Linear Algebra

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1 Vectors & Matrices

1.1 Vectors

Definition 1.1.1: Vector

A vector x $\in \mathbb{R}^n$:

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = (x_1, x_2, ..., x_n) \qquad \text{where each } x_i \in \mathbb{R}$$

Let $0 \in \mathbb{R}^n$ be $0 = \underbrace{(0, ..., 0)}_{n}$. Then for any x,y $\in \mathbb{R}^n$ and scalar c $\in \mathbb{R}$, define:

$$\mathrm{x} + \mathrm{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \qquad \mathrm{cx} = (cx_1, cx_2, ..., cx_n)$$

Also, define the length of x:

$$||x|| = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2} = (\textstyle\sum_{i=1}^n x_i^2)^{\frac{1}{2}} \in \mathbb{R}$$

A unit vector is a vector with length 1.

Theorem 1.1.2: Properties of Vectors

For any x,y,z $\in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$:

(a) Commutativity

$$\mathrm{x} + \mathrm{y} = \mathrm{y} + \mathrm{x}$$

Proof

$$\mathrm{x} + \mathrm{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) = (y_1 + x_1, y_2 + x_2, ..., y_n + x_n) = \mathrm{y} + \mathrm{x}$$

(b) Addititive Associativity

$$(\mathrm{x} + \mathrm{y}) + \mathrm{z} = \mathrm{x} + (\mathrm{y} + \mathrm{z})$$

Proof

$$\begin{aligned}(\mathrm{x} + \mathrm{y}) + \mathrm{z} &= (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) + \mathrm{z} \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, ..., x_n + y_n + z_n) \\ &= \mathrm{x} + (y_1 + z_1, y_2 + z_2, ..., y_n + z_n) = \mathrm{x} + (\mathrm{y} + \mathrm{z})\end{aligned}$$

(c) Additive Identity

There exists a unique $0_v \in \mathbb{R}^n$ such that for all x $\in \mathbb{R}^n$:

$$0_v + \mathrm{x} = \mathrm{x}$$

Moreover, $0_v = 0$. This only holds for $\mathbb{R}^n$ and not all vector spaces.

Proof

Suppose there are $0_{v_1}, 0_{v_2}$ where $0_{v_1} + \mathrm{x} = \mathrm{x}$ and $0_{v_2} + \mathrm{x} = \mathrm{x}$ for all x. Then:

$$0_{v_1} = 0_{v_2} + 0_{v_1} = 0_{v_1} + 0_{v_2} = 0_{v_2}$$

Thus, 0_v must be unique.

Since $0 + \mathrm{x} = (0 + x_1, ..., 0 + x_n) = (x_1, ..., x_n) = \mathrm{x}$, then $0_v = 0$.

(d) Additive Inverse

For any x, there exists a unique -x $\in \mathbb{R}^n$ such that:

$$\mathrm{x} + (\text{-x}) = 0_v$$

Moreover, $-x = (-1)x$.

Proof

Suppose there are $(-x)_a, (-x)_b$ where x + $(-x)_a = 0_v$ and x + $(-x)_b = 0_v$. Then:

$$(-x)_a = (-x)_a + 0_v = (-x)_a + x + (-x)_b = 0_v + (-x)_b = (-x)_b$$

Thus, $-x$ must be unique.

Since $(-1)x + \mathrm{x} = (-x_1 + x_1, ..., -x_n + x_n) = 0$, then $-x = (-1)x$.

(e) Distributivity

$c_1(x+y) = c_1x + c_1y \qquad (c_1+c_2)x = c_1x + c_2x$

Proof

$c_1(x+y) = (c_1x_1+c_1y_1, ..., c_1x_n+c_1y_n) = (c_1x_1, ..., c_1x_n) + (c_1y_1, ..., c_1y_n) = c_1x+c_1y$
$(c_1+c_2)x = (c_1x_1 + c_2x_1, ..., c_1x_n + c_2x_n)$
$= (c_1x_1, ..., c_1x_n) + (c_2x_1, ..., c_2x_n) = c_1x + c_2x$

(f) Multiplicative Associativity

$c_1(c_2x) = (c_1c_2)x$

Proof

$c_1(c_2)x = (c_1c_2x_1, ..., c_1c_2x_n) = (c_1c_2)x$

(g) Multiplicative Identity

1x = x

Proof

1x $= (1x_1, ..., 1x_n) = (x_1, ..., x_n) =$ x

Theorem 1.1.3: Rescaling to a Unit Vector

Let x $\in \mathbb{R}^n$ and c $\in \mathbb{R}$. Then, $|cx| = |c||x|$.
Thus, $\frac{x}{|x|}$ is a unit vector.

Proof

$|cx| = \sqrt{(cx_1)^2 + ... + (cx_n)^2} = \sqrt{c^2(x_1^2 + ... + x_n^2)} = |c|\sqrt{x_1^2 + ... + x_n^2} = |c||x|$
$|\frac{x}{|x|}| = \frac{1}{|x|}|x| = 1$

1.2 Matrices

Definition 1.2.1: Matrix

A $m \times n$ matrix A $\in M_{m\times n}(\mathbb{R})$:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{where each } a_{ij} \in \mathbb{R}$$

Let $0 \in M_{m\times n}(\mathbb{R})$: be a m $\times$ n matrix where the value of all entries are 0.
For A,B $\in M_{m\times n}(\mathbb{R})$ and scalar c $\in \mathbb{R}$, define:

$$\text{A+B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix} \qquad \text{cA} = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}$$

Theorem 1.2.2: Properties of Matrices

For any A,B,C $\in M_{m\times n}(\mathbb{R})$ and $c_1, c_2 \in \mathbb{R}$:

(a) Commutativity

A + B = B + A

Proof

$$\text{A + B} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & \dots & b_{1n}+a_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1}+a_{m1} & \dots & b_{mn}+a_{mn} \end{bmatrix} = \text{B + A}$$

(b) Addititive Associativity

(A + B) + C = A + (B + C)

Proof

$$(A+B)+C = \begin{bmatrix} a_{11}+b_{11}+c_{11} & \dots & a_{1n}+b_{1n}+c_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1}+c_{m1} & \dots & a_{mn}+b_{mn}+c_{mn} \end{bmatrix} = A+(B+C)$$

(c) Additive Identity

There exists a unique $0_M \in M_{m\times n}(\mathbb{R})$ such that for all A $\in M_{m\times n}(\mathbb{R})$:

0_M + A = A

Moreover, $0_M = 0$. This only holds for $M_{m\times n}(\mathbb{R})$ and not all vector spaces.

Proof

Suppose there are $0_{M_1}, 0_{M_2}$ where 0_{M_1} + A = A and 0_{M_2} + A = A for all A. Then:

$0_{M_1} = 0_{M_2} + 0_{M_1} = 0_{M_1} + 0_{M_2} = 0_{M_2}$

Thus, 0_M must be unique.

Since $0 + A = \begin{bmatrix} 0+a_{11} & \dots & 0+a_{1n} \\ \vdots & \ddots & \vdots \\ 0+a_{m1} & \dots & 0+a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A$, then $0_M = 0$.

(d) Additive Inverse

For any A, there exists a unique -A $\in M_{m\times n}(\mathbb{R})$ such that:

A + (-A) = 0_M

Moreover, $-A = (-1)A$.

Proof

Suppose there are $(-A)_a, (-A)_b$ where A + $(-A)_a = 0_M$ and A + $(-A)_b = 0_M$.

$(-A)_a = (-A)_a + 0_M = (-A)_a + A + (-A)_b = 0_M + (-A)_b = (-A)_b$

Thus, $-M$ must be unique.

Since $(-1)A + A = \begin{bmatrix} -a_{11}+a_{11} & \dots & -a_{1n}+a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{m1}+a_{m1} & \dots & -a_{mn}+a_{mn} \end{bmatrix} = 0$, then $-A = (-1)A$.

(e) Distributivity

$c_1(A+B) = c_1A + c_1B \qquad (c_1+c_2)A = c_1A + c_2A$

Proof

$$c_1(A+B) = \begin{bmatrix} c_1a_{11}+c_1b_{11} & \dots & c_1a_{1n}+c_1b_{1n} \\ \vdots & \ddots & \vdots \\ c_1a_{m1}+c_1b_{m1} & \dots & c_1a_{mn}+c_1b_{mn} \end{bmatrix} = c_1A + c_1B$$

$$(c_1+c_2)A = \begin{bmatrix} c_1a_{11}+c_2a_{11} & \dots & c_1a_{1n}+c_2a_{1n} \\ \vdots & \ddots & \vdots \\ c_1a_{m1}+c_2a_{m1} & \dots & c_1a_{mn}+c_2a_{mn} \end{bmatrix} = c_1A + c_2A$$

(f) Multiplicative Associativity

$c_1(c_2A) = (c_1c_2)A$

Proof

$$c_1(c_2)A = \begin{bmatrix} c_1c_2a_{11} & \dots & c_1c_2a_{1n} \\ \vdots & \ddots & \vdots \\ c_1c_2a_{m1} & \dots & c_1c_2a_{mn} \end{bmatrix} = (c_1c_2)A$$

(g) Multiplicative Identity
$1A = A$
Proof

$$1A = \begin{bmatrix} 1a_{11} & \dots & 1a_{1n} \\ \vdots & \ddots & \vdots \\ 1a_{m1} & \dots & 1a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A$$

1.3 System of Equations

Definition 1.3.1: Elementary Row Operations

There are three types of elementary row operations:

(a) Row Multiplication: Multiplying a row by a nonzero scalar

(b) Row Addition: Add a multiple of a row to another row

(c) Row Swapping: Swapping two rows

Note that for any elementary row operation, an entry in the i-th column can be be affected by another entry in the i-th column.

If for matrix A,B $\in M_{m\times n}(\mathbb{R})$, there is a sequence of elementary row operations that transforms A to B, then A and B are row equivalent.
Note if there is a sequence that transforms A to B, then performing the sequence in reverse will transform B to A so row equivalence between A and B is the same as row equivalence between B and A.

Definition 1.3.2: Reduced Row-Echelon Form: RREF

The reduced row-echelon form (rref) of matrix A $\in M_{m\times n}(\mathbb{R})$, rref(A) satisfies:

(a) If a row has nonzero entries, the first nonzero is 1

(b) If a row has a leading 1, then each row before it has a leading 1

(c) A column with a leading 1 has 0 for the other entries

For example:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition 1.3.3: System of Equations: Augmented Matrix

A m × n system of equations written as a m × (n+1) augmented matrix A $\in M_{m\times(n+1)}(\mathbb{R})$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \quad \Leftrightarrow \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{bmatrix}$$

If the i-th column of the rref(A) has a leading 1, then x_i is called a pivot variable else it is called a free variable. The rank of a matrix is equal to the number of pivot variables.
For example:

$$\begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{aligned} x_1 + 2x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 1 \\ x_2 \\ -1 \end{bmatrix}$$

Note a pivot variable, $\{x_1, x_3\}$, has a fixed value or its value depends on the free variables while the free variables, $\{x_2\}$, can be any value.

Theorem 1.3.4: Gauss-Jordan Elimination: Elementary row operations don't change solutions

Let m × n system of equations be the augmented matrix A $\in M_{m\times(n+1)}(\mathbb{R})$. By performing elementary row operations on A to get to rref(A), the solutions are unchanged.

Proof

Suppose the i-th row is multiplied by scalar c.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & \mid & b_1 \\ \vdots & \ddots & \vdots & \mid & \vdots \\ ca_{i1} & \dots & ca_{in} & \mid & cb_i \\ \vdots & \ddots & \vdots & \mid & \vdots \\ a_{m1} & \dots & a_{mn} & \mid & b_m \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \ldots \\ ca_{i1}x_1 + \ldots + ca_{in}x_n = cb_i \\ \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{array}$$

If $(x_1^*, ..., x_n^*)$ is a solution, then $a_{i1}x_1^* + ... + a_{in}x_n^* = b_i$ for any i $\in$ {1,...,m}:

$$ca_{i1}x_1^* + ... + ca_{in}x_n^* = c(a_{i1}x_1^* + ... + a_{in}x_n^*) = cb_i$$

If $(x_1^/, ..., x_n^/)$ is not a solution, then $a_{i1}x_1^/ + ... + a_{in}x_n^/ \neq b_i$ for any i $\in$ {1,...,m}:

$$ca_{i1}x_1^/ + ... + ca_{in}x_n^/ = c(a_{i1}x_1^/ + ... + a_{in}x_n^/) \neq cb_i$$

Thus, row multiplication does not change the solutions. Note c is nonzero since if c = 0, then any $(x_1, ..., x_n)$ satisfies $ca_{i1}x_1 + ca_{i2}x_2 + \ldots + ca_{in}x_n = 0 = cb_i$ which includes non-solutions.

Suppose the i-th row multiplied by c is added to the j-th row.

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & \mid & b_1 \\ \vdots & \ddots & \vdots & \mid & \vdots \\ ca_{i1}+a_{j1} & \dots & ca_{in}+a_{jn} & \mid & cb_i+b_j \\ \vdots & \ddots & \vdots & \mid & \vdots \\ a_{m1} & \dots & a_{mn} & \mid & b_m \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{l} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \ldots \\ (ca_{i1}+a_{j1})x_1 + \ldots + (ca_{in}+a_{jn})x_n = cb_i + b_j \\ \ldots \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{array}$$

If $(x_1^*, ..., x_n^*)$ is a solution, then $a_{i1}x_1^* + ... + a_{in}x_n^* = b_i$ for any i $\in$ {1,...,m}:

$$(ca_{i1}+a_{j1})x_1^* + ... + (ca_{in}+a_{jn})x_n^* = c(a_{i1}^* + ... + a_{in}^*) + (a_{j1}^* + ... + a_{jn}^*) = cb_i + b_j$$

If $(x_1^/, ..., x_n^/)$ is not a solution, then $a_{i1}x_1^/ + ... + a_{in}x_n^/ \neq b_i$ for any i $\in$ {1,...,m}:

$$(ca_{i1}+a_{j1})x_1^/ + ... + (ca_{in}+a_{jn})x_n^/ = c(a_{i1}^/ + ... + a_{in}^/) + (a_{j1}^/ + ... + a_{jn}^/) \neq cb_i + b_j$$

Thus, row addition does not change the solutions.

Suppose the i-th row is swapped with the j-th row. Note row swapping is the same as:

Add i-th row to j-th $\Rightarrow$ j'-th = i-th + j-th

Subtract i-th row by j'-th $\Rightarrow$ i'-th = -j-th

Add i'-th row to j'-th. Multiply the i'-th row by -1 $\Rightarrow$ j"-th = i-th i"-th = j-th

Since each step does not change solutions, then row swapping does not change solutions.

Theorem 1.3.5: Row equivalent matrices have the same solutions

If A,B $\in M_{m\times n}(\mathbb{R})$ are row equivalent, then Ax = 0 and Bx = 0 have the same solutions

Proof

If A and B are row equivalent, then the augmented matrices $[A \mid 0]$,$[B \mid 0] \in M_{m\times(n+1)}(\mathbb{R})$ are row equivalent. Then, there is a sequence of elementary row operations that transforms $[A \mid 0]$ to $[B \mid 0]$. By theorem 1.3.4, the solutions to $[A \mid 0]$ don't change when transforming to $[B \mid 0]$ so the solutions to $[A \mid 0]$ and $[B \mid 0]$ are the same.

Note Ax,Bx = 0 since if Ax = b where b $\neq$ 0 and Ax = c where c $\neq$ 0, then performing elementary row operations to change A to B might not change b to c. But if b = 0, then any elementary row operation will keep b as 0 since the entries in b can only be affected by other entries in b which are all 0. Thus, if also c = 0, then $[A \mid 0]$ and $[B \mid 0]$ will be row equivalent.

Theorem 1.3.6: The rref(A) is unique

Let matrix A $\in M_{m\times n}(\mathbb{R})$ be row equivalent to matrix B,C $\in M_{m\times n}(\mathbb{R})$ which are in reduced row-echelon form. Then, B = C.

Proof

Since A is row equivalent to B,C, then by theorem 1.3.5, Ax = 0 and Bx = 0 have the same solutions and Ax = 0 and Cx = 0 have the same solutions. Thus, Bx = 0 and Cx = 0 have the same solutions. The following proof will be a proof by induction.
Suppose A,B,C $\in M_{m\times 1}(\mathbb{R})$. Since B,C are in reduced row-echelon form, then B,C are either:

$$M_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since $M_1x = 0$ is only x = 0 and $M_2x = 0$ is any x $\in \mathbb{R}$ then either B,C = M_1 or B,C = M_2 since Bx = 0 and Cx = 0 has the same solutions. Thus, the base case holds true.
Suppose for some n $\in \mathbb{Z}_+$, any matrix M $\in M_{m\times n}(\mathbb{R})$ in reduced row echelon form is unique. Let A,B,C $\in M_{m\times(n+1)}(\mathbb{R})$ where A is row equivalent to B,C in reduced row echelon form. Let $A_n, B_n, C_n \in M_{m\times n}(\mathbb{R})$ be A,B,C without their (n+1)-th column. Since A is row equivalent to B,C, then A_n is row equivalent to B_n, C_n which are also in reduced row-echelon form since removing the last column of any rref is still a rref. Since $A_n \in M_{m\times n}(\mathbb{R})$ which is row equivalent to reduced row echelon matrices $B_n, C_n \in M_{m\times n}(\mathbb{R})$, then $B_n = \text{rref}(A_n) = C_n$. Thus, the first n columns of B,C are the same. Suppose the B $\neq$ C so only the (n+1)-th column can be different. Then there is a i $\in \{1,...,m\}$ where $b_{i(n+1)} \neq c_{i(n+1)}$. Let $(x_1^*, ..., x_{n+1}^*)$ be a solution.

$$b_{i1}x_1 + ... + b_{in}x_n + b_{i(n+1)}x_{n+1} = 0 \qquad c_{i1}x_1 + ... + c_{in}x_n + c_{i(n+1)}x_{n+1} = 0$$

Since $B_n = C_n$, then $b_{ij} = c_{ij}$ for i = {1,...,m} and j = {1,...,n}. Thus, $b_{i(n+1)}x_{n+1} = c_{i(n+1)}x_{n+1}$. Since $b_{i(n+1)} \neq c_{i(n+1)}$, then $x_{n+1} = 0$. Thus, x_{n+1} is a pivot variable so the (n+1)-th column of B,C have a leading 1. Thus, any other entry in the (n+1)-th column is 0. Since B and C are in reduced row-echelon form, then this pivot is right after any other pivots before it since this 1 is in the final column, but since the entries of B_n, C_n are the same, then this 1 is in the same row in C as in B. Thus, the (n+1)-th column of B and C are the same which contradicts the assumption that (n+1)-th column are different. Thus, B = C by induction.

Theorem 1.3.7: Number of solutions in a System of Equations

Let m × n system of equations be the augmented matrix A $\in M_{m\times(n+1)}(\mathbb{R})$. The system is called consistent if there is at least one solution and inconsistent if there are no solutions.

If the rref(A) contains the row [0...0 | 1], then the system has no solutions.
If there is at least one free variable, then there are infinitely many solutions and if all variables are pivots, then there is one solution.

Proof

Since a variable is either a pivot or free variable, then the rref(A) either:

- contains the row [0...0 | 1]
- doesn't contains the row [0...0 | 1] and have all pivot variables
- doesn't contains the row [0...0 | 1], but have all pivot variables

Since [0...0 | 1] implies $0 = 0x_1 + ... + 0x_n = 1$, then if rref(A) contains the row [0...0 | 1], there cannot be any solution regardless of pivot and free variables since no x = $(x_1, ..., x_n)$ will satisfy such a row. Now, suppose rref(A) doesn't contains the row [0...0 | 1].
Suppose the rref(A) have all pivot variables. Since pivot variables are fixed or depend on free variables which don't exist, then the pivot variables are all fixed and thus, unique.
Suppose the rref(A) has at least one free variable. Then at least one variable can be any real number and thus, there are infinitely many solutions.

Corollary 1.3.8: A unique solution must have as many equation as there are unknowns

Let m × n system of equations be the augmented matrix A $\in M_{m\times(n+1)}(\mathbb{R})$.

If there is a unique solution, then m $\geq$ n.

Proof

By theorem 1.3.7, a unique solution must have all pivot variables. If rref(A) has all pivots, then m $\geq$ n else there will be a column without a pivot.

Definition 1.3.9: Homogeneous & Inhomogeneous Equations

A m × n system of equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

can also be written in matrix form:

Ax = b

$$\text{where A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{R}), \quad \text{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad \text{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m$$

(a) Closed under Addition: For any $x_1, ...x_k \in \mathbb{R}^n$:

$$A(x_1 + ... + x_k) = Ax_1 + ... + Ax_k$$

(b) Closed under Scalar Multiplication For any x $\in \mathbb{R}^n$ and c $\in \mathbb{R}$:

$$A(cx) = cAx$$

For $x_1, ..., x_k \in \mathbb{R}^n$ and $c_1, ..., c_k \in \mathbb{R}$:

$$A(c_1x_1 + ... + c_kx_k)$$

$$\Leftrightarrow \begin{matrix} a_{11}(c_1x_{11} + ... + c_kx_{k1}) + a_{12}(c_1x_{12} + ... + c_kx_{k2}) + \ldots + a_{1n}(c_1x_{1n} + ... + c_kx_{kn}) \\ a_{21}(c_1x_{11} + ... + c_kx_{k1}) + a_{22}(c_1x_{12} + ... + c_kx_{k2}) + \ldots + a_{2n}(c_1x_{1n} + ... + c_kx_{kn}) \\ \ldots \\ a_{m1}(c_1x_{11} + ... + c_kx_{k1}) + a_{m2}(c_1x_{12} + ... + c_kx_{k2}) + \ldots + a_{mn}(c_1x_{1n} + ... + c_kx_{kn}) \end{matrix}$$

$$= \begin{matrix} c_1(a_{11}x_{11} + ... + a_{1n}x_{1n}) \\ c_1(a_{21}x_{11} + ... + a_{2n}x_{1n}) \\ \ldots \\ c_1(a_{m1}x_{11} + ... + a_{mn}x_{1n}) \end{matrix} + \ldots + \begin{matrix} c_k(a_{11}x_{k1} + ... + a_{1n}x_{kn}) \\ c_k(a_{21}x_{k1} + ... + a_{2n}x_{kn}) \\ \ldots \\ c_k(a_{m1}x_{k1} + ... + a_{mn}x_{kn}) \end{matrix} \Leftrightarrow c_1Ax_1 + ... + c_kAx_k$$

A Homogeneous equation is in the form:

Ax = 0 where A $\in M_{m\times n}(\mathbb{R})$, x $\in \mathbb{R}^n$, and 0 $\in \mathbb{R}^m$

A Inhomogeneous equation is in the form:

Ax = b where A $\in M_{m\times n}(\mathbb{R})$, x $\in \mathbb{R}^n$, and b $\neq 0 \in \mathbb{R}^m$

Theorem 1.3.10: Relationship between Homogeneous and Inhomogeneous

Let x_0 be a solution to Ax = b. Then any solution x_s of Ax = b:

$$x_s = x_0 + x^*$$

where x^* is a solution to Ax = 0

Proof

Let x_0 be a solution to Ax = b. Suppose x_s be a solution to Ax = b.

b $= Ax_s = A(x_0 + (x_s - x_0)) = Ax_0 + A(x_s - x_0) =$ b $+ A(x_s - x_0)$

Thus, $A(x_s - x_0) = 0$ so $x^* = x_s - x_0$ is a solution to Ax = 0.

Example

Find the solution(s), (x_1, x_2, x_3, x_4):

$x_1 + 2x_3 + 4x_4 = -8$

$x_2 - 3x_3 - x_4 = 6$

$3x_1 + 4x_2 - 6x_3 + 8x_4 = 0$

$-x_2 + 3x_3 + 4x_4 = -12$

What are the solutions if instead the equations are equal to 0?

$$\begin{bmatrix} 1 & 0 & 2 & 4 & \vert & -8 \\ 0 & 1 & -3 & -1 & \vert & 6 \\ 3 & 4 & -6 & 8 & \vert & 0 \\ 0 & -1 & 3 & 4 & \vert & -12 \end{bmatrix} \underset{\Rightarrow}{\text{Add -3(1st) to the (3rd)}} \begin{bmatrix} 1 & 0 & 2 & 4 & \vert & -8 \\ 0 & 1 & -3 & -1 & \vert & 6 \\ 0 & 4 & -12 & -4 & \vert & 24 \\ 0 & -1 & 3 & 4 & \vert & -12 \end{bmatrix}$$

$$\begin{matrix}\text{Add -4(2nd) to the (3rd)} \\ \text{Add (2nd) to the (4th)} \\ \Rightarrow\end{matrix} \begin{bmatrix} 1 & 0 & 2 & 4 & \vert & -8 \\ 0 & 1 & -3 & -1 & \vert & 6 \\ 0 & 0 & 0 & 0 & \vert & 0 \\ 0 & 0 & 0 & 3 & \vert & -6 \end{bmatrix} \begin{matrix}\text{Multiply (4th) by } \frac{1}{3} \\ \text{Add -4(4th) to (1st)} \\ \text{Add (4th) to (2nd)} \\ \Rightarrow\end{matrix} \begin{bmatrix} 1 & 0 & 2 & 0 & \vert & 0 \\ 0 & 1 & -3 & 0 & \vert & 4 \\ 0 & 0 & 0 & 0 & \vert & 0 \\ 0 & 0 & 0 & 1 & \vert & -2 \end{bmatrix}$$

Thus, the reduced row-echelon form of this matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & \vert & 0 \\ 0 & 1 & -3 & 0 & \vert & 4 \\ 0 & 0 & 0 & 1 & \vert & -2 \\ 0 & 0 & 0 & 0 & \vert & 0 \end{bmatrix}$$

The pivot variables are x_1, x_2, x_4 and the free variable is x_3 so the rank is 3.

The solutions to the system of equations are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 3x_3 + 4 \\ x_3 \\ -2 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} = x_3(-2, 3, 1, 0) + (0, 4, 0, -2)$$

Thus, $x_3(-2, 3, 1, 0)$ are the solutions when the equations equal to 0.

1.4 Linear Transformations

Definition 1.4.1: Linear Transformation

Function T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if for any x,y $\in \mathbb{R}^n$ and c $\in \mathbb{R}$:

(a) Closed under Addition: T(x+y) = T(x) + T(y)

(b) Closed under Scalar Multiplication: T(cx) = cT(x)

Note for any $x_1, ..., x_k \in \mathbb{R}^n$ and $c_1, ..., c_k \in \mathbb{R}$:

$$\begin{aligned} \mathrm{T}(c_1x_1 + ... + c_kx_k) &= \mathrm{T}(c_1x_1) + \mathrm{T}(c_2x_2 + ... + c_kx_k) \\ &= c_1\mathrm{T}(x_1) + \mathrm{T}(c_2x_2) + \mathrm{T}(c_3x_3 + ... + c_kx_k) \\ &= c_1\mathrm{T}(x_1) + c_2\mathrm{T}(x_2) + \mathrm{T}(c_3x_3) + \mathrm{T}(c_4x_4 + ... + c_kx_k) \\ &= ... = c_1\mathrm{T}(x_1) + ... + c_k\mathrm{T}(x_k) \end{aligned}$$

The standard vectors of $\mathbb{R}^n$ are $e_1, ..., e_n \in \mathbb{R}^n$ where:

$$e_i = (\underset{1}{0}, ..., \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, ..., \underset{n}{0})$$

Theorem 1.4.2: T(0) = 0

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then for $0 \in \mathbb{R}^n$:

$T(0) = 0 \in \mathbb{R}^m$

Proof

Let 0_n be the zero vector in $\mathbb{R}^n$ and 0_m be the zero vector in $\mathbb{R}^m$. Since $0(0_n) = 0_n$, then:

$T(0_n) = T(0(0_n)) = 0T(0_n) = 0_m$

Theorem 1.4.3: Every Linear transformation is a Matrix transformation

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then for any $x \in \mathbb{R}^n$:

$T(x) = Ax$

with $A \in M_{m\times n}(\mathbb{R})$ where $A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$ is called the standard matrix

Proof

Since any $x \in \mathbb{R}^n$ is $x = (x_1, ..., x_n) = x_1e_1 + ... + x_ne_n$, then:

$T(x) = T(x_1e_1 + ... + x_ne_n) = x_1T(e_1) + ... + x_nT(e_n)$

$$\Leftrightarrow \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax$$

Since T: $\mathbb{R}^n \to \mathbb{R}^m$ and each $e_i \in \mathbb{R}^n$, then each $T(e_i) \in \mathbb{R}^m$. Thus, $A \in M_{m\times n}(\mathbb{R})$.

Corollary 1.4.4: Linear Transformation: Scaling

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where for any $x \in \mathbb{R}^n$, the $T(x) = cx$ for some $c \in \mathbb{R}$. Then:

$$T(x) = \begin{bmatrix} c & 0 & \dots & 0 \\ 0 & c & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c \end{bmatrix} x$$

Proof

By theorem 1.4.3, $T(x) = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} x$. Since $T(e_1) = ce_1$, then:

$$\begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} ce_1 & ce_2 & \dots & ce_n \end{bmatrix} = \begin{bmatrix} c & 0 & \dots & 0 \\ 0 & c & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c \end{bmatrix}$$

Example

Let T: $\mathbb{R}^4 \to \mathbb{R}^4$ scales all $x \in \mathbb{R}^4$ by 2. Find T. Verify T((-1,2,1,3)) = (-2,4,2,6)

$$T(x) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} x \qquad T(\begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}) = \begin{bmatrix} 2*(-1)+0*2+0*1+0*3 \\ 0*(-1)+2*2+0*1+0*3 \\ 0*(-1)+0*2+2*1+0*3 \\ 0*(-1)+0*2+0*1+2*3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 2 \\ 6 \end{bmatrix}$$

Corollary 1.4.5: Linear Transformation: 2D Rotation

Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose for any x $\in \mathbb{R}^n$, the T rotates x counterclockwise by an angle of θ. Then:

$$\mathrm{T(x)} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathrm{x}$$

Proof

By theorem 1.4.3, T(x) $= \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$x. Since T rotates any x $\mathbb{R}^2$ counterclockwise by an angle of θ, then:

$$T(e_1) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Example

Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ rotate all x $\in \mathbb{R}^2$ by $\frac{\pi}{6}$ radians $= 30°$ degree counterclockwise. Find T. Find $\cos(75°)$ and $\sin(75°)$.

$$\mathrm{T(x)} = \begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} \mathrm{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \mathrm{x}$$

Note $75° = 45° + 30°$ so apply T on the unit vector at 45° which is $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$$\mathrm{T}(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}) = \begin{bmatrix} \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + -\frac{1}{2}\frac{\sqrt{2}}{2} \\ \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}-\sqrt{2}}{4} \\ \frac{\sqrt{6}+\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} \cos(75°) \\ \sin(75°) \end{bmatrix}$$

1.5 Invertibility

Definition 1.5.1: Product of Linear transformations

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ and S: $\mathbb{R}^m \to \mathbb{R}^t$ be linear transformations. Then for any x $\in \mathbb{R}^n$:

$$\mathrm{(ST)x} = \mathrm{S(Tx)}$$

is a linear transformation where ST: $\mathbb{R}^n \to \mathbb{R}^t$

Let $x_1, x_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$.

$$\begin{aligned} \mathrm{(ST)}(c_1x_1 + c_2x_2) &= \mathrm{S}(\mathrm{T}(c_1x_1 + c_2x_2)) = \mathrm{S}(c_1\mathrm{T}(x_1) + c_2\mathrm{T}(x_2)) \\ &= c_1\mathrm{S}(\mathrm{T}(x_1)) + c_2\mathrm{S}(\mathrm{T}(x_2)) = c_1\mathrm{(ST)}(x_1) + c_2\mathrm{(ST)}(x_2) \end{aligned}$$

Theorem 1.5.2: Product of Linear transformations are Matrix transformations

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ with T(x) = Ax and S: $\mathbb{R}^m \to \mathbb{R}^t$ with S(y) = By be linear transformations where A $= \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix}$ for A_i $\mathbb{R}^m$. Then for any x $\in \mathbb{R}^n$:

$$\mathrm{(ST)x} = \mathrm{(BA)x}$$

where BA $= \begin{bmatrix} BA_1 & BA_2 & \dots & BA_n \end{bmatrix}$

Proof

$$\mathrm{(ST)x} = \mathrm{B(Tx)} = \mathrm{B(Ax)} = \mathrm{B}(\begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix})$$

$$= \mathrm{B}(x_1A_1 + ... + x_nA_n) = x_1\mathrm{BA}_1 + ... + x_n\mathrm{BA}_n = \begin{bmatrix} BA_1 & \dots & BA_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Example

Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ rotate all x $\in \mathbb{R}^2$ by $\frac{\pi}{6}$ radians = 30° degree counterclockwise, then scale by 2. Find T.

The 30° degree counterclockwise rotation is $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and the scale by 2 transformation is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ as noted by examples under corollary 1.4.4 and 1.4.5.

$$T(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} x = \begin{bmatrix} 2\frac{\sqrt{3}}{2} + 0\frac{1}{2} & 2(-\frac{1}{2}) + 0\frac{\sqrt{3}}{2} \\ 0\frac{\sqrt{3}}{2} + 2\frac{1}{2} & 0(-\frac{1}{2}) + 2\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

Theorem 1.5.3: Properties of Matrix Products

(a) Associativity: For A $\in M_{m\times n}(\mathbb{R})$, B $\in M_{t\times m}(\mathbb{R})$, C $\in M_{s\times t}(\mathbb{R})$:

(CB)A = C(BA)

Proof

Let x $\in \mathbb{R}^n$:

$$(CB)Ax = (CB)(A_1x_1+...+A_nx_n) = CBA_1x_1+...+CBA_nx_n = CBAx = C(BA)x$$

(b) Distributivity: For A $\in M_{m\times n}(\mathbb{R})$, B $\in M_{m\times n}(\mathbb{R})$, C $\in M_{t\times m}(\mathbb{R})$:

C(A+B) = CA + CB

Proof

Let x $\in \mathbb{R}^n$:

$$\begin{aligned} C(A+B)x &= C((A_1 + B_1)x_1 + ... + (A_n + B_n)x_n) \\ &= C((A_1x_1 + ... + A_nx_n) + (B_1x_1 + ... + B_nx_n)) \\ &= (CA_1x_1 + ... + CA_nx_n) + (CB_1x_1 + ... + CB_nx_n) \\ &= CAx + CBx = (CA+CB)x \end{aligned}$$

(c) Distributivity: For A $\in M_{t\times m}(\mathbb{R})$, B $\in M_{t\times m}(\mathbb{R})$, C $\in M_{m\times n}(\mathbb{R})$:

(A+B)C = AC + BC

Proof

Let x $\in \mathbb{R}^n$:

$$\begin{aligned} (A+B)Cx &= (A+B)(C_1x_1 + ... + C_nx_n) \\ &= (A + B)C_1x_1 + ... + (A + B)C_nx_n \\ &= (AC_1x_1 + ... + AC_nx_n) + (BC_1x_1 + ... + BC_nx_n) \\ &= ACx + BCx = (AC+BC)x \end{aligned}$$

(d) Scalar Multiplication: For A $\in M_{m\times n}(\mathbb{R})$, B $\in M_{t\times m}(\mathbb{R})$ and c $\in \mathbb{R}$:

(cB)A = c(BA) = B(cA)

Proof

Let x $\in \mathbb{R}^n$:

$$\begin{aligned} (cB)Ax = (cB)(A_1x_1 + ... + A_nx_n) &= cBA_1x_1 + ... + cBA_nx_n = c(BA)x \\ &= BcA_1x_1 + ... + BcA_nx_n = B(cA)x \end{aligned}$$

Definition 1.5.4: Identity Matrix

The n × n identity matrix $I_n \in M_{n\times n}(\mathbb{R})$:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Definition 1.5.5: Elementary row operations are Matrix transformations

Elementary Row Operations as matrix are called elementary matrices.
Let A $\in M_{m \times n}(\mathbb{R})$. Then each elementary matrix B $\in M_{m \times m}(\mathbb{R})$ where:

Row Multiplication: Multiplying the i-th row by k

$$B = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & & i & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ i \\ \\ m \end{array} & \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array}$$

Row Addition: Adding the k times the j-th row to the i-th row

$$B = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & & i & & j & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ i \\ \\ j \\ \\ m \end{array} & \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array} \text{ or } \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & & j & & i & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ j \\ \\ i \\ \\ m \end{array} & \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array}$$

Row Swapping: Swapping the i-th and j-th row

$$B = \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & & i & & j & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ i \\ \\ j \\ \\ m \end{array} & \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array} \text{ or } \begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & & j & & i & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \\ j \\ \\ i \\ \\ m \end{array} & \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \end{array}$$

Then, BA is the matrix after the elementary row operation is applied to A.

Definition 1.5.6: Invertibility

Linear Transformation T: $\mathbb{R}^n \to \mathbb{R}^m$ is invertible if there exist a linear transformation S: $\mathbb{R}^m \to \mathbb{R}^n$ such that for all x $\in \mathbb{R}^n$ and y $\in \mathbb{R}^m$:

(ST)x = x (TS)y = y

Suppose T(x) = Ax where A $\in M_{m \times n}(\mathbb{R})$ and S(y) = By where B $\in M_{n \times m}(\mathbb{R})$.
Then if the property above holds true:

BAx = (ST)x = x = $I_{n \times n}$x ABy = (TS)y = y = $I_{m \times m}$x

BA = $I_{n \times n}$ AB = $I_{m \times m}$

If A is invertible, then B = A^{-1} is the inverse transformation of A.
Note if A is invertible, then A^{-1} is invertible since there is a A where $AA^{-1} = I_{m \times m}$ and $A^{-1}A = I_{n \times n}$ by the invertibility of A.
If A $\in M_{n \times n}(\mathbb{R})$, then A is called a square matrix.

Theorem 1.5.7: Only Square matrices can be Invertible

Let A $\in M_{m\times n}(\mathbb{R})$ and B $\in M_{n\times m}(\mathbb{R})$ such that n $\neq$ m.
Then, either AB $\neq I_{m\times m}$ or BA $\neq I_{n\times n}$.

Proof

Suppose n < m. Then for any x $\in \mathbb{R}^n$, by corollary 1.3.8, By = x does not have a unique y $\in \mathbb{R}^m$ so either y does not exist or there are infinitely many y.
If y does not exist, then for ABy = Ax, the y does not exist so AB $\neq I_{m\times m}$ else y = Ax. If By = x has infinitely many y, then there are y_1, y_2 where $y_1 \neq y_2$ such that By_1 = x = By_2 so AB $\neq I_{m\times m}$ else y_1 = ABy_1 = ABy_2 = y_2 contradicting $y_1 \neq y_2$. Thus, AB $\neq I_{m\times m}$.

Suppose n > m. Then for any y $\in \mathbb{R}^m$, by corollary 1.3.8, Ax = y does not have a unique x $\in \mathbb{R}^n$ so either x does not exist or there are infinitely many x.
If x does not exist, then for BAx = By, the x does not exist so BA $\neq I_{n\times n}$ else x = By. If Ax = y has infinitely many x, then there are x_1, x_2 where $x_1 \neq x_2$ such that Ax_1 = y = Ax_2 so BA $\neq I_{n\times n}$ else x_1 = BAx_1 = BAx_2 = x_2 contradicting $x_1 \neq x_2$. Thus, BA $\neq I_{n\times n}$.

Theorem 1.5.8: Determining Invertibility

A $\in M_{n\times n}(\mathbb{R})$ is invertible if and only if rref(A) = $I_{n\times n}$

Proof

Let A be invertible. Suppose rref(A) $\neq I_{n\times n}$. Then there is at least one free variable.
Then for any y $\in \mathbb{R}^n$, by theorem 1.3.7, Ax = y has infinitely many x $\in \mathbb{R}^n$ since x = $I_{n\times n}$x = $A^{-1}A$x = $A^{-1}y$ must exist due to the existence of A^{-1} by the invertibility of A. But then, there are x_1, x_2 where $x_1 \neq x_2$ such that Ax_1 = y = Ax_2 so $AA^{-1} \neq I_{n\times n}$ else $x_1 = A^{-1}Ax_1$ = $A^{-1}Ax_2 = x_2$ contradicting the invertibility of A. Thus, rref(A) = $I_{n\times n}$.

Let rref(A) = $I_{n\times n}$. Take the augmented matrix $[A \mid I_{n\times n}]$.
Since rref(A) = $I_{n\times n}$, then there is a sequence of elementary row operations that transforms A into $I_{n\times n}$ and thus, transform $[A \mid I_{n\times n}] = [I_{n\times n} \mid B]$ for some B $\in M_{n\times n}(\mathbb{R})$. Note

$[A \mid I_{n\times n}]$ $\Leftrightarrow$ Ax = $I_{n\times n}$y $[I_{n\times n} \mid B]$ $\Leftrightarrow$ $I_{n\times n}$x = By
(BA)x = B(Ax) = By = x (AB)y = A(By) = Ax = y $\Rightarrow$ A^{-1} = B

Corollary 1.5.9: Invertible n $\times$ n matrices have Rank n

A $\in M_{n\times n}(\mathbb{R})$ is invertible if and only if rank(A) = n

Proof

By theorem 1.5.8, A is invertible $\Leftrightarrow$ rref(A) = $I_{n\times n}$ $\Leftrightarrow$ A has n pivots (i.e. rank(A) = n).

Corollary 1.5.10: Invertible Matrices have unique solutions

A $\in M_{n\times n}(\mathbb{R})$ is invertible if and only if for any y $\in \mathbb{R}^n$, there is a unique x $\in \mathbb{R}^n$ where:
Ax = y
Thus, A is invertible if and only if Ax = 0 has the trivial solution, x = 0.

Proof

Suppose A is invertible. Then by corollary 1.5.9, rank(A) = n so A has n pivots. Then for augmented matrix, $[A \mid y]$, by theorem 1.3.7, there is one unique solution.

Suppose for any y $\in \mathbb{R}^n$, there is a unique x $\in \mathbb{R}^n$ where Ax = y. Then by theorem 1.3.7, A has n pivots so rank(A) = n. Then by corollary 1.5.9, A is invertible.

Suppose A is invertible. Since A0 = 0, then the only solution to Ax = 0 is x = 0.

Suppose Ax = 0 has only x = 0. Then, rref(A) has n pivots so rank(A) = n. Thus, by corollary 1.5.9, A is invertible.

Theorem 1.5.11: AB = $I_{n\times n}$ implies BA = $I_{n\times n}$

For A,B $\in M_{n\times n}(\mathbb{R})$, let AB = $I_{n\times n}$. Then A,B are invertible where:

A^{-1} = B $\qquad B^{-1}$ = A

Proof

Let x $\in \mathbb{R}^n$ be such that Bx = 0. Then, x = $I_{n\times n}$x = ABx = B0 = 0.
Then by corollary 1.5.10, B is invertible so B^{-1} exist where $B^{-1}B = I_{n\times n}$ and $BB^{-1} = I_{n\times n}$.

A = $AI_{n\times n}$ = A$I_{n\times n}$ = ABB^{-1} = $I_{n\times n}B^{-1} = B^{-1}$

Since B is invertible, then A = B^{-1} is invertible so $A^{-1}A = I_{n\times n}$ and $AA^{-1} = I_{n\times n}$.

$A^{-1} = A^{-1}I_{n\times n} = A^{-1}$AB = $I_{n\times n}$B = B

Theorem 1.5.12: Invertibility Equivalences

Let A $\in M_{n\times n}(\mathbb{R})$. Then the following are equivalent:

(a) A is invertible
(b) rref(A) = $I_{n\times n}$
(c) rank(A) = n
(d) For any y $\in \mathbb{R}^n$, then Ax = y has a unique solution x
(e) Ax = 0 has only the trivial soluton x = 0

Proof

$$(a) \Leftrightarrow \begin{cases} (b) & \text{theorem 1.5.8} \\ (c) & \text{corollary 1.5.9} \\ (d) & \text{corollary 1.5.10} \\ (e) & \text{corollary 1.5.10} \end{cases}$$

Theorem 1.5.13: Product of Invertible matrices is Invertible

Let A,B $\in M_{n\times n}(\mathbb{R})$ be invertible. Then, AB is invertible where:

$(AB)^{-1} = B^{-1}A^{-1}$

Proof

Since A and B are invertible, then there are $A^{-1}, B^{-1} \in M_{n\times n}(\mathbb{R})$ such that:

$A^{-1}A = I_{n\times n} \qquad AA^{-1} = I_{n\times n} \qquad B^{-1}B = I_{n\times n} \qquad BB^{-1} = I_{n\times n}$

Then AB is invertible since $(AB)^{-1}$ exist as $(AB)^{-1} = B^{-1}A^{-1}$:

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_{n\times n}B = B^{-1}B = I_{n\times n}$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_{n\times n}A^{-1} = AA^{-1} = I_{n\times n}$

Example

Let T: $\mathbb{R}^4 \to \mathbb{R}^4$ be T(x) = Ax where A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}$. Find $T^{-1}(1, 1, -1, -6)$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 2 & 4 & 7 & 11 & 0 & 1 & 0 & 0 \\ 3 & 7 & 14 & 25 & 0 & 0 & 1 & 0 \\ 4 & 11 & 25 & 50 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 13 & -3 & 0 & 1 & 0 \\ 0 & 3 & 13 & 34 & -4 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -7 & -22 & 7 & 0 & -2 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 13 & -3 & 0 & 1 & 0 \\ 0 & 0 & -2 & -5 & 5 & 0 & -3 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -7 & 7 & -2 & 0 \\ 0 & 0 & 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 7 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -6 & 9 & -5 & 1 \\ 0 & 1 & 0 & 0 & 9 & -1 & -5 & 2 \\ 0 & 0 & 1 & 0 & -5 & -5 & 9 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 & -3 & 1 \end{array}\right] = A^{-1} \Rightarrow \text{A is invertible}$$

$T^{-1}(1, 1, -1, -6) = A^{-1}(1, 1, -1, -6)$ = (2,1,-1,0)

2 Vector Space

2.1 Span & Independence

Definition 2.1.1: Vector Space

V is a vector space over $\mathbb{K}$ if for any u,v,w $\in$ V and a,b $\in$ $\mathbb{K}$:

(a) Commutativity

u + v = v + u

(b) Addititive Associativity

(u + v) + w = u + (v + w)

(c) Additive Identity

There exists a unique 0_v $\in$ V such that for all v $\in$ V:

0_v + v = v

(d) Additive Inverse

For any v, there exists a unique -v $\in$ V such that:

v + (-v) = 0_v

(e) Distributivity

a(u+v) = au + av   (a+b)u = au + bu

(f) Multiplicative Associativity

$a(bu) = (ab)u$

(g) Multiplicative Identity

1u = u

Theorem 2.1.2: 0v = 0_v, $a0_v$ = 0_v, (-1)v = -v

Let V be a vector space where v $\in$ V. Then:

0v = 0_v $a0_v = 0_v$ (-1)v = -v

Proof

Since 0v = (0+0)v = 0v + 0v, then:

0_v = 0v + (-0v) = 0v + 0v + (-0v) = 0v + 0_v = 0v

Since a0_v = a(0_v+0_v) = a0_v + a0_v, then:

0_v = a0_v + (-a0_v) = a0_v + a0_v + (-a0_v) = a0_v + 0_v = a0_v

Since 0_v = 0v = (-1+1)v = (-1)v + 1v = (-1)v + v, then:

-v = 0_v + (-v) = (-1)v + v + (-v) = (-1)v + 0_v = (-1)v

Definition 2.1.3: Linear Combination, Span, and Independence

x $\in \mathbb{R}^n$ is a linear combination of $v_1, ..., v_k \in \mathbb{R}^n$ if there are $c_1, ..., c_k \in \mathbb{R}$ such that:

x = $c_1v_1 + ... + c_kv_k$

The span of $v_1, ..., v_k \in \mathbb{R}^n$ is the set of all linear combinations of $v_1, ..., v_k$.

Also, $v_1, ..., v_k \in \mathbb{R}^n$ are linearly independent if none of the v_i are linear combinations of the other v_i's. Else, $v_1, ..., v_k$ are linearly dependent.

Theorem 2.1.4: Remove Linearly dependent vectors to get Linear independence

Let u $\in \mathbb{R}^n$ be a linear combination of $v_1, ..., v_k \in \mathbb{R}^n$. Then:

span($v_1, ..., v_k$) = span($v_1, ..., v_k$,u)

Thus, by removing vectors that are linear combinations (i.e. linearly dependent vectors), then the resulting set of vectors will be linearly independent and the span is unaffected.

Proof

Since u is a linear combination of $v_1, ..., v_k$, then there are $c_1, ..., c_k \in \mathbb{R}$ such that:

u $= c_1v_1 + ... + c_kv_k$

Let u_1 be a linear combination of $v_1, ..., v_k$,u. Then there are $a_1, ..., a_k, a \in \mathbb{R}$ such that:

$u_1 = a_1v_1+...+a_kv_k+au = a_1v_1+...+a_kv_k+a(c_1v_1+...+c_kv_k) = (a_1+ac_1)v_1+...+(a_k+ac_k)v_k$

Thus, $u_1 \in$ span($v_1, ..., v_k$) so span($v_1, ..., v_k$,u) $\subset$ span($v_1, ..., v_k$).

Let u_2 be a linear combination of $v_1, ..., v_k$. Then there are $b_1, ..., b_k, \in \mathbb{R}$ such that:

$u_2 = b_1v_1 + ... + b_kv_k = [(b_1 - c_1)v_1 + ... + (b_k - c_k)v_k] + [c_1v_1 + ... + c_kv_k]$

$= (b_1 - c_1)v_1 + ... + (b_k - c_k)v_k + u$

Thus, $u_2 \in$ span($v_1, ..., v_k$,u) so span($v_1, ..., v_k$) $\subset$ span($v_1, ..., v_k$,u).

Theorem 2.1.5: Condition for Linear independence

$v_1, ..., v_k \in \mathbb{R}^n$ are linearly independent if and only if the only solution $(c_1, ..., c_k)$:

$c_1v_1 + ... + c_kv_k = 0$

is $c_1 = ... = c_k = 0$.

Note regardless of $v_1, ..., v_k$, any $c_1v_1 + ... + c_kv_k = 0$ holds true when $(c_1, ..., c_k) = 0 \in \mathbb{R}^k$. $(c_1, ..., c_k) = 0$ is called the trivial solution. Any $(c_1, ..., c_k) \neq 0$ is a nontrivial solution.

Proof

Suppose $v_1, ..., v_k \in \mathbb{R}^n$ are linearly independent.

Then for any v_i, there are no $c_1, ..., c_{i-1}, c_{i+1}, ..., c_k \in \mathbb{R}$ such that:

$v_i = c_1v_1 + ... + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + ... + c_kv_k$

Thus, there are no $(c_1, ..., c_{i-1}, c_i = -1, c_{i+1}, ..., c_k)$ such that:

$0 = c_1v_1 + ... + c_{i-1}v_{i-1} + c_iv_i + c_{i+1}v_{i+1} + ... + c_kv_k$

The statement holds true if the equation was multiplied by any non-zero number. Thus, any $(c_1, ..., c_k)$ where at least one c_i is not 0 is not a solution. Since $(c_1, ..., c_k) = 0$ is a solution for $c_1v_1 + ... + c_kv_k = 0$, then for linearly independent $v_1, ..., v_k$, then $(c_1, ..., c_k) = 0$.

Suppose the solution, $(c_1, ..., c_k)$, to $c_1v_1 + ... + c_kv_k = 0$ is only $(c_1, ..., c_k) = 0$.

Suppose there is a linearly dependent vector, v_i. Then there are $a_1, ..., a_{i-1}, a_{i+1}, ..., a_k$ where:

$v_i = a_1v_1 + ... + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + ... + a_kv_k$

$0 = a_1v_1 + ... + a_{i-1}v_{i-1} + -v_i + a_{i+1}v_{i+1} + ... + a_kv_k$

Thus, $(a_1, ..., a_{i-1}, -1, a_{i+1}, ..., a_k)$ is a solution to $c_1v_1 + ... + c_kv_k = 0$ contradicting that $(c_1, ..., c_k) = 0$. Thus, there are no linearly dependent vectors.

Theorem 2.1.6: Extending the Span of Linearly independent vectors

Let $v_1, ..., v_k \in \mathbb{R}^n$ be linearly independent. If v $\in \mathbb{R}^n$ is not in the span($v_1, ..., v_k$), then $v_1, ..., v_k, v$ are linearly independent.

Proof

Let $c_1, ..., c_k, c \in \mathbb{R}$ be such that $c_1v_1 + ... + c_kv_k + cv = 0$. Suppose c $\neq$ 0. Then:

$c_1v_1 + ... + c_kv_k =$ -cv $\quad \Rightarrow \quad$ v $= \frac{-c_1}{c}v_1 + ... + \frac{-c_k}{c}v_k$

Then, v is a linear combination of $v_1, ..., v_k$ and thus, v is in the span($v_1, ..., v_k$) which is a contradiction. Thus, c = 0. Then:

$0 = c_1v_1 + ... + c_kv_k + cv = c_1v_1 + ... + c_kv_k$

Since $v_1, ..., v_k$ are linearly independent, then each $c_1 = ... = c_k = 0$. Thus, $(c_1, ..., c_k, c) = 0$ so $v_1, ..., v_k, v$ are linearly independent.

2.2 Subspaces: Image & Kernel

Definition 2.2.1: Subspaces

V $\subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if:

(a) Zero Vector Existence
$0 \in$ V

(b) Closed under Addition: If $v_1, v_2 \in$ V, then:
$v_1 + v_2 \in$ V

(c) Closed under Scalar Multiplication: If v $\in$ V and c $\in \mathbb{R}$, then:
cv $\in$ V

Theorem 2.2.2: Union of Subspaces's condition for Subspace

Let U,V $\subset \mathbb{R}^n$ be subspaces. Then, U $\cup$ V is a subspace if and only if U $\subset$ V or V $\subset$ U.

Proof

Suppose U $\cup$ V is a subspace. Suppose U $\not\subset$ V and V $\not\subset$ U. Then there is a u $\in$ U where u $\notin$ V and a v $\in$ V where v $\notin$ U. Thus, u,v $\in$ U $\cup$ V, but u+v $\notin$ U since v $\notin$ U and u+v $\notin$ V since u $\notin$ V. Thus, u+v $\notin$ U $\cup$ V contradicting V is a subspace. Thus, U $\subset$ V or V $\subset$ U.

If U $\subset$ V, then U $\cup$ V = V is a subspace. If V $\subset$ U, then U $\cup$ V = U is a subspace.

Theorem 2.2.3: Intersection of Subspaces is a Subspace

Let U,V $\subset \mathbb{R}^n$ be subspaces. Then, U $\cap$ V is a subspace.

Proof

Let x,y $\in$ U $\cap$ V and a,b $\in \mathbb{R}$. Then x,y $\in$ U,V. Since U and V are subspaces, then ax+by $\in$ U,V. Thus, ax+by $\in$ U $\cap$ V so U $\cap$ V is a subspace.

Definition 2.2.4: Image and Kernel

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation T(x) = Ax.
The image of T is the set of all Ax $\in \mathbb{R}^m$ where x $\in \mathbb{R}^n$:
im(T) = im(A) = $\{$ A(x) | x $\in \mathbb{R}^n$ $\}$
The kernel of T is the set of all x $\in \mathbb{R}^n$ such that Ax = 0:
ker(T) = ker(A) = $\{$ x | Ax = 0 $\}$

Theorem 2.2.5: Im(A) is a Subspace that spans the columns of A

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation T(x) = Ax.
Then, im(A) is a subspace and im(A) = span(A).

Proof

Since Ax = $A_1x_1 + ... + A_nx_n$ for x = $(x_1, ..., x_n) \in \mathbb{R}^n$ where $A_1, ..., A_n$ are the columns of A:
im(A) = $\{$ Ax | x $\in \mathbb{R}^n$ $\}$ = $\{ x_1A_1 + ... + x_nA_n \mid x_1, ..., x_n \in \mathbb{R} \}$ = span$(A_1, ..., A_n)$
Since A0 = 0, then 0 $\in$ im(A). Let u,v $\in$ im(A) and a,b $\in \mathbb{R}$.
Then there are $a_1, ..., a_n \in \mathbb{R}$ and $b_1, ..., b_n \in \mathbb{R}$ such that:
u = $a_1A_1 + ... + a_nA_n$ v = $b_1A_1 + ... + b_nA_n$
Thus, au+bv = $(aa_1 + bb_1)A_1 + ... + (aa_n + bb_n)A_n \in$ span$(A_1, ..., A_n)$ = im(A).

Theorem 2.2.6: ker(A) is a Subspace

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation T(x) = Ax. Then, ker(A) is a subspace.

Proof

Since A0 = 0, then 0 $\in$ ker(A). Let $x_1, x_2 \in$ ker(A) and a,b $\in \mathbb{R}$ so Ax_1 = Ax_2 = 0.
Then, A$(ax_1 + bx_2)$ = aA(x_1) + bA(x_2) = a0 + b0 = 0 so $ax_1 + bx_2 \in$ ker(A).

Theorem 2.2.7: Relationship between the Kernel and Linear independence

Let T: $\mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation T(x) = Ax where A = $\begin{bmatrix} A_1 & ... & A_m \end{bmatrix}$.
Then, $A_1, ..., A_m$ are linearly independent if and only if ker(A) = {0}.

Proof

Note Ax = $\begin{bmatrix} A_1 & ... & A_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1A_1 + ... + x_mA_m = 0$. Let $A_1, ..., A_m$ be linearly independent. Then, the only solution to $x_1A_1 + ... + x_mA_m = 0$ is $(x_1, ..., x_n) = 0$ so ker(A) = {0}.

Suppose ker(A) = {0}. Then, the only solution to $x_1A_1 + ... + x_mA_m$ = Ax = 0 is x = 0. By theorem 2.1.5, $A_1, ..., A_m$ are linearly independent.

2.3 Basis & Dimension

Definition 2.3.1: Basis & Dimension

Let set V $\subset \mathbb{R}^n$ be subspace of $\mathbb{R}^n$. If $v_1, ..., v_k \in \mathbb{R}^n$ are linearly independent and span V (i.e. span$(v_1, ..., v_k)$ = V), then $v_1, ..., v_k$ form a basis for V.
The dimension of V, dim(V), is the number of vectors in a basis of V.
Since $e_1, ..., e_n$, are linearly independent and any x $\in \mathbb{R}^n$, is x = $x_1e_1 + ... + x_ne_n$ so span$(e_1, ..., e_n) = \mathbb{R}^n$, then dim$(\mathbb{R}^n)$ = n. $e_1, ..., e_n$ are called the standard basis vectors.

Theorem 2.3.2: # Linearly independent vectors in V ≤ # vectors that span V

Let $v_1, ..., v_m \in$ V be linearly independent and $u_1, ..., u_k \in$ V span V, then m ≤ k

Proof

Let A = $\begin{bmatrix} v_1 & ... & v_m \end{bmatrix} \in M_{n\times m}(\mathbb{R})$ and B = $\begin{bmatrix} u_1 & ... & u_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$. Since im(B) = span(B) = V, then for $v_1, ..., v_m \in$ V, there are $c_1, ..., c_m \in \mathbb{R}^k$ such that $v_i = Bc_i$:

A = $\begin{bmatrix} v_1 & ... & v_m \end{bmatrix} = \begin{bmatrix} Bc_1 & ... & Bc_m \end{bmatrix}$ = B$\begin{bmatrix} c_1 & ... & c_m \end{bmatrix}$ = BC where C $\in M_{k\times m}(\mathbb{R})$

Thus, if Cx = 0, then Ax = BCx = B0 = 0 so ker(C) ⊂ ker(A). Since $v_1, ..., v_m \in$ V be linearly independent, then by theorem 2.2.7, ker(A) = {0} so ker(C) = {0}. Since Cx = 0 has a unique solution x = 0, then by corollary 1.3.8, k ≥ m.

Corollary 2.3.3: The Dimension is unique

Suppose $v_1, ..., v_m \in$ V and $u_1, ..., u_k \in$ V are bases for V, then dim(V) = m = k

Proof

Since $v_1, ..., v_m$ and $u_1, ..., u_k$ are bases, then $v_1, ..., v_m$ and $u_1, ..., u_k$ are linearly independent and span V. By theorem 2.3.2, m ≤ k and k ≤ m so m = k.

Theorem 2.3.4: Linear combinations of a Basis are unique

Let $v_1, ..., v_k \in \mathbb{R}^n$ form a basis for V $\subset \mathbb{R}^n$.
Then for any v ∈ V, there are unique $(c_1, ..., c_k)$ such that:

v = $c_1v_1 + ... + c_kv_k$

Proof

Since $v_1, ..., v_k$ form a basis for V, then V = span$(v_1, ..., v_k)$. Then for any v ∈ V, then v ∈ span$(v_1, ..., v_k)$. Thus, there are $c_1, ..., c_k \in \mathbb{R}$ such that v = $c_1v_1 + ... + c_kv_k$.
Let $a_1, ..., a_k \in \mathbb{R}$ such that v = $a_1v_1 + ... + a_kv_k$. Then:

$0 = (c_1 - a_1)v_1 + ... + (c_k - a_k)v_k$

Since $v_1, ..., v_k$ form a basis for V, then $v_1, ..., v_k$ are linearly independent. Thus, by theorem 2.1.5, $c_i - a_i = 0$ for i = {1,...,k} so $c_i = a_i$ for i = {1,...,k}. Thus, $(c_1, ..., c_k)$ must be unique.

Theorem 2.3.5: Connection between Span, Linear independence, and Basis

Let V $\subset \mathbb{R}^n$ where dim(V) = m. Then, m $\leq$ n and:

(a) If $u_1, ..., u_k \in$ V are linearly independent, then k $\leq$ m

(b) For $u_1, ..., u_k \in$ V, if span$(u_1, ..., u_k)$ = V, then k $\geq$ m

(c) $u_1, ..., u_m \in$ V are linearly independent if and only if span$(u_1, ..., u_m)$ = V

Proof

Since dim(V) = m, then there are $v_1, ..., v_m \in \mathbb{R}^n$ that are linearly independent and span V. Then for any v $\in$ V, there are $c_1, ..., c_m$ where:

$$\mathrm{v} = c_1v_1 + ... + c_mv_m \quad \Leftrightarrow \quad \begin{bmatrix} v_1 & ... & v_m \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \mathrm{v} \quad \Leftrightarrow \quad \mathrm{Ac} = \mathrm{v} \qquad \text{where A} \in M_{n\times m}(\mathbb{R})$$

By theorem 2.3.4, c is unique. Since A $\in M_{n\times m}(\mathbb{R})$, then by corollary 1.3.8, n $\geq$ m.

Let $u_1, ..., u_k$ be linearly independent. Since $v_1, ..., v_m$ span V, then by theorem 2.3.2, k $\leq$ m.

Let $u_1, ..., u_k$ span V. Since $v_1, ..., v_m$ are linearly independent, then by theorem 2.3.2, m $\leq$ k

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Suppose $u_1, ..., u_m \in$ V are linearly independent. Let v $\in$ V. If v $\notin$ span$(u_1, ..., u_m)$, then by theorem 2.1.6, $u_1, ..., u_m$,v are linearly independent contradicting that m+1 = dim(V) = m. Thus, any v $\in$ V is v $\in$ span$(u_1, ..., u_m)$ so V $\subset$ span$(u_1, ..., u_m)$. Since span$(u_1, ..., u_m) \subset$ V, then span$(u_1, ..., u_m)$ = V.

Suppose span$(u_1, ..., u_m)$ = V. Suppose there are $c_1, ..., c_m$ where not all $c_i = 0$ such that:

$c_1u_1 + ... + c_mu_m = 0$

If there is one $c_i \neq 0$, then $c_iu_i = c_1u_1 + ... + c_mu_m = 0$ implies $u_i = 0$ which is a dependent vector. Thus, there are at least two $c_i \neq 0$. Then, $u_i = \frac{-c_1}{c_i}u_1 + ... + \frac{-c_{i-1}}{c_i}u_{i-1} + \frac{-c_{i+1}}{c_i}u_{i+1} + ... + \frac{-c_m}{c_i}u_m$ where at least one $c_{j\neq i} \neq 0$ so u_i is a dependent vector. So there are less than m independent vectors contradicting that m = dim(V) < m. Thus, all $c_i = 0$ so $u_1, ..., u_m$ are independent.

Theorem 2.3.6: Determining Basis for Im(A) using rref

Let A = $\begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$ where each $v_i \in \mathbb{R}^n$.

Then the v_i that are linearly independent are the columns in rref(A) which contain pivots.

Thus, such v_i form a basis for im(A).

Proof

By theorem 2.2.5, im(A) = span(A) = span$(v_1, ..., v_k)$. Suppose the i-th column of rref($[A \mid 0]$) does not contain a pivot. Then, Ax = 0 where x $\in \mathbb{R}^k$ has a free variable at x_i so ker(A) has nonzero solutions. Thus, by theorem 2.2.7, $v_1, ..., v_k$ are linearly dependent. But, if all columns without pivots are removed are removed from rref(A) to make B, then Bx = 0 has only pivots and thus, by theorem 1.3.7, there is a unique solution and since B0 = 0, then ker(B) = {0} and thus, the columns with pivots are linearly independent.

If v_i are linearly dependent, then the sequence of elementary row operations to transform A into rref(A) transform the entries of the i-th column into 0 except possibly the first entry and thus, the i-th column does not have a pivot. By theorem 2.1.4, removing linearly dependent vectors will not change the span(A) = span$(v_1, ..., v_k)$ = im(A) so the resulting vectors with pivots will be linearly independent and span im(A) and thus, form a basis for im(A).

Corollary 2.3.7: dim(im(A)) = rank(A)

For any A $\in M_{m\times n}(\mathbb{R})$:

dim(im(A)) = rank(A)

Proof

Let A = $\begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$ where $A_i \in \mathbb{R}^m$. By theorem 2.3.6, the A_i that form a basis for im(A) are columns that contain pivots so dim(im(A)), the number of vectors in a basis for im(A) is the same as the number of pivots in rref(A), rank(A).

Theorem 2.3.8: Rank-Nullity Theorem

For any A $\in M_{m\times n}(\mathbb{R})$, the dim(ker(A)) is called the nullity of A where:

dim(im(A)) + dim(im(A)) = n

Proof

Note the number of free variables + the number of pivot variables = n.
By corollary 2.3.7, the number of pivot variables, rank(A) = dim(im(A)).
If the i-th column doesn't have a pivot, then the solutions to Ax = 0 are linear combinations of a vector v_i with 1 in the i-th row and 0 in any j-th row where the j-th column doesn't have a pivot so each v_i is linearly independent and span ker(A). Thus, the number of free variables is equal to the number of v_i, dim(ker(A)), so dim(im(A)) + dim(im(A)) = n.

2.4 Injectivity & Surjectivity

Definition 2.4.1: Injectivity and Surjectivity

Let T: V → W be a linear transformation.

T is injective if for any w ∈ T(V) = im(T), there is a unique v ∈ V such that T(v) = w

T is surjective if for any w ∈ W, there is a v ∈ V such that T(v) = w

Theorem 2.4.2: Connection between Invertibility, Injectivity, and Surjectivity

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where T(x) = Ax.

Then, T is invertible if and only if T is injective and surjective.

Proof

Suppose T is invertible. Let y $\in \mathbb{R}^n$. Then, by theorem 1.5.12, there is a unique x $\in \mathbb{R}^n$ such that Ax = T(x) = y. Thus, T is injective and surjective.

Suppose T is injective and surjective. Since T is surjective, then for any y $\in \mathbb{R}^n$, there is a x $\in \mathbb{R}^n$ such that Ax = T(x) = y. Since T is injective, then x is unique. Then, by theorem 1.5.12, T is invertible.

Theorem 2.4.3: Connection between Invertibility, Span, and Linear independence

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation T(x) = Ax where A = $\begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}$.

Then, T is invertible if and only if $A_1, ..., A_n$ are linearly independent and span $\mathbb{R}^n$.

Proof

Suppose T is invertible. By theorem 1.5.12, for any y $\in \mathbb{R}^n$, there is a unique x $\in \mathbb{R}^n$ where Ax = y. Thus, span$(A_1, ..., A_n)$ = im(A). Since dim$(\mathbb{R}^n)$ = n, then $A_1, ..., A_n$ span $\mathbb{R}^n$, then by theorem 2.3.5, $A_1, ..., A_n$ are linearly independent.

Theorem 2.4.4: Injectivity ⇔ Surjectivity

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where T(x) = Ax where A = $\begin{bmatrix} A_1 & ... & A_n \end{bmatrix}$. Then, T is surjective if and only if T is injective.

Proof

Suppose T is surjective. Then for any y $\in \mathbb{R}^n$, there is a x $\in \mathbb{R}^n$ where Ax = T(x) = y. Thus, span($A_1, ..., A_n$) = im(A) = $\mathbb{R}^n$. Since dim($\mathbb{R}^n$) = n, then by theorem 2.3.5, $A_1, ..., A_n$ are linearly independent. By theorem 2.4.3, T is invertible, then by theorem 2.4.2, T is injective.

Suppose T is injective. Then for any y $\in$ im(A), there is a unique x $\in \mathbb{R}^n$ where Ax = T(x) = y. Suppose there is a linearly dependent A_i. Then there are $c_1, ..., c_{i-1}, c_{i+1}, c_n$ where at least one $c_i \neq 0$ such that $A_i = c_1A_1 + ... + c_{i-1}A_{i-1} + c_{i+1}A_{i+1} + ... + c_nA_n$. Then:

$$\begin{aligned} \text{Ax} &= x_1A_1 + ... + x_nA_n \\ &= (x_1 + x_ic_1)A_1 + ... + (x_{i-1} + x_ic_{i-1})A_{i-1} + (x_{i+1} + x_ic_{i+1})A_{i+1} + ... + (x_n + x_ic_n)A_n \end{aligned}$$

If $x_i \neq 0$, then x = $(x_1, ..., x_n)$ is not the only solution to y = Ax contradicting that x is unique. Thus, $x_i = 0$. Similarily, if any other A_j is linearly dependent, then $x_j = 0$. Thus, Ax = $x_1A_1 + ... + x_nA_n = x_{i_1}A_{i_1} + ... + x_{i_k}A_{i_k}$ where $A_{i_1}, ..., A_{i_k}$ are the A_i that are linearly independent. Thus, for Ax = 0, then each $x_{i_1} = 0$. Since then all $x_i = 0$, then no A_i is linearly dependent so $A_1, ..., A_n$ are linearly independent. By theorem 2.4.3, T is invertible, then by theorem 2.4.2, T is surjective.

Theorem 2.4.5: Invertibility Equivalences Extended

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where T(x) = Ax where A = $\begin{bmatrix} A_1 & ... & A_n \end{bmatrix}$. Then the following are equivalent:

(a) A is invertible

(b) rref(A) = $I_{n\times n}$

(c) rank(A) = n

(d) For any y $\in \mathbb{R}^n$, then Ax = y has a unique solution x

(e) Ax = 0 has only the trivial soluton x = 0 (i.e ker(A) = {0})

(f) $A_1, ..., A_n$ are linearly independent

(g) im(A) = span($A_1, ..., A_n$) = $\mathbb{R}^n$

(h) T is injective

(i) T is surjective

Proof

$$(a) \Leftrightarrow \begin{cases} (b),(c),(d),(e) & \text{theorem 1.5.12} \\ (f) & \text{theorem 2.4.3 and 2.3.5.} \\ & \text{(a)} \Rightarrow \text{(f) only needs 2.4.3, but (a)} \Leftarrow \text{(f) needs 2.4.3, 2.3.5} \\ (g) & \text{theorem 2.4.3 and 2.3.5} \\ & \text{(a)} \Rightarrow \text{(g) only needs 2.4.3, but (a)} \Leftarrow \text{(g) needs 2.4.3, 2.3.5} \\ (h) & \text{theorem 2.4.2 and 2.4.4} \\ & \text{(a)} \Rightarrow \text{(h) only needs 2.4.2, but (a)} \Leftarrow \text{(h) needs 2.4.2, 2.4.4} \\ (i) & \text{theorem 2.4.2 and 2.4.4} \\ & \text{(a)} \Rightarrow \text{(i) only needs 2.4.2, but (a)} \Leftarrow \text{(i) needs 2.4.2, 2.4.4} \end{cases}$$

Example

Let T: $\mathbb{R}^4 \to \mathbb{R}^4$ be T(x) = Ax = $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$x.

Find if T is invertible, im(T), dim(im(T)), ker(T), and dim(ker(T)).

$$[A \mid I_{4\times4}] = \begin{bmatrix} 1 & 0 & 2 & 4 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 & | & 0 & 1 & 0 & 0 \\ 3 & 4 & -6 & 8 & | & 0 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & | & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 & | & 0 & 1 & 0 & 0 \\ 0 & 4 & -12 & -4 & | & -3 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 4 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & | & -3 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & | & 0 & 1 & 0 & 1 \end{bmatrix} \Rightarrow \text{rref(A)} \neq I_{4\times4} \Rightarrow \text{A is not invertible}$$

Since the 1st and 2nd column have pivots, then (1,0,3,0) and (0,1,4,-1) form a basis for im(T).

im(T) = $c_1(1,0,3,0) + c_2(0,1,4,-1)$ for $c_1, c_2 \in \mathbb{R}$

where (1,0,3,0), (0,1,4,-1) form a basis for im(T) so dim(im(T)) = 2

To find ker(T), (i.e. solving Ax = 0), replace the right matrix by a column with all 0 entries:

$$\begin{bmatrix} 1 & 0 & 2 & 4 & | & 0 \\ 0 & 1 & -3 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \Leftrightarrow \begin{matrix} x_1 + 2x_3 + 4x_4 = 0 \\ x_2 - 3x_3 - x_4 = 0 \end{matrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 - 4x_4 \\ 3x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

ker(T) = = $x_3(-2,3,1,0) + x_4(-4,1,0,1)$ for $x_3, x_4 \in \mathbb{R}$

where (-2,3,1,0), (-4,1,0,1) form a basis for ker(T) so dim(ker(T)) = 2

2.5 Coordinates

Definition 2.5.1: Coordinates

For V $\subset \mathbb{R}^n$, let $v_1, ..., v_k \in$ V be a basis for V. Then the coordinates of v $\in$ V relative to basis $\mathcal{B} = v_1, ..., v_k$ are $c_1, ..., c_k \in \mathbb{R}$ such that:

v = $c_1v_1 + ... + c_kv_k$

Then the coordinate vector $[v]_{\mathcal{B}} = (c_1, ..., c_k) \in \mathbb{R}^k$. Let B = $\begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$.

v = $c_1v_1 + ... + c_kv_k$ = B$[v]_{\mathcal{B}}$

Theorem 2.5.2: Properties of Coordinates

For V $\subset \mathbb{R}^n$, let $\mathcal{B} = v_1, ..., v_k \in$ V be a basis for V. For x,y $\in$ V and c $\in \mathbb{R}$:

(a) $[x+y]_{\mathcal{B}} = [x]_{\mathcal{B}} + [y]_{\mathcal{B}}$

(b) $[cx]_{\mathcal{B}} = c[x]_{\mathcal{B}}$

Proof

Let x = $a_1v_1 + ... + a_kv_k$ and y = $b_1v_1 + ... + b_kv_k$ for $a_1, ..., a_k, b_1, ..., b_k \in \mathbb{R}$. Let $c_1, c_2 \in \mathbb{R}$.

$[c_1x + c_2y]_{\mathcal{B}} = (c_1a_1 + c_2b_1, ..., c_1a_k + c_2b_k) = c_1(a_1, ..., c_1a_k) + c_2(b_1, ..., b_k) = c_1[x]_{\mathcal{B}} + c_2[y]_{\mathcal{B}}$

Theorem 2.5.3: Linear transformation by Coordinates

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where T(x) = Ax. Let $\mathcal{B} = v_1, ..., v_n$ be a basis for $\mathbb{R}^n$. Then for any x $\in \mathbb{R}^n$:

$$[T(x)]_{\mathcal{B}} = [Ax]_{\mathcal{B}} = A_{\mathcal{B}}[x]_{\mathcal{B}}$$

where the $\mathcal{B}$-matrix of T, $A_{\mathcal{B}} = \begin{bmatrix}[A(v_1)]_{\mathcal{B}} & ... & [A(v_n)_{\mathcal{B}}]\end{bmatrix} \in M_{n\times n}(\mathbb{R})$

Proof

Let x $= c_1v_1 + ... + c_nv_n$ for $c_1, ..., c_n \in \mathbb{R}$. Then $[x]_{\mathcal{B}} = (c_1, ..., c_n)$:

$$[T(x)]_{\mathcal{B}} = [c_1A(v_1) + ... + c_nA(v_n)]_{\mathcal{B}} = c_1[A(v_1)]_{\mathcal{B}} + ... + c_n[A(v_n)]_{\mathcal{B}}$$

$$= \begin{bmatrix}[A(v_1)]_{\mathcal{B}} & ... & [A(v_n)_{\mathcal{B}}]\end{bmatrix}\begin{bmatrix}c_1 \\ \vdots \\ c_n\end{bmatrix} = \begin{bmatrix}[A(v_1)]_{\mathcal{B}} & ... & [A(v_n)_{\mathcal{B}}]\end{bmatrix}[x]_{\mathcal{B}}$$

Theorem 2.5.4: Relationship between Ax and $A_{\mathcal{B}}x_{\mathcal{B}}$

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation where T(x) = Ax. Let $\mathcal{B} = v_1, ..., v_n$ be a basis for $\mathbb{R}^n$ where $[T(x)]_{\mathcal{B}} = A_{\mathcal{B}}[x]_{\mathcal{B}}$. Then:

$$\text{AB} = \text{B}A_{\mathcal{B}} \qquad A_{\mathcal{B}} = B^{-1}\text{AB}$$

where B $= \begin{bmatrix}v_1 & ... & v_n\end{bmatrix} \in M_{n\times n}(\mathbb{R})$ is invertible and $[x]_{\mathcal{B}} = B^{-1}$x

Proof

Let B $= \begin{bmatrix}v_1 & ... & v_n\end{bmatrix}$. Since x = B$[x]_{\mathcal{B}}$ and T(x) = B$[T(x)]_{\mathcal{B}}$, then:

$$\text{T(x)} = \text{Ax} = \text{AB}[x]_{\mathcal{B}} \qquad \text{T(x)} = \text{B}[T(x)]_{\mathcal{B}} = \text{B}A_{\mathcal{B}}[x]_{\mathcal{B}}$$

Thus, AB = B$A_{\mathcal{B}}$. Since $v_1, ..., v_n$ is a basis, then by theorem 2.4.5, B is invertible.
Since x = B$[x]_{\mathcal{B}}$, then $[x]_{\mathcal{B}} = B^{-1}$x

Corollary 2.5.5: Change of Bases

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $\mathcal{B} = v_1, ..., v_n$ be a basis for $\mathbb{R}^n$ where $[T(x)]_{\mathcal{B}} = A_{\mathcal{B}}[x]_{\mathcal{B}}$ and $\mathcal{B}' = v_1', ..., v_n'$ also be a basis for $\mathbb{R}^n$ where $[T(x)]_{\mathcal{B}'} = A_{\mathcal{B}'}[x]_{\mathcal{B}'}$.

$$A_{\mathcal{B}'} = S^{-1}A_{\mathcal{B}}S$$

where S $= \begin{bmatrix}v_1 & ... & v_n\end{bmatrix}^{-1}\begin{bmatrix}v_1' & ... & v_n'\end{bmatrix}$ is invertible and $[x]_{\mathcal{B}'} = S^{-1}[x]_{\mathcal{B}}$

Proof

Let T(x) = Ax. By theorem 2.5.4, for invertible B $= \begin{bmatrix}v_1 & ... & v_n\end{bmatrix}$ and B' $= \begin{bmatrix}v_1' & ... & v_n'\end{bmatrix}$:

$$\text{A} = \text{B}A_{\mathcal{B}}B^{-1} \qquad \text{A} = \text{B'}A_{\mathcal{B}'}(B')^{-1}$$

$$\text{B}A_{\mathcal{B}}B^{-1} = \text{B'}A_{\mathcal{B}'}(B')^{-1} \quad \Rightarrow \quad A_{\mathcal{B}'} = (B')^{-1}\text{B}A_{\mathcal{B}}B^{-1}B' = (B^{-1}B')^{-1}A_{\mathcal{B}}(B^{-1}B')$$

Since $v_1, ..., v_n$ and $u_1, ..., u_n$ are bases, then by theorem 1.5.13, $B^{-1}B'$ is invertible.
Since x = B$[x]_{\mathcal{B}}$ and x = B'$[x]_{\mathcal{B}'}$, then:

$$[x]_{\mathcal{B}'} = (B')^{-1}\text{x} = (B')^{-1}\text{B}[x]_{\mathcal{B}} = (B^{-1}B')^{-1}[x]_{\mathcal{B}}$$

Definition 2.5.6: Matrix Similarity

Let A,B $\in M_{n\times n}(\mathbb{R})$. Then A is similar to B if there is an invertible X $\in M_{n\times n}(\mathbb{R})$ where:

$$\text{AX} = \text{XB}$$

Theorem 2.5.7: Properties of Matrix Similarity

Let A,B,C $\in M_{n\times n}(\mathbb{R})$.

(a) Reflexivity

A is similar to A

Proof

A$I_{n\times n}$ = A = $I_{n\times n}$A

(b) Symmetry

If A is similar to B, then B is similar to A

Proof

Since A is similar to B, there is an invertible X $\in M_{n\times n}(\mathbb{R})$ where AX = XB.

BX^{-1} = X^{-1}XBX^{-1} = X^{-1}AXX^{-1} = X^{-1}A

(c) Transitvity

If A is similar to B and B is similar to C, then A is similar to C

Proof

Since A is similar to B, there is an invertible $X_1 \in M_{n\times n}(\mathbb{R})$ where AX_1 = X_1B.
Since B is similar to C, there is an invertible $X_2 \in M_{n\times n}(\mathbb{R})$ where BX_2 = X_2C.

AX_1X_2 = X_1BX_2 = X_1X_2C

Since X_1, X_2 are invertible, then by theorem 1.5.13, X_1X_2 is invertible.

Example

Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be be T(x) = Ax = $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}$x where $\mathcal{B}$ = (1,1,1), (1,2,3), and (1,3,6) is a basis for $\mathbb{R}^3$. Find $A_\mathcal{B}$. Find and verify $[T(-1,0,1)]_\mathcal{B}$.

$$A_\mathcal{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[(-1,0,1)]_\mathcal{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(-1,0,1)]_\mathcal{B} = A_\mathcal{B}[(-1,0,1)]_\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = (0,1,0)$$

Since $0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ = (1,2,3) and T(-1,0,1) = $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ = (1,2,3), then it is true that $[T(-1,0,1)]_\mathcal{B}$ = (0,1,0).

3 Orthogonality

3.1 Orthogonality

Definition 3.1.1: Dot Product

Let x,y $\in \mathbb{R}^n$. Then the dot product of x and y:

$x \cdot y = x_1 y_1 + ... + x_n y_n = \sum_{i=1}^n x_i y_i$

Theorem 3.1.2: Properties of the Dot Product

Let x,y,z $\in \mathbb{R}^n$ and c $\in \mathbb{R}$.

(a) Positive Definite

$x \cdot x \geq 0$ where $x \cdot x = 0$ if and only if x = 0

Proof

$x \cdot x = x_1 x_1 + ... + x_n x_n = x_1^2 + ... + x_n^2 \geq 0$
If $0 = x \cdot x = x_1^2 + ... + x_n^2$, then each $x_i = 0$ so x = 0.
If x = 0, then $x \cdot x = x_1^2 + ... + x_n^2 = 0 + ... + 0 = 0$.

(b) Symmetry

$x \cdot y = y \cdot x$

Proof

$x \cdot y = x_1 y_1 + ... + x_n y_n = y_1 x_1 + ... + y_n x_n = y \cdot x$

(c) Scalar Multiplication

$(cx) \cdot y = c(x \cdot y) = x \cdot (cy)$

Proof

$(cx) \cdot y = \sum_{i=1}^n c x_i y_i = c \sum_{i=1}^n x_i y_i = c(x \cdot y) = \sum_{i=1}^n x_i (c y_i) = x \cdot (cy)$

(d) Distributivity

$z \cdot (x + y) = (z \cdot x) + (z \cdot y) \qquad (x + y) \cdot z = (x \cdot z) + (y \cdot z)$

Proof

$z \cdot (x + y) = \sum_{i=1}^n z_i (x_i + y_i) = \sum_{i=1}^n z_i x_i + \sum_{i=1}^n z_i y_i = (z \cdot x) + (z \cdot y)$
$(x + y) \cdot z = \sum_{i=1}^n (x_i + y_i) z_i = \sum_{i=1}^n x_i z_i + \sum_{i=1}^n y_i z_i = (x \cdot z) + (y \cdot z)$

Theorem 3.1.3: Dot Product: Length Property

Let x $\in \mathbb{R}^n$. Then, $x \cdot x = |x|^2$.

Proof

$x \cdot x = x_1 x_1 + ... + x_n x_n = \sum_{i=1}^n x_i^2 = (\sqrt{\sum_{i=1}^n x_i^2})^2 = |x|^2$.

Theorem 3.1.4: Dot Product: Cancellation Property

Let x,y $\in \mathbb{R}^n$. Then, x = y if and only if $x \cdot z = y \cdot z$ for every z $\in \mathbb{R}^n$.

Proof

Suppose x = y. Then $x_i = y_i$ for i = {1,...,n}:

$x \cdot z = x_1 z_1 + ... + x_n z_n = y_1 z_1 + ... + y_n z_n = y \cdot z$

Suppose $x \cdot z = y \cdot z$ for every z $\in \mathbb{R}^n$. Then, $z \cdot (x - y) = 0$. Let z = $x - y$. Then:

$0 = z \cdot (x - y) = (x - y) \cdot (x - y)$

Thus, x - y = 0 so x = y.

Definition 3.1.5: Transpose

Let x $\in \mathbb{R}^n$ where x = $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. Then, the transpose of x:

$$x^T = \begin{bmatrix} x_1 & ... & x_n \end{bmatrix}$$

Let A $\in M_{m\times n}(\mathbb{R})$ where A = $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$. Then, the transpose of A:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \in M_{n\times m}(\mathbb{R})$$

Theorem 3.1.6: Properties of Transpose

Let A,B $\in M_{m\times n}(\mathbb{R})$ and c $\in \mathbb{R}$. Then:

(a) Addition

$$(A+B)^T = A^T + B^T$$

Proof

$$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \dots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \dots & a_{mn}+b_{mn} \end{bmatrix} = A^T + B^T$$

(b) Scalar Multiplication

$$(cA)^T = cA^T$$

Proof

$$(cA)^T = \begin{bmatrix} ca_{11} & ca_{21} & \dots & ca_{m1} \\ ca_{12} & ca_{22} & \dots & ca_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{1n} & ca_{2n} & \dots & ca_{mn} \end{bmatrix} = cA^T$$

Theorem 3.1.7: Dot Product: Transpose Property

Let x,y $\in \mathbb{R}^n$. Then:

$$x \cdot y = x^T y = y^T x$$

Proof

$$x \cdot y = x_1y_1 + ... + x_ny_n = \begin{bmatrix} x_1 & ... & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x^T y$$

$$x \cdot y = y \cdot x = y^T x$$

Theorem 3.1.8: Transpose of Matrix Product

Let A $\in M_{m\times n}(\mathbb{R})$ and B $\in M_{t\times m}(\mathbb{R})$. Then, $(BA)^T \in M_{n\times t}(\mathbb{R})$ where:

$(BA)^T = A^T B^T$

Proof

$$(BA)^T = (\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{t1} & b_{t2} & \dots & b_{tm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix})^T$$
$$= (\begin{bmatrix} \sum_{k=1}^m b_{1m}a_{m1} & \sum_{k=1}^m b_{1m}a_{m2} & \dots & \sum_{k=1}^m b_{1m}a_{mn} \\ \sum_{k=1}^m b_{2m}a_{m1} & \sum_{k=1}^m b_{2m}a_{m2} & \dots & \sum_{k=1}^m b_{2m}a_{mn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m b_{tm}a_{m1} & \sum_{k=1}^m b_{tm}a_{m2} & \dots & \sum_{k=1}^m b_{tm}a_{mn} \end{bmatrix})^T$$
$$= \begin{bmatrix} \sum_{k=1}^m b_{1m}a_{m1} & \sum_{k=1}^m b_{2m}a_{m1} & \dots & \sum_{k=1}^m b_{tm}a_{m1} \\ \sum_{k=1}^m b_{1m}a_{m2} & \sum_{k=1}^m b_{2m}a_{m2} & \dots & \sum_{k=1}^m b_{tm}a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m b_{1m}a_{mn} & \sum_{k=1}^m b_{2m}a_{mn} & \dots & \sum_{k=1}^m b_{tm}a_{mn} \end{bmatrix} \in M_{n\times t}(\mathbb{R})$$
$$= \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & \dots & b_{t1} \\ b_{12} & b_{22} & \dots & b_{t2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m} & b_{2m} & \dots & b_{tm} \end{bmatrix} = A^T B^T$$

Corollary 3.1.9: $(A^T)^{-1} = (A^{-1})^T$

For invertible A $\in M_{n\times n}(\mathbb{R})$, then A^T is invertible where:

$(A^T)^{-1} = (A^{-1})^T$

Proof

Since A is invertible, there exists a A^{-1} such that $A^{-1}A = I_{n\times n}$ and $AA^{-1} = I_{n\times n}$

$(A^{-1})^T A^T = (AA^{-1})^T = I_{n\times n}^T = I_{n\times n}$ $\qquad A^T(A^{-1})^T = (A^{-1}A)^T = I_{n\times n}^T = I_{n\times n}$

Thus, $(A^T)^{-1} = (A^{-1})^T$ so A^T is invertible.

Corollary 3.1.10: $(Ax)\cdot y = x \cdot (A^T y)$

For A $\in M_{m\times n}(\mathbb{R})$ and x $\in \mathbb{R}^n$, y $\in \mathbb{R}^m$, then $A^T \in M_{m\times n}(\mathbb{R})$ is unique such that:

$(Ax)\cdot y = x \cdot (A^T y)$

Proof

Since Ax $\in \mathbb{R}^m$, then:

$(Ax)\cdot y = (Ax)^T y = (x^T A^T)y = x^T(A^T y) = x\cdot(A^T y)$

Suppose there is a B $\in M_{n\times m}(\mathbb{R})$ such that $(Ax)\cdot y = x\cdot(By)$.

Then for any x $\in \mathbb{R}^n$, y $\in \mathbb{R}^m$:

$x\cdot(By) = (Ax)\cdot y = x\cdot(A^T y)$

By theorem 3.1.4, $By = A^T y$ so B = A^T.

Definition 3.1.11: Orthogonality

Vectors x,y $\in \mathbb{R}^n$ are orthogonal (i.e. perpendicular) if $x\cdot y = 0$

Theorem 3.1.12: Pythagorean Theorem in digher dimensions

Let x,y $\in \mathbb{R}^n$. Then the following are equivalent:

(a) $x \cdot y = 0$

(b) $|x|^2 + |y|^2 = |x+y|^2$

(c) If n = 2 or 3 and x,y $\neq$ 0, then x and y are perpendicular

Proof

Note $|x+y|^2 = (x+y)\cdot(x+y) = (x\cdot x) + (x\cdot y) + (y\cdot x) + (y\cdot y) = |x|^2 + |y|^2 + 2(x\cdot y)$.

Suppose $x \cdot y = 0$. Then, $|x|^2 + |y|^2 = |x+y|^2$.

Suppose $|x|^2 + |y|^2 = |x+y|^2$. Then $2(x\cdot y) = 0$ so $x \cdot y = 0$.

Suppose $|x|^2 + |y|^2 = |x+y|^2$. By the regular definiton of the pythagorean theorem in $\mathbb{R}^2, \mathbb{R}^3$, x, y, and x+y form a right triangle so x and y form a right angle and thus, perpendicular. Note if x = 0, then $|x|^2 + |y|^2 = |x+y|^2$ becomes $|y|^2 = |y|^2$ which is true, but has nothing to do with right triangles so exclude the case when x = 0. Similarily, exclude y = 0.

Theorem 3.1.13: $x \cdot y = |x||y|\cos(\theta)$

Let x,y $\in \mathbb{R}^n$. Then:

$x \cdot y = |x||y|\cos(\theta)$

where $\theta \in [,\pi]$ is the angle between x and y

Proof

Since x, y, and x-y form a triangle, by the Law of Cosine:

$|x-y|^2 = |x|^2 + |y|^2 - 2|x|\ |y|\cos(\theta)$

where $\theta \in [0,\pi]$ is the angle between x and y. Since:

$|x-y|^2 = (x-y)\cdot(x-y) = x\cdot x + y\cdot y - 2(x\cdot y) = |x|^2 + |y|^2 - 2(x\cdot y)$

then $x \cdot y = |x|\ |y|\cos(\theta)$.

3.2 Orthogonal Basis

Theorem 3.2.1: Orthogonal Projection

The orthogonal projection of x $\in \mathbb{R}^n$ onto y $\in \mathbb{R}^n$ is the component of x parallel to y:

$\text{proj}_y x = \frac{x\cdot y}{|y|^2} y$

Also, $\text{proj}_y x \in$ span(y) where $x - \text{proj}_y x$ and span(y) are orthogonal.

Proof

Since $\text{proj}_y x$ is parallel to y, let $\text{proj}_y x$ = cy for some c $\in \mathbb{R}$.

Let $x^\perp$ be the orthogonal component of x to y. Thus, x = $\text{proj}_y x + x^\perp$ = cy + $x^\perp$.

Since $x^\perp$ is orthogonal to y, then:

$x \cdot y = (cy + x^\perp)\cdot y = cy\cdot y + x^\perp \cdot y = cy \cdot y = c|y|^2$

Thus, c = $\frac{x\cdot y}{|y|^2}$ so $\text{proj}_y x$ = cy = $\frac{x\cdot y}{|y|^2} y \in$ span(y). Let a $\in \mathbb{R}$.

$(x - \text{proj}_y x)\cdot ay = (x - \frac{x\cdot y}{|y|^2} y)\cdot ay = (x\cdot ay) - \frac{x\cdot y}{|y|^2}(y\cdot ay)$

$= a(x\cdot y) - a\frac{x\cdot y}{|y|^2}|y|^2 = a(x\cdot y) - a(x\cdot y) = 0$

Thus, $x - \text{proj}_y x$ and span(y) are orthogonal so $x - \text{proj}_y x$ and $\text{proj}_y x$ are orthogonal.

Example

Let x = (-4,5,7) and y = (2,-4,1). Find the vector decomposition of x onto y.

Parallel component: $\text{proj}_y x = \frac{(-4*2+5*-4+7*1)}{2^2+(-4)^2+1^2}(2,-4,1) = \frac{-21}{21}(2,-4,1)$ = (-2,4,-1)

Orthogonal component: x - $\text{proj}_y x$ = (-4,5,7) - (-2,4,-1) = (-2,1,8)

Theorem 3.2.2: Cauchy-Schwarz Inequality

For x,y $\in \mathbb{R}^n$, $|x \cdot y| \leq |x||y|$

Proof

Let x = $\text{proj}_y x + x^\perp$ = cy + $x^\perp$ where $x^\perp$ is the orthogonal component of x to y and $\text{proj}_y x$ = cy is the parallel component of x to y. By theorem 3.2.1, then c = $\frac{x \cdot y}{|y|^2}$.

$$|x|^2 = |cy + x^\perp|^2 = |cy|^2 + |x^\perp|^2 = (\tfrac{x \cdot y}{|y|^2})^2|y|^2 + |x^\perp|^2$$

$$|x|^2|y|^2 = |y|^2(\tfrac{x \cdot y}{|y|^2})^2|y|^2 + |y|^2|x^\perp|^2 = (x \cdot y)^2 + |y|^2|x^\perp|^2$$

Since $|y|^2|x^\perp|^2 \geq 0$, then $(x \cdot y)^2 \leq |x|^2|y|^2$ so $|x \cdot y| \leq |x||y|$.

Corollary 3.2.3: Triangle Inequality

For x,y $\in \mathbb{R}^n$, $||x + y|| \leq ||x|| + ||y||$

Proof

$$|x + y|^2 = (x + y) \cdot (x + y) = x \cdot x + y \cdot y + 2(x \cdot y) = |x|^2 + |y|^2 + 2(x \cdot y)$$
$$\leq |x|^2 + |y|^2 + 2|x \cdot y| \leq |x|^2 + |y|^2 + 2|x|\ |y| = (|x| + |y|)^2$$

Definition 3.2.4: Orthonormal Vectors

Vectors $v_1, ..., v_k \in \mathbb{R}^n$ is orthogonal if:

$v_i \cdot v_j = 0$ for i $\neq$ j

Then, $v_1, ..., v_k \in \mathbb{R}^n$ is orthonormal if:

$v_i \cdot v_j = 0$ for i $\neq$ j $\qquad |v_i|^2 = v_i \cdot v_i = 1 \qquad \Leftrightarrow \qquad |v_i| = 1$ for i $\in$ {1,...,k}

Theorem 3.2.5: Orthogonal sets are Linearly independent

Let $v_1, ..., v_k \in \mathbb{R}^n$ be orthogonal. Then, $v_1, ..., v_k$ is linearly independent.

Proof

Let $c_1, ..., c_k \in \mathbb{R}$ such that $0 = c_1v_1 + ... + c_kv_k$. Since $v_i \cdot v_j = 0$ for i $\neq$ j, then:

$$0 = v_i \cdot 0 = v_i \cdot (c_1v_1 + ... + c_kv_k) = c_i(v_i \cdot v_i) = c_i|v_i|^2$$

Since $|v_i| > 0$, then $c_i = 0$. Since every $c_i = 0$, then $v_1, ..., v_k$ is linearly independent.

Theorem 3.2.6: Orthogonal Basis

For V $\subset \mathbb{R}^n$, let $v_1, ..., v_k \in \mathbb{R}^n$ be an orthogonal basis for V. Then, for each x $\in$ V:

x = $\text{proj}_{v_1} x + ... + \text{proj}_{v_k} x = (\frac{x \cdot v_1}{|v_1|^2})v_1 + ... + (\frac{x \cdot v_k}{|v_k|^2})v_k$

Then if $v_1, ..., v_k$ is an orthonormal basis, then:

x = $(x \cdot v_1)v_1 + ... + (x \cdot v_k)v_k$

Proof

Since $v_1, ..., v_k$ is a basis for V, then for any x $\in$ V, there are $c_1, ..., c_k \in \mathbb{R}$ such that:

x = $c_1v_1 + ... + c_kv_k$

Since $v_1, ..., v_k$ is orthogonal, then $v_i \cdot v_j = 0$ for i $\neq$ j. Thus:

$$v_i \cdot x = v_i \cdot (c_1v_1 + ... + c_kv_k) = c_i(v_i \cdot v_i) = c_i|v_i|^2 \qquad \Rightarrow \qquad c_i = \frac{x \cdot v_i}{|v_i|^2}$$

Since each $c_i = \frac{x \cdot v_i}{|v_i|^2}$, then:

x = $c_1v_1 + ... + c_kv_k = (\frac{x \cdot v_1}{|v_1|^2})v_1 + ... + (\frac{x \cdot v_k}{|v_k|^2})v_k = \text{proj}_{v_1} x + ... + \text{proj}_{v_k} x$

Additionally, if $v_1, ..., v_k$ is orthonormal, then each $|v_i| = 1$. Thus, x = $(x \cdot v_1)v_1 + ... + (x \cdot v_k)v_k$.

Definition 3.2.7: Orthogonal Complement

For V $\subset \mathbb{R}^n$, the orthogonal complement of V:

$V^\perp$ = { w $\in \mathbb{R}^n$: w · v = 0 for v $\in$ V }

Theorem 3.2.8: Orthogonal projection onto V $\subset \mathbb{R}^n$

Let V $\subset \mathbb{R}^n$. Then for any x $\in \mathbb{R}^n$:

x $= x^{||} + x^{\perp}$

where $x^{||} = \text{proj}_V x$, the orthogonal projection of x onto V, is the component of x in V and $x^{\perp}$ is the component of x orthogonal to V are both unique

Proof

Since $x^{||}$ is in V and $x^{\perp}$ is orthogonal to V, then $x^{||} \cdot x^{\perp} = 0$.
Let $v_1, ..., v_k \in \mathbb{R}^n$ be an orthogonal basis for V. Then by theorem 3.2.6, since $x^{||} \in$ V, then:

$x^{||} = \text{proj}_{v_1} x^{||} + ... + \text{proj}_{v_k} x^{||}$

Suppose x $= x_1^{||} + x_1^{\perp} = x_2^{||} + x_2^{\perp}$ for $x_1^{||}, x_2^{||}, x_1^{\perp}, x_2^{\perp} \in \mathbb{R}^n$. Then:

$(x_1^{||} - x_2^{||}) + (x_1^{\perp} - x_2^{\perp}) = x_1^{||} + x_1^{\perp} - x_2^{||} - x_2^{\perp} = x - x = 0$

Since $x_1^{||}, x_2^{||} \in$ V and $x_1^{\perp}, x_2^{\perp} \in V^{\perp}$, then:

$0 = 0 \cdot (x_1^{||} - x_2^{||}) = [(x_1^{||} - x_2^{||}) + (x_1^{\perp} - x_2^{\perp})] \cdot (x_1^{||} - x_2^{||}) = (x_1^{||} - x_2^{||}) \cdot (x_1^{||} - x_2^{||}) = |x_1^{||} - x_2^{||}|^2$

$|x_1^{||} - x_2^{||}| = 0 \quad \Rightarrow \quad x_1^{||} = x_2^{||}$

$0 = 0 \cdot (x_1^{\perp} - x_2^{\perp}) = [(x_1^{||} - x_2^{||}) + (x_1^{\perp} - x_2^{\perp})] \cdot (x_1^{\perp} - x_2^{\perp}) = (x_1^{\perp} - x_2^{\perp}) \cdot (x_1^{\perp} - x_2^{\perp}) = |x_1^{\perp} - x_2^{\perp}|^2$

$|x_1^{\perp} - x_2^{\perp}| = 0 \quad \Rightarrow \quad x_1^{\perp} = x_2^{\perp}$

Thus, $x^{||}, x^{\perp}$ are unique.

Theorem 3.2.9: Gram-Schmidt Process: Creating an Orthogonal Basis

For V $\subset \mathbb{R}^n$, let $u_1, ..., u_k \in \mathbb{R}^n$ be a basis for V. Then let $v_1, ..., v_k \in \mathbb{R}^n$:

$v_1 = u_1$

$v_2 = u_2 - \text{proj}_{v_1}(u_2)$

$v_3 = u_3 - \text{proj}_{v_1}(u_3) - \text{proj}_{v_2}(u_3)$

$\vdots$

$v_k = u_k - \text{proj}_{v_1}(u_k) - ... - \text{proj}_{v_{k-1}}(u_k)$

Then, $v_1, ..., v_k$ is an orthogonal basis for V.

Also, $\frac{v_1}{|v_1|}, ..., \frac{v_k}{|v_k|}$ is an orthonormal basis for V.

Proof

$v_1 \cdot v_2 = u_1 \cdot (u_2 - \text{proj}_{v_1}(u_2)) = u_1 \cdot (u_2 - \frac{u_2 \cdot u_1}{|u_1|^2} u_1) = u_1 \cdot u_2 - u_1 \cdot u_2 = 0$
Suppose for m $\leq$ k, then $v_1, ..., v_{m-1}$ are orthogonal.
Since $v_m = u_m - \text{proj}_{v_1}(u_m) - ... - \text{proj}_{v_{m-1}}(u_m)$, where by theorem 3.2.1, each $\text{proj}_{v_i}(u_m) = c_i v_i$ where $c_i = \frac{u_m \cdot v_i}{|v_i|^2}$ for i = {1,...,m-1}

$v_m = u_m - c_1 v_1 - ... - c_{m-1} v_{m-1}$

$v_i \cdot v_m = v_i \cdot (u_m - c_1 v_1 - ... - c_{m-1} v_{m-1}) = v_i \cdot u_m - c_i (v_i \cdot v_i) = v_i \cdot u_m - u_m \cdot v_i = 0$

Thus, v_m is orthgonal to any v_i for i = {1,...,m-1} so $v_1, ..., v_m$ is orthogonal.
Thus, by proof by induction, $v_1, ..., v_k$ is orthogonal.
Similarily, $v_1 = u_1 \in$ V. Suppose for m $\leq$ k, then $v_1, ..., v_{m-1} \in$ V.
Then, $v_m = u_m - c_1 v_1 - ... - c_{m-1} v_{m-1} \in$ V. Thus, by proof by induction, $v_1, ..., v_k \in$ V.
By theorem 3.2.5, $v_1, ..., v_k$ are linearly independent. Since $u_1, ..., u_k$ is a basis for V, then by corollary 2.3.3, dim(V) = k. Then, by theorem 2.3.5, $v_1, ..., v_k$ span V and thus, form a basis for V. Thus, $v_1, ..., v_k$ is an orthogonal basis for V.
Since each $|\frac{v_i}{|v_i|}| = \frac{1}{|v_i|}|v_i| = 1$, then $\frac{v_1}{|v_1|}, ..., \frac{v_k}{|v_k|}$ is an orthonormal basis for V.

Theorem 3.2.10: QR Factorization: Relationship between Basis and Orthonormal Basis

Let $u_1, ..., u_k \in \mathbb{R}^n$ be linearly independent such that A $= \begin{bmatrix} u_1 & ... & u_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$. Then, there is an orthogonal $v_1, ..., v_k \in \mathbb{R}^n$ where Q $= \begin{bmatrix} \frac{v_1}{|v_1|} & ... & \frac{v_k}{|v_k|} \end{bmatrix} \in M_{n\times k}(\mathbb{R})$ such that:

A = QR

where R $\in M_{k\times k}(\mathbb{R})$ is invertible with R $= \begin{bmatrix} |v_1| & \frac{u_2\cdot v_1}{|v_1|} & \frac{u_3\cdot v_1}{|v_1|} & \dots & \frac{u_k\cdot v_1}{|v_1|} \\ 0 & |v_2| & \frac{u_3\cdot v_2}{|v_2|} & \dots & \frac{u_k\cdot v_2}{|v_2|} \\ 0 & 0 & |v_3| & \dots & \frac{u_k\cdot v_3}{|v_3|} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & |v_k| \end{bmatrix}$

Proof

Let V = span$(u_1, ..., u_k)$. Then by theorem 2.3.5, $u_1, ..., u_k$ form a basis for V.
Thus, by theorem 3.2.9, there is an orthonormal basis $\frac{v_1}{|v_1|}, ..., \frac{v_k}{|v_k|}$ for V where $v_1 = u_1$ and $v_i = u_i - \text{proj}_{\text{span}(v_1,...,v_{i-1})}u_i$ for i = {2,...,k}.
By theorem 3.2.6, each $\text{proj}_{\text{span}(v_1,...,v_{i-1})}u_i = \frac{u_i\cdot v_1}{|v_1|^2}v_1 + ... + \frac{u_i\cdot v_{i-1}}{|v_{i-1}|^2}v_{i-1}$.
Thus, $u_i = \frac{u_i\cdot v_1}{|v_1|^2}v_1 + ... + \frac{u_i\cdot v_{i-1}}{|v_{i-1}|^2}v_{i-1} + v_i$. Then:

$$\begin{bmatrix} u_1 & ... & u_k \end{bmatrix} = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} 1 & \frac{u_2\cdot v_1}{|v_1|^2} & \frac{u_3\cdot v_1}{|v_1|^2} & \dots & \frac{u_k\cdot v_1}{|v_1|^2} \\ 0 & 1 & \frac{u_3\cdot v_2}{|v_2|^2} & \dots & \frac{u_k\cdot v_2}{|v_2|^2} \\ 0 & 0 & 1 & \dots & \frac{u_k\cdot v_3}{|v_3|^2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} \frac{v_1}{|v_1|} & ... & \frac{v_k}{|v_k|} \end{bmatrix} \begin{bmatrix} |v_1| & \frac{u_2\cdot v_1}{|v_1|} & \frac{u_3\cdot v_1}{|v_1|} & \dots & \frac{u_k\cdot v_1}{|v_1|} \\ 0 & |v_2| & \frac{u_3\cdot v_2}{|v_2|} & \dots & \frac{u_k\cdot v_2}{|v_2|} \\ 0 & 0 & |v_3| & \dots & \frac{u_k\cdot v_3}{|v_3|} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & |v_k| \end{bmatrix}$$

A = QR

By theorem 3.2.5, $v_1, ..., v_k$ are linearly independent so $|v_i| \neq 0$ else if $|v_i| = 0$, then $v_i = 0$ which is linearly dependent. Thus, rref(R) $= I_{n\times n}$ so R is invertible.

Theorem 3.2.11: Orthogonal projection matrix transformation

For V $\in \mathbb{R}^n$, let $v_1, ..., v_k \in \mathbb{R}^n$ be an orthonormal basis for V. Then for any x $\in \mathbb{R}^n$:

$\text{proj}_V x = AA^T$x

where A $= \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$

Proof

By theorem 3.2.8:

$$\text{proj}_V x = (x\cdot v_1)v_1 + ... + (x\cdot v_k)v_k = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} x\cdot v_1 \\ \vdots \\ x\cdot v_k \end{bmatrix} = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} v_1^T x \\ \vdots \\ v_k^T x \end{bmatrix}$$

$$= \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix} x = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} v_1 & ... & v_k \end{bmatrix}^T x$$

Theorem 3.2.12: Reflection matrix transformation

For V $\in \mathbb{R}^n$, let $v_1, ..., v_k \in \mathbb{R}^n$ be an orthonormal basis for V.
Then for any x $\in \mathbb{R}^n$, the reflection of x across V:

$\text{reflect}_V x = (2AA^T - I_{n\times n})$x

where A $= \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$

Proof

Note for x $= x^{||} + x^{\perp}$ where $x^{||}$ is the component of x parallel to V and $x^{||}$ is the component of x orthgonal to V, then $\text{reflect}_V x = x^{||} - x^{\perp}$. Thus, by theorem 3.2.11:

$\text{reflect}_V x = x^{||} - x^{\perp} = 2x^{||} - x = 2AA^T x - x = (2AA^T - I_{n\times n})$x

3.3 Orthogonal Transformations

Definition 3.3.1: Orthogonal Transformation

Linear Transformation T: $\mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if for any x $\in \mathbb{R}^n$:

$|T(x)| = |x|$

Theorem 3.3.2: Orthogonal transformation Equivalences

Let linear transformation T: $\mathbb{R}^n \to \mathbb{R}^n$ be T(x) = Ax for A $\in M_{n\times n}(\mathbb{R})$.
Then the following are equivalent:

(a) For any x $\in \mathbb{R}^n$, then $|T(x)| = |x|$

(b) For any x,y $\in \mathbb{R}^n$, then $T(x) \cdot T(y) = x \cdot y$

(c) $T(e_1), ..., T(e_n)$ is an orthonormal basis for $\mathbb{R}^n$

(d) $A^T A = I_{n\times n}$

(e) A is invertible where $A^{-1} = A^T$

Proof

Suppose for any x $\in \mathbb{R}$, then $|T(x)| = |x|$.
Thus, for x,y $\in \mathbb{R}^n$, then $|T(x) + T(y)| = |T(x+y)| = |x+y|$. Thus:

$$|T(x)+T(y)|^2 = |x+y|^2 = (x+y)\cdot(x+y) = |x|^2 + |y|^2 + 2(x\cdot y)$$

$$\begin{aligned}|T(x)+T(y)|^2 &= (T(x)+T(y))\cdot(T(x)+T(y))\\ &= |T(x)|^2 + |T(y)|^2 + 2(T(x)\cdot T(y)) = |x|^2 + |y|^2 + 2(T(x)\cdot T(y))\end{aligned}$$

Thus, $T(x)\cdot T(y) = x\cdot y$.

Suppose for any x,y $\in \mathbb{R}^n$, then $T(x)\cdot T(y) = x\cdot y$.

$$T(e_i)\cdot T(e_j) = e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Thus, $T(e_1), ..., T(e_n)$ are orthogonal. By theorem 3.2.5, $T(e_1), ..., T(e_n)$ are linearly independent. Since dim($\mathbb{R}^n$) = n, then by theorem 2.3.5, $T(e_1), ..., T(e_n)$ is a basis.

Suppose $T(e_1), ..., T(e_n)$ is an orthonormal basis for $\mathbb{R}^n$.

Since A = $\begin{bmatrix} T(e_1) & ... & T(e_n) \end{bmatrix}$ and $T(e_i)\cdot T(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, then:

$$\begin{aligned} A^T A &= \begin{bmatrix} T(e_1)^T \\ \vdots \\ T(e_n)^T \end{bmatrix} \begin{bmatrix} T(e_1) & ... & T(e_n) \end{bmatrix} \\ &= \begin{bmatrix} T(e_1)^T T(e_1) & T(e_1)^T T(e_2) & \dots & T(e_1)^T T(e_n) \\ T(e_2)^T T(e_1) & T(e_2)^T T(e_2) & \dots & T(e_2)^T T(e_n) \\ \vdots & \vdots & \ddots & \vdots \\ T(e_n)^T T(e_1) & T(e_n)^T T(e_2) & \dots & T(e_n)^T T(e_n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \end{aligned}$$

Suppose $A^T A = I_{n\times n}$.
Then by theorem 1.5.11, then A,A^T are invertible where $A^{-1} = A^T$.

Suppose A is invertible where $A^{-1} = A^T$.
Then for any x $\in \mathbb{R}^n$:

$$\begin{aligned}|T(x)|^2 &= T(x)\cdot T(x) = Ax\cdot Ax = (Ax)^T(Ax) = x^T A^T Ax \\ &= x^T A^{-1} Ax = x^T x = x \cdot x = |x|^2\end{aligned}$$

Thus, $|T(x)| = |x|$.

Corollary 3.3.3: Inverse and Transpose of an Orthogonal matrix is Orthogonal

Let A $\in M_{n\times n}(\mathbb{R})$ be orthogonal. Then, A^{-1}, A^T are orthogonal.

Proof

By theorem 3.3.2, A is invertible so A^{-1} is invertible. Since A^{-1} is invertible, then $|A^{-1}(x)|$ $= |x|$ so A^{-1} is orthogonal. Since $A^{-1} = A^T$, then A^T is orthogonal.

Corollary 3.3.4: Products of Orthogonal matrices are Orthogonal

Let A,B $\in M_{n\times n}(\mathbb{R})$ be orthogonal. Then, AB is orthogonal.

Proof

By theorem 3.3.2, A,B are invertible. Then by theorem 1.5.13, AB is invertible so AB is orthgonal.

4 Determinants

4.1 Determinant Function

Definition 4.1.1: Determinant Function

Function D: $M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a determinant function if for any A $\in M_{n\times n}(\mathbb{R})$ such that A $= \begin{bmatrix} A_1 & ... & A_n \end{bmatrix}$, then:

(a) Multilinearity: For any j $\in$ {1,...,n}, then T: $\mathbb{R}^n \to \mathbb{R}$:

T(x) = D$(A_1, ..., A_{i-1}, x, A_{i+1}, ..., A_n)$

is linear. So, for x,y $\in \mathbb{R}^n$ and c $\in \mathbb{R}$:

- T(x+y) = T(x) + T(y)
- T(cx) = cT(x)

(b) Alternating: If B $\in M_{n\times n}(\mathbb{R})$ is obtained by swapping two columns of A, then:

D(B) = -D(A)

Thus, if two columns are equal, then by swapping those columns, B = A.

D(A) = D(B) = -D(A) $\Rightarrow$ D(A) = 0

(c) Identity

D$(I_{n\times n})$ = 1

Since each $A_i = A_{i1}e_1 + ... + A_{in}e_n$, then by multilinearity:

$$\begin{aligned} \text{D(A)} &= \text{D}(A_1, A_2, ..., A_n) \\ &= \textstyle\sum_{i_1=1}^n A_{1i_1}\text{D}(e_{i_1}, A_2, ..., A_n) \\ &= \textstyle\sum_{i_1=1}^n \sum_{i_2=1}^n A_{1i_1}A_{2i_2}\text{D}(e_{i_1}, e_{i_2}, ..., A_n) \\ &\vdots \\ &= \textstyle\sum_{i_1=1}^n \cdots \sum_{i_n=1}^n A_{1i_1}...A_{ni_n}\text{D}(e_{i_1}, e_{i_2}, ..., e_{i_n}) \end{aligned}$$

By alternating, if $i_{k_1} = i_{k_2}$ for $k_1, k_3 \in$ {1,...,n}, then D$(e_{i_1}, e_{i_2}, ..., e_{i_n}) = 0$. Thus:

$$\text{D(A)} = \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} A_{1i_1}...A_{ni_n}\text{D}(e_{i_1}, e_{i_2}, ..., e_{i_n})$$

where each i_k is unique from {1,...,n}. Since each A_{ki_k} for k $\in$ {1,...,n} are different columns and each i_k is unique, then each A_{ki_k} is from a unique row and column from A:

$$\text{A} = \begin{bmatrix} A_1 & ... & A_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \qquad \Rightarrow \qquad A_{ki_k} = a_{ij}$$

Since $(e_{i_1}, e_{i_2}, ..., e_{i_n})$ is a rearrangement of $(e_1, ..., e_n)$, then by alternating:

$$\text{D}(e_{i_1}, e_{i_2}, ..., e_{i_n}) = (-1)^S\text{D}(e_1, ..., e_n) = (-1)^S\text{D}(I_{n\times n}) = (-1)^S$$

where S is the number of swaps to turn $(e_{i_1}, e_{i_2}, ..., e_{i_n})$ into $(e_1, ..., e_n)$. So, D(A) is unique:

$$\text{D(A)} = \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^S$$

Definition 4.1.2: Inversions

A $n \times n$ pattern P = $\{(i_1, j_1), ..., (i_n, j_n)\}$ where $\{i_1, ..., i_n\}$,$\{j_1, ..., j_n\}$ are rearrangements of {1,...,n}. Then, (i_{k_1}, j_{k_1}) and (i_{k_2}, j_{k_2}) where $j_{k_1} < j_{k_2}$ is an inversion if $i_{k_1} > i_{k_2}$.
Then, the signature of P:

sgn(P) = $(-1)^I$ where I is the total number of inversions

Theorem 4.1.3: D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$)

Let $e_{i_1}, e_{i_2}, ..., e_{i_n} \in \mathbb{R}^n$ be a rearrangement of $e_1, ..., e_n \in \mathbb{R}^n$. Let S be the number of swaps to turn $e_{i_1}, e_{i_2}, ..., e_{i_n}$ into $e_1, ..., e_n$ and I be the number of inversions in $(1, i_1), ..., (n, i_n)$.

S = I

Thus:

D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^S = (-1)^I$ = sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$)

Proof

Note D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^S$ and sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^I$. Suppose n = 2. Then, either:

D($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$): 0 swaps , 0 inversions

D($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$): 1 swap , 1 inversion

Thus, when n = 2, then S = I so D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^S = (-1)^I$ = sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$).
For k $\leq$ n, let $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$ have the same number of swaps as for inversions.
Then, for $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}, e_{i_k}$, the only possible new inversions compared to $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$ are e_{i_k} with each of $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$. Also, note the only new swaps are also e_{i_k} with each of $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$ since if $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$ is swapped into $e_{i_1^*}, e_{i_2^*}, ..., e_{i_{k-1^*}}$ which has no inversions, then e_{i_k} can swap with $e_{i_{k-1^*}}$ first if needed, then $e_{i_{k-2^*}}$ second if needed, and etc. Thus, each $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$ does not have to swap with one another, the only swaps are e_{i_k} with $e_{i_1}, e_{i_2}, ..., e_{i_{k-1}}$, and $e_{i_1^*}, e_{i_2^*}, ..., e_{i_{k-1^*}}, e_{i_{k^*}}$ can be reached with no inversions.
But, since D(e_{i_k}, e_{i_j}) = sgn(e_{i_k}, e_{i_j}) for j = {1,...,k-1}, then the total new inversions is the same as the total new swaps. Thus, by proof by induction, S = I so:

D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^S = (-1)^I$ = sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$)

Corollary 4.1.4: Determinant function redefined

For determinant function D: $M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, then for any A $\in M_{n\times n}(\mathbb{R})$:

D(A) = $\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^I$

where I is the number of inversions in $(1, i_1), ..., (n, i_n)$

Proof

D(A) = $\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^S$ where S is the number of swaps to turn $(e_{i_1}, e_{i_2}, ..., e_{i_n})$ into $(e_1, ..., e_n)$. Since D($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^S$, then by theorem 4.1.3, $(-1)^S$ = sgn($e_{i_1}, e_{i_2}, ..., e_{i_n}$) = $(-1)^I$ where I is the number of inversions in $(1, i_1), ..., (n, i_n)$. Thus:

D(A) = $\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^S = \sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^I$

Theorem 4.1.5: det() function

Function det: $M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ is a determinant function

$\det(A) = \sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^I$

where I is the number of inversions in $(1, i_1), ..., (n, i_n)$

Proof

For j $\in$ {1,...,n}, let x,y $\in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$:

$$\begin{aligned}
T(c_1x + c_2y) &= \det(A_1, ..., A_{j-1}, c_1x + c_2y, A_{j+1}, ..., A_n) \\
&= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{(j-1)i_{j-1}}(c_1x + c_2y)_{ji_j}a_{(j+1)i_{j+1}}...a_{ni_n}(-1)^I \\
&= c_1 \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{(j-1)i_{j-1}}x_{ji_j}a_{(j+1)i_{j+1}}...a_{ni_n}(-1)^I \\
&\quad + c_2 \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{(j-1)i_{j-1}}y_{ji_j}a_{(j+1)i_{j+1}}...a_{ni_n}(-1)^I \\
&= c_1\det(A_1, ..., A_{j-1}, x, A_{j+1}, ..., A_n) + c_2\det(A_1, ..., A_{j-1}, y, A_{j+1}, ..., A_n) \\
&= c_1T(x) + c_2T(y)
\end{aligned}$$

For A = $\begin{bmatrix} A_1 & ... & A_j & ... & A_k & ... & A_n \end{bmatrix}$, let B = $\begin{bmatrix} A_1 & ... & A_k & ... & A_j & ... & A_n \end{bmatrix}$.
By theorem 4.1.3, the I_B, number of inversions in B, is equal to S_B, the number of swaps in B to turn $e_{i_1}, ...e_{i_k}, ...e_{i_j}, ..., e_{i_n}$ into $e_1, ..., e_n$. Thus:

$$\begin{aligned}
\det(B) &= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ki_k}...a_{ji_j}...a_{ni_n}(-1)^{I_B} \\
&= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ji_j}...a_{ki_k}...a_{ni_n}(-1)^{S_B}
\end{aligned}$$

Note $e_{i_1}, ...e_{i_k}, ...e_{i_j}, ..., e_{i_n}$ can first swap into $e_{i_1}, ...e_{i_j}, ...e_{i_k}, ..., e_{i_n}$ and then perform swaps to turn into $e_1, ..., e_n$. Since the number of swaps to turn $e_{i_1}, ...e_{i_j}, ...e_{i_k}, ..., e_{i_n}$ into $e_1, ..., e_n$ is S_A, the number of swaps in A, then $S_B = S_A + 1$. Thus:

$$\begin{aligned}
\det(B) &= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ji_j}...a_{ki_k}...a_{ni_n}(-1)^{S_A+1} \\
&= -\textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ji_j}...a_{ki_k}...a_{ni_n}(-1)^{S_A} = -\det(A)
\end{aligned}$$

For $\det(I_{n\times n})$, the only nonzero $a_{1i_1}...a_{ni_n}$ is when each $a_{ki_k} = 1$ else $a_{ki_k} = 0$. Since there are no inversions from $e_1, ..., e_n$ to $e_1, ..., e_n$, then $\det(I_{n\times n}) = a_{1i_1}...a_{ni_n}(-1)^I = 1...1(-1)^0 = 1$.

Theorem 4.1.6: $\det(A^T) = \det(A)$

For A $\in M_{n\times n}(\mathbb{R})$:

$\det(A^T) = \det(A)$

Proof

Let $\det(A^T) = \sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^{I_{A^T}}$.
For $a_{k_1i_{k_1}}$,$a_{k_2i_{k_2}}$ where $i_{k_1} < i_{k_2}$, suppose $k_1 < k_2$. Then, there is no inversion for $a_{k_1i_{k_1}}$ and $a_{k_2i_{k_2}}$. Then for $a_{i_{k_1}k_1}$,$a_{i_{k_2}k_2}$ where $k_1 < k_2$ and $i_{k_1} < i_{k_2}$, then there is no inversion.
For $a_{k_1i_{k_1}}$,$a_{k_2i_{k_2}}$ where $i_{k_1} < i_{k_2}$, suppose $k_1 > k_2$. Then, there is an inversion for $a_{k_1i_{k_1}}$ and $a_{k_2i_{k_2}}$. Then for $a_{i_{k_2}k_2}$,$a_{i_{k_1}k_1}$ where $k_2 < k_1$ and $i_{k_2} > i_{k_1}$, then there is an inversion.
Thus, transpose perserves inversions so $I_{A^T} = I_A$.
Since each $a_{1i_1}...a_{ni_n}$ in A^T can be arrange in order by $i_1, ..., i_n$ into $a_{1^*i_1^*}...a_{n^*i_n^*}$ which when transposed is $a_{i_1^*1^*}...a_{i_n^*n^*}$ which is in A. Thus:

$$\begin{aligned}
\det(A^T) &= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^{I_{A^T}} \\
&= \textstyle\sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{i_1^*1^*}...a_{i_n^*n^*}(-1)^{I_A} = \det(A)
\end{aligned}$$

Theorem 4.1.7: Cofactor Expansion: Determing det(A) by parts

For A $\in M_{n\times n}(\mathbb{R})$, let let A_{ij} be A, but the i-th row and j-th column removed.
For any i-th row where i $\in$ {1,...,n}, then by fixing i:
det(A) $= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \text{det}(A_{ij})$
Or for any j-th column where j $\in$ {1,...,n}, then by fixing j:
det(A) $= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \text{det}(A_{ij})$

Proof

There are n possibles a_{ij} choices in the first column and by choosing any such one, then that row is eliminated for choice in the following columns. Thus, there are n-1 possible a_{ij} choices in the second column and by choosing any such one, then that row is also eliminated for choice in the following columns. Repeating the pattern, then there are n*(n-1)*(n-2)*...*1 = n! total unique $a_{1i_1}, ..., a_{ni_n}$ combinations. In the cofactor expansion, choose a fixed i. The case for a fixed j is analogous. For a fixed i, the cofactor expansion iterates through each of the n columns in row i so there are n unique a_{ij}. For each a_{ij}, the A_{ij} has the i-th row and j-th column removed so A_{ij} is a (n-1) by (n-1) matrix and thus, there are (n-1)! unique $a_{1i_1}, ..., a_{ni_n}$ combinations as proved earlier. Since each A_{ij} removes a different j-th column, then each $a_{1i_1}, ..., a_{ni_n}$ from different columns are unique. Thus, the n unique a_{ij} has (n-1)! unique $a_{1i_1}, ..., a_{ni_n}$ so there are n*(n-1)! = n! unique $a_{1i_1}, ..., a_{ni_n}$. Thus, the $a_{1i_1}, ..., a_{ni_n}$ in the cofactor expansion must be equivalent to the $a_{1i_1}, ..., a_{ni_n}$ in the original determinant.
For the fixed i, lets fixed j $\in$ {1,...,n}:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \cdots & a_{i,n} \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n} \end{bmatrix}$$

In the original determinant, each $a_{1i_1}, ..., a_{ni_n}$ associates $(-1)^{\#\text{ inversions } a_{1i_1}, ..., a_{ni_n}}$. As proven earlier, each $a_{1i_1}, ..., a_{ni_n}$ is expressed in the coefactor expansion. So for any $a_{1i_1}, ..., a_{ni_n}$ that contains a_{ij} with the fixed i,j, then from the $a_{ij}\text{det}(A_{ij})$ in the cofactor expansion, the det(A_{ij}) consists of the other a_{ij} in the $a_{1i_1}, ..., a_{ni_n}$ since none of the other a_{ij} can exist in row i or column j by definition of the determinant and thus, det(A_{ij}) must account for all the inversions exclusively between the other a_{ij}. To account for the inversions between the other a_{ij} and the fixed a_{ij}, refer to the matrix above. The only a_{ij} which contributes an inversion with the fixed a_{ij} must be in the lower left and upper right of the matrix by defintion of the determinant. Let A = $\#a_{ij}$ in upper left, B = $\#a_{ij}$ in upper right, C = $\#a_{ij}$ in lower left, and D = $\#a_{ij}$ in lower right. Since each $a_{1i_1}, ..., a_{ni_n}$ must have a a_{ij} in each row and column, then:
A+B = i - 1   A+C = j - 1 $\Rightarrow$ B+C = i + j - 2 - 2A
$(-1)^{I_A} = (-1)^{B+C} = (-1)^{i+j-2-2A} = (-1)^{i+j}(-1)^{-2}(-1)^{-2A} = (-1)^{i+j}$
Thus:
det(A) $= \sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}...a_{ni_n}(-1)^{I_A}$
$= \sum_{j=1}^{n} a_{ij}\text{det}(A_{ij})(-1)^{I_A} = \sum_{j=1}^{n} a_{ij}\text{det}(A_{ij})(-1)^{i+j}$

4.2 Properties of the Determinant

Theorem 4.2.1: Relationship between Determinant and Elementary row operations

Let A,B $\in M_{n\times n}(\mathbb{R})$.

(a) Row or Column Multiplication

Let B be obtained by multiplying the i-th row or j-th column of A by c $\in \mathbb{R}$. Then:

det(B) = c det(A)

Proof

Let B be A with the i-th column multiplied by c. Then:
det(B) = det$(B_1, ..., B_i, ...B_n)$ = det$(A_1, ..., cA_i, ...A_n)$
= c det$(A_1, ..., A_i, ...A_n)$ = c det(A)
Let B be A with the j-th column multiplied by c. Then:
det(B) = det(B^T) = c det(A^T) = c det(A)

(b) Row or Column Addition

Let B be obtained by adding to the i-th row by c $\in \mathbb{R}$ times the j-th row of A or by adding to the i-th column by c $\in \mathbb{R}$ times the j-th column of A. Then:

det(B) = det(A)

Proof

Let B be A with the i-th column added by c times the j-th column. Then:
det(B) = det$(B_1, ..., B_i, ..., B_j, ..., B_n)$ = det$(A_1, ..., A_i + cA_j, ..., A_j, ..., A_n)$
= det$(A_1, ..., A_i, ..., A_j, ..., A_n)$ + c det$(A_1, ..., A_j, ..., A_j, ..., A_n)$
= det(A) + c0 = det(A)
Let B be A with the i-th row added by c times the j-th row. Then:
det(B) = det(B^T) = det(A^T) = det(A)

(c) Row or Column Swapping

Let B be obtained swapping the i-th and j-th rows of A or by swapping the i-th and j-th columns of A. Then:

det(B) = -det(A)

Proof

Let B be A with the i-th and j-th columns swapped. Then:
det(B) = det$(B_1, ..., B_i, ..., B_j, ..., B_n)$ = det$(A_1, ..., A_j, ..., A_i, ..., A_n)$
= -det$(A_1, ..., A_i, ..., A_j, ..., A_n)$ = -det(A)
Let B be A with the i-th and j-th rows swapped. Then:
det(B) = det(B^T) = -det(A^T) = -det(A)

Theorem 4.2.2: Invertible A $\Leftrightarrow$ det(A) $\neq 0$

Let A $\in M_{n\times n}(\mathbb{R})$. Then, A is invertible if and only if det(A) $\neq 0$.

Proof

By theorem 2.4.5, A is invertible if and only if rref(A) = $I_{n\times n}$. Then, there is a sequence of elementary row operations that transformation A into $I_{n\times n}$. By theorem 4.2.1, then:
det(A) = $(-1)^S$det$(I_{n\times n})$ = $(-1)^S \neq 0$
where S is the number of row swaps in the sequence. Then if det(A) = 0, then A cannot be invertible. Thus, A is invertible if and only if det(A) $\neq 0$.

Theorem 4.2.3: det(AB) = det(A)det(B)

For A,B $\in M_{n\times n}(\mathbb{R})$, then det(AB) = det(A)det(B)

Proof

Suppose at least one of A,B is not invertible. Let A be not invertible.
Then by theorem 4.2.2, det(A) = 0. Since A is not invertible, then AB is not invertible.

det(AB) = 0 = 0det(B) = det(A)det(B)

Suppose A,B are invertible. By theorem 2.4.5, rref(A) = rref(B) = $I_{n\times n}$. Thus, there is a sequence S_1 of elementary row operations that transforms A into $I_{n\times n}$. Thus, sequence S_1 will transform AB into $I_{n\times n}$B = B. And then there is another sequence S_2 of elementary row operations that transforms B into $I_{n\times}$.

det(A) = $(-1)^{S_A}$ det(A) = $(-1)^{S_B}$

where S_A are the number of row swaps in S_1 and S_B are the number of row swaps in S_2.

det(AB) = $(-1)^{S_A}$det(B) = $(-1)^{S_A}(-1)^{S_B}$

Thus, det(AB) = det(A)det(B).

Corollary 4.2.4: det(A^k) = (det(A))^k

For A $\in M_{n\times n}(\mathbb{R})$, then det($A^k$) = (det(A))^k

Proof

det(A^k) = det(A)det(A^{k-1}) = det(A)det(A)det(A^{k-2}) = ... = (det(A))^k

Corollary 4.2.5: det(A^{-1}) = (det(A))$^{-1}$

For invertible A $\in M_{n\times n}(\mathbb{R})$, then det($A^{-1}$) = (det(A))$^{-1}$

Proof

Since A is invertible, then $A^{-1}A = I_{n\times n}$. Then:

1 = det($I_{n\times n}$) = det($A^{-1}A$) = det(A^{-1})det(A)

Thus, det(A^{-1}) = (det(A))$^{-1}$.

Corollary 4.2.6: Similar matrices have the same Determinant

For similar A,B $\in M_{n\times n}(\mathbb{R})$, then det(A) = det(B)

Proof

Since A,B are similar, then there is a invertible X $\in M_{n\times n}(\mathbb{R})$ such that AX = XB. Then:

det(A)det(X) = det(AX) = det(XB) = det(X)det(B)

Since X is invertible, then det(X) $\neq$ 0 so det(A) = det(B).

Corollary 4.2.7: Determinant of Orthogonal matrices

For orthogonal A $\in M_{n\times n}(\mathbb{R})$, then det(A) = $\pm$1.

Proof

Since A is orthogonal, then by theorem 3.3.2, $A^TA = I_{n\times n}$. Then:

1 = det($I_{n\times n}$) = det(A^TA) = det(A^T)det(A) = (det(A))^2

Thus, det(A) = $\pm$1.

4.3 Volume

Definition 4.3.1: Volume of Parallelotope

Let $v_1, ..., v_n \in \mathbb{R}^n$. Then a parallelotope, P($v_1, ..., v_n$), is a n-th order parallelogram.
For i = {2,...,n}, let $v_i^\perp = v_i$ - $\text{proj}_{\text{span}(v_1,...,v_{i-1})} v_i$. The volume of a parallelotope:

$$\text{Vol}_n(P(v_1, ..., v_n)) = |v_1|\ |v_2^\perp|\ |v_3^\perp|\ ...\ |v_n^\perp|$$

Theorem 4.3.2: $|\det(v_1, ..., v_n)| = \text{Vol}_n(P(v_1, ..., v_n))$

For $v_1, ..., v_n \in \mathbb{R}^n$, let A = $\begin{bmatrix} v_1 & ... & v_n \end{bmatrix} \in M_{n\times n}(\mathbb{R})$. Then:

$$\text{Vol}_n(P(v_1, ..., v_n)) = |\det(A)|$$

Proof

By theorem 3.2.9, $v_1, v_2^\perp, ..., v_n^\perp \in \mathbb{R}^n$ is orthogonal. Let B = $\begin{bmatrix} v_1 & v_2^\perp & ... & v_n^\perp \end{bmatrix}$. Thus:

$$B^T B = \begin{bmatrix} v_1^T \\ (v_2^\perp)^T \\ \vdots \\ (v_n^\perp)^T \end{bmatrix} \begin{bmatrix} v_1 & v_2^\perp & \dots & v_n^\perp \end{bmatrix} = \begin{bmatrix} v_1^T v_1 & v_1^T v_2^\perp & \dots & v_1^T v_n^\perp \\ (v_2^\perp)^T v_1 & (v_2^\perp)^T v_2^\perp & \dots & (v_2^\perp)^T v_n^\perp \\ \vdots & \vdots & \ddots & \vdots \\ (v_n^\perp)^T v_1 & (v_n^\perp)^T v_2^\perp & \dots & (v_n^\perp)^T v_n^\perp \end{bmatrix} = \begin{bmatrix} |v_1|^2 & 0 & \dots & 0 \\ 0 & |v_2^\perp|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |v_n^\perp|^2 \end{bmatrix}$$

Since each $v_i^\perp = v_i - (c_{i1}v_1 + ... + c_{i(i-1)}v_{i-1})$ for $c_{i1}, ..., c_{i(i-1)} \in \mathbb{R}$, then by theorem 4.2.1:

$$\det(\text{B}) = \det(v_1, v_2^\perp, v_3^\perp, ..., v_n^\perp) = \det(v_1, v_2, v_3^\perp, ..., v_n^\perp) = ... = \det(v_1, v_2, v_3, ..., v_n) = \det(\text{A})$$

Thus:

$$|\det(A)| = \sqrt{\det(A)\det(A)} = \sqrt{\det(A^T)\det(A)} = \sqrt{\det(A^T A)} = \sqrt{\det(B^T B)}$$
$$= \sqrt{|v_1|^2|v_2^\perp|^2...|v_n^\perp|^2} = |v_1||v_2^\perp|...|v_n^\perp| = \text{Vol}_n(P(v_1, ..., v_n))$$

Corollary 4.3.3: Expansion Factor: Linear transformation of a Parallelotope

Let linear transformation T: $\mathbb{R}^n \to \mathbb{R}^n$ be T(x) = Ax. Then for $v_1, ..., v_n \in \mathbb{R}^n$, T(P($v_1, ..., v_n$)) is a parallelotope where:

$$\text{Vol}_n(T(P(v_1, ..., v_n))) = |\det(A)|\text{Vol}_n(P(v_1, ..., v_n))$$

So, for T(x) = Ax, $|\det(A)|$ is called the expansion factor of T.

Proof

For any v $\in P(v_1, ..., v_n)$, then v = $c_1v_1 + ... + c_nv_n$ for $c_1, ..., c_n \in$ [0,1]. Thus:

$$T(P(v_1, ..., v_n)) = \{\ A(c_1v_1 + ... + c_nv_n)\ \} = \{\ c_1Av_1 + ... + c_nAv_n\ \}$$

Thus, T(P($v_1, ..., v_n$)) is the parallelotope, P($Av_1, ..., Av_n$). By theorem 4.3.2:

$$\text{Vol}_n(T(P(v_1, ..., v_n))) = |\det([Av_1\ ...\ Av_n])| = |\det(A)\det([v_1\ ...\ v_n])| = |\det(A)|\text{Vol}_n(P(v_1, ..., v_n))$$

Theorem 4.3.4: Cauchy-Binet Formula

For k $\leq$ n, let $v_1, ..., v_k \in \mathbb{R}^n$. Let A = $\begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \in M_{n\times k}(\mathbb{R})$. Then:

$\text{Vol}_k(P(v_1, ..., v_k)) = \sqrt{\det(A^T A)}$

Proof

Since each $v_i^{\perp} = v_i - (c_{i1}v_1 + ... + c_{i(i-1)}v_{i-1})$ for $c_{i1}, ..., c_{i(i-1)} \in \mathbb{R}$, then:

$$\text{B} = \begin{bmatrix} v_1 & v_2^{\perp} & ... & v_k^{\perp} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & ... & v_k \end{bmatrix} \begin{bmatrix} 1 & -c_{21} & -c_{31} & \dots & -c_{k1} \\ 0 & 1 & -c_{32} & \dots & -c_{k2} \\ 0 & 0 & 1 & \dots & -c_{k3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} = \text{AX}$$

Since det(X) = 1, then by theorem 4.2.2, X is invertible so $BX^{-1} = AXX^{-1} =$ A. By theorem 3.2.9, $v_1, v_2^{\perp}, ..., v_n^{\perp} \in \mathbb{R}^n$ is orthogonal. Thus:

$$B^T B = \begin{bmatrix} v_1^T \\ (v_2^{\perp})^T \\ \vdots \\ (v_k^{\perp})^T \end{bmatrix} \begin{bmatrix} v_1 & v_2^{\perp} & \dots & v_k^{\perp} \end{bmatrix} = \begin{bmatrix} v_1^T v_1 & v_1^T v_2^{\perp} & \dots & v_1^T v_k^{\perp} \\ (v_2^{\perp})^T v_1 & (v_2^{\perp})^T v_2^{\perp} & \dots & (v_2^{\perp})^T v_k^{\perp} \\ \vdots & \vdots & \ddots & \vdots \\ (v_k^{\perp})^T v_1 & (v_k^{\perp})^T v_2^{\perp} & \dots & (v_k^{\perp})^T v_k^{\perp} \end{bmatrix} = \begin{bmatrix} |v_1|^2 & 0 & \dots & 0 \\ 0 & |v_2^{\perp}|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |v_k^{\perp}|^2 \end{bmatrix}$$

$$\begin{aligned} \sqrt{\det(A^T A)} &= \sqrt{\det((BX^{-1})^T (BX^{-1}))} = \sqrt{\det((X^{-1})^T)\det(B^T B)\det(X^{-1})} \\ &= \sqrt{\det((X^{-1}))\det(B^T B)\det(X)^{-1}} = \sqrt{\det(B^T B)} \\ &= \sqrt{|v_1|^2 |v_2^{\perp}|^2 ... |v_k^{\perp}|^2} = |v_1||v_2^{\perp}|...|v_k^{\perp}| = \text{Vol}_k(P(v_1, ..., v_k)) \end{aligned}$$

5 Eigenvectors

5.1 Diagonalization

Definition 5.1.1: Diagonalizable Matrices

Matrix A $\in M_{n\times n}(\mathbb{R})$ is diagonal if:

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \qquad \text{where } a_{11}, ..., a_{nn} \in \mathbb{R}$$

Then A is diagonalizable if A is similar to a diagonal matrix A_D

Theorem 5.1.2: Diagonalizablility relationship between A and $A_\mathcal{B}$

Let A $\in M_{n\times n}(\mathbb{R})$. Then, A is diagonalizable if and only if for basis $\mathcal{B} = v_1, ..., v_n \in \mathbb{R}^n$, then $A_\mathcal{B}$ is diagonalizable.

Proof

By theorem 2.5.4, there is an invertible matrix B $= \begin{bmatrix} v_1 & ... & v_n \end{bmatrix} \in M_{n\times n}(\mathbb{R})$ such that:

AB = B$A_\mathcal{B}$

Thus, A is similar to $A_\mathcal{B}$

If A is diagonalizable, then A is similar to diagonal $A_D \in M_{n\times n}(\mathbb{R})$.
Then by theorem 2.5.7, $A_\mathcal{B}$ is similar to A_D and thus, $A_\mathcal{B}$ is diagonalizable.

If $A_\mathcal{B}$ is diagonalizable, then $A_\mathcal{B}$ is similar to diagonal A_D.
Then by theorem 2.5.7, A is similar to A_D and thus, A is diagonalizable.

Theorem 5.1.3: Diagonalizable A $\Leftrightarrow$ Av = λv for some v

A $\in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there is a basis $\mathcal{B} = v_1, ..., v_n \in \mathbb{R}^n$ where $A_\mathcal{B}$ is diagonal. Thus, for for some $\lambda_i \in \mathbb{R}$ where i = {1,...,n}:

$[Av_i]_\mathcal{B} = [\lambda_i v_i]_\mathcal{B}$

Proof

Suppose A is diagonalizable. Then there is an invertible B $= \begin{bmatrix} v_1 & ... & v_n \end{bmatrix}$, diagonal $A_D \in M_{n\times n}(\mathbb{R})$ such that AB = BA_D. By theorem 2.4.5, the $\mathcal{B} = v_1, ..., v_n$ form a basis for $\mathbb{R}^n$. By theorem 2.5.4, then $A_D = A_\mathcal{B}$ so $A_\mathcal{B}$ is a diagonal.

Suppose there is a basis $\mathcal{B} = v_1, ..., v_n \in \mathbb{R}^n$ where $A_\mathcal{B}$ is diagonal.
By theorem 2.5.4, then AB = B$A_\mathcal{B}$ where B is invertible so A is diagonalizable.

By theorem 2.5.3, then $A_\mathcal{B} = \begin{bmatrix} [A(v_1)]_\mathcal{B} & ... & [A(v_n)]_\mathcal{B} \end{bmatrix}$. Since $A_\mathcal{B}$ is a diagonal, then there are $\lambda_i \in \mathbb{R}$ for i = {1,...,n} such that:

$$[A(v_i)]_\mathcal{B} = \begin{bmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = \lambda_i e_i = \lambda_i [v_i]_\mathcal{B} = [\lambda_i v_i]_\mathcal{B}$$

Definition 5.1.4: Eigenvalues and Eigenvectors

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then, nonzero v $\in \mathbb{R}^n$ is an eigenvector of T if T(v) = λv for some eigenvalue $\lambda \in \mathbb{R}$ of T.
If eigenvectors $v_1, ..., v_n \in \mathbb{R}^n$ with eigenvalues $\lambda_1, ..., \lambda_n$ form a basis, then $v_1, ..., v_n$ is a eigenbasis for T.

Theorem 5.1.5: Diagonalizable $\Leftrightarrow$ Existence of Eigenbasis

A $\in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if there is an eigenbasis $\mathcal{B} = v_1, ..., v_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ for A. Then for B = $\begin{bmatrix} v_1 & ... & v_n \end{bmatrix}$, $A_{\mathcal{B}} = \begin{bmatrix} \lambda_1 e_1 & ... & \lambda_n e_n \end{bmatrix} \in M_{n\times n}(\mathbb{R})$:

AB = B$A_{\mathcal{B}}$

Proof

Suppose A is diagonalizable. Then, by theorem 5.1.3, there is a basis $\mathcal{B} = v_1, ..., v_n$ such that $A_{\mathcal{B}}$ is diagonal where $[Av_i]_{\mathcal{B}} = [\lambda_i v_i]_{\mathcal{B}}$ for some λ_i. Since $Av_i = \lambda_i v_i$, then v_i is an eigenvector with eigenvalue v_i. Thus, $\mathcal{B} = v_1, ..., v_n$ is an eigenbasis with eigenvalues $\lambda_1, ..., \lambda_n$ for A.

Suppose there is an eigenbasis $\mathcal{B} = v_1, ..., v_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ for A.
Since A$v_i = \lambda_i v_i$, then $A_{\mathcal{B}} = \begin{bmatrix} [Av_1] & ... & [Av_n] \end{bmatrix} = \begin{bmatrix} [\lambda_1 v_1] & ... & [\lambda_n v_n] \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 & ... & \lambda_n e_n \end{bmatrix}$, is diagonal. Thus, by theorem 5.1.3, then A is diagonalizable.

Since $v_1, ..., v_n$ is a basis, by theorem 2.5.4, then AB = B$A_{\mathcal{B}}$.

Theorem 5.1.6: Eigenvalues of an Orthogonal matrix

For orthogonal A $\in M_{n\times n}(\mathbb{R})$, the only possible eigenvalues are ± 1

Proof

Since A is orthogonal, then $|Ax| = |x|$. If $Ax = \lambda x$, then:

$|x| = |Ax| = |\lambda x| = |\lambda||x|$

Thus, $|\lambda| = 1$ so $\lambda = \pm 1$.

Theorem 5.1.7: Eigenvalues and Invertibility

A $\in M_{n\times n}(\mathbb{R})$ is invertible if and only if 0 is not an eigenvalue of A

Proof

Suppose A is invertible. Then by theorem 2.4.5, the only solution to Ax = 0 is x = 0. Thus, if $\lambda = 0$, then Ax = λx = 0 has only x = 0 $\neq$ 0 so $\lambda = 0$ is not an eigenvalue.

Suppose $\lambda = 0$ is not an eigenvalue of A. Then there are no nonzero x $\in \mathbb{R}^n$ such that Ax = λx = 0 so only x = 0. Thus, by theorem 2.4.5, A is invertible.

5.2 Characteristic Polynomial

Theorem 5.2.1: Determining Eigenvalues

$\lambda \in \mathbb{R}$ is an eigenvalue of A $\in M_{n\times n}(\mathbb{R})$ if and only if $\det(A - \lambda I_{n\times n}) = 0$

Proof

Suppose $\lambda \in \mathbb{R}$ is an eigenvalue of A. Then there is a nonzero v $\in \mathbb{R}^n$ such that Av = λv so $0 = Av - \lambda v = (A - \lambda I_{n\times n})v$. Since nonzero v $\in \ker(A - \lambda I_{n\times n})$, then $A - \lambda I_{n\times n}$ is not invertible. By theorem 4.2.3, then $\det(A - \lambda I_{n\times n}) = 0$.

Suppose $\det(A - \lambda I_{n\times n}) = 0$. By theorem 4.2.3, then $A - \lambda I_{n\times n}$ is not invertible so there is a nonzero v $\in \mathbb{R}^n$ such that $(A - \lambda I_{n\times n})v = 0$. Thus, Av = λv so λ is an eigenvalue.

Theorem 5.2.2: $\det(A - \lambda I_{n\times n}) = \det(A_{\mathcal{B}} - \lambda I_{n\times n})$

Let $\mathcal{B} = v_1, ..., v_n$ be a basis for $\mathbb{R}^n$. Then:

$\det(\text{A}) = \det(A_{\mathcal{B}})$ $\qquad \det(A - \lambda I_{n\times n}) = \det(A_{\mathcal{B}} - \lambda I_{n\times n})$

Proof

By theorem 2.5.4, AB = B$A_{\mathcal{B}}$ where B = $\begin{bmatrix} v_1 & ... & v_n \end{bmatrix}$ is invertible. Thus:

$$\det(\text{A}) = \det(BA_{\mathcal{B}}B^{-1}) = \det(\text{B})\det(A_{\mathcal{B}})\det(B^{-1}) = \det(\text{B})\det(A_{\mathcal{B}})(\det(\text{B}))^{-1} = \det(A_{\mathcal{B}})$$

For i = {1,..,n}:

$$[(A - \lambda I_{n\times n})]_{\mathcal{B}}[v_i]_{\mathcal{B}} = [(A - \lambda I_{n\times n})v_i]_{\mathcal{B}} = [Av_i]_{\mathcal{B}} - [\lambda v_i]_{\mathcal{B}} = A_{\mathcal{B}}[v_i]_{\mathcal{B}} - \lambda[v_i]_{\mathcal{B}} = (A_{\mathcal{B}} - \lambda I_{n\times n})[v_i]_{\mathcal{B}}$$

Thus, $\det(A - \lambda I_{n\times n}) = \det([A - \lambda I_{n\times n}]_{\mathcal{B}}) = \det(A_{\mathcal{B}} - \lambda I_{n\times n})$.

Definition 5.2.3: Characteristic Polynomial

Since the entries, a^*_{ij}, of $A - \lambda I_{n\times n}$ are the same as then entries of A except on its diagonal entries, $a^*_{ii} = a_{ii} - \lambda$, then $\det(A - \lambda I_{n\times n}) = \sum_{\{i_1,...,i_n\}=\{1,...,n\}} a_{1i_1}, ..., a_{ni_n}(-1)^I$ contains an arrangement $a_{1i_1}, ..., a_{ni_n}(-1)^I = (a_{11} - \lambda)(a_{22} - \lambda)...(a_{nn} - \lambda)$ which is a polynomial of degree n and the other arrangements contain less than n $(a_{ii} - \lambda)$ so $\det(A - \lambda I_{n\times n})$ is a polynomial of degree n with variable λ.

For A $\in M_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then $\det(A - \lambda I_{n\times n})$ is the characteristic polynomial of A.

Definition 5.2.4: Algebraic Multiplicity

Let A $\in M_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

Then, eigenvalue $\lambda^* \in \mathbb{R}$ of A has an algebraic multiplicity of k if:

$\det(A - \lambda I_{n\times n}) = (\lambda - \lambda^*)^k f(\lambda)$

where $f(\lambda)$ is a polynomial of degree n-k with $f(\lambda^*) \neq 0$. Then, almu(λ^*) = k $\geq$ 1.

Theorem 5.2.5: Relationship between Algebraic Multiplicity and Dimension

For A $\in M_{n\times n}(\mathbb{R})$, let $\lambda_1, ..., \lambda_k \in \mathbb{R}$ be distinct eigenvalues of A. Then k $\leq$ n where:

almu(λ_1) + ... + almu(λ_k) $\leq$ n

If $\lambda_1, ..., \lambda_k \in \mathbb{C}$, then k = n and almu$(\lambda_1)$ + ... + almu(λ_k) = n.

Proof

Since $\det(A - \lambda I_{n\times n})$ is a polynomial of degree n, then there are at most n real roots so k $\leq$ n. If $\lambda_1, ..., \lambda_k \in \mathbb{R}$ are distinct eigenvalues of A, then:

$\det(A - \lambda I_{n\times n}) = (\lambda - \lambda_1)^{m_1}...(\lambda - \lambda_k)^{m_k} f(\lambda)$

almu(λ_1) + ... + almu(λ_k) = $m_1 + ... + m_k \leq$ n

For $\lambda_1, ..., \lambda_k \in \mathbb{C}$, then:

$\det(A - \lambda I_{n\times n}) = (\lambda - \lambda_1)^{m_1}...(\lambda - \lambda_k)^{m_k}$

almu(λ_1) + ... + almu(λ_k) = $m_1 + ... + m_k$ = n

Corollary 5.2.6: Relationship between Algebraic Multiplicity and Determinant

Let $\lambda_1, ..., \lambda_k \in \mathbb{C}$ be distinct eigenvalues of A $\in M_{n\times n}(\mathbb{R})$ with almu(λ_i) $= m_i \in \mathbb{R}$. Then:

det(A) $= \lambda_1^{m_1} ... \lambda_k^{m_k}$

Proof

By theorem 5.2.5, almu(λ_1) + ... + almu(λ_k) = n. Thus:

$\det(A - \lambda I_{n\times n}) = (\lambda - \lambda_1)^{m_1} ... (\lambda - \lambda_k)^{m_k}$

Then for $\lambda = 0$:

det(A) $= \lambda_1^{m_1} ... \lambda_k^{m_k}$

5.3 Eigenspaces

Definition 5.3.1: Eigenspaces

For A $\in M_{n\times n}(\mathbb{R})$, let $\lambda \in \mathbb{R}$ be an eigenvalue. Then there is an eigenvector v $\neq 0 \in \mathbb{R}^n$ such that Av = λv so $0 = Av - \lambda v = (A - \lambda I_{n\times n})v$. Thus, v $\in \ker(A - \lambda I_{n\times n})$.

Then, the eigenspace of A for λ, the set of all eigenvectors v with eigenvalue λ:

$E_\lambda = \ker(A - \lambda I_{n\times n})$

Definition 5.3.2: Geometric Multiplicity

Let A $\in M_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

Then, the of geometric multiplicity of eigenvalue $\lambda^* \in \mathbb{R}$ of A:

gemu(λ^*) = dim(ker(E_{λ^*})) = dim(ker($A - \lambda^* I_{n\times n}$)) = nullity($A - \lambda^* I_{n\times n}$)

Since λ^* is an eigenvalue, then there is a eigenvector v $\neq 0 \in \ker(A - \lambda^* I_{n\times n})$ so dim(ker($A - \lambda^* I_{n\times n}$)) = gemu($\lambda$) ≥ 1.

Theorem 5.3.3: Relationship between Algebraic Multiplicity and Geometric Multiplicity

Let $\lambda^* \in \mathbb{R}$ be an eigenvalue of A $\in M_{n\times n}(\mathbb{R})$. Then:

gemu(λ^*) $\leq$ almu(λ^*)

Thus, if $\lambda_1, ..., \lambda_k \in \mathbb{R}$ are distinct eigenvalues of A, then:

gemu(λ_1) + ... + gemu(λ_k) $\leq$ n

Proof

Let gemu(λ^*) = m so dim(ker($A - \lambda^* I_{n\times n}$)) = m. Thus, let $v_1, ..., v_m$ form a basis for E_{λ^*}.
Then choose $u_1, ..., u_{n-m} \in \mathbb{R}^n$ such that $\mathcal{B} = v_1, ..., v_m, u_1, ..., u_{n-m}$ form a basis for $\mathbb{R}^n$.
Since A$v_i = \lambda^* v_i$ for i = {1,...,m}, then for $A_\mathcal{B} \in M_{n\times n}(\mathbb{R})$:

$$A_\mathcal{B} = \begin{bmatrix} \lambda^* & 0 & \dots & 0 & a_{11} & a_{12} & \dots & a_{1(n-m)} \\ 0 & \lambda^* & \dots & 0 & a_{21} & a_{22} & \dots & a_{2(n-m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^* & a_{m1} & a_{m2} & \dots & a_{m(n-m)} \\ 0 & 0 & \dots & 0 & a_{(m+1)1} & a_{(m+1)2} & \dots & a_{(m+1)(n-m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_{n1} & a_{n2} & \dots & a_{n(n-m)} \end{bmatrix} = \begin{bmatrix} \lambda^* I_{m\times n} & B_{m\times(n-m)} \\ 0_{(n-m)\times(n-m)} & D_{(n-m)\times(n-m)} \end{bmatrix}$$

where $[Au_j]_\mathcal{B} = (a_{1j}, ..., a_{nj})$ for j $\in$ {1,...,n-m}. Thus:

$\det(A_\mathcal{B} - \lambda I_{n\times n}) = (\lambda^* - \lambda)^m \det(D - \lambda I_{(n-m)\times(n-m)})$

By theorem 5.2.2, then $\det(A - \lambda I_{n\times n}) = \det(A_\mathcal{B} - \lambda I_{n\times n})$.
Thus, $\det(A - \lambda I_{n\times n}) = (\lambda^* - \lambda)^m \det(D - \lambda I_{(n-m)\times(n-m)})$ so gemu(λ^*) = m $\leq$ almu(λ^*).
Then for eigenvalues $\lambda_1, ..., \lambda_k \in \mathbb{R}$ for A, by theorem 5.2.5:

gemu(λ_1) + ... + gemu(λ_k) $\leq$ almu(λ_1) + ... + almu(λ_k) $\leq$ n

Theorem 5.3.4: Vandermonde Matrix

For $a_1, ..., a_n \in \mathbb{R}$, the Vandermonde Matrix, $\mathrm{V} \in M_{n\times n}(\mathbb{R})$:

$$\mathrm{V} = \begin{bmatrix} 1 & a_1 & a_1^2 & ... & a_1^{n-1} \\ 1 & a_2 & a_2^2 & ... & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & ... & a_n^{n-1} \end{bmatrix}$$

Then, $\det(\mathrm{V}) = \prod_{1\leq i<j\leq n} (a_j - a_i)$

Proof

Let $\mathrm{V} = V_0$. Let V_i be V with the first i-1 rows and last i-1 columns removed. Let W_i be V_i, but each j-th column from the last to the second is subtracted by a_i times the (j-1)-th column.

$$W_1 = \begin{bmatrix} 1 & a_1 - a_1 & a_1^2 - a_1a_1 & ... & a_1^{n-1} - a_1^{n-2}a_1 \\ 1 & a_2 - a_1 & a_2^2 - a_2a_1 & ... & a_2^{n-1} - a_2^{n-2}a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n - a_1 & a_n^2 - a_na_1 & ... & a_n^{n-1} - a_n^{n-2}a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & a_2 - a_1 & a_2(a_2 - a_1) & ... & a_2^{n-2}(a_2 - a_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n - a_1 & a_n(a_n - a_1) & ... & a_n^{n-2}(a_n - a_1) \end{bmatrix}$$

By theorem 4.2.1, $\det(W_1) = \det(V_1) = \det(\mathrm{V})$. Let A_i be W_i be the first row and column removed. By theorem 4.1.7, for $\det(W_1)$ across the first row, since only the first entry in the first row is nonzero, then $\det(W_1) = \det(A_1)$. But for each k-th row in A_i, $(a_{k+i} - a_i)$ is a factor for k = {1,...,n-i}. After factoring out each $(a_{k+1} - a_i)$, then A_i becomes V_{i+2}.

Thus, by theorem 4.2.1:

$$\det(V_1) = \det(W_1) = \det(A_1) = (a_2 - a_1)(a_3 - a_1)...(a_n - a_1)\det(V_2) = (\textstyle\prod_{i_1=2}^{n}(a_{i_1} - a_1))\det(V_2)$$

Repeating the process until V_n:

$$\det(V_1) = (\textstyle\prod_{i_1=2}^{n}(a_{i_1} - a_1))\det(V_2) = (\prod_{i_1=2}^{n}(a_{i_1} - a_1))\ (\prod_{i_2=3}^{n}(a_{i_2} - a_2))\det(V_3)$$
$$= ... = (\textstyle\prod_{i_1=2}^{n}(a_{i_1} - a_1))\ (\prod_{i_2=3}^{n}(a_{i_2} - a_2))\ ...(\prod_{i_{n-1}=n}^{n}(a_{i_{n-1}} - a_{n-1}))\det(V_n)$$

Since $V_n = [1]$, then $\det(V_n) = 1$. Thus:

$$\det(\mathrm{V}) = \det(V_1) = (\textstyle\prod_{i_1=2}^{n}(a_{i_1} - a_1))\ ...(\prod_{i_{n-1}=n}^{n}(a_{i_{n-1}} - a_{n-1})) = \prod_{1\leq i<j\leq n} (a_j - a_i)$$

Theorem 5.3.5: Eigenvectors from different Eigenspaces are Linearly independent

Let $\lambda_1, ..., \lambda_k \in \mathbb{R}$ be distinct eigenvalues of $\mathrm{A} \in M_{n\times n}(\mathbb{R})$ with eigenvector $v_i \in E_{\lambda_i}$. Then, $v_1, ..., v_k$ are linearly independent.

Proof

Let $c_1, ..., c_k \in \mathbb{R}$ be such that $c_1v_1 + ... + c_kv_k = 0$. Since $\mathrm{A}v_i = \lambda_iv_i$, then for m = {1,...,k}:

$$A^mv_i = A^{m-1}(\lambda_iv_i) = \lambda_iA^{m-1}(v_i) = \lambda_iA^{m-2}(\lambda_iv_i) = \lambda_i^2A^{m-2}(v_i) = ... = \lambda_i^mv_i$$

Thus, the system of equations:

$$0 = A0 = A(c_1v_1 + ... + c_kv_k) = c_1\lambda_1v_1 + ... + c_k\lambda_kv_k$$
$$0 = A^20 = A^2(c_1v_1 + ... + c_kv_k) = c_1\lambda_1^2v_1 + ... + c_k\lambda_k^2v_k$$
$$\vdots$$
$$0 = A^k0 = A^k(c_1v_1 + ... + c_kv_k) = c_1^k\lambda_1v_1 + ... + c_k^k\lambda_kv_k$$

as a matrix:

$$0_{n\times k} = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} c_1\lambda_1 & c_1\lambda_1^2 & \dots & c_1\lambda_1^k \\ c_2\lambda_2 & c_2\lambda_2^2 & \dots & c_2\lambda_2^k \\ \vdots & \vdots & \ddots & \vdots \\ c_k\lambda_k & c_k\lambda_k & \dots & c_k\lambda_k^k \end{bmatrix} = \begin{bmatrix} v_1 & ... & v_k \end{bmatrix} \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_k \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_1^2 & \dots & \lambda_1^k \\ \lambda_2 & \lambda_2^2 & \dots & \lambda_2^k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k & \lambda_k & \dots & \lambda_k^k \end{bmatrix} = \mathrm{VCM}$$

Since each i-th row has a factor of λ_i, by theorem 4.2.1 and 5.3.4, then:

$$\det(\mathrm{M}) = \lambda_1...\lambda_k \prod_{1\leq i<j\leq k} (\lambda_j - \lambda_i)$$

Since each $\lambda_i \neq 0$ are distinct, then $\det(\mathrm{M}) \neq 0$ so by theorem 4.2.2, then M is invertible.

$$0_{n\times k} = 0_{n\times k}M^{-1} = \mathrm{VCM}M^{-1} = \mathrm{VC} = \begin{bmatrix} c_1v_1 & ... & c_kv_k \end{bmatrix}$$

Since eigenvectors $v_i \neq 0$, then each $c_i = 0$. Thus, $v_1, ..., v_k$ are linearly independent.

Corollary 5.3.6: Diagonalizable $M_{n\times n}(\mathbb{R}) \Leftrightarrow \sum$ gemu(λ_i) = n

Let $\lambda_1, ..., \lambda_k \in \mathbb{R}$ be distinct eigenvalues of A $\in M_{n\times n}(\mathbb{R})$.

Then, A is diagonalizable if and only if gemu(λ_1) + ... + gemu(λ_k) = n.

Proof

Suppose A is diagonalizable. By theorem 5.1.5, there exists an eigenbasis $v_1, ..., v_n$ for A. Since each $v_i \in E_{\lambda_j}$ for some j = {1,...,k}, then n $\leq$ gemu(λ_1) + ... + gemu(λ_k). By theorem 5.3.3, then gemu(λ_1) + ... + gemu(λ_k) = n.

Suppose gemu(λ_1) + ... + gemu(λ_k) = n. Let eigenvectors $v_{i1}, ..., v_{im_i}$ form a basis for E_{λ_i} so $m_1 + ... + m_k$ = n. By theorem 5.3.5, eigenvectors from different eigenspaces are linearly independent so $v_{11}, ..., v_{1m_1}, v_{21}, ..., v_{2m_2}, ..., v_{k1}, ..., v_{km_k}$ are linearly independent. By theorem 2.3.5, $v_{11}, ..., v_{1m_1}, v_{21}, ..., v_{2m_2}, ..., v_{k1}, ..., v_{km_k}$ form a basis of eigenvectors. By theorem 5.1.5, then A is diagonalizable.

5.4 Symmetry

Definition 5.4.1: Orthogonally Diagonalizable

Matrix A $\in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable if there is an orthogonal matrix B such that $B^{-1}AB$ is diagonal

Since B is orthogonal, then by theorem 3.3.2, $B^{-1} = B^T$ so $B^{-1}AB = B^TAB$ is diagonal.

By theorem 5.1.5, then B = $\begin{bmatrix} v_1 & ... & v_n \end{bmatrix}$ where $v_1, ..., v_n$ is an eigenbasis for A. Since B is orthogonal, then by theorem 3.3.2, $v_1, ..., v_n$ is orthonormal. Thus, if A is orthogonally diagonalizable, there is an orthonormal eigenbasis for A.

Theorem 5.4.2: Eigenvectors from different Eigenspaces are Orthogonal

Let λ_1, λ_2 be distinct eigenvalues of symmetric A $\in M_{n\times n}(\mathbb{R})$.

For $x_1 \in E_{\lambda_1}$ and $x_2 \in E_{\lambda_2}$, then $x_1 \cdot x_2 = 0$

Proof

$$\begin{aligned} \lambda_1(x_1 \cdot x_2) &= \lambda_1 x_1 \cdot x_2 = Ax_1 \cdot x_2 = (Ax_1)^T x_2 = x_1^T A^T x_2 \\ &= x_1 \cdot A^T x_2 = x_1 \cdot Ax_2 = x_1 \cdot (\lambda_2 x_2) = \lambda_2(x_1 \cdot x_2) \end{aligned}$$

Since $(\lambda_1 - \lambda_2)(x_1 \cdot x_2) = 0$, but $\lambda_1 \neq \lambda_2$, then $x_1 \cdot x_2 = 0$.

Theorem 5.4.3: Symmetric $M_{n\times n}(\mathbb{R})$ has n real Eigenvalues

Let $\lambda_1, ...\lambda_k \in \mathbb{R}$ be distinct eigenvalues of symmetric A $\in M_{n\times n}(\mathbb{R})$ with almu$(\lambda_i) = m_i$.

Then, $m_1 + ... + m_k$ = n.

Proof

Let $p_1(\lambda)$: $\mathbb{R} \to \mathbb{R}$ be the characteristic polynomial of A and let $p_2(\lambda)$: $\mathbb{C} \to \mathbb{C}$ be the characteristic polynomial of A. Thus, $p_2(\lambda) = p_1(\lambda) = \det(A - \lambda I_{n\times n})$ if $\lambda \in \mathbb{R}$.

Let $p_2(\lambda) = (\lambda - \lambda_1)...(\lambda - \lambda_n)$. Let v = $v_1 + v_2 i$ be an eigenvector with eigenvalue $\lambda_i = a + bi$.

$$Av_1 + iAv_2 = A(v_1 + v_2 i) = Av = \lambda_i v = (a + bi)(v_1 + v_2 i) = (av_1 - bv_2) + (av_2 + bv_1)i$$

Since:

$$A\overline{v} = A(v_1 - v_2 i) = Av_1 - iAv_2 \qquad \overline{\lambda_i}\overline{v} = (a - bi)(v_1 - v_2 i) = (av_1 - bv_2) - (av_2 + bv_1)i$$

then $A\overline{v} = \overline{\lambda_i}\overline{v}$. Thus, $\overline{\lambda_i}$ is an eigenvalue of A with eigenvector $\overline{v}$. Then:

$$\overline{v}^T Av = \overline{v}^T(\lambda_i v) = \lambda_i(\overline{v}^T v) = \lambda_i |v|^2 \qquad \overline{v}^T Av = \overline{v}^T \overline{A^T} v = \overline{Av}^T v = \overline{\lambda_i v}^T v = \overline{\lambda_i}\,\overline{v}^T v = \overline{\lambda_i}|v|^2$$

Since $(\lambda_i - \overline{\lambda_i})|v|^2 = 0$, but $|v| > 0$ since v $\neq$ 0, then $a + bi = \lambda_i = \overline{\lambda_i} = a - bi$. Thus, b = 0 so all eigenvalues are real. Thus, any $p_1(\lambda) = p_2(\lambda) = (\lambda - \lambda_1)...(\lambda - \lambda_n)$ so A has n real eigenvalues when including their algebraic multiplicity.

Theorem 5.4.4: Spectral Theorem: Symmetric matrices have an orthonormal eigenbasis

Matrix A $\in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable if and only if A is symmetric

Proof

Suppose A is orthogonally diagonalizable. Thus, A has an $\mathcal{B} = v_1, ..., v_n \in \mathbb{R}^n$ is an orthonormal eigenbasis with eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$. Since A is diagonalizable, then by theorem 5.1.5, for B = $\begin{bmatrix} v_1 & ... & v_n \end{bmatrix}$ and $A_{\mathcal{B}} = \begin{bmatrix} \lambda_1 e_1 & ... & \lambda_n e_n \end{bmatrix}$:

AB = B$A_{\mathcal{B}}$

By theorem 3.3.2, B is invertible where $B^{-1} = B^T$. Thus, A = $BA_{\mathcal{B}}B^{-1} = BA_{\mathcal{B}}B^T$. Then:

$A^T = (BA_{\mathcal{B}}B^T)^T = (B^T)^T A_{\mathcal{B}}^T B^T = BA_{\mathcal{B}}B^T$ = A

Thus, A is symmetric.

Suppose A is symmetric. If n = 1, then let B = [1] which is orthogonal since $B^T B = I_{1\times 1}$ by theorem 3.3.2. Then, $B^{-1}AB = [1][a][1] = [a]$ is diagonal.

Suppose for k $\leq$ n, then any symmetric $M_{(k-1)\times(k-1)}(\mathbb{R})$ is orthogonally diagonalizable. Then for symmetric A $\in M_{k\times k}(\mathbb{R})$, by theorem 5.4.3, there is an eigenvalue λ with eigenvector v_1 such that $|v| = 1$. From v_1, let $v_1, u_2, ..., u_k$ be a basis for $\mathbb{R}^k$. Apply theorem 3.2.9 to get an orthonormal basis $v_1, ..., v_k$. Let B = $\begin{bmatrix} v_1 & ... & v_k \end{bmatrix}$ so B is orthogonal by 3.3.2 so $B^T = B^{-1}$. Note $B^T ABe_1 = B^T Av_1 = B^T(\lambda v_1) = \lambda(B^T v_1) = \lambda e_1$. Also, $(B^T AB) = B^T A^T (B^T)^T = B^T AB$ so $B^T AB$ is symmetric. Thus:

$$B^T AB = \begin{bmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & B^*_{(k-1)\times(k-1)} \end{bmatrix}$$

Since $B^* \in M_{(k-1)\times(k-1)}(\mathbb{R})$, then B^* is orthogonally diagonalizable so there is an orthogonal C $\in M_{(k-1)\times(k-1)}(\mathbb{R})$ such that D = $C^{-1}B^*C$ is diagonal. Let X = $\begin{bmatrix} 1 & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & C \end{bmatrix}$. Then:

$$\begin{aligned} X^T B^T ABX &= \begin{bmatrix} 1 & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & C^T \end{bmatrix} \begin{bmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & B^* \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & C \end{bmatrix} \\ &= \begin{bmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & C^T B^* C \end{bmatrix} = \begin{bmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & D \end{bmatrix} \end{aligned}$$

Since $X^T B^T ABX = (BX)^T A(BX)$ is diagonal where BX is orthogonal since B,X are orthogonal by corollary 3.3.4, then A $\in M_{k\times k}(\mathbb{R})$ is orthogonally diagonalizable. Thus, by proof by induction, A $\in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable.

5.5 Quadratic Forms

References

[1] Otto Bretscher, *Linear Algebra with Applications (4th Edition)*, ISBN-13: 978-0321796974