Fall Real Analysis

Azure

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Contents

1 The Real Number System

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** $\mathbb{Q}$ is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A $= \{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
But, A has no least upper bound in $\mathbb{Q}$, but A has a least upper bound in $\mathbb{R}$.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
Also, $\mathbb{R}$ exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x $<$ y
  - x $=$ y
  - x $>$ y
- Transitivity: If x $<$ y and y $<$ z, then x $<$ z.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.
A lower bound of E is a $\alpha \in$ S if x $\geq \alpha$ for all x $\in$ E.
If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.

Let E $\subset$ S be bounded from above. Least upper bound $\beta \in$ S exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E.
  Then β = sup(E).

Let E $\subset$ S be bounded from below. Greatest lower bound $\alpha \in$ S exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E.
  Then α = inf(E).

Example 1.2.5

Let S = $(1,2) \cup [3,4) \cup (5,6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup(E) exists, it may or may not exists at S.

If sup(E) exists, then sup(E) is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:

For every nonempty subset E $\subset$ S that is bounded from above:

sup(E) exists in S.

Example 1.3.2

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then z > y where z $\in$ A. So, y is not an upper bound.
  Since for any y, there is z > y where z $\in$ A, then sup(A) doesn't exists in $\mathbb{Q}$.
- If $y^2 - 2 > 0$, then z < y where z $\in$ B. So, y is an upper bound, but not sup(E).
  Since for any y, there is z < y where z $\in$ B, then inf(B) doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound or greatest lower bound property.

2 Day 2: Fields

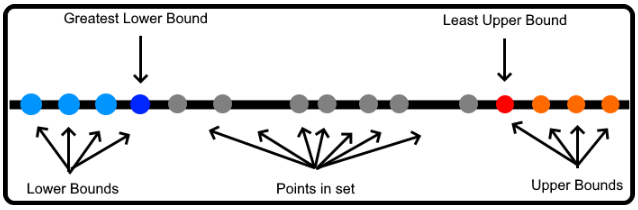
2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be a ordered set with the least upper bound property.
Let non-empty B $\subset$ S be bounded below.
Let L be the set of all lower bounds of B.
Then α = sup(L) exists in S.

Proof

L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, α = sup(L) exists in S.
We claim that α = inf(B).
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B since all upper bounds for L are in B. Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, inf(B) = α.



2.2 Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F
- x+y = y+x for all x,y $\in$ F
- (x+y)+z = x+(y+z) for all x,y,z $\in$ F
- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F
- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative Axioms

- If x,y $\in$ F, then xy $\in$ F
- yx = xy for all x,y $\in$ F
- (xy)z = x(yx) for all x,y,z $\in$ F
- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F
- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x}$ $\in$ F where x($\frac{1}{x}$) = 1

Distributive Law

x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Propositions 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0

Proof

From (a), let z = 0.

(c) If x+y = 0, then y = -x

Proof

From (a), let z = -x.

(d) -(-x) = x

Proof

From (c), let x = -x and y = x.

(e) If x $\neq$ 0 and xy = xz, then y = z

Proof

y = 1y = $\frac{1}{x}$xy = $\frac{1}{x}$xz = 1z = z

(f) If x $\neq$ 0 and xy = x, then y = 1

Proof

From (e), let z = 1.

(g) If x $\neq$ 0 and xy = 1, then y = $\frac{1}{x}$

Proof

From (e), let z = $\frac{1}{x}$.

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x

Proof

From (g), let x = $\frac{1}{x}$ and y = x.

(i) 0x = 0

Proof

Since 0x + 0x = (0+0)x = 0x = 0x + 0, then 0x = 0.

(j) If x,y $\neq$ 0, then xy $\neq$ 0

Proof

Suppose xy = 0, then 1 = $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}\frac{1}{x}$0 = 0.
0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)

Proof

xy + (-x)y = (x+(-x))y = 0y = 0. Then by part (c), (-x)y = -(xy).
xy + x(-y) = x(y+(-y)) = x0 = 0. Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy

Proof

By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].
By part (d), -[-(xy)] = xy.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x
- If x,y > 0, then xy > 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field since $i^2 = -1 \not> 1$.

Propositions 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

(a) If $x > 0$, then $-x < 0$ and vice versa

Proof

$-x = -x + 0 < -x + x = 0$

(b) If $x > 0$ and $y < z$, then $xy < xz$

Proof

Since $z-y > 0$, then $0 < x(z-y) = xz - xy$

(c) If $x < 0$ and $y < z$, then $xy > xz$

Proof

Since $-x > 0$ and $z-y > 0$, then $0 < -x(z-y) = xy - xz$

(d) If $x \neq 0$, $x^2 > 0$

Proof

If $x > 0$, then $x^2 = x \cdot x > 0$
If $x < 0$, then $(-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 > 0$

(e) If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

Since $(\frac{1}{y})y = 1 > 0$, then $(\frac{1}{y}) > 0$
Since $x < y$, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$

Theorem 2.3.3: $\mathbb{R}$ is an ordered field with $<$

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.
Also, $\mathbb{Q} \subset \mathbb{R}$ so $\mathbb{Q}$ is also an ordered field.

Theorem 2.3.4

For all x,y $\in \mathbb{R}$:

- Archimedean Property: If $x > 0$, there is $n \in \mathbb{Z}$ such that $nx > y$.

  Proof

  Fix $x > 0$. Suppose there is a y such that the property fails.
  Let $A = \{ nx: n = 1, 2, 3, \dots \}$.
  Then, A is nonempty and bounded from above by y.
  Then by the least upper bound property of $\mathbb{R}$, $\alpha = \sup(A)$ exists in $\mathbb{R}$.
  Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$.
  So $\alpha - x$ is not an upper bound of A.
  So there is a $mx \in A$ such that $mx > \alpha - x$.
  Then $\alpha < (m+1)x$, but $(m+1)x \in A$ contradicting α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

  Proof

  Since $x < y$, then $y-x > 0$. Then by the Archimedean Property, there exists a $n \in Z$ such that $n(y-x) > 1$. Thus, $ny > nx+1 > nx$
  By the well-ordering principle, there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$.
  Then, $m > nx \geq m-1$ so $nx+1 \geq m > nx$.
  Since $ny > nx+1 \geq m > nx$, then $y > m/n > x$.

3 Roots & Complex Field

3.1 nth Root

(a) If $0 < t \leq 1$, then $t^n \leq t$.

Proof

Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
Applying the process n times, then $t^n \leq t$.

(b) If $t \geq 1$, $t^n \geq t$.

Proof

Since $0 < 1 \leq t$, then $t \leq t^2$.
Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
Applying the process n times, $t \leq t^n$.

(c) If $0 < s < t$, then $s^n < t^n$.

Proof

$$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.
Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof

Uniqueness:
y is unique since if $y_1 < y_2$, then $x = {y_1}^n < {y_2}^n \neq x$.
Existence:
Let set $A = \{ t > 0 : t^n < x \}$.
$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < {t_1}^n < t_1 < x$ so $t_1 \in A$.
A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.
So x+1 is an upper bound of A.
Thus by the least upper bound property, $y = \sup(A)$ exists.
For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.
(Not an upper bound of A if < and not a least upper bound of A if >)
For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha)\ \underbrace{(\underset{\beta^{n-1}}{\beta^{n-1}} + \underset{<\beta^{n-1}}{\beta^{n-2}\alpha^1} + ... + \underset{<\beta^{n-1}}{\alpha^{n-1}})} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.
From inequality, let $\beta = y+h$ and $\alpha = y$
$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$
Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.
Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.
Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$
Thus, $t^n > x$ so $t \notin A$.
Thus, y-k is an upper bound of A which is a contradiction since $y = \sup(A)$.
Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 3.1.2: n-th root of product = product of n-th root

If a,b > 0 and n $\in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Proof

Let A = $a^{\frac{1}{n}}$ and B = $b^{\frac{1}{n}}$.
Then by theorem 3.1.1, since A is a solution to y_1^n = a, then A^n = a.
Similarly, B is a solution of y_2^n = b so B^n = b. Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 ... A_n B_1 B_2 ... B_n \\ &= A_1 A_2 ... B_1 A_n B_2 ... B_n = ... = A_1 B_1 A_2 ... A_{n-1} A_n B_3 ... B_n \\ &= ... = A_1 B_1 A_2 B_2 ... A_n B_n = (AB)^n \end{aligned}$$

Then again by theorem 3.1.1, there is a unique $(ab)^{\frac{1}{n}}$ = AB = $a^{\frac{1}{n}} b^{\frac{1}{n}}$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.
Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + ... + \frac{n_k}{10^k} \leq x$
Let E be the set of d_k for k = 0, 1, ... ∞. Then, x = sup(E).

3.3 Extended Reals

The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:

$-\infty < x < \infty$ for every $x \in \mathbb{R}$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If x > 0, then $x(\pm\infty) = \pm\infty$
- If x < 0, then $x(\pm\infty) = \mp\infty$

3.4 Complex Numbers

Definition 3.4.1: Complex

A complex number is an ordered pair (a,b) where a,b $\in \mathbb{R}$. For x,y $\in \mathbb{C}$

- x + y = (a,b) + (c,d) = (a + c , b + d)
- xy = (a,b) (c,d) = (ac - bd , ad + bc)
- $\frac{1}{x} = (a^2 + b^2)(a,-b)$

Thus, the axioms form a field where (0,0) = 0 and (1,0) = 1 and (0,1) = i.

Definition 3.4.2: Imaginary i

Let i = (0,1). Then, i^2 = -1.

Proof

i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1

Definition 3.4.3: Form a + bi

(a,b) = a + bi

Proof

(a,b) = (a,0) +(0,b) = (a,0) + (b,0)(0,1) = a + bi

Definition 3.4.4: Conjugate

Let conjugate: $\bar{z}$ = a - bi where Re(z) = a , Im(z) = b.

Let z = (a,b) and w = (c,d):

(a) $\overline{z+w} = \bar{z} + \overline{w}$

Proof

$\overline{z+w} = \overline{(a+c, b+d)}$ = (a+c,-b-d) = (a,-b) + (c,-d) = $\bar{z} + \overline{w}$

(b) $\overline{zw} = \bar{z}\ \overline{w}$

Proof

$\overline{zw} = \overline{(ac-bd, ad+bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\bar{z}\ \overline{w}$

(c) z + $\bar{z}$ = 2 Re(z)   z - $\bar{z}$ = 2i Im(z)

Proof

z + $\bar{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)
z - $\bar{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

(d) z$\bar{z}$ $\geq$ 0

Proof

z$\bar{z}$ = (a,b)(a,-b) = (a^2 + b^2 , -ab+ab) = a^2 + b^2 $\geq$ 0

Definition 3.4.5: Absolute Value

Let absolute value: | z | = $\sqrt{z\bar{z}}$

Let z = (a,b) and w = (c,d):

(a) If z $\neq$ 0, then | z | > 0.

Proof

$\sqrt{z\bar{z}} = \sqrt{a^2+b^2} \geq 0$ where | z | = 0 only if a,b = 0 so only if z = (0,0).

(b) | $\bar{z}$ | = | z |

Proof

| $\bar{z}$ | = $\sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$ = | z |

(c) | zw | = | z | | w |

Proof

| zw | = | (ac-bd,ad+bc) | = $\sqrt{(ac-bd)^2+(ad+bc)^2}$
= $\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$
= $\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$ = | z | | w |

(d) | Re(z) | $\leq$ | z |

Proof

| Re(z) | = | a | = $\sqrt{a^2} \leq \sqrt{a^2+b^2}$ = | z |

(e) | z+w | $\leq$ | z | + | w |

Proof

$|z+w|^2$ = (z+w)$\overline{(z+w)}$ = (z+w)($\bar{z}+\overline{w}$) = z$\bar{z}$ + z$\overline{w}$ + w$\bar{z}$ + w$\overline{w}$
= $|z|^2$ + $|w|^2$ + 2 Re(z$\overline{w}$) $\leq$ $|z|^2$ + $|w|^2$ + 2 $|z\overline{w}|$
= $|z|^2$ + $|w|^2$ + $2|z||w|$ = $(|z| + |w|)^2$

4 Euclidean Spaces & Cauchy-Schwarz

4.1 Euclidean Spaces

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples:

x = $(x_1, ..., x_k)$ for each $x_i \in \mathbb{R}$

with the properties:

- x+y = $(x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- cx = $(cx_1, ..., cx_k) \in \mathbb{R}^k$

So, $\mathbb{R}^n$ has a vector space structure. Similarly, for $\mathbb{C}^n$.

Definition 4.1.1: Inner Product for $\mathbb{R}^k$

$x \cdot y = x_1y_1 + ... + x_ky_k \in \mathbb{R}$

Definition 4.1.2: Norm

$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$

Definition 4.1.3: Extension to $\mathbb{C}^k$

For z,w $\in \mathbb{C}^n$

- $z \cdot w = z_1\overline{w_1} + ... + z_k\overline{w_k}$
- $z \cdot z = z_1\overline{z_1} + ... + z_k\overline{z_k} = |z_1|^2 + ... + |z_n|^2 = |z|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $b_1, ..., b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^n a_j(\overline{b_j})|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j(\overline{b_j})$.
If $B = 0$, then $b_1 = ... = b_n = 0$. Thus, $0 \leq A(0)$ holds true.
Suppose $B > 0$. Then:

$$\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j)(\overline{B}\,\overline{a_j} - \overline{C}\,\overline{b_j})$$
$$= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\,\overline{b_j}) = \sum B^2a_j\overline{a_j} - B\overline{C}a_j\overline{b_j} - BC\overline{a_j}b_j + C\overline{C}b_j\overline{b_j}$$
$$= B^2\sum |a_j|^2 - B\overline{C}\sum a_j\overline{b_j} - BC\sum \overline{a_j}b_j + |C|^2\sum |b_j|^2$$
$$= B^2A - B\overline{C}C - BC\overline{C} + |C|^2B = B^2A - 2|C|^2B + |C|^2B = B^2A - |C|^2B$$
$$= B(AB - |C|^2)$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.
Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i\overline{z_i}$, then $\sum z_i\overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$$|z \cdot w|^2 = |\sum z_i\overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2\,|w|^2$$

Thus, $|z \cdot w| \leq |z||w|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

(a) $|x| \geq 0$ where $|x| = 0$ only if x = 0

Proof

$|x| = \sqrt{\sum_{i=1}^{k} x_i^2} \geq 0$ where $|x| = 0$ only if $x_1 = ... = x_k = 0$

(b) $|\alpha x| = |\alpha||x|$

Proof

$|\alpha x| = \sqrt{\sum_{i=1}^{k} (\alpha x_i)^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^{k} x_i^2} = |\alpha||x|$

(c) $|x + y| \leq |x| + |y|$

Proof

$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2$
$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$

(d) $|x - y| \leq |x - z| + |y - z|$

Proof

$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$

5 Construction of $\mathbb{R}$: Theorem 2.3.3

There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Definition 5.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$
- If p $\in \alpha$ and q $\in \mathbb{Q} <$ p, then q $\in \alpha$
- If p $\in \alpha$, then p $<$ r $\in \mathbb{Q}$ for some r $\in \alpha$

Proposition 5.2: Order of $\mathbb{R} \rightarrow$ ordered set $\mathbb{R}$

Define $\alpha < \beta$ if α is a proper subset of β.

- If $\alpha \ngeq \beta$, then β is not a subset of α.
  Then there is a p $\in \beta$ such that p $\notin \alpha$.
  Then for any q $\in \alpha$, q $<$ p and thus, q $\in \beta$.
  Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.
- If $\alpha \nless \beta$ and $\alpha \ngtr \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
  If $\alpha \neq \beta$, there are p,q such that p $\in \alpha$, but p $\notin \beta$ and q $\in \beta$, but q $\notin \alpha$.
  But if p $\notin \beta$, then for any b $\in \beta$, b $<$ p. Thus, q $<$ p.
  Similarly, if q $\notin \alpha$, then for any a $\in \alpha$, a $<$ q. Thus, p $<$ q.
  Thus, there is a contradiction since p $>$ q and p $<$ q so $\alpha = \beta$.
- If $\alpha \nleq \beta$, then α is not a subset of β.
  Then there is a p $\in \alpha$ such that p $\notin \beta$.
  Then for any q $\in \beta$, q $<$ p and thus, q $\in \alpha$.
  Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.
- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ, then α is a proper subset of γ. Thus, $\alpha < \gamma$.

Thus, $\mathbb{R}$ is an ordered set with such an order $<$.

Proposition 5.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let A $\subset \mathbb{R}$ and β be an upper bound for A. Let γ be the union of all $\alpha \in$ A.
Thus, p $\in \gamma$ if and only if p $\in \alpha$ for some $\alpha \in$ A.
γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in$ A where α_0 is nonempty.
  Since α_0 is nonempty, then γ is nonempty.
  Since every $\alpha \in$ A is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. If q $<$ p, then q $\in \alpha_1$ so q $\in$ A.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. Thus, there is a r $\in \alpha_1$ such that r $>$ p so r $\in \gamma$. Thus, there is a r $\in \gamma$ where r $>$ p.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in$ A $\subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A.
Suppose $\delta < \gamma$. Then there is a s $\in \gamma$ such that s $\notin \delta$. Since s $\in \gamma$, then there is a $\alpha \in$ A such that s $\in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A.
Thus, $\gamma = \sup(\text{A})$.

Proposition 5.4: $\mathbb{R}$ is a field

If α, $\beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums r + s where r $\in \alpha$ and s $\in \beta$. Also, let 0* be the set of all negative rational numbers which is a cut since:

- 0* is nonempty and $0^* \neq \mathbb{Q}$
- If p $\in$ 0*, then any q $\in \mathbb{Q}$ < p is a negative rational and thus, q $\in$ 0*.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then for any p $\in$ 0*, there is a r $\in \mathbb{Q}$ where p < r < 0 so r is a negative rational so r $\in$ 0*.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take r' $\notin \alpha$, s' $\notin \beta$, then r' + s' > r + s for r $\in \alpha$, s $\in \beta$. Thus, r' + s' $\notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If p $\in \alpha + \beta$, then p = r + s where r $\in \alpha$ and s $\in \beta$.
  If q < p, then q - s < p - s = (r + s) - s = r so q - s $\in \alpha$.
  Since q - s $\in \alpha$ and s $\in \beta$, then (q - s) + s = q $\in \alpha + \beta$.
- If r $\in \alpha$, then there is a t $\in \alpha$ such that t > r. Let s $\in \beta$.
  Thus, for any p = r + s $\in \alpha + \beta$, there is a q = t + s $\in \alpha + \beta$ such that p = r + s < t + s = q.

$\alpha + \beta = \beta + \alpha$

If p = r + s $\in \alpha + \beta$ where r $\in \alpha$, s $\in \beta$, then s + r = r + s = p $\in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If r $\in \alpha$, s $\in \beta$, t $\in \gamma$, then r + s + t = (r + s) + t $\in (\alpha + \beta) + \gamma$ and r + s + t = r + (s + t) $\in \alpha + (\beta + \gamma)$.

$\alpha + 0^* = \alpha$

If r $\in \alpha$, s $\in$ 0*, then r + s < r. Thus, r + s $\in \alpha$. Thus, $\alpha + 0^* \subset \alpha$.
If p $\in \alpha$, there is a r $\in \alpha$ where r > p. Thus, p - r $\in$ 0*.
Since p = r + (p - r) $\in \alpha + 0^*$, then $\alpha \subset \alpha + 0^*$. Thus, $\alpha + 0^* = \alpha$.

There is a -α such that α + -α = 0*

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is r > 0 such that -p - r $\notin \alpha$.
$\beta \in \mathbb{R}$ since β is a cut:

- If s $\notin \alpha$ and p = -s - 1, then -p - 1 $\notin \alpha$. Thus, p $\in \beta$ so β is nonempty. If q $\in \alpha$, then -q $\notin \beta$ so $\beta \neq \mathbb{R}$.
- If p $\in \beta$, let r > 0 so -p - r $\notin \alpha$. If q < p, then -q - r > -p - r and thus, -q - r $\notin \alpha$ so q $\in \beta$.
- If p $\in \beta$, let t = p + (r/2). Then -t - (r/2) = -p - r $\notin \alpha$ and thus, t $\in \beta$ where p < t.

If r $\in \alpha$, s $\in \beta$, then s $\notin \alpha$. Thus, r < -s so r + s < 0. Thus, $\alpha + \beta \subset 0^*$.
Let v $\in$ 0* and let w = -v/2 so w > 0.
Thus, by the Achimedean property, there is an integer n such that nw $\in \alpha$, but (n+1)w $\notin \alpha$. Let p = -(n+2)w so -p - w = (n+1)w $\notin \alpha$ so p $\in \beta$.
Then, v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p $\in \alpha + \beta$.
Since v $\in$ 0*, then $0^* \subset \alpha + \beta$. Thus, $\alpha + \beta = 0^*$. Then, let -$\alpha = \beta$.

Thus, if α, β, $\gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
Thus, if $\alpha > 0^*$, then -α = -α + 0* < -$\alpha + \alpha$ = 0* so -α < 0*.
If α, $\beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that p $\leq$ rs for r $\in \alpha$, s $\in \beta$.
Define 1* as the set of all q < 1. Then all multiplication axioms holds with similar proofs as addition. Also, note since α, $\beta > 0^*$, then $\alpha\beta > 0^*$.
Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were α, β, γ >,< 0*.

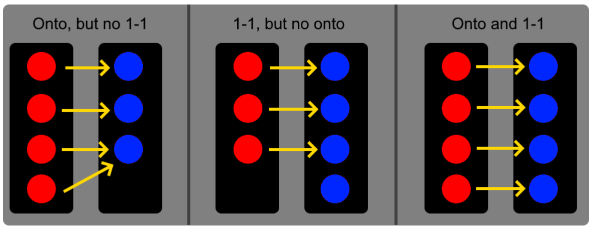
6 Cardinality

6.1 Cardinality

Definition 6.1.1: Onto and 1-1 Mapping

Suppose for every $x \in A$, there is an associated $f(x) \in B$.
Then f maps A into B = f: $A \to B$.

- If $f(A) = B$, then f maps A onto B.
- If for each $y \in B$, $f^{-1}(y)$ consist of at most one $x \in A$ where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B.



Definition 6.1.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function f: $A \to B$. (1-1 correspondence between A and B) Then:

$A \sim B$

If f: $A \to B$ is 1-1 and onto, then there is a f^{-1}: $B \to A$ that is 1-1 and onto.

Definition 6.1.3: Countability

- A is finite if $A \sim J_n = \{0, 1, ..., n\}$ for some $n \in \mathbb{N}$
- A is infinite if A is not finite
- A is countably infinite if $A \sim J = \mathbb{Z}_+$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite

Example 6.1.4

$\mathbb{Z}$ is countably infinite

Proof

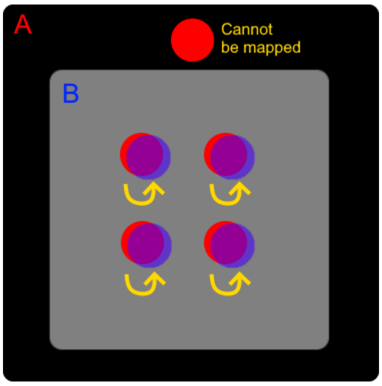
Let f: $\mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n-1}{2} & \text{n is odd} \end{cases}$$

So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto -1$, $4 \mapsto 2$, $5 \mapsto -2$, etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 6.1.5: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A.

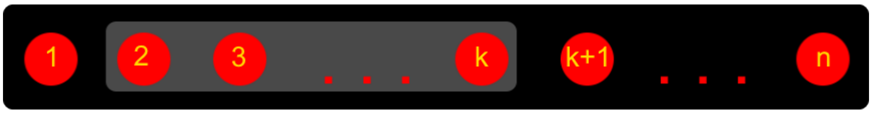


Theorem 6.1.6: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let E $\subset$ A be an infinite subset. For every distinct $x_i \in$ A, let x = $\{ x_1, x_2, ... \}$.
Let n_1 be smallest integer such that $\mathrm{x}_{n_1} \in$ E.
Then let n_2 be the smallest integer where $n_2 > n_1$ such that $\mathrm{x}_{n_2} \in$ E.
Repeat the process to create sequence f(k) = $\{ x_{n_1}, x_{n_2}, ..., x_{n_k}, ... \}$.
Thus, there is a 1-1 correspondence between E and $\mathbb{Z}_+$ so E is countably infinite.



6.2 Set of Sets

Definition 6.2.1: Union and Intersection

Let sets Ω,B be such that for each x $\in \Omega$, there is an associated $\mathrm{E}_x \subset$ B.

- E = $\cup_{x=1}^{n}$ E_x only if for every x $\in$ E, x $\in \mathrm{E}_x$ for at least one x $\in \Omega$.
- P = $\cap_{x=1}^{n}$ E_x only if for every x $\in$ P, x $\in \mathrm{E}_x$ for all x $\in \Omega$.

with properties:

(a) A $\cup$ B = B $\cup$ A — A $\cap$ B = B $\cap$ A

(b) (A $\cup$ B) $\cup$ C = A $\cup$ (B $\cup$ C) — (A $\cap$ B) $\cap$ C = A $\cap$ (B $\cap$ C)

(c) A $\subset$ A $\cup$ B — (A $\cap$ B) $\subset$ A

(d) If A $\subset$ B, then A $\cup$ B = B and A $\cap$ B = A

Proof

If x $\in$ A $\cup$ B, then x $\in$ A or/and x $\in$ B.

- If x $\in$ A, since A $\subset$ B, then x $\in$ B. Then, (A $\cup$ B) $\subset$ B.
- If x $\in$ B, then immediately (A $\cup$ B) $\subset$ B.

If x $\in$ B, then x $\in$ A $\cup$ B so B $\subset$ (A $\cup$ B). Thus, A $\cup$ B = B.

If x $\in$ A $\cap$ B, then x $\in$ A and x $\in$ B. Thus, (A $\cap$ B) $\subset$ A.
If x $\in$ A, since A $\subset$ B, then x $\in$ B so x $\in$ A $\cap$ B. Thus, A $\subset$ (A $\cap$ B).
Thus, A $\cap$ B = A.

(e) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Proof

If x ∈ A ∩ (B ∪ C), then x ∈ A and (x ∈ B or/and x ∈ C).
- If x ∈ B, then x ∈ (A ∩ B) so x ∈ (A ∩ B) ∪ (A ∩ C).
- If x ∈ C, then x ∈ (A ∩ C) so x ∈ (A ∩ B) ∪ (A ∩ C).

Thus, A ∩ (B ∪ C) ⊂ (A ∩ B) ∪ (A ∩ C).
If x ∈ (A ∩ B) ∪ (A ∩ C), then x ∈ A and (x ∈ B or/and x ∈ C).
Thus, (A ∩ B) ∪ (A ∩ C) ⊂ A ∩ (B ∪ C).
Thus, A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).

(f) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Proof

If x ∈ A ∪ (B ∩ C), then x ∈ A or/and (x ∈ B and x ∈ C).
- If x ∈ A, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).
- If x ∈ B,C, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).

If x ∈ (A ∪ B) ∩ (A ∪ C), then x ∈ A or/and (x ∈ B and x ∈ C).
Thus, (A ∪ B) ∩ (A ∪ C) ⊂ A ∪ (B ∩ C).
Thus, A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

Theorem 6.2.2: Union of countably infinite sets is countably infinite

If $E_1, E_2, \ldots$ are countably infinite sets, then S = $\cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

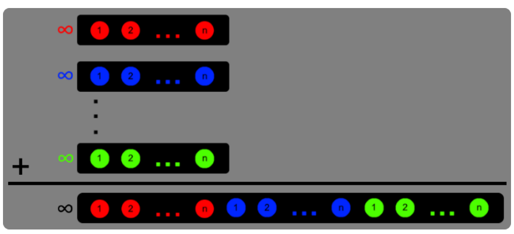
For each E_n, there is a sequence { x_{n1}, x_{n2}, ... }. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence S* = { x_{11} ; x_{21}, x_{12} ; x_{31}, x_{32}, x_{33} ; ... }.
Since S* ∼ S so S is at most countable and S is infinite since $E_1, E_2, \ldots$ are infinite, then S cannot be finite and thus, countably infinite.

Alternative Proof

For each E_n, let set $\widetilde{E_n}$ = E_n - $\cup_{m=1}^{\infty} E_m$ where m ≠ n. Thus, S = $\cup_{n=1}^{\infty} \widetilde{E_n}$.
Since each E_n is countably infinite, there exists a 1-1 mapping δ_n: $E_n \to \mathbb{Z}_+$.
Thus, for each $\widetilde{E_n}$, there is a 1-1 mapping δ_n: $\widetilde{E_n} \to$ A ⊂ $\mathbb{Z}_+$.
Let $p_1, p_2, \ldots$ be distinct primes. Since for s ∈ S, there exists a unique $\widetilde{E_i}$ such that s ∈ $\widetilde{E_i}$, then let f(s) = $p_1^{\delta_1(s)} p_2^{\delta_2(s)} \ldots$ where $p_k^{\delta_k(s)}$ = 1 if k ≠ i.
Then, by the Fundamental theorem of arithmetic, f maps s to a unique z ∈ $\mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable. Since any E_n ⊂ S is countably infinite, then S cannot be finite and thus, S is countably infinite.



Theorem 6.2.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B_n be the set of all n-tuples $(a_1,...,a_n)$ where $a_k \in$ A. Then B_n is countably infinite.

Proof

The base case B_1 is countably infinite since B_1 = A.
Suppose B_{n-1} is countably infinite. Then for every x ∈ B:
x = (b,a) b ∈ B_{n-1} and a ∈ A
Since for every fixed b, (b,a) ∼ A and thus, countably infinite.
Since B is a set of countably infinite sets, then B_n is countably infinite.

Definition 6.2.4: $\mathbb{Q}$ is countable

The set of rational numbers, $\mathbb{Q}$, is countably infinite.

Proof

Since elements of $\mathbb{Q}$ are of form $\frac{a}{b}$ which is a 2-tuple, then by the theorem 6.2.3, $\mathbb{Q}$ is countably infinite.

Alternative Proof

For every x ∈ $\mathbb{Q}$, let x = $(-1)^i \frac{p}{q}$ where p,q ∈ $\mathbb{Z}_+$.
Let f(x) = $2^i\ 3^p\ 5^q$. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to E ⊂ $\mathbb{Z}_+$.
Thus, $\mathbb{Q}$ is at most countable, but since p,q ∈ $\mathbb{Z}_+$, then $\mathbb{Q}$ cannot be finite and thus, is countably infinite.

Example 6.2.5: Sequences of 0 and 1 are uncountable

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences $s_1, s_2, \ldots$.
Then construct a sequence s as follows:
If the n-th digit in s_n is 1, then let the n-th digit of s be 0 and vice versa.
Thus. s differs from every s_n ∈ E so s ∉ E.
But, s ∈ A so E is a proper subset of A.
Thus, every countably infinite subset of A is a proper subset of A.
If A is countably infinite, then A is a proper subset of A which is a contradiction.

7 Metric Spaces & Closed/Open

7.1 Metric Spaces

Definition 7.1.1: Metric Spaces

A set X is a metric space if for ant p,q $\in$ X, there is an associated d(p,q) $\in \mathbb{R}$ such that:

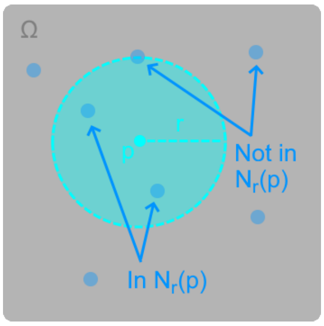
- d(p,q) $>$ 0 if p $\neq$ q
- d(p,q) = 0 if and only if p = q
- Symmetry: d(p,q) = d(q,p)
- Triangle Inequality: d(p,q) $\leq$ d(p,r) + d(r,q) for any r $\in$ X.

For euclidean spaces $\mathbb{R}^k$, d(x,y) = $|x-y|$ where x,y $\in \mathbb{R}^k$.

Definition 7.1.2: Types of Points and Sets

(a) Neighborhood

For p $\in \mathbb{R}^k$ and r $>$ 0, $N_r(p)$ is the set of all q $\in$ X where d(q,p) $<$ r



(b) Limit Points and Closed Sets

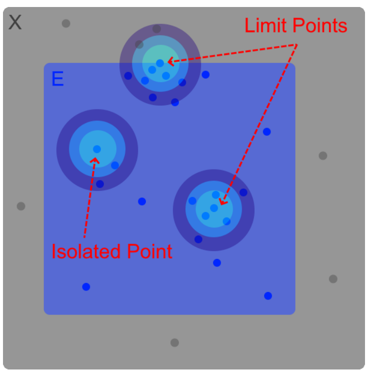
Closed set E contain all p $\in$ X where every $N_r(p)$ contain a q $\neq$ p $\in$ E

- Limit Points

  For point p $\in$ X, every $N_r(p)$ contains a q $\neq$ p $\in$ E
  The set of all limit points of E = E'
- Isolated Points

  If p $\in$ E is not a limit point of E
- Closed

  If every limit point p of E is a p $\in$ E



(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset$ E

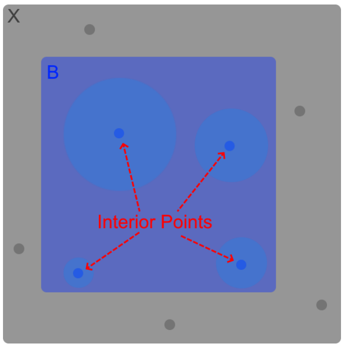
- Interior Point

  For p $\in$ X, there is a $N_r(p) \subset$ E

  The set of all interior points = E^o

- Open

  If every p $\in$ E is an interior point of E



(d) More about Sets

- Bounded

  If there is M $\in \mathbb{R}$, q $\in$ X such that d(p,q) < M for all p $\in$ E

- Complement

  From E, E^c is the set of all p $\in$ X such that p $\notin$ E

- Perfect

  If E is closed and if every p $\in$ E is a limit point of E

- Dense

  If every p $\in$ X is a limit point of E or/and p $\in$ E

- Boundary Point

  For p $\in$ X, if every $N_r(p)$ contains a x $\in$ E and y $\in E^c$

  The set of all boundary points = ∂E

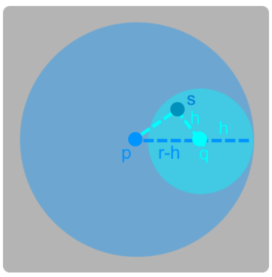
For a metric space X, {X,$\emptyset$} are both open and closed.

Theorem 7.1.3: $N_r(p)$ is open

Every neighborhood is an open set.

Proof

Let q $\in N_r(p)$. Then there is a h > 0 $\in \mathbb{R}$ such that d(q,p) = r - h.
Then for any s $\in N_h(q)$, d(s,p) $\leq$ d(s,q) + d(q,p) = h + (r - h) = r.
Thus, for any q $\in N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

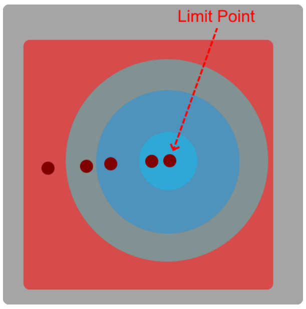


Theorem 7.1.4: If a set has a limit point, there are infinite q ∈ E in N$_r(p)$

If p is a limit point of set E, then every N$_r(p)$ contains infinitely many q ∈ E.

Proof

Suppose there is N$_{r_1}(p)$ which contains finitely many q = { q_1, ... , q_n }.
Let r = $\min_{m\in[1,n]}$ d(p,q_m). Then N$_r(p)$ contains no q ∈ E such that q ≠ p.
So, p is not a limit point of E which is a contradiction since p is a limit point of E.



Corollary 7.1.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since ∅ ∈ A, all finite set must be closed.

Proof

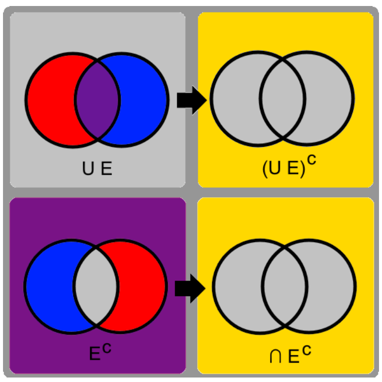
Let p be a limit point of finite set E. By theorem 7.1.4, then any N$_r(p)$ contain infinite q ∈ E so E is an infinite set which is a contradiction since E is finite.
So p cannot be limit point of E and thus, E has no limit points.

Theorem 7.1.6: De Morgan's Laws

Let $E_1, E_2, ...$ be a collection of sets. Then, (∪ E_x)^c = ∩ ($E_x{}^c$).

Proof

If p ∈ (∪ E_x)^c, then p ∉ (∪ E_x).
Thus, p ∉ E_x for any x so p ∈ E_x^c for all x. Thus, p ∈ ∩ (E_x^c) so (∪ E_x)^c ⊂ ∩ (E_x^c).
If p ∈ ∩ (E_x^c), then p ∈ E_x^c for all x.
Thus, p ∉ E_x for any x so p ∉ ∪ E_x. Thus, p ∈ (∪ E_x)^c so ∩ (E_x^c) ⊂ (∪ E_x)^c.
Thus, (∪ E_x)^c = ∩ (E_x^c).



Theorem 7.1.7: Open set → Closed complement

A set E is open if and only if E^c is closed.

Proof

Suppose E is open. Let x be a limit point of E^c.
Then for every r > 0, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.
Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.
Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.
Since $x \notin E$, x is not a limit point of E. Then there exists a r > 0 such that any p $\in N_r(x)$ is not in E. Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E. Since any $x \in E$ is an interior point of E, then E is open.

Corollary 7.1.8: Closed set → Open complement

A set F is closed if only only if F^c is open.

Proof

From theorem 7.1.7, let $E = F^c$.

Theorem 7.1.9: Union open → open and Intersection closed → closed

(a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup\, G_x$ is open.

Proof

If $p \in \cup\, G_x$, then $p \in G_x$ for at least one x. Let $\overline{x}$ be such an x.
Since $G_{\overline{x}}$ is open, then p is an interior point of $G_{\overline{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\overline{x}} \subset \cup\, G_x$. So p is an interior point of $\cup\, G_x$.
Since any $p \in \cup\, G_x$ is an interior point, then $\cup\, G_x$ is open.

(b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap\, F_x$ is closed.

Proof

By theorem 7.1.7, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup\, F_x^c$ is open.
Thus, again by theorem 7.1.7, $(\cup\, F_x^c)^c$ is closed.
By theorem 7.1.6, $(\cup\, F_x^c)^c = \cap\, (F_x^c)^c = \cap\, F_x$.

(c) If $G_1, ..., G_n$ is a finite collection of open sets, then $\cap_{x=1}^n\, G_x$ is open.

Proof

If $p \in \cap_{x=1}^n\, G_x$, then $p \in G_x$ for all G_x for x = {1, 2, ... , n}.
Since each G_x is open, then for any G_x, there is a $N_{r_x}(p) \subset G_x$.
Let $r = \min(r_1, r_2, ..., r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x.
So, $N_r(p) \subset \cap_{x=1}^n\, G_x$ and thus, p is an interior point of $\cap_{x=1}^n\, G_x$ so $\cap_{x=1}^n\, G_x$ is open.
Infinite + Closed: $G_i = (-1/i, 1/i)$ Infinite + Open: $G_i = (-i, i)$

(d) If $F_1, ..., F_n$ is a finite collection of closed sets, then $\cup_{x=1}^n\, F_x$ is closed.

Proof

By theorem 7.1.7, any F_x^c is open. Since $F_1^c, ..., F_n^c$ is a finite collection of open set, then by part (c), $\cap_{x=1}^n\, F_x^c$ is open.
Thus, again by theorem 7.1.7, $(\cap_{x=1}^n\, F_x^c)^c$ is closed.
By theorem 7.1.6, $(\cap_{x=1}^n\, F_x^c)^c = \cup_{x=1}^n\, (F_x^c)^c = \cup_{x=1}^n\, F_x$.
Infinite + Closed: $F_i = [-1/i, 1/i]$ Infinite + Open: $F_i = [1/i, \infty)$

Theorem 7.1.10: E' is closed

Let E ⊂ X. Then, (E')' ⊂ E'. Thus, E' is closed.

Proof

If x ∈ (E')', then for every $N_{r_1}(x)$, there is a y ≠ x where y ∈ E'.
Since y ∈ E', then for every $N_{r_2}(y)$, there is a z ≠ y where z ∈ E.
Let r = r_1 + r_2.
Then for every $N_r(x)$, there exists a z ≠ x where z ∈ E. Thus, x ∈ E' so (E')' ⊂ E'.

Theorem 7.1.11: E^o is open

Let E ⊂ X. Then, E^o is open.

Proof

If p ∈ E^o, there is a r > 0 such that $N_r(p)$ ⊂ E.
Then for 0 < s < r, $N_s(p)$ ⊂ $N_r(p)$ so any q ∈ $N_s(p)$ is q ∈ E^o.
Since any p ∈ E^o have a $N_s(p)$ ⊂ E^o, then E^o is open.

7.2 Intervals and Balls

Definition 7.2.1: Segments and Intervals

In ℝ, a segement is an open interval (a,b) = { x ∈ ℝ : a < x < b }
In ℝ, a interval is a closed interval [a,b] = { x ∈ ℝ : a ≤ x ≤ b }

Definition 7.2.2: Open Balls

In $\mathbb{R}^k$, an open ball of radius r > 0 centered at p is:
$N_r(p)$ = { x ∈ $\mathbb{R}^k$: $|x - p|$ < r } = { x ∈ $\mathbb{R}^k$: d(x,p) < r }
A closed ball has d(x,p) ≤ r.

Definition 7.2.3: Convex

E ⊂ $\mathbb{R}^k$ is convex if for all x,y ∈ E and t ∈ [0,1], tx + (1-t)y ∈ E.

Example 7.2.4: Balls are convex

Balls in $\mathbb{R}^k$ are convex.

Proof

Let x,y ∈ open ball $N_r(p)$. Let z = tx + (1-t)y for t ∈ [0,1].
Since $|x-p| < r$ and $|y-p| < r$:

$$\begin{aligned} |z-p| &= |tx+(1-t)y-p| = |tx+(1-t)y-tp+(t-1)p| \\ &= |t(x-p)+(1-t)(y-p)| \le t|(x-p)| + (1-t)|(y-p)| \\ &< \text{tr} + \text{(1-t)r} = \text{r} \end{aligned}$$

Thus, z ∈ $N_r(p)$ so balls are convex. Same proof applies to closed balls.

Definition 7.2.5: Dense

E ⊂ X is dense if every x ∈ X is either in E or a limit point of E.

Example 7.2.6: ℚ is dense in ℝ

Let X = ℝ. Then, E = ℚ is dense in ℝ.

Proof

Fix x ∈ ℝ and r > 0. There is a q ∈ ℚ such that x-r < q < x. So for any r > 0 and q ∈ ℚ, q ≠ x and q ∈ $N_r(x)$. Thus, every x ∈ ℝ is a limit point of ℚ.

8 Closure, Open Relative, & Compact

8.1 Closure

Definition 8.1.1: Closure

Let E $\subset$ metric space X and E' be the set of all limit points of E in X.

Then the closure of E: $\overline{E}$ = E $\cup$ E'

with the properties:

(a) $\overline{E}$ is closed

(b) E = $\overline{E}$ if and only if E is closed

(c) $\overline{E} \subset$ F for every closed F $\subset$ X such that E $\subset$ F

Proof

Suppose x $\in$ X, but x $\notin \overline{E}$. Thus, x $\in \overline{E}^c$.
Thus, there is a N$_r(x) \subset \overline{E}^c$ since else there is always a p $\in$ N$_r(x)$ where p $\in \overline{E}$ so x is a limit point of $\overline{E}$ so x $\in \overline{E}$. Thus, $\overline{E}^c$ is open so $\overline{E}$ is closed by theorem 7.1.7.

If E = $\overline{E}$, then by part (a), E is closed.
If E is closed, then E' $\subset$ E so E = E $\cup$ E' = $\overline{E}$.

If closed set F, then F' $\subset$ F and since E $\subset$ F, then E' $\subset$ F' $\subset$ F. Thus, $\overline{E} \subset$ F.

Theorem 8.1.2: sup(E) $\in \overline{E}$

Let non-empty set of real numbers, E, be bounded above. Let y = sup(E).
Then, y $\in \overline{E}$. Thus, y $\in$ E if E is closed and y $\notin$ E if E is open in $\mathbb{R}$.

Proof

If y $\in$ E, then y $\in \overline{E}$. Suppose y $\notin$ E.
For every h > 0, there exists a x $\in$ E such that y-h < x < y otherwise y-h is an upper bound for E which is a contradiction since y = sup(E).
Thus, y is a limit point of E so y $\in$ E'.
If E is closed, then y $\in$ E since y $\in$ E'. Also, y $\in \overline{E}$.
If E is open, then any N$_r(y) \not\subset$ E since N$_r(y)$ in $\mathbb{R}$ must contain a $\gamma >$ y so y $\notin$ E^o.

8.2 Open Relative

Definition 8.2.1: Open Relative

Suppose E $\subset$ Y $\subset$ metric space X.
Then E is open relative to Y if for each p $\in$ E, there is an r > 0
such that for any q $\in$ Y, then q $\in$ E if d(q,p) < r.

Theorem 8.2.2: E is open relative to Y ⊂ X if E = Y ∩ G and G is open in X

Suppose E ⊂ Y ⊂ X.

E is open relative to Y if and only if E = Y ∩ G for some open G ⊂ X.

Proof:

Suppose E is open relative to Y.
Then for each p ∈ E, there is a $r_p > 0$ such that for any q ∈ Y where d(p,q) < r_p, then q ∈ E.
Since Y ⊂ X, let V_p be the set of all q ∈ X such that d(p,q) < r_p and define G = $\cup_{p\in E} V_p$. Since V_p is open by theorem 7.1.3, then by theorem 7.1.9a, open G ⊂ X.
Since p ∈ V_p for all p ∈ E, then E ⊂ G ∩ Y. Also, by construction, then V_p ∩ Y ⊂ E so G ∩ Y ⊂ E. Thus, E = Y ∩ G.
If G is open in X and E = G ∩ Y, then every p ∈ E has a V_p ⊂ G.
Then, V_p ∩ Y ⊂ G ∩ Y = E so E is open relative to Y.

8.3 Compact Sets

Definition 8.3.1: Open Cover

An open cover of set E ⊂ X is a collection of open $G_1, G_2, \ldots$ ⊂ X such that E ⊂ ∪ G_i.

Definition 8.3.2: Compact

K ⊂ X is compact if every open cover of K contains a finite subcover.

If $G_1, G_2, \ldots$ is an open cover of K, then K ⊂ $\cup_{i=1}^n G_i$ for some n.

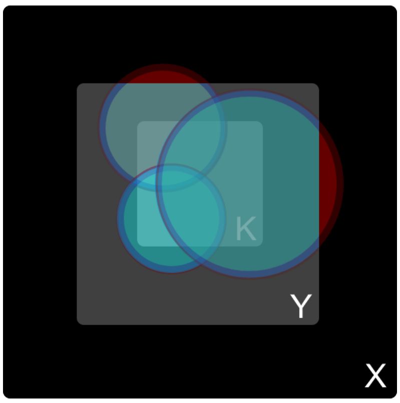
Theorem 8.3.3: A compact set is compact in every metric space

Suppose K ⊂ Y ⊂ X.

Then K is compact relative to X if and only if K is compact relative to Y.

Proof

Suppose K is compact relative to X.
Let $V_1, V_2, \ldots$ be sets open relative to Y such that K ⊂ ∪ V_x. Then by theorem 8.2.2 for each V_x, there is a G_x open relative to X where V_x = Y ∩ G_x.
Since K is compact relative to X, then there is a n such that K ⊂ $G_{x_1} \cup \ldots \cup G_{x_n}$.
Thus, K = K ∩ Y ⊂ $(\cup_{i=1}^n G_{x_i})$ ∩ Y = $(\cup_{i=1}^n G_{x_i}$ ∩ Y) = $\cup_{i=1}^n V_{x_i}$.
Since there are open $V_{x_1}, \ldots, V_{x_n}$ where K ⊂ $\cup_{i=1}^n V_{x_i}$ so K is compact relative to Y.
Suppose K is compact relative to Y.
Let open $G_1, G_2, \ldots$ ⊂ X such that X ⊂ ∪ G_x. For each G_x, let V_x = Y ∩ G_x ⊂ Y.
Since K is compact relative to Y, there is a n such that K ⊂ $\cup_{i=1}^n V_{x_i}$.
Thus, K ⊂ $\cup_{i=1}^n V_{x_i} = \cup_{i=1}^n$ (Y ∩ G_{x_i}) ⊂ $\cup_{i=1}^n G_{x_i}$ so K is compact relative to X.

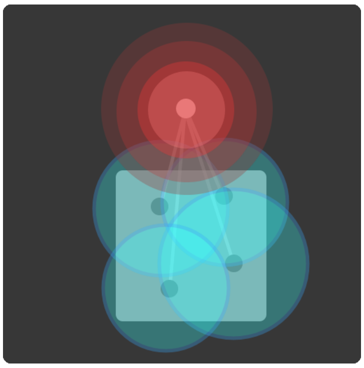


Theorem 8.3.4: A compact set is closed

Compact subsets of metric spaces are closed.

Proof

Let compact K $\subset$ X. Suppose p $\in$ X, but p $\notin$ K so p $\in$ K^c.
If q $\in$ K, let W_q be a neighborhood of q with r $< \frac{1}{2}$d(p,q). Let $V_{p,q}$ be a neighbohood of p with r $< \frac{1}{2}$d(p,q). Since K is compact, then there are finite points $q_1, ..., q_n$ such that K $\subset$ W where W = $W_{q_1} \cup ... \cup W_{q_n}$.
Let V = $V_{p,q_1} \cap ... \cap V_{p,q_n}$, then K $\cap$ V $\subset$ W $\cap$ V = $\emptyset$ so V $\subset$ K^c.
Since there is a neighborhood V for p $\in$ K^c where V $\subset$ K^c, then every p $\in$ K^c is an interior point so K^c is open. Then by theorem 7.1.7, K is closed.



Theorem 8.3.5: If closed E $\subset$ compact set K, E is compact

Closed subsets of compact sets are compact.

Proof

Suppose F $\subset$ K $\subset$ X where F is closed relative to X and K is compact.
Let $V_1, V_2, ...$ be an open cover for F. Let open set F^c be all k $\in$ K where k $\notin$ F.

$$K = F \cup F^c \subset V_1 \cup V_2 \cup ... \cup F^c$$

Thus, $V_1 \cup V_2 \cup ... \cup F^c$ is an open cover for K.
Since K is compact, there is a finite subcover Ω that covers K and thus, finite subcover Ω covers F $\cup$ F^c.
Remove F^c from Ω. Since finite subcover Ω - F^c covers F, then F is compact.

Corollary 8.3.6: Closed F $\cap$ compact K = compact

If F is closed and K is compact, then F $\cap$ K is compact.

Proof

Since K is compact, then K is closed by theorem 8.3.4.
Then, by 7.1.9b, F $\cap$ K is closed.
Since F $\cap$ K $\subset$ K, then by theorem 8.3.5, F $\cap$ K is compact.

Theorem 8.3.7: Nonempty $\cap_{i=1}^{n} K_i \to$ nonempty $\cap K_i$

For compact sets $K_1, K_2, ... \subset$ X where any intersection of finite K_i is nonempty, then $\cap K_i$ is nonempty.

Proof

Fix K_1. If there is a k $\in K_1$ where k $\in K_i$ for all i, then k $\in \cap K_i$ so $\cap K_i \neq \emptyset$.
Suppose for every k $\in K_1$, k $\notin K_i$ for some i.
Then for every k $\in K_1$, there is a K_i such that p $\notin K_i$ so p $\in K_i^c$.
Thus, $K_2^c, k_3^c, ...$ form an open cover for K_1.
Since K_1 is compact, there is a n where $K_1 \subset K_{i_1}^c \cup ... \cup K_{i_n}^c$.
But then, $K_1 \cap K_{i_1} \cap ... \cap K_{i_n} = \emptyset$ which is a contradiction.

Corollary 8.3.8: Nonempty K_i where $K_{i+1} \subset K_i \to$ nonempty $\cap K_i$

If $K_1, K_2, ...$ is a sequence of nonempty compact sets such that $K_{i+1} \subset K_i$, then $\cap K_i$ is nonempty.

Proof

Since each K_i is nonempty and if $i_1 < ... < i_n$, then $K_{i_1} \cap ... \cap K_{i_n} = K_{i_n}$ is nonempty, then by theorem 8.3.7, $\cap K_i$ is nonempty.

Theorem 8.3.9: Nonempty intervals I_n where $I_{n+1} \subset I_n \to$ nonempty $\cap I_n$

If $I_1, I_2, ...$ is a sequence of intervals in $\mathbb{R}^1$ such that $I_{n+1} \subset I_n$, then $\cap I_n$ is nonempty.

Proof

Let $I_n = [a_n,b_n]$ and thus, each I_n is nonempty. If $n_1 < ... < n_m$, then $I_{n_1} \cap ... \cap I_{n_m} = [a_{n_m},b_{n_m}]$ is nonempty. Thus, by theorem 8.3.7, $\cap I_n$ is nonempty.

Theorem 8.3.10: p $\in$ E' exists if infinite E $\subset$ compact K

If E is an infinite subset of compact set K, then E has a limit point in K.

Proof

If no p $\in$ K is a p $\in$ E', then each p would have a neighborhood V_p contains at most p $\in$ E if p $\in$ E. Thus, there is no finite subcover that covers E and thus, there is no finite subcover that covers K since E $\subset$ K which contradicts K is compact.

Definition 8.3.11: K-cells

The set of all x = $(x_1, ..., x_k) \in \mathbb{R}^k$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.

Theorem 8.3.12: K-cells are compact

Every k-cell is compact.

Proof

Let k-cell I consists of all x = $(x_1, ..., x_k)$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.
Let $\delta = \sqrt{\sum_{i=1}^{k}(b_i - a_i)^2}$. Thus, $|x - y| \leq \delta$ for $x, y \in$ I.
Suppose there exists an open cover $G_1, G_2, ...$ of I which contain no finite subcover.
Let $c_i = \frac{a_i+b_i}{2}$. Then each interval splits into $[a_i, c_i]$ and $[c_i, b_i]$ for i $\in$ [1,k] so there now exists 2^k k-cells Q_i whose union is I.
At least one Q_i cannot be covered else I would be covered. Then subdivide Q_i as before and repeating the process so $Q_{i+1} \subset Q_i$ and each are not covered.
However, there is a point x* $\in Q_{i_j}$ for all j such that $N_r(x^*) \subset G$ so Q_{i_1} is covered which is a contradiction.

Theorem 8.3.13: Heine-Borel Theorem

If a set E $\subset \mathbb{R}^k$ has one of the three properties, then it has the other two:

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

Proof

Suppose E is closed and bounded.
Then there exists a M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$ such that d(p,q) $<$ M for all p $\in$ E.
Thus, there is a k-cell K = $[-M+q_1,q_1+M] \times ... \times [-M+q_k,q_k+M]$ such that E $\subset$ K. Then by theorem 8.3.12, K is compact and thus by theorem 8.3.5, E is compact so (a) $\rightarrow$ (b).
Then by thereom 8.3.10, any infinite subset of E has a limit point in E so (b) $\rightarrow$ (c).
Suppose E is not bounded.
Then there exists p $\in$ E such that d(p,q) $>$ M for any M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$.
Let S $\subset$ E be such points p.
Then S is infinite else there is a maximal p and thus, p is bounded. Thus, S is infinite and contains no limit points in E since any d(p_1,p_2) $>$ M which contradicts that every infinite subset of E has a limit point in E. Thus, E is bounded.
Suppose E is not closed.
Then there exists a p $\in$ E', but p $\notin$ E. Since p is a limit point, then there is a q $\in$ E such that $\frac{1}{n+1} <$ d(q,p) $< \frac{1}{n}$ for n = {1, 2, ...}.
Let S $\subset$ E be such points q.
Thus, p is the only limit point of S since for r $< \frac{1}{n}$, any N$_r(q_i)$ contains no points of S other than q_i since d(q_i,q_j) $> \frac{1}{n}$ for any q_1,$q_2 \in$ S.
Thus, S is infinite, but the only p $\in$ S' is p $\notin$ E which contradicts that every infinite subset of E has a limit point in E. Thus, E is closed. So, (c) $\rightarrow$ (a).

Theorem 8.3.14: Weierstrass Theorem

Every bounded infinite set E $\subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof

Since E is bounded, then there exists a k-cell K such that E $\subset$ K. Since K is compact, then by theorem 8.3.10, E has a limit point in K and thus, in $\mathbb{R}^k$.

9 Perfect and Connected Sets

9.1 Perfect Sets

Definition 9.1.1: Perfect Set

E ⊂ X is perfect if E is closed and if every p ∈ E is p ∈ E'.

Theorem 9.1.2: Perfect sets are uncountable

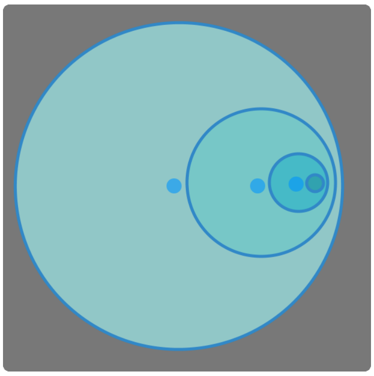
Let P be a nonempty perfect set in $\mathbb{R}^k$. Then, P is uncountable.

Proof

Since P has limit points, then by theorem 7.1.4, P is infinite.
Suppose P is countable. Then let $x_1, x_2, ... \in$ P.
Let V_i be a neighborhood of x_i where y $\in V_i$ for any y $\in \mathbb{R}^k$ such that $|y - x_i| <$ r.
Thus, the $\overline{V_i}$ is the set of all y $\in \mathbb{R}^k$ such that $|y - x_i| \leq$ r.
Since every x_i are limit points, then any $V_i \cap$ P is not empty where there is a V_{i+1}

(a) $\overline{V_{i+1}} \subset V_i$
(b) $x_i \notin \overline{V_{i+1}}$
(c) $V_{i+1} \cap$ P is nonempty

Let $K_i = \overline{V_i} \cap$ P. Since $\overline{V_i}$ is closed and bounded, then by theorem 8.3.11, $\overline{V_i}$ is compact. Since $x_i \notin K_{i+1}$, then no $x_i \in$ P is $x_i \in \cap K_i$. Since $K_n \subset$ P, then $\cap K_i$ is empty which contradicts corollary 8.3.8 since each K_i is nonempty and $K_{i+1} \subset K_i$.



Corollary 9.1.3: $\mathbb{R}$ is not countable

Every interval [a,b] is uncountable. Thus, $\mathbb{R}$ is uncountable.

Proof

Since [a,b] is closed and every p ∈ [a,b] is a limit point, then nonempty set [a,b] is perfect. Thus, by theorem 9.1.2, [a,b] is uncountable.

Definition 9.1.4: Cantor Sets

There exists perfect segments in $\mathbb{R}^1$ which contain no segment.
Let E_0 = [0,1].
For E_1, remove ($\frac{1}{3}$,$\frac{2}{3}$). Thus, E_1 = [0,$\frac{1}{3}$] ∪ [$\frac{2}{3}$,1].
For E_2, remove ($\frac{1}{9}$,$\frac{2}{9}$) and ($\frac{7}{9}$,$\frac{8}{9}$). Thus, E_2 = [0,$\frac{1}{9}$] ∪ [$\frac{2}{9}$,$\frac{3}{9}$] ∪ [$\frac{6}{9}$,$\frac{7}{9}$] ∪ [$\frac{8}{9}$,1].
Continuing such a sequence, the set of compact sets E_n are such that:

(a) $E_{n+1} \subset E_n$

(b) E_n is the union of 2^n intervals each of length 3^{-n}.

P = ∩ E_n is called the Cantor set. P is compact and nonempty.
Thus, any segment of form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ where k,m ∈ $\mathbb{Z}_+$ has no points in common with P. Since any segment (a,b) contain a segment of such a form since $3^{-m} < \frac{b-a}{6}$, then P contains no segment.
Let x ∈ P and segment S contain x. Let I_n be an interval of E_n containing x. Then choose a large enough n so $I_n \subset$ S.
Let x_n be an endpoint of I_n where $x_n \neq$ x and thus, x is a limit point. Since P is closed and every p ∈ P is p ∈ P', then P is perfect.

9.2 Connected Sets

Definition 9.2.1: Connected Set

A,B ⊂ X are separated if both A ∩ $\overline{B}$ and $\overline{A}$ ∩ B are empty.
E ⊂ X is connected if E is not the union of two nonempty separated sets.
Separated sets are disjoint, but disjoint sets need not be separated.

Theorem 9.2.2: All points between points in connected sets exists

E ⊂ $\mathbb{R}^1$ is connected if and only if:

If x,y ∈ E and x < z < y, then z ∈ E.

Proof

If there exists x,y ∈ E and z ∈ (x,y) such that z ∉ E, then E = $A_z \cup B_z$ where A_z = E ∩ $(-\infty, z)$ and B_z = E ∩ (z, ∞).
Since x ∈ A_z and y ∈ B_z, then A and B are nonempty. Since $A_z \subset (-\infty, z)$ and B_z = (z,∞), then A_z and B_z are separated. Thus, E is not connected.

Suppose E is not connected. Then, there are nonempty separated sets A and B such that A ∪ B = E. Pick x ∈ A, y ∈ B where x < y. Let z = sup(A ∩ [x,y]).
Since, z ∈ $\overline{A}$ so z ∉ B, then x ≤ z < y. If z ∉ A, then x < z < y so z ∉ E.
If z ∈ A, then z ∉ $\overline{B}$ and thus, there exists a z_1 such that z < z_1 < y and z_1 ∉ B.
Then, x < z_1 < y so z_1 ∉ E.

10 Convergent and Cauchy Sequences

10.1 Convergent Sequences

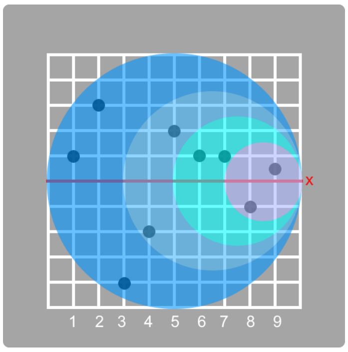
Definition 10.1.1: Convergent Sequence

A sequence $\{x_n\}$ in metric space X converge if there is a x $\in$ X such that:

For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for all n $\geq$ N, d(x_n,x) $< \epsilon$

Then, $\{x_n\}$ converges to x: $\lim_{n\to\infty} x_n =$ x

If $\{x_n\}$ does not converge, then it diverges.



Example 10.1.2

(a) Let $x_n = \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\lim_{n\to\infty} x_n = 0$

Proof

For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$. Then:

d(x_n,0) $= |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$

(b) Let $x_n = (-1)^n + \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\{x_n\}$ diverges.

Proof

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (-1)^n + \lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} (-1)^n$

Since $(-1)^n$ alternates between -1 and 1, then $\{x_n\}$ diverges.

Theorem 10.1.3: A convergent sequence is unique and bounded

(a) $\{p_n\}$ converges to p $\in$ X if and only if every N$_r(p)$ contains p_n for all, but finitely many n.

Proof

Suppose $p_n \to$ p. Then for N$_\epsilon(p)$, any q $\in$ X such that d(q,p) $< \epsilon$ is q $\in$ N$_\epsilon(p)$. Since $p_n \to$ p, there is a N such that for n $\geq$ N, d(p_n,p) $< \epsilon$.
Thus, for n $\geq$ N, $p_n \in$ N$_\epsilon(p)$.
Suppose every N$_r(p)$ contains p_n for all, but finitely many n.
For $\epsilon > 0$, let N$_\epsilon(p)$ be the set of all q $\in$ X such that d(p,q) $< \epsilon$. Thus, there exists a N such that $p_n \in$ N$_\epsilon(p)$ if n $\geq$ N.
Thus, d(p_n,p) $< \epsilon$ so $p_n \to$ p.

(b) If p,p' $\in$ X and $\{p_n\}$ converges to p and p', then p = p'.

Proof

For $\epsilon > 0$, there exists N,N' such that:

d(p_n,p) $< \frac{\epsilon}{2}$ for n $\geq$ N   d(p_n,p') $< \frac{\epsilon}{2}$ for n $\geq$ N'

Then for n $\geq$ max(N,N'), d(p,p') $\leq$ d(p,p_n) + d(p_n,p') $< \epsilon$.
Thus, p = p'.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

Proof

If $\{p_n\} \to$ p, there is a N such that for n > N, d(p_n,p) < 1.
Let r = max(d(p_1,p), ... , d(p_N,p), 1). Thus for all n, d(p_n,p) $\leq$ r.

(d) If E $\subset$ X and p $\in$ E', there is a $\{p_n\}$ in E such that p = $\lim_{n\to\infty} p_n$.

Proof

Since p $\in$ E', then for each n $\in \mathbb{Z}_+$, there is a $p_n \in$ E such that d(p_n,p) $< \frac{1}{n}$. For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$ so for n $\geq$ N, d(p_n,p) $< \frac{1}{n} \leq \frac{1}{N} < \epsilon$.
Thus, p = $\lim_{n\to\infty} p_n$.

Theorem 10.1.4: Arithmetic Operations for sequences

Suppose $\{s_n\}$,$\{t_n\} \in \mathbb{C}$ where $\lim_{n\to\infty} s_n$ = s and $\lim_{n\to\infty} t_n$ = t.

(a) $\lim_{n\to\infty} s_n + t_n$ = s + t

Proof

For $\epsilon > 0$, there exists N_1, N_2 such that

$|s_n - s| < \frac{\epsilon}{2}$ for n $\geq N_1$ $\quad$ $|t_n - t| < \frac{\epsilon}{2}$ for n $\geq N_2$

If N = max(N_1,N_2), then for n $\geq$ N:

$|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$

(b) $\lim_{n\to\infty} cs_n$ = cs and $\lim_{n\to\infty} c + s_n$ = c + s

Proof

For $\epsilon > 0$, there exists a N such that

$|s_n - s| < \frac{\epsilon}{|c|}$ for n $\geq$ N

$|cs_n - cs| \leq |c| \cdot |s_n - s| < \epsilon$

(c) $\lim_{n\to\infty} s_n t_n$ = st

Proof

Note $s_n t_n$ - st = $(s_n - s)(t_n - t)$ + t(s_n - s) + s(t_n - t).
For $\epsilon > 0$, there exists N_1,N_2 such that

$|s_n - s| < \sqrt{\epsilon}$ for n $\geq N_1$ $\quad$ $|t_n - t| < \sqrt{\epsilon}$ for n $\geq N_2$

If N = max(N_1,N_2), then for n $\geq$ N, $|(s_n - s)(t_n - t)| < \epsilon$.
Thus, $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$.

$$\lim_{n\to\infty} (s_n t_n - st) = \lim_{n\to\infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$$
$$= 0 + \text{t} \cdot 0 + \text{s} \cdot 0 = 0$$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ where $s_n, s \neq 0$

Proof

Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if n $\geq$ m so $|s_n| > \frac{1}{2}|s|$ for n $\geq$ m.
For $\epsilon > 0$, there is a N > m such that for n $\geq$ N, $|s_n - s| < \frac{1}{2}|s|^2\epsilon$.
Thus, for n $\geq$ N, $|\frac{1}{s_n} - \frac{1}{s}| = |\frac{s_n - s}{s_n s}| < \frac{2}{|s|^2}|s_n - s| < \epsilon$.

Theorem 10.1.5: Extension to $\mathbb{R}^k$

(a) Suppose $x_n \in \mathbb{R}^k$ and $x_n = (\alpha_{n_1}, \dots, \alpha_{n_k})$. Then $\{x_n\}$ converges to x = $(\alpha_1, \dots, \alpha_k)$ if and only if $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].

Proof

Suppose $\{x_n\}$ converges to x = $(\alpha_1, \dots, \alpha_k)$.
Since for any i $\in$ [1,k]:
$$|\alpha_{n_i} - \alpha_i| \leq \sqrt{|\alpha_{n_1} - \alpha_1|^2 + \dots + |\alpha_{n_k} - \alpha_k|^2} = |x_n - x| < \epsilon.$$
Then, $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$.

Suppose $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].
Then for $\epsilon > 0$, there is an N such that for n $\geq$ N:
$$|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}} \text{ for i} \in [1,k]$$
$$|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot (\frac{\epsilon}{\sqrt{k}})^2} = \epsilon$$

(b) Suppose $\{x_n\}$,$\{y_n\} \in \mathbb{R}^k$ and $\{\beta_n\} \in \mathbb{R}$ and $x_n \to$ x, $y_n \to$ y, $\beta_n \to \beta$.
$\lim_{n\to\infty} x_n + y_n$ = x+y $\quad$ $\lim_{n\to\infty} x_n \cdot y_n$ = x·y $\quad$ $\lim_{n\to\infty} \beta_n x_n = \beta$x

Proof

By part a, then $\lim_{n\to\infty} x_{n_i} + y_{n_i} = x_i + y_i$ so $\{x_n + y_n\} \to$ x+y.
Also, $\lim_{n\to\infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$ so $\{x_n \cdot y_n\} \to$ x·y.
Also, $\lim_{n\to\infty} \beta_i x_{n_i} = \beta_i x_i$ so $\{\beta_n x_n\} \to \beta x$.

10.2 Subsequences

Definition 10.2.1: Subsequence

For sequence $\{p_n\}$, let $\{n_k\} \in \mathbb{Z}_+$ where $n_k < n_{k+1}$.
Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.
If $\{p_{n_k}\}$ converges, then its limit is called a subsequential limit.

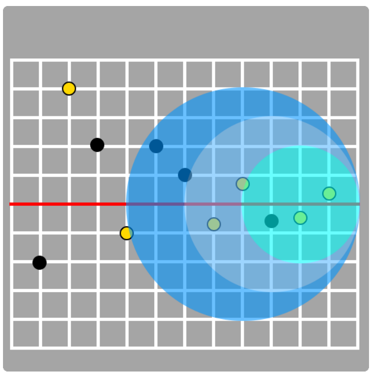
Theorem 10.2.2: $\{p_n\} \to$ p $\rightleftharpoons$ Every $\{p_{n_k}\} \to$ p

$\{p_n\}$ converges to p if and only if every subsequence converges to p.

Proof

Suppose $\{p_n\}$ converges to p.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, d(p_n,p) $< \epsilon$.
Let $\{p_{n_k}\} \subset \{p_n\}$. Then for $n_k \geq$ N, $|p_{n_k} - p| < \epsilon$. Thus, $\{p_{n_k}\} \to$ p.

Suppose every subsequence converges to p.
Since $\{p_n\}$ is a subsequence of itself, then $\{p_n\}$ converges to p.



Theorem 10.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \to$ p

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence converges to p $\in$ X.

Proof

Let E be the range of $\{p_n\}$.
If E is finite, there is a p $\in$ E and sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $p_{n_1} = p_{n_2} = ... =$ p. Thus, $\{p_{n_k}\} \to$ p.
If E is infinite, then by theorem 8.3.10, then there exists a p $\in$ E'.
Then there are n_k such that d(p_{n_k},p) $< \frac{1}{k}$. Thus, $\{p_{n_k}\} \to$ p.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof

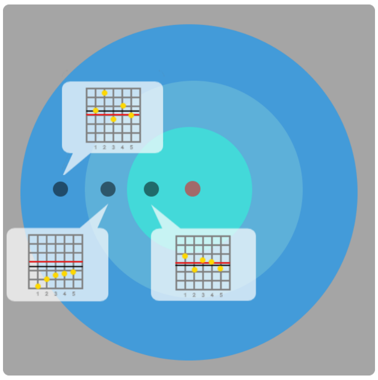
Let E be a bounded sequence in $\mathbb{R}^k$. Since E $\cup$ E' is bounded and closed, then by theorem 8.3.13, E $\cup$ E' is compact.
Thus by part a, E contains a convergent subsequence.

Theorem 10.2.4: The set of subsequential limits is closed

The subsequential limits of $\{p_n\}$ in metric space X form a closed subset of X.

Proof

Let E be the range of the set of all subsequential limits of $\{p_n\}$.
If E is empty, then E is closed. If E is finite, then E' is empty so E is closed.
Suppose E is infinite. Then, let q $\in$ E'.
Since q $\in$ E', there is a x $\in$ E where d(x,q) $< \frac{\epsilon}{2}$.
Since x $\in$ E, there is a $\{p_{n_k}\} \to$ x so there is a N such that for n $\geq$ N, d(p_{n_k},x) $< \frac{\epsilon}{2}$.
Thus, d(p_{n_k},q) $\leq$ d(p_{n_k},x) + d(x,q) $< \epsilon$ so q is a subsequential limit of $\{p_n\}$.
Thus, q $\in$ E so E is closed.



10.3 Cauchy Sequences

Definition 10.3.1: Metric Spaces

Sequence $\{p_n\} \in$ X is a Cauchy sequence if:

For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for all n,m $\geq$ N, d(p_n,p_m) $< \epsilon$

Let nonempty E $\subset$ X and S $\subset \mathbb{R}$ of d(p,q) where p,q $\in$ E.
Let sup(S) = diam(E). If $\{p_n\} \in$ X, and $p_N, p_{N+1}, ... \in E_N$, then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N\to\infty}$ diam(E_N) = 0.

Theorem 10.3.2: Cauchy sequences and its closure have the same diam

(a) If $\overline{E} \subset$ X, then diam($\overline{E}$) = diam(E).

Proof

Since E $\subset \overline{E}$, then diam(E) $\leq$ diam($\overline{E}$).
For $\epsilon > 0$, let p,q $\in$ E'.
Thus, there are p',q' $\in$ E such that d(p',p) $< \epsilon$ and d(q',q) $< \epsilon$. Thus:
d(p,q) $\leq$ d(p,p') + d(p',q') + d(q',q) $< 2\epsilon$ + d(p',q') $\leq 2\epsilon$ + diam(E).
Thus, diam($\overline{E}$) $\leq 2\epsilon$ + diam(E) so diam($\overline{E}$) = diam(E).

(b) If K_n is a sequence of compact sets of X such that $K_{n+1} \subset K_n$ and $\lim_{n\to\infty}$ diam(K_N) = 0, then $\cap$ K_n consist of only one point.

Proof

Let K = $\cap$ K_n. Since K_n is a sequence of compact sets, then by Corollary 8.3.8 , K is nonempty.
If K contains more than one point, then diam(K) > 0.
But since K $\subset K_n$, then diam(K) $\leq$ diam(K_n) which contradicts that diam(K_n) $\to 0$.

Theorem 10.3.3: Convergent sequences are cauchy sequences

(a) Every convergent sequence is a Cauchy sequence.

Proof

If $p_n \to$ p and $\epsilon > 0$, there is a N such that for all n $\geq$ N, d(p,p_n) $< \frac{\epsilon}{2}$.
Thus, for m,n $\geq$ N:
d(p_n,p_m) $\leq$ d(p_n,p) + d(p,p_m) $< \epsilon$.
Thus, $\{p_n\}$ is a Cauchy sequence.

(b) If $\{p_n\}$ is a Cauchy sequence in compact metric space X, then $\{p_n\}$ converges to some p $\in$ X.

Proof

Let $\{p_n\}$ be a Cauchy sequence in compact space X.
Let $p_N, p_{N+1}, ... \in E_N$.
Since $\{p_n\}$ is a Cauchy sequence, then $\lim_{N\to\infty}$ diam($\overline{E_N}$) = 0. Since $\overline{E_N}$ is closed in compact X, then by theorem 8.3.5, $\overline{E_N}$ is compact.
Since $E_{N+1} \subset E_N$, then $\overline{E_{N+1}} \subset \overline{E_N}$ and thus, by theorem 10.3.2b, then there is a unique p $\in \overline{E_N}$ for every N.
Since p $\in \overline{E_N}$, then d(p,q) $< \epsilon$ for every q $\in \overline{E_N}$ so every q $\in E_N$.
Then for $\epsilon > 0$, there is a N_0 such that for N $\geq N_0$, diam($\overline{E_N}$) $< \epsilon$.
Thus, d(p_n,p) $< \epsilon$ for n $\geq N_0$ so $\{p_n\} \to$ p.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

Proof

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Let $x_N, x_{N+1}, ... \in E_N$.
Then for some N, diam(E_N) < 1. Thus, the range of $\{x_n\}$ = $E_N \cup \{x_1, ..., x_{N-1}\}$. Thus, $\{x_n\}$ is bounded.
Thus, the $\overline{\{x_n\}}$ is closed and bounded so by theorem 8.3.13, $\overline{\{x_n\}}$ is compact. Thus, by part b, $\{x_n\}$ converges to some p $\in \mathbb{R}^k$.

Definition 10.3.4: Complete

A metric space where every Cauchy sequence converges is complete.
Thus, by theorem 10.3.3, all compact and Euclidean spaces are complete.

Definition 10.3.5: Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is:

(a) monotonically increasing if $s_n \leq s_{n+1}$

(b) monotonically decreasing if $s_n \geq s_{n+1}$

Theorem 10.3.6: Monotonic sequences converge if bounded

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof

Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$.
Suppose $\{s_n\}$ is bounded.
Let s = sup(E) so $s_n \leq$ s. For every $\epsilon > 0$, there is a N such that s - $\epsilon < s_N \leq$ s else s-ϵ would be an upper bound of E which contradicts s = sup(E).
Since $\{s_n\}$ increases, then for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s so $\{s_n\} \to$ s.

Suppose $\{s_n\}$ converges to s.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s.
Thus, $\{s_n\}$ is bounded from above.

Suppose $s_n \geq s_{n+1}$. Let E be the range of $\{s_n\}$.
Suppose $\{s_n\}$ is bounded.
Let s = inf(E) so $s_n \geq$ s. For every $\epsilon > 0$, there is a N such that s $\leq s_N <$ s + ϵ else s+ϵ would be a lower bound of E which contradicts s = inf(E).
Since $\{s_n\}$ decreases, then for n $\geq$ N, s $\leq s_n \leq s_N <$ s + ϵ so $\{s_n\} \to$ s.

Suppose $\{s_n\}$ converges to s.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s $\leq s_n \leq s_N <$ s + ϵ.
Thus, $\{s_n\}$ is bounded from below.

11 Limits and Special Sequences

11.1 Upper and Lower Limits

Definition 11.1.1: Infinite limits

Let $\{s_n\}$ be a sequence of real numbers such that:

For every real M, there is a N $\in \mathbb{Z}$ such that for n $\geq$ N, $s_n \geq$ M.

Then, $s_n \to +\infty$.

For every real M, there is a N $\in \mathbb{Z}$ such that for n $\geq$ N, $s_n \leq$ M.

Then, $s_n \to -\infty$.

Definition 11.1.2: Upper and Lower Limits

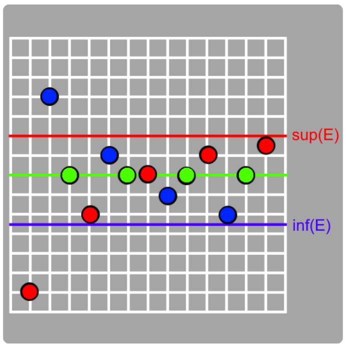
Let $\{s_n\} \subset \mathbb{R}$ and E contain all subsequential limits of $\{s_n\}$ plus possibly $\pm\infty$.

Then, the upper limit of $\{s_n\}$:

$s^* = \sup(E)$ $\lim_{n\to\infty} \sup(s_n) = s^*$

Then, the lower limit of $\{s_n\}$:

$s_* = \inf(E)$ $\lim_{n\to\infty} \inf(s_n) = s_*$



Theorem 11.1.3: Upper and Lower limits are unique

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and s^* be the upper limit of $\{s_n\}$. Then:

(a) $s^* \in$ E

Proof

If $s^* = +\infty$, then there is a $\{s_{n_k}\} \to +\infty$ so E is not bounded above.
If $s^* \in \mathbb{R}$, then E is bounded above so $s^* \in$ E'.
Then by theorem 10.2.4, $s^* \in$ E.
If $s^* = -\infty$, then there are no subsequential limits in E. Thus, for every M, there is a N such that for n $\geq$ N, $s_n \leq$ M so $-\infty \in$ E.

(b) If x $> s^*$, there is a N such that for n $\geq$ N, $s_n <$ x

Proof

Suppose there is a x $> s^*$ such that $s_n \geq$ x for infinitely many n.
Then, there is a y $\in$ E where y $\geq$ x $> s^*$ which contradicts $s^* = \sup(E)$.

(c) s^* is the only number that satisfies (a) and (b)

Proof

Suppose p,q satisfy part a and b where p $<$ q. Choose x where p $<$ x $<$ q. Since p satisfies b, then $s_n <$ x for n $\geq$ N. Thus, x is an upper bound for E so q $\notin E$ since q $>$ x contradicting that q satisfies part a.

The same properties are analogous for s_*.

Theorem 11.1.4: Inf & Sup of $s_n \leq t_n$

If $s_t \leq t_n$ for n $\geq$ fixed N, then

$\lim_{n\to\infty} \inf(s_n) \leq \lim_{n\to\infty} \inf(t_n)$

$\lim_{n\to\infty} \sup(s_n) \leq \lim_{n\to\infty} \sup(t_n)$

Proof

Let E_1 be the set of extended reals x such that $\{s_{n_k}\} \to$ x for some $\{s_{n_K}\}$.
Let E_2 be the set of extended reals y such that $\{t_{n_k}\} \to$ y for some $\{s_{n_K}\}$.
Let $s^* = \sup(E_1)$, $s_* = \inf(E_1)$, $t^* = \sup(E_2)$, and $t_* = \inf(E_2)$.
Since there is a N such that $s_n \leq t_n$ for n $\geq$ N, then:

x $\leftarrow \{s_N, s_{N+1}, ...\} \leq \{t_N, t_{N+1}, ...\} \to$ y

Thus, for n $\geq$ N, $\inf(s_n) \leq \inf(t_n)$ and $\sup(s_n) \leq \sup(t_n)$.

11.2 Special Sequences

Theorem 11.2.1: Special sequences

(a) If p > 0, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

Proof

For $\epsilon > 0$, let N > $\sqrt[p]{\frac{1}{\epsilon}}$.
Then for n $\geq$ N, $\lim_{n\to\infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}^p} = \epsilon$

(b) If p > 0, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

Proof

If p > 1, then let $x_n = \sqrt[n]{p}$ - 1 > 0.

p = $(x_n + 1)^n = x_n^n + nx_n^{n-1} + ... + nx_n + 1 \geq nx_n + 1$

Thus, $0 < x_n \leq \frac{p-1}{n}$ so $\{x_n\} \to 0$ and thus, $\{\sqrt[n]{p}\} \to 1$.
If p = 1, then $\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} 1 = 1$.
If 0 < p < 1, then $\frac{1}{p} > 1$. From the proof above for p > 1, $\{\sqrt[n]{\frac{1}{p}}\} \to 1$.
Thus, $\{\frac{1}{\sqrt[n]{p}}\} \to 1$ so $\{\sqrt[n]{p}\} \to 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Proof

Let $x_n = \sqrt[n]{n}$ - 1 $\geq$ 0.

n = $(x_n + 1)^n \geq \frac{n(n-1)}{2}x_n^2$

Thus, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ so $\{x_n\} \to 0$ and thus, $\{\sqrt[n]{n}\} \to 1$.

(d) If p > 0 and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

Proof

Let k $\in \mathbb{Z}$ such that k > α and k > 0. For n > 2k:

$(1+p)^n > \binom{n}{k}p^k = \frac{n(n-1)...(n-k+1)}{k!}p^k > \frac{n^k p^k}{2^k k!}$

Thus, $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$.
Since α - k < 0, then $\{n^{\alpha-k}\} \to 0$ so $\{\frac{n^\alpha}{(1+p)^n}\} \to 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Proof

From part d, let $\alpha = 0$.
Thus, $\lim_{n\to\infty} \frac{1}{(1+p)^n} = 0$ and since p > 0, then $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$.
Also, $-\lim_{n\to\infty} \frac{1}{(1+p)^n} = \lim_{n\to\infty} \frac{-1}{(1+p)^n} = 0$ so $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n >$ -1.

12 Series and Comparison Test

12.1 Series

Definition 12.1.1: Series

For sequence $\{a_n\}$, define $\sum_{n=p}^{q} a_n = a_p + a_{p+1} + ... + a_q$.
Then associate $\{a_n\}$ with a sequence $\{s_n\}$ such that $s_n = \sum_{k=1}^{n} a_k$.
Then $\{s_n\}$ is a series with partial sums s_n.
If $\{s_n\} \to$ s, then $\sum_{n=1}^{\infty} a_n =$ s is the sum of the convergent series.
Note $a_1 = s_1$ and $a_n = s_n$ - s_{n-1}.

Theorem 12.1.2: Cauchy Criterion for series

$\sum a_n$ converges if and only if:
For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$

Proof

Suppose $\sum_{k=1}^{n} a_k$ converges.
Then by theorem 10.3.3a, $\sum_{k=1}^{n} a_k$ is a Cauchy sequence.
Then for $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N:

$$d(\sum_{k=1}^{n} a_k , \sum_{k=1}^{m} a_k) = |\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k| = |\sum_{k=n}^{m} a_k| \leq \epsilon$$

Suppose for every $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$.

$$|\sum_{k=n}^{m} a_k| = |\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k| = d(\sum_{k=1}^{n} a_k , \sum_{k=1}^{m} a_k) \leq \epsilon$$

Thus, $\sum_{k=1}^{n} a_k$ is a Cauchy sequence and thus, convergent.

Theorem 12.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \to 0$

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof

Since $\sum a_n$ converges, then by theorem 12.1.2, for $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$. Then if m = n $\geq$ N, $|\sum_{k=n}^{m} a_k| = |a_n| \leq \epsilon$ so $\{a_n\} \to 0$.

Example 12.1.4: $\{a_n\} \to 0 \not\Rightarrow$ Convergent $\sum a_n$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + ... + \frac{1}{8}) + (\frac{1}{9} + ...) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$
Thus, $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$ which is unbounded and thus, not convergent.

Theorem 12.1.5: Convergent series $\rightleftharpoons$ Bounded sequence

A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.

Proof

Suppose $\sum a_n$ converges where $a_n \geq 0$.
Since $a_n \geq 0$, then $\{s_n\}$ is monotonic so by theorem 10.3.6, $\{s_n\}$ is bounded above.

Suppose $\{s_n\}$ is bounded where $a_n \geq 0$.
Since $\{s_n\}$ is monotonic and bounded, then by theorem 10.3.6, $\{s_n\}$ converges.

Theorem 12.1.6: Comparison Test

(a) If $|a_n| \leq c_n$ for n $\geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.

Proof

For $\epsilon > 0$, there exists a N $\geq N_0$ such that for m $\geq$ n $\geq$ N, $\sum_{k=n}^{m} c_k \leq \epsilon$.

$|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k \leq \epsilon$

Thus, $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for n $\geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof

Suppose $\sum a_n$ converges.
Then from part a, $\sum d_n$ converges which contradicts that $\sum a_n$ diverges.
Thus, $\sum a_n$ diverges.

12.2 Series of Nonnegative Terms

Theorem 12.2.1: Infinite Geometric Series

If x $\in$ [0,1), then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If x $\geq$ 1, the series diverges.

Proof

If x $\neq$ 1, then using the geometric series:

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

Let n $\to \infty$.
If x $\in$ [0,1), then by theorem 11.2.1e, $s_n = \frac{1}{1-x}\,(1 - x^{n+1}) = \frac{1}{1-x}$ (1 - 0) $= \frac{1}{1-x}$.
Also, by theorem 11.2.1e, if x $\geq$ 1, then the series diverges.

Theorem 12.2.2: Cauchy's Convergence Criterion

Suppose $0 \leq a_{i+1} \leq a_i$.
Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \text{ converges.}$$

Proof

Let $s_n = a_1 + a_2 + \ldots + a_n$ and $t_k = a_1 + 2a_2 + \ldots + 2^k a_{2^k}$.
For n $< 2^k$:

$$\begin{aligned} s_n &\leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + (a_{2^k} + \ldots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \ldots + 2^k a_{2^k} = t_k \end{aligned}$$

Thus, by the comparison test, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
For n $> 2^k$:

$$\begin{aligned} s_n &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \ldots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots + (a_{2^{k-1}+1} + \ldots + a_{2^k}) \\ &\geq \tfrac{1}{2}a_1 + a_2 + 2a_4 + \ldots + 2^{k-1} a_{2^k} = \tfrac{1}{2}t_k \end{aligned}$$

Thus, by the comparison test, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem 12.2.3: P-series

$\sum \frac{1}{n^p}$ converges if p > 1 and diverges if p ≤ 1.

Proof

If p ≤ 0, then by theorem 12.1.3, $\sum \frac{1}{n^p}$ diverges.
If p > 0, then by theorem 12.2.2, $\sum \frac{1}{n^p}$ converges only if $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges.
Since $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$, then by theorem 12.2.1, $\sum_{k=0}^{\infty} 2^{k(1-p)}$ converges if $2^{1-p} < 1$ so if 1-p < 0 so p > 1.

Theorem 12.2.4: Log P-series

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$ converges if p > 1 and diverges if p ≤ 1.

Proof

Since $\frac{1}{n(\log(n))^p}$ decreases, then by theorem 12.2.2,
$\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges.

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k}$$

Then by theorem 12.2.3, $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges if p > 1 and diverges if p ≤ 1.
Thus, $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if p > 1 and diverges and p ≤ 1.

Corollary 12.2.5: Log P-series extended

$\sum_{n=3}^{\infty} \frac{1}{n\log(n)(\log(\log(n)))^p}$ converges if p > 1 and diverges if p ≤ 1.

Proof

From theorem 12.2.4, replace n = log(n) and multiplying by $\frac{1}{n} \rightarrow \frac{1}{n\log(n)(\log(\log(n)))^p}$.
Since $\frac{1}{n\log(n)(\log(\log(n)))^p}$ decreases, by theorem 12.2.2 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k\log(2^k)(\log(\log(2^k)))^p}$:

$$\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k\log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$$

Since $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$ converges by theorem 12.2.4, $\sum_{n=3}^{\infty} \frac{1}{n\log(n)(\log(\log(n)))^p}$ converges.

12.3 The Number e

Definition 12.3.1: Summation equivalence to e

$s_m = \sum_{n=0}^{m} \frac{1}{n!} = 1 + \sum_{n=1}^{m} \frac{1}{n!} < 1 + \sum_{n=1}^{m} \frac{1}{2^{n-1}} < 3$
e = $\sum_{n=0}^{\infty} \frac{1}{n!}$

Theorem 12.3.2: Limit equivalence to e

$\lim_{n\to\infty} (1 + \frac{1}{n})^n$ = e

Proof

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $t_n = (1 + \frac{1}{n})^n$. Using the binomial theorem:

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n(n-1)\ldots(n-k+1))}{k!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!}(1)(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{k-1}{n})$$

Thus, $t_n \leq s_n$ so $\lim_{n\to\infty} \sup(t_n) \leq$ e.
If n ≥ m, then $t_n \geq \sum_{k=0}^{m} \frac{1}{k!}(1)(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{k-1}{n})$.
As n → ∞, then $\lim_{n\to\infty} \inf(t_n) \geq \sum_{k=0}^{m} \frac{1}{k!} = s_m$. As m → ∞, $\lim_{n\to\infty} \inf(t_n) \geq$ e.

Definition 12.3.3: Rapidity of convergence of e

0 < e - $s_n < \frac{1}{n!n}$

Proof

e - $s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} (1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots) = \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!n}$

Theorem 12.3.4: e is irrational

e is irrational

Proof

Suppose r is rational. Then let e = $\frac{p}{q}$ for p,q $\in \mathbb{Z}_+$.
Thus, by definition 12.3.3, 0 < e - s_q < $\frac{1}{q!q}$ so 0 < q!(e - s_q) < $\frac{1}{q}$.
Since e = $\frac{p}{q}$, then q!e is an integer and q!s_q = q!(1 + 1 + $\frac{1}{2!}$ + ... + $\frac{1}{q!}$) is an integer.
Thus, q!(e - s_q) is an integer which is between 0 and $\frac{1}{q}$ and thus, a contradiction.

12.4 Root and Ratio Tests

Theorem 12.4.1: Root Test

For $\sum a_n$, let $\alpha = \lim_{n\to\infty} \sup(\sqrt[n]{|a_n|})$.

(a) If $\alpha < 1$, $\sum a_n$ converges

(b) If $\alpha > 1$, $\sum a_n$ diverges

(c) If $\alpha = 1$, unclear

Proof

If $\alpha < 1$, choose β such that $\beta \in (\alpha,1)$ and N $\in \mathbb{Z}$ such that $\sqrt[n]{|a_n|} < \beta$ for n $\geq$ N.
Since $\beta \in (0,1)$, then by theorem 12.2.1, $\sum \beta^n$ converges. Then by the comparison test, $\sum a_n$ converges.

If $\alpha > 1$, then there is a a_{n_k} such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$.
Thus, $|a_n| > 1$ for infinitely many n so by theorem 12.1.3, $\sum a_n$ doesn't converge.

For $\alpha = 1$, both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have $\alpha = 1$, but $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges by theorem 12.2.3.

Theorem 12.4.2: Ratio Test

(a) $\sum a_n$ converges if $\lim_{n\to\infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$

(b) $\sum a_n$ diverges if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all n $\geq n_0$ for $n_0 \in \mathbb{Z}$

Proof

If $\lim_{n\to\infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$, there is a $\beta < 1$ and N such that for n $\geq$ N, $|\frac{a_{n+1}}{a_n}| < \beta$.
Then $|a_{N+1}| < \beta|a_N|$ so $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$.
Thus, $|a_{N+p}| < \beta^p|a_N|$ so $|a_n| < |a_N|\beta^{-N}\beta^n$.
Thus, by the comparison test, $\sum a_n$ converges.
If $|a_{n+1}| \geq |a_n| > 0$ for n $\geq n_0$, then by theorem 12.1.3, $\sum a_n$ diverges.

Theorem 12.4.3: Ratio convergence → Root convergence

$\lim_{n\to\infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n\to\infty} \inf(\sqrt[n]{c_n})$
$\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n\to\infty} \sup(\frac{c_{n+1}}{c_n})$

Proof

Let $\alpha = \lim_{n\to\infty} \inf(\frac{c_{n+1}}{c_n})$. If $\alpha = -\infty$, then $-\infty \leq \lim_{n\to\infty} \inf(\sqrt[n]{c_n})$ holds true.
If α is finite, there is a $\beta \leq \alpha$ and N such that for n $\geq$ N, $\frac{c_{n+1}}{c_n} \geq \beta$ so $c_{N+p} \geq \beta^p c_N$.
Then, $c_n \geq c_N\beta^{-N}\beta^n$ so $\sqrt[n]{c_n} \geq \sqrt[n]{c_N\beta^{-N}}\beta$. Thus, $\lim_{n\to\infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$.

Let $\alpha = \lim_{n\to\infty} \sup(\frac{c_{n+1}}{c_n})$. If $\alpha = \infty$, then $\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \infty$ holds true.
If α is finite, there is a $\beta \geq \alpha$ and N such that for n $\geq$ N, $\frac{c_{n+1}}{c_n} \leq \beta$ so $c_{N+p} \leq \beta^p c_N$.
Then, $c_n \leq c_N\beta^{-N}\beta^n$ so $\sqrt[n]{c_n} \leq \sqrt[n]{c_N\beta^{-N}}\beta$. Thus, $\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$.

13 Power Series

13.1 Power Series

Definition 13.1.1: Power series

For a sequence $\{c_n\} \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is a power series.
c_n are the coefficients and z $\in \mathbb{C}$.

Theorem 13.1.2: Radius of Convergence

For power series $\sum c_n z^n$, let $\alpha = \lim_{n\to\infty} \sup(\sqrt[n]{|c_n|})$ and R $= \frac{1}{\alpha}$.
Then $\sum c_n z^n$ converges if $|z| <$ R and diverges if $|z| >$ R.

Proof

Let $a_n = c_n z^n$. Using the root test,

$$\lim_{n\to\infty} \sup(\sqrt[n]{|a_n|}) = \lim_{n\to\infty} \sup(\sqrt[n]{|c_n z^n|})$$
$$= |z| \lim_{n\to\infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R}$$

Thus, $\sum c_n z^n$ converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$

13.2 Summation By Parts

Theorem 13.2.1: Summation by parts

For sequences $\{a_n\}$, $\{b_n\}$, let $A_n = \sum_{k=0}^{n} a_k$. Then for 0 $\leq$ p $\leq$ q:

$$\sum_{n=p}^{q} a_n b_n = (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$
$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$
$$= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p$$
$$= (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Theorem 13.2.2: Conditions for convergent $\sum a_n b_n$

Suppose for $\{a_n\}$, $\{b_n\}$:

- partial sums A_n of $\sum a_n$ form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n\to\infty} b_n = 0$

Then $\sum a_n b_n$ converges.

Proof

Since $\{A_n\}$ is bounded, $|A_n| \leq$ M for all n.
Since $\{b_n\}$ is monotonically decreasing and $\lim_{n\to\infty} b_n = 0$, then for $\epsilon > 0$, there is a N such that $b_N \leq \frac{\epsilon}{2M}$. Then for N $\leq$ p $\leq$ q:

$$|\sum_{n=p}^{q} a_n b_n| = (|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p|$$
$$\leq \text{M} \, |\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p| = 2\text{M}b_p \leq 2\text{M}b_N \leq \epsilon$$

Corollary 13.2.3: Convergent series of Alternating Sequences

Suppose for $\{c_n\}$:

- $|c_i| \geq |c_{i+1}|$
- $c_{2i-1} \geq 0$ and $c_{2i} \leq 0$
- $\lim_{n\to\infty} c_n = 0$

Then $\sum c_n$ converges.

Proof

From theorem 13.2.2, let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

Corollary 13.2.4: Convergent power series

Suppose for $\{c_n\}$:

- Radius of convergence of $\sum c_n z^n$ is 1
- $c_i \geq c_{i+1}$
- $\lim_{n\to\infty} c_n = 0$

Then $\sum c_n z^n$ converges at every point where $|z| = 1$ except possibly z = 1.

Proof

From theorem 13.2.2, let $a_n = z^n$ and $b_n = c_n$.
A_n of $\sum a_n$ form a bounded sequence since $|A_n| = |\sum_0^n z^n| = |\frac{1-z^{n+1}}{1-z}| \leq \frac{2}{|1-z|}$.

13.3 Absolute Convergence

Definition 13.3.1: Absolute convergence

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ converges non-absolutely.

Theorem 13.3.2: Absolute convergence → convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof

Since $\sum a_n$ converges absolutely, then for every $\epsilon > 0$, there is an integer N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$.
Thus, $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$ so $\sum a_n$ converges.

13.4 Addition & Multiplication of Series

Theorem 13.4.1: Addition and Scalar Multiplication

If $\sum a_n$ = A and $\sum b_n$ = B, then $\sum(a_n + b_n)$ = A + B and $\sum ca_n$ = cA.

Proof

Let $A_n = \sum_{k=0}^n a_k$ and $B_n = \sum_{k=0}^n b_k$.
Then $A_n + B_n = \sum_{k=0}^n a_k + b_k$ so $\lim_{n\to\infty} A_n + B_n$ = A + B.
Then $\lim_{n\to\infty} cA_n = \underbrace{A + ... + A}_{c} = \text{cA}$

Definition 13.4.2: Cauchy Product

For $\sum a_n$ and $\sum b_n$, let $c_n = \sum_{k=0}^n a_k b_{n-k}$ and the product as $\sum c_n$.

$$\begin{aligned}\sum_{n=0}^\infty a_n z^n \sum_{n=0}^\infty b_n z^n &= (a_0 + a_1 z + a_2 z^2 + ... + a_n z^n)\,(b_0 + b_1 z + b_2 z^2 + ... + b_n z^n) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + ...\end{aligned}$$

Theorem 13.4.3: Conditions $\sum c_n$ = AB

Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(b) $\sum_{n=0}^{\infty} a_n$ = A

(c) $\sum_{n=0}^{\infty} b_n$ = B

(d) $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then $\sum_{n=0}^{\infty} c_n$ = AB.

Proof

Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, and $\beta_n = B_n$ - B.

$$\begin{aligned} C_n &= a_0b_0 + (a_0b_1 + a_1b_0) + ... + (a_0b_n + ... + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + ... + a_nB_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + ... + a_n(B + \beta_0) \\ &= A_nB + a_0\beta_n + a_1\beta_{n-1} + ... + a_n\beta_0 \end{aligned}$$

Let $\gamma_n = a_0\beta_n + a_1\beta_{n-1} + ... + a_n\beta_0$ so $C_n = A_nB + \gamma_n$.

Since a_n converges absolutely, then $\sum_{n=0}^{\infty} |a_n| = \alpha$.

Since $\sum_{n=0}^{\infty} b_n$ = B, then $\beta_n \to 0$.

Then for $\epsilon > 0$, there is a N such that $|\beta_n| \leq \frac{\epsilon}{\alpha}$ for n $\geq$ N.

$$\begin{aligned} |\gamma_n| &\leq |\beta_0a_n + ... + \beta_Na_{n-N}| + |\beta_{N+1}a_{n-N-1} + ... + \beta_na_0| \\ &\leq |\beta_0a_n + ... + \beta_Na_{n-N}| + |a_{n-N-1} + ... + a_0|\frac{\epsilon}{\alpha} \\ &\leq |\beta_0a_n + ... + \beta_Na_{n-N}| + \alpha\frac{\epsilon}{\alpha} \end{aligned}$$

Thus, with a fixed N, since $a_n \to 0$, then $\lim_{n\to\infty} |\gamma_n| \leq \epsilon$ so $\lim_{n\to\infty} \gamma_n = 0$.

Thus, $\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_nB + \gamma_n$ = AB.

Theorem 13.4.4: By Cauchy Product, $\sum c_n$ = C implies C = AB

If $\sum a_n$ = A, $\sum b_n$ = B, $\sum c_n$ = C where $c_n = a_0b_n + ... + a_nb_0$, then C = AB.

13.5 Rearrangements

Definition 13.5.1: Rearrangements

Let $a'_n = a_{k_n}$. Then $\sum a'_n$ is a rearrangement of $\sum a_n$.

Theorem 13.5.2: Rearrangements can converge or diverge

Let $\sum a_n \in \mathbb{R}$ converge non-absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$.

Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that:

$$\lim_{n\to\infty} \inf(s'_n) = \alpha \qquad \lim_{n\to\infty} \sup(s'_n) = \beta$$

Proof

Let $p_n = \frac{|a_n| + a_n}{2}$ and $q_n = \frac{|a_n| - a_n}{2}$. Since $\sum |a_n|$ diverge, then $\sum p_n$ and $\sum q_n$ diverges. Let $P_1, P_2, P_3, ...$ be the nonnegative terms of $\sum a_n$ in order and $Q_1, Q_2, Q_3, ...$ be the absolute values of the negative terms of $\sum a_n$ in order. Thus, $\sum P_n$ and $\sum Q_n$ differ from $\sum p_n$ and $\sum q_n$ only by the zero terms and thus, are divergent.

Choose real-valued sequences $\{\alpha_n\} \to \alpha$, $\{\beta_n\} \to \beta$ such that $\alpha_n < \beta_n$ and $\beta_1 > 0$.

Let m_1,k_1 be the smallest integers such that:

$$P_1 + ... + P_{m_1} > \beta_1 \qquad P_1 + ... + P_{m_1} - Q_1 - ... - Q_{k_1} < \alpha_1$$

Let m_2,k_2 be the smallest integers such that:

$$P_1 + ... + P_{m_1} - Q_1 - ... - Q_{k_1} + P_{m_1+1} + ... + P_{m_2} < \beta_2$$

$$P_1 + ... + P_{m_1} - Q_1 - ... - Q_{k_1} + P_{m_1+1} + ... + P_{m_2} - Q_{k_1+1} - ... - Q_{k_2} < \alpha_2$$

Continuing such a process, then $\lim_{n\to\infty} \inf(s'_n) = \alpha$ and $\lim_{n\to\infty} \sup(s'_n) = \beta$.

Theorem 13.5.3: Absolute rearrangements converges uniquely

If $\sum a_n \in \mathbb{C}$ converges absolutely, then every rearrangement of $\sum a_n$ converges to the same sum.

Proof

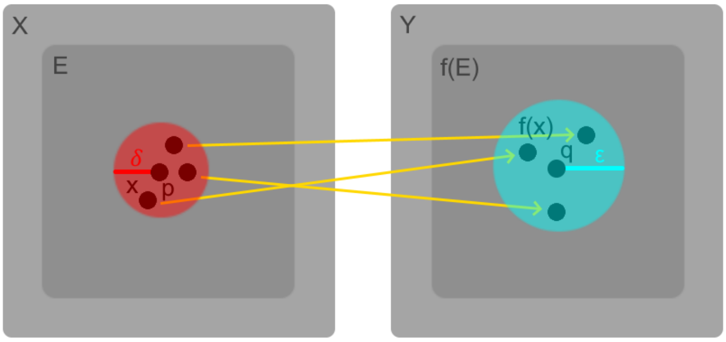
Let $\sum a'_n$ be a rearrangement with partial sums s'_n.
For $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N, $\sum_{i=n}^{m} |a_i| \leq \epsilon$.
Let p be the maximum index of $\{a_1, a_2, ..., a_N\}$ in a'_n and a_n.
Since if n > p, then $a_1, a_2, ..., a_N$ will cancel in $s_n - s'_n$ and thus, $|s_n - s'_n| \leq \epsilon$.
Thus, every $\{s'_n\}$ converges to $\{s_n\}$.

14 Continuity

14.1 Limits of Functions

Definition 14.1.1: Limits of functions

For metric spaces X,Y, let E $\subset$ X, f maps E into Y, and p $\in$ E'.
Then $\lim_{x \to p} f(x) = q$ if there is a q $\in$ Y such that:
For every $\epsilon > 0$, there is a $\delta > 0$ such that for all x $\in$ E
where $d_X(x,p) < \delta$, then $d_Y(f(x), q) < \epsilon$



Theorem 14.1.2: Sequence definition of $\lim_{x\to p} f(x) = q$

$\lim_{x\to p} f(x) = q$ if and only if $\lim_{n\to\infty} f(p_n) = q$ for every sequence $\{p_n\} \in$ E where $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.

Proof

Suppose $\lim_{x\to p} f(x) = q$.
For $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if x $\in$ E and $d_X(x,p) < \delta$.
Choose $\{p_n\} \in$ E such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.
Then for $\delta > 0$, there is N such that for n > N, then $d_X(p_n, p) < \delta$ so $d_Y(f(p_n), q) < \epsilon$.

Suppose $\lim_{x\to p} f(x) \neq q$. Then there is a $\epsilon > 0$ such that for every $\delta > 0$, there is a x $\in$ E where $d_Y(f(x), q) \geq \epsilon$, but $d_X(x,p) < \delta$. Let $\delta_n = \frac{1}{n}$ and thus, there is a $\{p_n\}$ where $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$, but $\lim_{n\to\infty} f(p_n) \neq q$.

Corollary 14.1.3: A limit of a function is unique

If f has a limit at p, this limit is unique.

Proof

If $\lim_{x\to p} f(x) = q$, then by theorem 14.1.2, $\lim_{n\to\infty} f(p_n) = q$ for every $\{p_n\} \in$ E where $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.
Thus, if there exists $\lim_{x\to p} f(x) = q'$, then there is a $\{p_n\} \in$ E where $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$, but $\lim_{n\to\infty} f(p_n) = q'$ which is a contradiction.

Theorem 14.1.4: Arithemtic operations on functions of limits

Let E $\subset$ X, p $\in$ E', and f(x),g(x) $\in \mathbb{C}$ so $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$.

(a) $\lim_{x\to p}(f+g)(x) = A + B$
(b) $\lim_{x\to p}(fg)(x) = AB$
(c) $\lim_{x\to p}(\frac{f}{g})(x) = \frac{A}{B}$

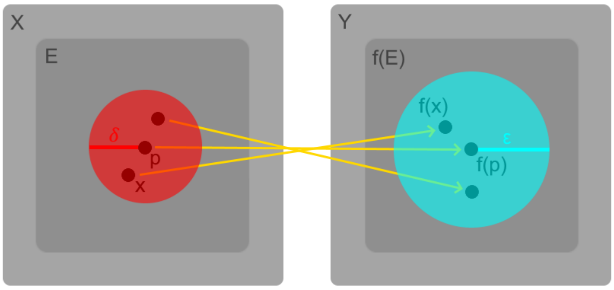
14.2 Continuous Functions

Definition 14.2.1: Continuous functions on a set

Suppose X,Y are metric spaces, E ⊂ X, p ∈ E, and f maps E into Y.
f is continuous at p if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all x ∈ E
where $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$

f(p) have to be defined to be continuous.
If f is continuous at every p ∈ E, then f is continuous on E.
f is continuous at isolated points since regardless of ϵ, there is a $\delta > 0$ such that $d_X(x, p) < \delta$ is x = p so $d_Y(f(x), f(p)) = 0 < \epsilon$.



Theorem 14.2.2: Continuity at p ⇌ lim f(p) = f(p)

Suppose E ⊂ X, p ∈ E, and f maps E into Y. Let p ∈ E'.
Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof

If f is continuous at p, then for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all x ∈ E where $d_X(x, p) < \delta$. Thus, $\lim_{x \to p} f(x) = f(p)$.

If $\lim_{x \to p} f(x) = f(p)$, then for every $\epsilon > 0$, there is a $\delta > 0$ where $d_Y(f(x), f(p)) < \epsilon$ for all x ∈ E where $d_X(x, p) < \delta$. Thus, f is continuous at p.

Theorem 14.2.3: Continuity Chain Rule

Suppose E ⊂ X, f: E → Y, g: f(E) → Z, and h: E → Z where h(x) = g(f(x)).
If f is continuous at p and g is continuous at f(p), then h is continuous at p.

Proof

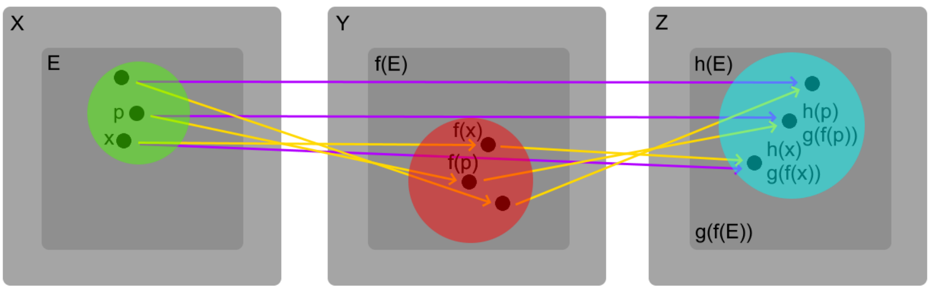
Since g is continuous at f(p), then for $\epsilon > 0$, there is a δ_1 such that:

$d_Z(g(y), g(f(p))) < \epsilon$ for $d_Y(y, f(p)) < \delta_1$ where y ∈ f(E)

Since f is continuous at p, there is a $\delta_2 > 0$ such that:

$d_Y(f(x), f(p)) < \delta_1$ for $d_X(x, p) < \delta_2$ where x ∈ E

Thus, $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$ for $d_X(x, p) < \delta_2$ where x ∈ E.
Thus, h is continuous at p.



Theorem 14.2.4: Continuous functions map open sets to open sets

f: X $\to$ Y is continuous on X if and only if:

$f^{-1}(V)$ is open in X for every open set V in Y.

Proof

Suppose f is continuous on X and V is an open set in Y.
Suppose p $\in$ X and f(p) $\in$ V. Since V is open, there exists $\epsilon > 0$ such that y $\in$ V if $d_Y(y, f(p)) < \epsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for $d_X(x,p) < \delta$. Thus, x $\in f^{-1}(V)$ for $d_X(x,p) < \delta$.

Suppose $f^{-1}(V)$ is open in X for every open V in Y.
Fix p $\in$ X and $\epsilon > 0$. Let V be the set of all y $\in$ Y such that $d_Y(y, f(p)) < \epsilon$ so V is open and thus, $f^{-1}(V)$ is open. Thus, there exists $\delta > 0$ such that x $\in f^{-1}(V)$ for $d_X(x,p) < \delta$. Since x $\in f^{-1}(V)$, then f(x) $\in$ V so $d_Y(f(x), f(p)) < \epsilon$.

Corollary 14.2.5: Continuous functions map closed sets to closed sets

f: X $\to$ Y is continuous on X if and only if:

$f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof

By theorem 14.2.4, f is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y. Let C = V^c. Since V is open, then C is closed.
Since $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$, then $f^{-1}(C)$ is closed since $f^{-1}(V)$ is open.

Theorem 14.2.6: Continuous functions

Let f,g be complex continuous functions on X.

Then f+g, fg, and $\frac{f}{g}$ where g $\neq$ 0 for all x $\in$ X are continuous on X.

Proof

If x is an isolated point, f+g, fg, and $\frac{f}{g}$ are continuous by definition. If x is a limit point, then by theorems 14.1.4 and 14.2.2, f+g, fg, and $\frac{f}{g}$ are continuous since

- $\lim_{x\to p}(f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} f(x) = f(p) + g(p)$
- $\lim_{x\to p}(fg)(x) = \lim_{x\to p} f(x) \lim_{x\to p} g(x) = f(p)g(p)$
- $\lim_{x\to p}(\frac{f}{g})(x) = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)} = \frac{f(p)}{g(p)}$

Theorem 14.2.7: Continuous functions on $\mathbb{R}^k$

(a) Let $f_1, ..., f_k$: X $\to \mathbb{R}$ and f: X $\to \mathbb{R}^k$ where f(x) = $(f_1(x), \dots, f_k(x))$.
Then f is continuous if and only if $f_1, ..., f_k$ are continuous.

(b) If f and g are continuous mappings of X into $\mathbb{R}^k$, then f + g and f $\cdot$ g are continuous on X.

Proof

Since $|f_i(x) - f_i(y)| \leq \sqrt{\sum_1^k |f_i(x) - f_i(y)|^2} = |f(x) - f(y)|$, then if f is continuous, then each f_i is continuous and vice versa.

Since f,g are continuous, then by part a, each f_i,g_i are continuous. Then by theorem 14.2.6, each f_i+g_i and $f_i g_i$ are continuous so by part a, f + g and f $\cdot$ g are continuous.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are $x_1 \cdot \ldots \cdot x_k$ where each x_i is continuous, rationals are polynomials divided by polynomials, and $||x| - |y|| \leq |x - y|$ implies $|x|$ is continuous.

15 Properties of Continuity

15.1 Continuity and Compactness

Definition 15.1.1: Bounded Functions

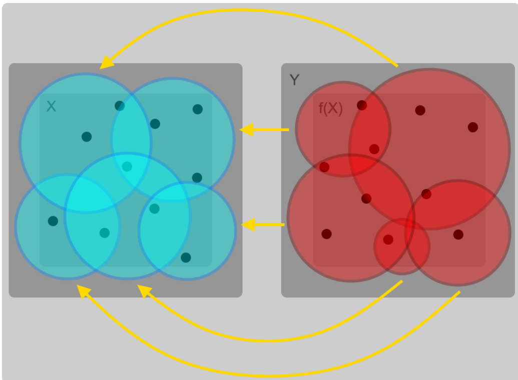
f: E $\to \mathbb{R}^k$ is bounded if there is a M $\in \mathbb{R}$ such that f(x) $\leq$ M for all x $\in$ E.

Theorem 15.1.2: Continuous functions from compact spaces are compact

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then f(X) is compact.

Proof

Let $\{V_\alpha\}$ be an open cover of f(X). Since f is continuous, then by theorem 14.2.4, each $f^{-1}(V_\alpha)$ is open.
Since X is compact, there is a n such that X $\subset f^{-1}(V_{\alpha_1}) \cup ... \cup f^{-1}(V_{\alpha_n})$.
Thus, f(X) $\subset V_{\alpha_1} \cup ... \cup V_{\alpha_n}$ so f(X) is compact.



Theorem 15.1.3: Continuous functions from compact to $\mathbb{R}^k$ are bounded

If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then f(X) is closed and bounded.

Proof

By theorem 15.1.2, f(X) is compact.
Then by theorem 8.3.13, f(X) is closed and bounded.

Theorem 15.1.4: Generalized extreme value theorem

Suppose f is a continuous real function of a compact metric space X such that M = $\sup_{x\in X}$ f(x) and m = $\inf_{x\in X}$ f(x).
Then there exists p,q $\in$ X such that f(p) = M and f(q) = m.

Proof

By theorem 15.1.3, f(X) is closed and bounded.
Let M = $\sup_{x\in X}$ f(x) and m = $\inf_{x\in X}$ f(x).
Since f(X) is bounded, then M,m $\in$ (f(X))' and since f(X) is closed, then M,m $\in$ f(X). Thus, there exists p,q $\in$ X such that f(p) = M and f(q) = m.

Theorem 15.1.5: If f is continuous 1-1, then f^{-1} is continuous

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then f^{-1} is a continuous mapping of Y onto X.

Proof

Let V be an open set in X.
Since V^c is closed and $V^c \subset$ compact set X, then by theorem 8.3.5, V^c is compact.
Thus by theorem 15.1.2, f(V^c) is a compact subset of Y so f(V^c) is closed.
Since f is 1-1 and onto, f(V^c) = $(f(V))^c$ so f(V) is open. Since from any open set V in X, f(V) is open in Y, then by theorem 14.2.4, f^{-1} is continuous.

Definition 15.1.6: Uniformly Continuous

Let f: X $\to$ Y. Then f is uniformly continuous on X if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all p,q $\in$ X where $d_X(p,q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

Theorem 15.1.7: Continuous functions from compact are uniformly continuous

Let f be a continuous mapping of a compact metric space X into metric space Y. Then f is uniformly continuous on X.

Proof

For $\epsilon > 0$, since f is continuous, then for each p $\in$ X, there is a $\phi(p)$ such that for all q $\in$ X where $d_X(q,p) < \phi(p)$, $d_Y(f(q), f(p)) < \frac{\epsilon}{2}$.
Let J(p) be the set of all q $\in$ X where $d_X(q,p) < \frac{1}{2}\phi(p)$.
Since the set of all J(p) is an open cover of X and since X is compact, then there is a n such that X $\subset$ J(p_1) $\cup$ ... $\cup$ J(p_n). Let $\delta = \frac{1}{2}$ min($\phi(p_1), ..., \phi(p_n)$) > 0.
Then for p,q $\in$ X where $d_X(p,q) < \delta$, there is a m where $1 \leq m \leq n$ such that p $\in$ J(p_m) so $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. Thus:

$$d_X(q, p_m) \leq d_X(q,p) + d_X(p, p_m) < \delta + \tfrac{1}{2}\phi(p_m) \leq \phi(p_m)$$
$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

Theorem 15.1.8: Continuous functions from noncompact $\not\to$ uniformly continuous

Let E be a noncompact set in $\mathbb{R}^1$.

(a) There exists a continuous function which is not bounded.

(b) There exists a continuous, bounded function which is has no maximum.

(c) If E is bounded, there exists a continuous function which is not uniformly continuous.

Proof

Suppose E is bounded so there is a $x_0 \in$ E', but $x_0 \notin$ E.
Consider f(x) = $\frac{1}{x - x_0}$ which is continuous on E, but unbounded.
For $\epsilon > 0$ and $\delta > 0$, there is a x $\in$ E such that $|x - x_0| < \delta$. Take t close enough to x_0 so $|f(t) - f(x_0)| > \epsilon$, but $|t - x| < \delta$. Thus, f is not uniformly continuous.

Consider g(x) = $\frac{1}{1+(x-x_0)^2}$ which is continuous on E and bounded since g(x) $\in$ (0,1).
Since $\sup_{x \in E}$ g(x) = 1, but g(x) < 1 for all x $\in$ E, then g has no maximum on E.

15.2 Continuity and Connectedness

Theorem 15.2.1: Continuous functions map connected to connected

If f is a continuous mapping of X into Y and E is a connected subset of X, then f(E) is connected.

Proof

Suppose f(E) = A $\cup$ B where A and B are nonempty separated subsets of Y.
Let G = E $\cap$ $f^{-1}(A)$ and H = E $\cap$ $f^{-1}(B)$. Then E = G $\cup$ H.
Since A $\subset$ $\overline{A}$, G $\subset$ $f^{-1}(\overline{A})$. Since f is continuous, then $f^{-1}(\overline{A})$ is closed so $\overline{G} \subset f^{-1}(\overline{A})$. Thus, f($\overline{G}$) $\subset$ $\overline{A}$.
Since f(H) = B and $\overline{A}$ $\cap$ B is empty, $\overline{G}$ $\cap$ H is empty. Similarily, G $\cap$ $\overline{H}$ is empty so G and H are separated which contradicts that E = G $\cup$ H is connected.

Theorem 15.2.2: Generalized Intermediate Value Theorem

Let f be a continuous real function on [a,b]. If f(a) $<$ c $<$ f(b), then there exists x $\in$ (a,b) such that f(x) = c.

Proof

Since [a,b] is connected, then by theorem 15.2.1, f([a,b]) is a connected subset of $\mathbb{R}^1$.
Thus, by theorem 9.2.2, any c where f(a) $<$ c $<$ f(b) is c $\in$ f(x) for some x $\in$ [a,b].

16 Discontinuities

16.1 Discontinuities

Definition 16.1.1: Right and left Limits

Let f be defined on (a,b).

Then for any x where x $\in$ [a,b), f(x+) = q if:
f(t_n) $\to$ q as n $\to \infty$ for all sequences $\{t_n\}$ in (x,b) such that $t_n \to$ x.

Then for any x where x $\in$ (a,b], f(x-) = q if:
f(t_n) $\to$ q as n $\to \infty$ for all sequences $\{t_n\}$ in (a,x) such that $t_n \to$ x.

Then $\lim_{t \to x}$ f(t) exists if and only if f(x-) = f(x+) = $\lim_{t \to x}$ f(t).

Definition 16.1.2: Types of discontinuities

Let f be defined on (a,b).
If f is discontinuous at x, but f(x+) and f(x-) exists, then f have a simple discontinuity of the first kind else it is a discontinuity of the second kind.
Thus, a simple discontinuity is either:

- f(x-) $\neq$ f(x+)
- f(x-) = f(x+) $\neq$ f(x)

16.2 Monotonic Functions

Definition 16.2.1: Monotonic

Let f be real on (a,b).
f is monotonically increasing if f(x) $\leq$ f(y) for $a < x < y < b$.
f is monotonically decreasing if f(x) $\geq$ f(y) for $a < x < y < b$.

Theorem 16.2.2: Right and left limits of monotonics on (a,b)

Let f be monotonically increasing on (a,b).
Then f(x+) and f(x-) exists at every x $\in$ (a,b) where:
$\sup_{t \in (a,x)}$ f(t) = f(x-) $\leq$ f(x) $\leq$ f(x+) = $\inf_{t \in (x,b)}$ f(t)
Furthermore, for $a < x < y < b$, f(x+) $\leq$ f(y-).
Properties analogous for monotonically decreasing functions.

Proof

Since f is monotonically increasing, then for t $\in$ (a,x), f(t) is bounded above by f(x) and thus, by the least upper bounded property, $\sup_{t \in (a,x)}$ f(t) exists.
For $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_{t \in (a,x)}$ f(t) - ϵ < f(x - δ) $\leq$ $\sup_{t \in (a,x)}$ f(t) for a < x - δ < x. Since f(x - δ) $\leq$ f(t) $\leq$ $\sup_{t \in (a,x)}$ f(t) for t $\in$ (x-δ,x), then $|f(t) - \sup_{t \in (a,x)} f(t)| < \epsilon$ for t $\in$ (x-δ,x) so f(x-) = $\sup_{t \in (a,x)}$ f(t).
For $\epsilon > 0$, there exists a $\delta > 0$ such that $\inf_{t \in (x,b)}$ f(t) < f(x + δ) $\leq$ $\inf_{t \in (x,b)}$ f(t) + ϵ for x < x + δ < b. Since f(x + δ) $\geq$ f(t) $\geq$ $\inf_{t \in (x,b)}$ f(t) for t $\in$ (x,x+δ), then $|f(t) - \inf_{t \in (x,b)} f(t)| < \epsilon$ for t $\in$ (x,x+δ) so f(x+) = $\inf_{t \in (x,b)}$ f(t).
Thus, $\sup_{t \in (a,x)}$ f(t) = f(x-) $\leq$ f(x) $\leq$ f(x+) = $\inf_{t \in (x,b)}$ f(t).

If $a < x < y < b$, then:
f(x+) = $\inf_{t \in (x,b)}$ f(t) = $\inf_{t \in (x,y)}$ f(t) $\leq$ $\sup_{t \in (x,y)}$ f(t) = $\sup_{t \in (a,y)}$ f(t) = f(y-)

Corollary 16.2.3: Monotonics can only have simple discontinuities

Monotonic functions have no discontinuities of the second kind.

Proof

By theorem 16.2.2, f(x-) and f(x+) exists and thus, f can only have simple discontinuities and not discontinuities of the second kind.

Theorem 16.2.4: Discontinuities of monotonics is at most countable

Let f be monotonic on (a,b).
Then the set of points of (a,b) where f is discontinuous is at most countable.

Proof

Suppose f is increasing. Let E be the set of points where f is discontinuous. Then for x $\in$ E, there is a rational r(x) where f(x-) < r(x) < f(x+).
Then for $x_1 < x_2$, by theorem 16.2.2, f(x_1+) $\leq$ f(x_2-). Then:

$$f(x_1-) < r(x_1) < f(x_1+) \leq f(x_2-) < r(x_2) < f(x_2+)$$

Thus, r(x_1) $\neq$ r(x_2) if $x_1 \neq x_2$.
Since there is a 1-1 correspondence between E and a subset of rational numbers which is countable, then E is at most countable.
If f is decreasing, proof is analogous.

16.3 Infinite Limits \ Limits at Infinity

Definition 16.3.1: Neighborhoods in extended reals

For any real c, a neighborhood of $+\infty$ = (c,$+\infty$).
For any real c, a neighborhood of $-\infty$ = ($-\infty$,c).

Definition 16.3.2: Infinite Limits

Let real function f be defined on E $\subset \mathbb{R}$.
Then f(t) $\to$ A as t $\to$ x where A and x are extended reals if:
For every neighborhood U of A, there is a neighborhood V of x such that V $\cap$ E $\neq \emptyset$ and f(t) $\in$ U for all t $\in$ V $\cap$ E where t $\neq$ x.

Theorem 16.3.2: Arithmetic operations on functions of infinite limits

Let f,g be defined on E $\subset \mathbb{R}$ where f(t) $\to$ A and g(t) $\to$ B as t $\to$ x.

(a) If f(t) $\to$ A', then A' = A.

(b) (f+g)(t) $\to$ A + B

(c) (fg)(t) $\to$ AB

(d) $\frac{f}{g}$(t) $\to \frac{A}{B}$

17 Differentiation

17.1 Derivative of a function

Definition 17.1.1: Derivative

Let f be defined on any x ∈ [a,b].

$\phi(t) = \frac{f(t)-f(x)}{t-x}$ for t ≠ x

The derivative of f at x:

$f'(x) = \lim_{t\to x} \phi(t)$

if the limit exist as defined by definition 14.1.1.

If f' is defined at x, then f is differentiable at x.

Theorem 17.1.2: Differentiability → Continuity

Let f be defined on [a,b].

If f is differentiable at x ∈ [a,b], then f is continuous at x.

Proof

As t → x:

f(t) - f(x) $= \frac{f(t)-f(x)}{t-x} \cdot (t-x) \to f'(x)\cdot 0 = 0$

Theorem 17.1.3: Arithmetic operations on differentiation

Suppose f,g are defined on [a,b] and differentiable on x ∈ [a,b]. Then f+g, fg, and $\frac{f}{g}$ are differentiable at x:

(a) (f+g)'(x) = f'(x) + g'(x)

$$\lim_{t\to x} \frac{(f+g)(t)-(f+g)(x)}{t-x} = \lim_{t\to x} \frac{f(t)-f(x)+g(t)-g(x)}{t-x}$$
$$= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x} = f'(x) + g'(x)$$

(b) (fg)'(x) = f'(x)g(x) + f(x)g'(x)

$$\lim_{t\to x} \frac{(fg)(t)-(fg)(x)}{t-x} = \lim_{t\to x} \frac{f(t)g(t)-f(x)g(x)}{t-x}$$
$$= \lim_{t\to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x}$$
$$= \lim_{t\to x} \frac{[f(t)-f(x)]g(t)}{t-x} + \lim_{t\to x} \frac{f(x)[g(t)-g(x)}{t-x}$$
$$= f'(x)g(x) + f(x)g'(x)$$

(c) $(\frac{f}{g})$'(x) $= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

$$\lim_{t\to x} \frac{(\frac{f}{g})(t)-(\frac{f}{g})(x)}{t-x} = \lim_{t\to x} \frac{\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}}{t-x} = \lim_{t\to x} \frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)(t-x)}$$
$$= \lim_{t\to x} \frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{g(t)g(x)(t-x)}$$
$$= \lim_{t\to x} \frac{[f(t)-f(x)]g(x)}{g(t)g(x)(t-x)} + \lim_{t\to x} \frac{f(x)[g(x)-g(t)]}{g(t)g(x)(t-x)}$$
$$= \frac{f'(x)g(x)}{g^2(x)} + \frac{f(x)[-g'(x)]}{g^2(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

Theorem 17.1.4: Differentiation Chain Rule

Suppose f is continuous on [a,b], f'(x) exists at x ∈ [a,b], g is defined on interval I containing f([a,b]), and g is differentiable at f(x).

If h(t) = g(f(t)), then h is differentiable at x and h'(x) = g'(f(x)) · f'(x)

Proof

Since f is differentiable at x and g is differentiable at f(x), then:

f(t) - f(x) = (t-x) [f'(x) + u(t)] for t ∈ [a,b] and $\lim_{t\to x}$ u(t) = 0

g(s) - g(f(x)) = (s-f(x)) [g'(f(x)) + v(s)] for s ∈ I and $\lim_{s\to f(x)}$ v(s) = 0

Thus:

$$\begin{aligned}\lim_{t\to x} \frac{h(t)-h(x)}{t-x} &= \lim_{t\to x} \frac{g(f(t))-g(f(x))}{t-x} \\ &= \lim_{t\to x} \frac{(f(t)-f(x))[g'(f(x))+v(f(t))]}{t-x} \\ &= \lim_{t\to x} \frac{(t-x)[f'(t)+u(t)][g'(f(x))+v(f(t))]}{t-x} \\ &= g'(f(x))\cdot f'(x) + f'(x)\cdot 0 + g'(f(x))\cdot 0 + 0\cdot 0 = g'(f(x)) \cdot f'(x)\end{aligned}$$

17.2 Mean Value Theorems

Definition 17.2.1: Local Extrema

Let real-valued f ∈ X.

Then f has a local maximum at p ∈ X if:

There is $\delta > 0$ such that for all q ∈ X where $d_X(q,p) < \delta$, f(q) ≤ f(p).

Then f has a local minimum at p ∈ X if:

There is $\delta > 0$ such that for all q ∈ X where $d_X(q,p) < \delta$, f(q) ≥ f(p).

Theorem 17.2.2: Derivative at local extrema is 0

Let f be defined on [a,b].

If f has a local maximum at x ∈ (a,b) and f'(x) exists, then f'(x) = 0.

If f has a local minimum at x ∈ (a,b) and f'(x) exists, then f'(x) = 0.

Proof

Suppose x is a local maximum.

Then there is a $\delta > 0$ such that for all t ∈ (a,b) where $|t - x| < \delta$, then f(t) ≤ f(x).

Then for $t < x$, $\frac{f(t)-f(x)}{t-x} \geq 0$. Thus, $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ = f'(x) ≥ 0.

For $t > x$, $\frac{f(t)-f(x)}{t-x} \leq 0$. Thus, $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ = f'(x) ≤ 0.

Since f'(x) exists, then f'(x) = 0.

Proof is analogous for local minimum.

Theorem 17.2.3: Generalized Mean Value Thereom

If f,g are continuous real functions on [a,b] and differentiable on (a,b), then there is a x ∈ (a,b) such that [f(b) - f(a)] · g'(x) = [g(b) - g(a)] · f'(x).

Proof

Let h(t) = [f(b) - f(a)] · g(t) - [g(b) - g(a)] · f(t) for t ∈ [a,b].

Since f,g are continuous on [a,b] and differentiable on (a,b), then h is continuous on [a,b] and differentiable on (a,b). Also, h(a) = f(b)g(a) - f(a)g(b) = h(b).

If h is constant, then h'(x) = 0 and thus, theorem holds true for every x ∈ (a,b).

If h(t) > h(a) for some t ∈ (a,b), let x ∈ [a,b] where h attains a local maximum. If h(t) < h(a) for some t ∈ (a,b), let x ∈ [a,b] where h attains a local minimum. Then by theorem 17.2.2, h'(x) = 0 and thus, theorem holds true at local extrema.

Theorem 17.2.4: Mean Value Thereom

If f is a real continuous function on [a,b] and differentiable on (a,b), then there is a x $\in$ (a,b) such that f(b) - f(a) = (b-a) f'(x).

Proof

From thereom 17.2.3, let g(x) = x.

Theorem 17.2.5: Sign of derivative determines increasing/decreasing

Suppose f is differentiable on (a,b).

(a) If f'(x) $\geq$ 0 for all x $\in$ (a,b), then f is monotonically increasing.

(b) If f'(x) = 0 for all x $\in$ (a,b), then f is constant.

(c) If f'(x) $\leq$ 0 for all x $\in$ (a,b), then f is monotonically decreasing

Proof

From theorem 17.2.4, f(x_2) - f(x_1) = $(x_2 - x_1)$ f'(x) for x $\in$ (x_1, x_2) $\subset$ (a,b).
If f'(x) $\geq$ 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) $\geq$ 0. Since f(x_2) $\geq$ f(x_1) for $x_2 > x_1$, then f is monotonically increasing.
If f'(x) = 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) = 0. Since f(x_2) = f(x_1) for $x_2 > x_1$, then f is constant.
If f'(x) $\leq$ 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) $\leq$ 0. Since f(x_2) $\leq$ f(x_1) for $x_2 > x_1$, then f is monotonically decreasing.

17.3 Continuity of Derivatives

Theorem 17.3.1: Intermediate values of derivatives exists

Suppose f is a real differentiable function on [a,b] and $f'(a) < \lambda < f'(b)$.
Then there is a x $\in$ (a,b) such that f'(x) = λ.
Statement holds true if $f'(a) > f'(b)$.

Proof

Suppose $f'(a) < \lambda < f'(b)$. Let g(t) = f(t) - λt.
Since f(t),t are differentiable on [a,b], then g(t) is differentiable on [a,b].
Then g'(a) = f'(a) - λ < 0 so g(t_1) < g(a) for some t_1 $\in$ (a,b).
Also, g'(b) = f'(b) - λ > 0 so g(t_2) < g(b) for some t_2 $\in$ (a,b).
Thus, there is a x where g(x) is a local minimum so g'(x) = 0 and thus, f'(x) = λ.

Corollary 17.3.2: Differentiable functions have no simple discontinuities

If f is differentiable on [a,b], then f' cannot have simple discontinuities on [a,b].

Proof

By theorem 17.3.1, f'(x) exists for any x $\in$ [a,b].

17.4 L'Hospital's Rule

Theorem 17.4.1: L'Hospital's Rule

Suppose f,g are real and differentiable in (a,b) and g'(x) $\neq$ 0 for all x $\in$ (a,b).
Suppose $\lim_{x\to a} \frac{f'(x)}{g'(x)} \to A$. If either:

- $\lim_{x\to a}$ f(x) $\to$ 0 and $\lim_{x\to a}$ g(x) $\to$ 0
- $\lim_{x\to a}$ g(x) $\to \infty$ or $\lim_{x\to a}$ g(x) $\to -\infty$

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} \to$ A.
Statement holds true if x $\to$ b.

Proof

Consider the case $-\infty \leq A < \infty$.
Choose q such that $A < q$ and r such that $A < r < q$.
Thus, there is a c $\in$ (a,b) such that $a < x < c$ for $\frac{f'(x)}{g'(x)} < r$.
For $a < x < y < c$, then by theorem 17.2.3, there is a t $\in$ (x,y) such that:

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$$

If $\lim_{x\to a}$ f(x) $\to$ 0 and $\lim_{x\to a}$ g(x) $\to$ 0, then as $x \to a$, $\frac{f(y)}{f(x)} \leq r < q$ for $y \in (a,c)$.
If $\lim_{x\to a}$ g(x) $\to \infty$, then keeping y fixed, choose $c_1 \in (a,y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Thus:

$$\frac{g(x)-g(y)}{g(x)} \cdot \frac{f(x)-f(y)}{g(x)-g(y)} < \frac{g(x)-g(y)}{g(x)} \cdot r \text{ for } x \in (a,c_1)$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

Thus as $x \to a$, there is a $c_2 \in (a,c_1)$ such that $\frac{f(x)}{g(x)} < r < q$ for $x \in (a,c_2)$.
Proof is analogous if $\lim_{x\to a}$ g(x) $\to -\infty$.
Thus, $\lim_{x\to a} \frac{f(x)}{g(x)} \to$ A.

17.5 Derivative of Higher Order

Definition 17.5.1: Derivative of Higher Order

If f has a derivative f' on an interval and f' is differentiable, then the derivative of f' is f", the second derivative of f. Then, $f^{(n)}$ is the nth derivative of f.
For $f^{(n)}(x)$ to exist at x, $f^{(n-1)}(t)$ must exist in a neighborhood of x and $f^{(n-1)}$ must be differentiable at x.
If $f^{(n-1)}$ exist in a neighborhood of x, then $f^{(n-2)}$ must be differentiable in that neighborhood and so on until f is differentiable on that neighborhood.

17.6 Taylor's Theorem

Theorem 17.6.1: Taylor's Theorem

Suppose f is a real function on [a,b], n $\in \mathbb{Z}_+$, $f^{(n-1)}$ is continuous on [a,b], $f^n(t)$ exists at every t $\in$ (a,b).

Let $\alpha, \beta \in$ [a,b] be distinct and P(t) = $\sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t-\alpha)^k$.

Then there exists a x between α and β such that f(β) = P(β) + $\frac{f^n(x)}{n!}(\beta-\alpha)^n$

Proof

Let M be the number defined by $f(\beta) = P(\beta) + M(\beta-\alpha)^n$.
Let $g(t) = f(t) - P(t) - M(t-\alpha)^n$ for $t \in [\alpha, \beta]$. Thus, $g^{(n)}(t) = f^{(n)}(t) - n!M$.
Also since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for k = [0,n-1], then $g(\alpha) = g'(\alpha) = ... = g^{(n-1)}(\alpha) = 0$.
Since the choice of M gives $g(\beta) = 0$, then by the Mean Value Theorem, $g'(x_1) = 0$ for some x_1 between α and β.
Since $g'(\alpha) = 0$, then $g''(x_2) = 0$ for some x_2 between α and x_1.
Thus, $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1} so x_n is between α and β.
Thus, there exists an $x_n \in (\alpha, \beta)$ such that:

$$0 = g^{(n)}(x_n) = f^{(n)}(x_n) - n!M$$
$$M = \frac{f^n(x_n)}{n!}$$

17.7 Differentiation of Vector-Valued Functions

Definition 17.7.1: Extending derivative to Vector-Valued Functions

For vector-valued function f: t $\in$ [a,b] $\to \mathbb{R}^k$, the derivative of f at x:

f'(x) = $\lim_{t \to x} |\frac{f(t)-f(x)}{t-x}|$

if the limit exist as defined by definition 14.1.1.

If f = $(f_1, ..., f_k)$, then f' = $(f_1', ..., f_k')$ and f is differentiable at x if and only if $f_1, ..., f_k$ are differentiable at x.

Thus, by theorem 14.2.7, these theorems hold true for vector-valued functions:

- 17.1.2: If f is differentiable at x, then f is continuous at x.
- 17.1.3a\b: If f,g are differentiable at x, then f+g,f·g are differentiable at x.

However, theorem 17.2.4: Mean Value Theorem and theorem 17.4.1: L'Hospital's Rule does not always hold true since theorem 17.1.3c, dividing vectors by vectors, is not defined for vector-valued functions.

Theorem 17.7.2: Mean Value Theorem for $\mathbb{R}^k$

Suppose f is a continuous mapping of [a,b] into $\mathbb{R}^k$ and f is differentiable on (a,b). Then there is a x $\in$ (a,b) such that $|f(b) - f(a)| \leq$ (b-a) $|f'(x)|$

Proof

Let z = f(b) - f(a) and define $\phi(t) = z \cdot f(t)$ for t $\in$ [a,b].
Then $\phi(t)$ is real-valued continuous on [a,b] and differentiable on (a,b).
Then by the Mean Value Theorem, for some x $\in$ (a,b):

$$\phi(b) - \phi(a) = \text{(b-a) } \phi'(x) = \text{(b-a) } z \cdot f'(x)$$

Since $\phi(b) - \phi(a) = z \cdot f(b) - z \cdot f(a) = z \cdot z = |z|^2$, then by the Schwarz Inequality:

$$|z|^2 = \text{(b-a) } |z \cdot f'(x)| \leq \text{(b-a) } |z||f'(x)|$$
$$|z| \leq \text{(b-a) } |f'(x)|$$
$$|f(b) - f(a)| \leq \text{(b-a) } |f'(x)|$$

References

[1] Walter Rudin, *Principles of Mathematical Analysis (3rd Edition)*, ISBN-13: 978-0070542358