{bmatrix} -a_{11} + a_{11} & \cdots & -a_{1n} + a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1} + a_{m1} & \cdots & -a_{mn} + a_{mn} \end{bmatrix} = 0, \text{ then } -A = (-1)A.$$

(e) Distributivity
$$c_1(A + B) = c_1 A + c_1 B \qquad (c_1 + c_2)A = c_1 A + c_2 A$$
Proof

$$c_1(A + B) = \begin{bmatrix} c_1 a_{11} + c_1 b_{11} & \cdots & c_1 a_{1n} + c_1 b_{1n} \\ \vdots & \ddots & \vdots \\ c_1 a_{m1} + c_1 b_{m1} & \cdots & c_1 a_{mn} + c_1 b_{mn} \end{bmatrix} = c_1 A + c_1 B$$

$$(c_1 + c_2)A = \begin{bmatrix} c_1 a_{11} + c_2 a_{11} & \cdots & c_1 a_{1n} + c_2 a_{1n} \\ \vdots & \ddots & \vdots \\ c_1 a_{m1} + c_2 a_{m1} & \cdots & c_1 a_{mn} + c_2 a_{mn} \end{bmatrix} = c_1 A + c_2 A$$

(f) Multiplicative Associativity
$$c_1(c_2)A = (c_1 c_2)A$$
Proof

$$c_1(c_2)A = \begin{bmatrix} c_1 c_2 a_{11} & \cdots & c_1 c_2 a_{1n} \\ \vdots & \ddots & \vdots \\ c_1 c_2 a_{m1} & \cdots & c_1 c_2 a_{mn} \end{bmatrix} = (c_1 c_2)A$$

(g) Multiplicative Identity
$$1A = A$$
Proof

$$1A = \begin{bmatrix} 1a_{11} & \dots & 1a_{1n} \\ \vdots & \ddots & \vdots \\ 1a_{m1} & \dots & 1a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A$$

1.3   System of Equations

Definition 1.3.1: Elementary Row Operations

There are three types of elementary row operations:

(a) Row Swapping: Swapping two rows

(b) Row Multiplication: Multiplying a row by a nonzero scalar

(c) Row Addition: Add a multiple of a row to another row

Note that for any elementary row operation, an entry in the i-th column can be be affected by another entry in the i-th column.

If for matrix A,B $\in M_{m \times n}(\mathbb{R})$, there is a sequence of elementary row operations that transforms A to B, then A and B are row equivalent.

Note if there is a sequence that transforms A to B, then performing the sequence in reverse will transform B to A so row equivalence between A and B is the same as row equivalence between B and A.

Definition 1.3.2: Reduced Row-Echelon Form: RREF

The reduced row-echelon form (rref) of matrix A $\in M_{m \times n}(\mathbb{R})$, rref(A) satisfies:

(a) If a row has nonzero entries, the first nonzero is 1

(b) If a row has a leading 1, then each row before it has a leading 1

(c) A column with a leading 1 has 0 for the other entries

For example:
$$\begin{bmatrix} ① & 2 & 0 & 1 \\ 0 & 0 & ① & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 1.3.3: System of Equations: Augmented Matrix

A m $\times$ n system of equations written as a m $\times$ (n+1) augmented matrix A $\in M_{m \times (n+1)}(\mathbb{R})$:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \quad \Leftrightarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{bmatrix}$$

If the i-th column of the rref(A) has a leading 1, then x_i is called a pivot variable else it is called a free variable. For example:

$$\begin{bmatrix} ① & 2 & 0 & | & 1 \\ 0 & 0 & ① & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{l} x_1 + 2x_2 = 1 \\ x_3 = -1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 1 \\ x_2 \\ -1 \end{bmatrix}$$

Note a pivot variable, $\{x_1, x_3\}$, has a fixed value or its value depends on the free variables while the free variables, $\{x_2\}$, can be any value.

Theorem 1.3.4: Gauss-Jordan Elimination: Elementary Row Operations don't change solutions

Let m × n system of equations be the augmented matrix $A \in M_{m \times (n+1)}(\mathbb{R})$. By performing elementary row operations on A to get to rref(A), the solutions are unchanged.

Proof

Suppose the i-th row is multiplied by scalar c.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & | & b_1 \\ \vdots & \ddots & \vdots & | & \vdots \\ ca_{i1} & \dots & ca_{in} & | & cb_i \\ \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & \dots & a_{mn} & | & b_m \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \dots \\ ca_{i1}x_1 + \dots + ca_{in}x_n = cb_i \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array}$$

If $(x_1^*, \dots, x_n^*)$ is a solution, then $a_{i1}x_1^* + \dots + a_{in}x_n^* = b_i$ for any i ∈ {1,...,m}:

$$ca_{i1}x_1^* + \dots + ca_{in}x_n^* = c(a_{i1}x_1^* + \dots + a_{in}x_n^*) = cb_i$$

If $(x_1', \dots, x_n')$ is not a solution, then $a_{i1}x_1' + \dots + a_{in}x_n' \neq b_i$ for any i ∈ {1,...,m}:

$$ca_{i1}x_1' + \dots + ca_{in}x_n' = c(a_{i1}x_1' + \dots + a_{in}x_n') \neq cb_i$$

Thus, row multiplication does not change the solutions. Note c is nonzero since if c = 0, then any $(x_1, \dots, x_n)$ satisfies $ca_{i1}x_1 + ca_{i2}x_2 + \dots + ca_{in}x_n = 0 = cb_i$ which includes non-solutions.

Suppose the i-th row multiplied by c is added to the j-th row.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & | & b_1 \\ \vdots & \ddots & \vdots & | & \vdots \\ ca_{i1}+a_{j1} & \dots & ca_{in}+a_{jn} & | & cb_i+b_j \\ \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & \dots & a_{mn} & | & b_m \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \dots \\ (ca_{i1}+a_{j1})x_1 + \dots + (ca_{in}+a_{jn})x_n = cb_i+b_j \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array}$$

If $(x_1^*, \dots, x_n^*)$ is a solution, then $a_{i1}x_1^* + \dots + a_{in}x_n^* = b_i$ for any i ∈ {1,...,m}:

$$(ca_{i1}+a_{j1})x_1^* + \dots + (ca_{in}+a_{jn})x_n^* = c(a_{i1}^* + \dots + a_{in}^*) + (a_{j1}^* + \dots + a_{jn}^*) = cb_i + b_j$$

If $(x_1', \dots, x_n')$ is not a solution, then $a_{i1}x_1' + \dots + a_{in}x_n' \neq b_i$ for any i ∈ {1,...,m}:

$$(ca_{i1}+a_{j1})x_1' + \dots + (ca_{in}+a_{jn})x_n' = c(a_{i1}' + \dots + a_{in}') + (a_{j1}' + \dots + a_{jn}') \neq cb_i + b_j$$

Thus, row addition does not change the solutions.

Suppose the i-th row is swapped with the j-th row. Note row swapping is the same as:

Add i-th row to j-th $\qquad \Rightarrow \qquad$ j'-th = i-th + j-th

Subtract i-th row by j'-th $\qquad \Rightarrow \qquad$ i'-th = -j-th

Add i'-th row to j'-th. Multiply the i'-th row by -1 $\qquad \Rightarrow \qquad$ j''-th = i-th $\qquad$ i''-th = j-th

Since each step does not change solutions, then row swapping does not change solutions.

Theorem 1.3.5: Row equivalent matrices have the same solutions

If $A, B \in M_{m \times n}(\mathbb{R})$ are row equivalent, then Ax = 0 and Bx = 0 have the same solutions

Proof

If A and B are row equivalent, then the augmented matrices $[A \mid 0], [B \mid 0] \in M_{m \times (n+1)}(\mathbb{R})$ are row equivalent. Then, there is a sequence of elementary row operations that transforms $[A \mid 0]$ to $[B \mid 0]$. By theorem 1.3.4, the solutions to $[A \mid 0]$ don't change when transforming to $[B \mid 0]$ so the solutions to $[A \mid 0]$ and $[B \mid 0]$ are the same.

Note Ax,Bx = 0 since if Ax = b where b ≠ 0 and Ax = c where c ≠ 0, then performing elementary row operations to change A to B might not change b to c. But if b = 0, then any elementary row operation will keep b as 0 since the entries in b can only be affected by other entries in b which are all 0. Thus, if also c = 0, then $[A \mid 0]$ and $[B \mid 0]$ will be row equivalent.

Theorem 1.3.6: The rref(A) is unique

Let matrix A $\in M_{m \times n}(\mathbb{R})$ be row equivalent to matrix B,C $\in M_{m \times n}(\mathbb{R})$ which are in reduced row-echelon form. Then, B = C.

Proof

Since A is row equivalent to B,C, then by theorem 1.3.5, Ax = 0 and Bx = 0 have the same solutions and Ax = 0 and Cx = 0 have the same solutions. Thus, Bx = 0 and Cx = 0 have the same solutions. The following proof will be a proof by induction.
Suppose A,B,C $\in M_{m \times 1}(\mathbb{R})$. Since B,C are in reduced row-echelon form, then B,C are either:

$$M_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $M_1 x = 0$ is only x = 0 and $M_2 x = 0$ is any x $\in \mathbb{R}$ then either B,C = M_1 or B,C = M_2 since Bx = 0 and Cx = 0 has the same solutions. Thus, the base case holds true.
Suppose for some n $\in \mathbb{Z}_+$, any matrix M $\in M_{m \times n}(\mathbb{R})$ in reduced row echelon form is unique. Let A,B,C $\in M_{m \times (n+1)}(\mathbb{R})$ where A is row equivalent to B,C in reduced row echelon form. Let $A_n, B_n, C_n \in M_{m \times n}(\mathbb{R})$ be A,B,C without their (n+1)-th column. Since A is row equivalent to B,C, then A_n is row equivalent to B_n, C_n which are also in reduced row-echelon form since removing the last column of any rref is still a rref. Since $A_n \in M_{m \times n}(\mathbb{R})$ which is row equivalent to reduced row echelon matrices $B_n, C_n \in M_{m \times n}(\mathbb{R})$, then $B_n = \text{rref}(A_n) = C_n$. Thus, the first n columns of B,C are the same. Suppose the B $\neq$ C so only the (n+1)-th column can be different. Then there is a i $\in \{1,...,m\}$ where $b_{i(n+1)} \neq c_{i(n+1)}$. Let $(x_1^*, ..., x_{n+1}^*)$ be a solution.
    $b_{i1} x_1 + ... + b_{in} x_n + b_{i(n+1)} x_{n+1} = 0$ $c_{i1} x_1 + ... + c_{in} x_n + c_{i(n+1)} x_{n+1} = 0$
Since $B_n = C_n$, then $b_{ij} = c_{ij}$ for i = {1,...,m} and j = {1,...,n}. Thus, $b_{i(n+1)} x_{n+1} = c_{i(n+1)} x_{n+1}$. Since $b_{i(n+1)} \neq c_{i(n+1)}$, then $x_{n+1} = 0$. Thus, x_{n+1} is a pivot variable so the (n+1)-th column of B,C have a leading 1. Thus, any other entry in the (n+1)-th column is 0. Since B and C are in reduced row-echelon form, then this pivot is right after any other pivots before it since this 1 is in the final column, but since the entries of B_n, C_n are the same, then this 1 is in the same row in C as in B. Thus, the (n+1)-th column of B and C are the same which contradicts the assumption that (n+1)-th column are different. Thus, B = C by induction.

Theorem 1.3.7: Number of Solutions in a System of Equations

Let m $\times$ n system of equations be the augmented matrix A $\in M_{m \times (n+1)}(\mathbb{R})$. The system is called consistent if there is at least one solution and inconsistent if there are no solutions.

If the rref(A) contains the row [0...0 | 1], then the system has no solutions.
If there is at least one free variable, then there are infinitely many solutions and if all variables are pivots, then there is one solution.

Proof

Since a variable is either a pivot or free variable, then the rref(A) either:
    - contains the row [0...0 | 1]
    - doesn't contains the row [0...0 | 1] and have all pivot variables
    - doesn't contains the row [0...0 | 1], but have all pivot variables
Since [0...0 | 1] implies $0 = 0x_1 + ... + 0x_n = 1$, then if rref(A) contains the row [0...0 | 1], there cannot be any solution regardless of pivot and free variables since no x = $(x_1, ..., x_n)$ will satisfy such a row. Now, suppose rref(A) doesn't contains the row [0...0 | 1].
Suppose the rref(A) have all pivot variables. Since pivot variables are fixed or depend on free variables which don't exist, then the pivot variables are all fixed and thus, unique.
Suppose the rref(A) has at least one free variable. Then at least one variable can be any real number and thus, there are infinitely many solutions.

Corollary 1.3.8: A unique solution must have as many equation as there are unknowns

Let m × n system of equations be the augmented matrix A $\in M_{m\times(n+1)}(\mathbb{R})$.

If there is a unique solution, then m $\geq$ n.

Proof

By theorem 1.3.7, a unique solution must have all pivot variables. If rref(A) has all pivots, then m $\geq$ n else there will be a column without a pivot.

Definition 1.3.9: Homogeneous & Inhomogeneous Equations

A m × n system of equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

can also be written in matrix form:

Ax = b

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{R}), \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$

(a) Closed under Addition: For any $x_1, \ldots x_k \in \mathbb{R}^n$:

$$A(x_1 + \ldots + x_k) = Ax_1 + \ldots + Ax_k$$

(b) Closed under Scalar Multiplication For any $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

$$A(cx) = cAx$$

For $x_1, \ldots, x_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$:

$$A(c_1x_1 + \ldots + c_kx_k)$$

$$\Leftrightarrow \begin{array}{l} a_{11}(c_1x_{11} + \ldots + c_kx_{k1}) + a_{12}(c_1x_{12} + \ldots + c_kx_{k2}) + \ldots + a_{1n}(c_1x_{1n} + \ldots + c_kx_{kn}) \\ a_{21}(c_1x_{11} + \ldots + c_kx_{k1}) + a_{22}(c_1x_{12} + \ldots + c_kx_{k2}) + \ldots + a_{2n}(c_1x_{1n} + \ldots + c_kx_{kn}) \\ \ldots \\ a_{m1}(c_1x_{11} + \ldots + c_kx_{k1}) + a_{m2}(c_1x_{12} + \ldots + c_kx_{k2}) + \ldots + a_{mn}(c_1x_{1n} + \ldots + c_kx_{kn}) \end{array}$$

$$= \begin{array}{l} c_1(a_{11}x_{11} + \ldots + a_{1n}x_{1n}) \\ c_1(a_{21}x_{11} + \ldots + a_{2n}x_{1n}) \\ \ldots \\ c_1(a_{m1}x_{11} + \ldots + a_{mn}x_{1n}) \end{array} + \ldots + \begin{array}{l} c_k(a_{11}x_{k1} + \ldots + a_{1n}x_{kn}) \\ c_k(a_{21}x_{k1} + \ldots + a_{2n}x_{kn}) \\ \ldots \\ c_k(a_{m1}x_{k1} + \ldots + a_{mn}x_{kn}) \end{array} \Leftrightarrow c_1Ax_1 + \ldots + c_kAx_k$$

A Homogeneous equation is in the form:

Ax = 0   where A $\in M_{m\times n}(\mathbb{R})$, x $\in \mathbb{R}^n$, and 0 $\in \mathbb{R}^m$

A Inhomogeneous equation is in the form:

Ax = b   where A $\in M_{m\times n}(\mathbb{R})$, x $\in \mathbb{R}^n$, and b $\neq 0 \in \mathbb{R}^m$

Theorem 1.3.10: Relationship between Homogeneous and Inhomogeneous

Let x_0 be a solution to Ax = b. Then any solution x_s of Ax = b:

$$x_s = x_0 + x^*$$

where x^* is a solution to Ax = 0

Proof

Let x_0 be a solution to Ax = b. Suppose x_s be a solution to Ax = b.

b $= Ax_s = A(x_0 + (x_s - x_0)) = Ax_0 + A(x_s - x_0) =$ b $+ A(x_s - x_0)$

Thus, $A(x_s - x_0) = 0$ so $x^* = x_s - x_0$ is a solution to Ax = 0.

Example

Find the solution(s), (x_1, x_2, x_3, x_4):

$$x_1 + 2x_3 + 4x_4 = -8$$

$$x_2 - 3x_3 - x_4 = 6$$

$$3x_1 + 4x_2 - 6x_3 + 8x_4 = 0$$

$$-x_2 + 3x_3 + 4x_4 = -12$$

What are the solutions if instead the equations are equal to 0?

$$\begin{bmatrix} 1 & 0 & 2 & 4 & | & -8 \\ 0 & 1 & -3 & -1 & | & 6 \\ 3 & 4 & -6 & 8 & | & 0 \\ 0 & -1 & 3 & 4 & | & -12 \end{bmatrix} \xrightarrow[\Rightarrow]{\text{Add -3(1st) to the (3rd)}} \begin{bmatrix} 1 & 0 & 2 & 4 & | & -8 \\ 0 & 1 & -3 & -1 & | & 6 \\ 0 & 4 & -12 & -4 & | & 24 \\ 0 & -1 & 3 & 4 & | & -12 \end{bmatrix}$$

$$\xrightarrow[\Rightarrow]{\substack{\text{Add -4(2nd) to the (3rd)} \\ \text{Add (2nd) to the (4th)}}} \begin{bmatrix} 1 & 0 & 2 & 4 & | & -8 \\ 0 & 1 & -3 & -1 & | & 6 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 3 & | & -6 \end{bmatrix} \xrightarrow[\Rightarrow]{\substack{\text{Multiply (4th) by } \frac{1}{3} \\ \text{Add -4(4th) to (1st)} \\ \text{Add (4th) to (2nd)}}} \begin{bmatrix} 1 & 0 & 2 & 0 & | & 0 \\ 0 & 1 & -3 & 0 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & -2 \end{bmatrix}$$

Thus, the reduced row-echelon form of this matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & | & 0 \\ 0 & 1 & -3 & 0 & | & 4 \\ 0 & 0 & 0 & 1 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The pivot variables are x_1, x_2, x_4 and the free variable is x_3. The solutions are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 3x_3 + 4 \\ x_3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} = x_3(-2, 3, 1, 0) + (0, 4, 0, -2)$$

Thus, $x_3(-2, 3, 1, 0)$ are the solutions when the equations equal to 0.

2   Vector Space

3   Orthogonality

4 Eigenvectors

References

[1] Otto Bretscher, *Linear Algebra with Applications (4th Edition)*, ISBN-13: 978-0321796974