

# Fall Real Analysis

Azure

Fall 2021

## Contents

<b>1</b>	<b>Day 1: The Real Number System</b>	<b>4</b>
1.1	Number Systems . . . . .	4
1.2	Real Number System . . . . .	4
1.3	Least Upper Bound Property . . . . .	5
<b>2</b>	<b>Day 2: Fields and Order</b>	<b>6</b>
2.1	Greatest Upper Bound Property . . . . .	6
2.2	Fields . . . . .	6
2.3	Ordered Fields . . . . .	7
<b>3</b>	<b>Day 3: Roots and the Complex Field</b>	<b>9</b>
3.1	$n$ th Root . . . . .	9
3.2	Decimals . . . . .	10
3.3	Extended Reals . . . . .	10
3.4	Complex Numbers . . . . .	10
<b>4</b>	<b>Day 4: Euclidean Spaces &amp; Cauchy-Schwarz</b>	<b>12</b>
4.1	Euclidean Spaces . . . . .	12
4.2	Cauchy-Schwarz . . . . .	12
<b>5</b>	<b>Day 5: Existence of <math>\mathbb{R}</math></b>	<b>14</b>
<b>6</b>	<b>Day 6: Cardinality</b>	<b>16</b>
6.1	Cardinality . . . . .	16
6.2	Set of Sets . . . . .	17
<b>7</b>	<b>Day 7: Metric Spaces and Closed/Open</b>	<b>20</b>
7.1	Metric Spaces . . . . .	20
7.2	Intervals and Balls . . . . .	24
<b>8</b>	<b>Day 8: Open Relative and Compact</b>	<b>25</b>
8.1	Closure . . . . .	25
8.2	Open Relative . . . . .	25
8.3	Compact Sets . . . . .	26
<b>9</b>	<b>Day 9: Perfect &amp; Connected Sets</b>	<b>30</b>
9.1	Perfect Sets . . . . .	30
9.2	Connected Sets . . . . .	31

<b>10 Day 10: Convergence &amp; Cauchy Sequences</b>	<b>32</b>
10.1 Convergent Sequences . . . . .	32
10.2 Subsequences . . . . .	34
10.3 Cauchy Sequences . . . . .	35
<b>11 Day 11: Limits &amp; Special Sequences</b>	<b>37</b>
11.1 Upper and Lower Limits . . . . .	37
11.2 Special Sequences . . . . .	38
<b>12 Day 12: Series &amp; Comparison Test</b>	<b>39</b>
12.1 Series . . . . .	39
12.2 Series of Nonnegative Terms . . . . .	40
12.3 The Number $e$ . . . . .	41
12.4 Root and Ratio Tests . . . . .	42
<b>13 Day 13: Power Series</b>	<b>43</b>
13.1 Power Series . . . . .	43
13.2 Summation By Parts . . . . .	43
13.3 Absolute Convergence . . . . .	44
13.4 Addition & Multiplication of Series . . . . .	44
13.5 Rearrangements . . . . .	45
<b>14 Day 14: Continuity</b>	<b>47</b>
14.1 Limits of Functions . . . . .	47
14.2 Continuous Functions . . . . .	47
<b>15 Day 15: Continuity Properties</b>	<b>50</b>

# 1 The Real Number System

## 1.1 Number Systems

Natural :  $\mathbb{N} = \{1, 2, 3, \dots\}$

Integer :  $\mathbb{Z} = \{-2, -1, 0, 1, 2, \dots\}$

Rational :  $\mathbb{Q} = \frac{p}{q}$  where  $p, q \in \mathbb{N}$

\*\*\*  $\mathbb{Q}$  is countable, but fails to have the least upper bound property \*\*\*

### Example 1.1.1

Let  $\alpha \in \mathbb{R}$  where  $\alpha^2 = 2$ . Then  $\alpha$  cannot be rational.

#### Proof

Let  $\alpha = \frac{p}{q}$  where  $p$  and  $q$  cannot both be even.

Let set  $A = \{x \in \mathbb{Q} \text{ for } x^2 < 2\}$  where  $A \neq \emptyset$  and 2 is an upper bound for  $A$ .

But,  $A$  has no least upper bound in  $\mathbb{Q}$ , but  $A$  has a least upper bound in  $\mathbb{R}$ .

## 1.2 Real Number System

$\mathbb{R}$  is the unique ordered field with the least upper bound property.

Also,  $\mathbb{R}$  exists and unique.

### Definition 1.2.1: Order

Let  $S$  be a set. An order on  $S$  is a relation  $<$  satisfying two axioms:

- **Trichotomy**: For all  $x, y \in S$ , only one holds true:
  - $x < y$
  - $x = y$
  - $x > y$
- **Transitivity**: If  $x < y$  and  $y < z$ , then  $x < z$ .

### Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

### Definition 1.2.3: Bounds

Let  $S$  be an ordered set and  $E \subset S$ .

An upper bound of  $E$  is a  $\beta \in S$  if  $x \leq \beta$  for all  $x \in E$ .

If such a  $\beta$  exists, then  $E$  is bounded from above.

A lower bound of  $E$  is a  $\alpha \in S$  if  $x \geq \alpha$  for all  $x \in E$ .

If such a  $\alpha$  exists, then  $E$  is bounded from below.

**Definition 1.2.4: Infimum & Supremum**

Let  $S$  be an ordered set.

Let  $E \subset S$  be bounded from above. Least upper bound  $\beta \in S$  exists if:

- $\beta$  is an upper bound for  $E$
- If  $\gamma < \beta$ , then  $\gamma$  is not an upper bound for  $E$ .  
Then  $\beta = \sup(E)$ .

Let  $E \subset S$  be bounded from below. Greatest lower bound  $\alpha \in S$  exists if:

- $\alpha$  is a lower bound for  $E$
- If  $\gamma > \alpha$ , then  $\gamma$  is not a lower bound for  $E$ .  
Then  $\alpha = \inf(E)$ .

**Example 1.2.5**

Let  $S = (1, 2) \cup [3, 4) \cup (5, 6)$  with the order  $<$  from  $\mathbb{R}$ . For subsets  $E$  of  $S$ :

- $E = (1, 2)$  is bounded above and  $\sup(E) = 2$
- $E = (5, 6)$  is not bounded above so  $\sup(E) = \text{DNE}$
- $E = [3, 4)$  is bounded below  $\inf(E) = 3$  and  $\sup(E) = \text{DNE}$

**Observations on the Least Upper Bound**

If  $\sup(E)$  exists, it may or may not exist at  $S$ .

If  $\sup(E)$  exists, then  $\sup(E)$  is unique. If  $\gamma \neq \alpha$ , then  $\gamma < \alpha$  or  $\gamma > \alpha$ .

**1.3 Least Upper Bound Property****Theorem 1.3.1: Least Upper Bound Property**

An ordered set  $S$  has a least upper bound property if:

For every nonempty subset  $E \subset S$  that is bounded from above:  
 $\sup(E)$  exists in  $S$ .

**Example 1.3.2**

$\mathbb{Q}$  doesn't have a least upper bound property. For example,  $z = \sqrt{2}$ .

**Proof**

Let  $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$ , then take  $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$ .

Let set  $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$  and set  $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If  $y^2 - 2 < 0$ , then  $z > y$  where  $z \in A$ . So,  $y$  is not an upper bound.  
Since for any  $y$ , there is  $z > y$  where  $z \in A$ , then  $\sup(A)$  doesn't exist in  $\mathbb{Q}$ .
- If  $y^2 - 2 > 0$ , then  $z < y$  where  $z \in B$ . So,  $y$  is an upper bound, but not  $\sup(E)$ .  
Since for any  $y$ , there is  $z < y$  where  $z \in B$ , then  $\inf(B)$  doesn't exist in  $\mathbb{Q}$ .

Thus,  $\mathbb{Q}$  doesn't have the least upper bound or greatest lower bound property.

## 2 Day 2: Fields

### 2.1 Greatest Upper Bound Property

**Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound**

Let  $S$  be an ordered set with the least upper bound property.

Let non-empty  $B \subset S$  be bounded below.

Let  $L$  be the set of all lower bounds of  $B$ .

Then  $\alpha = \sup(L)$  exists in  $S$ .

**Proof**

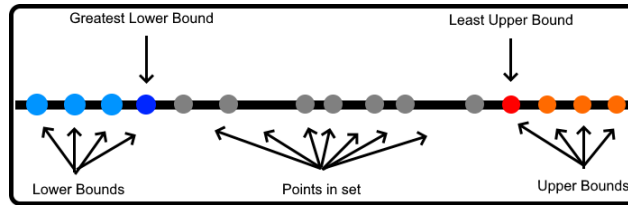
$L$  is non-empty since  $B$  is bounded from below.

Thus, by the least upper bound property of  $S$ ,  $\alpha = \sup(L)$  exists in  $S$ .

We claim that  $\alpha = \inf(B)$ .

If  $\gamma < \alpha$ , then  $\gamma$  is not an upper bound for  $L$  so  $\gamma \notin B$  since all upper bounds for  $L$  are in  $B$ . Thus, for every  $x \in B$ ,  $\alpha \leq x$ .

If  $\gamma \geq \alpha$ , then  $\gamma$  is an upper bound of  $L$  so  $\gamma \in B$ . Thus,  $\inf(B) = \alpha$ .



### 2.2 Fields

Addition Axioms

- If  $x, y \in F$ , then  $x+y \in F$
- $x+y = y+x$  for all  $x, y \in F$
- $(x+y)+z = x+(y+z)$  for all  $x, y, z \in F$
- There exists  $0 \in F$  such that  $0+x = x$  for all  $x \in F$
- For every  $x \in F$ , there is  $-x \in F$  where  $x+(-x) = 0$

Multiplicative Axioms

- If  $x, y \in F$ , then  $xy \in F$
- $yx = xy$  for all  $x, y \in F$
- $(xy)z = x(yz)$  for all  $x, y, z \in F$
- There exists  $1 \neq 0 \in F$  such that  $1x = x$  for all  $x \in F$
- If  $x \neq 0 \in F$ , there is  $\frac{1}{x} \in F$  where  $x(\frac{1}{x}) = 1$

Distributive Law

$$x(y+z) = xy + xz \text{ hold for all } x, y, z \in F.$$

**Propositions 2.2.1**

- (a) If  $x+y = x+z$ , then  $y = z$

**Proof**

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If
- $x+y = x$
- , then
- $y = 0$

ProofFrom (a), let  $z = 0$ .

- (c) If
- $x+y = 0$
- , then
- $y = -x$

ProofFrom (a), let  $z = -x$ .

- (d)
- $-(-x) = x$

ProofFrom (c), let  $x = -x$  and  $y = x$ .

- (e) If
- $x \neq 0$
- and
- $xy = xz$
- , then
- $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If
- $x \neq 0$
- and
- $xy = x$
- , then
- $y = 1$

ProofFrom (e), let  $z = 1$ .

- (g) If
- $x \neq 0$
- and
- $xy = 1$
- , then
- $y = \frac{1}{x}$

ProofFrom (e), let  $z = \frac{1}{x}$ .

- (h) If
- $x \neq 0$
- , then
- $\frac{1}{1/x} = x$

ProofFrom (g), let  $x = \frac{1}{x}$  and  $y = x$ .

- (i)
- $0x = 0$

ProofSince  $0x + 0x = (0+0)x = 0x = 0x + 0$ , then  $0x = 0$ .

- (j) If
- $x, y \neq 0$
- , then
- $xy \neq 0$

Proof

Suppose  $xy = 0$ , then  $1 = \frac{1}{y}\frac{1}{x}xy = \frac{1}{y}\frac{1}{x}0 = 0$ .  
 $0 = 1$  is a contradiction.

- (k)
- $(-x)y = -(xy) = x(-y)$

Proof

$xy + (-x)y = (x+(-x))y = 0y = 0$ . Then by part (c),  $(-x)y = -(xy)$ .  
 $xy + x(-y) = x(y+(-y)) = x0 = 0$ . Then by part (c),  $x(-y) = -(xy)$ .

- (l)
- $(-x)(-y) = xy$

Proof

By part (k), then  $(-x)(-y) = -[x(-y)] = -[-(xy)]$ .  
 By part (d),  $-[-(xy)] = xy$ .

## 2.3 Ordered Fields

An ordered field  $F$  is a field  $F$  which is also an ordered set for all  $x, y, z \in F$ .

- If  $y < z$ , then  $y+x < z+x$
- If  $x, y > 0$ , then  $xy > 0$

**Definition 2.3.1:**  $\mathbb{Q}$  and  $\mathbb{R}$  are ordered fields

$\mathbb{Q}$ ,  $\mathbb{R}$  are ordered fields, but  $\mathbb{C}$  is not an ordered field since  $i^2 = -1 \not> 1$ .

**Propositions 2.3.2**

Let  $F$  be an ordered field. For all  $x, y, z \in F$ .

- (a) If  $x > 0$ , then  $-x < 0$  and vice versa

**Proof**

$$-x = -x + 0 < -x + x = 0$$

- (b) If  $x > 0$  and  $y < z$ , then  $xy < xz$

**Proof**

$$\text{Since } z - y > 0, \text{ then } 0 < x(z - y) = xz - xy$$

- (c) If  $x < 0$  and  $y < z$ , then  $xy > xz$

**Proof**

$$\text{Since } -x > 0 \text{ and } z - y > 0, \text{ then } 0 < -x(z - y) = xy - xz$$

- (d) If  $x \neq 0$ ,  $x^2 > 0$

**Proof**

$$\text{If } x > 0, \text{ then } x^2 = x \cdot x > 0$$

$$\text{If } x < 0, \text{ then } (-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 > 0$$

- (e) If  $0 < x < y$ , then  $0 < 1/y < 1/x$

**Proof**

$$\text{Since } (\frac{1}{y})y = 1 > 0, \text{ then } (\frac{1}{y}) > 0$$

$$\text{Since } x < y, \text{ then } \frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$$

**Theorem 2.3.3:**  $\mathbb{R}$  is an ordered field with  $<$ 

There exists a unique ordered field  $\mathbb{R}$  with the least upper bound property.

Also,  $\mathbb{Q} \subset \mathbb{R}$  so  $\mathbb{Q}$  is also an ordered field.

**Theorem 2.3.4**

For all  $x, y \in \mathbb{R}$ :

- **Archimedean Property:** If  $x > 0$ , there is  $n \in \mathbb{Z}$  such that  $nx > y$ .

**Proof**

Fix  $x > 0$ . Suppose there is a  $y$  such that the property fails.

Let  $A = \{ nx : n = 1, 2, 3, \dots \}$ .

Then,  $A$  is nonempty and bounded from above by  $y$ .

Then by the least upper bound property of  $\mathbb{R}$ ,  $\alpha = \sup(A)$  exists in  $\mathbb{R}$ .

Since  $x > 0$ , then  $-x < 0$  so  $\alpha - x < \alpha - 0 = \alpha$ .

So  $\alpha - x$  is not an upper bound of  $A$ .

So there is a  $mx \in A$  such that  $mx > \alpha - x$ .

Then  $\alpha < (m+1)x$ , but  $(m+1)x \in A$  contradicting  $\alpha$  is an upper bound for  $A$ .

- **$\mathbb{Q}$  is dense in  $\mathbb{R}$ :** If  $x < y$ , there is a  $p \in \mathbb{Q}$  such that  $x < p < y$ .

**Proof**

Since  $x < y$ , then  $y - x > 0$ . Then by the Archimedean Property, there exists a  $n \in \mathbb{Z}$  such that  $n(y - x) > 1$ . Thus,  $ny > nx + 1 > nx$

By the well-ordering principle, there is a smallest  $m \in \mathbb{Z}_+$  such that  $m > nx$ .

Then,  $m > nx \geq m - 1$  so  $nx + 1 \geq m > nx$ .

Since  $ny > nx + 1 \geq m > nx$ , then  $y > m/n > x$ .



### 3 Roots & Complex Field

#### 3.1 nth Root

- (a) If  $0 < t \leq 1$ , then  $t^n \leq t$ .

**Proof**

Since  $t > 0$  and  $t \leq 1$ , then  $t^2 \leq t$ .  
 Since  $t^2 \leq t$ , then  $t^3 \leq t^2$  so  $t^3 \leq t^2 \leq t$ .  
 Applying the process  $n$  times, then  $t^n \leq t$ .

- (b) If  $t \geq 1$ ,  $t^n \geq t$ .

**Proof**

Since  $0 < 1 \leq t$ , then  $t \leq t^2$ .  
 Since  $t \leq t^2$ , then  $t^2 \leq t^3$  so  $t \leq t^2 \leq t^3$ .  
 Applying the process  $n$  times,  $t \leq t^n$ .

- (c) If  $0 < s < t$ , then  $s^n < t^n$ .

**Proof**

$$\underbrace{s \cdot s \cdot \dots \cdot s}_n < t \cdot s \cdot \dots \cdot s < t \cdot t \cdot \dots \cdot s < \dots < \underbrace{t \cdot \dots \cdot t}_n$$

**Theorem 3.1.1:**  $y^n = x$  has a unique  $y$

Fix  $n \in \mathbb{Z}_+$ . For every  $x > 0$ , there exists a unique  $y \in \mathbb{R}$  such that  $y^n = x$ .

Also, such a  $y$  is written as  $y = \sqrt[n]{x} = x^{\frac{1}{n}}$ .

**Proof**

**Uniqueness:**

$y$  is unique since if  $y_1 < y_2$ , then  $x = y_1^n < y_2^n \neq x$ .

**Existence:**

Let set  $A = \{ t > 0 : t^n < x \}$ .

$A \neq \emptyset$  since let  $t_1 = \frac{x}{x+1} < 1$  so  $t_1 < x$  and thus,  $0 < t_1^n < t_1 < x$  so  $t_1 \in A$ .

$A$  is bounded above since if  $t \geq x+1$ , then  $t > 1$  so  $t^n \geq t \geq x+1 > x$  so  $t \notin A$ .

So  $x+1$  is an upper bound of  $A$ .

Thus by the least upper bound property,  $y = \sup(A)$  exists.

For  $y^n = x$ , show  $y^n < x$  and  $y^n > x$  cannot hold true.

\*\*\* (Not an upper bound of  $A$  if  $<$  and not a least upper bound of  $A$  if  $>$ ) \*\*\*

For  $0 < \alpha < \beta$ :

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\beta^{n-1} + \beta^{n-2}\alpha^1 + \dots + \alpha^{n-1})}_{\substack{\beta^{n-1} < \beta^{n-1} < \beta^{n-1}}} < (\beta - \alpha)n\beta^{n-1}$$

Suppose  $y^n < x$ . Pick  $0 < h < 1$  and  $h < \frac{x - y^n}{n(y+1)^{n-1}}$ .

From inequality, let  $\beta = y+h$  and  $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus,  $(y+h)^n < x$  so  $y+h \in A$  and thus, not an upper bound of  $A$  which is a contradiction since  $y = \sup(A)$ .

Suppose  $y^n > x$ . Pick  $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$ .

Consider  $t \geq y-k$ , then:  $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus,  $t^n > x$  so  $t \notin A$ .

Thus,  $y-k$  is an upper bound of  $A$  which is a contradiction since  $y = \sup(A)$ .

Since  $y^n < x$  and  $y^n > x$ , then  $y^n = x$ .

**Corollary 3.1.2: n-th root of product = product of n-th root**

If  $a, b > 0$  and  $n \in \mathbb{Z}_+$ , then  $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$ .

**Proof**

Let  $A = a^{\frac{1}{n}}$  and  $B = b^{\frac{1}{n}}$ .

Then by **theorem 3.1.1**, since  $A$  is a solution to  $y_1^n = a$ , then  $A^n = a$ .

Similarly,  $B$  is a solution of  $y_2^n = b$  so  $B^n = b$ . Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 \dots A_n B_1 B_2 \dots B_n \\ &= A_1 A_2 \dots B_1 A_n B_2 \dots B_n = \dots = A_1 B_1 A_2 \dots A_{n-1} A_n B_3 \dots B_n \\ &= \dots = A_1 B_1 A_2 B_2 \dots A_n B_n = (AB)^n \end{aligned}$$

Then again by **theorem 3.1.1**, there is a unique  $(ab)^{\frac{1}{n}} = AB = a^{\frac{1}{n}} b^{\frac{1}{n}}$ .

**3.2 Decimals**

Let  $n_0$  be the largest integer such that  $n_0 \leq x$  for  $x > 0 \in \mathbb{R}$ .

Then let  $n_k$  be the largest integer such that  $d_k = n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x$

Let  $E$  be the set of  $d_k$  for  $k = 0, 1, \dots, \infty$ . Then,  $x = \sup(E)$ .

**3.3 Extended Reals**

The extended real number system consist of  $\mathbb{R}$  and  $\pm\infty$  such that:

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}$$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If  $x > 0$ , then  $x(\pm\infty) = \pm\infty$
- If  $x < 0$ , then  $x(\pm\infty) = \mp\infty$

**3.4 Complex Numbers****Definition 3.4.1: Complex**

A complex number is an ordered pair  $(a, b)$  where  $a, b \in \mathbb{R}$ . For  $x, y \in \mathbb{C}$

- $x + y = (a, b) + (c, d) = (a + c, b + d)$
- $xy = (a, b)(c, d) = (ac - bd, ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)^{-1}(a, -b)$

Thus, the axioms form a field where  $(0, 0) = 0$  and  $(1, 0) = 1$  and  $(0, 1) = i$ .

**Definition 3.4.2: Imaginary i**

Let  $i = (0, 1)$ . Then,  $i^2 = -1$ .

**Proof**

$$i^2 = (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1$$

**Definition 3.4.3: Form  $a + bi$** 

$$(a, b) = a + bi$$

**Proof**

$$(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi$$

**Definition 3.4.4: Conjugate**

Let conjugate:  $\bar{z} = a - bi$  where  $\text{Re}(z) = a$ ,  $\text{Im}(z) = b$ .

Let  $z = (a,b)$  and  $w = (c,d)$ :

(a)  $\overline{z+w} = \bar{z} + \bar{w}$

**Proof**

$$\overline{z+w} = \overline{(a+c, b+d)} = (a+c, -b-d) = (a, -b) + (c, -d) = \bar{z} + \bar{w}$$

(b)  $\overline{zw} = \bar{z} \bar{w}$

**Proof**

$$\overline{zw} = \overline{(ac-bd, ad+bc)} = (ac-bd, -ad-bc) = (a, -b)(c, -d) = \bar{z} \bar{w}$$

(c)  $z + \bar{z} = 2 \text{Re}(z)$        $z - \bar{z} = 2i \text{Im}(z)$

**Proof**

$$\begin{aligned} z + \bar{z} &= (a,b) + (a,-b) = (2a, 0) = 2 \text{Re}(z) \\ z - \bar{z} &= (a,b) - (a,-b) = (0, 2b) = (0, 2) b = 2i \text{Im}(z) \end{aligned}$$

(d)  $z\bar{z} \geq 0$

**Proof**

$$z\bar{z} = (a,b)(a,-b) = (a^2 + b^2, -ab+ab) = a^2 + b^2 \geq 0$$

**Definition 3.4.5: Absolute Value**

Let absolute value:  $|z| = \sqrt{z\bar{z}}$

Let  $z = (a,b)$  and  $w = (c,d)$ :

(a) If  $z \neq 0$ , then  $|z| > 0$ .

**Proof**

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0 \text{ where } |z| = 0 \text{ only if } a, b = 0 \text{ so only if } z = (0,0).$$

(b)  $|\bar{z}| = |z|$

**Proof**

$$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

(c)  $|zw| = |z| |w|$

**Proof**

$$\begin{aligned} |zw| &= |(ac-bd, ad+bc)| = \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z| |w| \end{aligned}$$

(d)  $|\text{Re}(z)| \leq |z|$

**Proof**

$$|\text{Re}(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

(e)  $|z+w| \leq |z| + |w|$

**Proof**

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + |w|^2 + 2 \text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

## 4 Euclidean Spaces & Cauchy-Schwarz

### 4.1 Euclidean Spaces

For each positive integer  $k$ , let  $\mathbb{R}^k$  be the set of all ordered  $k$ -tuples:

$$x = (x_1, \dots, x_k) \quad \text{for each } x_i \in \mathbb{R}$$

with the properties:

- $x+y = (x_1 + y_1, \dots, x_k + y_k) \in \mathbb{R}^k$
- $cx = (cx_1, \dots, cx_k) \in \mathbb{R}^k$

So,  $\mathbb{R}^n$  has a vector space structure. Similarly, for  $\mathbb{C}^n$ .

**Definition 4.1.1: Inner Product for  $\mathbb{R}^k$**

$$x \cdot y = x_1 y_1 + \dots + x_k y_k \in \mathbb{R}$$

**Definition 4.1.2: Norm**

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$$

**Definition 4.1.3: Extension to  $\mathbb{C}^k$**

For  $z, w \in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + \dots + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + \dots + z_k \overline{z_k} = |z_1|^2 + \dots + |z_n|^2 = |z|^2$

### 4.2 Cauchy-Schwarz

**Theorem 4.2.1: Cauchy-Schwarz**

If  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  and  $b_1, \dots, b_n \in \mathbb{C}$ , then:

$$|\sum_{j=1}^n \alpha_j (b_j)|^2 \leq \sum_{j=1}^n |\alpha_j|^2 \sum_{j=1}^n |b_j|^2$$

**Proof**

Let  $A = \sum |a_j|^2$  and  $B = \sum |b_j|^2$  and  $C = \sum a_j \overline{b_j}$ .

If  $B = 0$ , then  $b_1 = \dots = b_n = 0$ . Thus,  $0 \leq A(0)$  holds true.

Suppose  $B > 0$ . Then:

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) \\ &= \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B \overline{C} a_j \overline{b_j} - B C \overline{a_j} b_j + C \overline{C} b_j \overline{b_j} \\ &= B^2 \sum |a_j|^2 - B \overline{C} \sum a_j \overline{b_j} - B C \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B \overline{C} C - B C \overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

Since  $|Ba_j - Cb_j| \geq 0$ , then  $B(AB - |C|^2) \geq 0$ .

Since  $B > 0$ , then  $AB - |C|^2 \geq 0$  so  $AB \geq |C|^2$ .

**Definition 4.2.2: Consequence of the Cauchy-Schwarz**

Since  $|z_i|^2 = z_i \overline{z_i}$ , then  $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$ . Thus:

$$|z \cdot w|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2 |w|^2$$

Thus,  $|z \cdot w| \leq |z| |w|$ .

## Propositions 4.2.3

Let  $x, y, z \in \mathbb{R}^k$  where  $\alpha \in \mathbb{R}$ :

- (a)  $|x| \geq 0$  where  $|x| = 0$  only if  $x = 0$

Proof

$$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = \dots = x_k = 0$$

- (b)  $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2} \sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$$

- (c)  $|x + y| \leq |x| + |y|$

Proof

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

- (d)  $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

## 5 Construction of $\mathbb{R}$ : **Theorem 2.3.3**

There exists an ordered field  $\mathbb{R}$  which has the least upper bound property.  
Also,  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield.

### Definition 5.1: Cuts

Define a cut as any set  $\alpha \subset \mathbb{Q}$  with the properties:

- $\alpha$  is not empty and  $\alpha \neq \mathbb{Q}$
- If  $p \in \alpha$  and  $q \in \mathbb{Q} < p$ , then  $q \in \alpha$
- If  $p \in \alpha$ , then  $p < r \in \mathbb{Q}$  for some  $r \in \alpha$

### Proposition 5.2: Order of $\mathbb{R} \rightarrow$ ordered set $\mathbb{R}$

Define  $\alpha < \beta$  if  $\alpha$  is a proper subset of  $\beta$ .

- If  $\alpha \not\subseteq \beta$ , then  $\beta$  is not a subset of  $\alpha$ .  
Then there is a  $p \in \beta$  such that  $p \notin \alpha$ .  
Then for any  $q \in \alpha$ ,  $q < p$  and thus,  $q \in \beta$ .  
Thus,  $\alpha \subset \beta$  and since  $\alpha \neq \beta$ , then  $\alpha < \beta$ .
- If  $\alpha \not\subseteq \beta$  and  $\alpha \not\supseteq \beta$ , then either  $\alpha = \beta$  or  $\alpha \neq \beta$ .  
If  $\alpha \neq \beta$ , there are  $p, q$  such that  $p \in \alpha$ , but  $p \notin \beta$  and  $q \in \beta$ , but  $q \notin \alpha$ .  
But if  $p \notin \beta$ , then for any  $b \in \beta$ ,  $b < p$ . Thus,  $q < p$ .  
Similarly, if  $q \notin \alpha$ , then for any  $a \in \alpha$ ,  $a < q$ . Thus,  $p < q$ .  
Thus, there is a contradiction since  $p > q$  and  $p < q$  so  $\alpha = \beta$ .
- If  $\alpha \not\subseteq \beta$ , then  $\alpha$  is not a subset of  $\beta$ .  
Then there is a  $p \in \alpha$  such that  $p \notin \beta$ .  
Then for any  $q \in \beta$ ,  $q < p$  and thus,  $q \in \alpha$ .  
Thus,  $\beta \subset \alpha$  and since  $\alpha \neq \beta$ , then  $\beta < \alpha$ .
- If  $\alpha < \beta$  and  $\beta < \gamma$ , then since  $\alpha$  is a proper subset of  $\beta$  and  $\beta$  is a proper subset of  $\gamma$ , then  $\alpha$  is a proper subset of  $\gamma$ . Thus,  $\alpha < \gamma$ .

Thus,  $\mathbb{R}$  is an ordered set with such an order  $<$ .

### Proposition 5.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let  $A \subset \mathbb{R}$  and  $\beta$  be an upper bound for  $A$ . Let  $\gamma$  be the union of all  $\alpha \in A$ .  
Thus,  $p \in \gamma$  if and only if  $p \in \alpha$  for some  $\alpha \in A$ .

$\gamma$  defines a cut since:

- Since  $A$  is nonempty, there exists a  $\alpha_0 \in A$  where  $\alpha_0$  is nonempty.  
Since  $\alpha_0$  is nonempty, then  $\gamma$  is nonempty.  
Since every  $\alpha \in A$  is  $\alpha < \beta$ , then  $\gamma < \beta$  so  $\gamma \subset \beta$  and thus,  $\gamma \neq \mathbb{Q}$ .
- If  $p \in \gamma$ , then  $p \in \alpha_1$  for some  $\alpha_1 \in A$ . If  $q < p$ , then  $q \in \alpha_1$  so  $q \in A$ .
- If  $p \in \gamma$ , then  $p \in \alpha_1$  for some  $\alpha_1 \in A$ . Thus, there is a  $r \in \alpha_1$  such that  $r > p$  so  $r \in \gamma$ . Thus, there is a  $r \in \gamma$  where  $r > p$ .

Since  $\gamma$  defines a cut, then  $\gamma \in \mathbb{R}$ . Since every  $\alpha \in A \subset \gamma$ , then  $\alpha \leq \gamma$  so  $\gamma$  is an upper bound for  $A$ .

Suppose  $\delta < \gamma$ . Then there is a  $s \in \gamma$  such that  $s \notin \delta$ . Since  $s \in \gamma$ , then there is a  $\alpha \in A$  such that  $s \in \alpha$ . Since  $\delta < \alpha$ , then  $\delta$  is not an upper bound of  $A$ .

Thus,  $\gamma = \sup(A)$ .

**Proposition 5.4:  $\mathbb{R}$  is a field**

If  $\alpha, \beta \in \mathbb{R}$ , define  $\alpha + \beta$  as the set of all sums  $r + s$  where  $r \in \alpha$  and  $s \in \beta$ . Also, let  $0^*$  be the set of all negative rational numbers which is a cut since:

- $0^*$  is nonempty and  $0^* \neq \mathbb{Q}$
- If  $p \in 0^*$ , then any  $q \in \mathbb{Q} < p$  is a negative rational and thus,  $q \in 0^*$ .
- Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , then for any  $p \in 0^*$ , there is a  $r \in \mathbb{Q}$  where  $p < r < 0$  so  $r$  is a negative rational so  $r \in 0^*$ .

$\alpha + \beta \in \mathbb{R}$  since  $\alpha + \beta$  is a cut:

- $\alpha + \beta$  is non-empty since  $\alpha, \beta$  are non-empty. Take  $r' \notin \alpha, s' \notin \beta$ , then  $r' + s' > r + s$  for  $r \in \alpha, s \in \beta$ . Thus,  $r' + s' \notin \alpha + \beta$  so  $\alpha + \beta \neq \mathbb{Q}$ .
- If  $p \in \alpha + \beta$ , then  $p = r + s$  where  $r \in \alpha$  and  $s \in \beta$ .  
If  $q < p$ , then  $q - s < p - s = (r + s) - s = r$  so  $q - s \in \alpha$ .  
Since  $q - s \in \alpha$  and  $s \in \beta$ , then  $(q - s) + s = q \in \alpha + \beta$ .
- If  $r \in \alpha$ , then there is a  $t \in \alpha$  such that  $t > r$ . Let  $s \in \beta$ .  
Thus, for any  $p = r + s \in \alpha + \beta$ , there is a  $q = t + s \in \alpha + \beta$  such that  $p = r + s < t + s = q$ .

$\alpha + \beta = \beta + \alpha$

If  $p = r + s \in \alpha + \beta$  where  $r \in \alpha, s \in \beta$ , then  $s + r = r + s = p \in \beta + \alpha$ .

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If  $r \in \alpha, s \in \beta, t \in \gamma$ , then  $r + s + t = (r + s) + t \in (\alpha + \beta) + \gamma$  and  $r + s + t = r + (s + t) \in \alpha + (\beta + \gamma)$ .

$\alpha + 0^* = \alpha$

If  $r \in \alpha, s \in 0^*$ , then  $r + s < r$ . Thus,  $r + s \in \alpha$ . Thus,  $\alpha + 0^* \subset \alpha$ .

If  $p \in \alpha$ , there is a  $r \in \alpha$  where  $r > p$ . Thus,  $p - r \in 0^*$ .

Since  $p = r + (p - r) \in \alpha + 0^*$ , then  $\alpha \subset \alpha + 0^*$ . Thus,  $\alpha + 0^* = \alpha$ .

There is a  $-\alpha$  such that  $\alpha + -\alpha = 0^*$

Fix  $\alpha \in \mathbb{R}$ . Let set  $\beta$  be all  $p$  where there is  $r > 0$  such that  $-p - r \notin \alpha$ .

$\beta \in \mathbb{R}$  since  $\beta$  is a cut:

- If  $s \notin \alpha$  and  $p = -s - 1$ , then  $-p - 1 \notin \alpha$ . Thus,  $p \in \beta$  so  $\beta$  is nonempty. If  $q \in \alpha$ , then  $-q \notin \beta$  so  $\beta \neq \mathbb{R}$ .
- If  $p \in \beta$ , let  $r > 0$  so  $-p - r \notin \alpha$ . If  $q < p$ , then  $-q - r > -p - r$  and thus,  $-q - r \notin \alpha$  so  $q \in \beta$ .
- If  $p \in \beta$ , let  $t = p + (r/2)$ . Then  $-t - (r/2) = -p - r \notin \alpha$  and thus,  $t \in \beta$  where  $p < t$ .

If  $r \in \alpha, s \in \beta$ , then  $s \notin \alpha$ . Thus,  $r < -s$  so  $r + s < 0$ . Thus,  $\alpha + \beta \subset 0^*$ .

Let  $v \in 0^*$  and let  $w = -v/2$  so  $w > 0$ .

Thus, by the Archimedean property, there is an integer  $n$  such that  $nw \in \alpha$ , but  $(n+1)w \notin \alpha$ . Let  $p = -(n+2)w$  so  $-p - w = (n+1)w \notin \alpha$  so  $p \in \beta$ .

Then,  $v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p \in \alpha + \beta$ .

Since  $v \in 0^*$ , then  $0^* \subset \alpha + \beta$ . Thus,  $\alpha + \beta = 0^*$ . Then, let  $-\alpha = \beta$ .

Thus, if  $\alpha, \beta, \gamma \in \mathbb{R}$  and  $\beta < \gamma$ , then  $\alpha + \beta < \alpha + \gamma$ .

Thus, if  $\alpha > 0^*$ , then  $-\alpha = -\alpha + 0^* < -\alpha + \alpha = 0^*$  so  $-\alpha < 0^*$ .

If  $\alpha, \beta \in \mathbb{R}_+$ , define  $\alpha\beta$  as the set of all  $p$  such that  $p \leq rs$  for  $r \in \alpha, s \in \beta$ .

Define  $1^*$  as the set of all  $q < 1$ . Then all multiplication axioms holds with similar proofs as addition. Also, note since  $\alpha, \beta > 0^*$ , then  $\alpha\beta > 0^*$ .

Also,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$  holds through cases were  $\alpha, \beta, \gamma >, < 0^*$ .

## 6 Cardinality

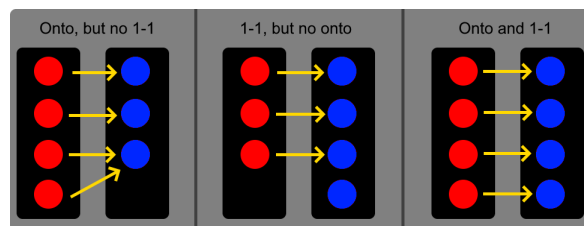
### 6.1 Cardinality

#### Definition 6.1.1: Onto and 1-1 Mapping

Suppose for every  $x \in A$ , there is an associated  $f(x) \in B$ .

Then  $f$  maps  $A$  into  $B = f: A \rightarrow B$ .

- If  $f(A) = B$ , then  $f$  maps  $A$  onto  $B$ .
- If for each  $y \in B$ ,  $f^{-1}(y)$  consist of at most one  $x \in A$  where  $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$  for  $y_1 \neq y_2$ , then  $f$  is a 1-1 mapping of  $A$  into  $B$ .



#### Definition 6.1.2: 1-1 Correspondence

Sets  $A$  and  $B$  are equivalent (have the same cardinality) if there is a 1-1 onto function  $f: A \rightarrow B$ . (1-1 correspondence between  $A$  and  $B$ ) Then:

$$A \sim B$$

If  $f: A \rightarrow B$  is 1-1 and onto, then there is a  $f^{-1}: B \rightarrow A$  that is 1-1 and onto.

#### Definition 6.1.3: Countability

- $A$  is finite if  $A \sim J_n = \{0, 1, \dots, n\}$  for some  $n \in \mathbb{N}$
- $A$  is infinite if  $A$  is not finite
- $A$  is countably infinite if  $A \sim J = \mathbb{Z}_+$
- $A$  is uncountable if  $A$  is not finite or countably infinite
- $A$  is at most countable if  $A$  is finite or countably infinite

#### Example 6.1.4

$\mathbb{Z}$  is countably infinite

#### Proof

Let  $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}$

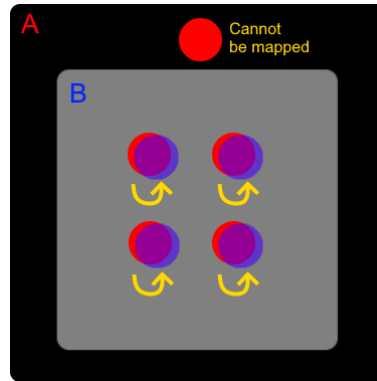
$$f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n-1}{2} & n \text{ is odd} \end{cases}$$

So  $1 \mapsto 0$ ,  $2 \mapsto 1$ ,  $3 \mapsto -1$ ,  $4 \mapsto 2$ ,  $5 \mapsto -2$ , etc. Thus,  $\mathbb{Z} \sim \mathbb{Z}_+$ .



**Definition 6.1.5: Pigeonhole Principle**

If  $A$  is finite,  $A$  is not equivalent to any proper set of  $A$ .

**Theorem 6.1.6: Infinite subsets of countable sets are countable**

An infinite subset  $E$  of a countably infinite set  $A$  is countably infinite.

**Proof**

Let  $E \subset A$  be an infinite subset. For every distinct  $x_i \in A$ , let  $x = \{x_1, x_2, \dots\}$ .  
 Let  $n_1$  be smallest integer such that  $x_{n_1} \in E$ .  
 Then let  $n_2$  be the smallest integer where  $n_2 > n_1$  such that  $x_{n_2} \in E$ .  
 Repeat the process to create sequence  $f(k) = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ .  
 Thus, there is a 1-1 correspondence between  $E$  and  $\mathbb{Z}_+$  so  $E$  is countably infinite.

**6.2 Set of Sets****Definition 6.2.1: Union and Intersection**

Let sets  $\Omega, B$  be such that for each  $x \in \Omega$ , there is an associated  $E_x \subset B$ .

- $E = \bigcup_{x=1}^n E_x$  only if for every  $x \in E$ ,  $x \in E_x$  for at least one  $x \in \Omega$ .
- $P = \bigcap_{x=1}^n E_x$  only if for every  $x \in P$ ,  $x \in E_x$  for all  $x \in \Omega$ .

with properties:

- |   |   |
|---|---|
| (a) $A \cup B = B \cup A$                                     | (a) $A \cap B = B \cap A$                   |
| (b) $(A \cup B) \cup C = A \cup (B \cup C)$                   | (b) $(A \cap B) \cap C = A \cap (B \cap C)$ |
| (c) $A \subset A \cup B$                                      | (c) $(A \cap B) \subset A$                  |
| (d) If $A \subset B$ , then $A \cup B = B$ and $A \cap B = A$ |   |

**Proof**

If  $x \in A \cup B$ , then  $x \in A$  or/and  $x \in B$ .

- If  $x \in A$ , since  $A \subset B$ , then  $x \in B$ . Then,  $(A \cup B) \subset B$ .
- If  $x \in B$ , then immediately  $(A \cup B) \subset B$ .

If  $x \in B$ , then  $x \in A \cup B$  so  $B \subset (A \cup B)$ . Thus,  $A \cup B = B$ .

If  $x \in A \cap B$ , then  $x \in A$  and  $x \in B$ . Thus,  $(A \cap B) \subset A$ .

If  $x \in A$ , since  $A \subset B$ , then  $x \in B$  so  $x \in A \cap B$ . Thus,  $A \subset (A \cap B)$ .  
 Thus,  $A \cap B = A$ .

$$(e) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof

If  $x \in A \cap (B \cup C)$ , then  $x \in A$  and ( $x \in B$  or/and  $x \in C$ ).

- If  $x \in B$ , then  $x \in (A \cap B)$  so  $x \in (A \cap B) \cup (A \cap C)$ .
- If  $x \in C$ , then  $x \in (A \cap C)$  so  $x \in (A \cap B) \cup (A \cap C)$ .

Thus,  $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ .

If  $x \in (A \cap B) \cup (A \cap C)$ , then  $x \in A$  and ( $x \in B$  or/and  $x \in C$ ).

Thus,  $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ .

Thus,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

$$(f) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

If  $x \in A \cup (B \cap C)$ , then  $x \in A$  or/and ( $x \in B$  and  $x \in C$ ).

- If  $x \in A$ , then  $x \in (A \cup B)$  and  $x \in (A \cup C)$  so  $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ .
- If  $x \in B, C$ , then  $x \in (A \cup B)$  and  $x \in (A \cup C)$  so  $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ .

If  $x \in (A \cup B) \cap (A \cup C)$ , then  $x \in A$  or/and ( $x \in B$  and  $x \in C$ ).

Thus,  $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ .

Thus,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

### Theorem 6.2.2: Union of countably infinite sets is countably infinite

If  $E_1, E_2, \dots$  are countably infinite sets, then  $S = \cup_{n=1}^{\infty} E_n$  is countably infinite.

Proof

For each  $E_n$ , there is a sequence  $\{x_{n1}, x_{n2}, \dots\}$ . Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence  $S^* = \{x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; \dots\}$ .

Since  $S^* \sim S$  so  $S$  is at most countable and  $S$  is infinite since  $E_1, E_2, \dots$  are infinite, then  $S$  cannot be finite and thus, countably infinite.

Alternative Proof

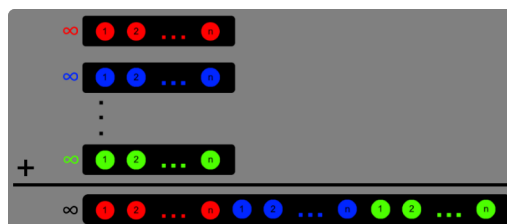
For each  $E_n$ , let set  $\widetilde{E}_n = E_n - \cup_{m=1}^{\infty} E_m$  where  $m \neq n$ . Thus,  $S = \cup_{n=1}^{\infty} \widetilde{E}_n$ .

Since each  $E_n$  is countably infinite, there exists a 1-1 mapping  $\delta_n: E_n \rightarrow \mathbb{Z}_+$ .

Thus, for each  $\widetilde{E}_n$ , there is a 1-1 mapping  $\delta_n: \widetilde{E}_n \rightarrow A \subset \mathbb{Z}_+$ .

Let  $p_1, p_2, \dots$  be distinct primes. Since for  $s \in S$ , there exists a unique  $\widetilde{E}_i$  such that  $s \in \widetilde{E}_i$ , then let  $f(s) = p_1^{\delta_1(s)} p_2^{\delta_2(s)} \dots$  where  $p_k^{\delta_k(s)} = 1$  if  $k \neq i$ .

Then, by the Fundamental theorem of arithmetic,  $f$  maps  $s$  to a unique  $z \in \mathbb{Z}_+$  and thus,  $f$  is a 1-1 function so  $S$  is at most countable. Since any  $E_n \subset S$  is countably infinite, then  $S$  cannot be finite and thus,  $S$  is countably infinite.



**Theorem 6.2.3: The set of countable n-tuples are countable**

Let  $A$  be a countably infinite set and  $B_n$  be the set of all  $n$ -tuples  $(a_1, \dots, a_n)$  where  $a_k \in A$ . Then  $B_n$  is countably infinite.

**Proof**

The base case  $B_1$  is countably infinite since  $B_1 = A$ .

Suppose  $B_{n-1}$  is countably infinite. Then for every  $x \in B$ :

$$x = (b, a) \quad b \in B_{n-1} \text{ and } a \in A$$

Since for every fixed  $b$ ,  $(b, a) \sim A$  and thus, countably infinite.

Since  $B$  is a set of countably infinite sets, then  $B_n$  is countably infinite.

**Definition 6.2.4:  $\mathbb{Q}$  is countable**

The set of rational numbers,  $\mathbb{Q}$ , is countably infinite.

**Proof**

Since elements of  $\mathbb{Q}$  are of form  $\frac{a}{b}$  which is a 2-tuple, then by the **theorem 6.2.3**,  $\mathbb{Q}$  is countably infinite.

**Alternative Proof**

For every  $x \in \mathbb{Q}$ , let  $x = (-1)^i \frac{p}{q}$  where  $p, q \in \mathbb{Z}_+$ .

Let  $f(x) = 2^i 3^p 5^q$ . Then by the Fundamental theorem of arithmetic,  $f$  is a 1-1 mapping of  $x$  to  $E \subset \mathbb{Z}_+$ .

Thus,  $\mathbb{Q}$  is at most countable, but since  $p, q \in \mathbb{Z}_+$ , then  $\mathbb{Q}$  cannot be finite and thus, is countably infinite.

**Example 6.2.5: Sequences of 0 and 1 are uncountable**

Let  $A$  be the set of all sequences whose elements are digits 0 and 1. Then  $A$  is uncountable.

**Proof: Cantor's Diagonalization Proof**

Let set  $E$  be a countably infinite subset of  $A$  which consist of sequences  $s_1, s_2, \dots$ . Then construct a sequence  $s$  as follows:

If the  $n$ -th digit in  $s_n$  is 1, then let the  $n$ -th digit of  $s$  be 0 and vice versa.

Thus,  $s$  differs from every  $s_n \in E$  so  $s \notin E$ .

But,  $s \in A$  so  $E$  is a proper subset of  $A$ .

Thus, every countably infinite subset of  $A$  is a proper subset of  $A$ .

If  $A$  is countably infinite, then  $A$  is a proper subset of  $A$  which is a contradiction.

## 7 Metric Spaces & Closed/Open

### 7.1 Metric Spaces

#### Definition 7.1.1: Metric Spaces

A set  $X$  is a metric space if for any  $p, q \in X$ , there is an associated  $d(p, q) \in \mathbb{R}$  such that:

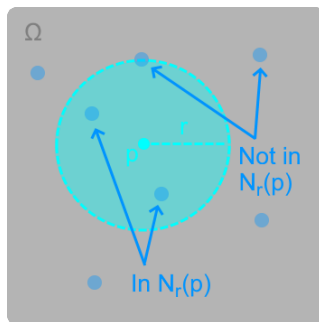
- $d(p, q) > 0$  if  $p \neq q$
- $d(p, q) = 0$  if and only if  $p = q$
- **Symmetry**:  $d(p, q) = d(q, p)$
- **Triangle Inequality**:  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ .

For euclidean spaces  $\mathbb{R}^k$ ,  $d(x, y) = |x - y|$  where  $x, y \in \mathbb{R}^k$ .

#### Definition 7.1.2: Types of Points and Sets

##### (a) Neighborhood

For  $p \in \mathbb{R}^k$  and  $r > 0$ ,  $N_r(p)$  is the set of all  $q \in X$  where  $d(q, p) < r$



##### (b) Limit Points and Closed Sets

Closed set  $E$  contains all  $p \in X$  where every  $N_r(p)$  contains a  $q \neq p \in E$

##### • Limit Points

For point  $p \in X$ , every  $N_r(p)$  contains a  $q \neq p \in E$

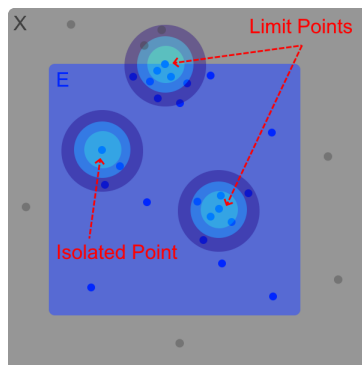
The set of all limit points of  $E = E'$

##### • Isolated Points

If  $p \in E$  is not a limit point of  $E$

##### • Closed

If every limit point  $p$  of  $E$  is a  $p \in E$



## (c) Interior Points and Open Sets

Open set  $E$  contains all its  $p$  which has a  $N_r(p) \subset E$

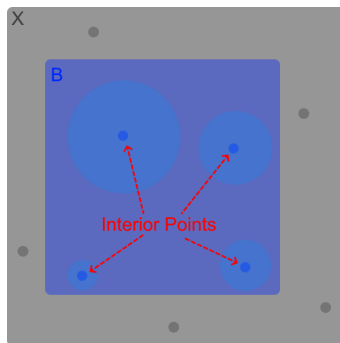
- Interior Point

For  $p \in X$ , there is a  $N_r(p) \subset E$

The set of all interior points =  $E^\circ$

- Open

If every  $p \in E$  is an interior point of  $E$



## (d) More about Sets

- Bounded

If there is  $M \in \mathbb{R}$ ,  $q \in X$  such that  $d(p, q) < M$  for all  $p \in E$

- Complement

From  $E$ ,  $E^c$  is the set of all  $p \in X$  such that  $p \notin E$

- Perfect

If  $E$  is closed and if every  $p \in E$  is a limit point of  $E$

- Dense

If every  $p \in X$  is a limit point of  $E$  or/and  $p \in E$

- Boundary Point

For  $p \in X$ , if every  $N_r(p)$  contains a  $x \in E$  and  $y \in E^c$

The set of all boundary points =  $\partial E$

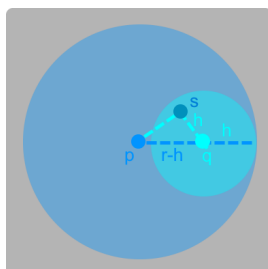
For a metric space  $X$ ,  $\{X, \emptyset\}$  are both open and closed.

**Theorem 7.1.3:  $N_r(p)$  is open**

Every neighborhood is an open set.

**Proof**

Let  $q \in N_r(p)$ . Then there is a  $h > 0 \in \mathbb{R}$  such that  $d(q, p) = r - h$ .  
 Then for any  $s \in N_h(q)$ ,  $d(s, p) \leq d(s, q) + d(q, p) = h + (r - h) = r$ .  
 Thus, for any  $q \in N_r(p)$ , there exists a  $N_h(q) \subset N_r(p)$ .

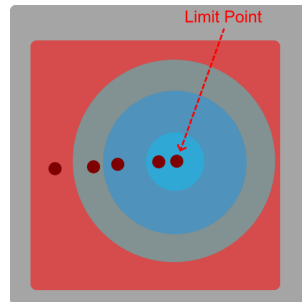


**Theorem 7.1.4:** If a set has a limit point, there are infinite  $q \in E$  in  $N_r(p)$

If  $p$  is a limit point of set  $E$ , then every  $N_r(p)$  contains infinitely many  $q \in E$ .

**Proof**

Suppose there is  $N_{r_1}(p)$  which contains finitely many  $q = \{q_1, \dots, q_n\}$ .  
 Let  $r = \min_{m \in [1, n]} d(p, q_m)$ . Then  $N_r(p)$  contains no  $q \in E$  such that  $q \neq p$ .  
 So,  $p$  is not a limit point of  $E$  which is a contradiction since  $p$  is a limit point of  $E$ .



**Corollary 7.1.5:** Limit points do not exist in finite sets

A finite set  $E$  has no limit points. Since  $\emptyset \in A$ , all finite set must be closed.

**Proof**

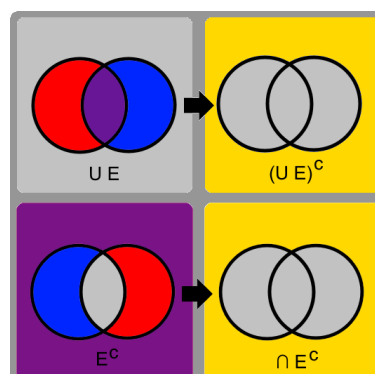
Let  $p$  be a limit point of finite set  $E$ . By **theorem 7.1.4**, then any  $N_r(p)$  contain infinite  $q \in E$  so  $E$  is an infinite set which is a contradiction since  $E$  is finite.  
 So  $p$  cannot be limit point of  $E$  and thus,  $E$  has no limit points.

**Theorem 7.1.6:** De Morgan's Laws

Let  $E_1, E_2, \dots$  be a collection of sets. Then,  $(\cup E_x)^c = \cap (E_x^c)$ .

**Proof**

If  $p \in (\cup E_x)^c$ , then  $p \notin (\cup E_x)$ .  
 Thus,  $p \notin E_x$  for any  $x$  so  $p \in E_x^c$  for all  $x$ . Thus,  $p \in \cap (E_x^c)$  so  $(\cup E_x)^c \subset \cap (E_x^c)$ .  
 If  $p \in \cap (E_x^c)$ , then  $p \in E_x^c$  for all  $x$ .  
 Thus,  $p \notin E_x$  for any  $x$  so  $p \notin \cup E_x$ . Thus,  $p \in (\cup E_x)^c$  so  $\cap (E_x^c) \subset (\cup E_x)^c$ .  
 Thus,  $(\cup E_x)^c = \cap (E_x^c)$ .



**Theorem 7.1.7: Open set  $\rightarrow$  Closed complement**

A set  $E$  is open if and only if  $E^c$  is closed.

**Proof**

Suppose  $E$  is open. Let  $x$  be a limit point of  $E^c$ .

Then for every  $r > 0$ ,  $N_r(x)$  must contain a  $p \in E^c$  such that  $p \neq x$ .

Then,  $N_r(x) \not\subset E$  so  $x$  is not an interior point of  $E$  and thus,  $x \notin E$  so  $x \in E^c$ .

Since any limit point  $x$  of  $E^c$  is a  $x \in E^c$ , then  $E^c$  is closed.

Suppose  $E^c$  is closed. Let  $x \in E$ .

Since  $x \notin E^c$ ,  $x$  is not a limit point of  $E^c$ . Then there exists a  $r > 0$  such that any  $p \in N_r(x)$  is not in  $E^c$ . Thus, every  $p \in N_r(x)$  is  $p \in E$  so  $N_r(x) \subset E$  and thus,  $x$  is an interior point of  $E$ . Since any  $x \in E$  is an interior point of  $E$ , then  $E$  is open.

**Corollary 7.1.8: Closed set  $\rightarrow$  Open complement**

A set  $F$  is closed if and only if  $F^c$  is open.

**Proof**

From **theorem 7.1.7**, let  $E = F^c$ .

**Theorem 7.1.9: Union open  $\rightarrow$  open and Intersection closed  $\rightarrow$  closed**

- (a) If  $\{G_x\}$  is a finite or infinite collection of open sets, then  $\cup G_x$  is open.

**Proof**

If  $p \in \cup G_x$ , then  $p \in G_x$  for at least one  $x$ . Let  $\bar{x}$  be such an  $x$ .

Since  $G_{\bar{x}}$  is open, then  $p$  is an interior point of  $G_{\bar{x}}$  and thus, there is a  $N_r(p)$  such that  $N_r(p) \subset G_{\bar{x}} \subset \cup G_x$ . So  $p$  is an interior point of  $\cup G_x$ . Since any  $p \in \cup G_x$  is an interior point, then  $\cup G_x$  is open.

- (b) If  $\{F_x\}$  is a finite or infinite collection of closed sets, then  $\cap F_x$  is closed.

**Proof**

By **theorem 7.1.7**, any  $F_x^c$  is open. Since  $\{F_x^c\}$  is a finite or infinite collection of open set, then by part (a),  $\cup F_x^c$  is open.

Thus, again by **theorem 7.1.7**,  $(\cup F_x^c)^c$  is closed.

By **theorem 7.1.6**,  $(\cup F_x^c)^c = \cap (F_x^c)^c = \cap F_x$ .

- (c) If  $G_1, \dots, G_n$  is a finite collection of open sets, then  $\cap_{x=1}^n G_x$  is open.

**Proof**

If  $p \in \cap_{x=1}^n G_x$ , then  $p \in G_x$  for all  $G_x$  for  $x = \{1, 2, \dots, n\}$ .

Since each  $G_x$  is open, then for any  $G_x$ , there is a  $N_{r_x}(p) \subset G_x$ .

Let  $r = \min(r_1, r_2, \dots, r_n)$ . Thus,  $p \in N_r(p) \subset N_{r_x}(p)$  for all  $x$ .

So,  $N_r(p) \subset \cap_{x=1}^n G_x$  and thus,  $p$  is an interior point of  $\cap_{x=1}^n G_x$  so  $\cap_{x=1}^n G_x$  is open.

**Infinite + Closed:**  $G_i = (-1/i, 1/i)$

**Infinite + Open:**  $G_i = (-i, i)$

- (d) If  $F_1, \dots, F_n$  is a finite collection of closed sets, then  $\cup_{x=1}^n F_x$  is closed.

**Proof**

By **theorem 7.1.7**, any  $F_x^c$  is open. Since  $F_1^c, \dots, F_n^c$  is a finite collection of open set, then by part (c),  $\cap_{x=1}^n F_x^c$  is open.

Thus, again by **theorem 7.1.7**,  $(\cap_{x=1}^n F_x^c)^c$  is closed.

By **theorem 7.1.6**,  $(\cap_{x=1}^n F_x^c)^c = \cup_{x=1}^n (F_x^c)^c = \cup_{x=1}^n F_x$ .

**Infinite + Closed:**  $F_i = [-1/i, 1/i]$

**Infinite + Open:**  $F_i = [1/i, \infty)$

**Theorem 7.1.10:  $E'$  is closed**

Let  $E \subset X$ . Then,  $(E')' \subset E'$ . Thus,  $E'$  is closed.

**Proof**

If  $x \in (E')'$ , then for every  $N_{r_1}(x)$ , there is a  $y \neq x$  where  $y \in E'$ .  
 Since  $y \in E'$ , then for every  $N_{r_2}(y)$ , there is a  $z \neq y$  where  $z \in E$ .  
 Let  $r = r_1 + r_2$ .  
 Then for every  $N_r(x)$ , there exists a  $z \neq x$  where  $z \in E$ . Thus,  $x \in E'$  so  $(E')' \subset E'$ .

**Theorem 7.1.11:  $E^\circ$  is open**

Let  $E \subset X$ . Then,  $E^\circ$  is open.

**Proof**

If  $p \in E^\circ$ , there is a  $r > 0$  such that  $N_r(p) \subset E$ .  
 Then for  $0 < s < r$ ,  $N_s(p) \subset N_r(p)$  so any  $q \in N_s(p)$  is  $q \in E^\circ$ .  
 Since any  $p \in E^\circ$  have a  $N_s(p) \subset E^\circ$ , then  $E^\circ$  is open.

## 7.2 Intervals and Balls

**Definition 7.2.1: Segments and Intervals**

In  $\mathbb{R}$ , a **segment** is an open interval  $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

In  $\mathbb{R}$ , a **interval** is a closed interval  $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

**Definition 7.2.2: Open Balls**

In  $\mathbb{R}^k$ , an **open ball** of radius  $r > 0$  centered at  $p$  is:

$$N_r(p) = \{x \in \mathbb{R}^k : |x - p| < r\} = \{x \in \mathbb{R}^k : d(x,p) < r\}$$

A **closed ball** has  $d(x,p) \leq r$ .

**Definition 7.2.3: Convex**

$E \subset \mathbb{R}^k$  is **convex** if for all  $x, y \in E$  and  $t \in [0,1]$ ,  $tx + (1-t)y \in E$ .

**Example 7.2.4: Balls are convex**

Balls in  $\mathbb{R}^k$  are convex.

**Proof**

Let  $x, y \in$  open ball  $N_r(p)$ . Let  $z = tx + (1-t)y$  for  $t \in [0,1]$ .  
 Since  $|x - p| < r$  and  $|y - p| < r$ :  

$$\begin{aligned} |z - p| &= |tx + (1-t)y - p| = |tx + (1-t)y - tp + (t-1)p| \\ &= |t(x-p) + (1-t)(y-p)| \leq t|x-p| + (1-t)|y-p| \\ &< tr + (1-t)r = r \end{aligned}$$
  
 Thus,  $z \in N_r(p)$  so balls are convex. Same proof applies to closed balls.

**Definition 7.2.5: Dense**

$E \subset X$  is **dense** if every  $x \in X$  is either in  $E$  or a limit point of  $E$ .

**Example 7.2.6:  $\mathbb{Q}$  is dense in  $\mathbb{R}$** 

Let  $X = \mathbb{R}$ . Then,  $E = \mathbb{Q}$  is dense in  $\mathbb{R}$ .

**Proof**

Fix  $x \in \mathbb{R}$  and  $r > 0$ . There is a  $q \in \mathbb{Q}$  such that  $x-r < q < x$ . So for any  $r > 0$  and  $q \in \mathbb{Q}$ ,  $q \neq x$  and  $q \in N_r(x)$ . Thus, every  $x \in \mathbb{R}$  is a limit point of  $\mathbb{Q}$ .



## 8 Closure, Open Relative, & Compact

### 8.1 Closure

#### Definition 8.1.1: Closure

Let  $E \subset$  metric space  $X$  and  $E'$  be the set of all limit points of  $E$  in  $X$ .

Then the closure of  $E$ :  $\overline{E} = E \cup E'$

with the properties:

- (a)  $\overline{E}$  is closed
- (b)  $E = \overline{E}$  if and only if  $E$  is closed
- (c)  $\overline{E} \subset F$  for every closed  $F \subset X$  such that  $E \subset F$

#### Proof

Suppose  $x \in X$ , but  $x \notin \overline{E}$ . Thus,  $x \in \overline{E}^c$ .

Thus, there is a  $N_r(x) \subset \overline{E}^c$  since else there is always a  $p \in N_r(x)$  where  $p \in \overline{E}$  so  $x$  is a limit point of  $\overline{E}$  so  $x \in \overline{E}$ . Thus,  $\overline{E}^c$  is open so  $\overline{E}$  is closed by [theorem 7.1.7](#).

If  $E = \overline{E}$ , then by part (a),  $E$  is closed.

If  $E$  is closed, then  $E' \subset E$  so  $E = E \cup E' = \overline{E}$ .

If closed set  $F$ , then  $F' \subset F$  and since  $E \subset F$ , then  $E' \subset F' \subset F$ . Thus,  $\overline{E} \subset F$ .

#### Theorem 8.1.2: $\sup(E) \in \overline{E}$

Let non-empty set of real numbers,  $E$ , be bounded above. Let  $y = \sup(E)$ .

Then,  $y \in \overline{E}$ . Thus,  $y \in E$  if  $E$  is closed and  $y \notin E$  if  $E$  is open in  $\mathbb{R}$ .

#### Proof

If  $y \in E$ , then  $y \in \overline{E}$ . Suppose  $y \notin E$ .

For every  $h > 0$ , there exists a  $x \in E$  such that  $y-h < x < y$  otherwise  $y-h$  is an upper bound for  $E$  which is a contradiction since  $y = \sup(E)$ .

Thus,  $y$  is a limit point of  $E$  so  $y \in E'$ .

If  $E$  is closed, then  $y \in E$  since  $y \in E'$ . Also,  $y \in \overline{E}$ .

If  $E$  is open, then any  $N_r(y) \not\subset E$  since  $N_r(y)$  in  $\mathbb{R}$  must contain a  $\gamma > y$  so  $y \notin E^\circ$ .

### 8.2 Open Relative

#### Definition 8.2.1: Open Relative

Suppose  $E \subset Y \subset$  metric space  $X$ .

Then  $E$  is open relative to  $Y$  if for each  $p \in E$ , there is an  $r > 0$  such that for any  $q \in Y$ , then  $q \in E$  if  $d(q,p) < r$ .

**Theorem 8.2.2:**  $E$  is open relative to  $Y \subset X$  if  $E = Y \cap G$  and  $G$  is open in  $X$

Suppose  $E \subset Y \subset X$ .

$E$  is open relative to  $Y$  if and only if  $E = Y \cap G$  for some open  $G \subset X$ .

**Proof:**

Suppose  $E$  is open relative to  $Y$ .

Then for each  $p \in E$ , there is a  $r_p > 0$  such that for any  $q \in Y$  where  $d(p, q) < r_p$ , then  $q \in E$ .

Since  $Y \subset X$ , let  $V_p$  be the set of all  $q \in X$  such that  $d(p, q) < r_p$  and define  $G = \bigcup_{p \in E} V_p$ . Since  $V_p$  is open by **theorem 7.1.3**, then by **theorem 7.1.9a**, open  $G \subset X$ . Since  $p \in V_p$  for all  $p \in E$ , then  $E \subset G \cap Y$ . Also, by construction, then  $V_p \cap Y \subset E$  so  $G \cap Y \subset E$ . Thus,  $E = Y \cap G$ .

If  $G$  is open in  $X$  and  $E = G \cap Y$ , then every  $p \in E$  has a  $V_p \subset G$ .

Then,  $V_p \cap Y \subset G \cap Y = E$  so  $E$  is open relative to  $Y$ .

## 8.3 Compact Sets

**Definition 8.3.1: Open Cover**

An open cover of set  $E \subset X$  is a collection of open  $G_1, G_2, \dots \subset X$  such that  $E \subset \bigcup G_i$ .

**Definition 8.3.2: Compact**

$K \subset X$  is compact if every open cover of  $K$  contains a finite subcover.

If  $G_1, G_2, \dots$  is an open cover of  $K$ , then  $K \subset \bigcup_{i=1}^n G_i$  for some  $n$ .

**Theorem 8.3.3:** A compact set is compact in every metric space

Suppose  $K \subset Y \subset X$ .

Then  $K$  is compact relative to  $X$  if and only if  $K$  is compact relative to  $Y$ .

**Proof**

Suppose  $K$  is compact relative to  $X$ .

Let  $V_1, V_2, \dots$  be sets open relative to  $Y$  such that  $K \subset \bigcup V_x$ . Then by **theorem 8.2.2** for each  $V_x$ , there is a  $G_x$  open relative to  $X$  where  $V_x = Y \cap G_x$ .

Since  $K$  is compact relative to  $X$ , then there is a  $n$  such that  $K \subset G_{x_1} \cup \dots \cup G_{x_n}$ .

Thus,  $K = K \cap Y \subset (\bigcup_{i=1}^n G_{x_i}) \cap Y = (\bigcup_{i=1}^n (G_{x_i} \cap Y)) = \bigcup_{i=1}^n V_{x_i}$ .

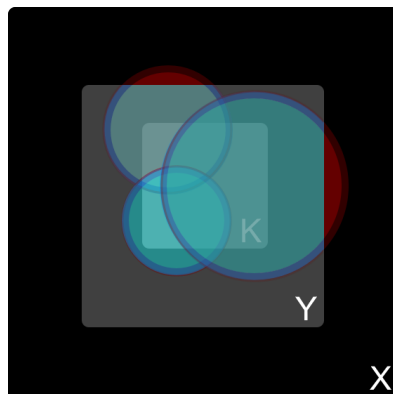
Since there are open  $V_{x_1}, \dots, V_{x_n}$  where  $K \subset \bigcup_{i=1}^n V_{x_i}$  so  $K$  is compact relative to  $Y$ .

Suppose  $K$  is compact relative to  $Y$ .

Let open  $G_1, G_2, \dots \subset X$  such that  $X \subset \bigcup G_x$ . For each  $G_x$ , let  $V_x = Y \cap G_x \subset Y$ .

Since  $K$  is compact relative to  $Y$ , there is a  $n$  such that  $K \subset \bigcup_{i=1}^n V_{x_i}$ .

Thus,  $K \subset \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (Y \cap G_{x_i}) \subset \bigcup_{i=1}^n G_{x_i}$  so  $K$  is compact relative to  $X$ .

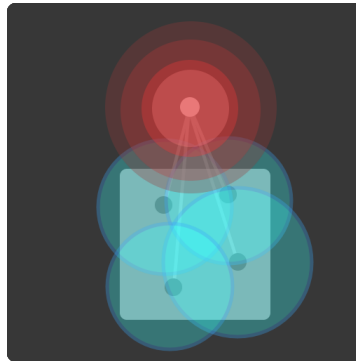


**Theorem 8.3.4: A compact set is closed**

Compact subsets of metric spaces are closed.

**Proof**

Let compact  $K \subset X$ . Suppose  $p \in X$ , but  $p \notin K$  so  $p \in K^c$ .  
 If  $q \in K$ , let  $W_q$  be a neighborhood of  $q$  with  $r < \frac{1}{2}d(p,q)$ . Let  $V_{p,q}$  be a neighborhood of  $p$  with  $r < \frac{1}{2}d(p,q)$ . Since  $K$  is compact, then there are finite points  $q_1, \dots, q_n$  such that  $K \subset W$  where  $W = W_{q_1} \cup \dots \cup W_{q_n}$ .  
 Let  $V = V_{p,q_1} \cap \dots \cap V_{p,q_n}$ , then  $K \cap V \subset W \cap V = \emptyset$  so  $V \subset K^c$ .  
 Since there is a neighborhood  $V$  for  $p \in K^c$  where  $V \subset K^c$ , then every  $p \in K^c$  is an interior point so  $K^c$  is open. Then by **theorem 7.1.7**,  $K$  is closed.

**Theorem 8.3.5: If closed  $E \subset$  compact set  $K$ ,  $E$  is compact**

Closed subsets of compact sets are compact.

**Proof**

Suppose  $F \subset K \subset X$  where  $F$  is closed relative to  $X$  and  $K$  is compact.  
 Let  $V_1, V_2, \dots$  be an open cover for  $F$ . Let open set  $F^c$  be all  $k \in K$  where  $k \notin F$ .  
 $K = F \cup F^c \subset V_1 \cup V_2 \cup \dots \cup F^c$   
 Thus,  $V_1 \cup V_2 \cup \dots \cup F^c$  is an open cover for  $K$ .  
 Since  $K$  is compact, there is a finite subcover  $\Omega$  that covers  $K$  and thus, finite subcover  $\Omega$  covers  $F \cup F^c$ .  
 Remove  $F^c$  from  $\Omega$ . Since finite subcover  $\Omega - F^c$  covers  $F$ , then  $F$  is compact.

**Corollary 8.3.6: Closed  $F \cap$  compact  $K =$  compact**

If  $F$  is closed and  $K$  is compact, then  $F \cap K$  is compact.

**Proof**

Since  $K$  is compact, then  $K$  is closed by **theorem 8.3.4**.  
 Then, by **7.1.9b**,  $F \cap K$  is closed.  
 Since  $F \cap K \subset K$ , then by **theorem 8.3.5**,  $F \cap K$  is compact.

**Theorem 8.3.7: Nonempty  $\cap_{i=1}^n K_i \rightarrow$  nonempty  $\cap K_i$** 

For compact sets  $K_1, K_2, \dots \subset X$  where any intersection of finite  $K_i$  is nonempty, then  $\cap K_i$  is nonempty.

**Proof**

Fix  $K_1$ . If there is a  $k \in K_1$  where  $k \in K_i$  for all  $i$ , then  $k \in \cap K_i$  so  $\cap K_i \neq \emptyset$ .  
 Suppose for every  $k \in K_1$ ,  $k \notin K_i$  for some  $i$ .  
 Then for every  $k \in K_1$ , there is a  $K_i$  such that  $k \notin K_i$  so  $k \in K_i^c$ .  
 Thus,  $K_2^c, K_3^c, \dots$  form an open cover for  $K_1$ .  
 Since  $K_1$  is compact, there is a  $n$  where  $K_1 \subset K_{i_1}^c \cup \dots \cup K_{i_n}^c$ .  
 But then,  $K_1 \cap K_{i_1} \cap \dots \cap K_{i_n} = \emptyset$  which is a contradiction.

**Corollary 8.3.8: Nonempty  $K_i$  where  $K_{i+1} \subset K_i \rightarrow$  nonempty  $\cap K_i$** 

If  $K_1, K_2, \dots$  is a sequence of nonempty compact sets such that  $K_{i+1} \subset K_i$ , then  $\cap K_i$  is nonempty.

**Proof**

Since each  $K_i$  is nonempty and if  $i_1 < \dots < i_n$ , then  $K_{i_1} \cap \dots \cap K_{i_n} = K_{i_n}$  is nonempty, then by **theorem 8.3.7**,  $\cap K_i$  is nonempty.

**Theorem 8.3.9: Nonempty intervals  $I_n$  where  $I_{n+1} \subset I_n \rightarrow$  nonempty  $\cap I_n$** 

If  $I_1, I_2, \dots$  is a sequence of intervals in  $\mathbb{R}^1$  such that  $I_{n+1} \subset I_n$ , then  $\cap I_n$  is nonempty.

**Proof**

Let  $I_n = [a_n, b_n]$  and thus, each  $I_n$  is nonempty. If  $n_1 < \dots < n_m$ , then  $I_{n_1} \cap \dots \cap I_{n_m} = [a_{n_m}, b_{n_m}]$  is nonempty. Thus, by **theorem 8.3.7**,  $\cap I_n$  is nonempty.

**Theorem 8.3.10:  $p \in E'$  exists if infinite  $E \subset$  compact  $K$** 

If  $E$  is an infinite subset of compact set  $K$ , then  $E$  has a limit point in  $K$ .

**Proof**

If no  $p \in K$  is a  $p \in E'$ , then each  $p$  would have a neighborhood  $V_p$  contains at most  $p \in E$  if  $p \in E$ . Thus, there is no finite subcover that covers  $E$  and thus, there is no finite subcover that covers  $K$  since  $E \subset K$  which contradicts  $K$  is compact.

**Definition 8.3.11: K-cells**

The set of all  $x = (x_1, \dots, x_k) \in \mathbb{R}^k$  where  $x_i \in [a_i, b_i]$  for fixed  $a_i, b_i \in \mathbb{R}$ .

**Theorem 8.3.12: K-cells are compact**

Every  $k$ -cell is compact.

**Proof**

Let  $k$ -cell  $I$  consists of all  $x = (x_1, \dots, x_k)$  where  $x_i \in [a_i, b_i]$  for fixed  $a_i, b_i \in \mathbb{R}$ .  
 Let  $\delta = \sqrt{\sum_{i=1}^k (b_i - a_i)^2}$ . Thus,  $|x - y| \leq \delta$  for  $x, y \in I$ .  
 Suppose there exists an open cover  $G_1, G_2, \dots$  of  $I$  which contain no finite subcover.  
 Let  $c_i = \frac{a_i + b_i}{2}$ . Then each interval splits into  $[a_i, c_i]$  and  $[c_i, b_i]$  for  $i \in [1, k]$  so there now exists  $2^k$   $k$ -cells  $Q_i$  whose union is  $I$ .  
 At least one  $Q_i$  cannot be covered else  $I$  would be covered. Then subdivide  $Q_i$  as before and repeating the process so  $Q_{i+1} \subset Q_i$  and each are not covered.  
 However, there is a point  $x^* \in Q_{i_j}$  for all  $j$  such that  $N_r(x^*) \subset G$  so  $Q_{i_1}$  is covered which is a contradiction.

**Theorem 8.3.13: Heine-Borel Theorem**

If a set  $E \subset \mathbb{R}^k$  has one of the three properties, then it has the other two:

- (a)  $E$  is closed and bounded
- (b)  $E$  is compact
- (c) Every infinite subset of  $E$  has a limit point in  $E$

**Proof**

Suppose  $E$  is closed and bounded.

Then there exists a  $M \in \mathbb{R}$  and  $q \in \mathbb{R}^k$  such that  $d(p, q) < M$  for all  $p \in E$ .

Thus, there is a  $k$ -cell  $K = [-M + q_1, q_1 + M] \times \dots \times [-M + q_k, q_k + M]$  such that  $E \subset K$ . Then by [theorem 8.3.12](#),  $K$  is compact and thus by [theorem 8.3.5](#),  $E$  is compact so (a)  $\rightarrow$  (b).

Then by [theorem 8.3.10](#), any infinite subset of  $E$  has a limit point in  $E$  so (b)  $\rightarrow$  (c). Suppose  $E$  is not bounded.

Then there exists  $p \in E$  such that  $d(p, q) > M$  for any  $M \in \mathbb{R}$  and  $q \in \mathbb{R}^k$ .

Let  $S \subset E$  be such points  $p$ .

Then  $S$  is infinite else there is a maximal  $p$  and thus,  $p$  is bounded. Thus,  $S$  is infinite and contains no limit points in  $E$  since any  $d(p_1, p_2) > M$  which contradicts that every infinite subset of  $E$  has a limit point in  $E$ . Thus,  $E$  is bounded.

Suppose  $E$  is not closed.

Then there exists a  $p \in E'$ , but  $p \notin E$ . Since  $p$  is a limit point, then there is a  $q \in E$  such that  $\frac{1}{n+1} < d(q, p) < \frac{1}{n}$  for  $n = \{1, 2, \dots\}$ .

Let  $S \subset E$  be such points  $q$ .

Thus,  $p$  is the only limit point of  $S$  since for  $r < \frac{1}{n}$ , any  $N_r(q_i)$  contains no points of  $S$  other than  $q_i$  since  $d(q_i, q_j) > \frac{1}{n}$  for any  $q_1, q_2 \in S$ .

Thus,  $S$  is infinite, but the only  $p \in S'$  is  $p \notin E$  which contradicts that every infinite subset of  $E$  has a limit point in  $E$ . Thus,  $E$  is closed. So, (c)  $\rightarrow$  (a).

**Theorem 8.3.14: Weierstrass Theorem**

Every bounded infinite set  $E \subset \mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .

**Proof**

Since  $E$  is bounded, then there exists a  $k$ -cell  $K$  such that  $E \subset K$ . Since  $K$  is compact, then by [theorem 8.3.10](#),  $E$  has a limit point in  $K$  and thus, in  $\mathbb{R}^k$ .

## 9 Perfect and Connected Sets

### 9.1 Perfect Sets

#### Definition 9.1.1: Perfect Set

$E \subset X$  is perfect if  $E$  is closed and if every  $p \in E$  is  $p \in E'$ .

#### Theorem 9.1.2: Perfect sets are uncountable

Let  $P$  be a nonempty perfect set in  $\mathbb{R}^k$ . Then,  $P$  is uncountable.

#### Proof

Since  $P$  has limit points, then by [theorem 7.1.4](#),  $P$  is infinite.

Suppose  $P$  is countable. Then let  $x_1, x_2, \dots \in P$ .

Let  $V_i$  be a neighborhood of  $x_i$  where  $y \in V_i$  for any  $y \in \mathbb{R}^k$  such that  $|y - x_i| < r$ .

Thus, the  $\bar{V}_i$  is the set of all  $y \in \mathbb{R}^k$  such that  $|y - x_i| \leq r$ .

Since every  $x_i$  are limit points, then any  $V_i \cap P$  is not empty where there is a  $V_{i+1}$

(a)  $\bar{V}_{i+1} \subset V_i$

(b)  $x_i \notin \bar{V}_{i+1}$

(c)  $V_{i+1} \cap P$  is nonempty

Let  $K_i = \bar{V}_i \cap P$ . Since  $\bar{V}_i$  is closed and bounded, then by [theorem 8.3.11](#),  $\bar{V}_i$  is compact. Since  $x_i \notin K_{i+1}$ , then no  $x_i \in P$  is  $x_i \in \cap K_i$ . Since  $K_n \subset P$ , then  $\cap K_i$  is nonempty which contradicts [corollary 8.3.8](#) since each  $K_i$  is empty and  $K_{i+1} \subset K_i$ .

#### Corollary 9.1.3: $\mathbb{R}$ is not countable

Every interval  $[a, b]$  is uncountable. Thus,  $\mathbb{R}$  is uncountable.

#### Proof

Since  $[a, b]$  is closed and every  $p \in [a, b]$  is a limit point, then nonempty set  $[a, b]$  is perfect. Thus, by [theorem 9.1.2](#),  $[a, b]$  is uncountable.

#### Definition 9.1.4: Cantor Sets

There exists perfect segments in  $\mathbb{R}^1$  which contain no segment.

Let  $E_0 = [0, 1]$ .

For  $E_1$ , remove  $(\frac{1}{3}, \frac{2}{3})$ . Thus,  $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ .

For  $E_2$ , remove  $(\frac{1}{9}, \frac{2}{9})$  and  $(\frac{7}{9}, \frac{8}{9})$ . Thus,  $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ .

Continuing such a sequence, the set of compact sets  $E_n$  are such that:

(a)  $E_{n+1} \subset E_n$

(b)  $E_n$  is the union of  $2^n$  intervals each of length  $3^{-n}$ .

$P = \cap E_n$  is called the Cantor set.  $P$  is compact and nonempty.

Thus, any segment of form  $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$  where  $k, m \in \mathbb{Z}_+$  has no points in common with  $P$ . Since any segment  $(a, b)$  contain a segment of such a form since  $3^{-m} < \frac{b-a}{6}$ , then  $P$  contains no segment.

Let  $x \in P$  and segment  $S$  contain  $x$ . Let  $I_n$  be an interval of  $E_n$  containing  $x$ . Then choose a large enough  $n$  so  $I_n \subset S$ .

Let  $x_n$  be an endpoint of  $I_n$  where  $x_n \neq x$  and thus,  $x$  is a limit point. Since  $P$  is closed and every  $p \in P$  is  $p \in P'$ , then  $P$  is perfect.

## 9.2 Connected Sets

### Definition 9.2.1: Connected Set

$A, B \subset X$  are separated if both  $A \cap \overline{B}$  and  $\overline{A} \cap B$  are empty.

$E \subset X$  is connected if  $E$  is not the union of two nonempty separated sets.

Separated sets are disjoint, but disjoint sets need not be separated.

### Theorem 9.2.2: All points between points in connected sets exists

$E \subset \mathbb{R}^1$  is connected if and only if:

If  $x, y \in E$  and  $x < z < y$ , then  $z \in E$ .

#### Proof

If there exists  $x, y \in E$  and  $z \in (x, y)$  such that  $z \notin E$ , then  $E = A_z \cup B_z$  where  $A_z = E \cap (-\infty, z)$  and  $B_z = E \cap (z, \infty)$ .

Since  $x \in A_z$  and  $y \in B_z$ , then  $A$  and  $B$  are nonempty. Since  $A_z \subset (-\infty, z)$  and  $B_z = (z, \infty)$ , then  $A_z$  and  $B_z$  are separated. Thus,  $E$  is not connected.

Suppose  $E$  is not connected. Then, there are nonempty separated sets  $A$  and  $B$  such that  $A \cup B = E$ . Pick  $x \in A$ ,  $y \in B$  where  $x < y$ . Let  $z = \sup(A \cap [x, y])$ .

Since,  $z \in \overline{A}$  so  $z \notin B$ , then  $x \leq z < y$ . If  $z \notin A$ , then  $x < z < y$  so  $z \notin E$ .

If  $z \in A$ , then  $z \notin \overline{B}$  and thus, there exists a  $z_1$  such that  $z < z_1 < y$  and  $z_1 \notin B$ . Then,  $x < z_1 < y$  so  $z_1 \notin E$ .

## 10 Convergent and Cauchy Sequences

### 10.1 Convergent Sequences

#### Definition 10.1.1: Convergent Sequence

A sequence  $\{x_n\}$  in metric space  $X$  converge if there is a  $x \in X$  such that:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for all  $n \geq N$ ,  $d(x_n, x) < \epsilon$

Then,  $\{x_n\}$  converges to  $x$ :  $\lim_{n \rightarrow \infty} x_n = x$

If  $\{x_n\}$  does not converge, then it diverges.

#### Example 10.1.2

- (a) Let  $x_n = \frac{1}{n}$  in  $\mathbb{R}^2$ . Then,  $\lim_{n \rightarrow \infty} x_n = 0$

##### Proof

For  $\epsilon > 0$ , there is a  $\frac{1}{N} < \epsilon$ . Then:  
 $d(x_n, 0) = |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$

- (b) Let  $x_n = (-1)^n + \frac{1}{n}$  in  $\mathbb{R}^2$ . Then,  $\{x_n\}$  diverges.

##### Proof

$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (-1)^n + \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} (-1)^n$   
 Since  $(-1)^n$  alternates between -1 and 1, then  $\{x_n\}$  diverges.

#### Theorem 10.1.3: A convergent sequence is unique

- (a)  $\{p_n\}$  converges to  $p \in X$  if and only if every  $N_r(p)$  contains  $p_n$  for all, but finitely many  $n$ .

##### Proof

Suppose  $p_n \rightarrow p$ . Then for  $N_\epsilon(p)$ , any  $q \in X$  such that  $d(q, p) < \epsilon$  is  $q \in N_\epsilon(p)$ . Since  $p_n \rightarrow p$ , there is a  $N$  such that for  $n \geq N$ ,  $d(p_n, p) < \epsilon$ .  
 Thus, for  $n \geq N$ ,  $p_n \in N_\epsilon(p)$ .  
 Suppose every  $N_r(p)$  contains  $p_n$  for all, but finitely many  $n$ .  
 For  $\epsilon > 0$ , let  $N_\epsilon(p)$  be the set of all  $q \in X$  such that  $d(p, q) < \epsilon$ . Thus, there exists a  $N$  such that  $p_n \in N_\epsilon(p)$  if  $n \geq N$ .  
 Thus,  $d(p_n, p) < \epsilon$  so  $p_n \rightarrow p$ .

- (b) If  $p, p' \in X$  and  $\{p_n\}$  converges to  $p$  and  $p'$ , then  $p = p'$ .

##### Proof

For  $\epsilon > 0$ , there exists  $N, N'$  such that:  
 $d(p_n, p) < \frac{\epsilon}{2}$  for  $n \geq N$        $d(p_n, p') < \frac{\epsilon}{2}$  for  $n \geq N'$   
 Then for  $n \geq \max(N, N')$ ,  $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$ .  
 Thus,  $p = p'$ .

- (c) If  $\{p_n\}$  converges, then  $\{p_n\}$  is bounded.

##### Proof

If  $\{p_n\} \rightarrow p$ , there is a  $N$  such that for  $n > N$ ,  $d(p_n, p) < 1$ .  
 Let  $r = \max(1, d(p_1, p), \dots, d(p_N, p))$ . Thus for all  $n$ ,  $d(p_n, p) \leq r$ .

- (d) If  $E \subset X$  and  $p \in E'$ , there is a  $\{p_n\}$  in  $E$  such that  $p = \lim_{n \rightarrow \infty} p_n$ .

##### Proof

Since  $p \in E'$ , then for each  $n \in \mathbb{Z}_+$ , there is a  $p_n \in E$  such that  $d(p_n, p) < \frac{1}{n}$ . For  $\epsilon > 0$ , there is a  $\frac{1}{N} < \epsilon$  so for  $n \geq N$ ,  $d(p_n, p) < \frac{1}{n} < \frac{1}{N} < \epsilon$ .  
 Thus,  $p = \lim_{n \rightarrow \infty} p_n$ .



**Theorem 10.1.4: Arithmetic Operations for Sequences**

Suppose  $\{s_n\}, \{t_n\} \in \mathbb{C}$  where  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ .

- (a)  $\lim_{n \rightarrow \infty} s_n + t_n = s + t$

Proof

For  $\epsilon > 0$ , there exists  $N_1, N_2$  such that

$$|s_n - s| < \frac{\epsilon}{2} \text{ for } n \geq N_1 \quad |t_n - t| < \frac{\epsilon}{2} \text{ for } n \geq N_2$$

If  $N = \max(N_1, N_2)$ , then for  $n \geq N$ :

$$|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$$

- (b)  $\lim_{n \rightarrow \infty} cs_n = cs$  and  $\lim_{n \rightarrow \infty} c + s_n = c + s$

Proof

For  $\epsilon > 0$ , there exists a  $N$  such that

$$|s_n - s| < \frac{\epsilon}{c} \text{ for } n \geq N$$

$$|cs_n - cs| \leq c \cdot |s_n - s| < \epsilon$$

- (c)  $\lim_{n \rightarrow \infty} s_n t_n = st$

Proof

Note  $s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$ .

For  $\epsilon > 0$ , there exists  $N_1, N_2$  such that

$$|s_n - s| < \sqrt{\epsilon} \text{ for } n \geq N_1 \quad |t_n - t| < \sqrt{\epsilon} \text{ for } n \geq N_2$$

If  $N = \max(N_1, N_2)$ , then for  $n \geq N$ ,  $|(s_n - s)(t_n - t)| < \epsilon$ .

Thus,  $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} (s_n t_n - st) &= \lim_{n \rightarrow \infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t) \\ &= 0 + t \cdot 0 + s \cdot 0 = 0 \end{aligned}$$

- (d)  $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$  where  $s_n, s \neq 0$

Proof

Choose  $m$  such that  $|s_n - s| < \frac{1}{2}|s|$  if  $n \geq m$  so  $|s_n| > \frac{1}{2}|s|$  for  $n \geq m$ .

For  $\epsilon > 0$ , there is a  $N > m$  such that for  $n \geq N$ ,  $|s_n - s| < \frac{1}{2}|s|^2\epsilon$ .

Thus, for  $n \geq N$ ,  $|\frac{1}{s_n} - \frac{1}{s}| = |\frac{s_n - s}{s_n s}| < \frac{2}{|s|^2}|s_n - s| < \epsilon$ .

**Theorem 10.1.5: Extension to  $\mathbb{R}^k$** 

- (a) Suppose  $x_n \in \mathbb{R}^k$  and  $x_n = (\alpha_{n_1}, \dots, \alpha_{n_k})$ . Then  $\{x_n\}$  converges to  $x = (\alpha_1, \dots, \alpha_k)$  if and only if  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$  for  $i \in [1, k]$ .

Proof

Suppose  $\{x_n\}$  converges to  $x = (\alpha_1, \dots, \alpha_k)$ .

Since for any  $i \in [1, k]$ ,  $|\alpha_{n_i} - \alpha_i| \leq |x_n - x| < \epsilon$ . Then,  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ .

Suppose  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$  for  $i \in [1, k]$ .

Then for  $\epsilon > 0$ , there is an  $N$  such that for  $n \geq N$ :

$$|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}} \text{ for } i \in [1, k]$$

$$|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \epsilon$$

- (b) Suppose  $\{x_n\}, \{y_n\} \in \mathbb{R}^k$  and  $\{\beta_n\} \in \mathbb{R}$  and  $x_n \rightarrow x$ ,  $y_n \rightarrow y$ ,  $\beta_n \rightarrow \beta$ .  
 $\lim_{n \rightarrow \infty} x_n + y_n = x + y \quad \lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y \quad \lim_{n \rightarrow \infty} \beta_n x_n = \beta x$

Proof

By part a, then  $\lim_{n \rightarrow \infty} x_{n_i} + y_{n_i} = x_i + y_i$  so  $\{x_n + y_n\} \rightarrow x + y$ .

Also,  $\lim_{n \rightarrow \infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$  so  $\{x_n \cdot y_n\} \rightarrow x \cdot y$ .

Also,  $\lim_{n \rightarrow \infty} \beta_i x_{n_i} = \beta_i x_i$  so  $\{\beta_n x_n\} \rightarrow \beta x$ .

## 10.2 Subsequences

### Definition 10.2.1: Subsequence

For sequence  $\{p_n\}$ , let  $\{n_k\} \in \mathbb{Z}_+$  where  $n_k < n_{k+1}$ .

Then  $\{p_{n_k}\}$  is a subsequence of  $\{p_n\}$ .

If  $\{p_{n_k}\}$  converges, then its limit is called a subsequential limit.

### Theorem 10.2.2: $\{p_n\} \rightarrow p \iff \{p_{n_k}\} \rightarrow p$

$\{p_n\}$  converges to  $p$  if and only if every subsequence converges to  $p$ .

#### Proof

Suppose  $\{p_n\}$  converges to  $p$ .

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $|p_n - p| < \epsilon$ .

Let  $\{p_{n_k}\}$  be a subsequence of  $\{p_n\}$ .

Then for  $n_k \geq N$ ,  $|p_{n_k} - p| < \epsilon$ . Thus, every  $\{p_{n_k}\} \rightarrow p$ .

Suppose every subsequence converges to  $p$ .

Since  $\{p_n\}$  is a subsequence of itself, then  $\{p_n\}$  converges to  $p$ .

### Theorem 10.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \rightarrow p$

- (a) If  $\{p_n\}$  is a sequence in a compact metric space  $X$ , then some subsequence converges to  $p \in X$ .

#### Proof

Let  $E$  be the range of  $\{p_n\}$ .

If  $E$  is finite, there is a  $p \in E$  and sequence  $\{n_k\}$  with  $n_k < n_{k+1}$  such that  $p_{n_1} = p_{n_2} = \dots = p$ . Thus,  $\{p_{n_k}\} \rightarrow p$ .

If  $E$  is infinite, then by **theorem 8.3.10**, then there exists a  $p \in E'$ .

Then there are  $n_k$  such that  $d(p_{n_k}, p) < \frac{1}{k}$ . Thus,  $\{p_{n_k}\} \rightarrow p$ .

- (b) Every bounded sequence in  $\mathbb{R}^k$  contains a convergent subsequence.

#### Proof

Let  $E$  be a bounded sequence in  $\mathbb{R}^k$ . Since  $E \cup E'$  is bounded and closed, then by **theorem 8.3.13**,  $E \cup E'$  is compact.

Thus by part a,  $E$  contains a convergent subsequence.

### Theorem 10.2.4: The set of subsequential limits is closed

The subsequential limits of  $\{p_n\}$  in metric space  $X$  form a closed subset of  $X$ .

#### Proof

Let  $E$  be the range of the set of all subsequential limits of  $\{p_n\}$ .

If  $E$  is empty, then  $E$  is closed. If  $E$  is finite, then  $E'$  is empty so  $E$  is closed.

Suppose  $E$  is infinite. Then, let  $q \in E'$ .

Choose  $n_1$  so  $p_{n_1} \neq q$ . Let  $\frac{\epsilon}{2} = d(p_{n_1}, q)$ .

Since  $q \in E'$ , there is a  $x \in E$  where  $d(x, q) < \frac{\epsilon}{2}$ .

Since  $x \in E$ , there is a  $\{p_{n_k}\} \rightarrow x$  so there is a  $N$  such that for  $n \geq N$ ,  $d(p_{n_k}, x) < \frac{\epsilon}{2}$ .

Thus,  $d(p_{n_k}, q) \leq d(p_{n_k}, x) + d(x, q) < \epsilon$  so  $q$  is a subsequential limit of  $\{p_n\}$ .

Thus,  $q \in E$  so  $E$  is closed.

## 10.3 Cauchy Sequences

### Definition 10.3.1: Metric Spaces

Sequence  $\{p_n\} \in X$  is a Cauchy sequence if:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for all  $n, m \geq N$ ,  $d(p_n, p_m) < \epsilon$

Let nonempty  $E \subset X$  and  $S \subset \mathbb{R}$  of  $d(p, q)$  where  $p, q \in E$ .

Let  $\sup(S) = \text{diam}(E)$ . If  $\{p_n\} \in X$ , and  $p_N, p_{N+1}, \dots \in E_N$ , then  $\{p_n\}$  is a Cauchy sequence if and only if  $\lim_{N \rightarrow \infty} \text{diam}(E_N) = 0$ .

### Theorem 10.3.2: Cauchy sequences and its closure have the same diam

- (a) If  $\overline{E} \subset X$ , then  $\text{diam}(\overline{E}) = \text{diam}(E)$ .

#### Proof

Since  $E \subset \overline{E}$ , then  $\text{diam}(E) \leq \text{diam}(\overline{E})$ .

For  $\epsilon > 0$ , let  $p, q \in E'$ .

Thus, there are  $p', q' \in E$  such that  $d(p', p) < \epsilon$  and  $d(q', q) < \epsilon$ . Thus:  
 $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \leq 2\epsilon + \text{diam}(E)$ .  
 Thus,  $\text{diam}(\overline{E}) \leq 2\epsilon + \text{diam}(E)$  so  $\text{diam}(\overline{E}) = \text{diam}(E)$ .

- (b) If  $K_n$  is a sequence of compact sets of  $X$  such that  $K_{n+1} \subset K_n$  and  $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$ , then  $\cap K_n$  consist of only one point.

#### Proof

Let  $K = \cap K_n$ . Since  $K_n$  is a sequence of compact sets, then by [Corollary 8.3.8](#),  $K$  is nonempty.

If  $K$  contains more than one point, then  $\text{diam}(K) > 0$ .

But since  $K \subset K_n$ , then  $\text{diam}(K) \leq \text{diam}(K_n)$  which contradicts that  $\text{diam}(K_n) \rightarrow 0$ .

### Theorem 10.3.3: Cauchy sequences are convergent

- (a) In every metric space, every convergent sequence is a Cauchy sequence.

#### Proof

If  $p_n \rightarrow p$  and  $\epsilon > 0$ , there is a  $N$  such that for all  $n \geq N$ ,  $d(p, p_n) < \frac{\epsilon}{2}$ .  
 Thus, for  $m, n \geq N$ :

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon.$$

Thus,  $\{p_n\}$  is a Cauchy sequence.

- (b) If  $\{p_n\}$  is a Cauchy sequence in compact metric space  $X$ , then  $\{p_n\}$  converges to some  $p \in X$ .

#### Proof

Let  $\{p_n\}$  be a Cauchy sequence in compact space  $X$ .

Let  $p_N, p_{N+1}, \dots \in E_N$ .

Since  $\{p_n\}$  is a Cauchy sequence, then  $\lim_{N \rightarrow \infty} \text{diam}(\overline{E_N}) = 0$ . Since  $\overline{E_N}$  is closed in compact  $X$ , then by [theorem 8.3.5](#),  $\overline{E_N}$  is compact.

Since  $E_{N+1} \subset E_N$ , then  $\overline{E_{N+1}} \subset \overline{E_N}$  and thus, by [theorem 10.3.2b](#), then there is a unique  $p \in \overline{E_N}$  for every  $N$ .

Since  $p \in \overline{E_N}$ , then  $d(p, q) < \epsilon$  for every  $q \in \overline{E_N}$  so every  $q \in E_N$ .

Then for  $\epsilon > 0$ , there is a  $N_0$  such that for  $N \geq N_0$ ,  $\text{diam}(\overline{E_N}) < \epsilon$ .

Thus,  $d(p_n, p) < \epsilon$  for  $n \geq N_0$  so  $\{p_n\} \rightarrow p$ .

(c) In  $\mathbb{R}^k$ , every Cauchy sequence converges.

**Proof**

Let  $\{x_n\}$  be a Cauchy sequence in  $\mathbb{R}^k$ . Let  $x_N, x_{N+1}, \dots \in E_N$ . Then for some  $N$ ,  $\text{diam}(E_N) < 1$ . Thus, the range of  $\{x_n\} = E_N \cup \{x_1, \dots, x_{N-1}\}$ . Thus,  $\{x_n\}$  is bounded. Thus, the  $\overline{\{x_n\}}$  is closed and bounded so by **theorem 8.3.13**,  $\overline{\{x_n\}}$  is compact. Thus, by part b,  $\{x_n\}$  converges to some  $p \in \mathbb{R}^k$ .

#### Definition 10.3.4: Complete

A metric space where every Cauchy sequence converges is complete.

Thus, by **theorem 10.3.3**, all compact and Euclidean metric spaces are complete.

#### Definition 10.3.5: Monotonic Sequences

A sequence  $\{s_n\}$  of real numbers is:

- (a) monotonically increasing if  $s_n \leq s_{n+1}$
- (b) monotonically decreasing if  $s_n \geq s_{n+1}$

#### Theorem 10.3.6: Monotonic sequences converge if bounded

Suppose  $\{s_n\}$  is monotonic. Then  $\{s_n\}$  converges if and only if it is bounded.

**Proof**

Suppose  $s_n \leq s_{n+1}$ . Let  $E$  be the range of  $\{s_n\}$ .  
 Suppose  $\{s_n\}$  is bounded.  
 Let  $s = \sup(E)$  so  $s_n \leq s$ . For every  $\epsilon > 0$ , there is a  $N$  such that  $s - \epsilon < s_N \leq s$  else  $s - \epsilon$  would be an upper bound of  $E$  which contradicts  $s = \sup(E)$ .  
 Since  $\{s_n\}$  increases, then for  $n \geq N$ ,  $s - \epsilon < s_N \leq s_n \leq s$  so  $\{s_n\} \rightarrow s$ .  
 Suppose  $\{s_n\}$  converges to  $s$ .  
 Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $s - \epsilon < s_N \leq s_n \leq s$ .  
 Thus,  $\{s_n\}$  is bounded from above.  
 Suppose  $s_n \geq s_{n+1}$ . Let  $E$  be the range of  $\{s_n\}$ .  
 Suppose  $\{s_n\}$  is bounded.  
 Let  $s = \inf(E)$  so  $s_n \geq s$ . For every  $\epsilon > 0$ , there is a  $N$  such that  $s \leq s_N < s + \epsilon$  else  $s + \epsilon$  would be a lower bound of  $E$  which contradicts  $s = \inf(E)$ .  
 Since  $\{s_n\}$  decreases, then for  $n \geq N$ ,  $s \leq s_n \leq s_N < s + \epsilon$  so  $\{s_n\} \rightarrow s$ .  
 Suppose  $\{s_n\}$  converges to  $s$ .  
 Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $s \leq s_n < s + \epsilon$ .  
 Thus,  $\{s_n\}$  is bounded from below.

## 11 Limits and Special Sequences

### 11.1 Upper and Lower Limits

#### Definition 11.1.1: No limit

Let  $\{s_n\}$  be a sequence of real numbers such that:

For every real  $M$ , there is a  $N \in \mathbb{Z}$  such that for  $n \geq N$ ,  $s_n \geq M$ .

Then,  $s_n \rightarrow +\infty$ .

For every real  $M$ , there is a  $N \in \mathbb{Z}$  such that for  $n \geq N$ ,  $s_n \leq M$ .

Then,  $s_n \rightarrow -\infty$ .

#### Definition 11.1.2: Upper and Lower Limits

Let  $\{s_n\}$  be a sequence of real numbers.

Let  $E$  contain all subsequential limits of  $\{s_n\}$  plus possibly  $\pm\infty$ .

Then, the upper limit of  $\{s_n\}$ :

$$s^* = \sup(E) \quad \lim_{n \rightarrow \infty} \sup(s_n) = s^*$$

Then, the lower limit of  $\{s_n\}$ :

$$s_* = \inf(E) \quad \lim_{n \rightarrow \infty} \inf(s_n) = s_*$$

#### Theorem 11.1.3: Upper and Lower limits are unique

Let  $\{s_n\}$  be a sequence of real numbers. Let  $E$  be the set of subsequential limits and  $s^*$  be the upper limit of  $\{s_n\}$ . Then:

(a)  $s^* \in E$

##### Proof

If  $s^* = +\infty$ , then there is a  $\{s_{n_k}\} \rightarrow +\infty$  so  $E$  is not bounded above.

If  $s^* \in \mathbb{R}$ , then  $E$  is bounded above so  $s^* \in E$ .

Then by [theorem 10.2.4](#),  $s^* \in E$ .

If  $s^* = -\infty$ , then there are no subsequential limits in  $E$ . Thus, for every  $M$ , there is a  $N$  such that for  $n \geq N$ ,  $s_n \leq M$  so  $-\infty \in E$ .

(b) If  $x > s^*$ , there is a  $N$  such that for  $n \geq N$ ,  $s_n < x$

##### Proof

Suppose there is a  $x > s^*$  such that  $s_n \geq x$  for infinitely many  $n$ .

Then, there is a  $y \in E$  where  $y \geq x > s^*$  which contradicts  $s^* = \sup(E)$ .

(c)  $s^*$  is the only number that satisfies (a) and (b)

##### Proof

Suppose  $p, q$  satisfy part a and b where  $p < q$ .

Choose  $x$  where  $p < x < q$ . Since  $p$  satisfies b, then  $s_n < x$  for  $n \geq N$ .

Thus,  $q$  cannot satisfy a.

The same properties are analogous for  $s_*$ .

**Theorem 11.1.4: Inf & Sup of  $s_n \leq t_n$** 

If  $s_n \leq t_n$  for  $n \geq$  fixed  $N$ , then

$$\lim_{n \rightarrow \infty} \inf(s_n) \leq \lim_{n \rightarrow \infty} \inf(t_n)$$

$$\lim_{n \rightarrow \infty} \sup(s_n) \leq \lim_{n \rightarrow \infty} \sup(t_n)$$

**Proof**

Let  $E_1$  be the set of extended reals  $x$  such that  $\{s_{n_k}\} \rightarrow x$  for some  $\{s_{n_k}\}$ .

Let  $E_2$  be the set of extended reals  $y$  such that  $\{t_{n_k}\} \rightarrow y$  for some  $\{s_{n_k}\}$ .

Let  $s^* = \sup(E_1)$ ,  $s_* = \inf(E_1)$ ,  $t^* = \sup(E_2)$ , and  $t_* = \inf(E_2)$ .

Since there is a  $N$  such that  $s_n \leq t_n$  for  $n \geq N$ , then:

$$x \leftarrow \{s_N, s_{N+1}, \dots\} \leq \{t_N, t_{N+1}, \dots\} \rightarrow y$$

Thus, for  $n \geq N$ ,  $\inf(s_n) \leq \inf(t_n)$  and  $\sup(s_n) \leq \sup(t_n)$ .

**11.2 Special Sequences****Theorem 11.2.1: Special Sequences**

- (a) If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

**Proof**

For  $\epsilon > 0$ , let  $N > \sqrt[p]{\frac{1}{\epsilon}}$ .

Then for  $n \geq N$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}} = \epsilon$

- (b) If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$

**Proof**

If  $p > 1$ , then let  $x_n = \sqrt[p]{p} - 1 > 0$ .

$$p = (x_n + 1)^n = x_n^n + nx_n^{n-1} + \dots + nx_n + 1 \geq nx_n + 1$$

Thus,  $0 < x_n \leq \frac{p-1}{n}$  so  $\{x_n\} \rightarrow 0$  and thus,  $\{\sqrt[p]{p}\} \rightarrow 1$ .

If  $p = 1$ , then  $\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} 1 = 1$ .

If  $0 < p < 1$ , then  $\frac{1}{p} > 1$ . From the proof above for  $p > 1$ ,  $\{\sqrt[p]{\frac{1}{p}}\} \rightarrow 1$ .

Thus,  $\{\frac{1}{\sqrt[p]{p}}\} \rightarrow 1$  so  $\{\sqrt[p]{p}\} \rightarrow 1$ .

- (c)  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

**Proof**

Let  $x_n = \sqrt[n]{n} - 1 \geq 0$ .

$$n = (x_n + 1)^n \geq \frac{n(n-1)}{2} x_n^2$$

Thus,  $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$  so  $\{x_n\} \rightarrow 0$  and thus,  $\{\sqrt[n]{n}\} \rightarrow 1$ .

- (d) If  $p > 0$  and  $\alpha \in \mathbb{R}$ , then  $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$

**Proof**

Let  $k \in \mathbb{Z}$  such that  $k > \alpha$  and  $k > 0$ . For  $n > 2k$ :

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

Thus,  $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$ .

Since  $\alpha - k < 0$ , then  $\{n^{\alpha-k}\} \rightarrow 0$  so  $\{\frac{n^\alpha}{(1+p)^n}\} \rightarrow 0$ .

- (e) If  $|x| < 1$ , then  $\lim_{n \rightarrow \infty} x^n = 0$

**Proof**

From part d, let  $\alpha = 0$ .

Thus,  $\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = 0$  and since  $p > 0$ , then  $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$ .

Also,  $-\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = \lim_{n \rightarrow \infty} \frac{-1}{(1+p)^n} = 0$  so  $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n > -1$ .

## 12 Series and Comparison Test

### 12.1 Series

#### Definition 12.1.1: Series

For sequence  $\{a_n\}$ , define  $\sum_{n=p}^q a_n = a_p + a_{p+1} + \dots + a_q$ .

Then associate  $\{a_n\}$  with a sequence  $\{s_n\}$  such that  $s_n = \sum_{k=1}^n a_k$ .

Then  $\{s_n\}$  is a series with partial sums  $s_n$ .

If  $\{s_n\} \rightarrow s$ , then  $\sum_{n=1}^{\infty} a_n = s$  is the sum of the convergent series.

Note  $a_1 = s_1$  and  $a_n = s_n - s_{n-1}$ .

#### Theorem 12.1.2: Cauchy Criterion Redefined

$\sum a_n$  converges if and only if:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$

#### Proof

Suppose  $\sum_{k=1}^n a_k$  converges.

Then by [theorem 10.3.3a](#),  $\sum_{k=1}^n a_k$  is a Cauchy sequence.

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ :

$$d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = |\sum_{k=n}^m a_k| \leq \epsilon$$

Suppose for every  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$ .

$$|\sum_{k=n}^m a_k| = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) \leq \epsilon$$

Thus,  $\sum_{k=1}^n a_k$  is a Cauchy sequence and thus, convergent.

#### Theorem 12.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \rightarrow 0$

If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

#### Proof

Since  $\sum a_n$  converges, then by [theorem 12.1.2](#), for  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$ . Then if  $m = n \geq N$ ,  $|\sum_{k=n}^m a_k| = |a_n| \leq \epsilon$  so  $\{a_n\} \rightarrow 0$ .

#### Example 12.1.4: $\{a_n\} \rightarrow 0 \not\Rightarrow$ Convergent $\sum a_n$

$\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

#### Proof

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \dots + \frac{1}{8}) + (\frac{1}{9} + \dots) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Thus,  $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$  which is unbounded and thus, not convergent.

#### Theorem 12.1.5: Convergent series $\Leftrightarrow$ Bounded sequence

A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.

#### Proof

Suppose  $\sum a_n$  converges where  $a_n \geq 0$ .

Since  $a_n \geq 0$ , then  $\{s_n\}$  is monotonic so by [theorem 10.3.6](#),  $\{s_n\}$  is bounded above.

Suppose  $\{s_n\}$  is bounded where  $a_n \geq 0$ .

Since  $\{s_n\}$  is monotonic and bounded, then by [theorem 10.3.6](#),  $\{s_n\}$  converges.

**Theorem 12.1.6: Comparison Test**

- (a) If  $|a_n| \leq c_n$  for  $n \geq N_0$  and  $\sum c_n$  converges, then  $\sum a_n$  converges.

**Proof**

For  $\epsilon > 0$ , there exists a  $N \geq N_0$  such that for  $m \geq n \geq N$ ,  $\sum_{k=n}^m c_k \leq \epsilon$ .  
 $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \epsilon$   
 Thus,  $\sum a_n$  converges.

- (b) If  $a_n \geq d_n \geq 0$  for  $n \geq N_0$  and  $\sum d_n$  diverges, then  $\sum a_n$  diverges.

**Proof**

Suppose  $\sum a_n$  converges.  
 Then from part a,  $\sum d_n$  converges which contradicts that  $\sum a_n$  diverges.  
 Thus,  $\sum a_n$  diverges.

**12.2 Series of Nonnegative Terms****Theorem 12.2.1: Infinite Geometric Series**

If  $x \in [0,1)$ , then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If  $x \geq 1$ , the series diverges.

**Proof**

If  $x \neq 1$ , then using the geometric series:

$$s_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$$

Let  $n \rightarrow \infty$ .

If  $x \in [0,1)$ , then by **theorem 11.2.1e**,  $s_n = \frac{1}{1-x} (1 - x^{n+1}) = \frac{1}{1-x} (1 - 0) = \frac{1}{1-x}$ .

Also, by **theorem 11.2.1e**, if  $x \geq 1$ , then the series diverges.

**Theorem 12.2.2: Cauchy's Convergence Criterion**

Suppose  $0 \leq a_{i+1} \leq a_i$ .

Then the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots \text{ converges.}$$

**Proof**

Let  $s_n = a_1 + a_2 + \dots + a_n$  and  $t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$ .

For  $n < 2^k$ :

$$\begin{aligned} s_n &\leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \dots + 2^k a_{2^k} = t_k \end{aligned}$$

Thus, by the **comparison test**, if  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

For  $n > 2^k$ :

$$\begin{aligned} s_n &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k \end{aligned}$$

Thus, by the **comparison test**, if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  converges.



**Theorem 12.2.3: P-series**

$\sum \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof**

If  $p \leq 0$ , then by [theorem 12.1.3](#),  $\sum \frac{1}{n^p}$  diverges.  
 If  $p > 0$ , then by [theorem 12.2.2](#),  $\sum \frac{1}{n^p}$  converges only if  $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$  converges.  
 Since  $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{k(1-p)}$ , then by [theorem 12.2.1](#),  $\sum_{k=0}^{\infty} 2^{k(1-p)}$  converges if  $2^{1-p} < 1$  so if  $1-p < 0$  so  $p > 1$ .

**Theorem 12.2.4: Log P-series**

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof**

Since  $\frac{1}{n(\log(n))^p}$  decreases, then by [theorem 12.2.2](#),  
 $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p}$  converges.  
 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)^p} = \frac{1}{\log(2)^p} \sum_{k=1}^{\infty} \frac{1}{k}$   
 Then by [theorem 12.2.3](#),  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .  
 Thus,  $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Corollary 12.2.5: Log P-series extended**

$\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof**

From [theorem 12.2.4](#), replace  $n = \log(n)$  and multiplying by  $\frac{1}{n} \rightarrow \frac{1}{n \log(n)(\log(\log(n)))^p}$ .  
 Since  $\frac{1}{n \log(n)(\log(\log(n)))^p}$  decreases, by [theorem 12.2.2](#)  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)(\log(\log(2^k)))^p}$ :  
 $\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k \log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$   
 Since  $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$  converges by [theorem 12.2.4](#),  $\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$  converges.

**12.3 The Number e****Definition 12.3.1: Summation equivalence to e**

$$s_m = \sum_{n=0}^m \frac{1}{n!} = 1 + \sum_{n=1}^m \frac{1}{n!} < 1 + \sum_{n=1}^m \frac{1}{2^{n-1}} < 3$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

**Theorem 12.3.2: Limit equivalence to e**

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

**Proof**

Let  $s_n = \sum_{k=0}^n \frac{1}{k!}$  and  $t_n = \left(1 + \frac{1}{n}\right)^n$ . Using the binomial theorem:  
 $t_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$   
 Thus,  $t_n \leq s_n$  so  $\lim_{n \rightarrow \infty} \sup(t_n) \leq e$ .  
 If  $n \geq m$ , then  $t_n \geq \sum_{k=0}^m \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$ .  
 As  $n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \inf(t_n) \geq \sum_{k=0}^m \frac{1}{k!} = s_m$ . As  $m \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} \inf(t_n) \geq e$ .

**Definition 12.3.3: Rapidity of convergence of e**

$$0 < e - s_n < \frac{1}{n!n}$$

**Proof**

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) = \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!n}$$

**Theorem 12.3.4: e is irrational**

e is irrational

**Proof**

Suppose r is rational. Then let  $e = \frac{p}{q}$  for  $p, q \in \mathbb{Z}_+$ .  
 Thus, by [definition 12.3.3](#),  $0 < e - s_q < \frac{1}{q!q}$  so  $0 < q!(e - s_q) < \frac{1}{q}$ .  
 Since  $e = \frac{p}{q}$ , then  $q!e$  is an integer and  $q!s_q = q!(1 + \frac{1}{2!} + \dots + \frac{1}{q!})$  is an integer.  
 Thus,  $q!(e - s_q)$  is an integer which is between 0 and  $\frac{1}{q}$  and thus, a contradiction.

**12.4 Root and Ratio Tests****Theorem 12.4.1: Root Test**

For  $\sum a_n$ , let  $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|})$ .

- (a) If  $\alpha < 1$ ,  $\sum a_n$  converges
- (b) If  $\alpha > 1$ ,  $\sum a_n$  diverges
- (c) If  $\alpha = 1$ , unclear

**Proof**

If  $\alpha < 1$ , choose  $\beta$  such that  $\beta \in (\alpha, 1)$  and  $N \in \mathbb{Z}$  such that  $\sqrt[n]{|a_n|} < \beta$  for  $n \geq N$ .  
 Since  $\beta \in (0, 1)$ , then by [theorem 12.2.1](#),  $\sum \beta^n$  converges. Then by the [comparison test](#),  $\sum a_n$  converges.  
 If  $\alpha > 1$ , then there is a  $a_{n_k}$  such that  $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$ .  
 Thus,  $|a_n| > 1$  for infinitely many  $n$  so by [theorem 12.1.3](#),  $\sum a_n$  doesn't converge.  
 For  $\alpha = 1$ , both  $\sum \frac{1}{n}$  and  $\sum \frac{1}{n^2}$  have  $\alpha = 1$ , but  $\sum \frac{1}{n}$  diverges and  $\sum \frac{1}{n^2}$  converges by [theorem 12.2.3](#).

**Theorem 12.4.2: Ratio Test**

- (a)  $\sum a_n$  converges if  $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$
- (b)  $\sum a_n$  diverges if  $|\frac{a_{n+1}}{a_n}| \geq 1$  for all  $n \geq n_0$  for  $n_0 \in \mathbb{Z}$

**Proof**

If  $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$ , there is a  $\beta < 1$  and  $N$  such that for  $n \geq N$ ,  $|\frac{a_{n+1}}{a_n}| < \beta$ .  
 Then  $|a_{N+1}| < \beta|a_N|$  so  $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$ .  
 Thus,  $|a_{N+p}| < \beta^p|a_N|$  so  $|a_n| < |a_N|\beta^{-N}\beta^n$ .  
 Thus, by the [comparison test](#),  $\sum a_n$  converges.  
 If  $|a_{n+1}| \geq |a_n| > 0$  for  $n \geq n_0$ , then by [theorem 12.1.3](#),  $\sum a_n$  diverges.

**Theorem 12.4.3: Ratio convergence  $\rightarrow$  Root convergence**

$$\lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$$

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$$

**Proof**

Let  $\alpha = \lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n})$ . If  $\alpha = -\infty$ , then  $-\infty \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$  holds true.  
 If  $\alpha$  is finite, there is a  $\beta \leq \alpha$  and  $N$  such that for  $n \geq N$ ,  $\frac{c_{n+1}}{c_n} \geq \beta$  so  $c_{N+p} \geq \beta^p c_N$ .  
 Then,  $c_n \geq c_N \beta^{-N} \beta^n$  so  $\sqrt[n]{c_n} \geq \sqrt[n]{c_N \beta^{-N} \beta^n}$ . Thus,  $\lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$ .  
 Let  $\alpha = \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$ . If  $\alpha = \infty$ , then  $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \infty$  holds true.  
 If  $\alpha$  is finite, there is a  $\beta \geq \alpha$  and  $N$  such that for  $n \geq N$ ,  $\frac{c_{n+1}}{c_n} \leq \beta$  so  $c_{N+p} \leq \beta^p c_N$ .  
 Then,  $c_n \leq c_N \beta^{-N} \beta^n$  so  $\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N} \beta^n}$ . Thus,  $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$ .

## 13 Power Series

### 13.1 Power Series

#### Definition 13.1.1: Power Series

For a sequence  $\{c_n\} \in \mathbb{C}$ , the series  $\sum_{n=0}^{\infty} c_n z^n$  is a power series.  
 $c_n$  are the coefficients and  $z \in \mathbb{C}$ .

#### Theorem 13.1.2: Radius of Convergence

For power series  $\sum c_n z^n$ , let  $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$  and  $R = \frac{1}{\alpha}$ .  
 Then  $\sum c_n z^n$  converges if  $|z| < R$  and diverges if  $|z| > R$ .

#### Proof

Let  $a_n = c_n z^n$ . Using the **root test**,  

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|}) = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n z^n|})$$

$$= |z| \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R}$$
 Thus,  $\sum c_n z^n$  converges if  $\frac{|z|}{R} < 1$  and diverges if  $\frac{|z|}{R} > 1$

### 13.2 Summation By Parts

#### Theorem 13.2.1: Summation By Parts

For sequences  $\{a_n\}$ ,  $\{b_n\}$ , let  $A_n = \sum_{k=0}^n a_k$ . Then for  $0 \leq p \leq q$ :  

$$\sum_{n=p}^q a_n b_n = (\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

#### Proof

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p}^q A_{n-1} b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p \\ &= (\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p \end{aligned}$$

#### Theorem 13.2.2: Conditions for convergent $\sum a_n b_n$

Suppose for  $\{a_n\}$ ,  $\{b_n\}$ :

- partial sums  $A_n$  of  $\sum a_n$  form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n \rightarrow \infty} b_n = 0$

Then  $\sum a_n b_n$  converges.

#### Proof

Since  $\{A_n\}$  is bounded,  $|A_n| \leq M$  for all  $n$ .  
 Since  $\{b_n\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ , then for  $\epsilon > 0$ , there is a  $N$  such that  $b_N \leq \frac{\epsilon}{2M}$ . Then for  $N \leq p \leq q$ :  

$$\begin{aligned} |\sum_{n=p}^q a_n b_n| &= |(\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p| \\ &\leq M |\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p| = 2M b_p \leq 2M b_N \leq \epsilon \end{aligned}$$

**Corollary 13.2.3: Convergent Series of Alternating Sequences**

Suppose for  $\{c_n\}$ :

- $|c_i| \geq |c_{i+1}|$
- $c_{2i-1} \geq 0$  and  $c_{2i} \leq 0$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then  $\sum c_n$  converges.

**Proof**

From **theorem 13.2.2**, let  $a_n = (-1)^{n+1}$  and  $b_n = |c_n|$ .

**Corollary 13.2.4: Convergent Power Series**

Suppose for  $\{c_n\}$ :

- Radius of convergence of  $\sum c_n z^n$  is 1
- $c_i \geq c_{i+1}$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then  $\sum c_n z^n$  converges at every point where  $|z| = 1$  except possibly  $z = 1$ .

**Proof**

From **theorem 13.2.2**, let  $a_n = z^n$  and  $b_n = c_n$ .  
 $A_n$  of  $\sum a_n$  form a bounded sequence since  $|A_n| = |\sum_0^n z^n| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|}$ .

**13.3 Absolute Convergence****Definition 13.3.1: Absolute Convergence**

$\sum a_n$  converges absolutely if  $\sum |a_n|$  converges.

If  $\sum a_n$  converges, but  $\sum |a_n|$  diverges, then  $\sum a_n$  converges non-absolutely.

**Theorem 13.3.2: Absolute convergence  $\rightarrow$  convergence**

If  $\sum a_n$  converges absolutely, then  $\sum a_n$  converges.

**Proof**

Since  $\sum a_n$  converges absolutely, then for every  $\epsilon > 0$ , there is an integer  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$ .  
 Thus,  $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$  so  $\sum a_n$  converges.

**13.4 Addition & Multiplication of Series****Theorem 13.4.1: Addition and Scalar Multiplication**

If  $\sum a_n = A$  and  $\sum b_n = B$ , then  $\sum (a_n + b_n) = A + B$  and  $\sum ca_n = cA$ .

**Proof**

Let  $A_n = \sum_{k=0}^n a_k$  and  $B_n = \sum_{k=0}^n b_k$ .  
 Then  $A_n + B_n = \sum_{k=0}^n a_k + b_k$  so  $\lim_{n \rightarrow \infty} A_n + B_n = A + B$ .  
 Then  $\lim_{n \rightarrow \infty} cA_n = \underbrace{A + \dots + A}_c = cA$

**Definition 13.4.2: Cauchy Product**

For  $\sum a_n$  and  $\sum b_n$ , let  $c_n = \sum_{k=0}^n a_k b_{n-k}$  and the product as  $\sum c_n$ .

$$\begin{aligned} \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n) (b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \end{aligned}$$

**Theorem 13.4.3: Conditions  $\sum c_n = AB$** 

Suppose

(a)  $\sum_{n=0}^{\infty} a_n$  converges absolutely

(b)  $\sum_{n=0}^{\infty} a_n = A$

(c)  $\sum_{n=0}^{\infty} b_n = B$

(d)  $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then  $\sum_{n=0}^{\infty} c_n = AB$ .**Proof**Let  $A_n = \sum_{k=0}^n a_k$ ,  $B_n = \sum_{k=0}^n b_k$ ,  $C_n = \sum_{k=0}^n c_k$ , and  $\beta_n = B_n - B$ .

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0)$$

$$= a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0$$

$$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \dots + a_n (B + \beta_0)$$

$$= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$$

Let  $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$  so  $C_n = A_n B + \gamma_n$ .Since  $a_n$  converges absolutely, then  $\sum_{n=0}^{\infty} |a_n| = \alpha$ .Since  $\sum_{n=0}^{\infty} b_n = B$ , then  $\beta_n \rightarrow 0$ .Then for  $\epsilon > 0$ , there is a  $N$  such that  $|\beta_n| \leq \frac{\epsilon}{\alpha}$  for  $n \geq N$ .

$$|\gamma_n| \leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \dots + \beta_n a_0|$$

$$\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |a_{n-N-1} + \dots + a_0| \frac{\epsilon}{\alpha}$$

$$\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + \alpha \frac{\epsilon}{\alpha}$$

Thus, with a fixed  $N$ , since  $a_n \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} |\gamma_n| \leq \epsilon$  so  $\lim_{n \rightarrow \infty} \gamma_n = 0$ .Thus,  $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} A_n B + \gamma_n = AB$ .**Theorem 13.4.4: By Cauchy Product,  $\sum c_n = C$  implies  $C = AB$** If  $\sum a_n = A$ ,  $\sum b_n = B$ ,  $\sum c_n = C$  where  $c_n = a_0 b_n + \dots + a_n b_0$ , then  $C = AB$ .**13.5 Rearrangements****Definition 13.5.1: Rearrangements**Let  $a'_n = a_{k_n}$ . Then  $\sum a'_n$  is a rearrangement of  $\sum a_n$ .**Theorem 13.5.2: Rearrangements can converge or diverge**Let  $\sum a_n \in \mathbb{R}$  converge non-absolutely. Suppose  $-\infty \leq \alpha \leq \beta \leq \infty$ .Then there exists a rearrangement  $\sum a'_n$  with partial sums  $s'_n$  such that:

$$\lim_{n \rightarrow \infty} \inf(s'_n) = \alpha \quad \lim_{n \rightarrow \infty} \sup(s'_n) = \beta$$

**Proof**Let  $p_n = \frac{|a_n| + a_n}{2}$  and  $q_n = \frac{|a_n| - a_n}{2}$ . Since  $\sum |a_n|$  diverge, then  $\sum p_n$  and  $\sum q_n$  diverges.Let  $P_1, P_2, P_3, \dots$  be the nonnegative terms of  $\sum a_n$  in order and  $Q_1, Q_2, Q_3, \dots$  be the absolute values of the negative terms of  $\sum a_n$  in order. Thus,  $\sum P_n$  and  $\sum Q_n$  differ from  $\sum p_n$  and  $\sum q_n$  only by the zero terms and thus, are divergent.Choose real-valued sequences  $\{\alpha_n\} \rightarrow \alpha$ ,  $\{\beta_n\} \rightarrow \beta$  such that  $\alpha_n < \beta_n$  and  $\beta_1 > 0$ .Let  $m_1, k_1$  be the smallest integers such that:

$$P_1 + \dots + P_{m_1} > \beta_1 \quad P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} < \alpha_1$$

Let  $m_2, k_2$  be the smallest integers such that:

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} < \beta_2$$

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} < \alpha_2$$

Continuing such a process, then  $\lim_{n \rightarrow \infty} \inf(s'_n) = \alpha$  and  $\lim_{n \rightarrow \infty} \sup(s'_n) = \beta$ .

**Theorem 13.5.3: Absolute rearrangements converges uniquely**

If  $\sum a_n \in \mathbb{C}$  converges absolutely, then every rearrangement of  $\sum a_n$  converges to the same sum.

**Proof**

Let  $\sum a'_n$  be a rearrangement with partial sums  $s'_n$ .  
For  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ ,  $\sum_{i=n}^m |a_i| \leq \epsilon$ .  
Let  $p$  be the maximum index of  $\{a_1, a_2, \dots, a_N\}$  in  $a'_n$  and  $a_n$ .  
Since if  $n > p$ , then  $a_1, a_2, \dots, a_N$  will cancel in  $s_n - s'_n$  and thus,  $|s_n - s'_n| \leq \epsilon$ .  
Thus, every  $\{s'_n\}$  converges to  $\{s_n\}$ .

## 14 Continuity

### 14.1 Limits of Functions

#### Definition 14.1.1: Limits of Functions

For metric spaces  $X, Y$ , let  $E \subset X$ ,  $f$  maps  $E$  into  $Y$ , and  $p \in E$ .

Then  $\lim_{x \rightarrow p} f(x) = q$  if there is a  $q \in Y$  such that:

For every  $\epsilon > 0$ , there is a  $\delta > 0$  where  $d_Y(f(x), q) < \epsilon$   
for all  $x \in E$  where  $d_X(x, p) < \delta$

#### Theorem 14.1.2: Sequence definition of $\lim_{x \rightarrow p} f(x) = q$

$\lim_{x \rightarrow p} f(x) = q$  if and only if  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every sequence  $\{p_n\} \in E$   
where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

#### Proof

Suppose  $\lim_{x \rightarrow p} f(x) = q$ .

For  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), q) < \epsilon$  if  $x \in E$  and  $d_X(x, p) < \delta$ .

Choose  $\{p_n\} \in E$  such that  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

Thus, for  $n > N$ ,  $d_Y(f(p_n), q) < \epsilon$ .

Suppose  $\lim_{x \rightarrow p} f(x) \neq q$ .

Then there is a  $\epsilon > 0$  such that for every  $\delta > 0$ , there is a  $x \in E$  where  $d_Y(f(x), q) \geq \epsilon$ , but  $d_X(x, p) < \delta$ . Let  $\delta_n = \frac{1}{n}$  and thus, there is a  $\{p_n\}$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ , but  $\lim_{n \rightarrow \infty} f(p_n) \neq q$ .

#### Corollary 14.1.3: A limit of a function is unique

If  $f$  has a limit at  $p$ , this limit is unique.

#### Proof

If  $\lim_{x \rightarrow p} f(x) = q$ , then by **theorem 14.1.2**,  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every  $\{p_n\} \in E$   
where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

Thus, if there exists  $\lim_{x \rightarrow p} f(x) = q'$ , then there is a  $\{p_n\} \in E$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ , but  $\lim_{n \rightarrow \infty} f(p_n) = q'$  which is a contradiction.

#### Theorem 14.1.4: Arithmetic operations on functions of limits

Suppose  $E \subset X$ ,  $p \in E$ , and  $f(x), g(x) \in \mathbb{C}$  on  $E$  where:

$$\lim_{x \rightarrow p} f(x) = A \quad \lim_{x \rightarrow p} g(x) = B.$$

- (a)  $\lim_{x \rightarrow p} (f + g)(x) = A + B$
- (b)  $\lim_{x \rightarrow p} (fg)(x) = AB$
- (c)  $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$

### 14.2 Continuous Functions

#### Definition 14.2.1: Continuous functions on a set

Suppose  $X, Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ .

$f$  is continuous at  $p$  if:

For every  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), f(p)) < \epsilon$   
for all  $x \in E$  where  $d_X(x, p) < \delta$ .

If  $f$  is continuous at every  $p \in E$ , then  $f$  is continuous on  $E$ .

$f$  is continuous at isolated points since regardless of  $\epsilon$ , there is a  $\delta > 0$  such  
that  $d_X(x, p) < \delta$  is  $x = p$  so  $d_Y(f(x), f(p)) = 0 < \epsilon$ .

**Theorem 14.2.2: Continuity at limit points**

Suppose  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Let  $p \in E'$ .

Then  $f$  is continuous at  $p$  if and only if  $\lim_{x \rightarrow p} f(x) = f(p)$ .

**Proof**

If  $f$  is continuous at  $p$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), f(p)) < \epsilon$  for all  $x \in E$  where  $d_X(x, p) < \delta$ . Thus,  $\lim_{x \rightarrow p} f(x) = f(p)$ .

If  $\lim_{x \rightarrow p} f(x) = f(p)$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  where  $d_Y(f(x), f(p)) < \epsilon$  for all  $x \in E$  where  $d_X(x, p) < \delta$ . Thus,  $f$  is continuous at  $p$ .

**Theorem 14.2.3: Continuity with Chain Rule**

Suppose  $E \subset X$ ,  $f: E \rightarrow Y$ ,  $g: f(E) \rightarrow Z$ , and  $h: E \rightarrow Z$  where  $h(x) = g(f(x))$ .

If  $f$  is continuous at  $p$  and  $g$  is continuous at  $f(p)$ , then  $h$  is continuous at  $p$ .

**Proof**

Since  $g$  is continuous at  $f(p)$ , then for  $\epsilon > 0$ , there is a  $\delta_1$  such that:

$$d_Z(g(y), g(f(p))) < \epsilon \text{ for } d_Y(y, f(p)) < \delta_1 \text{ where } y \in f(E)$$

Since  $f$  is continuous at  $p$ , there is a  $\delta_2 > 0$  such that:

$$d_Y(f(x), f(p)) < \delta_1 \text{ for } d_X(x, p) < \delta_2 \text{ where } x \in E$$

Thus:

$$d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon \text{ for } d_X(x, p) < \delta_2 \text{ where } x \in E$$

Thus,  $h$  is continuous at  $p$ .

**Theorem 14.2.4: Continuous functions map open sets to open sets**

$f: X \rightarrow Y$  is continuous on  $X$  if and only if:

$$f^{-1}(V) \text{ is open in } X \text{ for every open set } V \text{ in } Y.$$

**Proof**

Suppose  $f$  is continuous on  $X$  and  $V$  is an open set in  $Y$ .

Suppose  $p \in X$  and  $f(p) \in V$ . Since  $V$  is open, there exists  $\epsilon > 0$  such that  $y \in V$  if  $d_Y(y, f(p)) < \epsilon$ . Since  $f$  is continuous at  $p$ , there exists  $\delta > 0$  such that  $d_Y(f(x), f(p)) < \epsilon$  for  $d_X(x, p) < \delta$ . Thus,  $x \in f^{-1}(V)$  for  $d_X(x, p) < \delta$ .

Suppose  $f^{-1}(V)$  is open in  $X$  for every open  $V$  in  $Y$ .

Fix  $p \in X$  and  $\epsilon > 0$ . Let  $V$  be the set of all  $y \in Y$  such that  $d_Y(y, f(p)) < \epsilon$  so  $V$  is open and thus,  $f^{-1}(V)$  is open. Thus, there exists  $\delta > 0$  such that  $x \in f^{-1}(V)$  for  $d_X(x, p) < \delta$ . Since  $x \in f^{-1}(V)$ , then  $f(x) \in V$  so  $d_Y(f(x), f(p)) < \epsilon$ .

**Corollary 14.2.5: Continuous functions map closed sets to closed sets**

$f: X \rightarrow Y$  is continuous on  $X$  if and only if:

$$f^{-1}(C) \text{ is closed in } X \text{ for every closed set } C \text{ in } Y.$$

**Proof**

By **theorem 14.2.4**,  $f$  is continuous if and only if  $f^{-1}(V)$  is open in  $X$  for every open set  $V$  in  $Y$ . Let  $C = V^c$ . Since  $V$  is open, then  $C$  is closed.

Since  $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$ , then  $f^{-1}(C)$  is closed since  $f^{-1}(V)$  is open.



**Theorem 14.2.6: Continuous functions**

Let  $f, g$  be complex continuous functions on  $X$ .

Then  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  where  $g \neq 0$  for all  $x \in X$  are continuous on  $X$ .

**Proof**

If  $x$  is an isolated point,  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  are continuous by definition. If  $x$  is a limit point, then by **theorems 14.1.4 and 14.2.2**,  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  are continuous since

- $\lim_{x \rightarrow p} (f + g)(x) = \lim_{x \rightarrow p} f(x) + \lim_{x \rightarrow p} g(x) = f(p) + g(p)$
- $\lim_{x \rightarrow p} (fg)(x) = \lim_{x \rightarrow p} f(x) \lim_{x \rightarrow p} g(x) = f(p)g(p)$
- $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \rightarrow p} f(x)}{\lim_{x \rightarrow p} g(x)} = \frac{f(p)}{g(p)}$

**Theorem 14.2.7: Continuous Functions on  $\mathbb{R}^k$** 

- (a) Let  $f_1, \dots, f_k$  be real functions on  $X$  and  $f: X \rightarrow \mathbb{R}^k$  where  $f(x) = (f_1(x), \dots, f_k(x))$  for  $x \in X$ .

Then  $f$  is continuous if and only if  $f_1, \dots, f_k$  are continuous.

- (b) If  $f$  and  $g$  are continuous mappings of  $X$  into  $\mathbb{R}^k$ , then  $f+g$  and  $f \cdot g$  are continuous on  $X$ .

**Proof**

Since  $|f_i(x) - f_i(y)| \leq |f(x) - f(y)|$ , then if  $f$  is continuous, then each  $f_i$  is continuous and vice versa.

By **theorem 14.2.6**,  $f+g$  and  $f \cdot g$  are continuous since each  $f_i, g_i$  are continuous by part a.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are  $x_1 \cdot \dots \cdot x_k$  where each  $x_i$  is continuous, rationals are polynomials divided by polynomials, and  $||x| - |y|| \leq |x - y|$  implies  $|x|$  is continuous.

## 15 Properties of Continuity

## References