Fall Real Analysis

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Contents

1   The Real Number System

1.1   Number Systems

$$\text{Natural}: \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer}: \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational}: \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** Q is countable, but fails to have the least upper bound property ***

Example 1.1.1
    Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.
Proof
Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A = $\{x \in \mathbb{Q} \text{ for } x^2 < 2\}$ where $A \neq \emptyset$ and 2 is an upper bound for A.
Then, A has no least upper bound in $\mathbb{Q}$, but A has a least upper bound in $\mathbb{R}$.

1.2   Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
Also, $\mathbb{R}$ exists and unique.

Definition 1.2.1: Order
    Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:

  - x < y
  - x = y
  - x > y

- Transitivity: If x < y and y < z, then x < z.

Definition 1.2.2: Ordered Set
    An ordered set is a set with an order.

Definition 1.2.3: Bounds
    Let S be an ordered set and E $\subset$ S.
    An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
        If such a β exists, then E is bounded from above.
    A lower bound of E is a $\alpha \in$ S if x $\geq \alpha$ for all x $\in$ E.
        If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum
Let S be an ordered set.
Let $E \subset S$ be bounded from above. Least upper bound $\beta \in S$ exists if:

- β is an upper bound for E

- If $\gamma < \beta$, then γ is not an upper bound for E.

Then $\beta = \sup(E)$.

Let $E \subset S$ be bounded from below. Greatest lower bound $\alpha \in S$ exists if:

- α is a lower bound for E

- If $\gamma > \alpha$, then γ is not a lower bound for E.

Then $\alpha = \inf(E)$.

Example 1.2.5
Let $S = (1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and $\sup(E) = 3$

- E = (5,6) is not bounded above so $\sup(E) = $ DNE

- E = [3,4) is bounded below $\inf(E) = 3$ and $\sup(E) = $ DNE

Observations on the Least Upper Bound
If $\sup(E)$ exists, it may or may not exists at S.
If $\sup(E)$ exists, then $\sup(E)$ is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3   Least Upper Bound Property

Theorem 1.3.1
An ordered set S has a least upper bound property if:
For every nonempty subset $E \subset S$ that is bounded from above:
$\sup(E)$ exists in S.

Example 1.3.2
$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.
Proof
Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set $A = \{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then $z > y$ where $z \in A$. So, y is not a upper bound.

  Since for any y, there is $z > y$ where $z \in A$, then $\sup(A)$ doesn't exists in $\mathbb{Q}$.

- If $y^2 - 2 > 0$, then $z < y$ where $z \in B$. So, y is an upper bound, but not $\sup(E)$.

  Since for any y, there is $z < y$ where $z \in B$, then $\inf(B)$ doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound.

2   Day 2: Fields

2.1   Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound implies Greatest Upper Bound
Let S be a ordered set with the least upper bound property.
Let non-empty B $\subset$ S be bounded below.
Let L be the set of all lower bounds of B.
Then $\alpha = \sup(L)$ exists in S.
Proof
L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, $\alpha = \sup(L)$ exists in S.
We claim that $\alpha = \inf(B)$.
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B.
Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, $\inf(B) = \alpha$.

2.2   Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F

- x+y = y+x for all x,y $\in$ F

- (x+y)+z = x+(y+z) for all x,y,z $\in$ F

- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F

- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative xioms

- If x,y $\in$ F, then xy $\in$ F

- yx = xy for all x,y $\in$ F

- (xy)z = x(yx) for all x,y,z $\in$ F

- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F

- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x} \in$ F where x$(\frac{1}{x})$ = 1

Distributive Law
x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Definition 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0

    Proof

        From (a), let z = 0.

(c) If x+y = 0, then y = -x

    Proof

        From (a), let z = -x.

(d) -(-x) = x

    Proof

        From (c), let x = -x and y = x.

(e) If $x \neq 0$ and xy = xz, then y = z

    Proof

        $y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$

(f) If $x \neq 0$ and xy = x, then y = 1

    Proof

        From (e), let z = 1.

(g) If $x \neq 0$ and xy = 1, then $y = \frac{1}{x}$

    Proof

        From (e), let $z = \frac{1}{x}$.

(h) If $x \neq 0$, then $\frac{1}{1/x} = x$

    Proof

        From (g), let $x = \frac{1}{x}$ and y = x.

(i) 0x = 0

    Proof

        Since 0x + 0x = (0+0)x = 0x, then 0x = 0.

(j) If $x,y \neq 0$, then $xy \neq 0$

    Proof

        Suppose xy = 0, then $\frac{1}{y}\frac{1}{x}xy = \frac{1}{y}1y = \frac{1}{y}y = 1$.

        xy = 0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)

    Proof

        xy + (-x)y = (x+(-x))y = 0y = 0.

        Then by part (c), (-x)y = -(xy).

        Similarly, xy + x(-y) = x(y+(-y)) = x0 = 0.

        Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy

Proof

By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].

By part (d), -[-(xy)] = xy.

2.3   Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x

- If x,y > 0, then xy > 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field.

Definition 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

- If x > 0, -x < 0 and vice versa

Proof

-x = (-x) + 0 < (-x) + x = 0

- If x > 0 and y < z, then xy < xz

Proof

Since z-y > 0, then 0 < x(z-y) = xz - xy

- If x < 0 and y < z, then xy > xz

Proof

Since -x > 0 and z-y > 0, then 0 < -x(z-y) = xy - xz

- If x $\neq$ 0, $x^2 > 0$

Proof

If x > 0, then $x^2 = x \cdot x > 0$

If x < 0, then $x^2 = (-x) \cdot (-x) > 0$

- If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

Since $(\frac{1}{y})y = 1 > 0$, then $(\frac{1}{y}) > 0$

Since x < y, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$

Theorem 2.3.3: $\mathbb{R}$ is a ordered field with <

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$.

Theorem 2.3.4

For all x,y $\in \mathbb{R}$:

- **Archimedean Property**: If x > 0, there is n $\in \mathbb{Z}$ such that nx > y.

  Proof

  Fix x > 0. Suppose there is a y such that the property fails.

  Let A = { nx: n = 1, 2, 3, ... }.

  Then, A is nonempty and bounded from above by y.

  Then by the least upper bound property by $\mathbb{R}$, $\alpha = \sup(A)$ exists in $\mathbb{R}$.

  Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$.

  So $\alpha - x$ is not an upper bound of A.

  So there is a mx $\in$ A such that mx $> \alpha - x$

  But then $\alpha < $ (m+1)x where (m+1)x $\in$ A which contradicts α is an upper bound for A.

- **$\mathbb{Q}$ is dense in $\mathbb{R}$**: If x < y, there is a p $\in \mathbb{Q}$ such that x < p < y.

  Proof

  Since x < y, then y-x > 0. Then by the Archimedean Property, there exists a n $\in$ Z such that n(y-x) > 1. Thus, ny > nx+1 > nx

  By the well-ordering principle, there is a smallest m $\in \mathbb{Z}_+$ such that m > nx.

  Then, m > nx $\geq$ m-1 so nx+1 $\geq$ m > nx.

  Since ny > nx+1 $\geq$ m > nx, then y > m/n > x.

3   Roots & Complex

3.1   nth Root

- If $0 < t \leq 1$, then $t^n \leq t$.

  Proof

  Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.

  Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.

  Applying the process n times, then $t^n \leq t$.

- If $t \geq 1$, $t^n \geq t$.

  Proof

  Since $0 < 1 \leq t$, then $t \leq t^2$.

  Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.

  Applying the process n times, $t \leq t^n$.

- If $0 < s < t$, $s^n < t^n$.

  Proof

  $$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix n. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = y_1{}^n < y_2{}^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ and $< x$ and thus, $0 < t_1{}^n < t_1 < x$ so $t_1 \in A$.

A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

So x+1 is an upper bound of A.

Thus by the least upper bound property, $y = \sup(A)$ exists.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

(Not an upper bound of A if $<$ and not a least upper bound of A if $>$)

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\underbrace{\beta^{n-1}}_{\beta^{n-1}} + \underbrace{\beta^{n-2}\alpha^1}_{<\beta^{n-1}} + ... + \underbrace{\alpha^{n-1}}_{<\beta^{n-1}})} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.

From inequality, let $\beta = y+h$ and $\alpha = y$

$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.

Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus, $t^n > x$ so $t \notin A$.

Thus, y-k is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

3.2   Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.
Then let n_k be the largest integer such that:

$\quad d_k = n_0 + \frac{n_1}{10} + ... + \frac{n_k}{10^k} \leq x$

Let E be the set of d_k for $k = 0, 1, ... \infty$. Then, $x = \sup(E)$.

3.3   Extended Reals

The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:

$\quad -\infty < x < \infty \qquad$ for every $x \in \mathbb{R}$

with the properties:

- $x \pm \infty = \pm\infty$

- $x \, / \pm\infty = 0$

- If $x > 0$, then $x(\pm\infty) = \pm\infty$

- If $x < 0$, then $x(\pm\infty) = \mp\infty$

3.4   Complex Numbers

Definition 3.3.1: Complex
$\qquad$ A complex number is an ordered pair (a,b) where $a,b \in \mathbb{R}$. For $x,y \in \mathbb{C}$

- $x + y = (a,b) + (c,d) = (a + c , b + d)$

- $xy = (a,b) (c,d) = (ac - bd , ad + bc)$

- $\frac{1}{x} = (a^2 + b^2)(a,-b)$

Thus, the axioms form a field where $(0,0) = 0$ and $(1,0) = 1$ and $(0,1) = i$.

Definition 3.3.2: Imaginary i
$\qquad$ Let $i = (0,1)$. Then, $i^2 = -1$.
Proof
$i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1$

Definition 3.3.3: Form a + bi
$\qquad$ $(a,b) = a + bi$
Proof
$(a,b) = (a,0) + (0,b) = (a,0) + (b,0)(0,1) = a + bi$

Definition 3.3.4: Conjugate
$\qquad$ Let conjugate: $\bar{z} = a - bi$ where $\text{Re}(z) = a$, $\text{Im}(z) = b$

$\qquad$ Let $z = (a,b)$ and $w = (c,d)$:

- $\overline{z + w} = \bar{z} + \bar{w}$

    Proof

    $\overline{z + w} = \overline{(a + c, b + d)} = (a+c,-b-d) = (a,-b) + (c,-d) = \bar{z} + \bar{w}$

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- $\overline{zw} = \overline{z}\ \overline{w}$

  Proof

  $\overline{zw} = \overline{(ac - bd, ad + bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\overline{z}\ \overline{w}$

- z + $\overline{z}$ = 2 Re(z)   z - $\overline{z}$ = 2i Im(z)

  Proof

  z + $\overline{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)

  z - $\overline{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

- $z\overline{z} \geq 0$

  Proof

  $z\overline{z}$ = (a,b)(a,-b) = ($a^2 + b^2$, -ab+ab) = $a^2 + b^2 \geq 0$

Definition 3.3.5: Absolute Value
Let absolute value: $| z | = \sqrt{z\overline{z}}$

Let z = (a,b) and w = (c,d):

- If z $\neq$ 0, then $| z | > 0$.

  Proof

  $\sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \geq 0$ where $| z | = 0$ only if a,b = 0 so only if z = (0,0).

- $| \overline{z} | = | z |$

  Proof

  $| \overline{z} | = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = | z |$

- $| zw | = | z | | w |$

  Proof

  $| zw | = | $ (ac-bd,ad+bc) $ | = \sqrt{(ac - bd)^2 + (ad + bc)^2}$

  $= \sqrt{a^2 c^2 + b^2 d^2 + a^2 d^2 + b^2 c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)}$

  $= \sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2} = | z | | w |$

- $| Re(z) | \leq | z |$

  Proof

  $| Re(z) | = | a | = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = | z |$

- $| z+w | \leq | z | + | w |$

  Proof

  $|z + w|^2 = $ (z+w)$\overline{(z + w)}$ = (z+w)$(\overline{z} + \overline{w})$ = $z\overline{z}$ + $z\overline{w}$ + w$\overline{z}$ + w$\overline{w}$

  $= |z|^2 + |w|^2 + 2$ Re(z$\overline{w}$) $\leq |z|^2 + |w|^2 + 2$ $|z\overline{w}|$ $= |z|^2 + |w|^2 + 2|z||w|$

  $= (|z| + |w|)^2$

4   Temp

References