Real Analysis

Azure

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1   Ordered Sets and Fields

1.1   Ordered Sets and Bounds

Definition 1.1.1: Ordered Set

An order is:
- Trichotomy: For all x,y $\in$ S, only one holds true:
  - $x < y$
  - $x = y$
  - $x > y$
- Transitivity: If $x < y$ and $y < z$, then $x < z$.

An ordered set is a set with an order.

Definition 1.1.2: Bounds

Let S be an ordered set and E $\subset$ S.

An upper bound of E is a $\beta \in$ S such that for $x \leq \beta$ for all $x \in$ E.

  If such a β exists, then E is bounded from above.

A lower bound of E is a $\alpha \in$ S such that for $x \geq \alpha$ for all $x \in$ E.

  If such a α exists, then E is bounded from below.

Definition 1.1.3: Infimum & Supremum

Let S be an ordered set.

Let E $\subset$ S be bounded from above. Least upper bound $\beta \in$ S exists if:
- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E.

Then $\beta = \sup(E)$.

Let E $\subset$ S be bounded from below. Greatest lower bound $\alpha \in$ S exists if:
- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E.

Then $\alpha = \inf(E)$.

Even if $\sup(E)$ exists, it may or may not exists at S.

If $\sup(E)$ exists, then $\sup(E)$ is unique. Statement also holds true for $\inf(E)$.

Example

Let $S = (1,2) \cup [3,4) \cup (5,6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:
- E = (1,2) is bounded above with $\sup(E) = 3$ and not bounded below.
- E = (5,6) is not bounded above or below so $\inf(E)$, $\sup(E)$ = DNE.
- E = [3,4) is bounded below with $\inf(E) = 3$, but $\sup(E)$ = DNE.

1.2 Least Upper Bound Property

Theorem 1.2.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:

For every nonempty subset $E \subset S$ that is bounded from above:

$\sup(E)$ exists in S.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set $A = \{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then $z > y$ where $z \in A$. So, y is not an upper bound.
  Since for any y, there is $z > y$ where $z \in A$, then $\sup(A)$ doesn't exists in $\mathbb{Q}$.

- If $y^2 - 2 > 0$, then $z < y$ where $z \in B$. So, y is an upper bound, but not $\sup(E)$.
  Since for any y, there is $z < y$ where $z \in B$, then $\inf(B)$ doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound or greatest lower bound property.

Example

$\mathbb{Q}$ doesn't have a least upper bound property. Take for example, $\sqrt{2}$.

Let $x^2 = 2$. If x was rational, there is a rational $\frac{p}{q}$ where $x = \frac{p}{q}$ where both p and q are not even.

$(\frac{p}{q})^2 = 2 \qquad \Rightarrow \qquad p^2 = 2q^2$

Since $2q^2$ is even, then p^2 is even so p is even. Thus, p is divisible by 2 so p^2 is divisible by 4 so q^2 is divisible by 2 so q is even. Thus, both p and q must be even which is a contradiction so $x = \sqrt{2}$ cannot be rational.

So if $\sqrt{2} < \frac{a}{b}$ for some rational $\frac{a}{b}$, there is always another rational $\frac{p}{q}$:

$\sqrt{2} < \frac{p}{q} < \frac{a}{b}$

and there will never be a rational $\frac{p}{q}$ such that $\sqrt{2} = \frac{p}{q}$ since $\sqrt{2}$ is not rational.

Theorem 1.2.2: Least Upper Bound + Lower Bound implies Greatest Lower Bound

Let S be a ordered set with the least upper bound property.

Let non-empty $B \subset S$ be bounded below.

Let L be the set of all lower bounds of B.

Then $\alpha = \sup(L)$ exists in S.

Proof

L is non-empty since B is bounded from below.

Thus, by the least upper bound property of S, $\alpha = \sup(L)$ exists in S.

We claim that $\alpha = \inf(B)$.

If $\gamma < \alpha$, then γ is not an upper bound for L so $y \notin B$ since all upper bounds for L are in B. Thus, for every $x \in B$, $\alpha \leq x$.

If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in B$. Thus, $\inf(B) = \alpha$.
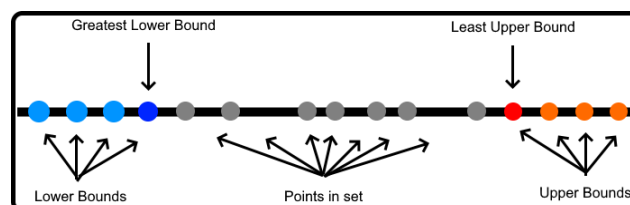
1.3   Fields

Definition 1.3.1: Fields Axioms
    (a) Addition Axioms
- If $x, y \in F$, then $x + y \in F$
- $x + y = y + x$ for all $x, y \in F$
- $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$
- There exists $0 \in F$ such that $0 + x = x$ for all $x \in F$
- For every $x \in F$, there is $-x \in F$ where $x + (-x) = 0$

    (b) Multiplicative Axioms
- If $x, y \in F$, then $xy \in F$
- $yx = xy$ for all $x, y \in F$
- $(xy)z = x(yx)$ for all $x, y, z \in F$
- There exists $1 \neq 0 \in F$ such that $1x = x$ for all $x \in F$
- If $x \neq 0 \in F$, there is $\frac{1}{x} \in F$ where $x(\frac{1}{x}) = 1$

    (c) Distributive Law
        $x(y+z) = xy + xz$ hold for all $x, y, z \in F$

Theorem 1.3.2: Consequences of the Field Axioms
    (a) If $x+y = x+z$, then $y = z$
      Proof

> $y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$

    (b) If $x+y = x$, then $y = 0$
      Proof

> From (a), let $z = 0$

    (c) If $x+y = 0$, then $y = -x$
      Proof

> From (a), let $z = -x$

    (d) $-(-x) = x$
      Proof

> From (c), let $x = -x$ and $y = x$

    (e) If $x \neq 0$ and $xy = xz$, then $y = z$
      Proof

> $y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$

    (f) If $x \neq 0$ and $xy = x$, then $y = 1$
      Proof

> From (e), let $z = 1$

    (g) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$
      Proof

> From (e), let $z = \frac{1}{x}$

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x
Proof

> From (g), let x = $\frac{1}{x}$ and y = x

(i) 0x = 0
Proof

> Since 0x + 0x = (0+0)x = 0x = 0x + 0, then 0x = 0

(j) If x,y $\neq$ 0, then xy $\neq$ 0
Proof

> Suppose xy = 0, then 1 = $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}\frac{1}{x}$0 = 0.
> 0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)
Proof

> xy + (-x)y = (x+(-x))y = 0y = 0. Then by part (c), (-x)y = -(xy).
> xy + x(-y) = x(y+(-y)) = x0 = 0. Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy
Proof

> By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)]. By part (d), -[-(xy)] = xy.

1.4   Ordered Fields

Definition 1.4.1: Ordered Field

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x

- If x,y > 0, then xy > 0

$\mathbb{Q}$,$\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field since i^2 = -1 $\not>$ 0.

Theorem 1.4.2: Properties of the Ordered Field

(a) If x > 0, then -x < 0 and vice versa
Proof

> -x = -x + 0 < -x + x = 0

(b) If x > 0 and y < z, then xy < xz
Proof

> Since z-y > 0, then 0 < x(z-y) = xz - xy

(c) If x < 0 and y < z, then xy > xz
Proof

> Since -x > 0 and z-y > 0, then 0 < -x(z-y) = xy - xz

(d) If x $\neq$ 0, x^2 > 0
Proof

> If x > 0 $\Rightarrow$ x^2 = x $\cdot$ x > 0. If x < 0 $\Rightarrow$ $(-x)^2$ = (-x) $\cdot$ (-x) = x $\cdot$ x = x^2 > 0

(e) If $0 < x < y$, then $0 < 1/y < 1/x$
Proof

> $(\frac{1}{y})$y = 1 > 0 so $\frac{1}{y}$ > 0. Since x < y, then $\frac{1}{y}$ = $(\frac{1}{y})(\frac{1}{x})$x < $(\frac{1}{y})(\frac{1}{x})$y = $\frac{1}{x}$.

Theorem 1.4.3: $\mathbb{R}$ is an ordered field

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$ so $\mathbb{Q}$ is also an ordered field.

Proof

> The proof in Day 5 is a construction of $\mathbb{R}$ by defining a specific order $<$.

Theorem 1.4.4: $\mathbb{Q}$ is dense in $\mathbb{R}$

(a) Archimedean Property: For x,y $\in \mathbb{R}$, if x > 0, there is n $\in \mathbb{Z}$ where nx > y.

Proof

> Fix x > 0. Let A = { nx: n = 1,2,... }. Suppose there is a y where nx $\leq$ y. Then, A is nonempty and bounded from above by y. By the least upper bound property of $\mathbb{R}$, $\alpha = \sup(A)$ exists in $\mathbb{R}$.
>
> Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$. So $\alpha - x$ is not an upper bound of A. So there is a mx $\in$ A such that mx > $\alpha - x$. Then $\alpha <$ (m+1)x, but (m+1)x $\in$ A contradicting α is an upper bound for A.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$: For x,y $\in \mathbb{R}$, if x < y, there is a p $\in \mathbb{Q}$ where x < p < y.

Proof

> Since x < y, then y-x > 0. Then by the Archimedean Property, there exists n $\in$ Z such that n(y-x) > 1. Thus, ny > nx+1 > nx.
>
> Since there is a smallest m $\in \mathbb{Z}_+$ such that m > nx, then m > nx $\geq$ m-1 so nx+1 $\geq$ m > nx. Since ny > nx+1 $\geq$ m > nx, then y > m/n > x.

2   Roots, Complex Field, & Euclidean Spaces

2.1   nth Root

Theorem 2.1.1: nth Root

    (a) If $0 < t \leq 1$, then $t^n \leq t$.

       Proof

> Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
> Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
> Applying the process n times, then $t^n \leq t$.

    (b) If $t \geq 1$, $t^n \geq t$.

       Proof

> Since $0 < 1 \leq t$, then $t \leq t^2$.
> Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
> Applying the process n times, $t \leq t^n$.

    (c) If $0 < s < t$, then $s^n < t^n$.

       Proof

> $$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$$

Theorem 2.1.2: $y^n = x$ has a unique y

    Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

    Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof

> Uniqueness:
> y is unique since if $y_1 < y_2$, then $x = y_1^n < y_2^n \neq x$.
> Existence:
> Let set $A = \{\, t > 0 : t^n < x \,\}$.
> $A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < t_1{}^n < t_1 < x$ so $t_1 \in A$.
> A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.
> So $x+1$ is an upper bound of A.
> Thus by the least upper bound property, $y = \sup(A)$ exists.
> For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.
> ***(Not an upper bound of A if < and not a least upper bound of A if >)***
> For $0 < \alpha < \beta$:
> $$\beta^n - \alpha^n = (\beta - \alpha)\ (\underbrace{\beta^{n-1}}_{\beta^{n-1}} + \underbrace{\beta^{n-2}\alpha^1 + ... }_{<\beta^{n-1}} + \underbrace{\alpha^{n-1}}_{<\beta^{n-1}}) < (\beta - \alpha)n\beta^{n-1}$$
> Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$.
>   From inequality, let $\beta = y+h$ and $\alpha = y$
> $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$
>   Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.
> Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.
>   Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$
>   Thus, $t^n > x$ so $t \notin A$.
>   Thus, $y-k$ is an upper bound of A which is a contradiction since $y = \sup(A)$.
> Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 2.1.3: n-th root of product = product of n-th root
    If a,b > 0 and n $\in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$
Proof

> Let A $= a^{\frac{1}{n}}$, B $= b^{\frac{1}{n}}$. By theorem 2.1.2, since A is a root for $y_1^n = a$, then $A^n = a$.
> Similarly, B is a solution of $y_2^n = b$ so $B^n = b$. Thus:
> $$ab = A^n B^n = A_1 A_2 ... A_n B_1 B_2 ... B_n$$
> $$= A_1 A_2 ... B_1 A_n B_2 ... B_n = ... = A_1 B_1 A_2 ... A_{n-1} A_n B_2 ... B_n$$
> $$= ... = A_1 B_1 A_2 B_2 ... A_n B_n = (AB)^n$$
> Then again by theorem 2.1.2, there is a unique $(ab)^{\frac{1}{n}} = AB = a^{\frac{1}{n}}b^{\frac{1}{n}}$.

2.2   Decimals

Definition 2.2.1: Decimals
    Let n_0 be the largest integer such that $n_0 \leq x$ for x $> 0 \in \mathbb{R}$.
    Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + ... + \frac{n_k}{10^k} \leq x$
    Let E be the set of d_k for k $= 0, 1, ... \infty$. Then, x $= \sup(E)$.

2.3   Extended Reals

Definition 2.3.1: Extended Reals
    The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:
        $-\infty < x < \infty$ for every x $\in \mathbb{R}$
    with the properties:
    - x $\pm \infty = \pm\infty$
    - x $/ \pm\infty = 0$
    - If x > 0, then $x(\pm\infty) = \pm\infty$. If x < 0, then $x(\pm\infty) = \mp\infty$

2.4   Complex Numbers

Definition 2.4.1: Complex Number
    A complex number is an ordered pair (a,b) where a,b $\in \mathbb{R}$. For x,y $\in \mathbb{C}$
        - x + y = (a,b) + (c,d) = (a + c , b + d)
        - xy = (a,b) (c,d) = (ac - bd , ad + bc)
        - $\frac{1}{x} = (a^2 + b^2)(a,-b)$
    Thus, the axioms form a field where (0,0) = 0 and (1,0) = 1 and (0,1) = i.

Theorem 2.4.2: Imaginary i and Form a + bi
    Let i = (0,1). Then, i^2 = -1.
    Then, (a,b) = a + bi
Proof

> i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1
> (a,b) = (a,0) +(0,b) = (a,0) + (b,0)(0,1) = a + bi

Definition 2.4.3: Conjugate

Let conjugate: $\bar{z}$ = a - bi where Re(z) = a , Im(z) = b.
Let z = (a,b) and w = (c,d):

(a) $\overline{z + w} = \bar{z} + \bar{w}$

Proof

$\overline{z + w} = \overline{(a + c, b + d)}$ = (a+c,-b-d) = (a,-b) + (c,-d) = $\bar{z} + \bar{w}$

(b) $\overline{zw} = \bar{z}\,\bar{w}$

Proof

$\overline{zw} = \overline{(ac - bd, ad + bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\bar{z}\,\bar{w}$

(c) z + $\bar{z}$ = 2 Re(z)   z - $\bar{z}$ = 2i Im(z)

Proof

z + $\bar{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)
z - $\bar{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

(d) $z\bar{z} \geq 0$

Proof

$z\bar{z}$ = (a,b)(a,-b) = (a^2 + b^2 , -ab+ab) = a^2 + b$^2 \geq 0$

Definition 2.4.4: Absolute Value

Let absolute value: $| z | = \sqrt{z\bar{z}}$
Let z = (a,b) and w = (c,d):

(a) If $z \neq 0$, then $| z | > 0$.

Proof

$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0$ where $| z | = 0$ only if a,b = 0 so only if z = (0,0).

(b) $| \bar{z} | = | z |$

Proof

$| \bar{z} | = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = | z |$

(c) $| zw | = | z | | w |$

Proof

$| zw | = | \text{(ac-bd,ad+bc)} | = \sqrt{(ac - bd)^2 + (ad + bc)^2}$
$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)}$
$= \sqrt{a^2 + b^2}\,\sqrt{c^2 + d^2} = | z | | w |$

(d) $| \text{Re(z)} | \leq | z |$

Proof

$| \text{Re(z)} | = | a | = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = | z |$

(e) $| \text{z+w} | \leq | z | + | w |$

Proof

$|z + w|^2 = \text{(z+w)}\overline{(z + w)} = \text{(z+w)}(\bar{z} + \bar{w}) = z\bar{z} + \text{z}\bar{w} + \text{w}\bar{z} + \text{w}\bar{w}$
$= |z|^2 + |w|^2 + 2\,\text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2\,|z\bar{w}|$
$= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$

2.5   Euclidean Spaces

Definition 2.5.1: Euclidean Spaces

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples:
$$\mathrm{x} = (x_1, ..., x_k) \qquad \text{for each } x_i \in \mathbb{R}$$
with the properties:

- $\mathrm{x+y} = (x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- $\mathrm{cx} = (cx_1, ..., cx_k) \in \mathbb{R}^k$

So, $\mathbb{R}^n$ has a vector space structure. Similarly, for $\mathbb{C}^n$.

Definition 2.5.2: Inner Product for $\mathbb{R}^k$

$$x \cdot y = x_1 y_1 + ... + x_k y_k \in \mathbb{R}$$

Definition 2.5.3: Norm

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{k} x_i^2}$$

Definition 2.5.4: Extension to $\mathbb{C}^k$

For z,w $\in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + ... + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + ... + z_k \overline{z_k} = |z_1|^2 + ... + |z_k|^2 = |z|^2$

2.6   Cauchy-Schwarz

Theorem 2.6.1: Cauchy-Schwarz

If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $b_1, ..., b_n \in \mathbb{C}$, then:
$$|\sum_{j=1}^{n} a_j (\overline{b_j})|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j (\overline{b_j})$.
If $B = 0$, then $b_1 = ... = b_n = 0$. Thus, $0 \leq A(0)$ holds true.
Suppose $B > 0$. Then:
$$\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j)(\overline{B}\,\overline{a_j} - \overline{C}\,\overline{b_j})$$
$$= \sum (Ba_j - Cb_j)(\overline{B a_j} - \overline{C}\,\overline{b_j}) = \sum B^2 a_j \overline{a_j} - B\overline{C} a_j \overline{b_j} - BC \overline{a_j} b_j + C\overline{C} b_j \overline{b_j}$$
$$= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j \overline{b_j} - BC \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2$$
$$= B^2 A - B\overline{C}C - BC\overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B$$
$$= B(AB - |C|^2)$$
Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.
Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Corollary 2.6.2: $|z \cdot w| \leq |z||w|$

Since $|z_i|^2 = z_i \overline{z_i}$, then $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:
$$|z \cdot w|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2\,|w|^2$$
Thus, $|z \cdot w| \leq |z||w|$.

Theorem 2.6.3: Properties of $\mathbb{R}^k$

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

(a) $|x| \geq 0$ where $|x| = 0$ only if x = 0

Proof

$$|x| = \sqrt{\sum_{i=1}^{k} x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = ... = x_k = 0$$

(b) $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^{k}(\alpha x_i)^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^{k} x_i^2} = |\alpha||x|$$

(c) $|x + y| \leq |x| + |y|$

Proof

$$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2$$
$$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

(d) $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

3   Construction of $\mathbb{R}$

There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Definition 5.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$

- If p $\in \alpha$ and q $\in \mathbb{Q} <$ p, then q $\in \alpha$

- If p $\in \alpha$, then p $<$ r $\in \mathbb{Q}$ for some r $\in \alpha$

Proposition 5.2: Order of $\mathbb{R} \to$ ordered set $\mathbb{R}$

Define $\alpha < \beta$ if α is a proper subset of β.

- If $\alpha \not\supseteq \beta$, then β is not a subset of α.
  Then there is a p $\in \beta$ such that p $\notin \alpha$.
  Then for any q $\in \alpha$, q $<$ p and thus, q $\in \beta$.
  Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.

- If $\alpha \not< \beta$ and $\alpha \not> \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
  If $\alpha \neq \beta$, there are p,q such that p $\in \alpha$, but p $\notin \beta$ and q $\in \beta$, but q $\notin \alpha$.
  But if p $\notin \beta$, then for any b $\in \beta$, b $<$ p. Thus, q $<$ p.
  Similarly, if q $\notin \alpha$, then for any a $\in \alpha$, a $<$ q. Thus, p $<$ q.
  Thus, there is a contradiction since p $>$ q and p $<$ q so $\alpha = \beta$.

- If $\alpha \not\leq \beta$, then α is not a subset of β.
  Then there is a p $\in \alpha$ such that p $\notin \beta$.
  Then for any q $\in \beta$, q $<$ p and thus, q $\in \alpha$.
  Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.

- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ, then α is a proper subset of γ. Thus, $\alpha < \gamma$.

Thus, $\mathbb{R}$ is an ordered set with such an order $<$.

Proposition 5.3: Least Upper Bound of $\mathbb{R} \to$ Least Upper Bound Property

Let A $\subset \mathbb{R}$ and β be an upper bound for A. Let γ be the union of all $\alpha \in$ A. Thus, p $\in \gamma$ if and only if p $\in \alpha$ for some $\alpha \in$ A.
γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in$ A where α_0 is nonempty.
  Since α_0 is nonempty, then γ is nonempty.
  Since every $\alpha \in$ A is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.

- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. If q $<$ p, then q $\in \alpha_1$ so q $\in$ A.

- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. Thus, there is a r $\in \alpha_1$ such that r $>$ p so r $\in \gamma$. Thus, there is a r $\in \gamma$ where r $>$ p.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in$ A $\subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A.
Suppose $\delta < \gamma$. Then there is a s $\in \gamma$ such that s $\notin \delta$. Since s $\in \gamma$, then there is a $\alpha \in$ A such that s $\in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A. Thus, $\gamma = \sup($A$)$.

Proposition 5.4: $\mathbb{R}$ is a field

If α, $\beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums r + s where r $\in \alpha$ and s $\in \beta$.
Also, let 0* be the set of all negative rational numbers which is a cut since:

- 0* is nonempty and $0^* \neq \mathbb{Q}$
- If p $\in$ 0*, then any q $\in \mathbb{Q}$ < p is a negative rational and thus, q $\in$ 0*.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then for any p $\in$ 0*, there is a r $\in \mathbb{Q}$ where p < r < 0 so r is a negative rational so r $\in$ 0*.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take r' $\notin \alpha$, s' $\notin \beta$, then r' + s' > r + s for r $\in \alpha$, s $\in \beta$. Thus, r' + s' $\notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If p $\in \alpha + \beta$, then p = r + s where r $\in \alpha$ and s $\in \beta$.
  If q < p, then q - s < p - s = (r + s) - s = r so q - s $\in \alpha$.
  Since q - s $\in \alpha$ and s $\in \beta$, then (q - s) + s = q $\in \alpha + \beta$.
- If r $\in \alpha$, then there is a t $\in \alpha$ such that t > r. Let s $\in \beta$.
  Thus, for any p = r + s $\in \alpha + \beta$, there is a q = t + s $\in \alpha + \beta$ such that p = r + s < t + s = q.

$\alpha + \beta = \beta + \alpha$

If p = r + s $\in \alpha + \beta$ where r $\in \alpha$, s $\in \beta$, then s + r = r + s = p $\in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If r $\in \alpha$, s $\in \beta$, t $\in \gamma$, then r + s + t = (r + s) + t $\in (\alpha + \beta) + \gamma$ and r + s + t = r + (s + t) $\in \alpha + (\beta + \gamma)$.

$\alpha + 0^* = \alpha$

If r $\in \alpha$, s $\in$ 0*, then r + s < r. Thus, r + s $\in \alpha$. Thus, $\alpha + 0^* \subset \alpha$.
If p $\in \alpha$, there is a r $\in \alpha$ where r > p. Thus, p - r $\in$ 0*.
Since p = r + (p - r) $\in \alpha + 0^*$, then $\alpha \subset \alpha + 0^*$. Thus, $\alpha + 0^* = \alpha$.

There is a $-\alpha$ such that $\alpha + -\alpha = 0^*$

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is r > 0 such that -p - r $\notin \alpha$.
$\beta \in \mathbb{R}$ since β is a cut:

- If s $\notin \alpha$ and p = -s - 1, then -p - 1 $\notin \alpha$. Thus, p $\in \beta$ so β is nonempty. If q $\in \alpha$, then -q $\notin \beta$ so $\beta \neq \mathbb{R}$.
- If p $\in \beta$, let r > 0 so -p - r $\notin \alpha$. If q < p, then -q - r > -p - r and thus, -q - r $\notin \alpha$ so q $\in \beta$.
- If p $\in \beta$, let t = p + (r/2). Then -t - (r/2) = -p - r $\notin \alpha$ and thus, t $\in \beta$ where p < t.

If r $\in \alpha$, s $\in \beta$, then s $\notin \alpha$. Thus, r < -s so r + s < 0. Thus, $\alpha + \beta \subset 0^*$.
Let v $\in$ 0* and let w = -v/2 so w > 0.
Thus, by the Achimedean property, there is an integer n such that nw $\in \alpha$, but (n+1)w $\notin \alpha$. Let p = -(n+2)w so -p - w = (n+1)w $\notin \alpha$ so p $\in \beta$.
Then, v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p $\in \alpha + \beta$.
Since v $\in$ 0*, then $0^* \subset \alpha + \beta$. Thus, $\alpha + \beta = 0^*$. Then, let $-\alpha = \beta$.

Thus, if α, β, $\gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
Thus, if $\alpha > 0^*$, then $-\alpha = -\alpha + 0^* < -\alpha + \alpha = 0^*$ so $-\alpha < 0^*$.
If α, $\beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that p $\leq$ rs for r $\in \alpha$, s $\in \beta$.
Define 1* as the set of all q < 1. Then all multiplication axioms holds with similar proofs as addition. Also, note since α, $\beta > 0^*$, then $\alpha\beta > 0^*$.
Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were α, β, γ >,< 0*.

4   Cardinality

4.1   Cardinality

Definition 4.1.1: Onto and 1-1 Mapping

Suppose for every x ∈ A, there is an associated f(x) ∈ B.
Then f maps A into B = f: A → B.

- If f(A) = B, then f maps A onto B.
- If for each y ∈ B, $f^{-1}(y)$ consist of at most one x ∈ A where $f^{-1}(y_1) = x_1$ ≠ $x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B.
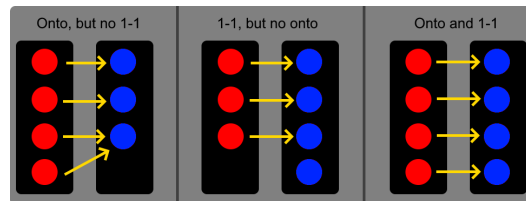
Definition 4.1.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function f: A → B. (1-1 correspondence between A and B) Then, A ∼ B.
If f: A → B is 1-1 and onto, then there is a f^{-1}: B → A that is 1-1 and onto.

Definition 4.1.3: Countability

- A is finite if A ∼ J_n = {0, 1, ..., n} for some n ∈ ℕ
- A is infinite if A is not finite
- A is countably infinite if A ∼ J = $\mathbb{Z}_+$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite

Example

ℤ is countably infinite

Proof

Let f(n): $\mathbb{Z}_+ \to \mathbb{Z} = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n-1}{2} & \text{n is odd} \end{cases}$
So 1 ↦ 0 , 2 ↦ 1, 3 ↦ -1, 4 ↦ 2, 5 ↦ -2 , etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 4.1.4: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A.

Theorem 4.1.5: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite

Proof

Let E $\subset$ A be an infinite subset. For every distinct $x_i \in$ A, let $\{ x_1, x_2, ... \} \in$ A.
Let n_1 be smallest integer such that $x_{n_1} \in$ E.
Then let n_2 be the smallest integer where $n_2 > n_1$ such that $x_{n_2} \in$ E.
Repeat the process to create sequence f(k) = $\{ x_{n_1}, x_{n_2}, ..., x_{n_k}, ... \}$.
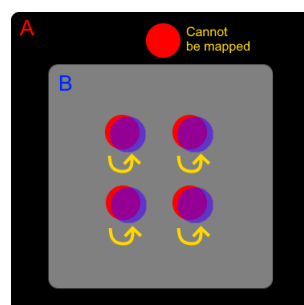Thus, there is a 1-1 correspondence between E and $\mathbb{Z}_+$ so E is countably infinite.

4.2   Set of Sets

Definition 4.2.1: Union and Intersection

Let sets Ω,B be such that for each x $\in \Omega$, there is an associated $E_x \subset$ B.

- E = $\cup_{x=1}^{n}$ E_x only if for every x $\in$ E, x $\in E_x$ for at least one x $\in \Omega$.

- P = $\cap_{x=1}^{n}$ E_x only if for every x $\in$ P, x $\in E_x$ for all x $\in \Omega$.

with properties:

(a) A $\cup$ B = B $\cup$ A   A $\cap$ B = B $\cap$ A

(b) (A $\cup$ B) $\cup$ C = A $\cup$ (B $\cup$ C)   (A $\cap$ B) $\cap$ C = A $\cap$ (B $\cap$ C)

(c) A $\subset$ A $\cup$ B   (A $\cap$ B) $\subset$ A

(d) If A $\subset$ B, then A $\cup$ B = B and A $\cap$ B = A

Proof

If x $\in$ A $\cup$ B, then x $\in$ A or/and x $\in$ B.
- If x $\in$ A, since A $\subset$ B, then x $\in$ B. Then, (A $\cup$ B) $\subset$ B.

- If x $\in$ B, then immediately (A $\cup$ B) $\subset$ B.
If x $\in$ B, then x $\in$ A $\cup$ B so B $\subset$ (A $\cup$ B). Thus, A $\cup$ B = B.

If x $\in$ A $\cap$ B, then x $\in$ A and x $\in$ B. Thus, (A $\cap$ B) $\subset$ A.
If x $\in$ A, since A $\subset$ B, then x $\in$ B so x $\in$ A $\cap$ B. Thus, A $\subset$ (A $\cap$ B).
Thus, A $\cap$ B = A.

(e) A $\cap$ (B $\cup$ C) = (A $\cap$ B) $\cup$ (A $\cap$ C)

Proof

If x $\in$ A $\cap$ (B $\cup$ C), then x $\in$ A and (x $\in$ B or/and x $\in$ C).
- If x $\in$ B, then x $\in$ (A $\cap$ B) so x $\in$ (A $\cap$ B) $\cup$ (A $\cap$ C).

- If x $\in$ C, then x $\in$ (A $\cap$ C) so x $\in$ (A $\cap$ B) $\cup$ (A $\cap$ C).
Thus, A $\cap$ (B $\cup$ C) $\subset$ (A $\cap$ B) $\cup$ (A $\cap$ C).
If x $\in$ (A $\cap$ B) $\cup$ (A $\cap$ C), then x $\in$ A and (x $\in$ B or/and x $\in$ C).
Thus, (A $\cap$ B) $\cup$ (A $\cap$ C) $\subset$ A $\cap$ (B $\cup$ C).
Thus, A $\cap$ (B $\cup$ C) = (A $\cap$ B) $\cup$ (A $\cap$ C).

(f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

<u>Proof</u>

> If $x \in A \cup (B \cap C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).
> - If $x \in A$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
> - If $x \in B,C$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
>
> If $x \in (A \cup B) \cap (A \cup C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).
> Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
> Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 4.2.2: Union of countably infinite sets is countably infinite

If $E_1, E_2, \ldots$ are countably infinite sets, then $S = \cup_{n=1}^{\infty} E_n$ is countably infinite.

<u>Proof</u>

> For each E_n, there is a sequence $\{ x_{n1}, x_{n2}, \ldots \}$. Then construct an array as such:
> $$\begin{pmatrix} x_{11} & x_{12} & \ldots \\ x_{21} & x_{22} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$
> Take elements diagonally, then sequence $S^* = \{ x_{11} ; x_{21}, x_{12} ; x_{31}, x_{32}, x_{33} ; \ldots \}$.
> Since $S^* \sim S$ so S is at most countable and S is infinite since $E_1, E_2, \ldots$ are infinite, then S cannot be finite and thus, countably infinite.

<u>Alternative Proof</u>

> For each E_n, let set $\widetilde{E_n} = E_n - \cup_{m=1}^{\infty} E_m$ where $m \neq n$. Thus, $S = \cup_{n=1}^{\infty} \widetilde{E_n}$.
> Since each E_n is countably infinite, there exists a 1-1 mapping δ_n: $E_n \to \mathbb{Z}_+$.
> Thus, for each $\widetilde{E_n}$, there is a 1-1 mapping δ_n: $\widetilde{E_n} \to A \subset \mathbb{Z}_+$.
> Let $p_1, p_2, \ldots$ be distinct primes. Since for $s \in S$, there exists a unique $\widetilde{E_i}$ such that $s \in \widetilde{E_i}$, then let $f(s) = p_1^{\delta_1(s)} p_2^{\delta_2(s)} \ldots$ where $p_k^{\delta_k(s)} = 1$ if $k \neq i$.
> Then, by the Fundamental theorem of arithmetic, f maps s to a unique $z \in \mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable. Since any $E_n \subset S$ is countably infinite, then S cannot be finite and thus, S is countably infinite.
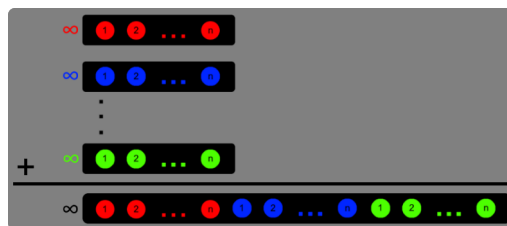
Theorem 4.2.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B_n be the set of all n-tuples $(a_1,\ldots,a_n)$ where $a_k \in A$. Then B_n is countably infinite.

<u>Proof</u>

> The base case B_1 is countably infinite since $B_1 = A$.
> Suppose B_{n-1} is countably infinite. Then for every $x \in B$:
> $\qquad$ x = (b,a) $\qquad\qquad$ $b \in B_{n-1}$ and $a \in A$
> Since for every fixed b, $(b,a) \sim A$ and thus, countably infinite.
> Since B is a set of countably infinite sets, then B_n is countably infinite.

Theorem 4.2.4: $\mathbb{Q}$ is countable

The set of rational numbers, $\mathbb{Q}$, is countably infinite

Proof

Since elements of $\mathbb{Q}$ are of form $\frac{a}{b}$ which is a 2-tuple, then by the theorem 4.2.3, $\mathbb{Q}$ is countably infinite.

Alternative Proof

For every x $\in \mathbb{Q}$, let x $= (-1)^i \ \frac{p}{q}$ where p,q $\in \mathbb{Z}_+$.

Let f(x) $= 2^i \ 3^p \ 5^q$. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to E $\subset \mathbb{Z}_+$.

Thus, $\mathbb{Q}$ is at most countable, but since p,q $\in \mathbb{Z}_+$, then $\mathbb{Q}$ cannot be finite and thus, is countably infinite.

Example

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences $s_1, s_2, ...$

Then construct a sequence s as follows:

If the n-th digit in s_n is 1, then let the n-th digit of s be 0 and vice versa.

Thus. s differs from every $s_n \in$ E so s $\notin$ E.

But, s $\in$ A so E is a proper subset of A.

Thus, every countably infinite subset of A is a proper subset of A.

If A is countably infinite, then A is a proper subset of A which is a contradiction.

5   Metric Spaces & Closed/Open

5.1   Metric Spaces

Definition 5.1.1: Metric Spaces

A set X is a metric space if for ant p,q $\in$ X, there is an associated d(p,q) $\in$ $\mathbb{R}$ such that:

- d(p,q) $>$ 0   if p $\neq$ q
- d(p,q) = 0 if and only if p = q
- Symmetry: d(p,q) = d(q,p)
- Triangle Inequality: d(p,q) $\leq$ d(p,r) + d(r,q)   for any r $\in$ X.

For euclidean spaces $\mathbb{R}^k$, d(x,y) = $|x - y|$ where x,y $\in$ $\mathbb{R}^k$.

Definition 5.1.2: Types of Points and Sets

For metric space X and set E $\subset$ X:

(a) Neighborhood

For p $\in$ X and r $>$ 0, $N_r(p)$ is the set of all q $\in$ X where d(q,p) $<$ r
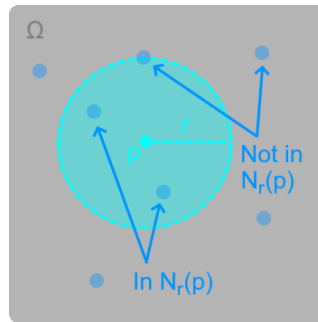
(b) Limit Points and Closed Sets

Closed set E contain all p $\in$ X where every $N_r(p)$ contain a q $\neq$ p $\in$ E

- Limit Points

  For point p $\in$ X, every $N_r(p)$ contains a q $\neq$ p $\in$ E

  The set of all limit points of E = E'

- Isolated Points

  If p $\in$ E is not a limit point of E

- Closed

  If every limit point p of E is a p $\in$ E
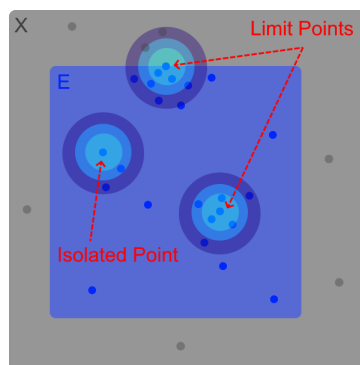
(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset E$

- Interior Point

    For p $\in$ X, there is a $N_r(p) \subset E$

    The set of all interior points = E^o

- Open

    If every p $\in$ E is an interior point of E
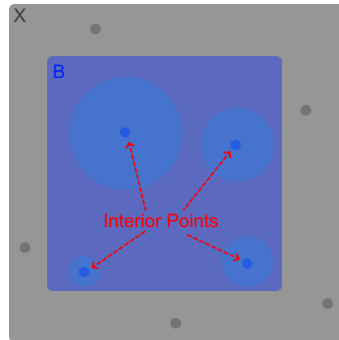
(d) More about Sets

- Bounded

    If there is M $\in$ $\mathbb{R}$, q $\in$ X such that d(p,q) < M for all p $\in$ E

- Complement

    From E, E^c is the set of all p $\in$ X such that p $\notin$ E

- Perfect

    If E is closed and if every p $\in$ E is a limit point of E

- Dense

    If every p $\in$ X is a limit point of E or/and p $\in$ E

- Boundary Point

    For p $\in$ X, if every $N_r(p)$ contains a x $\in$ E and y $\in$ E^c

    The set of all boundary points = ∂E

For a metric space X, {X,$\emptyset$} are both open and closed.

Theorem 5.1.3: $N_r(p)$ is open

Every neighborhood is an open set

Proof

Let q $\in$ $N_r(p)$. Then there is a h > 0 $\in$ $\mathbb{R}$ such that d(q,p) = r - h.
Then for any s $\in$ $N_h(q)$, d(s,p) $\leq$ d(s,q) + d(q,p) = h + (r - h) = r.
Thus, for any q $\in$ $N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.
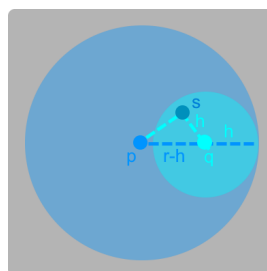
Theorem 5.1.4: If a set has a limit point, there are infinite q $\in$ E in $N_r(p)$

If p is a limit point of set E, then every $N_r(p)$ contains infinitely many q $\in$ E.

Proof

Suppose there is $N_{r_1}(p)$ which contains finitely many q = { q_1, ... , q_n }.
Let r = $\min_{m \in [1,n]}$ d(p,q_m). Then $N_r(p)$ contains no q $\in$ E such that q $\neq$ p.
So, p is not a limit point of E which is a contradiction since p is a limit point of E.
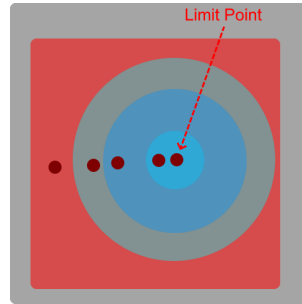
Corollary 5.1.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since $\emptyset \in$ E, all finite set must be closed.

Proof

Let p be a limit point of finite set E. By theorem 5.1.4, then any $N_r(p)$ contain infinite
q $\in$ E so E is an infinite set which is a contradiction since E is finite.
So p cannot be limit point of E and thus, E has no limit points. Since finite set E
contains all its limit points because there are no limit points, then E is closed.

Theorem 5.1.6: De Morgan's Laws

Let $E_1, E_2, ...$ be a collection of sets. Then, $(\cup \ E_x)^c = \cap \ (E_x{}^c)$.

Proof

If p $\in (\cup \ E_x)^c$, then p $\notin (\cup \ E_x)$.
Thus, p $\notin E_x$ for any x so p $\in E_x^c$ for all x. Thus, p $\in \cap \ (E_x^c)$ so $(\cup \ E_x)^c \subset \cap \ (E_x^c)$.
If p $\in \cap \ (E_x^c)$, then p $\in E_x^c$ for all x.
Thus, p $\notin E_x$ for any x so p $\notin \cup \ E_x$. Thus, p $\in (\cup \ E_x)^c$ so $\cap \ (E_x^c) \subset (\cup \ E_x)^c$.
Thus, $(\cup \ E_x)^c = \cap \ (E_x^c)$.
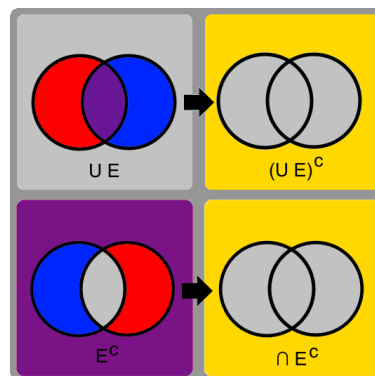
Theorem 5.1.7: Open set $\to$ Closed complement

A set E is open if and only if E^c is closed

Proof

Suppose E is open. Let x be a limit point of E^c.
Then for every r > 0, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.
Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.
Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.
Since $x \notin E$, x is not a limit point of E. Then there exists a r > 0 such that any $p \in N_r(x)$ is not in E. Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E. Since any $x \in E$ is an interior point of E, then E is open.

Corollary 5.1.8: Closed set $\to$ Open complement

A set F is closed if only only if F^c is open.

Proof

From theorem 5.1.7, let $E = F^c$

Theorem 5.1.9: Union open $\to$ open and Intersection closed $\to$ closed
   (a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup \, G_x$ is open.
   Proof

   If $p \in \cup \, G_x$, then $p \in G_x$ for at least one x. Let $\overline{x}$ be such an x.
   Since $G_{\overline{x}}$ is open, then p is an interior point of $G_{\overline{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\overline{x}} \subset \cup \, G_x$. So p is an interior point of $\cup \, G_x$.
   Since any $p \in \cup \, G_x$ is an interior point, then $\cup \, G_x$ is open.

   (b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap \, F_x$ is closed.
   Proof

   By theorem 5.1.7, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup \, F_x^c$ is open.
   Thus, again by theorem 5.1.7, $(\cup \, F_x^c)^c$ is closed.
   By theorem 5.1.6, $(\cup \, F_x^c)^c = \cap \, (F_x^c)^c = \cap \, F_x$.

   (c) If $G_1, ..., G_n$ is a finite collection of open sets, then $\cap_{x=1}^n \, G_x$ is open.
   Proof

   If $p \in \cap_{x=1}^n \, G_x$, then $p \in G_x$ for all G_x for x = {1, 2, ... , n}.
   Since each G_x is open, then for any G_x, there is a $N_{r_x}(p) \subset G_x$.
   Let $r = \min(r_1, r_2, ..., r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x.
   So, $N_r(p) \subset \cap_{x=1}^n \, G_x$ and thus, p is an interior point of $\cap_{x=1}^n \, G_x$ so $\cap_{x=1}^n \, G_x$ is open.

   Infinite + Closed: $G_i = (-1/i, 1/i)$ Infinite + Open: $G_i = (-i, i)$

   (d) If $F_1, ..., F_n$ is a finite collection of closed sets, then $\cup_{x=1}^n \, F_x$ is closed.
   Proof

   By theorem 5.1.7, any F_x^c is open. Since $F_1^c, ..., F_n^c$ is a finite collection of open set, then by part (c), $\cap_{x=1}^n \, F_x^c$ is open.
   Thus, again by theorem 5.1.7, $(\cap_{x=1}^n \, F_x^c)^c$ is closed.
   By theorem 5.1.6, $(\cap_{x=1}^n \, F_x^c)^c = \cup_{x=1}^n \, (F_x^c)^c = \cup_{x=1}^n \, F_x$.

   Infinite + Closed: $F_i = [-1/i, 1/i]$ Infinite + Open: $F_i = [1/i, \infty)$

Theorem 5.1.10: E' is closed

Let E $\subset$ X. Then, (E')' $\subset$ E'. Thus, E' is closed.

Proof

If x $\in$ (E')', then for every $N_{r_1}(x)$, there is a y $\neq$ x where y $\in$ E'.
Since y $\in$ E', then for every $N_{r_2}(y)$, there is a z $\neq$ y where z $\in$ E.
Let r $= r_1 + r_2$.
Then for every $N_r(x)$, there exists a z $\neq$ x where z $\in$ E. Thus, x $\in$ E' so (E')' $\subset$ E'.

Theorem 5.1.11: E^o is open

Let E $\subset$ X. Then, E^o is open.

Proof

If p $\in E^o$, there is a r > 0 such that $N_r(p) \subset$ E.
Then for $0 < s < r$, $N_s(p) \subset N_r(p)$ so any q $\in N_s(p)$ is q $\in E^o$.
Since any p $\in E^o$ have a $N_s(p) \subset E^o$, then E^o is open.

5.2   Intervals and Balls

Definition 5.2.1: Segments and Intervals

In $\mathbb{R}$, a segement is an open interval (a,b) = { x $\in \mathbb{R}$: a $<$ x $<$ b }
In $\mathbb{R}$, a interval is a closed interval [a,b] = { x $\in \mathbb{R}$: a $\leq$ x $\leq$ b }

Definition 5.2.2: Open Balls

In $\mathbb{R}^k$, an open ball of radius r > 0 centered at p is:
$$N_r(p) = \{ \text{ x} \in \mathbb{R}^k : |x - p| < \text{r} \} = \{ \text{ x} \in \mathbb{R}^k : \text{d(x,p)} < \text{r} \}$$
A closed ball has d(x,p) $\leq$ r.

Definition 5.2.3: Convex

E $\subset \mathbb{R}^k$ is convex if for all x,y $\in$ E and t $\in$ [0,1], tx + (1-t)y $\in$ E.

Example

Balls in $\mathbb{R}^k$ are convex

Let x,y $\in$ open ball $N_r(p)$. Let z = tx + (1-t)y for t $\in$ [0,1].
Since $|x - p| < r$ and $|y - p| < r$:
$$|z - p| = |tx + (1 - t)y - p| = |tx + (1 - t)y - tp + (t - 1)p|$$
$$= |t(x - p) + (1 - t)(y - p)| \leq t|(x - p)| + (1 - t)|(y - p)|$$
$$< \text{tr} + \text{(1-t)r} = \text{r}$$
Thus, z $\in N_r(p)$ so balls are convex. Same proof applies to closed balls.

Definition 5.2.4: Dense

E $\subset$ X is dense if every x $\in$ X is either in E or a limit point of E.

Example

Let X $= \mathbb{R}$. Then, E $= \mathbb{Q}$ is dense in $\mathbb{R}$.

Fix x $\in \mathbb{R}$ and r > 0. There is a q $\in \mathbb{Q}$ such that x-r $<$ q $<$ x. So for any r > 0 and
q $\in \mathbb{Q}$, q $\neq$ x and q $\in N_r(x)$. Thus, every x $\in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

6   Closure, Open Relative, & Compact

6.1   Closure

Definition 6.1.1: Closure

Let E $\subset$ metric space X and E' be the set of all limit points of E in X.

Then the closure of E: $\overline{E}$ = E $\cup$ E'

with the properties:

(a) $\overline{E}$ is closed

Proof

> Suppose x $\in$ X, but x $\notin$ $\overline{E}$. Thus, x $\in$ $\overline{E}^c$.
> Thus, there is a $N_r(x) \subset \overline{E}^c$ since else there is always a p $\in N_r(x)$ where p $\in \overline{E}$ so x is a limit point of $\overline{E}$ so x $\in \overline{E}$. Thus, $\overline{E}^c$ is open so $\overline{E}$ is closed by theorem 5.1.7.

(b) E = $\overline{E}$ if and only if E is closed

Proof

> If E = $\overline{E}$, then by part (a), E is closed.
> If E is closed, then E' $\subset$ E so E = E $\cup$ E' = $\overline{E}$.

(c) $\overline{E} \subset$ F for every closed F $\subset$ X such that E $\subset$ F

Proof

> If closed set F, then F' $\subset$ F. Since E $\subset$ F, then E' $\subset$ F' $\subset$ F so $\overline{E} \subset$ F.

Theorem 6.1.2: sup(E) $\in \overline{E}$

Let non-empty set of real numbers, E, be bounded above. Let y = sup(E).

Then, y $\in \overline{E}$. Thus, y $\in$ E if E is closed and y $\notin$ E if E is open in $\mathbb{R}$.

Proof

> If y $\in$ E, then y $\in \overline{E}$. Suppose y $\notin$ E.
> For every h > 0, there exists a x $\in$ E such that y-h < x < y otherwise y-h is an upper bound for E which is a contradiction since y = sup(E).
> Thus, y is a limit point of E so y $\in$ E'.
> If E is closed, then y $\in$ E since y $\in$ E'. Also, y $\in \overline{E}$.
> If E is open, then any $N_r(y) \not\subset$ E since $N_r(y)$ in $\mathbb{R}$ must contain a γ > y so y $\notin E^o$.

6.2   Open Relative

Definition 6.2.1: Open Relative

Suppose E $\subset$ Y $\subset$ metric space X.

Then E is open relative to Y if for each p $\in$ E:

There is an r > 0 such that for any q $\in$ Y where d(q,p) < r, then q $\in$ E.

Theorem 6.2.2: E is open relative to Y $\subset$ X if E = Y $\cap$ G and G is open in X

Suppose E $\subset$ Y $\subset$ X.

E is open relative to Y if and only if E = Y $\cap$ G for some open G $\subset$ X.

Proof

Suppose E is open relative to Y.
Then for each p $\in$ E, there is a $r_p > 0$ such that for any q $\in$ Y where d(p,q) $< r_p$, then q $\in$ E.
Since Y $\subset$ X, let V_p be the set of all q $\in$ X such that d(p,q) $< r_p$ and define G = $\cup_{p \in E} V_p$. Since V_p is open by theorem 5.1.3, then by theorem 5.1.9a, open G $\subset$ X.
Since p $\in V_p$ for all p $\in$ E, then E $\subset$ G $\cap$ Y. Also, by construction, then $V_p \cap$ Y $\subset$ E so G $\cap$ Y $\subset$ E. Thus, E = Y $\cap$ G.
If G is open in X and E = G $\cap$ Y, then every p $\in$ E has a $V_p \subset$ G.
Then, $V_p \cap$ Y $\subset$ G $\cap$ Y = E so E is open relative to Y.

6.3   Compact Sets

Definition 6.3.1: Open Cover

An open cover of set E $\subset$ X is a collection of open $G_1, G_2, ... \subset$ X
such that E $\subset \cup G_i$.

Definition 6.3.2: Compact

K $\subset$ X is compact if every open cover of K contains a finite subcover.

If $G_1, G_2, ...$ is an open cover of K, then K $\subset \cup_{i=1}^{n} G_i$ for some n.

Theorem 6.3.3: A compact set is compact in every metric space

Suppose K $\subset$ Y $\subset$ X.

Then K is compact relative to X if and only if K is compact relative to Y.

Proof

Suppose K is compact relative to X.
Let $V_1, V_2, ...$ be sets open relative to Y such that K $\subset \cup V_x$. Then by theorem 6.2.2 for each V_x, there is a G_x open relative to X where V_x = Y $\cap G_x$.
Since K is compact relative to X, then there is a n such that K $\subset G_{x_1} \cup ... \cup G_{x_n}$.
Thus, K = K $\cap$ Y $\subset (\cup_{i=1}^{n} G_{x_i}) \cap$ Y = $(\cup_{i=1}^{n} G_{x_i} \cap$ Y) = $\cup_{i=1}^{n} V_{x_i}$.
Since there are open $V_{x_1}, ..., V_{x_n}$ where K $\subset \cup_{i=1}^{n} V_{x_i}$ so K is compact relative to Y.
Suppose K is compact relative to Y.
Let open $G_1, G_2, ... \subset$ X such that X $\subset \cup G_x$. For each G_x, let V_x = Y $\cap G_x \subset$ Y.
Since K is compact relative to Y, there is a n such that K $\subset \cup_{i=1}^{n} V_{x_i}$.
Thus, K $\subset \cup_{i=1}^{n} V_{x_i} = \cup_{i=1}^{n} (Y \cap G_{x_i}) \subset \cup_{i=1}^{n} G_{x_i}$ so K is compact relative to X.
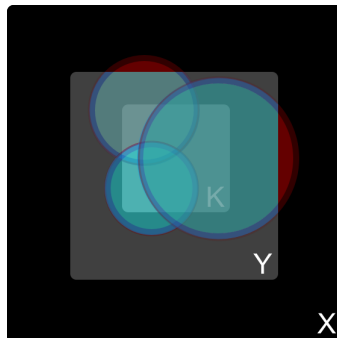
Theorem 6.3.4: A compact set is closed

Compact subsets of metric spaces are closed

Proof

Let compact $K \subset X$. Suppose $p \in X$, but $p \notin K$ so $p \in K^c$.

If $q \in K$, let W_q be a neighborhood of q with $r < \frac{1}{2}d(p,q)$. Let $V_{p,q}$ be a neighbohood of p with $r < \frac{1}{2}d(p,q)$. Since K is compact, then there are finite points $q_1, ..., q_n$ such that $K \subset W$ where $W = W_{q_1} \cup ... \cup W_{q_n}$.

Let $V = V_{p,q_1} \cap ... \cap V_{p,q_n}$, then $K \cap V \subset W \cap V = \emptyset$ so $V \subset K^c$.

Since there is a neighborhood V for $p \in K^c$ where $V \subset K^c$, then every $p \in K^c$ is an interior point so K^c is open. Then by theorem 5.1.7, K is closed.
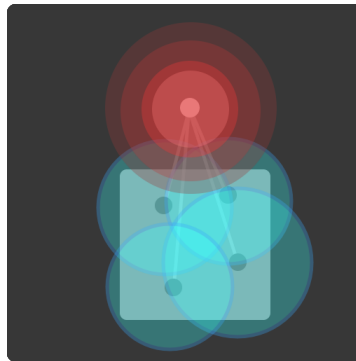
Theorem 6.3.5: If closed E ⊂ compact set K, E is compact

Closed subsets of compact sets are compact

Proof

Suppose $F \subset K \subset X$ where F is closed relative to X and K is compact.

Let $V_1, V_2, ...$ be an open cover for F. Let open set F^c be all $k \in K$ where $k \notin F$.

$K = F \cup F^c \subset V_1 \cup V_2 \cup ... \cup F^c$

Thus, $V_1 \cup V_2 \cup ... \cup F^c$ is an open cover for K.

Since K is compact, there is a finite subcover Ω that covers K and thus, finite subcover Ω covers $F \cup F^c$.

Remove F^c from Ω. Since finite subcover Ω - F^c covers F, then F is compact.

Corollary 6.3.6: Closed F ∩ compact K = compact

If F is closed and K is compact, then $F \cap K$ is compact

Proof

Since K is compact, then K is closed by theorem 6.3.4.

Then, by 5.1.9b, $F \cap K$ is closed.

Since $F \cap K \subset K$, then by theorem 6.3.5, $F \cap K$ is compact.

Theorem 6.3.7: Nonempty $\cap_{i=1}^{n} K_i \to$ nonempty $\cap K_i$

For compact sets $K_1, K_2, ... \subset$ X where any intersection of finite K_i is nonempty, then $\cap K_i$ is nonempty

Proof

Fix K_1. If there is a k $\in K_1$ where k $\in K_i$ for all i, then k $\in \cap K_i$ so $\cap K_i \neq \emptyset$.
Suppose for every k $\in K_1$, k $\notin K_i$ for some i.
Then for every k $\in K_1$, there is a K_i such that p $\notin K_i$ so p $\in K_i^c$.
Thus, $K_2^c, k_3^c, ...$ form an open cover for K_1. Since K_1 is compact, there is a n where $K_1 \subset K_{i_1}^c \cup ... \cup K_{i_n}^c$. But then, $K_1 \cap K_{i_1} \cap ... \cap K_{i_n} = \emptyset$ which is a contradiction.

Corollary 6.3.8: Nonempty K_i where $K_{i+1} \subset K_i \to$ nonempty $\cap K_i$

If $K_1, K_2, ...$ is a sequence of nonempty compact sets such that $K_{i+1} \subset K_i$, then $\cap K_i$ is nonempty

Proof

Since each K_i is nonempty and if $i_1 < ... < i_n$, then $K_{i_1} \cap ... \cap K_{i_n} = K_{i_n}$ is nonempty, then by theorem 6.3.7, $\cap K_i$ is nonempty.

Theorem 6.3.9: Nonempty intervals I_n where $I_{n+1} \subset I_n \to$ nonempty $\cap I_n$

If $I_1, I_2, ...$ is a sequence of intervals in $\mathbb{R}^1$ such that $I_{n+1} \subset I_n$, then $\cap I_n$ is nonempty.

Proof

Let $I_n = [a_n, b_n]$ and thus, each I_n is nonempty. If $n_1 < ... < n_m$, then $I_{n_1} \cap ... \cap I_{n_m} = [a_{n_m}, b_{n_m}]$ is nonempty. Thus, by theorem 6.3.7, $\cap I_n$ is nonempty.

Theorem 6.3.10: p $\in$ E' exists if infinite E $\subset$ compact K

If E is an infinite subset of compact set K, then E has a limit point in K

Proof

If no p $\in$ K is a p $\in$ E', then each p would have a neighbohood V_p contains at most p $\in$ E if p $\in$ E. Thus, there is no finite subcover that covers E and thus, there is no finite subcover that covers K since E $\subset$ K which contradicts K is compact.

Definition 6.3.11: K-cells

The set of all x $= (x_1, ..., x_k) \in \mathbb{R}^k$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$

Theorem 6.3.12: K-cells are compact

Every k-cell is compact

Proof

Let k-cell I consists of all x $= (x_1, ..., x_k)$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.
Let $\delta = \sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}$. Thus, $|x - y| \leq \delta$ for $x, y \in$ I.
Suppose there exists an open cover $G_1, G_2, ...$ of I which contain no finite subcover.
Let $c_i = \frac{a_i + b_i}{2}$. Then each interval splits into $[a_i, c_i]$ and $[c_i, b_i]$ for i $\in$ [1,k] so there now exists 2^k k-cells Q_i whose union is I.
At least one Q_i cannot be covered else I would be covered. Then subdivide Q_i as before and repeating the process so $Q_{i+1} \subset Q_i$ and each are not covered.
However, there is a point x* $\in Q_{i_j}$ for all j such that $N_r(x^*) \subset$ G so Q_{i_1} is covered which is a contradiction.

Theorem 6.3.13: Heine-Borel Theorem

If a set E $\subset \mathbb{R}^k$ has one of the three properties, then it has the other two:
(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

Proof

Suppose E is closed and bounded.
Then there exists a M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$ such that d(p,q) < M for all p $\in$ E.
Thus, there is a k-cell K = $[-M + q_1, q_1 + M] \times ... \times [-M + q_k, q_k + M]$ such that E $\subset$ K. Then by theorem 6.3.12, K is compact and thus by theorem 6.3.5, E is compact so (a) $\rightarrow$ (b).
Then by thereom 6.3.10, any infinite subset of E has a limit point in E so (b) $\rightarrow$ (c).
Suppose E is not bounded.
Then there exists p $\in$ E such that d(p,q) > M for any M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$.
Let S $\subset$ E be such points p.
Then S is infinite else there is a maximal p and thus, p is bounded. Thus, S is infinite and contains no limit points in E since any $d(p_1, p_2) > M$ which contradicts that every infinite subset of E has a limit point in E. Thus, E is bounded.
Suppose E is not closed.
Then there exists a p $\in$ E', but p $\notin$ E. Since p is a limit point, then there is a q $\in$ E such that $\frac{1}{n+1} <$ d(q,p) $< \frac{1}{n}$ for n = {1, 2, ...}.
Let S $\subset$ E be such points q.
Thus, p is the only limit point of S since for r $< \frac{1}{n}$, any $N_r(q_i)$ contains no points of S other than q_i since $d(q_i, q_j) > \frac{1}{n}$ for any $q_1, q_2 \in$ S.
Thus, S is infinite, but the only p $\in$ S' is p $\notin$ E which contradicts that every infinite subset of E has a limit point in E. Thus, E is closed. So, (c) $\rightarrow$ (a).

Theorem 6.3.14: Weierstrass Theorem

Every bounded infinite set E $\subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof

Since E is bounded, then there exists a k-cell K such that E $\subset$ K. Since K is compact, then by theorem 6.3.10, E has a limit point in K and thus, in $\mathbb{R}^k$.

7 Perfect and Connected Sets

7.1 Perfect Sets

Definition 7.1.1: Perfect Set

E $\subset$ X is perfect if E is closed and if every p $\in$ E is p $\in$ E'

Theorem 7.1.2: Perfect sets are uncountable

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then, P is uncountable.

Proof

Since P has limit points, then by theorem 5.1.4, P is infinite.
Suppose P is countable. Then let $x_1, x_2, ... \in$ P.
Let V_i be a neighborhood of x_i where y $\in V_i$ for any y $\in \mathbb{R}^k$ such that $|y - x_i| <$ r.
Thus, the $\overline{V_i}$ is the set of all y $\in \mathbb{R}^k$ such that $|y - x_i| \leq$ r.
Since every x_i are limit points, then any $V_i \cap$ P is not empty where there is a V_{i+1}
  (a) $\overline{V_{i+1}} \subset V_i$
  (b) $x_i \notin \overline{V_{i+1}}$
  (c) $V_{i+1} \cap$ P is nonempty
Let $K_i = \overline{V_i} \cap$ P. Since $\overline{V_i}$ is closed and bounded, then by theorem 6.3.11, $\overline{V_i}$ is compact. Since $x_i \notin K_{i+1}$, then no $x_i \in$ P is $x_i \in \cap K_i$. Since $K_n \subset$ P, then $\cap K_i$ is empty which contradicts corollary 6.3.8 since each K_i is nonempty and $K_{i+1} \subset K_i$.
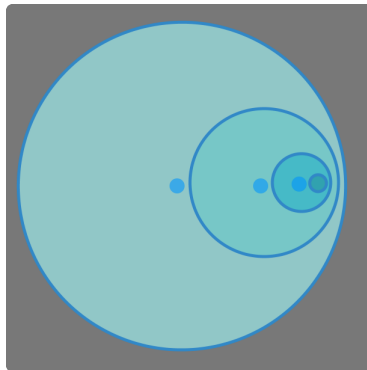
Corollary 7.1.3: $\mathbb{R}$ is not countable

Every interval [a,b] is uncountable. Thus, $\mathbb{R}$ is uncountable.

Proof

Since [a,b] is closed and every p $\in$ [a,b] is a limit point, then nonempty set [a,b] is perfect. Thus, by theorem 7.1.2, [a,b] is uncountable.

Definition 7.1.4: Cantor Set

There exists perfect segments in $\mathbb{R}^1$ which contain no segment.

Let $E_0 = [0,1]$.

For E_1, remove $(\frac{1}{3},\frac{2}{3})$. Thus, $E_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$.

For E_2, remove $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$. Thus, $E_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{3}{9}] \cup [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},1]$.

Continuing such a sequence, the set of compact sets E_n are such that:

(a) $E_{n+1} \subset E_n$

(b) E_n is the union of 2^n intervals each of length 3^{-n}.

P $= \cap\ E_n$ is called the Cantor set. P is compact and nonempty.

Thus, any segment of form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ where k,m $\in \mathbb{Z}_+$ has no points in common with P. Since any segment (a,b) contain a segment of such a form since $3^{-m} < \frac{b-a}{6}$, then P contains no segment.

Let x $\in$ P and segment S contain x. Let I_n be an interval of E_n containing x. Then choose a large enough n so $I_n \subset$ S.

Let x_n be an endpoint of I_n where $x_n \neq$ x and thus, x is a limit point. Since P is closed and every p $\in$ P is p $\in$ P', then P is perfect.

7.2   Connected Sets

Definition 7.2.1: Connected Set

A,B $\subset$ X are separated if both A $\cap\ \overline{B}$ and $\overline{A} \cap$ B are empty.

E $\subset$ X is connected if E is not the union of two nonempty separated sets.

Separated sets are disjoint, but disjoint sets need not be separated.

Theorem 7.2.2: All points between points in connected sets exists

E $\subset \mathbb{R}^1$ is connected if and only if:

If x,y $\in$ E and x < z < y, then z $\in$ E.

Proof

If there exists x,y $\in$ E and z $\in$ (x,y) such that z $\notin$ E, then E $= A_z \cup B_z$ where $A_z =$ E $\cap\ (-\infty, z)$ and $B_z =$ E $\cap\ (z, \infty)$.

Since x $\in A_z$ and y $\in B_z$, then A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, then A_z and B_z are separated. Thus, E is not connected.

Suppose E is not connected. Then, there are nonempty separated sets A and B such that A $\cup$ B = E. Pick x $\in$ A, y $\in$ B where x < y. Let z = sup(A $\cap$ [x,y]).

Since, z $\in \overline{A}$ so z $\notin$ B, then x $\leq$ z < y. If z $\notin$ A, then x < z < y so z $\notin$ E.

If z $\in$ A, then z $\notin \overline{B}$ and thus, there exists a z_1 such that z $< z_1 <$ y and $z_1 \notin$ B. Then, x $< z_1 <$ y so $z_1 \notin$ E.

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8   Convergent and Cauchy Sequences

8.1   Convergent Sequences

Definition 8.1.1: Convergent Sequence

A sequence $\{x_n\}$ in metric space X converge if there is a $x \in X$ such that:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for all $n \geq N$, $d(x_n,\text{x}) < \epsilon$

Then, $\{x_n\}$ converges to x: $\lim_{n \to \infty} x_n = \text{x}$

If $\{x_n\}$ does not converge, then it diverges.
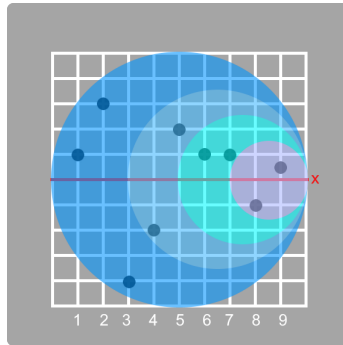
Example

(a) Let $x_n = \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\lim_{n \to \infty} x_n = 0$

Proof

> For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$. Then:
> $$d(x_n,0) = |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

(b) Let $x_n = (-1)^n + \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\{x_n\}$ diverges.

Proof

> $\lim_{n \to \infty} x_n = \lim_{n \to \infty} (-1)^n + \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} (-1)^n$
> Since $(-1)^n$ alternates between -1 and 1, then $\{x_n\}$ diverges.

Theorem 8.1.2: A convergent sequence is unique and bounded

(a) $\{p_n\}$ converges to $p \in X$ if and only if every $N_r(p)$ contains p_n for all, but finitely many n.

Proof

> Suppose $p_n \to p$. Then for $N_\epsilon(p)$, any $q \in X$ such that $d(q,p) < \epsilon$ is $q \in N_\epsilon(p)$. Since $p_n \to p$, there is a N such that for $n \geq N$, $d(p_n,p) < \epsilon$.
> Thus, for $n \geq N$, $p_n \in N_\epsilon(p)$.
> Suppose every $N_r(p)$ contains p_n for all, but finitely many n.
> For $\epsilon > 0$, let $N_\epsilon(p)$ be the set of all $q \in X$ such that $d(p,q) < \epsilon$. Thus, there exists a N such that $p_n \in N_\epsilon(p)$ if $n \geq N$.
> Thus, $d(p_n,p) < \epsilon$ so $p_n \to p$.

(b) If p,p' $\in$ X and $\{p_n\}$ converges to p and p', then p = p'.

Proof

> For $\epsilon > 0$, there exists N,N' such that:
> $$d(p_n,\text{p}) < \frac{\epsilon}{2} \text{ for n} \geq \text{N} \qquad d(p_n,\text{p'}) < \frac{\epsilon}{2} \text{ for n} \geq \text{N'}$$
> Then for $n \geq \max(N,N')$, $d(p,p') \leq d(p,p_n) + d(p_n,p') < \epsilon$.
> Thus, p = p'.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
   Proof

> If $\{p_n\} \to$ p, there is a N such that for n > N, $d(p_n,\text{p}) < 1$.
> Let r = max($d(p_1,\text{p})$, ... , $d(p_N,\text{p})$, 1). Thus for all n, $d(p_n,\text{p}) \le$ r.

(d) If E $\subset$ X and p $\in$ E', there is a $\{p_n\}$ in E such that p = $\lim_{n\to\infty} p_n$.
   Proof

> Since p $\in$ E', then for each n $\in \mathbb{Z}_+$, there is a $p_n \in$ E such that $d(p_n,\text{p}) < \frac{1}{n}$.
> For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$ so for n $\ge$ N, $d(p_n,\text{p}) < \frac{1}{n} \le \frac{1}{N} < \epsilon$.
> Thus, p = $\lim_{n\to\infty} p_n$.

Theorem 8.1.3: Arithmetic Operations for sequences

Suppose $\{s_n\},\{t_n\} \in \mathbb{C}$ where $\lim_{n\to\infty} s_n =$ s and $\lim_{n\to\infty} t_n =$ t.

(a) $\lim_{n\to\infty} s_n + t_n =$ s + t
   Proof

> For $\epsilon > 0$, there exists N_1, N_2 such that
> $\quad |s_n - s| < \frac{\epsilon}{2}$ for n $\ge N_1 \qquad |t_n - t| < \frac{\epsilon}{2}$ for n $\ge N_2$
> If N = $\max(N_1,N_2)$, then for n $\ge$ N:
> $\quad |s_n + t_n - s + t| \le |s_n - s| + |t_n - t| < \epsilon$

(b) $\lim_{n\to\infty} cs_n =$ cs and $\lim_{n\to\infty} c + s_n =$ c + s
   Proof

> For $\epsilon > 0$, there exists a N such that
> $\quad |s_n - s| < \frac{\epsilon}{|c|} \qquad$ for n $\ge$ N
> $\quad |cs_n - cs| \le |c| \cdot |s_n - s| < \epsilon$

(c) $\lim_{n\to\infty} s_n t_n =$ st
   Proof

> Note $s_n t_n$ - st = $(s_n - s)(t_n - t) + t(s_n$ - s$) + s(t_n$ - t$)$.
> For $\epsilon > 0$, there exists N_1,N_2 such that
> $\qquad |s_n - s| < \sqrt{\epsilon}$ for n $\ge N_1 \qquad |t_n - t| < \sqrt{\epsilon}$ for n $\ge N_2$
> If N = $\max(N_1,N_2)$, then for n $\ge$ N, $|(s_n - s)(t_n - t)| < \epsilon$.
> Thus, $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$.
> $\qquad \lim_{n\to\infty} (s_n t_n - st) = \lim_{n\to\infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$
> $\qquad\qquad\qquad = 0 + t \cdot 0 + s \cdot 0 = 0$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ where $s_n, s \ne 0$
   Proof

> Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if n $\ge$ m so $|s_n| > \frac{1}{2}|s|$ for n $\ge$ m.
> For $\epsilon > 0$, there is a N > m such that for n $\ge$ N, $|s_n - s| < \frac{1}{2}|s|^2\epsilon$.
> Thus, for n $\ge$ N, $|\frac{1}{s_n} - \frac{1}{s}| = |\frac{s_n - s}{s_n s}| < \frac{2}{|s|^2}|s_n - s| < \epsilon$.

Theorem 8.1.4: Extension to $\mathbb{R}^k$

(a) Suppose $x_n \in \mathbb{R}^k$ and $x_n = (\alpha_{n_1}, \ldots, \alpha_{n_k})$. Then $\{x_n\}$ converges to x $=$ $(\alpha_1, \ldots, \alpha_k)$ if and only if $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].

Proof

> Suppose $\{x_n\}$ converges to x $= (\alpha_1, \ldots, \alpha_k)$.
> Since for any i $\in$ [1,k]:
> $$|\alpha_{n_i} - \alpha_i| \leq \sqrt{|\alpha_{n_1} - \alpha_1|^2 + \ldots + |\alpha_{n_k} - \alpha_k|^2} = |x_n - x| < \epsilon.$$
> Then, $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$.
>
> Suppose $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].
> Then for $\epsilon > 0$, there is an N such that for n $\geq$ N:
> $$|\alpha_{n_i} - \alpha_i| < \tfrac{\epsilon}{\sqrt{k}} \text{ for i} \in [1,k]$$
> $$|x_n - x| = \sqrt{\sum_{i=1}^{k} |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot (\tfrac{\epsilon}{\sqrt{k}})^2} = \epsilon$$

(b) Suppose $\{x_n\}, \{y_n\} \in \mathbb{R}^k$ and $\{\beta_n\} \in \mathbb{R}$ and $x_n \to$ x, $y_n \to$ y, $\beta_n \to \beta$.
$\lim_{n\to\infty} x_n + y_n =$ x+y $\lim_{n\to\infty} x_n \cdot y_n =$ x·y $\lim_{n\to\infty} \beta_n x_n = \beta$x

Proof

> By part a, then $\lim_{n\to\infty} x_{n_i} + y_{n_i} = x_i + y_i$ so $\{x_n + y_n\} \to$ x+y.
> Also, $\lim_{n\to\infty} \sum_{i=1}^{k} x_{n_i} y_{n_i} = \sum_{i=1}^{k} x_i y_i$ so $\{x_n \cdot y_n\} \to$ x·y.
> Also, $\lim_{n\to\infty} \beta_i x_{n_i} = \beta_i x_i$ so $\{\beta_n x_n\} \to \beta x$.

8.2   Subsequences

Definition 8.2.1: Subsequence

For sequence $\{p_n\}$, let $\{n_k\} \in \mathbb{Z}_+$ where $n_k < n_{k+1}$.
Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.
If $\{p_{n_k}\}$ converges, then its limit is called a subsequential limit.

Theorem 8.2.2: $\{p_n\} \to$ p $\rightleftharpoons$ Every $\{p_{n_k}\} \to$ p

$\{p_n\}$ converges to p if and only if every subsequence converges to p

Proof

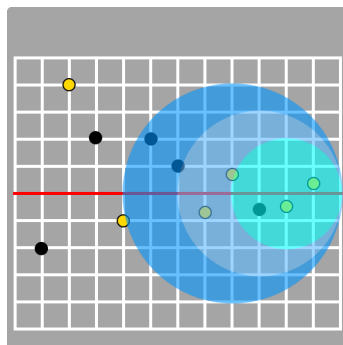> Suppose $\{p_n\}$ converges to p.
> Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, d(p_n,p) $< \epsilon$.
> Let $\{p_{n_k}\} \subset \{p_n\}$. Then for $n_k \geq$ N, $|p_{n_k} - p| < \epsilon$. Thus, $\{p_{n_k}\} \to$ p.
>
> Suppose every subsequence converges to p.
> Since $\{p_n\}$ is a subsequence of itself, then $\{p_n\}$ converges to p.

Theorem 8.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \to$ p

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence converges to p $\in$ X.

Proof

> Let E be the range of $\{p_n\}$.
> If E is finite, there is a p $\in$ E and sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $p_{n_1} = p_{n_2} = ... =$ p. Thus, $\{p_{n_k}\} \to$ p.
> If E is infinite, then by theorem 6.3.10, then there exists a p $\in$ E'.
> Then there are n_k such that d(p_{n_k},p) $< \frac{1}{k}$. Thus, $\{p_{n_k}\} \to$ p.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof

> Let E be a bounded sequence in $\mathbb{R}^k$. Since E $\cup$ E' is bounded and closed, then by theorem 6.3.13, E $\cup$ E' is compact.
> Thus by part a, E contains a convergent subsequence.

Theorem 8.2.4: The set of subsequential limits is closed

The subsequential limits of $\{p_n\}$ in metric space X form a closed subset of X

Proof

> Let E be the range of the set of all subsequential limits of $\{p_n\}$.
> If E is empty, then E is closed. If E is finite, then E' is empty so E is closed.
> Suppose E is infinite. Then, let q $\in$ E'.
> Since q $\in$ E', there is a x $\in$ E where d(x,q) $< \frac{\epsilon}{2}$.
> Since x $\in$ E, there is a $\{p_{n_k}\} \to$ x so there is a N such that for n $\geq$ N, d(p_{n_k},x) $< \frac{\epsilon}{2}$.
> Thus, d(p_{n_k},q) $\leq$ d(p_{n_k},x) + d(x,q) $< \epsilon$ so q is a subsequential limit of $\{p_n\}$.
> Thus, q $\in$ E so E is closed.
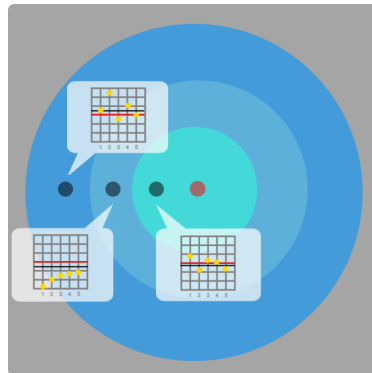
8.3   Cauchy Sequences

Definition 8.3.1: Metric Spaces

Sequence $\{p_n\} \in$ X is a Cauchy sequence if:

For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for all n,m $\geq$ N, d(p_n,p_m) $< \epsilon$

Let nonempty E $\subset$ X and S $\subset \mathbb{R}$ of d(p,q) where p,q $\in$ E.

Let sup(S) = diam(E). If $\{p_n\} \in$ X, and $p_N, p_{N+1}, ... \in E_N$, then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N \to \infty}$ diam(E_N) = 0.

Theorem 8.3.2: Cauchy sequences and its closure have the same diam

(a) If $\overline{E} \subset X$, then $\text{diam}(\overline{E}) = \text{diam}(E)$.

Proof

> Since $E \subset \overline{E}$, then $\text{diam}(E) \leq \text{diam}(\overline{E})$.
> For $\epsilon > 0$, let p,q $\in$ E'.
> Thus, there are p',q' $\in$ E such that $d(p',p) < \epsilon$ and $d(q',q) < \epsilon$. Thus:
> $d(p,q) \leq d(p,p') + d(p',q') + d(q',q) < 2\epsilon + d(p',q') \leq 2\epsilon + \text{diam}(E)$.
> Thus, $\text{diam}(\overline{E}) \leq 2\epsilon + \text{diam}(E)$ so $\text{diam}(\overline{E}) = \text{diam}(E)$.

(b) If K_n is a sequence of compact sets of X such that $K_{n+1} \subset K_n$ and $\lim_{n \to \infty} \text{diam}(K_N) = 0$, then $\cap K_n$ consist of only one point.

Proof

> Let $K = \cap K_n$. Since K_n is a sequence of compact sets, then by corollary 6.3.8, K is nonempty.
> If K contains more than one point, then $\text{diam}(K) > 0$. But since $K \subset K_n$, then $\text{diam}(K) \leq \text{diam}(K_n)$ which contradicts that $\text{diam}(K_n) \to 0$.

Theorem 8.3.3: Convergent sequences are cauchy sequences

(a) Every convergent sequence is a Cauchy sequence.

Proof

> If $p_n \to p$ and $\epsilon > 0$, there is a N such that for all $n \geq N$, $d(p,p_n) < \frac{\epsilon}{2}$. Thus, for m,n $\geq$ N:
> $\quad d(p_n,p_m) \leq d(p_n,p) + d(p,p_m) < \epsilon$.
> Thus, $\{p_n\}$ is a Cauchy sequence.

(b) If $\{p_n\}$ is a Cauchy sequence in compact metric space X, then $\{p_n\}$ converges to some p $\in$ X.

Proof

> Let $\{p_n\}$ be a Cauchy sequence in compact space X.
> Let $p_N, p_{N+1}, ... \in E_N$.
> Since $\{p_n\}$ is a Cauchy sequence, then $\lim_{N \to \infty} \text{diam}(\overline{E_N}) = 0$. Since $\overline{E_N}$ is closed in compact X, then by theorem 6.3.5, $\overline{E_N}$ is compact.
> Since $E_{N+1} \subset E_N$, then $\overline{E_{N+1}} \subset \overline{E_N}$ and thus, by theorem 8.3.2b, then there is a unique p $\in \overline{E_N}$ for every N.
> Since p $\in \overline{E_N}$, then $d(p,q) < \epsilon$ for every q $\in \overline{E_N}$ so every q $\in E_N$.
> Then for $\epsilon > 0$, there is a N_0 such that for $N \geq N_0$, $\text{diam}(\overline{E_N}) < \epsilon$.
> Thus, $d(p_n,p) < \epsilon$ for $n \geq N_0$ so $\{p_n\} \to$ p.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.

Proof

> Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Let $x_N, x_{N+1}, ... \in E_N$.
> Then for some N, $\text{diam}(E_N) < 1$. Thus, the range of $\{x_n\} = E_N \cup \{x_1, ..., x_{N-1}\}$. Thus, $\{x_n\}$ is bounded.
> Thus, the $\overline{\{x_n\}}$ is closed and bounded so by theorem 6.3.13, $\overline{\{x_n\}}$ is compact.
> Thus, by part b, $\{x_n\}$ converges to some p $\in \mathbb{R}^k$.

Definition 8.3.4: Complete

A metric space where every Cauchy sequence converges is complete.

Thus, by theorem 8.3.3, all compact and Euclidean spaces are complete.

Definition 8.3.5: Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is:
(a) monotonically increasing if $s_n \leq s_{n+1}$
(b) monotonically decreasing if $s_n \geq s_{n+1}$

Theorem 8.3.6: Monotonic sequences converge if bounded

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded

Proof

Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$.

Suppose $\{s_n\}$ is bounded.

Let s = sup(E) so $s_n \leq$ s. For every $\epsilon > 0$, there is a N such that s - $\epsilon < s_N \leq$ s else s-ϵ would be an upper bound of E which contradicts s = sup(E).

Since $\{s_n\}$ increases, then for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s so $\{s_n\} \to$ s.

Suppose $\{s_n\}$ converges to s.

Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s.

Thus, $\{s_n\}$ is bounded from above.

Suppose $s_n \geq s_{n+1}$. Let E be the range of $\{s_n\}$.

Suppose $\{s_n\}$ is bounded.

Let s = inf(E) so $s_n \geq$ s. For every $\epsilon > 0$, there is a N such that s $\leq s_N <$ s + ϵ else s+ϵ would be a lower bound of E which contradicts s = inf(E).

Since $\{s_n\}$ decreases, then for n $\geq$ N, s $\leq s_n \leq s_N <$ s + ϵ so $\{s_n\} \to$ s.

Suppose $\{s_n\}$ converges to s.

Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s $\leq s_n \leq s_N <$ s + ϵ.

Thus, $\{s_n\}$ is bounded from below.

9   Limits and Special Sequences

9.1   Upper and Lower Limits

Definition 9.1.1: Infinite limits

Let $\{s_n\}$ be a sequence of real numbers such that:

For every real M, there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \geq M$.
Then, $s_n \to +\infty$.
For every real M, there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \leq M$.
Then, $s_n \to -\infty$.

Definition 9.1.2: Upper and Lower Limits

Let $\{s_n\} \subset \mathbb{R}$ and E contain all subsequential limits of $\{s_n\}$ plus possibly $\pm\infty$.
Then, the upper limit of $\{s_n\}$:

$s^* = \sup(E) \qquad \lim_{n\to\infty} \sup(s_n) = s^*$

Then, the lower limit of $\{s_n\}$:

$s_* = \inf(E) \qquad \lim_{n\to\infty} \inf(s_n) = s_*$
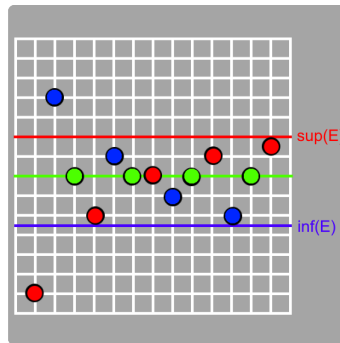
Theorem 9.1.3: Upper and Lower limits are unique

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and s^* be the upper limit of $\{s_n\}$. Then:

(a) $s^* \in E$

Proof

> If $s^* = +\infty$, then there is a $\{s_{n_k}\} \to +\infty$ so E is not bounded above.
> If $s^* \in \mathbb{R}$, then E is bounded above so $s^* \in E'$.
> Then by theorem 8.2.4, $s^* \in E$.
> If $s^* = -\infty$, then there are no subsequential limits in E. Thus, for every M, there is a N such that for $n \geq N$, $s_n \leq M$ so $-\infty \in E$.

(b) If $x > s^*$, there is a N such that for $n \geq N$, $s_n < x$

Proof

> Suppose there is a $x > s^*$ such that $s_n \geq x$ for infinitely many n.
> Then, there is a $y \in E$ where $y \geq x > s^*$ which contradicts $s^* = \sup(E)$.

(c) s^* is the only number that satisfies (a) and (b)

Proof

> Suppose p,q satisfy part a and b where $p < q$. Choose x where $p < x < q$. Since p satisfies b, then $s_n < x$ for $n \geq N$. Thus, x is an upper bound for E so $q \notin E$ since $q > x$ contradicting that q satisfies part a.

The same properties are analogous for s_*.

Theorem 9.1.4: Inf & Sup of $s_n \leq t_n$

If $s_t \leq t_n$ for n $\geq$ fixed N, then

$\lim_{n\to\infty} \inf(s_n) \leq \lim_{n\to\infty} \inf(t_n)$

$\lim_{n\to\infty} \sup(s_n) \leq \lim_{n\to\infty} \sup(t_n)$

Proof

Let E_1 be the set of extended reals x such that $\{s_{n_k}\} \to$ x for some $\{s_{n_K}\}$.

Let E_2 be the set of extended reals y such that $\{t_{n_k}\} \to$ y for some $\{s_{n_K}\}$.

Let $s^* = \sup(E_1)$, $s_* = \inf(E_1)$, $t^* = \sup(E_2)$, and $t_* = \inf(E_2)$.

Since there is a N such that $s_n \leq t_n$ for n $\geq$ N, then:

$\quad$ x $\leftarrow \{s_N, s_{N+1}, ...\} \leq \{t_N, t_{N+1}, ...\} \to$ y

Thus, for n $\geq$ N, $\inf(s_n) \leq \inf(t_n)$ and $\sup(s_n) \leq \sup(t_n)$.

9.2   Special Sequences

Theorem 9.2.1: Special sequences

(a) If p > 0, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$

Proof

For $\epsilon > 0$, let $N > \sqrt[p]{\frac{1}{\epsilon}}$. Then for n $\geq$ N, $\lim_{n\to\infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}^p} = \epsilon$

(b) If p > 0, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$

Proof

If p > 1, then let $x_n = \sqrt[n]{p}$ - 1 > 0.

$\quad p = (x_n + 1)^n = x_n^n + nx_n^{n-1} + ... + nx_n + 1 \geq nx_n + 1$

Thus, $0 < x_n \leq \frac{p-1}{n}$ so $\{x_n\} \to 0$ and thus, $\{\sqrt[n]{p}\} \to 1$.

If p = 1, then $\lim_{n\to\infty} \sqrt[n]{p} = \lim_{n\to\infty} 1 = 1$.

If 0 < p < 1, then $\frac{1}{p} > 1$. From the proof above for p > 1, $\{\sqrt[n]{\frac{1}{p}}\} \to 1$.

Thus, $\{\frac{1}{\sqrt[n]{p}}\} \to 1$ so $\{\sqrt[n]{p}\} \to 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Proof

Let $x_n = \sqrt[n]{n}$ - 1 $\geq$ 0 Then, n $= (x_n + 1)^n \geq \frac{n(n-1)}{2}x_n^2$.

Thus, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ so $\{x_n\} \to 0$ and thus, $\{\sqrt[n]{n}\} \to 1$.

(d) If p > 0 and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

Proof

Let k $\in \mathbb{Z}$ such that k > α and k > 0. For n > 2k:

$\quad (1+p)^n > \binom{n}{k}p^k = \frac{n(n-1)...(n-k+1)}{k!}p^k > \frac{n^k p^k}{2^k k!}$

Thus, $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k}n^{\alpha-k}$.

Since α - k < 0, then $\{n^{\alpha-k}\} \to 0$ so $\{\frac{n^\alpha}{(1+p)^n}\} \to 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

Proof

From part d, let $\alpha = 0$.

Thus, $\lim_{n\to\infty} \frac{1}{(1+p)^n} = 0$ and since p > 0, then $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$.

Also, $-\lim_{n\to\infty} \frac{1}{(1+p)^n} = \lim_{n\to\infty} \frac{-1}{(1+p)^n} = 0$ so $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n > $ -1.

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10   Series and Convergence Tests

10.1   Series

Definition 10.1.1: Series
For sequence $\{a_n\}$, define $\sum_{n=p}^{q} a_n = a_p + a_{p+1} + ... + a_q$.
Then associate $\{a_n\}$ with a sequence $\{s_n\}$ such that $s_n = \sum_{k=1}^{n} a_k$.
Then $\{s_n\}$ is a series with partial sums s_n.
If $\{s_n\} \to$ s, then $\sum_{n=1}^{\infty} a_n =$ s is the sum of the convergent series.
Note $a_1 = s_1$ and $a_n = s_n$ - s_{n-1}.

Theorem 10.1.2: Cauchy Criterion for series
$\sum a_n$ converges if and only if:
For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$
Proof

Suppose $\sum_{k=1}^{n} a_k$ converges.
Then by theorem 8.3.3a, $\sum_{k=1}^{n} a_k$ is a Cauchy sequence.
Then for $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N:
$\quad$ d$(\sum_{k=1}^{n} a_k , \sum_{k=1}^{m} a_k) = |\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k| = |\sum_{k=n}^{m} a_k| \leq \epsilon$

Suppose for every $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$.
$\quad |\sum_{k=n}^{m} a_k| = |\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k| = d(\sum_{k=1}^{n} a_k , \sum_{k=1}^{m} a_k) \leq \epsilon$
Thus, $\sum_{k=1}^{n} a_k$ is a Cauchy sequence and thus, convergent.

Theorem 10.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \to 0$
If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Proof

Since $\sum a_n$ converges, then by theorem 10.1.2, for $\epsilon > 0$, there is a N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| \leq \epsilon$. Then if m $=$ n $\geq$ N, $|\sum_{k=n}^{m} a_k| = |a_n| \leq \epsilon$ so $\{a_n\} \to 0$.

Example
$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + ... + \frac{1}{8}) + (\frac{1}{9} + ...) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$
Thus, $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$ which is unbounded and thus, not convergent.

Theorem 10.1.4: Convergent series $\rightleftharpoons$ Bounded sequence
A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.
Proof

Suppose $\sum a_n$ converges where $a_n \geq 0$.
Since $a_n \geq 0$, then $\{s_n\}$ is monotonic so by theorem 8.3.6, $\{s_n\}$ is bounded above.

Suppose $\{s_n\}$ is bounded where $a_n \geq 0$.
Since $\{s_n\}$ is monotonic and bounded, then by theorem 8.3.6, $\{s_n\}$ converges.

Theorem 10.1.5: Comparison Test

(a) If $|a_n| \leq c_n$ for n $\geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.
Proof

> For $\epsilon > 0$, there exists a N $\geq N_0$ such that for m $\geq$ n $\geq$ N, $\sum_{k=n}^m c_k \leq \epsilon$.
> $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \epsilon$
> Thus, $\sum a_n$ converges.

(b) If $a_n \geq d_n \geq 0$ for n $\geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.
Proof

> Suppose $\sum a_n$ converges.
> Then from part a, $\sum d_n$ converges which contradicts that $\sum a_n$ diverges.
> Thus, $\sum a_n$ diverges.

10.2   Series of Nonnegative Terms

Theorem 10.2.1: Infinite Geometric Series

If x $\in$ [0,1), then:
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
If x $\geq$ 1, the series diverges.
Proof

> If x $\neq$ 1, then using the geometric series $s_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$. Let n $\to \infty$.
> If x $\in$ [0,1), then by theorem 9.2.1e, $s_n = \frac{1}{1-x}(1 - x^{n+1}) = \frac{1}{1-x}(1 - 0) = \frac{1}{1-x}$.
> Also, by theorem 9.2.1e, if x $\geq$ 1, then the series diverges.

Theorem 10.2.2: Cauchy's Convergence Criterion

Suppose $0 \leq a_{i+1} \leq a_i$.
Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series
$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + ...$ converges.
Proof

> Let $s_n = a_1 + a_2 + ... + a_n$ and $t_k = a_1 + 2a_2 + ... + 2^k a_{2^k}$. For n $< 2^k$:
> $s_n \leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + ... + a_{2^k}$
> $\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + ... + (a_{2^k} + ... + a_{2^{k+1}-1})$
> $\leq a_1 + 2a_2 + 4a_4 + ... + 2^k a_{2^k} = t_k$
> By comparison test, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. For n $> 2^k$:
> $s_n \geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + ... + a_{2^k}$
> $= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ... + (a_{2^{k-1}+1} + ... + a_{2^k})$
> $\geq \frac{1}{2}a_1 + a_2 + 2a_4 + ... + 2^{k-1} a_{2^k} = \frac{1}{2}t_k$
> By comparison test, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem 10.2.3: P-series

$\sum \frac{1}{n^p}$ converges if p > 1 and diverges if p ≤ 1
Proof

> If p ≤ 0, then by theorem 10.1.3, $\sum \frac{1}{n^p}$ diverges.
> If p > 0, then by theorem 10.2.2, $\sum \frac{1}{n^p}$ converges only if $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges.
> Since $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$, then by theorem 10.2.1, $\sum_{k=0}^{\infty} 2^{k(1-p)}$ converges
> if $2^{1-p} < 1$ so if 1-p < 0 so p > 1.

Theorem 10.2.4: Log P-series

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$ converges if p > 1 and diverges if p ≤ 1.

Proof

Since $\frac{1}{n(\log(n))^p}$ decreases, then by theorem 10.2.2,

$\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges.

$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k}$

Then by theorem 10.2.3, $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges if p > 1 and diverges if p ≤ 1.

Thus, $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if p > 1 and diverges and p ≤ 1.

Corollary 10.2.5: Log P-series extended

$\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges if p > 1 and diverges if p ≤ 1

Proof

From theorem 10.2.4, replace n = log(n) and multiplying by $\frac{1}{n} \rightarrow \frac{1}{n \log(n)(\log(\log(n)))^p}$.

Since $\frac{1}{n \log(n)(\log(\log(n)))^p}$ decreases, by theorem 10.2.2 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)(\log(\log(2^k)))^p}$:

$\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k \log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$

Since $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$ converges by theorem 10.2.4, $\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges.

10.3   The Number e

Definition 10.3.1: Summation equivalence to e

$s_m = \sum_{n=0}^{m} \frac{1}{n!} = 1 + \sum_{n=1}^{m} \frac{1}{n!} < 1 + \sum_{n=1}^{m} \frac{1}{2^{n-1}} < 3$

$e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Theorem 10.3.2: Limit equivalence to e

$\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$

Proof

Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ and $t_n = (1 + \frac{1}{n})^n$. Using the binomial theorem:

$t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n(n-1)...(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!}(1)(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{k-1}{n})$

Thus, $t_n \leq s_n$ so $\lim_{n \to \infty} \sup(t_n) \leq e$.

If n ≥ m, then $t_n \geq \sum_{k=0}^{m} \frac{1}{k!}(1)(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{k-1}{n})$.

As n → ∞, then $\lim_{n \to \infty} \inf(t_n) \geq \sum_{k=0}^{m} \frac{1}{k!} = s_m$. As m → ∞, $\lim_{n \to \infty} \inf(t_n) \geq e$.

Theorem 10.3.3: Rapidity of convergence of e

$0 < e - s_n < \frac{1}{n!n}$

Proof

$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + ...\right) = \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!n}$

Theorem 10.3.4: e is irrational

e is irrational

Proof

Suppose r is rational. Then let e $= \frac{p}{q}$ for p,q $\in \mathbb{Z}_+$.

Thus, by theorem 10.3.3, $0 < $ e $- s_q < \frac{1}{q!q}$ so $0 < q!(e - s_q) < \frac{1}{q}$.

Since e $= \frac{p}{q}$, then q!e is an integer and $q!s_q = q!(1 + 1 + \frac{1}{2!} + ... + \frac{1}{q!})$ is an integer.

Thus, $q!(e - s_q)$ is an integer which is between 0 and $\frac{1}{q}$ and thus, a contradiction.

10.4   Root and Ratio Tests

Theorem 10.4.1: Root Test

For $\sum a_n$, let $\alpha = \lim_{n\to\infty} \sup(\sqrt[n]{|a_n|})$.

   (a) If $\alpha < 1$, $\sum a_n$ converges

   (b) If $\alpha > 1$, $\sum a_n$ diverges

   (c) If $\alpha = 1$, unclear

Proof

If $\alpha < 1$, choose β such that $\beta \in (\alpha,1)$ and N $\in \mathbb{Z}$ such that $\sqrt[n]{|a_n|} < \beta$ for n $\geq$ N.
Since $\beta \in (0,1)$, then by theorem 10.2.1, $\sum \beta^n$ converges. Then by the comparison test, $\sum a_n$ converges.

If $\alpha > 1$, then there is a a_{n_k} such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$.
Thus, $|a_n| > 1$ for infinitely many n so by theorem 10.1.3, $\sum a_n$ doesn't converge.

$\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$ have $\alpha = 1$, but $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges by theorem 10.2.3.

Theorem 10.4.2: Ratio Test

   (a) $\sum a_n$ converges if $\lim_{n\to\infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$

   (b) $\sum a_n$ diverges if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all n $\geq n_0$ for $n_0 \in \mathbb{Z}$

Proof

If $\lim_{n\to\infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$, there is a $\beta < 1$ and N such that for n $\geq$ N, $|\frac{a_{n+1}}{a_n}| < \beta$.
Then $|a_{N+1}| < \beta|a_N|$ so $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$.
Thus, $|a_{N+p}| < \beta^p|a_N|$ so $|a_n| < |a_N|\beta^{-N}\beta^n$.
Thus, by the comparison test, $\sum a_n$ converges.
If $|a_{n+1}| \geq |a_n| > 0$ for n $\geq n_0$, then by theorem 10.1.3, $\sum a_n$ diverges.

Theorem 10.4.3: Ratio convergence $\to$ Root convergence

$\lim_{n\to\infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n\to\infty} \inf(\sqrt[n]{c_n})$

$\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n\to\infty} \sup(\frac{c_{n+1}}{c_n})$

Proof

Let $\alpha = \lim_{n\to\infty} \inf(\frac{c_{n+1}}{c_n})$. If $\alpha = -\infty$, then $-\infty \leq \lim_{n\to\infty} \inf(\sqrt[n]{c_n})$ holds true.
If α is finite, there is a $\beta \leq \alpha$ and N such that for n $\geq$ N, $\frac{c_{n+1}}{c_n} \geq \beta$ so $c_{N+p} \geq \beta^p c_N$.
Then, $c_n \geq c_N\beta^{-N}\beta^n$ so $\sqrt[n]{c_n} \geq \sqrt[n]{c_N\beta^{-N}}\beta$. Thus, $\lim_{n\to\infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$.

Let $\alpha = \lim_{n\to\infty} \sup(\frac{c_{n+1}}{c_n})$. If $\alpha = \infty$, then $\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \infty$ holds true.
If α is finite, there is a $\beta \geq \alpha$ and N such that for n $\geq$ N, $\frac{c_{n+1}}{c_n} \leq \beta$ so $c_{N+p} \leq \beta^p c_N$.
Then, $c_n \leq c_N\beta^{-N}\beta^n$ so $\sqrt[n]{c_n} \leq \sqrt[n]{c_N\beta^{-N}}\beta$. Thus, $\lim_{n\to\infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$.

10.5   Power Series

Definition 10.5.1: Power series

For a sequence $\{c_n\} \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is a power series.
c_n are the coefficients and $z \in \mathbb{C}$.

Theorem 10.5.2: Radius of Convergence

For power series $\sum c_n z^n$, let $\alpha = \lim_{n \to \infty} \sup(\sqrt[n]{|c_n|})$ and R $= \frac{1}{\alpha}$.
Then $\sum c_n z^n$ converges if $|z| < $ R and diverges if $|z| > $ R.

Proof

Let $a_n = c_n z^n$. Using the root test,
$$\lim_{n \to \infty} \sup(\sqrt[n]{|a_n|}) = \lim_{n \to \infty} \sup(\sqrt[n]{|c_n z^n|})$$
$$= |z| \lim_{n \to \infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R}$$
Thus, $\sum c_n z^n$ converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$

10.6   Summation By Parts

Theorem 10.6.1: Summation by parts

For sequences $\{a_n\}$, $\{b_n\}$, let $A_n = \sum_{k=0}^{n} a_k$. Then for $0 \leq$ p $\leq$ q:
$$\sum_{n=p}^{q} a_n b_n = (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Proof

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n$$
$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$
$$= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p$$
$$= (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Theorem 10.6.2: Conditions for convergent $\sum a_n b_n$

Suppose for $\{a_n\}$, $\{b_n\}$:
- partial sums A_n of $\sum a_n$ form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n \to \infty} b_n = 0$

Then $\sum a_n b_n$ converges.

Proof

Since $\{A_n\}$ is bounded, $|A_n| \leq$ M for all n.
Since $\{b_n\}$ is monotonically decreasing and $\lim_{n \to \infty} b_n = 0$, then for $\epsilon > 0$, there is a
N such that $b_N \leq \frac{\epsilon}{2M}$. Then for N $\leq$ p $\leq$ q:
$$|\sum_{n=p}^{q} a_n b_n| = (|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p|$$
$$\leq \text{M } |\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p| = 2M b_p \leq 2M b_N \leq \epsilon$$

Corollary 10.6.3: Convergent series of Alternating Sequences

Suppose for $\{c_n\}$:

- $|c_i| \geq |c_{i+1}|$

- $c_{2i-1} \geq 0$ and $c_{2i} \leq 0$

- $\lim_{n \to \infty} c_n = 0$

Then $\sum c_n$ converges.

Proof

From theorem 10.6.2, let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

Corollary 10.6.4: Convergent power series at radius of convergence

Suppose for $\{c_n\}$:

- Radius of convergence of $\sum c_n z^n$ is 1

- $c_i \geq c_{i+1}$

- $\lim_{n \to \infty} c_n = 0$

Then $\sum c_n z^n$ converges at every point where $|z| = 1$ except possibly z = 1.

Proof

From theorem 10.6.2, let $a_n = z^n$ and $b_n = c_n$.
A_n of $\sum a_n$ form a bounded sequence since $|A_n| = |\sum_0^n z^n| = |\frac{1-z^{n+1}}{1-z}| \leq \frac{2}{|1-z|}$.

10.7   Absolute Convergence

Definition 10.7.1: Absolute convergence

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ converges non-absolutely.

Theorem 10.7.2: Absolute convergence $\to$ convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges

Proof

Since $\sum a_n$ converges absolutely, then for every $\epsilon > 0$, there is an integer N such that for m $\geq$ n $\geq$ N, $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$.
Thus, $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$ so $\sum a_n$ converges.

10.8   Addition & Multiplication of Series

Theorem 10.8.1: Addition and Scalar Multiplication

If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$ and $\sum c a_n = cA$.

Proof

Let $A_n = \sum_{k=0}^n a_k$ and $B_n = \sum_{k=0}^n b_k$.
Then $A_n + B_n = \sum_{k=0}^n a_k + b_k$ so $\lim_{n \to \infty} A_n + B_n = A + B$.
Then $\lim_{n \to \infty} c A_n = \underbrace{A + ... + A}_{c} = cA$

Definition 10.8.2: Cauchy Product

For $\sum a_n$ and $\sum b_n$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ and the product as $\sum c_n$.

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = (a_0 + a_1 z + a_2 z^2 + ... + a_n z^n)(b_0 + b_1 z + b_2 z^2 + ... + b_n z^n)$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + ...$$

Theorem 10.8.3: Conditions $\sum c_n = AB$

Suppose
- $\sum_{n=0}^{\infty} a_n$ converges absolutely
- $\sum_{n=0}^{\infty} a_n = A$
- $\sum_{n=0}^{\infty} b_n = B$
- $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then $\sum_{n=0}^{\infty} c_n = AB$.

Proof

Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$, and $\beta_n = B_n$ - B.
$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + ... + (a_0 b_n + ... + a_n b_0)$$
$$= a_0 B_n + a_1 B_{n-1} + ... + a_n B_0$$
$$= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + ... + a_n(B + \beta_0)$$
$$= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + ... + a_n \beta_0$$

Let $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + ... + a_n \beta_0$ so $C_n = A_n B + \gamma_n$.

Since a_n converges absolutely, then $\sum_{n=0}^{\infty} |a_n| = \alpha$.

Since $\sum_{n=0}^{\infty} b_n = B$, then $\beta_n \to 0$.

Then for $\epsilon > 0$, there is a N such that $|\beta_n| \leq \frac{\epsilon}{\alpha}$ for $n \geq N$.
$$|\gamma_n| \leq |\beta_0 a_n + ... + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + ... + \beta_n a_0|$$
$$\leq |\beta_0 a_n + ... + \beta_N a_{n-N}| + |a_{n-N-1} + ... + a_0|\frac{\epsilon}{\alpha}$$
$$\leq |\beta_0 a_n + ... + \beta_N a_{n-N}| + \alpha\frac{\epsilon}{\alpha}$$

Thus, with a fixed N, since $a_n \to 0$, then $\lim_{n\to\infty} |\gamma_n| \leq \epsilon$ so $\lim_{n\to\infty} \gamma_n = 0$.

Thus, $\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_n B + \gamma_n = AB$.

Theorem 10.8.4: By Cauchy Product, $\sum c_n = C$ implies C = AB

If $\sum a_n = A$, $\sum b_n = B$, $\sum c_n = C$ where $c_n = a_0 b_n + ... + a_n b_0$, then C = AB.

11   Continuity

11.1   Limits of Functions

Definition 11.1.1: Limits of functions

For metric spaces X,Y, let $E \subset X$, f maps E into Y, and $p \in E'$.

Then $\lim_{x \to p} f(x) = q$ if there is a $q \in Y$ such that:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in E$
where $d_X(x, p) < \delta$, then $d_Y(f(x), q) < \epsilon$
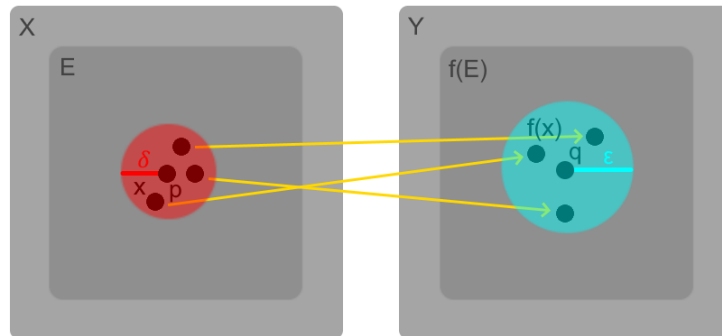
Theorem 11.1.2: Sequence definition of $\lim_{x \to p} f(x) = q$

$\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\} \in E$
where $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

Proof

Suppose $\lim_{x \to p} f(x) = q$.
For $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $d_X(x, p) < \delta$.
Choose $\{p_n\} \in E$ such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.
Then for $\delta > 0$, there is N such that for $n > N$, then $d_X(p_n, p) < \delta$ so $d_Y(f(p_n), q) < \epsilon$.

Suppose $\lim_{x \to p} f(x) \neq q$. Then there is a $\epsilon > 0$ such that for every $\delta > 0$, there is a $x \in E$ where $d_Y(f(x), q) \geq \epsilon$, but $d_X(x, p) < \delta$. Let $\delta_n = \frac{1}{n}$ and thus, there is a $\{p_n\}$ where $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$, but $\lim_{n \to \infty} f(p_n) \neq q$.

Corollary 11.1.3: A limit of a function is unique

If f has a limit at p, this limit is unique.

Proof

If $\lim_{x \to p} f(x) = q$, then by theorem 11.1.2, $\lim_{n \to \infty} f(p_n) = q$ for every $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.
Thus, if there exists $\lim_{x \to p} f(x) = q'$, then there is a $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$, but $\lim_{n \to \infty} f(p_n) = q'$ which is a contradiction.

Theorem 11.1.4: Arithemtic operations on functions of limits

Let $E \subset X$, $p \in E'$, and f(x),g(x) $\in \mathbb{C}$ so $\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$.

(a) $\lim_{x \to p}(f + g)(x) = A + B$

(b) $\lim_{x \to p}(fg)(x) = AB$

(c) $\lim_{x \to p}(\frac{f}{g})(x) = \frac{A}{B}$

11.2   Continuous Functions

Definition 11.2.1: Continuous functions on a set

Suppose X,Y are metric spaces, E ⊂ X, p ∈ E, and f maps E into Y.

f is continuous at p if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all x ∈ E
where $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$

f(p) have to be defined to be continuous.

If f is continuous at every p ∈ E, then f is continuous on E.

f is continuous at isolated points since regardless of ϵ, there is a $\delta > 0$ such that $d_X(x, p) < \delta$ is x = p so $d_Y(f(x), f(p)) = 0 < \epsilon$.
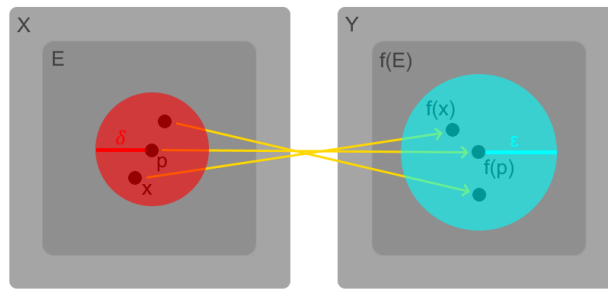
Theorem 11.2.2: Continuity at p $\rightleftharpoons$ lim f(p) = f(p)

Suppose E ⊂ X, p ∈ E, and f maps E into Y. Let p ∈ E'.

Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof

If f is continuous at p, then for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all x ∈ E where $d_X(x, p) < \delta$. Thus, $\lim_{x \to p} f(x) = f(p)$.

If $\lim_{x \to p} f(x) = f(p)$, then for every $\epsilon > 0$, there is a $\delta > 0$ where $d_Y(f(x), f(p)) < \epsilon$ for all x ∈ E where $d_X(x, p) < \delta$. Thus, f is continuous at p.

Theorem 11.2.3: Continuity Chain Rule

Suppose E ⊂ X, f: E → Y, g: f(E) → Z, and h: E → Z where h(x) = g(f(x)).

If f is continuous at p and g is continuous at f(p), then h is continuous at p.

Proof

Since g is continuous at f(p), then for $\epsilon > 0$, there is a δ_1 such that:
  $d_Z(g(y), g(f(p))) < \epsilon$ for $d_Y(y, f(p)) < \delta_1$ where y ∈ f(E)
Since f is continuous at p, there is a $\delta_2 > 0$ such that:
  $d_Y(f(x), f(p)) < \delta_1$ for $d_X(x, p) < \delta_2$ where x ∈ E
Thus, $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$ for $d_X(x, p) < \delta_2$ where x ∈ E.
Thus, h is continuous at p.
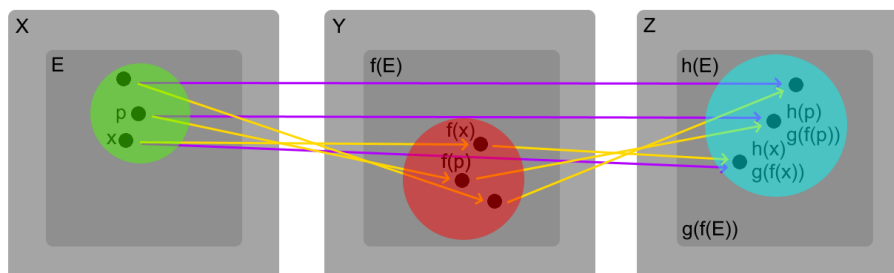
Theorem 11.2.4: Continuous functions map open sets to open sets
>   f: X → Y is continuous on X if and only if:
>   > $f^{-1}(V)$ is open in X for every open set V in Y.

Proof

Suppose f is continuous on X and V is an open set in Y.
Suppose p ∈ X and f(p) ∈ V. Since V is open, there exists $\epsilon > 0$ such that y ∈ V if $d_Y(y, f(p)) < \epsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for $d_X(x, p) < \delta$. Thus, x ∈ $f^{-1}(V)$ for $d_X(x, p) < \delta$.

Suppose $f^{-1}(V)$ is open in X for every open V in Y.
Fix p ∈ X and $\epsilon > 0$. Let V be the set of all y ∈ Y such that $d_Y(y, f(p)) < \epsilon$ so V is open and thus, $f^{-1}(V)$ is open. Thus, there exists $\delta > 0$ such that x ∈ $f^{-1}(V)$ for $d_X(x, p) < \delta$. Since x ∈ $f^{-1}(V)$, then f(x) ∈ V so $d_Y(f(x), f(p)) < \epsilon$.

Corollary 11.2.5: Continuous functions map closed sets to closed sets
>   f: X → Y is continuous on X if and only if:
>   > $f^{-1}(C)$ is closed in X for every closed set C in Y.

Proof

By theorem 11.2.4, f is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y. Let $C = V^c$. Since V is open, then C is closed.
Since $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$, then $f^{-1}(C)$ is closed since $f^{-1}(V)$ is open.

Theorem 11.2.6: Continuous functions
>   Let f,g be complex continuous functions on X.
>   Then f+g, fg, and $\frac{f}{g}$ where $g \neq 0$ for all x ∈ X are continuous on X.

Proof

If x is an isolated point, f+g, fg, and $\frac{f}{g}$ are continuous by definition. If x is a limit point, then by theorems 11.1.4 and 11.2.2, f+g, fg, and $\frac{f}{g}$ are continuous since

- $\lim_{x \to p}(f + g)(x) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) = f(p) + g(p)$
- $\lim_{x \to p}(fg)(x) = \lim_{x \to p} f(x) \lim_{x \to p} g(x) = f(p)g(p)$
- $\lim_{x \to p}(\frac{f}{g})(x) = \frac{\lim_{x \to p} f(x)}{\lim_{x \to p} g(x)} = \frac{f(p)}{g(p)}$

Theorem 11.2.7: Continuous functions on $\mathbb{R}^k$
>   (a) Let $f_1, ..., f_k$: X → $\mathbb{R}$ and f: X → $\mathbb{R}^k$ where f(x) = $(f_1(x), ... , f_k(x))$.
>   Then f is continuous if and only if $f_1, ..., f_k$ are continuous.
>   (b) If f and g are continuous mappings of X into $\mathbb{R}^k$, then f + g and f · g are continuous on X.

Proof

Since $|f_i(x) - f_i(y)| \leq \sqrt{\sum_1^k |f_i(x) - f_i(y)|^2} = |f(x) - f(y)|$, then if f is continuous, then each f_i is continuous and vice versa.

Since f,g are continuous, then by part a, each f_i, g_i are continuous. Then by theorem 11.2.6, each $f_i + g_i$ and $f_i g_i$ are continuous so by part a, f + g and f · g are continuous.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are $x_1 \cdot ... \cdot x_k$ where each x_i is continuous, rationals are polynomials divided by polynomials, and $||x| - |y|| \leq |x - y|$ implies $|x|$ is continuous.

11.3   Continuity and Compactness

Definition 11.3.1: Bounded Functions

f: E $\to \mathbb{R}^k$ is bounded if there is a M $\in \mathbb{R}$ such that f(x) $\leq$ M for all x $\in$ E.

Theorem 11.3.2: Continuous functions from compact spaces are compact

Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then f(X) is compact.

Proof

Let $\{V_\alpha\}$ be an open cover of f(X). Since f is continuous, then by theorem 11.2.4, each $f^{-1}(V_\alpha)$ is open. Since X is compact, there is n where $X \subset f^{-1}(V_{\alpha_1}) \cup ... \cup f^{-1}(V_{\alpha_n})$. Thus, $f(X) \subset V_{\alpha_1} \cup ... \cup V_{\alpha_n}$ so f(X) is compact.
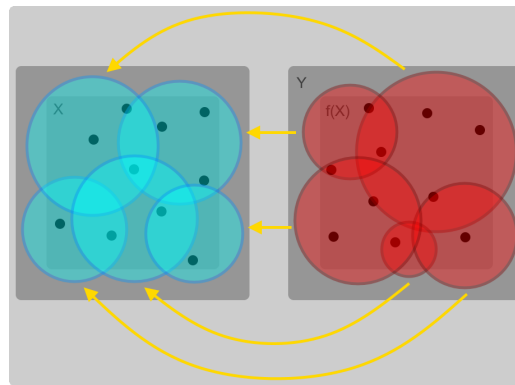
Theorem 11.3.3: Continuous functions from compact to $\mathbb{R}^k$ are bounded

If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then f(X) is closed and bounded.

Proof

By theorem 11.2.2, f(X) is compact. By theorem 6.3.13, f(X) is closed and bounded.

Theorem 11.3.4: Generalized extreme value theorem

Suppose f is a continuous real function of a compact metric space X such that M = $\sup_{x \in X}$ f(x) and m = $\inf_{x \in X}$ f(x).

Then there exists p,q $\in$ X such that f(p) = M and f(q) = m.

Proof

By theorem 11.3.3, f(X) is closed and bounded. Let M = $\sup_{x \in X}$ f(x), m = $\inf_{x \in X}$ f(x).

Since f(X) is bounded, then M,m $\in$ (f(X))' and since f(X) is closed, then M,m $\in$ f(X). Thus, there exists p,q $\in$ X such that f(p) = M and f(q) = m.

Theorem 11.3.5: If f is continuous 1-1, then f^{-1} is continuous

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then f^{-1} is a continuous mapping of Y onto X.

Proof

Let V be an open set in X.
Since V^c is closed and $V^c \subset$ compact set X, then by theorem 6.3.5, V^c is compact.
Thus by theorem 11.3.2, $f(V^c)$ is a compact subset of Y so $f(V^c)$ is closed.
Since f is 1-1 and onto, $f(V^c) = (f(V))^c$ so f(V) is open. Since from any open set V in X, f(V) is open in Y, then by theorem 11.2.4, f^{-1} is continuous.

Definition 11.3.6: Uniformly Continuous

Let f: $X \to Y$. Then f is uniformly continuous on X if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all p,q $\in$ X
where $d_X(p, q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

Theorem 11.3.7: Continuous functions on compact are uniformly continuous

Let f be a continuous mapping of a compact metric space X into metric space
Y. Then f is uniformly continuous on X.

Proof

For $\epsilon > 0$, since f is continuous, then for each p $\in$ X, there is a $\phi(p)$ such that for all
q $\in$ X where $d_X(q, p) < \phi(p)$, $d_Y(f(q), f(p)) < \frac{\epsilon}{2}$.
Let J(p) be the set of all q $\in$ X where $d_X(q, p) < \frac{1}{2}\phi(p)$.
Since the set of all J(p) is an open cover of X and since X is compact, then there is a
n such that $X \subset J(p_1) \cup ... \cup J(p_n)$. Let $\delta = \frac{1}{2}$ min($\phi(p_1), ..., \phi(p_n)$) > 0.
Then for p,q $\in$ X where $d_X(p, q) < \delta$, there is a m where $1 \le m \le n$ such that p $\in$
$J(p_m)$ so $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. Thus:

$$d_X(q, p_m) \le d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m)$$
$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Theorem 11.3.8: Continuous functions from noncompact $\nrightarrow$ uniformly continuous

Let E be a noncompact set in $\mathbb{R}^1$.

(a) There exists a continuous function which is not bounded.

(b) There exists a continuous, bounded function which is has no maximum.

(c) If E is bounded, there exists a continuous function which is not uniformly
continuous.

Proof

Suppose E is bounded so there is a $x_0 \in$ E', but $x_0 \notin$ E.
Consider f(x) $= \frac{1}{x-x_0}$ which is continuous on E, but unbounded.
For $\epsilon > 0$ and $\delta > 0$, there is a x $\in$ E such that $|x - x_0| < \delta$. Take t close enough to
x_0 so $|f(t) - f(x_0)| > \epsilon$, but $|t - x| < \delta$. Thus, f is not uniformly continuous.

Consider g(x) $= \frac{1}{1+(x-x_0)^2}$ which is continuous on E and bounded since g(x) $\in$ (0,1).
Since $\sup_{x \in E}$ g(x) $= 1$, but g(x) < 1 for all x $\in$ E, then g has no maximum on E.

11.4   Continuity and Connectedness

Theorem 11.4.1: Continuous functions map connected to connected

If f is a continuous mapping of X into Y and E is a connected subset of X,
then f(E) is connected.

Proof

Suppose f(E) = A $\cup$ B where A and B are nonempty separated subsets of Y.
Let G = E $\cap$ $f^{-1}(A)$ and H = E $\cap$ $f^{-1}(B)$. Then E = G $\cup$ H.
Since A $\subset \overline{A}$, G $\subset f^{-1}(\overline{A})$. Since f is continuous, then $f^{-1}(\overline{A})$ is closed so $\overline{G} \subset f^{-1}(\overline{A})$.
Thus, f($\overline{G}$) $\subset \overline{A}$.
Since f(H) = B and $\overline{A} \cap$ B is empty, $\overline{G} \cap$ H is empty. Similarly, G $\cap \overline{H}$ is empty so
G and H are separated which contradicts that E = G $\cup$ H is connected.

Theorem 11.4.2: Generalized Intermediate Value Theorem

Let f be a continuous real function on [a,b]. If $f(a) < c < f(b)$, then there exists $x \in (a,b)$ such that $f(x) = c$.

Proof

Since [a,b] is connected, then by theorem 11.4.1, f([a,b]) is a connected subset of $\mathbb{R}^1$. Thus, by theorem 7.2.2, any c where $f(a) < c < f(b)$ is $c \in f(x)$ for some $x \in [a,b]$.

11.5   Discontinuities

Definition 11.5.1: Right and Left Limits

Let f be defined on (a,b).

Then for any x where $x \in [a,b)$, $f(x+) = q$ if:

$f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$.

Then for any x where $x \in (a,b]$, $f(x-) = q$ if:

$f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (a,x) such that $t_n \to x$.

Then $\lim_{t \to x} f(t)$ exists if and only if $f(x-) = f(x+) = \lim_{t \to x} f(t)$.

Definition 11.5.2: Types of discontinuities

If f is discontinuous at x, but $f(x+)$ and $f(x-)$ exists, then f have a simple discontinuity of the first kind else it is a discontinuity of the second kind.

Thus, a simple discontinuity is either:

- $f(x-) \neq f(x+)$
- $f(x-) = f(x+) \neq f(x)$

11.6   Monotonic Functions

Definition 11.6.1: Monotonic

f: (a,b) $\to \mathbb{R}$ is monotonically increasing if $f(x) \leq f(y)$ for $a < x < y < b$.
f: (a,b) $\to \mathbb{R}$ is monotonically decreasing if $f(x) \geq f(y)$ for $a < x < y < b$.

Theorem 11.6.2: Right and Left Limits of monotonics on (a,b)

Let f be monotonically increasing on (a,b).

Then $f(x+)$ and $f(x-)$ exists at every $x \in (a,b)$ where:

$\sup_{t \in (a,x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x,b)} f(t)$

Furthermore, for $a < x < y < b$, $f(x+) \leq f(y-)$.

Properties analogous for monotonically decreasing functions.

Proof

Since f is monotonically increasing, then for $t \in (a,x)$, f(t) is bounded above by f(x) and thus, by the least upper bounded property, $\sup_{t \in (a,x)} f(t)$ exists.
For $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_{t \in (a,x)} f(t) - \epsilon < f(x - \delta) \leq \sup_{t \in (a,x)} f(t)$ for $a < x - \delta < x$. Since $f(x - \delta) \leq f(t) \leq \sup_{t \in (a,x)} f(t)$ for $t \in (x-\delta,x)$, then $|f(t) - \sup_{t \in (a,x)} f(t)| < \epsilon$ for $t \in (x-\delta,x)$ so $f(x-) = \sup_{t \in (a,x)} f(t)$.
For $\epsilon > 0$, there exists a $\delta > 0$ such that $\inf_{t \in (x,b)} f(t) < f(x + \delta) \leq \inf_{t \in (x,b)} f(t) + \epsilon$ for $x < x + \delta < b$. Since $f(x + \delta) \geq f(t) \geq \inf_{t \in (x,b)} f(t)$ for $t \in (x,x+\delta)$, then $|f(t) - \inf_{t \in (x,b)} f(t)| < \epsilon$ for $t \in (x,x+\delta)$ so $f(x+) = \inf_{t \in (x,b)} f(t)$.
Thus, $\sup_{t \in (a,x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x,b)} f(t)$.
If $a < x < y < b$, then:

$f(x+) = \inf_{t \in (x,b)} f(t) = \inf_{t \in (x,y)} f(t) \leq \sup_{t \in (x,y)} f(t) = \sup_{t \in (a,y)} f(t) = f(y-)$

Corollary 11.6.3: Monotonics can only have simple discontinuities

Monotonic functions have no discontinuities of the second kind

Proof

By theorem 11.6.2, f(x-) and f(x+) exists and thus, f can only have simple discontinuities and not discontinuities of the second kind.

Theorem 11.6.4: Discontinuities of monotonics is at most countable

Let f be monotonic on (a,b).

Then the set of points of (a,b) where f is discontinuous is at most countable.

Proof

Suppose f is increasing. Let E be the set of points where f is discontinuous. Then for $x \in E$, there is a rational r(x) where f(x-) < r(x) < f(x+).
Then for $x_1 < x_2$, by theorem 11.6.2, $f(x_1+) \leq f(x_2-)$. Then:
$$f(x_1\text{-}) < r(x_1) < f(x_1+) \leq f(x_2\text{-}) < r(x_2) < f(x_2+)$$
Thus, $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.
Since there is a 1-1 correspondence between E and a subset of rational numbers which is countable, then E is at most countable.
If f is decreasing, proof is analogous.

11.7   Infinite Limits / Limits at Infinity

Definition 11.7.1: Neighborhoods in extended reals

For any real c, a neighborhood of $+\infty = (c, +\infty)$.
For any real c, a neighborhood of $-\infty = (-\infty, c)$.

Definition 11.7.2: Infinite Limits

Let real function f be defined on $E \subset \mathbb{R}$.
Then $f(t) \to A$ as $t \to x$ where A and x are extended reals if:
For every neighborhood U of A, there is a neighborhood V of x such that
$V \cap E \neq \emptyset$ and $f(t) \in U$ for all $t \in V \cap E$ where $t \neq x$.

Theorem 11.7.3: Arithmetic operations on functions of infinite limits

Let f,g be defined on $E \subset \mathbb{R}$ where $f(t) \to A$ and $g(t) \to B$ as $t \to x$.
(a) If $f(t) \to A'$, then A' = A.
(b) $(f+g)(t) \to A + B$
(c) $(fg)(t) \to AB$
(d) $\frac{f}{g}(t) \to \frac{A}{B}$

12   Differentiation

12.1   Derivative of a function

Definition 12.1.1: Derivative

Let f be defined on any x $\in$ [a,b].

$\phi(t) = \frac{f(t)-f(x)}{t-x}$ for t $\neq$ x

The derivative of f at x:

$f'(x) = \lim_{t \to x} \phi(t)$

if the limit exist as defined by definition 11.1.1.

If f' is defined at x, then f is differentiable at x.

Theorem 12.1.2: Differentiability $\to$ Continuity

Let f be defined on [a,b].

If f is differentiable at x $\in$ [a,b], then f is continuous at x.

Proof

As t $\to$ x:

f(t) - f(x) $= \frac{f(t)-f(x)}{t-x} \cdot (t-x) \to f'(x) \cdot 0 = 0$

Theorem 12.1.3: Arithmetic operations on differentiation

Suppose f,g are defined on [a,b] and differentiable on x $\in$ [a,b]. Then f+g, fg, and $\frac{f}{g}$ are differentiable at x:

(a) (f+g)'(x) = f'(x) + g'(x)

Proof

$\lim_{t \to x} \frac{(f+g)(t)-(f+g)(x)}{t-x} = \lim_{t \to x} \frac{f(t)-f(x)+g(t)-g(x)}{t-x}$
$= \lim_{t \to x} \frac{f(t)-f(x)}{t-x} + \lim_{t \to x} \frac{g(t)-g(x)}{t-x} = $ f'(x) + g'(x)

(b) (fg)'(x) = f'(x)g(x) + f(x)g'(x)

Proof

$\lim_{t \to x} \frac{(fg)(t)-(fg)(x)}{t-x} = \lim_{t \to x} \frac{f(t)g(t)-f(x)g(x)}{t-x}$
$= \lim_{t \to x} \frac{f(t)g(t)-f(x)g(t)+f(x)g(t)-f(x)g(x)}{t-x}$
$= \lim_{t \to x} \frac{[f(t)-f(x)]g(t)}{t-x} + \lim_{t \to x} \frac{f(x)[g(t)-g(x)]}{t-x}$
$= $ f'(x)g(x) + f(x)g'(x)

(c) $(\frac{f}{g})$'(x) $= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

Proof

$\lim_{t \to x} \frac{(\frac{f}{g})(t)-(\frac{f}{g})(x)}{t-x} = \lim_{t \to x} \frac{\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}}{t-x} = \lim_{t \to x} \frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)(t-x)}$
$= \lim_{t \to x} \frac{f(t)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{g(t)g(x)(t-x)}$
$= \lim_{t \to x} \frac{[f(t)-f(x)]g(x)}{g(t)g(x)(t-x)} + \lim_{t \to x} \frac{f(x)[g(x)-g(t)]}{g(t)g(x)(t-x)}$
$= \frac{f'(x)g(x)}{g^2(x)} + \frac{f(x)[-g'(x)]}{g^2(x)} = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

Theorem 12.1.4: Differentiation Chain Rule
    Suppose f is continuous on [a,b], f'(x) exists at x $\in$ [a,b], g is defined on interval
    I containing f([a,b]), and g is differentiable at f(x).
    If h(t) = g(f(t)), then h is differentiable at x and h'(x) = g'(f(x)) $\cdot$ f'(x)
Proof

Since f is differentiable at x and g is differentiable at f(x), then:
    f(t) - f(x) = (t-x) [f'(x) + u(t)]   for t $\in$ [a,b] and $\lim_{t \to x} u(t) = 0$
    g(s) - g(f(x)) = (s-f(x)) [g'(f(x)) + v(s)]   for s $\in$ I and $\lim_{s \to f(x)} v(s) = 0$
Thus:
$$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{t - x}$$
$$= \lim_{t \to x} \frac{(f(t) - f(x))[g'(f(x)) + v(f(t))]}{t - x}$$
$$= \lim_{t \to x} \frac{(t - x)[f'(t) + u(t)][g'(f(x)) + v(f(t))]}{t - x}$$
$$= g'(f(x)) \cdot f'(x) + f'(x) \cdot 0 + g'(f(x)) \cdot 0 + 0 \cdot 0 = g'(f(x)) \cdot f'(x)$$

12.2   Mean Value Theorems

Definition 12.2.1: Local Extrema
    Let real-valued f $\in$ X.
    Then f has a local maximum at p $\in$ X if:
        There is $\delta > 0$ such that for all q $\in$ X where $d_X(q, p) < \delta$, f(q) $\leq$ f(p).
    Then f has a local minimum at p $\in$ X if:
        There is $\delta > 0$ such that for all q $\in$ X where $d_X(q, p) < \delta$, f(q) $\geq$ f(p).

Theorem 12.2.2: Derivative at local extrema is 0
    Let f be defined on [a,b].
    If f has a local maximum at x $\in$ (a,b) and f'(x) exists, then f'(x) = 0.
    If f has a local minimum at x $\in$ (a,b) and f'(x) exists, then f'(x) = 0.
Proof

Suppose x is a local maximum.
Then there is a $\delta > 0$ such that for all t $\in$ (a,b) where $|t - x| < \delta$, then f(t) $\leq$ f(x).
Then for $t < x$, $\frac{f(t) - f(x)}{t - x} \geq 0$. Thus, $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ = f'(x) $\geq$ 0.
For $t > x$, $\frac{f(t) - f(x)}{t - x} \leq 0$. Thus, $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ = f'(x) $\leq$ 0.
Since f'(x) exists, then f'(x) = 0.
Proof is analogous for local minimum.

Theorem 12.2.3: Generalized Mean Value Thereom
    If f,g are continuous real functions on [a,b] and differentiable on (a,b), then
    there is a x $\in$ (a,b) such that [f(b) - f(a)] $\cdot$ g'(x) = [g(b) - g(a)] $\cdot$ f'(x).
Proof

Let h(t) = [f(b) - f(a)] $\cdot$ g(t) - [g(b) - g(a)] $\cdot$ f(t) for t $\in$ [a,b].
Since f,g are continuous on [a,b] and differentiable on (a,b), then h is continuous on
[a,b] and differentiable on (a,b). Also, h(a) = f(b)g(a) - f(a)g(b) = h(b).
If h is constant, then h'(x) = 0 and thus, theorem holds true for every x $\in$ (a,b).
If h(t) > h(a) for some t $\in$ (a,b), let x $\in$ [a,b] where h attains a local maximum. If
h(t) < h(a) for some t $\in$ (a,b), let x $\in$ [a,b] where h attains a local minimum. Then
by theorem 12.2.2, h'(x) = 0 and thus, theorem holds true at local extrema.

Theorem 12.2.4: Mean Value Thereom
   If f is a real continuous function on [a,b] and differentiable on (a,b), then there
   is a x $\in$ (a,b) such that f(b) - f(a) = (b-a) f'(x).
Proof

From thereom 12.2.3, let g(x) = x.

Theorem 12.2.5: Sign of derivative determines increasing/decreasing
   Suppose f is differentiable on (a,b).
 (a) If f'(x) $\geq$ 0 for all x $\in$ (a,b), then f is monotonically increasing.
 (b) If f'(x) = 0 for all x $\in$ (a,b), then f is constant.
 (c) If f'(x) $\leq$ 0 for all x $\in$ (a,b), then f is monotonically decreasing
Proof

From theorem 12.2.4, f(x_2) - f(x_1) = ($x_2 - x_1$) f'(x) for x $\in$ (x_1, x_2) $\subset$ (a,b).
If f'(x) $\geq$ 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) $\geq$ 0. Since f(x_2) $\geq$ f(x_1) for $x_2 > x_1$,
then f is monotonically increasing.
If f'(x) = 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) = 0. Since f(x_2) = f(x_1) for $x_2 > x_1$,
then f is constant.
If f'(x) $\leq$ 0 for all x $\in$ (a,b), then f(x_2) - f(x_1) $\leq$ 0. Since f(x_2) $\leq$ f(x_1) for $x_2 > x_1$,
then f is monotonically decreasing.

12.3   Continuity of Derivatives

Theorem 12.3.1: Intermediate values of derivatives exists
   Suppose f is a real differentiable function on [a,b] and $f'(a) < \lambda < f'(b)$.
   Then there is a x $\in$ (a,b) such that f'(x) = λ.
   Statement holds true if $f'(a) > f'(b)$.
Proof

Suppose $f'(a) < \lambda < f'(b)$. Let g(t) = f(t) - λt.
Since f(t),t are differentiable on [a,b], then g(t) is differentiable on [a,b].
Then g'(a) = f'(a) - λ < 0 so g(t_1) < g(a) for some $t_1 \in$ (a,b).
Also, g'(b) = f'(b) - λ > 0 so g(t_2) < g(b) for some $t_2 \in$ (a,b).
Thus, there is a x where g(x) is a local minimum so g'(x) = 0 and thus, f'(x) = λ.

Corollary 12.3.2: Differentiable functions have no simple discontinuities
   If f is differentiable on [a,b], then f' cannot have simple discontinuities on [a,b].
Proof

By theorem 12.3.1, f'(x) exists for any x $\in$ [a,b].

12.4   L'Hospital's Rule

Theorem 12.4.1: L'Hospital's Rule

Suppose f,g are real and differentiable on (a,b) and g'(x) $\neq$ 0 for all x $\in$ (a,b).
Suppose $\lim_{x \to a} \frac{f'(x)}{g'(x)} \to A$. If either:

- $\lim_{x \to a}$ f(x) $\to$ 0 and $\lim_{x \to a}$ g(x) $\to$ 0

- $\lim_{x \to a}$ g(x) $\to \infty$ or $\lim_{x \to a}$ g(x) $\to -\infty$

Then, $\lim_{x \to a} \frac{f(x)}{g(x)} \to A$.
Statement holds true if x $\to$ b.

Proof

Consider the case $-\infty \leq A < \infty$.
Choose q such that $A < q$ and r such that $A < r < q$.
Thus, there is a c $\in$ (a,b) such that $a < x < c$ for $\frac{f'(x)}{g'(x)} < r$.
For $a < x < y < c$, then by theorem 12.2.3, there is a t $\in$ (x,y) such that:
$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$
If $\lim_{x \to a}$ f(x) $\to$ 0 and $\lim_{x \to a}$ g(x) $\to$ 0, then as $x \to a$, $\frac{f(y)}{f(x)} \leq r < q$ for $y \in (a,c)$.
If $\lim_{x \to a}$ g(x) $\to \infty$, then keeping y fixed, choose $c_1 \in (a,y)$ such that $g(x) > g(y)$
and $g(x) > 0$ if $a < x < c_1$. Thus:
$\frac{g(x)-g(y)}{g(x)} \cdot \frac{f(x)-f(y)}{g(x)-g(y)} < \frac{g(x)-g(y)}{g(x)} \cdot r$ for $x \in (a,c_1)$
$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$
Thus as $x \to a$, there is a $c_2 \in (a,c_1)$ such that $\frac{f(x)}{g(x)} < r < q$ for $x \in (a,c_2)$.
Proof is analogous if $\lim_{x \to a}$ g(x) $\to -\infty$.
Thus, $\lim_{x \to a} \frac{f(x)}{g(x)} \to A$.

12.5   Derivative of Higher Order

Definition 12.5.1: Derivative of Higher Order

If f has a derivative f' on an interval and f' is differentiable, then the derivative
of f' is f", the second derivative of f. Then, $f^{(n)}$ is the nth derivative of f.
For $f^{(n)}(x)$ to exist at x, $f^{(n-1)}(t)$ must exist in a neighborhood of x and $f^{(n-1)}$
must be differentiable at x.
If $f^{(n-1)}$ exist in a neighborhood of x, then $f^{(n-2)}$ must be differentiable in
that neighborhood and so on until f is differentiable on that neighborhood.

12.6   Taylor's Theorem

Theorem 12.6.1: Taylor's Theorem

Suppose f is a real function on [a,b], $n \in \mathbb{Z}_+$, $f^{(n-1)}$ is continuous on [a,b], $f^n(t)$ exists at every $t \in (a,b)$.

Let $\alpha, \beta \in$ [a,b] be distinct and P(t) = $\sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t - \alpha)^k$.

Then there exists a x between α and β such that f(β) = P(β) + $\frac{f^n(x)}{n!}(\beta - \alpha)^n$

Proof

Let M be the number defined by $f(\beta) = P(\beta) + M(\beta - \alpha)^n$.
Let $g(t) = f(t) - P(t) - M(t - \alpha)^n$ for $t \in [\alpha, \beta]$. Thus, $g^{(n)}(t) = f^{(n)}(t) - n!M$.
Also since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for k = [0,n-1], then $g(\alpha) = g'(\alpha) = ... = g^{(n-1)}(\alpha) = 0$.
Since the choice of M gives $g(\beta) = 0$, then by the Mean Value Theorem, $g'(x_1) = 0$ for some x_1 between α and β.
Since $g'(\alpha) = 0$, then $g''(x_2) = 0$ for some x_2 between α and x_1.
Thus, $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1} so x_n is between α and β.
Thus, there exists an $x_n \in (\alpha, \beta)$ such that:
$\quad 0 = g^{(n)}(x_n) = f^{(n)}(x_n) - n!M$
$\quad M = \frac{f^n(x_n)}{n!}$

12.7   Differentiation of Vector-Valued Functions

Definition 12.7.1: Extending derivative to Vector-Valued Functions

For vector-valued function f: $t \in$ [a,b] $\to \mathbb{R}^k$, the derivative of f at x:
$\quad$ f'(x) = $\lim_{t \to x} |\frac{f(t)-f(x)}{t-x}|$
if the limit exist as defined by definition 14.1.1.

If f = $(f_1, ..., f_k)$, then f' = $(f_1', ..., f_k')$ and f is differentiable at x if and only if $f_1, ..., f_k$ are differentiable at x.

Thus, by theorem 11.2.7, these theorems hold true for vector-valued functions:
- 12.1.2: If f is differentiable at x, then f is continuous at x.

- 12.1.3a: If f,g are differentiable at x, then f+g,f·g are differentiable at x.

However, theorem 12.2.4: Mean Value Theorem and theorem 12.4.1: L'Hospital's Rule does not always hold true since theorem 12.1.3b/c, multiplying/dividing vectors by vectors, is not defined for vector-valued functions.

Theorem 12.7.2: Mean Value Theorem for $\mathbb{R}^k$

Suppose f is a continuous mapping of [a,b] into $\mathbb{R}^k$ and f is differentiable on (a,b). Then there is a x $\in$ (a,b) such that $|f(b) - f(a)| \leq$ (b-a) $|f'(x)|$

Proof

Let z = f(b) - f(a) and define $\phi(t) = z \cdot f(t)$ for $t \in$ [a,b].
Then $\phi(t)$ is real-valued continuous on [a,b] and differentiable on (a,b).
Then by the Mean Value Theorem, for some x $\in$ (a,b):
$\quad \phi(b) - \phi(a) =$ (b-a) $\phi'(x) =$ (b-a) $z \cdot f'(x)$
Since $\phi(b) - \phi(a) = z \cdot f(b) - z \cdot f(a) = z \cdot z = |z|^2$, then by the Schwarz Inequality:
$\quad |z|^2 =$ (b-a) $|z \cdot f'(x)| \leq$ (b-a) $|z||f'(x)|$
$\quad |z| \leq$ (b-a) $|f'(x)|$
$\quad |f(b) - f(a)| \leq$ (b-a) $|f'(x)|$

13   Riemann-Stieltjes Integral

13.1   Riemann-Stieltjes Integral

Definition 13.1.1: Riemann Integral

For [a,b], let a $= x_0 \leq x_1 \leq ... \leq x_n =$ b and $\Delta x_i = x_i - x_{i-1}$.
Suppose real f is bounded. Then for each partition P, $\{x_0, ..., x_n\}$,
let $m_i = \inf$ f($[x_{i-1}, x_i]$) and $M_i = \sup$ f($[x_{i-1}, x_i]$).
Then let L(P,f) $= \sum_{i=1}^n m_i \Delta x_i$ and U(P,f) $= \sum_{i=1}^n M_i \Delta x_i$. Thus, over all P:

Lower Riemann Integral: $\underline{\int_a^b}$ f(x) dx $= \sup$ L(P,f)

Upper Riemann Integral: $\overline{\int_a^b}$ f(x) dx $= \inf$ U(P,f)

If $\underline{\int_a^b} f(x) dx = \overline{\int_a^b} f(x) dx$, then f is Riemann-integrable (f $\in \mathscr{R}$) and $\int_a^b f(x) dx$.

Since f is bounded, there are m,M such that m $\leq$ f(x) $\leq$ M.
Thus, m(b-a) $\leq$ L(P,f) $\leq$ U(P,f) $\leq$ M(b-a).
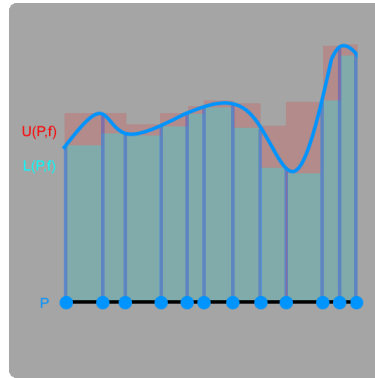
Definition 13.1.2: Riemann-Stieltjes Integral

Let α be monotonically increasing on [a,b].
Then for each partition P, let $\Delta \alpha_i = \alpha(x_i)$ - $\alpha(x_{i-1})$.
For real bounded f, let L(P,f,α) $= \sum_{i=1}^n m_i \Delta \alpha_i$ and U(P,f,α) $= \sum_{i=1}^n M_i \Delta \alpha_i$.
Thus, $\underline{\int_a^b}$ f(x) dα(x) $= \sup$ L(P,f,α) and $\overline{\int_a^b}$ f(x) dα(x) $= \inf$ U(P,f,α).

If $\underline{\int_a^b}$ f(x) dα(x) $= \overline{\int_a^b}$ f(x) dα(x), then f $\in \mathscr{R}(\alpha)$ with value $\int_a^b$ f(x) dα(x).

Definition 13.1.3: Refinement

Partition Q is a refinement of P if P $\subset$ Q.
For partitions P_1, P_2, then Q $= P_1 \cup P_2$ is the common refinement.

Theorem 13.1.4: Refinements monotonically increase L(P,f) & decrease U(P,f)

If Q is a refinement of P, then:

$$L(P,f,\alpha) \leq L(Q,f,\alpha) \leq U(Q,f,\alpha) \leq U(P,f,\alpha)$$

Proof

Since Q is a refinement of P, then $P \subset Q$.
Suppose $Q = P \cup \{x*\}$ where $P = \{x_0, ..., x_n\}$ and $Q = \{x_0, ..., x_{k-1}, x*, x_k, .., x_n\}$.
Regardless of anymore points, the process below will be analogous.

$$L(P,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta\alpha_i + m_{[x_{k-1},x_k]}[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ m_{[x_{k-1},x_k]}[\alpha(x_k) - \alpha(x*)] + \sum_{i=k+1}^{n} m_i \Delta\alpha_i$$

$$L(Q,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta\alpha_i + m_{[x_{k-1},x*]}[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ m_{[x*,x_k]}[\alpha(x_k) - \alpha(x_*)] + \sum_{i=k+1}^{n} m_i \Delta\alpha_i$$

Since $[x_{k-1}, x*], [x*, x_k] \subset [x_{k-1}, x_k]$, then $m_{[x_{k-1},x_k]} \leq m_{[x_{k-1},x*]}, m_{[x*,x_k]}$. Thus:

$$L(Q,f,\alpha) \text{ - } L(P,f,\alpha) = (m_{[x_{k-1},x*]} - m_{[x_{k-1},x_k]})[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ (m_{[x_{k*},x_k]} - m_{[x_{k-1},x_k]})[\alpha(x_k) - \alpha(x*)] \geq 0.$$

$$U(P,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta\alpha_i + M_{[x_{k-1},x_k]}[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ M_{[x_{k-1},x_k]}[\alpha(x_k) - \alpha(x*)] + \sum_{i=k+1}^{n} M_i \Delta\alpha_i$$

$$U(Q,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta\alpha_i + M_{[x_{k-1},x*]}[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ M_{[x*,x_k]}[\alpha(x_k) - \alpha(x_*)] + \sum_{i=k+1}^{n} M_i \Delta\alpha_i$$

Since $[x_{k-1}, x*], [x*, x_k] \subset [x_{k-1}, x_k]$, then $M_{[x_{k-1},x_k]} \geq M_{[x_{k-1},x*]}, M_{[x*,x_k]}$. Thus:

$$U(Q,f,\alpha) \text{ - } U(P,f,\alpha) = (M_{[x_{k-1},x*]} - M_{[x_{k-1},x_k]})[\alpha(x*) - \alpha(x_{k-1})]$$
$$+ (M_{[x_{k*},x_k]} - M_{[x_{k-1},x_k]})[\alpha(x_k) - \alpha(x*)] \leq 0.$$
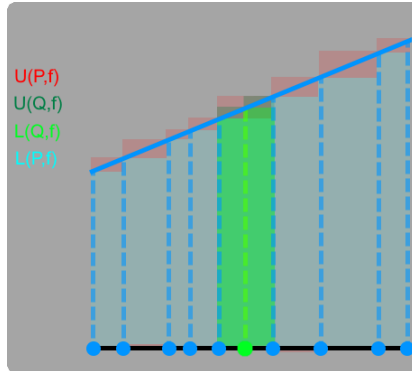
Theorem 13.1.5: Lower Riemann Integral $\leq$ Upper Riemann Integral

$$\underline{\int}_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$$

Proof

For partitions P_1, P_2, let L(P_1, f, α) and U(P_2, f, α). Let $P = P_1 \cup P_2$. Thus:

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$$

Thus, over all partitions for P_1, $\underline{\int}_a^b f d\alpha \leq U(P_2, f, \alpha)$

Thus, over all partitions for P_2, $\underline{\int}_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$

Theorem 13.1.6: Riemann-Integrability ϵ Definition

$f \in \mathscr{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$

Proof

If $f \in \mathscr{R}(\alpha)$, then $\underline{\int}_a^f d\alpha = \overline{\int}_a^b f d\alpha = \int_a^b f d\alpha$. For $\epsilon > 0$, there exists partitions P_1, P_2:

$\int_a^b f d\alpha$ - $L(P_1, f, \alpha) < \frac{\epsilon}{2}$ $U(P_2, f, \alpha)$ - $\int_a^b f d\alpha < \frac{\epsilon}{2}$

Then for partition $P = P_1 \cup P_2$, then:

$\int_a^b f d\alpha$ - $L(P, f, \alpha) \leq \int_a^b f d\alpha$ - $L(P_1, f, \alpha) < \frac{\epsilon}{2}$

$U(P, f, \alpha)$ - $\int_a^b f d\alpha \leq U(P_2, f, \alpha)$ - $\int_a^b f d\alpha < \frac{\epsilon}{2}$

Thus, $U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$.

For $\epsilon > 0$, there is a partition P such that $U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$.

Since $L(P, f, \alpha) \leq \underline{\int}_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha \leq U(P, f, \alpha)$, then $\overline{\int}_a^b f d\alpha$ - $\underline{\int}_a^b f d\alpha < \epsilon$.

Theorem 13.1.7: Properties of Riemann-Integrability

(a) If $f \in \mathscr{R}(\alpha)$, then $U(Q, f, \alpha)$ - $L(Q, f, \alpha) < \epsilon$ for every refinement of P, Q

Proof

By theorem 13.1.6, for $\epsilon > 0$, there is a P such that:

$U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$.

Then by theorem 13.1.4, for any refinement of P, Q, then:

$U(Q, f, \alpha)$ - $L(Q, f, \alpha) < \epsilon$.

(b) If $f \in \mathscr{R}(\alpha)$ where $P = \{x_0, .., x_n\}$ and $s_i, t_i \in [x_{i-1}, x_i]$, then:

$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta \alpha_i < \epsilon$

Proof

By theorem 13.1.6, for $\epsilon > 0$, there is a P such that:

$U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$

$\sum_{i=1}^n M_i \Delta \alpha_i$ - $\sum_{i=1}^n m_i \Delta \alpha_i < \epsilon$

Since $s_i, t_i \in [x_{i-1}, x_i]$, then $m_i \leq f(s_i), f(t_i) \leq M_i$.

Thus, $|f(s_i) - f(t_i)| \leq M_i - m_i$.

$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta \alpha_i \leq \sum_{i=1}^n M_i - m_i \Delta \alpha_i \leq \epsilon$

(c) If $f \in \mathscr{R}(\alpha)$ where $P = \{x_0, .., x_n\}$ and $t_i \in [x_{i-1}, x_i]$, then:

$|\sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha| < \epsilon$

Proof

Since sup $L(P, f, \alpha) = \underline{\int}_a^b f d\alpha = \int_a^b f d\alpha = \overline{\int}_a^b f d\alpha = $ inf $U(P, f, \alpha)$, then:

$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha)$

Since $t_i \in [x_{i-1}, x_i]$, then $m_i \leq f(t_i) \leq M_i$. Thus:

$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta \alpha_i \leq \sum_{i=1}^n f(t_i) \Delta \alpha_i$

$\leq \sum_{i=1}^n M_i \Delta \alpha_i = U(P, f, \alpha)$

Thus, $|\sum_{i=1}^n f(t_i) \Delta \alpha_i - \int_a^b f d\alpha| \leq U(P, f, \alpha)$ - $L(P, f, \alpha) < \epsilon$.

13.2   Riemann-Integrable Functions

Theorem 13.2.1: Continuous functions are Riemann-Integrable

If f is continuous on [a,b], then $f \in \mathscr{R}(\alpha)$

Proof

For $\epsilon > 0$, choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$. Since f is continuous and [a,b] is compact, then f is uniformly continuous. Thus, for $\eta > 0$, there is a $\delta > 0$ such that for all x,t $\in$ [a,b] where $|x - t| < \delta$, then $|f(x) - f(t)| < \eta$. For partition P of [a,b] such that $\Delta x_i < \delta$ for all i={1,...,n}, then $M_i - m_i \leq \eta$ for each i. Thus:
U(P, f, α) - L$(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \leq \sum_{i=1}^{n}\eta\Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \epsilon$

Theorem 13.2.2: Monotonic functions are Riemann-Integrable

If f is monotonic on [a,b] and α is continuous on [a,b], then $f \in \mathscr{R}(\alpha)$

Proof

Since α is continuous on [a,b], let $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$ where n $\in \mathbb{Z}_+$
Let partition P = $\{\alpha(x_0), ..., \alpha(x_n)\}$. Suppose f is monotonically increasing. Thus:
U(P, f, α) - L$(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}(M_i - m_i)$
$= \frac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)]$
For $\epsilon > 0$, there exists a n such that $\frac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)] < \epsilon$ so $f \in \mathscr{R}(\alpha)$.
If f is monotonically decreasing, then $\sum_{i=1}^{n}(M_i - m_i) = \sum_{i=1}^{n}[f(x_{i-1}) - f(x_i)]$.

Theorem 13.2.3: Bounded functions with finite discontinuities are Riemann-Integrable

If f is bounded on [a,b] with finitely many discontinuities and α is continuous at every discontinuity, then $f \in \mathscr{R}(\alpha)$

Proof

Since f is bounded, let M = sup $|f(x)|$ and E be the set of discontinuities of f.
Since E is finite and α is continuous over E, then for $\epsilon > 0$, there are finitely many disjoint $[u_j, v_j]$ where $\sum[\alpha(v_j) - \alpha(u_j)] < \epsilon$ which cover E.
Let K = [a,b] $\setminus \cup (u_j, v_j)$ which is compact. Since f is continuous over compact K, then f is uniformly continuous over K. Thus, for $\epsilon > 0$, there is a $\delta > 0$ such that for s,t $\in$ K where $|s - t| < \delta$, then $|f(s) - f(t)| < \epsilon$.
Let partition P = $\{x_0, ..., x_n\}$ of [a,b] where each $\Delta x_i < \delta$ and if x $\in (u_j, v_j) \notin$ P, but $u_j, v_j \in$ P. Thus, $M_i - m_i \leq 2M$ for each i and $M_i - m_i \leq \epsilon$ unless x_{i-1} is a u_j, then:
U(P, f, α) - L$(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \sum_{K}(M_i - m_i)\Delta\alpha_i + \sum_{K^c}(M_i - m_i)\Delta\alpha_i$
$\leq \epsilon\sum_{K}\Delta\alpha_i + 2M\sum_{K^c}\Delta\alpha_i \leq [\alpha(b) - \alpha(a)]\epsilon + 2M\epsilon$
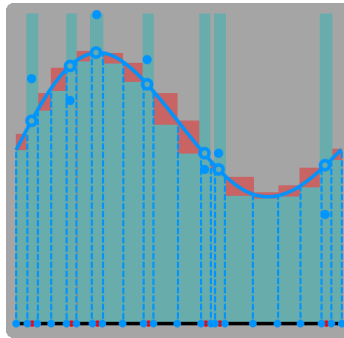
Theorem 13.2.4: Composite of continuous-integrable functions are Riemann-Integrable

If $f \in \mathscr{R}(\alpha)$ on [a,b] where $f \in$ [m,M] and ϕ is continuous on [m,M] such that $h(x) = \phi(f(x))$, then $h \in \mathscr{R}(\alpha)$

<u>Proof</u>

Since ϕ is continuous and [m,M] is compact, then ϕ is uniformly continuous. Thus, for $\epsilon > 0$, there is a $0 < \delta < \epsilon$ such that for all s,t $\in$ [m,M] where $|s - t| \leq \delta$, then $|\phi(s) - \phi(t)| < \epsilon$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition P = $\{x_0, ..., x_n\}$ such that:

   U(P, f, α) - L$(P, f, \alpha) < \delta^2$

For each i={1,...,n}, let i $\in$ A if $M_i - m_i < \delta$ and i $\in$ B if $M_i - m_i \geq \delta$.

Let $m_i^* = \inf \phi(f([x_{i-1}, x_i]))$ and $M_i^* = \sup \phi(f([x_{i-1}, x_i]))$.

For A, since $M_i - m_i < \delta$, then $M_i^* - m_i^* \leq \epsilon$.

For B, $M_i^* - m_i^* \leq 2K$ where K = $\sup_{[m,M]} |\phi|$.

   $\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B}(M_i - m_i)\Delta\alpha_i < \delta^2$

   $\sum_{i \in B} \Delta\alpha_i \leq \delta < \epsilon$

Thus:

   U(P, h, α) - L$(P, h, \alpha) = \sum_{i \in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B}(M_i^* - m_i^*)\Delta\alpha_i$

   $\leq \epsilon \sum_{i \in A} \Delta\alpha_i + 2K \sum_{i \in B} \Delta\alpha_i$

   $\leq \epsilon[\alpha(b) - \alpha(a)] + 2K\epsilon < \epsilon[\alpha(b) - \alpha(a) + 2K]$

13.3   Integral Properties

Theorem 13.3.1: Integral Additive Properties

(a) If $f_1, f_2 \in \mathscr{R}(\alpha)$ on [a,b] and constant c, then $f_1 + f_2, cf_1 \in \mathscr{R}(\alpha)$ and

   $\int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$

   $\int_a^b cf_1 \, d\alpha = c \int_a^b f_1 \, d\alpha$

<u>Proof</u>

Since $f_1, f_2 \in \mathscr{R}(\alpha)$, then there are partitions P_1, P_2 such that for $\epsilon > 0$:

   U(P_1, f_1, α) - L$(P_1, f_1, \alpha) < \frac{\epsilon}{2}$ U(P_2, f_2, α) - L$(P_2, f_2, \alpha) < \frac{\epsilon}{2}$

Thus for partition P = $P_1 \cup P_2$:

   U(P, f_1, α) + U(P, f_2, α) - L(P, f_1, α) - L$(P, f_2, \alpha) < \epsilon$

   U$(P, f_1 + f_2, \alpha)$ - L$(P, f_1 + f_2, \alpha) < \epsilon$

For any partition Q:

   L(Q, f_1, α) + L$(Q, f_2, \alpha) \leq$ L$(Q, f_1 + f_2, \alpha) \leq$ U$(Q, f_1 + f_2, \alpha)$

                    $\leq$ U(Q, f_1, α) + U(Q, f_2, α)

Thus, $f_1 + f_2 \in \mathscr{R}(\alpha)$ where:

   $\int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha = \underline{\int}_a^b f_1 \, d\alpha + \underline{\int}_a^b f_2 \, d\alpha \leq \underline{\int}_a^b f_1 + f_2 \, d\alpha$

       $= \int_a^b f_1 + f_2 \, d\alpha = \overline{\int}_a^b f_1 + f_2 \, d\alpha$

       $\leq \overline{\int}_a^b f_1 \, d\alpha + \overline{\int}_a^b f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$

Proof for cf_1 is analogous by replacing $\frac{\epsilon}{2}$ with $\frac{\epsilon}{c}$.

(b) If $f_1, f_2 \in \mathscr{R}(\alpha)$ and $f_1(x) \leq f_2(x)$ on [a,b], then $\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$

<u>Proof</u>

Since $f_1, f_2 \in \mathscr{R}(\alpha)$, then by part a, $0 \leq \int_a^b f_2 - f_2 d\alpha = \int_a^b f_2 d\alpha$ - $\int_a^b f_1 d\alpha$.

(c) If $f \in \mathscr{R}(\alpha)$ on [a,b] and c $\in$ (a,b), then f $\in \mathscr{R}(\alpha)$ on [a,c],[c,b] and
$\int_a^c f \, d\alpha + \int_c^b f \, d\alpha = \int_a^b f \, d\alpha$
Proof

> Since $f \in \mathscr{R}(\alpha)$ on [a,b], there is a partition P of [a,b] such that for $\epsilon > 0$:
> $\quad$ U(P, f, α) - L$(P, f, \alpha) < \epsilon$
> For partition P of [a,b], let refinement of P, Q = P $\cup$ {c}. Thus:
> $\quad$ L$(P, f, \alpha) \le$ L$(Q, f, \alpha) \le$ U$(Q, f, \alpha) \le$ U(P, f, α)
> Thus, let A = (P < c) $\cup$ c $\in$ [a,c] and B = c $\cup$ (c < P) $\in$ (c,b):
> $\quad$ L$(Q, f, \alpha) = \sum_Q m_q \Delta\alpha_q$
> $\qquad\qquad \le \sum_A m_a \Delta\alpha_a + \sum_B m_b \Delta\alpha_b =$ L$(A, f, \alpha) +$ L(B, f, α)
> $\quad$ U$(Q, f, \alpha) = \sum_Q M_q \Delta\alpha_q$
> $\qquad\qquad \ge \sum_A M_a \Delta\alpha_a + \sum_B M_b \Delta\alpha_b =$ U$(A, f, \alpha) +$ U(B, f, α)
> Since Q is a refinement of P, then U(Q, f, α) - L$(Q, f, \alpha) < \epsilon$. Thus:
> $\quad 0 \le$ U$(A, f, \alpha) +$ U(B, f, α) - L(A, f, α) - L$(B, f, \alpha) < \epsilon$
> $\quad$ U(A, f, α) - L$(A, f, \alpha) < \epsilon \qquad$ U(B, f, α) - L$(B, f, \alpha) < \epsilon$
> Thus, $f \in \mathscr{R}(\alpha)$ on [a,c],[c,b] where:
> $\quad \underline{\int}_a^b f \, d\alpha \le \underline{\int}_a^c f \, d\alpha + \underline{\int}_c^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha$
> $\qquad\qquad = \overline{\int}_a^c f \, d\alpha + \overline{\int}_c^b f \, d\alpha \le \overline{\int}_a^b f \, d\alpha$
> Since $\underline{\int}_a^b f \, d\alpha, \overline{\int}_a^b f \, d\alpha = \int_a^b f \, d\alpha$, then $\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha$.

(d) If $f \in \mathscr{R}(\alpha_1), \mathscr{R}(\alpha_2)$ and constant c, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$, $f \in \mathscr{R}(c\alpha_1)$ and
$\int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$
$\int_a^b f \, d(c\alpha_1) = c \int_a^b f \, d\alpha_1$
Proof

> Since $f \in \mathscr{R}(\alpha_1), \mathscr{R}(\alpha_2)$, then there are partitions P_1, P_2 where for $\epsilon > 0$:
> $\quad$ U(P_1, f, α_1) - L$(P_1, f, \alpha_1) < \frac{\epsilon}{2} \qquad$ U(P_2, f, α_2) - L$(P_2, f, \alpha_2) < \frac{\epsilon}{2}$
> Thus, for partition P = $P_1 \cup P_2$:
> $\quad \sum_{i=1}^n (M_i - m_i)\Delta\alpha_{1i} < \frac{\epsilon}{2} \qquad\qquad \sum_{i=1}^n (M_i - m_i)\Delta\alpha_{2i} < \frac{\epsilon}{2}$
> $\qquad\qquad \sum_{i=1}^n (M_i - m_i)(\Delta\alpha_{1i} + \Delta\alpha_{2i}) < \epsilon$
> $\qquad\qquad$ U$(P, f, \alpha_1 + \alpha_2)$ - L$(P, f, \alpha_1 + \alpha_2) < \epsilon$
> For any partition Q:
> $\quad$ L$(Q, f, \alpha_1) +$ L$(Q, f, \alpha_2) \le$ L$(Q, f, \alpha_1 + \alpha_2)$
> $\qquad\qquad\qquad\qquad\qquad \le$ U$(Q, f, \alpha_1 + \alpha_2)$
> $\qquad\qquad\qquad\qquad\qquad \le$ U$(Q, f, \alpha_1) +$ U(Q, f, α_2)
> Thus, $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ where:
> $\quad \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2 = \underline{\int}_a^b f \, d\alpha_1 + \underline{\int}_a^b f \, d\alpha_2 \le \underline{\int}_a^b f \, d(\alpha_1 + \alpha_2)$
> $\qquad\qquad = \int_a^b f \, d(\alpha_1 + \alpha_2) = \overline{\int}_a^b f \, d(\alpha_1 + \alpha_2)$
> $\qquad\qquad \le \overline{\int}_a^b f \, d\alpha_1 + \overline{\int}_a^b f \, d\alpha_2 = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$
> Proof for $c\alpha_1$ is analogous by replacing $\frac{\epsilon}{2}$ with $\frac{\epsilon}{c}$.

Theorem 13.3.2: Integral Multiplicative Properties
  (a) If f,g $\in \mathscr{R}(\alpha)$ on [a,b], then fg $\in \mathscr{R}(\alpha)$
      Proof

> Since f,g $\in \mathscr{R}(\alpha)$, then f+g,f-g $\in \mathscr{R}(\alpha)$.
> By theorem 13.2.4, let $\phi(t) = t^2$ which is continuous so
> $\phi(f + g) = (f + g)^2, \phi(f - g) = (f - g)^2 \in \mathscr{R}(\alpha)$.
> Thus, 4fg $= (f + g)^2 - (f - g)^2 \in \mathscr{R}(\alpha)$.

  (b) If f $\in \mathscr{R}(\alpha)$ on [a,b], then $|f| \in \mathscr{R}(\alpha)$ where $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$
      Proof

> By theorem 13.2.4, let $\phi(t) = |t|$ which is continuous so $|f| \in \mathscr{R}(\alpha)$.
> Then choose c $= \pm 1$ such that $c \int f d\alpha \geq 0$. Then:
> $\quad |\int f d\alpha| = c \int f d\alpha = \int c f d\alpha \leq \int |f| d\alpha$

13.4   Change of Variable

Definition 13.4.1: Unit Step Function

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Theorem 13.4.2: Integrating f over I centered at s
      If f is bounded on [a,b] and continuous at s $\in$ (a,b) where $\alpha(x) = I$(x-s), then:
      $\int_a^b f \, d\alpha = $ f(s)

Intuition

> If x < s < y, then $\Delta I = I(y - s)$ - $I(x - s) = 1 - 0 = 1$ else $\Delta I = 0$.
> So, $f(x)d\alpha(x) \approx f(x)\Delta I$ have only $f(s)\Delta I = f(s)$ since the others $\Delta I = 0$.

Proof

> For partition P $= \{x_0, x_1, x_2, x_3\}$ where $x_1 = $ s:
> $\quad$ L$(P, f, \alpha) = m_2 \qquad$ U$(P, f, \alpha) = M_2$
> Since f is continuous at s, then for $\epsilon > 0$, there is a $\delta > 0$ where for all x $\in$ [s,s+δ],
> then $|f(x) - f(s)| < \frac{\epsilon}{2}$. Thus, $M_2 - m_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ so $\int f d\alpha$ exist where:
> $\quad f(s) - m_2 < \frac{\epsilon}{2}$ so $\underline{\int} f d\alpha = $ f(s) $\qquad M_2 - f(s) < \frac{\epsilon}{2}$ so $\overline{\int} f d\alpha = $ f(s)
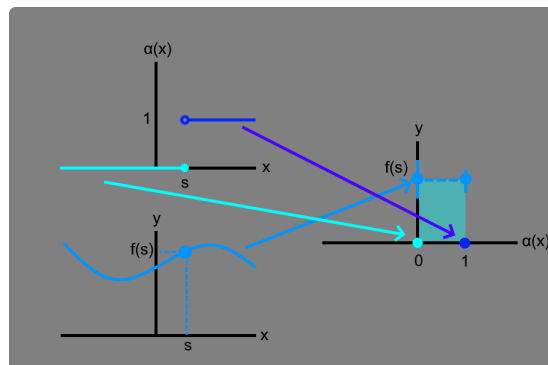
Theorem 13.4.3: Integrating f over a step function

If $c_n \geq 0$ where $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b), and $\alpha(x) = \sum c_n I(x - s_n)$. Then for continuous f on [a,b]:
$$\int_a^b f d\alpha = \sum c_n f(s_n)$$

Intuition

Similar to theorem 13.4.2, but over a step function. The $\{s_n\}$ determines where the steps are and the $\{\sum c_n\}$ determines the value at each step.
Thus, $f(x)d\alpha(x)$ have only:
$$f(s_n) \cdot (\text{value}_{\text{current step}} - \text{value}_{\text{previous step}}) = f(s_n) \cdot (\sum c_n - \sum c_{n-1}) = f(s_n) \cdot c_n$$

Proof

Since $\alpha(x) = \sum c_n I(x - s_n) \leq \sum c_n$, then by the comparison test, $\alpha(x)$ converges.
Since $c_n, I(x - s_n) \geq 0$, then $\alpha(x)$ is monotonic.
Since $a < s_n$ for any n, then $\alpha(a) = \sum c_n I(a - s_n) = \sum c_n 0 = 0$.
Since $b > s_n$ for any n, then $\alpha(a) = \sum c_n I(b - s_n) = \sum c_n 1 = \sum c_n$.
Since $\sum c_n$ converges, then for $\epsilon > 0$, there is a N such that $\sum_{n=N+1}^{\infty} c_n < \epsilon$.
Let $\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$ and $\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$. By theorem 13.4.2:
$$\int_a^b f d\alpha_1 = \int_a^b f d(\sum_{n=1}^{N} c_n I(x - s_n)) = \sum_{n=1}^{N} c_n f(s_n)$$
$$|\int_a^b f d\alpha_2| = \sum_{n=N+1}^{\infty} c_n f(s_n) \leq \sum_{n=N+1}^{\infty} c_n \sup(|f(x)|) = \sup(|f(x)|)\epsilon$$
Thus, $\int f d\alpha = \int f d(\alpha_1 + \alpha_2) = \int f d\alpha_1 + \int f d\alpha_2 = \sum_{n=1}^{N} c_n f(s_n) + \sup(|f(x)|)\epsilon$

Theorem 13.4.4: $\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$

If $\alpha' \in \mathscr{R}$ on [a,b] and f is real, bounded on [a,b], then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. Then:
$$\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$$

Intuition

If α is differentiable on [x,y], then by the Mean Value Theorem, there is a t $\in$ [x,y]:
$$\alpha(x) - \alpha(y) = \alpha'(t) \cdot (x - y)$$
Since $d\alpha \approx \Delta\alpha(x) = \alpha'(t)\Delta x \approx \alpha'(x) \, dx$, then $\int_a^b f \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$.

Proof

Since $\alpha' \in \mathscr{R}$, then $\epsilon > 0$, there is a partition $P = \{x_0, ..., x_n\}$ such that:
   U(P, α') - L$(P, \alpha') < \epsilon$
By the Mean Value Theorem, there are $t_i \in [x_{i-1}, x_i]$ such that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$.
Then for $s_i \in [x_{i-1}, x_i]$:
   $\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \leq$ U(P, α') - L$(P, \alpha') < \epsilon$
Let M = $\sup(|f(x)|)$. Since $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$, then:
   $|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i|$
   $= |\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i|$
   $\leq$ M$|\sum_{i=1}^{n} \alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} \alpha'(s_i)\Delta x_i|$ = Mϵ
Thus:

$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq$ U$(P, f\alpha') + $ Mϵ	$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \geq$ L$(P, f\alpha') + $ Mϵ
U$(P, f, \alpha) \leq$ U$(P, f\alpha') + $ Mϵ	L$(P, f, \alpha) \geq$ L$(P, f\alpha') + $ Mϵ
$|\overline{\int} f d\alpha$ - $\overline{\int} f\alpha' \, dx| <$ Mϵ	$|\underline{\int} f d\alpha$ - $\underline{\int} f\alpha' \, dx| <$ Mϵ

Thus, $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$.

Theorem 13.4.5: Integral Change of Variable: $\int_a^b f(x) \, dx = \int_A^B f(\phi(y))\phi'(y) \, dy$

Let strictly increasing continuous ϕ: [A,B] $\to$ [a,b] and $f \in \mathscr{R}(\alpha)$ on [a,b].
Let $\beta(y) = \alpha(\phi(y))$ and $g(y) = f(\phi(y))$ for y $\in$ [A,B]. Then g $\in \mathscr{R}(\beta)$ where:
$\int_A^B g \, d\beta = \int_a^b f \, d\alpha$

Intuition

Partition of [a,b] = $\{x_0, ..., x_n\}$ $\sim$ partition of [A,B] = $\{y_0, ..., y_n\}$ where $x_i = \phi(y_i)$.
Thus, $g(y)d\beta(y) \approx f(\phi(y))\Delta\alpha(\phi(y)) = f(x)\Delta\alpha(x) \approx f(x)d\alpha$.

Proof

Since $f \in \mathscr{R}(\alpha)$, then for $\epsilon > 0$, there is a partition P such that:
U(P, f, α) - L$(P, f, \alpha) < \epsilon$
For partition P = $\{x_0, ..., x_n\}$ of [a,b], there is a partition Q = $\{y_0, ..., y_n\}$ of [A,B]
where $x_i = \phi(y_i)$. Thus:
L(Q, g, β) = L$(Q, f(\phi(y)), \alpha(\phi(y)))$ = L$(P, f(x), \alpha(x))$ = L(P, f, α)
U(Q, g, β) = U$(Q, f(\phi(y)), \alpha(\phi(y)))$ = U$(P, f(x), \alpha(x))$ = U(P, f, α)
Thus, U(Q, g, β) - L(Q, g, β) = U(P, f, α) - L$(P, f, \alpha) < \epsilon$ so $g \in \mathscr{R}(\beta)$ and
$\int_A^B g \, d\beta = \int_a^b f \, d\alpha$.

Let $\alpha(x)$ = x. Then $\beta(y) = \phi(y)$. If $\beta' \in \mathscr{R}$ on [A,B], then by theorem 13.4.5:
$\int_a^b f(x) \, dx = \int_a^b f \, d\alpha = \int_A^B g \, d\beta = \int_A^B g(y)\beta'(y) \, dy = \int_A^B f(\phi(y))\phi'(y) \, dy$

13.5   Fundamental Theorem of Calculus

Theorem 13.5.1: If F(x) = $\int f(x)dx$, then F'(x) = f(x)

Let $f \in \mathscr{R}$ on [a,b]. For x $\in$ [a,b], let F(x) = $\int_a^x$ f(t) dt.
Then F is continuous on [a,b] and if f is continuous at $x_0 \in$ [a,b], then F is
differentiable at x_0 where F'(x_0) = f(x_0).

Intuition

If f is integrable, then $|F(x) - F(y)| = |\int_x^y f(t)dt| < \epsilon$ if x and y are close enough.
If f is continuous at $x_0 \in [t, y]$, then for close enough t,y:
$|\frac{F(y)-F(t)}{y-t} - f(x_0)| = |\frac{1}{y-t}\int_t^y [f(x) - f(x_0)]| < \epsilon$

Proof

Since $f \in \mathscr{R}$, then f is bounded. Let $|f(t)| \leq$ M for any t $\in$ [a,b]. Then for $\epsilon > 0$,
there is a $\frac{\epsilon}{M} > \delta > 0$ such that for all x,y $\in$ [a,b] where $|y - x| < \delta$, then:
$|F(y) - F(x)| = |\int_a^y f(t)dt - \int_a^x f(t)dt| = |\int_x^y f(t)dt| \leq M|y - x| < M\delta < \epsilon$
Thus, F is uniformly continuous on [a,b].
Suppose f is continuous at x_0. Then for $\epsilon > 0$, there is a $\delta > 0$ such that for all t $\in$
[a,b] where $|t - x_0| < \delta$, then $|f(t) - f(x_0)| < \epsilon$.
Thus, for s,t $\in [x_0 - \delta, x_0 + \delta]$ where $s < x_0 < t$:
$|\frac{F(t)-F(s)}{t-s} - f(x_0)| = |\frac{1}{t-s}\int_s^t f(x)dx - f(x_0)|$
$= |\frac{1}{t-s}\int_s^t f(x)dx - \frac{1}{t-s}(t-s)f(x_0)|$
$= |\frac{1}{t-s}\int_s^t f(x)dx - \frac{1}{t-s}\int_s^t f(x_0)dx|$
$= |\frac{1}{t-s}\int_s^t [f(x) - f(x_0)]dx| < |\frac{1}{t-s}(t-s)\epsilon| = \epsilon$
Thus, F'(x_0) = f(x_0).

Theorem 13.5.2: Fundamental Theorem of Calculus: $\int_a^b$ f(x) dx = F(b) - F(a)

  If $f \in \mathscr{R}$ on [a,b] and there is a differentiable F on [a,b] such that F' = f, then
   $\int_a^b$ f(x) dx = F(b) - F(a)

Intuition

Since F is differentiable, then by the Mean Value Theorem, there is a t $\in$ [x,y]   F(y) - F(x) = $(y - x) \cdot F'(t) = (y - x) \cdot f(t)$ Thus, $\int_a^b f(x)dx \approx \sum f(t)\Delta x = \sum [F(x_i) - F(x_{i-1})]$ = F(b) - F(a)

Proof

Since $f \in \mathscr{R}$, then for $\epsilon > 0$, there is a partition P = $\{x_0, ..., x_n\}$ of [a,b] such that:
  U(P,f) - L(P,f) $< \epsilon$
Since there is a differentiable F on [a,b], then F is differentiable over any $[x_{i-1}, x_i]$.
Then by the Mean Value Theorem, there are $t_i \in (x_{i-1}, x_i)$ such that:
  F(x_i) - F(x_{i-1}) = $(x_i - x_{i-1})$ F'(t_i) = Δx_i f(t_i)
Thus, $\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}$ [F(x_i) - F(x_{i-1})] = F(b) - F(a).
Since $\sum_{i=1}^{n} f(t_i)\Delta x_i \leq \sum_{i=1}^{n} \sup(f([x_{i-1}, x_i]))\Delta x_i$ = U(P,f), then:
  $|$[F(b) - F(a)] $- \int_a^b f(x)dx| = |\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx| \leq$ U(P,f) - L(P,f) $< \epsilon$

Theorem 13.5.3: Integration by Parts

  Suppose F,G are differentiable on [a,b] and F' = f, G' = g $\in \mathscr{R}$. Then:
   $\int_a^b$ F(x)g(x) dx = F(b)G(b) - F(a)G(a) - $\int_a^b$ f(x)G(x) dx

Intuition

By the derivative product rule, $(HG)' = H'G + HG'$. Then: $\int$ H'G dx = $\int$ (HG)' - HG' dx = $[HG]_a^b$ - $\int$ HG' dx

Proof

Let H(x) = F(x)G(x) where H'(x) = f(x)G(x) + F(x)g(x).
Since F,G are differentiable and thus, continuous, then F,G $\in \mathscr{R}$.
Thus, H' $\in \mathscr{R}$. Then by theorem 13.5.2:
  $\int_a^b$ H'(x) dx = H(b) - H(a)
  $\int_a^b$ f(x)G(x) + F(x)g(x) dx = H(b) - H(a)
  $\int_a^b$ F(x)g(x) dx = H(b) - H(a) - $\int_a^b$ f(x)G(x) dx

13.6   Integration of Vector-Valued Functions

Definition 13.6.1: Integration of Vector-Valued Functions

Let real $f_1, ..., f_k$ be defined on [a,b] where f = $(f_1, ..., f_k)$.

Then, let $f \in \mathscr{R}(\alpha)$ if each $f_i \in \mathscr{R}(\alpha)$ where $\int_a^b$ f dα = $(\int_a^b f_i$ dα,...,$\int_a^b f_k$ dα).

Thus, all these theorems hold true for vector-valued functions:

(a) Theorem 13.3.1a

If $f_1, f_2 \in \mathscr{R}(\alpha)$ and constant c, then:

$f_1 + f_2 \in \mathscr{R}(\alpha)$ with $\int_a^b f_1 + f_2$ dα = $\int_a^b f_1$ dα + $\int_a^b f_2$ dα

$cf_1 \in \mathscr{R}(\alpha)$ with $\int_a^b cf_1$ dα = $c \int_a^b f_1$ dα

(b) Theorem 13.3.1c

If $f \in \mathscr{R}(\alpha)$ on [a,b] where c $\in$ (a,b), then f $\in \mathscr{R}(\alpha)$ on [a,c],[c,b] where:

$\int_a^b$ f dα = $\int_a^c$ f dα + $\int_c^b$ f dα

(c) Theorem 13.3.1e

If $f \in \mathscr{R}(\alpha_1), \mathscr{R}(\alpha_2)$ and constant c, then:

$f \in \mathscr{R}(\alpha_1 + \alpha_2)$ with $\int_a^b$ f d$(\alpha_1 + \alpha_2)$ = $\int_a^b$ f dα_1 + $\int_a^b$ f dα_2

$f \in \mathscr{R}(c\alpha_1)$ with $\int_a^b$ f d$(c\alpha_1)$ = $c \int_a^b$ f dα_1

(d) Theorem 13.4.4

If $\alpha' \in \mathscr{R}$ on [a,b], then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$.

$\int_a^b$ f(x) dα = $\int_a^b$ f(x)α'(x) dx

(e) Theorem 13.5.2

If $f \in \mathscr{R}$ and there is a differentiable F on [a,b] such that F' = f, then:

$\int_a^b$ f(x) dx = F(b) - F(a)

Theorem 13.6.2: $|\int f d\alpha| \leq \int |f| d\alpha$

If f: [a,b] $\rightarrow \mathbb{R}^k$ where $f \in \mathscr{R}(\alpha)$, then $|f| \in \mathscr{R}(\alpha)$ where:

$|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$

Proof

For f = $(f_1, ..., f_k)$, then $|f| = (f_1^2 + ... + f_k^2)^{\frac{1}{2}}$.

Since $f \in \mathscr{R}(\alpha)$, then each $f_i \in \mathscr{R}(\alpha)$ so $f_1^2 + ... + f_k^2 \in \mathscr{R}(\alpha)$.

Since $x^{\frac{1}{2}}$ is continuous on [0,∞), then by theorem 13.2.4, $|f| = (f_1^2 + ... + f_k^2)^{\frac{1}{2}} \in \mathscr{R}(\alpha)$.

Let y = $(y_1, ..., y_k)$ where each $y_i = \int f_i d\alpha$. Thus, y = $\int f d\alpha$ where:

$|y|^2 = \sum_1^k y_i^2 = \sum_1^k (y_i \int f_i d\alpha) = \int (\sum y_i f_i) d\alpha$

By the Schwarz inequality, $\sum y_i f_i(t) \leq |y||f(t)|$. Thus:

$|y|^2 = \int (\sum y_i f_i) d\alpha \leq \int |y||f| d\alpha$

$|\int_a^b f d\alpha| = |y| \leq \int |f| d\alpha$

13.7   Line Integrals

Definition 13.7.1: Rectifiable Curves

A curve in $\mathbb{R}^k$ is a continuous γ: [a,b] $\rightarrow \mathbb{R}^k$.

If γ is 1-1, then γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is a closed curve.

For partition P = $\{x_0, ...x_n\}$ and curve γ on [a,b], let:

$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$

Then the length of γ is defined:

$\Lambda(\gamma) = \sup(\Lambda(P, \gamma))$

If $\Lambda(\gamma) < \infty$, then γ is rectifiable.

Theorem 13.7.2: Line Integral of $\gamma = \int_a^b |\gamma'(x)| \, dx$

If γ' is continuous on [a,b], then γ is rectifiable where

$\Lambda(\gamma) = \int_a^b |\gamma'(x)| \, dx$

Proof

Since γ is differentiable, then by theorem 13.5.2, for a $\leq x_{i-1} < x_i \leq$ b:

$|\gamma(x_i) - \gamma(x_{i-1})| = |\int_{x_{i-1}}^{x_i} \gamma'(x) \, dx| \leq \int_{x_{i-1}}^{x_i} |\gamma'(x)| \, dx$

Thus, for any partition P = $\{x_0, ..., x_n\}$:

$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^n (\int_{x_{i-1}}^{x_i} |\gamma'(x)| \, dx) = \int_a^b |\gamma'(x)| \, dx$

$\Lambda(\gamma) \leq \int_a^b |\gamma'(x)| \, dx$

Since γ' is continuous on compact [a,b], then γ' is uniformly continuous. Thus, for $\epsilon >$ 0, there is a $\delta > 0$ such that for all s,t $\in$ [a,b] where $|s-t| < \delta$, then $|\gamma'(s) - \gamma'(t)| < \epsilon$. Then for partition P where each $\Delta x_i < \delta$ and x $\in [x_{i-1}, x_i]$:

$|\gamma'(x)| \leq |\gamma'(x_i)| + \epsilon$

Then:

$\int_{x_{i-1}}^{x_i} |\gamma'(x)| \, dx \leq (|\gamma'(x_i)| + \epsilon) \, \Delta x_i = |\gamma'(x_i)|\Delta x_i + \epsilon \Delta x_i$

$= |\int_{x_{i-1}}^{x_i} [\gamma'(x) + \gamma'(x_i) - \gamma'(x)] \, dx | + \epsilon \Delta x_i$

$\leq |\int_{x_{i-1}}^{x_i} \gamma'(x) \, dx | + |\int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(x)] \, dx | + \epsilon \Delta x_i$

$\leq |\gamma(x_i) - \gamma(x_{i-1})| + \epsilon \Delta x_i + \epsilon \Delta x_i$

Thus:

$\int_a^b |\gamma'(x)| \, dx = \int_{x_0}^{x_1} |\gamma'(x)| \, dx + ... + \int_{x_{n-1}}^{x_n} |\gamma'(x)| \, dx$

$\leq \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon(\text{b-a}) = \Lambda(P, \gamma) + 2\epsilon(\text{b-a})$

Since $\int_a^b |\gamma'(x)| \, dx \leq \Lambda(\gamma) + 2\epsilon(\text{b-a}) \leq \int_a^b |\gamma'(x)| \, dx + 2\epsilon(\text{b-a})$, then:

$\Lambda(\gamma) = \int_a^b |\gamma'(x)| \, dx$.

14   Sequences and Series of Functions

14.1   Pointwise Convergence of Functions

Definition 14.1.1: Sequences and Series of Functions
  Suppose $\{f_n\}$ is a sequence of functions defined on set E.
  If $\{f_n(x)\}$ converges for any x $\in$ E, then:
    f(x) = $\lim_{n\to\infty} f_n(x)$ for x $\in$ E
  So for x $\in$ E and $\epsilon > 0$, there is a N_x such that for n $\geq N_x$:
    $|f_n(x) - f(x)| < \epsilon$

  If $\sum f_n(x)$ converges for every x $\in$ E, then:
    f(x) = $\sum_{n=1}^{\infty} f_n(x)$ for x $\in$ E

14.2   Uniform Convergence of Functions

Definition 14.2.1: Uniform Convergence
  $\{f_n\}$ converges uniformly on E to a function f if for all x $\in$ E:
    For $\epsilon > 0$, there is a N $\in \mathbb{Z}$ where for n $\geq$ N, then $|f_n(x) - f(x)| \leq \epsilon$

  $\sum f_n(X)$ converges uniformly if $\{s_n\}$ converges uniformly on E
  where $\sum_{i=1}^{n} f_i(x) = s_n(x)$:
    For $\epsilon > 0$, there is a N $\in \mathbb{Z}$ where for m $\geq$ n $\geq$ N, then $|\sum_{i=n}^{m} f_i(x)| \leq \epsilon$

Theorem 14.2.2: Cauchy Criterion for sequence of functions
  $\{f_n\}$ converges uniformly on E if and only if:
    For $\epsilon > 0$, there is a N $\in \mathbb{Z}$ where for n,m $\geq$ N and every x $\in$ E, then:
      $|f_n(x) - f_m(x)| \leq \epsilon$
Intuition

Convergent sequences are Cauchy and Cauchy sequences in $\mathbb{R}$ are convergent.

Proof

If $\{f_n\}$ converges uniformly on E, then for $\epsilon > 0$, there is a N where for n,m $\geq$ N:
  $|f_n(x) - f(x)| \leq \frac{\epsilon}{2}$ $|f_m(x) - f(x)| \leq \frac{\epsilon}{2}$
  $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

If for $\epsilon > 0$, there is a N $\in \mathbb{Z}$ where for n,m $\geq$ N and every x $\in$ E so
$|f_n(x) - f_m(x)| \leq \epsilon$, then $\{f_n\}$ is a Cauchy sequence in $\mathbb{R}^k$ and thus, converges.
Then there is a f(x) where f(x) = $\lim_{m\to\infty} f_m(x)$. Thus:
  $|f_n(x) - f(x)| \leq |f_n(x) - \lim_{m\to\infty} f_m(x)| \leq \epsilon$

Theorem 14.2.3: Connection between Convergence and Uniform Convergence

Suppose for x $\in$ E, $\lim_{n\to\infty} f_n(x) = $ f(x). Let $M_n = \sup_{x\in E}(|f_n(x) - f(x)|)$.

Then $\{f_n\}$ converges uniformly to f on E if and only if $\lim_{n\to\infty} M_n = 0$.

Intuition

Pointwise convergence implies for any particular x_0 and $\epsilon > 0$ so $|f_n(x_0) - f(x_0)| < \epsilon$. Uniform convergence implies for every x and $\epsilon > 0$ so $|f_n(x) - f(x)| < \epsilon$.

Thus, uniform convergence implies pointwise convergence, but pointwise convergence might not imply uniform convergence since for n $\geq N_1$, $|f_n(x_0) - f(x_0)| < \epsilon$, but there might always exist $x_1 \neq x_0$ where $|f_n(x_1) - f(x_1)| \not< \epsilon$ until $N_2 > N_1$. If $\sup_{x\in E}(|f_n(x) - f(x)|) \to 0$, then x_1 cannot exist and thus, pointwise implies uniform.

Proof

If $\{f_n\}$ converges uniformly to f on E, then for $\epsilon > 0$, there is a N where for n $\geq$ N:
$\quad |f_n(x) - f(x)| \leq \epsilon \qquad$ for all x $\in$ E
Thus, $M_n = \sup_{x\in E}(|f_n(x) - f(x)|) \leq \epsilon$ so $\lim_{n\to\infty} M_n \leq \epsilon$.

If $\lim_{n\to\infty} M_n = 0$, then for $\epsilon > 0$, there is a N where for n $\geq$ N so $\lim_{n\to\infty} M_n \leq \epsilon$.
Since $\lim_{n\to\infty} f_n(x) = $ f(x) for x $\in$ E, there is a N_x for each x where for n $\geq N_x$:
$\quad |f_n(x) - f(x)| \leq \epsilon$
Since there is a N such that for n $\geq$ N so $M_n = \sup_{x\in E}(|f_n(x) - f(x)|) \leq \epsilon$, then there is $\sup_{x\in E}(\{N_x\}) = $ N such that for all x $\in$ E where n $\geq$ N:
$\quad |f_n(x) - f(x)| \leq \sup_{x\in E}(|f_n(x) - f(x)|) = M_n \leq \epsilon$

Theorem 14.2.4: Condition for Uniform Convergence for Series

For $\{f_n\}$ defined on E, suppose $|f_n(x)| \leq M_n$ for any x $\in$ E.

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on E.

Proof

If $\sum M_n$ converges, then for $\epsilon > 0$, there is a N where for m $\geq$ n $\geq$ N:
$\quad |\sum_{i=n}^{m} f_i(x)| \leq \sum_{i=n}^{m} |f_i(x)| \leq \sum_{i=n}^{m} M_n \leq \epsilon$

14.3   Uniform Convergence and Continuity

Theorem 14.3.1: $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$

      Suppose $\{f_n\}$ converges uniformly to f on a set E.

      Let x $\in$ E' where $\lim_{t \to x} f_n(t) = A_n$.

      Then $\{A_n\}$ converges where $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$.

Intuition

Since $\{f_n\}$ converges uniformly so for any t, then $\lim_{n \to \infty} f_n(t) = f(t)$.
For t near x, then $\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} f(t)$.

Note uniform convergence is essential since $f_n \to f$ and $f_n(t) \to f(t)$ for any t including t near x. Since pointwise convergence possibly $f_n(t) \not\to f(t)$ for some t near x, then continuity possibly might not hold.

Proof

Since $\{f_n\}$ converges uniformly, then for $\epsilon > 0$, there is a N where for m,n $\geq$ N and every t $\in$ E, then $|f_n(t) - f_m(t)| \leq \epsilon$. Then for t $\to$ x:

    $|A_n - A_m| = |\lim_{t \to x} f_n(t) - \lim_{t \to x} f_m(t)| \leq \epsilon$

Thus, $\{A_n\}$ is a Cauchy Sequence in $\mathbb{R}^k$ so $\{A_n\}$ converges to A $= \lim_{n \to \infty} A_n$.
Since $\{A_n\}$ converges to A, then for $\epsilon > 0$, there is a N_1 where for n $\geq N_1$:

    $|A - A_n| \leq \frac{\epsilon}{3}$

Since $\{f_n\}$ converges uniformly to f, then for $\epsilon > 0$, there is a N_2 where for n $\geq N_2$:

    $|f(t) - f_n(t)| \leq \frac{\epsilon}{3}$.

Since there is a r such that for t $\in N_r(x)$, then:

    $|f_n(t) - \lim_{t \to x} f_n(t)| = |f_n(t) - A_n| \leq \frac{\epsilon}{3}$

Thus, for t $\to$ x, $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \leq \epsilon$.
Thus, $\lim_{t \to x} f(t) = $ A $= \lim_{n \to \infty} A_n$.

Theorem 14.3.2: Uniform Convergence perserve Continuity

      If $\{f_n\}$ converges uniformly to f on E where each f_n is continuous on E, then

      f is continuous on E.

Intuition

If each f_n is continuous:

    $\lim_{t \to x} f(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{n \to \infty} f_n(x) = f(x)$

Proof

Since $\{f_n\}$ converges uniformly to f, then by theorem 14.3.1, for any x $\in$ E':

    $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$

Since each f_n is continuous, then:

    $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{t \to x} f(t)$

    $\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{n \to \infty} f_n(x) = f(x)$

Theorem 14.3.3: Decreasing, continuous sequence over compact converges uniformly

Suppose K is compact and

(a) $\{f_n\}$ is a sequence of continuous functions on K

(b) $\{f_n\}$ converges pointwise to a continuous f on K

(c) $f_n(x) \geq f_{n+1}(x)$ for all x $\in$ K

Then f_n converges uniformly to f on K.

Proof

Let $g_n = f_n - f$ so g_n is continuous where $g_n \geq g_{n+1}$.
Thus, $\lim_{n \to \infty} g_n(x) = 0$ pointwise. For $\epsilon > 0$, let $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
Since g_n is continuous and the set of $g_n(x) \geq \epsilon$ is closed, then K_n is closed. Since closed $K_n \subset$ compact K, then K_n is compact.
Since $g_n \geq g_{n+1}$, then $K_{n+1} \subset K_n$. For any x $\in$ K, $\lim_{n \to \infty} g_n(x) = 0$ so there is a N_x such that x $\notin K_n$ if n $> N_x$. Thus, any x $\notin \cap_{n=1}^{\infty} K_n$ so $\cap_{n=1}^{\infty} K_n = \emptyset$.
Since $\cap_{n=1}^{\infty} K_n = \emptyset$, then K_n is empty for some N.
Thus, $0 \leq g_n(x) < \epsilon$ for all x $\in$ K where n $\geq$ N.

Definition 14.3.4: Supremum Norm

$\mathscr{C}(X)$ is the set of all complex, continuous, bounded functions in metric X.

If X is compact, then bounded is not needed

Then for each $f \in \mathscr{C}(X)$, associate a supremum norm:

$$||f|| = \sup_{x \in X} |f(x)| < \infty$$

(a) $||f(x)|| = 0$ if and only if $f(x) = 0$ for every x $\in$ X

(b) Since $|f + g| \leq |f| + |g| \leq ||f|| + ||g||$, then $||f + g|| \leq ||f|| + ||g||$

Then for $f, g \in \mathscr{C}(X)$, let distance $||f - g||$ and thus, $\mathscr{C}(X)$ is a metric space.

By theorem 14.2.3, $\{f_n\} \to$ f on $\mathscr{C}(X)$ if and only if $\{f_n\} \to$ f uniformly on X.

Theorem 14.3.5: $\mathscr{C}(X)$ is a complete metric space

$\mathscr{C}(X)$ is a complete metric space

Intuition

A Cauchy sequence $\{f_n\}$ is uniformly convergent to f.
Since $\mathscr{C}(X)$ contain continuous functions, then f is continuous.
Since functions in $\mathscr{C}(X)$ are bounded, then f is bounded.

Proof

Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$.
Since $\{f_n\} \in \mathscr{C}(X)$, then each f_n is continuous and bounded.
Then for $\epsilon > 0$, there is a N such that for n,m $\geq$ N, then:
    $|f_n - f_m| \leq ||f_n - f_m|| \leq \epsilon$
Then by theorem 14.2.2, $\{f_n\}$ converges uniformly to f.
Since each f_n is continuous and $\{f_n\}$ converges uniformly to f, then by theorem 14.3.2, f is continuous on $\mathscr{C}(X)$.
Since $\{f_n\}$ converges uniformly to f, there is a N where for n $\geq$ N:
    $|f - f_n(x)| \leq \epsilon$
Since each f_n is bounded, then f is bounded. Since f is continuous and bounded, then f $\in \mathscr{C}(X)$. Thus, every Cauchy sequence $\{f_n\}$ converges to f $\in \mathscr{C}(X)$.

14.4   Uniform Convergence and Integration

Theorem 14.4.1: Uniform Convergence perserves Integrability

If $\{f_n\} \in \mathscr{R}(\alpha)$ converges uniformly to f on [a,b], then $f \in \mathscr{R}(\alpha)$ on [a,b] where:
$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$

Intuition

Since f_n is integrable, then $\int_a^b f_n \, d\alpha$ exist and since $\{f_n\}$ uniformly converges, then for $\epsilon > 0$, $|f - f_n| < \epsilon$. Thus, for a large enough n, $\int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha$.

Proof

Since $\{f_n\}$ converges uniformly to f, then for $\epsilon > 0$:
$|f - f_n| < \epsilon$ $\qquad \rightarrow \qquad$ $f_n - \epsilon < f < f_n + \epsilon$
Then:
$\int_a^b f_n - \epsilon \, d\alpha < \underline{\int}_a^b f \, d\alpha \leq \overline{\int}_a^b f \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$
Thus,
$\overline{\int}_a^b f \, d\alpha$ - $\underline{\int}_a^b f \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$ - $\int_a^b f_n - \epsilon \, d\alpha = 2\epsilon[\alpha(b) - \alpha(a)]$
So, $\int_a^b f \, d\alpha$ exists and since $f_n \in \mathscr{R}(\alpha)$ where $\int_a^b f_n - \epsilon \, d\alpha < \int_a^b f_n d\alpha < \int_a^b f_n + \epsilon \, d\alpha$:
$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$

Theorem 14.4.2: Uniform Convergence perserves Integrability for series

If $f_n \in \mathscr{R}(\alpha)$ on [a,b] and f(x) = $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then:
$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$

Proof

Since $f_n \in \mathscr{R}(\alpha)$, then f(x) $\in \mathscr{R}(\alpha)$. Since f(x) converges uniformly, then by thereom 14.4.1, then $\int_a^b f \, d\alpha = \lim_{N\to\infty} \sum_{n=1}^{N} \int_a^b f_n \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$.

14.5   Uniform Convergence and Differentiation

Theorem 14.5.1: Uniform Convergence of Derivatives perserve Differentiability

Suppose $\{f_n\}$ are differentiable on [a,b] such that $\{f_n(x_0)\}$ converges for some $x_0 \in$ [a,b]. If $\{f'_n\}$ converges uniformly on [a,b], then $\{f_n\}$ converges uniformly to f on [a,b] where:

$$f'(x) = \lim_{n \to \infty} f'_n(x) \qquad \text{for x} \in \text{[a,b]}$$

Intuition

Since $\{f'_n\}$ converges uniformly, for t near x, then by the Mean Value Theorem:

$\frac{f_n(t) - f_n(x)}{t-x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$

Since $\{f'_n\}$ converges uniformly, by the Mean Value Theorem, there is a t $\in [x_1, x_2]$:

$|[f_n(x_2) - f_m(x_2)] - [f_n(x_1) - f_m(x_1)]| = (x_2 - x_1)|f'_n(t) - f'_m(t)| < \epsilon$

Thus, $\{f_n - f_m\}$ converges uniformly so if $\{f_n\}$ converges for some x_0:

$[f_n(x) - f_m(x)] = |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] + [f_n(x_0) - f_m(x_0)]| \le \epsilon$

Thus, $\{f_n\}$ converges uniformly which preserves continuity so for t near x as n $\to \infty$:

$f'(x) = \frac{f(t) - f(x)}{t-x} = \frac{f_n(t) - f_n(x)}{t-x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$

Note uniform convergence of $\{f'_n\}$ gives $\frac{f_n(t) - f_n(x)}{t-x} = \frac{(t-x)f'_n(x)}{t-x}$. Then uniform convergence of $\{f'_n\}$ with convergent $f_n(x_0)$ leads to uniform convergence of $\{f_n\}$ which gives $\frac{f(t) - f(x)}{t-x} = \frac{f_n(t) - f_n(x)}{t-x}$.

Proof

Since $f_n(x_0)$ converges for some $x_0 \in$ [a,b], then for $\epsilon > 0$, there is a N_1 such that for $n_1, m_1 \ge N_1$:

$|f_{n_1}(x_0) - f_{m_1}(x_0)| < \frac{\epsilon}{2}$

Since f'_n converges uniformly, then there is a N_2 such that for $n_2, m_2 \ge N_2$:

$|f'_{n_2}(t) - f'_{m_2}(t)| < \frac{\epsilon}{2(b-a)}$

Let N = max(N_1, N_2). Then for n,m $\ge$ N:

$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \qquad\qquad |f'_n(t) - f'_m(t)| < \frac{\epsilon}{2(b-a)}$

Since f_n is differentiable, then $f_n - f_m$ is differentiable. Then by the Mean Value Theorem, there is a x $\in$ (a,b) such that:

$|[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)]| \le |x - t||f'_n(t) - f'_m(t)| < |x - t|\frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2}$

Thus, for n,m $\ge$ N:

$|f_n(x) - f_m(x)| \le |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| + |f_n(x_0) - f_m(x_0)| < \epsilon$

Thus, $\{f_n\}$ converges uniformly to f(x) = $\lim_{n \to \infty} f_n(x)$ where:

$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x} \qquad\qquad \phi(t) = \frac{f(t) - f(x)}{t-x}$

Since $\lim_{t \to x} |\phi_n(t) - \phi_m(t)| < \frac{\epsilon}{2(b-a)}$, then:

$\lim_{n \to \infty} \phi_n(t) = \frac{f(t) - f(x)}{t-x} = \phi(t)$

Since $\{\phi_n(t)\}$ converges uniformly to $\phi(t)$, then by theorem 14.3.1:

$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} \lim_{t \to x} \phi_n(t) = \lim_{n \to \infty} f'_n(x)$

There exists a real continuous function on $\mathbb{R}$ which is nowhere differentiable.

Proof

Let $\phi(x) = |x|$ for x $\in$ [-1,1]. Then to extend to all real x, let $\phi(x+2) = \phi(x)$.
Then ϕ is continuous on $\mathbb{R}$ where for s,t $\in \mathbb{R}$, $|\phi(s) - \phi(t)| \leq |s - t|$.
Let f(x) $= \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$.
Since f(x) $\leq \sum_{n=0}^{\infty} (\frac{3}{4})^n$, then f(x) converges uniformly and since $\phi(x)$ is continuous,
then f(x) is continuous. Then for a fixed x and positive integer m, choose $\delta_m = \pm\frac{1}{2}4^{-m}$
such that no integer lies in $(4^m x, 4^m(x + \delta_m))$. Let $\gamma_n = \frac{\phi(4^n(x+\delta_n)) - \phi(4^n x)}{\delta_m}$.
For n > m, $4^n \delta_m$ is even so $\gamma_n = 0$. For n $\in$ [0,m], $|\gamma_n| \leq \frac{|4^n \delta_n|}{\delta_m} = 4^m < 4^n$.
Since $|\gamma_m| = 4^m$, then:
$\quad |\frac{f(x+\delta_m) - f(x)}{\delta_m}| = |\sum_{n=0}^{m}(\frac{3}{4})^n \gamma_n| + |\sum_{n=m+1}^{\infty}(\frac{3}{4})^n \gamma_n| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1)$
As m $\to \infty$, then $\delta_m \to 0$, but $|\frac{f(x+\delta_m) - f(x)}{\delta_m}| \to \infty$ so f is not differentiable at any x.

14.6   Equicontinuous Families of Functions

Definition 14.6.1: Boundedness

Let $\{f_n\}$ be defined on set E.
$\{f_n\}$ is pointwise bounded on E if for x $\in$ E and every n, there is a ϕ where:
$\quad |f_n(x)| < \phi(x)$
$\{f_n\}$ is uniformly bounded on E if for every n and x $\in$ E, there is a M where:
$\quad |f_n(x)| < M$

Definition 14.6.2: Equicontinuous

A family of complex functions, $\mathscr{F}$, defined on set E $\subset$ X is equicontinuous if
for every $\epsilon > 0$, there is a $\delta > 0$ such that for all x,y $\in$ E and f $\in \mathscr{F}$ where
$d(x,y) < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem 14.6.3: Pointwise bounded $\{f_n\}$ over countable sets have convergent $\{f_{n_k}\}$

If $\{f_n\}$ are pointwise bounded, complex functions on countable set E, then
$\{f_n\}$ has subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every x $\in$ E.

Intuition

Any $\{f_{n_k}\} \subset \{f_n\}$ is pointwise bounded so there is a convergent subsequence for a
particular x. Let $\{f_{n_{k_1}}\}$ be a convergent subsequence for x_1. Then find a subsequence
$\{f_{n_{k_2}}\} \subset \{f_{n_{k_1}}\}$ which converges for x_2. Continue the process until every x.

Proof

For each $x_i \in$ E, let $\{x_i\}$. For x_1, $\{f_n(x_1)\}$ is piecewise bounded so there exists a
subsequence $\{f_{1,k}(x_1)\}$ which converges as k $\to \infty$.
Since $\{f_{1,k}\}$ is piecewise bounded since $\{f_{1,k}\} \subset \{f_n\}$, then there is a subsequence
$\{f_{2,k}\} \subset \{f_{1,k}\}$ such that $\{f_{2,k}(x_2)\}$ converges as k $\to \infty$. Then continuing the pattern:
$\quad S_1$: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ ...
$\quad S_2$: $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ ...
$\quad S_3$: $f_{3,1}$ $f_{3,2}$ $f_{3,3}$ ...
$\quad$...
Thus, $\{f_{n,n}(x_i)\}$ converges as n $\to \infty$ for every $x_i \in$ E.

Theorem 14.6.4: Uniform convergent $\{f_n\}$ where $f_n \in \mathscr{C}(K)$ is equicontinuous

> If K is a compact metric space where $f_n \in \mathscr{C}(K)$ and $\{f_n\}$ converges uniformly
> on K, then $\{f_n\}$ is equicontinuous on K.

Intuition

Since $\{f_n\}$ converges uniformly, then there is a N where for n > N, then $|f_n - f_N| < \epsilon$.
Since $\{f_n\}$ is continuous over compact K, then $\{f_n\}$ is uniformly continuous. So for
d(x,y) < δ, then:
$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\epsilon$$

Proof

Since $\{f_n\}$ converges uniformly, then for $\epsilon > 0$, there is a N such that for n > N:
$$||f_n - f_N|| < \tfrac{\epsilon}{3}$$
Since f_i for i $\in$ [1,N] is continuous over compact K, then f_i is uniformly continuous
so there is a $\delta > 0$ such that for all x,y where d(x,y) < δ, then $|f_i(x) - f_i(y)| < \tfrac{\epsilon}{3}$.
Then for n > N and d(x,y) < δ:
$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \epsilon$$
Thus, for $\epsilon > 0$, there is a $\delta > 0$ such that for all f_n and x,y $\in$ K where d(x,y) < δ,
$|f_n(x) - f_n(y)| < \epsilon$. So, $\{f_n\}$ is equicontinuous.

Theorem 14.6.5: Pointwise bounded and equicontinuous $\{f_n\}$ over compact K is
uniformly bounded and have uniformly convergent $\{f_{n_k}\}$

> If K is compact where $\{f_n\} \in \mathscr{C}(K)$ is pointwise bounded and equicontinuous:
>   (a) $\{f_n\}$ is uniformly bounded on K
>   (b) $\{f_n\}$ contains a uniformly convergent subsequence

Intuition

Since $\{f_n\}$ is equicontinuous, for d(x,y) < δ, then $|f_n(x) - f_n(y)| < \epsilon$.
Since $\{f_n\}$ is pointwise bounded on compact K, there are finite $x_0, .., x_n$ such that
d(x,x_i) < δ so $|f_n(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i)| < \epsilon + M$.
For a countable dense subset of K, the countability gives a convergent subsequence
$\{g_n\}$ and the dense gives d(x,x_i) < δ for finite $x_1, ..., x_m$ so:
$$|g_n(x) - g_m(y)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)| < \epsilon.$$

Proof

Since f_n is equicontinuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for x,y $\in$ K
where d(x,y) < δ, then $|f_n(x) - f_n(y)| < \epsilon$.
Since K is compact, there are finite $p_1, ..., p_r \in$ K so for any x $\in$ K, there is at least
one p_i so d(x,p_i) < δ. Since $\{f_i\}$ is pointwise bounded, there is a M_i so $|f_n(p_i)| < M_i$.
Let M = $\max(M_1, ..., M_r)$. So, $|f_n(x)| < |f_n(x) - f_n(p_i)| + |f_n(p_i)| < \epsilon + M_i < \epsilon + M$.
Thus, $\{f_n\}$ is uniformly bounded on K.

Let countable dense E $\subset$ K. By theorem 14.6.3, $\{f_n\}$ has a convergent subsequence
$\{f_{n_i}(x)\}$ for every x $\in$ E. Let $V(x,\delta) = \{$y $\in$ K : d(x,y) < $\delta\}$ so $|f_n(x) - f_n(y)| < \tfrac{\epsilon}{3}$.
Since E is dense in compact K, there are finitely many $x_1, ..., x_m \in$ E such that:
    K $\subset V(x_1,\delta) \cup ... \cup V(x_m,\delta)$.
Since $\{f_{n_i}(x)\}$ converges for every x $\in$ E, there is a N where for $n_i, n_j \geq$ N, s $\in$ [1,m]:
    $|f_{n_i}(x_s) - f_{n_j}(x_s)| < \tfrac{\epsilon}{3}$
Thus, for any x $\in$ K, there is a $x_s \in$ E such that:
    $|f_{n_i}(x) - f_{n_j}(x)| \leq |f_{n_i}(x) - f_{n_i}(x_s)| + |f_{n_i}(x_s) - f_{n_j}(x_s)| + |f_{n_j}(x_s) - f_{n_j}(x)| < \epsilon$
Thus, $\{f_n\}$ contains a subsequence that uniformly converges.

14.7   Stone-Weierstrass Theorem

Theorem 14.7.1: There are polynomials that converge uniformly to continuous f

For complex continuous f on [a,b], there is a sequence of polynomials $\{P_n\}$ that converges uniformly to f(x).

Proof

Let [a,b] = [0,1] where f(0) = f(1) = 0 and f(x) = 0 if $x \notin [0,1]$.
Thus, f is uniformly continuous over $\mathbb{R}$.
Let $Q_n(x) = c_n(1-x^2)^n$ where c_n is chosen so $\int_{-1}^{1} Q_n(x)\ dx = 1$. Since:

$$\int_{-1}^{1} (1-x^2)^n\ dx = 2\int_0^1 (1-x^2)^n\ dx \geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n\ dx \geq 2\int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2\ dx$$
$$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

so $c_n < \sqrt{n}$. Thus for $\delta > 0$, $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$ so $Q_n \to 0$ on $|x| \in [\delta, 1]$.
Let $P_n(x) = \int_{-1}^{1}$ f(x+t)$Q_n(t)$ dt for x $\in$ [0,1]. Since $P_n(x) = \int_{-x}^{1-x}$ f(x+t)$Q_n(t)$ dt = $\int_0^1$ f(t)$Q_n(t - x)$ dt which is a polynomial so $\{P_n\}$ is a sequence of polynomials.
Since f is uniformly continuous, for $\epsilon > 0$, there is a $\delta > 0$ such that for $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let M = sup($|f(x)|$). Then:

$$|P_n(x) - f(x)| \leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)\ dt$$
$$\leq 2M\int_{-1}^{-\delta} Q_n(t)\ dt + \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\ dt + 2M\int_{\delta}^{1} Q_n(t)\ dt$$
$$\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} < \epsilon \qquad \text{for a large enough n}$$

Corollary 14.7.2: There are polynomials that converges uniformly to $|x|$

For [-a,a], there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and $P_n(x)$ converges uniformly to $|x|$.

Proof

By Theorem 14.7.1, there is a $\{P_n^*\}$ of real polynomials that converges uniformly to $|x|$. Since $P_n^*(0) \to |0| = 0$, let $P_n(x) = P_n^*(x)$ - $P_n^*(0)$.

Definition 14.7.3: Algebra, Uniformly Closed, and Uniform Closure

A family of complex functions on E, $\mathscr{A}$, is an algebra if for f,g $\in \mathscr{A}$, then:

(a) f+g $\in \mathscr{A}$

(b) fg $\in \mathscr{A}$

(c) cf $\in \mathscr{A}$ for complex constant c

$\mathscr{A}$ is uniformly closed if:
For any $f_n \in \mathscr{A}$ where f_n uniformly converges to f, then $f \in \mathscr{A}$

Let the uniform closure, $\mathscr{B}$, be the set of all uniformly convergent f from $\mathscr{A}$.

Theorem 14.7.4: Bounded algebra implies Uniformly closed uniform closure

For algebra $\mathscr{A}$ of bounded functions, $\mathscr{B}$ is a uniformly closed algebra.

Proof

If f,g $\in \mathscr{B}$, there are uniformly convergent $\{f_n\}$, $\{g_n\}$ where $f_n \to$ f, $g_n \to$ g and $f_n, g_n \in \mathscr{A}$. Since f_n, g_n are bounded and $\mathscr{A}$ is an algebra, then uniformly convergent:
$$f_n + g_n \to \text{f+g} \qquad f_n g_n \to \text{fg} \qquad cf_n \to \text{cf}$$
Thus, $f + g, fg, cf \in \mathscr{B}$ so $\mathscr{B}$ is a uniformly closed algebra.

Definition 14.7.5: Separate Points

For family of functions, $\mathscr{A}$, separate points on E:
If for every pair of distinct $x_1, x_2 \in$ E, there is a f $\in \mathscr{A}$ where f$(x_1) \neq$ f(x_2).

$\mathscr{A}$ vanishes at no point of E:
If for each x $\in$ E, there is a g $\in \mathscr{A}$ such that g(x) $\neq 0$

Theorem 14.7.6: Non-vashing, separate algebra contain all values

Suppose algebra $\mathscr{A}$ separates points and vanishes at no points on E. If x_1, x_2 are distinct points, then for constants c_1, c_2, there is a f $\in \mathscr{A}$ where:
f$(x_1) = c_1$ and f$(x_2) = c_2$.

Proof

Since $\mathscr{A}$ separates points and vanishes at no points on E, then there are g,h,k $\in \mathscr{A}$:
$\quad g(x_1) \neq g(x_2) \qquad h(x_1) \neq 0 \qquad k(x_2) \neq 0$
Let u $= k(g - g(x_1))$ and v $= h(g - g(x_2))$ so u,v $\in \mathscr{A}$ where $u(x_1) = v(x_2) = 0$ and $u(x_2), v(x_1) \neq 0$. Then, f $= \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$ have $f(x_1) = c_1$ and $f(x_2) = c_2$.

Theorem 14.7.7: Stone-Weierstrass Theorem

If algebra of real continuous functions on compact K, $\mathscr{A}$, separates points and vanishes at no points on K, then $\mathscr{B}$ consist of all real continuous functions.

Proof

Claim: If f $\in \mathscr{B}$, then $|f| \in \mathscr{B}$.
Let a $= \sup(|f(x)|)$. By Corollary 14.7.2, for $\epsilon > 0$, there are $c_1, ..., c_n$ such that:
$\quad |\sum_{i=1}^{n} c_i y^i - |y|| < \epsilon \qquad$ for y $\in$ [-a,a]
Since $\mathscr{B}$ is an algebra, then g $= \sum_{i=1}^{n} c_i f^i \in \mathscr{B}$. Thus:
$\quad |g(x) - |f(x)|| < \epsilon \qquad$ for x $\in$ K
Since β is uniformly closed, then $|f(x)| \in \mathscr{B}$.

Claim: If f,g $\in \mathscr{B}$, then min(f,g), max(f,g) $\in \mathscr{B}$.
Since:
$\quad \max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2} \qquad \min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2}$
then max(f,g), min(f,g) $\in \mathscr{B}$.

Claim: For real, continuous f on K and $\epsilon > 0$, there exist $g_x \in \mathscr{B}$ where $g_x(x) =$ f(x) and $g_x(t) > f(t) - \epsilon$ for t $\in$ K.
Since $\mathscr{A} \subset \mathscr{B}$ where $\mathscr{A}$ separates points and vanishes at no points on E, then $\mathscr{B}$ is the same. Then by theorem 14.7.6, for y $\in$ K, there is a $h_y \in \mathscr{B}$ where:
$\quad h_y(x) =$ f(x) $\qquad h_y(y) =$ f(y)
Since h_y is continuous, there is an open set J_y such that $h_y(t) > f(t) - \epsilon$ for t $\in J_y$.
Since K is compact, there are finite $y_1, ..., y_n$ such that K $\subset J_{y_1} \cup ... \cup J_{y_n}$.
Let $g_x = \max(h_{y_1}, ..., h_{y_n})$ so $g_x \in \mathscr{B}$ where $g_x(t) > f(t) - \epsilon$ for t $\in$ K.

Claim: For real, continuous f on K and $\epsilon > 0$, there is a h $\in \mathscr{B}$ where $|h(x) - f(x)| < \epsilon$.
Since g_x is continuous, there is an open set V_x where $g_x(t) < f(t) + \epsilon$ for t $\in V_x$.
Since K is compact, there are finite $x_1, ..., x_m$ such that K $\subset V_{x_1} \cup ... \cup V_{x_m}$.
Let h $= \min(g_{x_1}, ..., g_{x_m})$ so h $\in \mathscr{B}$ where h(t) $> f(t) - \epsilon$. But, h(t) $< f(t) + \epsilon$ so $|h(x) - f(x)| < \epsilon$. Since $\mathscr{B}$ is uniformly closed, then the theorem holds true.

Definition 14.7.8: Self-Adjoint

$\mathscr{A}$ is self-adjoint if for every f $\in \mathscr{A}$, then $\overline{f} \in \mathscr{A}$

Theorem 14.7.9: Stone-Weierstrass for complex functions

If self-adjoint algebra of complex continuous functions on compact K, $\mathscr{A}$, separates points and vanishes at no points on K, then $\mathscr{B}$ consist of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$.

Proof

Let $\mathscr{A}_R$ be the set of all real functions on K in $\mathscr{A}$.

If f $\in \mathscr{A}$ and f $= u + iv$ for real u,v then $2u = f + \overline{f} \in \mathscr{A}_R$.

If $x_1 \neq x_2$, there exists f $\in \mathscr{A}$ such that f$(x_1) = 1$ and f$(x_2) = 0$ so u$(x_1) \neq$ u(x_2) so $\mathscr{A}_R$ separates points.

If x $\in$ K, then g(x) $\neq 0$ for some g $\in \mathscr{A}$ and there is a complex λ such that $\lambda g(x) > 0$. If f $= \lambda g$, then u(x) > 0 so $\mathscr{A}_R$ vanishes at no point of K.

Then by theorem 14.7.7, every real continuous function on K lies in $\mathscr{B}_{\mathscr{A}_R}$ and since $\mathscr{B}_{\mathscr{A}_R} \subset \mathscr{B}$, then every real continuous function lies in $\mathscr{B}$. If f is complex continuous where f $=$ u+iv, then f $\in \mathscr{B}$ since u,v $\in \mathscr{B}$.

15   Special Functions

15.1   Power Series

Power series, $f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$

If f(x) converges for $|x - a| < R$ for some R, then f is expanded in a power series about x = a.

Theorem 15.1.2: Convergent Power Series are differentiable

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, then f(x) converges uniformly on $[-R + \epsilon, R - \epsilon]$ for any $\epsilon > 0$.

Then, f is continuous and differentiable in $(-R, R)$ where:

$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$

Proof

For $\epsilon > 0$ and $|x| \leq R - \epsilon$:

$|c_n x^n| \leq |c_n (R - \epsilon)^n|$

Since $\sum c_n (R - \epsilon)^n$ converges absolutely in $[-R + \epsilon, R - \epsilon]$, then f(x) uniformly converges on $[-R + \epsilon, R - \epsilon]$.

Since $\lim_{n \to \infty} \sqrt[n]{n} = 1$, then:

$\lim_{n \to \infty} \sup(\sqrt[n]{n|c_n|}) = \lim_{n \to \infty} \sup(\sqrt[n]{|c_n|})$

so f(x) and f'(x) have the same interval of convergence so f'(x) uniformly converges on $[-R + \epsilon, R - \epsilon]$. Since f'(x) exists, then f is differentiable and thus, continuous.

Corollary 15.1.3: Power Series have infinite derivatives

On $(-R, R)$, f has derivatives of all orders:

$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)...(n-k+1) c_n x^{n-k}$

$f^{(k)}(0) = k! c_k$

Proof

By theorem 15.1.2, apply derivative k times.

Theorem 15.1.4: Continuity of Power Series at endpoints

Suppose $\sum c_n$ converges where $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for x $\in$ (-1,1).

Then $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$.

Proof

Let $s_n = c_0 + ... + c_n$.

$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = \sum_{n=0}^{m} s_n x^n - \sum_{n=0}^{m} s_{n-1} x^n$

$= \sum_{n=0}^{m} s_n x^n - \sum_{n=0}^{m-1} s_n x^{n+1} = (1 - x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m$

Since $|x| < 1$, then as m $\to \infty$, then $s_m x^m \to 0$. Let $s = \lim_{n \to \infty} s_n$.

Thus, for $\epsilon > 0$, there is a N such that for n > N, then $|s - s_n| < \frac{\epsilon}{2}$.

Since $(1 - x) \sum_{n=0}^{\infty} x^n = (1 - x) \frac{1}{1-x} = 1$, then:

$|f(x) - s| = |(1 - x) \sum_{n=0}^{\infty} (s_n - s) x^n| \leq (1 - x) \sum_{n=0}^{N} |s_n - s||x|^n + \frac{\epsilon}{2}$

Then choose $\delta > 0$ such that $(1 - x) \sum_{n=0}^{N} |s_n - s| < \frac{\epsilon}{2}$ for x > 1 - δ. Thus:

$|\lim_{x \to 1} f(x) - s| < \epsilon$

Corollary 15.1.5: Cauchy Product

If $\sum a_n \to A$, $\sum b_n \to B$, and $\sum c_n \to C$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then:

C = AB

Proof

For $x \in (0,1)$, let:

$\quad f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n \qquad h(x) = \sum_{n=0}^{\infty} c_n x^n$

Then f,g,h absolutely converges. Note fg = h.

By theorem 15.1.4, then $AB = \lim_{x \to 1} f(x)g(x) = \lim_{x \to 1} h(x) = C$.

Theorem 15.1.6: $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{1=1}^{\infty} a_{i,j}$

Suppose $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ where $\sum b_i$ converges, then:

$\quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$

Proof

Let countable set E contain points x_n where $x_n \to x_0$. Let:

$\quad f_i(x_n) = \sum_{j=1}^{n} a_{i,j} \qquad f_i(x_0) = \sum_{j=1}^{\infty} a_{i,j} \qquad g(x) = \sum_{i=1}^{\infty} f_i(x)$

Thus, each f_i is continuous at x_0. Since $|f_i(x)| \le b_i$, then g(x) converges uniformly so g is continuous at x_0. Thus:

$\quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} \sum_{i=1}^{\infty} f_i(x_n)$

$\quad = \lim_{n \to \infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{i,j} = \lim_{n \to \infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$

Theorem 15.1.7: Extension to Taylor's Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ where $a \in$ (-R,R), then f is expanded in a power series about x = a which converges in $|x - a| < R - |a|$ where:

$\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Proof

$f(x) = \sum_{n=0}^{\infty} c_n[(x-a)+a]^n = \sum_{n=0}^{\infty} c_n \sum_{m=0}^{n} \binom{n}{m} a^{n-m}(x-a)^m$

$\quad = \sum_{m=0}^{\infty} [\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m}](x-a)^m$

Then by corollary 15.1.3, $\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} = \frac{f^{(m)}(a)}{m!}$ so $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Theorem 15.1.8: Equivalent Power Series have the same coefficients

If $\sum a_n x^n$, $\sum b_n x^n$ converge in S = (-R,R), let E be the set of all $x \in$ S where $\sum a_n x^n = \sum b_n x^n$. If E has a limit point in S, then $a_n = b_n$ for all n.

Proof

Let $c_n = a_n - b_n$ and $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Then f(x) = 0 on E.

Let A = E' and B = S \ E'. Thus, B is open. If $x_0 \in$ A, then:

$\quad f(x) = \sum_{n=0}^{\infty} d_n(x-x_0)^n \qquad |x - x_0| < R - |x_0|$

Suppose $d_n \ne 0$ for some n. Let k be the smallest integer where $d_k \ne 0$. Then:

$\quad f(x) = (x-x_0)^k g(x) \qquad |x - x_0| < R - |x_0|$ and $g(x) = \sum_{m=0}^{\infty} d_{k+m}(x-x_0)^m$

Since g is continuous at x_0 and $g(x_0) = d_k \ne 0$, there is a $\delta > 0$ such that $g(x) \ne 0$ for $|x - x_0| < \delta$.

Thus, $f(x) \ne 0$ if $|x - x_0| < \delta$ which contradicts that x_0 is a limit point of E. Thus, d_n = 0 for all n so f(x) = 0 for all x so A is open. Thus, A and B are disjoint and thus, are separated. Since S = A $\cup$ B and S is connected, then either A or B is empty. Since A cannot be empty, then B is empty so A = S. Since f is continuous in S, then A $\subset$ S so E = S so c_n = 0 for all n.

15.2   Exponential and Logarithmic Functions

Definition 15.2.1: Exponential Function

Define $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

By the ratio test:

$$\lim_{n\to\infty} \sup(|\tfrac{a_{n+1}}{a_n}|) = \lim_{n\to\infty} \sup(|\tfrac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}|) = \lim_{n\to\infty} \sup(|\tfrac{z}{n+1}|) = 0 < 1$$

Thus, $E(x)$ converges. Then by corollary 15.1.5:

$$E(x)E(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y)$$

As a result, $E(x)E(-x) = E(0) = 1$. As a consequence:

(a) $E(x) \neq 0$ for all x

(b) If $x > 0$, then $E(x) > 0$ and thus, $E(x) > 0$ for all $x \in \mathbb{R}$

(c) $\lim_{x\to\infty} E(x) \to \infty$ so $\lim_{x\to-\infty} E(x) \to 0$ for $x \in \mathbb{R}$

(d) For $0 < x < y$, $E(x) < E(y)$ so $E(-y) = \frac{1}{E(y)} < \frac{1}{E(x)} = E(-x)$ so $E(x)$ is strictly increasing on $\mathbb{R}$

(e) $E'(x) = \lim_{h\to0} \frac{E(x+h)-E(x)}{h} = \lim_{h\to0} \frac{E(x)E(h)-E(x)}{h}$
$= E(x) \lim_{h\to0} \frac{E(h)-1}{h} = E(x) (\lim_{h\to0} \frac{E(h)}{h} - \lim_{h\to0} \frac{1}{h})$
$= E(x) (\lim_{h\to0} \frac{1}{h} + 1 - \lim_{h\to0} \frac{1}{h}) = E(x)$

(f) For $n > 0 \in \mathbb{Z}$:
$$E(n) = \underbrace{E(1)...E(1)}_{n} = e^n$$

For $p = \frac{n}{m} > 0 \in \mathbb{Q}$:
$[E(p)]^m = E(mp) = E(n) = e^n$ so $E(p) = e^{n/m} = e^p$
Since $E(-p) = \frac{1}{E(p)} = e^{-p}$, then $E(p) = e^p$ hold for all $p \in \mathbb{Q}$.
For $x \in \mathbb{R}$, let $e^x = \sup_{x>p}(e^p)$ for $p \in \mathbb{Q}$. Since $E(x)$ is continuous and monotonically increasing, for every $\epsilon > 0$, there is a $\delta > 0$ where $|x - p| < \delta$, then $|\sup_{x>p}(e^p) - e^p| < \epsilon$. Thus:
$$e^x = \sup_{x>p}(e^p) = \lim_{p\to x} E(p) = E(x).$$

Theorem 15.2.2: Properties of e^x

(a) e^x is continuous and differentiable for all $x \in \mathbb{R}$

(b) $(e^x)' = e^x$

(c) e^x is strictly increasing where $e^x > 0$

(d) $e^{x+y} = e^x e^y$

(e) $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$

(f) $\lim_{x\to\infty} x^n e^{-x} = 0$ for every $n > 0$

Proof

Part (a) is proved by convergent power series while parts (c) to (e) is proved by properties of $E(x)$ above. Since $e^x > \frac{x^{n+1}}{(n+1)!}$ for $x > 0$ and every $n \in \mathbb{Z}_+$, then:

$0 \leq \lim_{x\to\infty} x^n e^{-x} < \lim_{x\to\infty} \frac{(n+1)!}{x} = 0$

Thus, $\lim_{x\to\infty} x^n e^{-x} = 0$ for every $n \in \mathbb{Z}_+$. Since $x^n e^{-x}$ is continuous and $n \in \mathbb{Z}_+$ is dense in $\mathbb{R}_+$, then $\lim_{x\to\infty} x^n e^{-x} = 0$ for every $n > 0$.

Definition 15.2.3: Logarithmic Function

Since $y = E(x)$ is strictly increasing on $\mathbb{R}$, then $E(x)$ is injective and thus, there exist an inverse function $L(y)$ which is also strictly increasing. Since $E(x)$ is differentiable, then $L(y)$ is also differentiable. Then:

$\quad E(L(y)) = y \qquad$ where $y > 0$
$\quad L(E(x)) = x \qquad$ where $x \in \mathbb{R}$

Then:

$\quad L'(E(x))E'(x) = L'(y)E(x) = L'(y)y = 1 \qquad \Rightarrow \qquad L'(y) = \frac{1}{y}$

Since for $x = 0$ have $y = E(0) = 1$, then $L(1) = 0$. Thus:

$\quad L(y) = \int_1^y L'(t) \, dt = \int_1^y \frac{1}{t} \, dt$

As a consequence:

(a) Let $y_1 = E(x_1)$ and $y_2 = E(x_2)$, then:

$\qquad L(y_1 y_2) = L(E(x_1)E(x_2)) = L(E(x_1+x_2)) = x_1 + x_2 = L(y_1) + L(y_2)$

(b) Let $\log(y) = L(y)$. Then:

$\qquad$ Since $\lim_{x \to \infty} E(x) = \infty$, then $\lim_{y \to \infty} L(y) = \infty$.
$\qquad$ Since $\lim_{x \to -\infty} E(x) = 0$, then $\lim_{y \to 0} L(y) = -\infty$.

(c) For $n \in \mathbb{Z}$:

$\qquad$ If $n \geq 0$, $E(nL(y)) = E(\underbrace{L(y) + ... + L(y)}_{n}) = E(L(y^n)) = y^n$

$\qquad$ If $n < 0$, $E(nL(y)) = E(-(\underbrace{L(y) + ... + L(y)}_{-n})) = [E(L(y^{-n}))]^{-1} = y^n$

$\qquad$ For $p = \frac{a}{b} \in \mathbb{Q}$ where $b > 0$, let $t^b = y$:

$\qquad E(pL(y)) = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(y))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(t^b))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}bL(t))^n}{n!}$
$\qquad \quad = \sum_{n=0}^{\infty} \frac{(aL(t))^n}{n!} = \sum_{n=0}^{\infty} \frac{(L(t^a))^n}{n!} = t^a = y^{\frac{a}{b}} = y^p$

$\qquad$ For $c \in \mathbb{R}$, let $y^c = \sup_{c>p}(E(pL(y)))$. Since $E(x), L(y)$ are continuous and monotonically increasing, then for every $\epsilon > 0$, there is a $\delta > 0$ where $|c - p| < \delta$, then $|\sup_{c>p}(E(pL(y)) - E(pL(y))| < \epsilon$. Thus:

$\qquad y^c = \sup_{c>p}(E(pL(y)) = \lim_{p \to c} E(pL(y)) = E(cL(y))$

(d) For $y \in \mathbb{C}$ and $c \neq 0 \in \mathbb{R}$:

$\qquad (y^c)' = E'(cL(y))cL'(y) = E(cL(y))c\frac{1}{y} = y^c c\frac{1}{y} = cy^{c-1}$

$\qquad$ Thus:

$\qquad$ If $c \neq -1$, then $\int y^c \, dy = \int \frac{1}{c+1}(y^{c+1})' \, dy = \frac{1}{c+1}y^{c+1}$
$\qquad$ If $c = -1$, then $\int y^{-1} \, dy = \int L'(y) \, dy = L(y) = \log(y)$

(e) $\lim_{y \to \infty} y^{-c} \log(y) = 0$ for every $c > 0$

$\qquad$ For $\epsilon \in (0, c)$ and $y > 1$:

$\qquad y^{-c} \log(y) = y^{-c} \int_1^y t^{-1} \, dt < y^{-c} \int_1^y t^{\epsilon-1} \, dt = y^{-c}\frac{y^\epsilon - 1}{\epsilon} < \frac{1}{y^{c-\epsilon}\epsilon}$
$\qquad 0 \leq \lim_{y \to \infty} y^{-c} \log(y) < \lim_{y \to \infty} \frac{1}{y^{c-\epsilon}\epsilon} = 0$

15.3   Trigonometric Function

Definition 15.3.1: Trigonometric Function

Define for x $\in \mathbb{C}$:

$C(x) = \frac{1}{2}[E(ix) + E(-ix)]$ $S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$

Since $E(\overline{x}) = \sum_{n=0}^{\infty} \frac{\overline{x}^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{x^n}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = \overline{E(x)}$, then for x $\in \mathbb{R}$:

$C(x), S(x) \in \mathbb{R}$

Also, $E(ix) = C(x) + iS(x)$. Then:

(a) $|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(0) = 1$ so $|E(ix)| = 1$

(b) $C(0) = \frac{1}{2}[E(0) + E(0)] = 1$
$S(0) = \frac{1}{2i}[E(0) - E(0)] = 0$

(c) $C'(x) = \frac{1}{2}[E'(ix)i + E'(-ix)(-i)] = \frac{1}{2}[E(ix)i - E(-ix)i] = -S(x)$
$S'(x) = \frac{1}{2i}[E'(ix)i - E'(-ix)(-i)] = \frac{1}{2i}[E(ix)i + E(-ix)i] = C(x)$

(d) There exists positive numbers such that $C(x) = 0$.
If the claim is false, since C is continuous and $C(0) = 1$, then $S'(x) = C(x) > 0$. Then $S(x)$ is strictly increasing and since $S(0) = 0$, then $S(x) > 0$ for x > 0. Then for $0 < x < y$:

$S(x)(y-x) < \int_x^y S(t) \, dt = \int_x^y -C'(t) \, dt = C(x) - C(y)$
$\leq |C(x) - C(y)| \leq |C(x)| + |C(y)| = 2$

But if $S(x) > 0$, then $S(x)(y-x) \not< 2$ for a large enough y for any $S(x)$. Thus by contradiction, there are positive numbers where $C(x) = 0$.

Since the set of zeros of a continuous function is closed, there exists a smallest positive number x_0 such that $C(x_0) = 0$. Let $\pi = 2x_0$. Then, $C(\frac{\pi}{2}) = C(x_0) = 0$ and since $|E(ix)| = |C(x) + iS(x)| = 1$, then $S(\frac{\pi}{2}) = \pm 1$. Since $C(x)$ is continuous where $C(0) = 1$ and $C(\frac{\pi}{2}) = 0$, then $S'(x) = C(x) > 0$ for x $\in (0, \frac{\pi}{2})$ where $S(0) = 0$ so $S(\frac{\pi}{2}) = 1$. Thus, $E(\frac{\pi}{2}i) = C(\frac{\pi}{2}) + iS(\frac{\pi}{2}) = 0 + i1 = i$. Then:

$-1 = i^2 = E(\frac{\pi}{2}i)E(\frac{\pi}{2}i) = E(\frac{\pi}{2}i + \frac{\pi}{2}i) = E(\pi i)$
$1 = (-1)^2 = E(\pi i)E(\pi i) = E(\pi i + \pi i) = E(2\pi i)$
$E(z) = E(z)1 = E(z)E(2\pi i) = E(z + 2\pi i)$

Theorem 15.3.2: Properties of C(x) and S(x)

(a) E is periodic with period $2\pi i$
Proof

> Since $E(z) = E(z + 2\pi i)$, E has period $2\pi i$.

(b) C(x) and S(x) are periodic with period 2π
Proof

> Since $C(x) = \frac{1}{2}[E(ix) + E(-ix)]$ and $S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$ where $E(x)$ have period $2\pi i$ so $C(x)$ and $S(x)$ have period 2π.

(c) If t $\in (0, 2\pi)$, then $E(it) \neq 1$
Proof

> If t $\in (0, \frac{\pi}{2})$ where $E(it) = x + iy$, then $x, y \in (0,1)$.
> Note $E(4it) = [E(it)]^4 = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2)$.
> If $E(4it)$ is real, then $x^2 - y^2 = 0$. Thus, since $|E(ix)| = 1$, then $x^2 + y^2 = 1$ so $x^2 = y^2 = \frac{1}{2}$ and thus, $E(4it) = -1 \neq 1$.

(d) For z $\in \mathbb{C}$ where $|z| = 1$, there is a unique t $\in [0,2\pi)$ such that E(it) = z
   Proof

> By part (c), for $0 \leq t_1 < t_2 < 2\pi$:
> $E(it_2)[E(it_1)]^{-1} = E(it_2)[E(-it_1)] = E(it_2\text{-}it_1) \neq 1$
> Thus, t $\in [0,2\pi)$ must be unique. Let fixed z = x + iy where $|z| = 1$.
> For x,y ≥ 0, since C(x) decreases from 1 to 0 on $[0,\frac{\pi}{2}]$, then C(t) = x for some
> t $\in [0,\frac{\pi}{2}]$. Since $|E(it)| = C(t)^2 + S(t)^2 = 1$ and $x^2 + y^2 = 1$, then S(t) = y
> so E(it) = x + yi = z.
>
> If x < 0, y ≥ 0, fix -iz instead of z and thus, E(it) = -iz for some t $\in [0,\frac{\pi}{2}]$.
> Since $E(\frac{\pi}{2}i) = i$, then z = -iz(i) = $E(it)E(\frac{\pi}{2}i) = E(i(t+\frac{\pi}{2}))$.
>
> If x,y < 0, fix -z instead of z and thus, E(it) = -z for some t $\in [0,\frac{\pi}{2}]$.
> Since $E(\pi i) = -1$, then z = -z(-1) = $E(it)E(\pi i) = E(i(t+\pi))$.
>
> If x ≥ 0, y < 0, fix iz instead of z and thus, E(it) = iz for some t $\in [0,\frac{\pi}{2}]$.
> Then z = iz(-1)(i) = $E(it)E(\pi i)E(\frac{\pi}{2}i) = E(i(t+\frac{3\pi}{2}))$.

Definition 15.3.3: Unit Curve
   Let $\gamma(t)$ = E(it) for t $\in [0,2\pi]$.
   By theorem 15.3.2(d) and E(z) = E(z+2πi), then $\gamma(t)$ is a simple closed curve
   whose range is the unit circle. Since $\gamma'(t)$ = iE'(it) = iE(it), the length of γ:
      $\Lambda(\gamma) = \int_0^{2\pi} |\gamma'(t)|$ dt = 2π
   Thus, $\pi = 2x_0$ defined earlier have the same geometric significance as π. Then
   consider the triangle with vertices at:
      $z_1 = 0$ $z_2 = C(t_0)$ $z_3 = \gamma(t_0) = (C(t_0),S(t_0))$
   Thus, C(t) = cos(t) and S(t) = sin(t).

15.4   Algebraic Completeness of the Complex Field

Theorem 15.4.1: Every complex polynomial has a complex root
   For $a_0, ..., a_n \neq 0 \in \mathbb{C}$ where n ≥ 1, let P(z) = $\sum_{k=0}^{n} a_k z^k$.
   Then P(z) = 0 for some z $\in \mathbb{C}$.
Proof

> Assume $a_n = 1$. Let $\mu = \inf(|P(z)|)$. If $|z| = R$, then:
> $|P(z)| \geq R^n(1 - |a_{n-1}|R^{-1} - ... - |a_0|R^{-n})$
> Thus, $\lim_{R\to\infty} |P(z)| = \infty$ so there is a R_0 such that $|R(z)| > \mu$ if $|z| > R_0$.
> Since $|P|$ is continuous, then for a closed $N_{R_0}(0)$, by the Extreme Value Theorem:
> $|P(z_0)| = \mu$ for some z_0
> Suppose $\mu \neq 0$. Let polynomial Q(z) = $\frac{P(z+z_0)}{P(z_0)}$ where Q(0) = 1, Q(z) $\geq$ 1 for all z.
> Then there is a smallest integer k $\leq$ n so $b_k \neq 0$ so Q(z) = $1 + b_k z^k + ... + b_n z^n$.
> By theorem 15.3.2(d), there is a $\theta \in \mathbb{R}$ such that $e^{ik\theta}b_k = -|b_k|$.
> If r > 0 and $r^k|b_k| < 1$, then $|1 + b_k r^k e^{ik\theta}| = 1 - r^k|b_k|$. Thus:
> $|Q(re^{i\theta})| = |1 + b_k r^k e^{i\theta k} + b_{k+1}r^{k+1}e^{i\theta k+1} + ... + b_n r^n e^{i\theta n}|$
> $\leq |1 + b_k r^k e^{i\theta k}| + |b_{k+1}r^{k+1}e^{i\theta k+1}| + ... + |b_n r^n e^{i\theta n}|$
> $= 1 - r^k|b_k| + r^{k+1}|b_{k+1}| + ... + r^n|b_n| = 1 - r^k(|b_k| - r|b_{k+1}| - ... - r^{n-k}|b_n|)$
> Thus, for a sufficiently small r, $|Q(re^{i\theta})| < 1$ which contradicts Q(z) $\geq$ 1 for all z.
> Thus, $\mu = 0$ so there is a z_0 such that $|P(z_0)| = \mu = 0$ so P(z_0) = 0.

15.5   Fourier Series

A trigonometric polynomial is a finite sum where for $x \in \mathbb{R}$:

$$f(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] = \sum_{n=-N}^{N} c_n e^{inx}$$

A trigonometric series is then:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Thus:

(a) $f(x)$ has period of 2π

(b) Since $(\frac{1}{in} e^{inx})' = e^{inx}$ where $\frac{1}{in} e^{inx}$ have period of 2π, then for $n \in \mathbb{Z}$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}\ dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tfrac{1}{in} e^{inx})'\ dx = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, ... \end{cases}$$

(c) For $m \in \{-N, -N+1, ..., N\}$, then:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx}\ dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sum_{n=-N}^{N} c_n e^{inx} e^{-imx}]\ dx$$
$$= \sum_{n=-N}^{N} [\frac{1}{2\pi} \int_{-\pi}^{\pi} c_n e^{inx} e^{-imx}\ dx] = c_m$$

(d) If $f(x)$ is real, then:

$$\overline{c_m} = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} dx = c_{-m}$$

Thus, $f(x)$ is real if and only if $c_{-n} = \overline{c_n}$ for $n = \{0, 1, ..., N\}$.

If $f(x)$ is integrable on $[-\pi, \pi]$, then c_m are called the Fourier coefficients and $f(x)$ is a Fourier series of f.

Let $\{\phi_n\}$ be a sequence of complex functions on [a,b] such that:

$$\int_a^b \phi_n(x) \overline{\phi_m(x)}\ dx = 0 \qquad \text{for } m \neq n$$

Then, $\{\phi_n\}$ is an orthogonal system of functions on [a,b]. Additionally, if:

$$\int_a^b \phi_n(x) \overline{\phi_n(x)}\ dx = \int_a^b |\phi_n(x)|^2\ dx = 1$$

for all n, then $\{\phi_n\}$ is orthonormal.

If $\{\phi_n\}$ is orthonormal on [a,b] and $c_n = \int_a^b f(t) \overline{\phi_n(t)}\ dt$ for $n = \{1, 2, ...\}$, then c_n is the n-th Fourier coefficient of f relative to $\{\phi_n\}$. Then:

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Theorem 15.5.3: Fourier Series of f is the best approximation to f

For orthonormal $\{\phi_n\}$ on [a,b], let n-th partial sum of the Fourier series of f, $\sum_{m=1}^{n} c_m\phi_m(x) = s_n(x)$. Suppose $f \in \mathscr{R}$ and $t_n(x) = \sum_{m=1}^{n} \gamma_m\phi_m(x)$. Then:

$$\int_a^b |f - s_n|^2 \, dx \leq \int_a^b |f - t_n|^2 \, dx$$

where

$$\int_a^b |f - s_n|^2 \, dx = \int_a^b |f - t_n|^2 \, dx$$

if and only if $\gamma_m = c_m$ for every m = {1,...,n}.

Also, $\int |s_n(x)|^2 \, dx \leq \int |f(x)|^2 \, dx$.

<u>Proof</u>

$\int f(x)\overline{t_n(x)} \, dx = \int f(x) \sum [\overline{\gamma_m}\overline{\phi_m(x)}] \, dx = \sum [\int f(x)\overline{\gamma_m}\overline{\phi_m(x)} \, dx] = \sum c_m\overline{\gamma_m}$

Since $\{\phi_n\}$ is orthonormal, then:

$\int |t_n(x)|^2 \, dx = \int t_n(x)\overline{t_n(x)} \, dx = \int [\sum_m \gamma_m\phi_m(x)][\sum_k \overline{\gamma_k}\overline{\phi_k(x)}] \, dx$
$= \sum_m \sum_k [\int \gamma_m\phi_m(x)\overline{\gamma_k}\phi_k(x) \, dx] = \sum |\gamma_m|^2$

Thus:

$\int |f(x) - t_n(x)|^2 \, dx = \int |f(x)|^2 \, dx - \int f(x)\overline{t_n(x)} \, dx - \int \overline{f(x)}t_n(x) \, dx + \int |t_n(x)|^2 \, dx$
$= \int |f(x)|^2 \, dx - \sum c_m\overline{\gamma_m} - \sum \overline{c_m}\gamma_m + \sum |\gamma_m|^2$
$= \int |f(x)|^2 \, dx - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2$

Thus, $\int |f(x) - t_n(x)|^2 \, dx$ is minimized if and only if $\gamma_m = c_m$ for every m = {1,...,n}.

Let $\gamma_m = c_m$ and since $\int |f(x) - s_n(x)|^2 \, dx \geq 0$, then:

$\int |f(x) - s_n(x)|^2 \, dx = \int |f(x)|^2 \, dx - \sum |c_m|^2$
$\int |s_n(x)|^2 \, dx = \sum |c_m|^2 \leq \int |f(x)|^2 \, dx$

Theorem 15.5.4: Bessel Inequality

For $\{\phi_n\}$ is orthonormal on [a,b] and $f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x)$, if $f \in \mathscr{R}$, then:

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 \, dx \qquad \text{and} \qquad \lim_{n\to\infty} c_n = 0$$

<u>Proof</u>

Since $\{\phi_n\}$ is orthonormal on [a,b] and $f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x)$, then by theorem 15.5.3, for any integer n > 1:

$\sum_{m=1}^{n} |c_m|^2 \leq \int_a^b |f(x)|^2 \, dx$

Thus, as $n \to \infty$, then $\sum_{m=1}^{\infty} |c_m|^2 \leq \int_a^b |f(x)|^2 \, dx$.

Since $\sum_{m=1}^{\infty} |c_m|^2$ is monotonically increasing and bounded above, then $\sum_{m=1}^{\infty} |c_m|^2$ converges and thus, $\lim_{n\to\infty} c_n = 0$.

Definition 15.5.5: Trigonometric Series

Consider functions $f \in \mathscr{R}$ on $[-\pi, \pi]$ with period 2π. Let $\phi_n(x) = e^{inx}$ which is orthogonal and orthonormal when $\phi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$.

Thus, the N-th partial sum of the Fourier series of f is:

$s_N(f; x) = \sum_{n=-N}^{N} c_n e^{inx}$

where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} \, dt$. Then by theorem 15.5.3:

$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(f; x)|^2 \, dx = \sum_{n=-N}^{N} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$

From the Dirichlet kernel, $D_N(x) = \sum_{n=-N}^{N} e^{inx}$:

$(e^{ix} - 1)D_N(x) = \sum_{n=-N}^{N}[e^{i(n+1)x} - e^{inx}] = e^{i(N+1)x} - e^{-iNx}$

$D_N(x) = \frac{e^{-\frac{1}{2}ix}(e^{i(N+1)x} - e^{-iNx})}{e^{-\frac{1}{2}ix}(e^{ix} - 1)} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}}$

$= \frac{2i \sin((N+\frac{1}{2})x)}{2i \sin(\frac{1}{2}x)} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$

Since e^{inx} is periodic for 2π for each $n \in$ [-N,N], then $D_N(x)$ is periodic for 2π. Thus, since f is also periodic for 2π, then:

$s_N(f; x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N}[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt]e^{inx}$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)[\sum_{n=-N}^{N} e^{in(x-t)}]dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)D_N(x-t)dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t)dt$

Theorem 15.5.6: If f is continuous at some x, then Fourier Series of f converges to f

If for some x, there are $\delta > 0$ and M such that $|f(x+t) - f(x)| \le M|t|$ for all $t \in (-\delta, \delta)$, then:

$\lim_{N \to \infty} s_N(f; x) = f(x)$

Proof

Let $g(t) = \frac{f(x-t) - f(x)}{\sin(\frac{1}{2}t)}$ for $t \in [-\pi, \pi]$ where $g(0) = 0$. Then by definition 15.5.1(b):

$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sum_{n=-N}^{N} e^{inx}] \, dx = 1$

Thus:

$s_N(f; x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t) \, dt - f(x)$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)D_N(t) \, dt - f(x)\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t) \, dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)]D_N(t) \, dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\sin((N+\frac{1}{2})t) \, dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)[\sin(Nt)\cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)\cos(Nt)] \, dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t)\cos(\frac{1}{2}t)]\sin(Nt) \, dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t)\sin(\frac{1}{2}t)]\cos(Nt) \, dt$

Since $g(t)$ and $\cos(\frac{1}{2}t), \sin(\frac{1}{2}t)$ are bounded on $[-\pi, \pi]$, then $g(t)\cos(\frac{1}{2}t)$ and $g(t)\sin(\frac{1}{2}t)$ are bounded on $[-\pi, \pi]$. As $N \to \infty$, then $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t)\cos(\frac{1}{2}t)]\sin(Nt)$ dt $= 0$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t)\sin(\frac{1}{2}t)]\cos(Nt)$ dt $= 0$ so $\lim_{N \to \infty} s_N(f; x) = f(x)$.

Corollary 15.5.7: Localization Theorem

If $f(x) = 0$ for all x in some segment J, then for every $x \in J$:

$\lim_{N \to \infty} s_N(f; x) = 0$

Proof

Let J = (a,b). Then for $x \in$ J, choose δ such that $(x - \delta, x + \delta) \subset$ J.

Thus for any $t \in (-\delta, \delta)$, then $|f(x+t) - f(x)| = |0 - 0| = 0$.

Then by theorem 15.5.6, for every $x \in$ J, $\lim_{N \to \infty} s_N(f; x) = f(x) = 0$.

Corollary 15.5.8: Equivalent functions on (a,b) have similar Fourier Series on (a,b)

If f(t) = g(t) for all t in some neighborhood of x, then:

$$\lim_{N\to\infty} [s_N(f;x) - s_N(g;x)] = 0$$

Proof

Since f(t) - g(t) = 0 for all $t \in (x - \delta, x + \delta)$, then by corollary 15.5.7, then:

$\lim_{N\to\infty} s_N(f - g;x) = 0$

The Fourier series for f-g:

$s_N(f - g;x) = \sum_{n=-N}^{N} c_n e^{inx}$ where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi}(f - g)(t)e^{-int}$ dt

The Fourier series for f and g:

$s_N(f;x) = \sum_{n=-N}^{N} a_n e^{inx}$ where $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}$ dt

$s_N(g;x) = \sum_{n=-N}^{N} b_n e^{inx}$ where $b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)e^{-int}$ dt

Then $s_N(f - g;x) = s_N(f;x)$ - $s_N(g;x)$ and thus:

$\lim_{N\to\infty} [s_N(f;x)$ - $s_N(g;x)] = \lim_{N\to\infty} s_N(f - g;x) = 0$

Theorem 15.5.9: There are Fourier Series that converge uniformly to continuous f

If f is continuous with period 2π, then for $\epsilon > 0$, there is a trigonometric polynomial P such that for all $x \in \mathbb{R}$:

$$|P(x) - f(x)| < \epsilon$$

Proof

Since f(x) has a period of 2π, then for a fixed $x \in \mathbb{R}$, f(x) on $\mathbb{R}$ can be defined on compact [x,x+2π] which is the complex unit circle T by a mapping of $x \to e^{ix}$.
The set of trigonometric polynomials, P(x) = $\sum_{n=-N}^{N} c_n e^{inx}$ for constants $c_n \in \mathbb{C}$ and integer N ≥ 0, is an algebra $\mathscr{A}$ since for $P_1(x) = \sum_{n=-N_1}^{N_1} a_n e^{inx}$ and $P_2(x) = \sum_{n=-N_2}^{N_2} b_n e^{inx}$, let N = $\max(N_1, N_2)$ and $a_n, b_n = 0$ if n ≥ N_1, N_2 respectively:

$P_1(x) + P_2(x) = \sum_{n=-N}^{N}(a_n + b_n)e^{inx}$ so $P_1(x) + P_2(x) \in \mathscr{A}$

$P_1(x)P_2(x) = \sum_{n=-2N}^{n=2N} d_n e^{inx}$ where $d_n = \sum_{k=-N}^{N} a_k b_{n-k}$ so $P_1(x)P_2(x) \in \mathscr{A}$

$cP_1(x) = \sum_{n=-N_1}^{N_1}(ca_n)e^{inx}$ where $ca_n \in \mathbb{C}$ so $cP_1(x) \in \mathscr{A}$

Also, $\mathscr{A}$ is self-adjoint since:

$\overline{P_1(x)} = \sum_{n=-N_1}^{N_1} \overline{a_n}e^{-inx} = \sum_{n=-N_1}^{N_1} \overline{a_{-n}}e^{inx}$ where $\overline{a_{-n}} \in \mathbb{C}$ so $\overline{P_1(x)} \in \mathscr{A}$

Also, $\mathscr{A}$ separates points on T since any two points on T are distinct and $\mathscr{A}$ vanishes at no point of T since (0,0) does not exist on the complex unit circle. For $\pi > \epsilon > 0$, since the mapping $x \to e^{ix}$ is 1-1 from [x+ε,x+2π − ε], then $\mathscr{A}$ separates points and vanishes at no point on [x+ε,x+2π − ε].
Thus, by theorem 14.7.9, then $\mathscr{B}$, the set of all uniformly convergent P(x) from $\mathscr{A}$, consist of all complex continuous f on [x+ε,x+2π − ε].
So there is a P(x) such that P(x) converges uniformly to f so for all t ∈ [x,x+2π], then $|P(t) - f(t)| < \epsilon$. Since f has a period of 2π, then for all $x \in \mathbb{R}$, then $|P(t) - f(t)| < \epsilon$.

Definition 15.5.10: L^p Space

For p $\geq$ 1, let $L^p = \{$ f: [a,b] $\to \mathbb{C} \mid ||f||_p = [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} < \infty \}$.

For complex f,g $\in \mathscr{R}$:

(a) Holder's Inequality: If $\frac{1}{p} + \frac{1}{q} = 1$ where p,g $\geq$ 1, then $||fg||_1 \leq ||f||_p ||g||_q$

Proof

Claim: If a,b $\geq$ 0, then ab $\leq \frac{a^p}{p} + \frac{b^q}{q}$ and equality only if $a^p = b^q$.

Take y = f(x) = x^{p-1} for x $\in$ [0,a] and x = $f^{-1}(y) = \sqrt[p-1]{y}$ for y $\in$ [0,b].

The total area is $\int_0^a x^{p-1}dx + \int_0^b y^{\frac{1}{p-1}}dy = \frac{a^p}{p} + \frac{p-1}{p}b^{\frac{p}{p-1}} = \frac{a^p}{p} + \frac{b^q}{q}$.

Graphing each function on their respective axes, it is shown that regardless if $a^{p-1} > b$ or $a^{p-1} < b$, the total area is greater than ab and equality holds only if a^{p-1} = b so $b^q = a^{(p-1)q} = a^{(p-1)\frac{p}{p-1}} = a^p$.

$$\frac{1}{||f||_p ||g||_q}||fg||_1 = \frac{1}{||f||_p ||g||_q} \int |fg| \text{ dx} = \frac{1}{||f||_p ||g||_q} \int |f||g| \text{ dx}$$
$$= \int \frac{|f|}{||f||_p} \frac{|g|}{||g||_q} \text{ dx} \leq \int \frac{|f|^p}{||f||_p^p p} + \frac{|g|^q}{||g||_q^q q} \text{ dx}$$
$$= \frac{1}{||f||_p^p p} \int |f|^p \text{ dx} + \frac{1}{||g||_q^q q} \int |g|^q \text{ dx}$$
$$= \frac{1}{||f||_p^p p}||f||_p^p + \frac{1}{||g||_q^q q}||g||_q^q = \frac{1}{p} + \frac{1}{q} = 1$$

Since a = $\frac{|f|}{||f||_p}$ and b = $\frac{|g|}{||g||_q}$, then equality holds only if $\frac{|f|^p}{||f||_p^p} = \frac{|g|^q}{||g||_q^q}$.

(b) Minkowski's Inequality: $||f + g||_p \leq ||f||_p + ||g||_p$

Proof

Since f,g $\in \mathscr{R}$, then $|f + g|^p \in \mathscr{R}$. By Holder's Inequality:

$$||f + g||_p^p = \int_a^b |f(x) + g(x)|^p dx = \int_a^b |f(x) + g(x)||f(x) + g(x)|^{p-1}dx$$
$$\leq \int_a^b (|f(x)| + |g(x)|)|f(x) + g(x)|^{p-1}dx$$
$$\leq \int_a^b |f(x)||f(x) + g(x)|^{p-1}dx + \int_a^b |g(x)||f(x) + g(x)|^{p-1}dx$$
$$\leq ([\int_a^b |f(x)|^p dx]^{\frac{1}{p}} + [\int_a^b |g(x)|^p dx]^{\frac{1}{p}})(\int_a^b |f(x) + g(x)|^{p-1(\frac{p}{p-1})}dx)^{1-\frac{1}{p}}$$
$$= (||f||_p + ||g||_p)||f + g||_p^{p-1}$$

Theorem 15.5.11: For integrable f, there are continuous g where f-g $\in L^2$

Let f $\in \mathscr{R}$ on [a,b]. Then for $\epsilon > 0$, there is a continuous g where:

g(a) = f(a)   g(b) = f(b) $||f(x) - g(x)||_2 < \epsilon$

Proof

Since f $\in \mathscr{R}$, then $|f(x)| < M$. For $\epsilon > 0$, there is a partition P =$\{x_0, ..., x_n\}$ of [a,b]:
    U(P,f) - L(P,f) = $\sum_{i=1}^n (M_i - m_i)\Delta x_i < \frac{\epsilon^2}{2M}$

Let g(t) = $\frac{x_i - t}{\Delta x_i}f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i}f(x_i)$ for t $\in [x_{i-1}, x_i]$ which is continuous on [a,b] since:
    $g(x_i+)$ = $f(x_i) = g(x_i-)$ $\Rightarrow$ $g(x_i) = f(x_i)$ so g(a) = f(a) , g(b) = f(b)

Thus, for t $\in [x_{i-1}, x_i]$:
    $|f(t) - g(t)| = |f(t) - \frac{x_i - t}{\Delta x_i}f(x_{i-1}) - \frac{t - x_{i-1}}{\Delta x_i}f(x_i)|$
    $= |\frac{x_i - t}{\Delta x_i}[f(t) - f(x_{i-1})] + \frac{t - x_{i-1}}{\Delta x_i}[f(t) - f(x_i)]|$
    $\leq |\frac{x_i - t}{\Delta x_i}||f(t) - f(x_{i-1})| + |\frac{t - x_{i-1}}{\Delta x_i}||f(t) - f(x_i)| = M_i - m_i$

Since g is continuous, then g $\in \mathscr{R}$ and thus, $|f(x) - g(x)|^2 \in \mathscr{R}$. Thus:
    $||f(x) - g(x)||_2 = [\int_a^b |f(x) - g(x)|^2 dx]^{\frac{1}{2}} = \lim_{n\to\infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t) - g(t)|^2 dt]^{\frac{1}{2}}$
    $\leq \lim_{n\to\infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i)^2 dt]^{\frac{1}{2}} \leq \lim_{n\to\infty} [\sum_{i=1}^n 2M \int_{x_{i-1}}^{x_i} (M_i - m_i)dt]^{\frac{1}{2}}$
    $= \lim_{n\to\infty} [2M \sum_{i=1}^n (M_i - m_i)\Delta x_i]^{\frac{1}{2}} < \lim_{n\to\infty} [2M \frac{\epsilon^2}{2M}]^{\frac{1}{2}} = \epsilon$

Theorem 15.5.12: Parseval's Theorem
  For f,g $\in \mathscr{R}$ with period of 2π where:
  $$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$$
  then:
  $$\lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2 \, dx = 0$$
  $$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}$$
  $$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof

Since $f \in \mathscr{R}$ on $[x, x+2\pi]$ for a fixed $x \in \mathbb{R}$, where $f(x) = f(x+2\pi)$, then by theorem 15.5.11, for $\epsilon > 0$, there is a continuous h such that:
  $$||f(x) - h(x)||_2 < \epsilon$$
Also, h(x) = f(x) and h(x+2π) = f(x+2π) for any $x \in \mathscr{R}$, and since $f(x) = f(x+2\pi)$, then h has a period of 2π. Then by theorem 15.5.9, there is a trigonometric polynomial P(x) such that for all $x \in \mathbb{R}$:
  $$|h(x) - P(x)| < \epsilon \qquad \Rightarrow \qquad ||h(x) - P(x)||_2 = \left[\int_x^{x+2\pi} |h(x) - P(x)|^2 dx\right]^{\frac{1}{2}} < \sqrt{2\pi}\epsilon$$
Then by theorem 15.5.3:
  $$||h(x) - s_N(h;x)||_2 \leq ||h(x) - P(x)||_2 < \sqrt{2\pi}\epsilon$$
  $$||s_N(h;x) - s_N(f;x)||_2 = ||s_N(h-f;x)||_2 \leq ||h(x) - f(x)||_2 < \epsilon$$
Thus:
  $$||f(x) - s_N(f;x)||_2 \leq ||f(x) - h(x)||_2 + ||h(x) - s_N(h;x)||_2 + ||s_N(h;x) - s_N(f;x)||_2$$
  $$< (2 + \sqrt{2\pi})\epsilon$$

Note $\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f;x)\overline{g(x)}dx = \sum_{n=-N}^{N}\left[c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{g(x)}dx\right] = \sum_{n=-N}^{N} c_n\overline{\gamma_n}$.
By Holder's Inequality:
  $$|\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx - \int_{-\pi}^{\pi} s_N(f;x)\overline{g(x)}dx|$$
  $$\leq \int_{-\pi}^{\pi} |f(x) - s_N(f;x)||g(x)|dx$$
  $$\leq \left[\int_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2 dx\right]^{\frac{1}{2}}\left[\int_{-\pi}^{\pi} |g(x)|^2 dx\right]^{\frac{1}{2}}$$
  $$= ||f(x) - s_N(f;x)||_2 ||g(x)||_2$$
Since $g \in \mathscr{R}$, then $|g|^2 \in \mathscr{R}$ and thus, $||g(x)||_2$ is bounded.
Since $\lim_{N \to \infty} ||f(x) - s_N(f;x)||_2 = 0$, then:
  $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_N(f;x)\overline{g(x)}dx = \lim_{N \to \infty}\sum_{n=-N}^{N} c_n\overline{\gamma_n} = \sum_{n=-\infty}^{\infty} c_n\overline{\gamma_n}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{f(x)}dx = \sum_{n=-\infty}^{\infty} c_n\overline{c_n} = \sum_{n=-\infty}^{\infty} |c_n|^2$$

16   Multivariable Functions

16.1   Linear Transformations

Definition 16.1.1: Vector Spaces

    (a) Vector Space

        A nonempty set $X \subset \mathbb{R}^n$ is a vector space if for all x,y $\in$ X and scalar c:

            x+y $\in$ X   cx $\in$ X

        Null vector 0 is also defined as $0 = (0,...,0) \in \mathbb{R}^k$.

    (b) Linear Combinations and Span

        For scalars $c_1, ..., c_k$, a linear combination of $x_1, ..., x_k \in \mathbb{R}^n$:

            $c_1 x_1 + ... + c_k x_k$

        The span of $x_1, ..., x_k$ is the set of all linear combinations of $x_1, ..., x_k$.

    (c) Independence and Dimension

        If $c_1 x_1 + ... + c_k x_k = 0$ only if $c_1 = ... = c_k = 0$, then $x_1, ..., x_k$ are independent.
        Any independent set does not contain 0 since $c0 + c_1 x_1 + ... + c_k x_k = 0$ holds
        true for $c, 0, ..., 0$ where c is any number, not just $0, 0, ..., 0$.

        If vector space X have r independent vectors, but not r+1 independent
        vectors, then dim(X) = r. The set $\{0\}$ has dimension 0.

    (d) Basis

        If $x_1, .., x_k \in X$ are independent and spans X, then $x_1, .., x_k$ is a basis of X.
        Thus, for every x $\in$ X:

            Since $x_1, .., x_k$ spans X, there exists $c_1, ..., c_k$ such that $x = c_1 x_1 + ... + c_k x_k$.
            Since $x_1, .., x_k$ are independent, then such $c_1, ..., c_k$ are unique else there
            are $a_1, ..., a_k$ where at least one $a_i \neq c_i$ such that:

                $x = a_1 x_1 + ... + a_k x_k$ $\Rightarrow$ $0 = x - x = (a_1 - c_1)x_1 + ... + (a_k - c_k)x_k$
            where at least one $(a_i - c_i) \neq 0$ contradicting $x_1, ..., x_k$ are independent.
        The $c_1, ..., c_k$ are called the coordinates of x with respect to basis $x_1, ..., x_k$.

    (e) Standard Basis of $\mathbb{R}^k$

        Let $e_i = (\underset{1}{0}, ..., \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, ..., \underset{k}{0}) \in \mathbb{R}^k$.

        Thus, $e_1, ..., e_k$ is a basis for $\mathbb{R}^k$ where any $x = (x_1, ..., x_k) = x_1 e_1 + ... + x_k e_k$.

Theorem 16.1.2: dim(X) $\leq$ (# vectors that span X)

    If vector space X is spanned by r vectors, then dim(X) $\leq$ r.

Proof

> If dim(X) > r, then there are at minimum r+1 independent vectors that spans X
> which contradicts that X is spanned by r vectors.
> Let X be spanned by $x_1, ..., x_r \neq 0$. If $x_1, ..., x_r$ are independent, then dim(X) = r.
> If $x_1, ..., x_r$ are not independent, then there is at least two $c_k \neq 0$ where:
> $0 = c_1 x_1 + ... + c_r x_r$
> since if only one $c_k \neq 0$, then $0 = c_1 x_1 + ... + c_r x_r = c_k x_k$ which implies $x_k = 0$ since
> $c_k \neq 0$ which is a contradiction. Thus, for $c_k, c_{i_1}, ..., c_{i_n} \neq 0$:
> $0 = c_1 x_1 + ... + c_r x_r = c_k x_k + c_{i_1} x_{i_1} + ... + c_{i_n} x_{i_n}$ $\Rightarrow$ $x_k = \frac{-c_{i_1}}{c_k} x_{i_1} + ... + \frac{-c_{i_n}}{c_k} x_{i_n}$
> Remove x_k from $x_1, ..., x_r$ and repeat the process until all x_i are independent and
> thus, dim(X) = r - (# x_i removed) < r.

Corollary 16.1.3: $\dim(X)$ = (# vectors in a basis)

      If $x_1, ..., x_n$ is a basis for X, then $\dim(X)$ = n.

      Thus, $\dim(\mathbb{R}^n)$ = n.

Proof

Since $x_1, ..., x_n$ is a basis for X, then $x_1, ..., x_n$ spans X and are independent.
Since $x_1, ..., x_n$ span X, then by theorem 16.1.2, then $\dim(X) \leq$ n. Since $x_1, ..., x_n$ are independent, then $\dim(X) \geq$ n since there might be another x_i independent to $x_1, ..., x_n$ and another and so on. Thus, $\dim(\mathbb{R}^n)$ = n.
Since $e_1, ..., e_n$ is a basis for $\mathbb{R}^n$, then $\dim(\mathbb{R}^n)$ = n.

Theorem 16.1.4: Properties of Basis

      For vector space X where $\dim(X)$ = n:

        (a) n vectors span X if and only if the n vectors are independent

        (b) X has a basis where every basis have only n vectors

        (c) For independent $x_1, ..., x_r$ where r $\in$ {1,...,n}, X has a basis with $x_1, ..., x_r$

Intuition

$x_1, ..., x_m$ can span X, but not independent since there might be a x_i that is dependent on the other x_i (aka $x_i = a_i x_i + ... + a_{i-1}x_{i-1} + a_{i+1}x_{i+1} + ... + a_m x_m$).
$x_1, ..., x_k$ can be independent, but not span X since there might be another x that is independent to each x_i (aka x $\neq b_1 x_1 + ... + b_k x_k$ for any $b_1, ..., b_k$).
So to get a basis, either remove the dependent elements from $x_1, ..., x_m$ to get independent or add independent elements to $x_1, ..., x_k$ to get a span of X. Simply, a basis has a set amount of vectors, but $x_1, ..., x_m$ has too much while $x_1, ..., x_k$ has too few.

Proof

Let $x_1, ..., x_n$ span X. If $x_1, ..., x_n$ are not independent, then remove x_i until $x_1, ..., x_k$ are independent as performed in theorem 16.1.2. Thus, $\dim(X)$ = k < n which is a contradiction and thus, $x_1, ..., x_n$ are independent.
For independent $x_1, ..., x_n$, add $y_1, ..., y_k \in$ X so $x_1, ..., x_n, y_1, ..., y_k$ span X. Since $\dim(X)$ = n, then $x_1, ..., x_n, y_1, ..., y_k$ are not independent. Since any non-independent set can remove elements in its span until it is independent and thus, preserves its span as performed in theorem 16.1.2, then each y_i can be removed to reach independent $x_1, ..., x_n$ which still spans X.

By part (a), any n independent vectors spans X so thus, forms a basis for X. For $x_1, ..., x_k$ where k > n, since $\dim(X)$ = n, then $x_1, ..., x_k$ is non-independent and is thus, not a basis. For $x_1, ..., x_k$ where k < n, since $\dim(X)$ = n, there is a x $\in$ X such that $x_1, ..., x_k, x$ are independent. Then x $\neq c_1 x_1 + ... + c_k x_k$ for any $c_1, ..., c_k$ else
    x = $c_1 x_1 + ... + c_k x_k$ $\Rightarrow$ 0 = $c_1 x_1 + ... + c_k x_k + -x$
so $x_1, ..., x_k, x$ are not independent. Thus, there is a x $\in$ X that is not in the span of $x_1, ..., x_k$ so $x_1, ..., x_k$ does not span X.

For independent $x_1, ..., x_r$, since $\dim(X)$ = n, there are $x_{r+1}, ..., x_n$ such that $x_1, ..., x_n$ are independent. By part (a), $x_1, ..., x_n$ spans X so $x_1, ..., x_n$ forms a basis which contain $x_1, ..., x_r$.

Definition 16.1.5: Linear Transformation

A mapping A of vector space X into vector space Y is a linear transformation if for all $x_1, x_2 \in$ X and scalar c:

$$A(x_1 + x_2) = Ax_1 + Ax_2 \qquad A(cx_1) = cAx_1$$

Since A0 + A0 = A(0+0) = A0, then A0 = 0.

If $x_1, ..., x_n$ is a basis for X, then for any x $\in$ X, there is a unique set of $c_1, ..., c_n$ where x $= c_1x_1 + ... + c_nx_n$ such that:

$$Ax = A(c_1x_1 + ... + c_nx_n) = c_1Ax_1 + ... + c_nAx_n$$

Linear transformation that maps X into X are linear operators.
Additionally, if A is 1-1 and maps X onto X, then A is invertible.
Thus, there is a A^{-1} such that:

$$A^{-1}(Ax) = x \qquad \text{for all x} \in X$$

Since A maps X onto X, for any x $\in$ X, then Ax = y $\in$ X.
Thus, for all y $\in$ X, then x $= A^{-1}(Ax) = A^{-1}y$. Thus:

$$A(A^{-1}y) = Ax = y$$

Also, for any $x_1, x_2 \in$ X and scalars c_1, c_2 where $Ax_1 = y_1$ and $Ax_2 = y_2$:

$$A^{-1}(c_1y_1 + c_2y_2) = A^{-1}(c_1Ax_1 + c_2Ax_2) = A^{-1}(A(c_1x_1 + c_2x_2))$$
$$= c_1x_1 + c_2x_2 = c_1A^{-1}(y_1) + c_2A^{-1}(y_2)$$

So, A^{-1} is a linear transformation.

Theorem 16.1.6: Linear Operators imply 1-1 $\rightleftharpoons$ onto

Linear operator A preserves independence if and only if A is 1-1.
Thus, linear operator A is 1-1 if and only if A(X) = X.

Proof

Let $x_1, ..., x_n$ be a basis for X where each $Ax_i = y_i \in$ X. So for any y $\in$ A(X), there is x $\in$ X where x $= c_1x_1 + ... + c_nx_n$ for a unique set of $c_1, ..., c_n$ such that:

$$y = Ax = A(c_1x_1 + ... + c_nx_n) = c_1Ax_1 + ... + c_nAx_n = c_1y_1 + ... + c_ny_n$$

If A is 1-1, then there is only one such x so in respect to $y_1, ..., y_n$, then any y $= k_1y_1 + ... + k_ny_n$ must have $k_1 = c_1, ..., k_n = c_n$. Thus, for y = 0, since 0 = A0 and $x_1, ..., x_n$ are independent, then $c_1 = ... = c_n = 0$ so $y_1, ..., y_n$ are independent. If A is not 1-1, then there is y where there are at least two distinct such x so in respect to $y_1, ..., y_n$, then y $= k_1y_1 + ... + k_ny_n$ holds true for at least 2 distinct $k_1, ..., k_n$ so $y_1, ..., y_n$ are not independent. Thus, A is 1-1 if and only if $y_1, ..., y_n$ is independent. By theorem 16.1.4a, $y_1, ..., y_n$ span X so A(X) = X if and only if $y_1, ..., y_n$ are independent.

Definition 16.1.7: Operations of Linear Transformatons

    Let L(X,Y) be the set of all linear transformation of X into Y.

    Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$.

(a) If $A_1, A_2 \in$ L(X,Y) and c_1, c_2 are scalars, then for any x $\in$ X, define:
$$(c_1 A_1 + c_2 A_2)\text{x} = c_1 A_1 x + c_2 A_2 x$$

(b) For vector space Z, if A $\in$ L(X,Y) and B $\in$ L(Y,Z), then for any x $\in$ X, define:
$$(BA)\text{x} = B(A\text{x}) \in \text{L(X,Z)}$$

(c) For A $\in$ L($\mathbb{R}^n, \mathbb{R}^m$), define the norm:
$$||A|| = \sup(|Ax| \mid \text{x} \in \mathbb{R}^n \text{ where } |x| \leq 1)$$

(d) $|Ax| = |A(|x|\frac{x}{|x|})| = |A(\frac{x}{|x|})| \ |x| \leq \sup(|A(\frac{x}{|x|})|) \ |x| = ||A|| \ |x|$

    If there is a λ such that $|Ax| \leq \lambda|x|$ for all x $\in \mathbb{R}^n$, then $||A|| \leq \lambda|1| = \lambda$.

(e) For A,B $\in$ L($\mathbb{R}^n, \mathbb{R}^m$), the distance between A and B is defined $||A - B||$

Theorem 16.1.8: Operations of Norms of Linear Transformations

(a) If A $\in$ L($\mathbb{R}^n, \mathbb{R}^m$), then $||A|| < \infty$. Thus, A is uniformly continuous.

Proof

> For $|x| \leq 1$:
> $$|Ax| = |A(x_1 e_1 + ... + x_n e_n)| \leq |x_1||Ae_1| + ... + |x_n||Ae_n|$$
> $$\leq |Ae_1| + ... + |Ae_n| = \text{M}$$
> Thus, $||Ax|| \leq |Ae_1| + ... + |Ae_n| = \text{M} < \infty$.
> Let $|x - y| < \epsilon$ and thus, $|Ax - Ay| = |A(x - y)| \leq ||A|| \ |x - y| < M\epsilon$ so A is uniformly continuous.

(b) If A,B $\in$ L($\mathbb{R}^n, \mathbb{R}^m$) and c is a scalar, then:
$$||A + B|| \leq ||A|| + ||B|| \qquad ||cA|| = |c| \ ||A||$$

Proof

> For $|x| \leq 1$, $|(A + B)x| \leq |Ax + Bx| \leq |Ax| + |Bx| \leq ||A|| + ||B||$.
> Thus, $||A + B|| \leq ||A|| + ||B||$. Since $|cAx| = |c||Ax|$, then $||cA|| = |c| \ ||A||$.
> Also, for the distance between A and B, by part a:
> $$||A - B|| \leq ||A + B|| \leq ||A|| + ||B|| \leq M_1 + M_2$$

(c) If A $\in$ L($\mathbb{R}^n, \mathbb{R}^m$) and B $\in$ L($\mathbb{R}^m, \mathbb{R}^k$), then:
$$||BA|| \leq ||B|| \ ||A||$$

Proof

> For $|x| \leq 1$, $|BAx| = |B(Ax)| \leq ||B|| \ |Ax| \leq ||B|| \ ||A|| \ |x| \leq ||B|| \ ||A||$.
> Thus, $||BA|| \leq ||B|| \ ||A||$.

Theorem 16.1.9: Operations of Norms of Invertible Linear Operators

(a) If $A \in \Omega$ and $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ where $||B - A|| \, ||A^{-1}|| < 1$, then $B \in \Omega$

Proof

> $\frac{1}{||A^{-1}||}|x| = \frac{1}{||A^{-1}||}|A^{-1}Ax| \leq \frac{1}{||A^{-1}||}||A^{-1}|| \, |Ax|$
> $\qquad = |Ax| \leq |(A - B)x| + |Bx| \leq ||A - B|| \, |x| + |Bx|$
> Thus, $|Bx| \geq (\frac{1}{||A^{-1}||} - ||A - B||) \, |x| \geq \frac{2}{||A^{-1}||}|x| \geq 0$ so $Bx \neq 0$ if $x \neq 0$ so B
> is 1-1. Then by theorem 16.1.4a, B spans $\mathbb{R}^n$ so B is invertible so $B \in \Omega$.

(b) $\Omega \subset L(\mathbb{R}^n, \mathbb{R}^n)$ is open and the mapping T: $A \to A^{-1}$ is continuous on Ω

Proof

> Since $||B - A|| < \frac{1}{||A^{-1}||}$ for any $B \in \Omega$, then for every $B \in \Omega$, there exist an
> open subset of $L(\mathbb{R}^n, \mathbb{R}^n)$ that contains B so Ω is open.
> Since
> $\qquad |y| = |BB^{-1}y| \geq (\frac{1}{||A^{-1}||} - ||A - B||) \, |B^{-1}y|$
> $\qquad\qquad \geq (\frac{1}{||A^{-1}||} - ||A - B||) \, ||B^{-1}|| \, |y|$
> then $\frac{1}{\frac{1}{||A^{-1}||} - ||A - B||} \geq ||B^{-1}||$. Thus, by theorem 16.1.8:
> $\qquad ||B^{-1} - A^{-1}|| = ||B^{-1}(A - B)A^{-1}||$
> $\qquad\qquad\qquad \leq ||B^{-1}|| \, ||A - B|| \, ||A^{-1}|| \leq \frac{||A-B|| \, ||A^{-1}||}{\frac{1}{||A^{-1}||} - ||A-B||}$
> Since $\lim_{B \to A} ||A - B|| \to 0$ so $||B^{-1} - A^{-1}|| \to$, then T is continuous on Ω.

Definition 16.1.10: Matrices

Let $x_1, ..., x_n$ be a basis for X and $y_1, ..., y_m$ be a basis for Y.

Then every A $\in$ L(X,Y) determines a set of numbers a_{ij} such that:

$\qquad Ax_j = \sum_{i=1}^{m} a_{ij}y_i \qquad$ for j = {1,...,n}

Thus, A can be represented by an m by n matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

Since the a_{ij} of Ax_j are from the j-th column [A], then Ax_j is called the column vector of [A]. Thus, the span(A) is the span of the column vectors of [A].

For any x $\in$ X, there is a unique set of $c_1, ..., c_n$ such that x $= c_1 x_1 + ... + c_n x_n$:

$$[Ax] = \begin{bmatrix} (y_1)\ \{ & \overbrace{a_{11}}^{c_1} & \overbrace{a_{12}}^{c_2} & ... & \overbrace{a_{1n}}^{c_n} \\ (y_2)\ \{ & a_{21} & a_{22} & ... & a_{2n} \\ & \vdots & \vdots & \ddots & \vdots \\ (y_m)\ \{ & a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

$\quad$ Ax $= A(\sum_{j=1}^{n} c_j x_j)$
$\qquad = \sum_{j=1}^{n} c_j Ax_j$
$\qquad = \sum_{j=1}^{n} c_j \sum_{i=1}^{m} a_{ij}y_i$
$\qquad = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}c_j y_i$
$\qquad = \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij}c_j]\ y_i$

So $[\sum_{j=1}^{n} a_{1j}c_j], ..., [\sum_{j=1}^{n} a_{mj}c_j]$ are Ax's coordinates in respect to $y_1, ..., y_m$.

Let A $\in$ L(X,Y) and B $\in$ L(Y,Z). Then, BA $\in$ L(X,Z).

Let $z_1, ..., z_p$ be a basis for Z where:

$\qquad By_i = \sum_{k=1}^{p} b_{ki}z_k \qquad (BA)x_j = \sum_{k=1}^{p} c_{kj}z_k$

Thus, B as a p by m matrix and BA as a p by n matrix can be represented:

$$[B] = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1m} \\ b_{21} & b_{22} & ... & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & ... & b_{pm} \end{bmatrix} \qquad [BA] = \begin{bmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & ... & c_{pn} \end{bmatrix}$$

$\quad (BA)x_j = B(Ax_j) = B(\sum_{i=1}^{m} a_{ij}y_i)$
$\qquad = \sum_{i=1}^{m} a_{ij}By_i$
$\qquad = \sum_{i=1}^{m} a_{ij} \sum_{k=1}^{p} b_{ki}z_k$
$\qquad = \sum_{i=1}^{m} \sum_{k=1}^{p} b_{ki}a_{ij}z_k$
$\qquad = \sum_{k=1}^{p} [\sum_{i=1}^{m} b_{ki}a_{ij}]\ z_k$

Thus, $c_{kj} = \sum_{i=1}^{m} b_{ki}a_{ij}$ for j = {1,...,n} and k = {1,...,p}.

So to get matrix [BA] from [B] and [A]:

$$\begin{bmatrix} b_{11} & b_{12} & ... & b_{1m} \\ b_{21} & b_{22} & ... & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & ... & b_{pm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} b_{1i}a_{i1} & \sum_{i=1}^{m} b_{1i}a_{i2} & ... & \sum_{i=1}^{m} b_{1i}a_{in} \\ \sum_{i=1}^{m} b_{2i}a_{i1} & \sum_{i=1}^{m} b_{2i}a_{i2} & ... & \sum_{i=1}^{m} b_{2i}a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} b_{pi}a_{i1} & \sum_{i=1}^{m} b_{pi}a_{i2} & ... & \sum_{i=1}^{m} b_{pi}a_{in} \end{bmatrix}$$

For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, since $Ax = \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij}c_j] \, e_i$ where $x = \sum_{j=1}^{n} c_j e_j$, then by the Cauchy-Schwarz Inequality:

$$|Ax|^2 = \sum_{i=1}^{m} [\sum_{j=1}^{n} a_{ij}c_j]^2$$
$$\leq \sum_{i=1}^{m} [(\sum_{j=1}^{n} a_{ij}^2) (\sum_{j=1}^{n} c_j^2)]$$
$$= [\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2] (\sum_{j=1}^{n} c_j^2)$$
$$= [\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2] \, |x|^2$$

Thus, for $|x| \leq 1$, then $||A|| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$.

Theorem 16.1.11: A linear transformation of continuous functions is continuous

If each a_{ij} is a continuous function on S and for each $p \in S$, then $A_p \in L(\mathbb{R}^n, \mathbb{R}^m)$ with entries $a_{ij}(p)$, then the mapping T: $S \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

Proof

Since each $a_{i,j}$ is continuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for t,p $\in$ S where $|t - p| < \delta$, then $|a_{i,j}(t) - a_{i,j}(p)| < \frac{\epsilon}{\sqrt{mn}}$. Thus, for $|t - p| < \delta$:

$$||A_p - A_t|| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij}(p) - a_{ij}(t))^2} < \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (\frac{\epsilon}{\sqrt{mn}})^2} = \epsilon$$

16.2   Differentiation

Definition 16.2.1: Derivative Extended to Higher Dimensions

First, let's redefine the derivative such that it can be extended to higher dimensions. For f: $(a,b) \subset \mathbb{R} \to \mathbb{R}^m$, let f'(x) = y $\in \mathbb{R}^m$ such that:

f(x+h) - f(x) = yh + r(h)   where $\lim_{h \to 0} \frac{r(h)}{h} = 0$

Since y: h $\to$ hy is a linear transformation from $\mathbb{R}$ to $\mathbb{R}^m$, then f'(x) $\in L(\mathbb{R}, \mathbb{R}^m)$.

Now for derivatives in higher dimensions.
Let f: x $\in$ open $E \subset \mathbb{R}^n \to \mathbb{R}^m$.
If there is an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that for any h $\in$ E:

f(x+h) - f(x) = Ah + $r_A(h)$ where $\lim_{h \to 0} \frac{|r_A(h)|}{|h|} = 0$

then f is differentiable at x. Then differential of f at x, f'(x) = A.
If f is differentiable at every x $\in$ E, then f is differentiable on E.

Theorem 16.2.2: The derivative of a function is unique

Let f: x $\in$ open $E \subset \mathbb{R}^n \to \mathbb{R}^m$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that for any h $\in$ E:

f(x+h) - f(x) = Ah + $r_A(h)$ where $\lim_{h \to 0} \frac{|r_A(h)|}{|h|} = 0$

Suppose A = A_1 and A = A_2 satisfies such conditions. Then $A_1 = A_2$.

Proof

For any h $\in \mathbb{R}^n$:

$$|(A_2 - A_1)h| = |[f(x+h) - f(x) - r_{A_1}(h)] - [f(x+h) - f(x) - r_{A_2}(h)]|$$
$$= |r_{A_2}(h) - r_{A_1}(h)|$$
$$\leq |r_{A_2}(h)| + |r_{A_1}(h)|$$

Since $A_1, A_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$, for any t where h is fixed, then:

$$|(A_2 - A_1)(th)| \leq |r_{A_2}(th)| + |r_{A_1}(th)|$$
$$|t||(A_2 - A_1)h| \leq |r_{A_2}(th)| + |r_{A_1}(th)|$$
$$|(A_2 - A_1)h| \leq \frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|}$$

So as t $\to$ 0, then $\frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|} \to 0 + 0 = 0$. Thus, $A_1 = A_2$.

Theorem 16.2.3: Derivative of a Linear Transformation
If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $x \in \mathbb{R}^n$, then:
$$A'(x) = A$$

Proof

Since $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then let f(x) = Ax.

f(x+h) - f(x) = A(x+h) - Ax = Ax + Ah - Ax = Ah

Thus, $r_A(h) = 0$ so $\lim_{h \to 0} \frac{|r_A(h)|}{|h|} = \lim_{h \to 0} 0 = 0$. Thus, A'(x) = f'(x) = A.

Theorem 16.2.4: Chain Rule in Higher Dimensions
Let f: open $E \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in E$ and g: $f(E) \subset$ open H $\subset \mathbb{R}^m \to \mathbb{R}^k$ be differentiable at $f(x_0)$.

Then F: $E \to \mathbb{R}^k$ where F(x) = g(f(x)) is differetiable at x_0 such that:
$$F'(x_0) = g'(f(x_0))\ f'(x_0)$$

Proof

Since f is differentiable at x_0 and g is differentiable at $f(x_0)$, then there is a $A = f'(x_0)$ and $B = g'(f(x_0))$ such that:

$f(x_0+h) - f(x_0) = Ah + r_A(h)$ where $\lim_{h \to 0} \frac{|r_A(h)|}{|h|} = 0$

$g(f(x_0)+k) - g(f(x_0)) = Bk + r_B(k)$ where $\lim_{k \to 0} \frac{|r_B(k)|}{|k|} = 0$

Let k = $f(x_0+h) - f(x_0)$. Thus:

$F(x_0+h) - F(x_0) - BAh = g(f(x_0+h)) - g(f(x_0)) - BAh$
$$= g(f(x_0)+k) - g(f(x_0)) - BAh = Bk + r_B(k) - BAh$$
$$= B(k - Ah) + r_B(k) = B(f(x_0+h) - f(x_0) - Ah) + r_B(k)$$
$$= Br_A(h) + r_B(k)$$

$\frac{|F(x_0+h)-F(x_0)-BAh|}{|h|} = \frac{|Br_A(h)+r_B(k)|}{|h|} \leq \frac{|Br_A(h)|+|r_B(k)|}{|h|} \leq \frac{||B||\ |r_A(h)|+|r_B(k)|}{|h|}$

Since f is differentiable at x_0, then f is continuous at x_0 and thus, $\lim_{h \to 0} k = 0$.

Since $\lim_{h \to 0} \frac{|r_A(h)|}{|h|} = 0$ and $\lim_{k \to 0} \frac{|r_A(k)|}{|k|} = 0$, then:

$\lim_{h \to 0} \frac{|F(x_0+h)-F(x_0)-BAh|}{|h|} \leq \lim_{h \to 0} ||B|| \frac{|r_A(h)|}{|h|} + \lim_{h \to 0} \frac{|r_B(k)|}{|h|} = 0 + 0 = 0$

Thus, $F'(x_0) = BA = g'(f(x_0))\ f'(x_0)$.

Definition 16.2.5: Partial Derivatives: Derivatives along the standard basis
Let f: open $E \subset \mathbb{R}^n \to \mathbb{R}^m$. The components of f are the $f_1, ..., f_m \in \mathbb{R}$ such that for $x \in E$, then $f(x) = \sum_{i=1}^{m} f_i(x)e_i$.

Since $e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, then f(x) $\cdot e_i = [\sum_{i=1}^{m} f_i(x)e_i] \cdot e_i = f_i(x)$.

Then for $x \in E$ and $i \in \{1,...,m\}$ and $j \in \{1,...,n\}$, let the partial derivative $\frac{\partial f_i}{\partial x_j} = D_j f_i$ be the derivative of f_i with respect to x_j. Then for $t \in \mathbb{R}$:

$$f_i(x+te_j) - f_i(x) = D_j f_i(te_j) + r_{D_j f_i}(te_j) \qquad \text{where } \lim_{t \to 0} \frac{|r_{D_j f_i}(te_j)|}{|t|} = 0$$

Theorem 16.2.6: Derivative of f is the sum of all partial derivatives

Let f: open $E \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x $\in$ E. Then the partial derivatives $(D_j f_i)(x)$ exists such that for j $\in$ {1,...,n}:

$$f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x)e_i$$

Proof

For a fixed j, since f is differentiable at x, then:

$\quad$ f(x+te_j) - f(x) = f'(x)(te_j) + r(te_j) $\qquad$ where $\lim_{t \to 0} \frac{|r(te_j)|}{|t|} = 0$

Then f'(x) exist where:

$\quad \lim_{t \to 0} \frac{f(x+te_j)-f(x)}{t} = \lim_{t \to 0} \frac{f'(x)(te_j)}{t} + \frac{r(te_j)}{t} = \lim_{t \to 0} t\frac{f'(x)e_j}{t} = f'(x)e_j$

Since f(x) = $\sum_{i=1}^{m} f_i(x)e_i$, then:

$\quad \lim_{t \to 0} \frac{f(x+te_j)-f(x)}{t} = \lim_{t \to 0} \sum_{i=1}^{m} \frac{f_i(x+te_j)-f_i(x)}{t}e_i = f'(x)e_j$

Since f'(x) exist and $\lim_{t \to 0} \frac{f_i(x+te_j)-f_i(x)}{t} = D_j f_i(x)$, then each $D_j f_i(x)$ exists where:

$\quad$ f'(x)e_j = $\sum_{i=1}^{m} \lim_{t \to 0} \frac{f_i(x+te_j)-f_i(x)}{t}e_i = \sum_{i=1}^{m} (D_j f_i)(x)e_i$

Definition 16.2.7: Matrix of the Differential of f

By theorem 16.2.6, f'(x)e_j = $\sum_{i=1}^{m} (D_j f_i)(x)e_i$ where $(D_j f_i)(x)$ is the derivative of the component f_i in respect to x_j for j = {1,...,n}.

Since $f'(x)e_j$ is the j-th column of [f'(x)], then:

$$[f'(x)] = \left[\sum_{i=1}^{m}(D_1 f_i)(x)e_i \quad \sum_{i=1}^{m}(D_2 f_i)(x)e_i \quad ... \quad \sum_{i=1}^{m}(D_n f_i)(x)e_i \right]$$

where each $\sum_{i=1}^{m}(D_j f_i)(x)e_i$ is a column vector at the j-th column.

Since each $\sum_{i=1}^{m}(D_j f_i)(x)e_i$ has a coordinate of $(D_j f_i)(x)$ for e_i where each $e_i = (\underset{1}{0}, ..., \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, ..., \underset{m}{0}) \in \mathbb{R}^m$, then:

$$[f'(x)] = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & ... & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & ... & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & ... & (D_n f_m)(x) \end{bmatrix}$$

Thus, for x $\in \mathbb{R}^n$ where x = $x_1 e_1 + ... + x_n e_n$, then:

$\quad$ f'(x)x = $f'(x)[\sum_{j=1}^{n} x_j e_j]$

$\qquad = \sum_{j=1}^{n} x_j f'(x)e_j$

$\qquad = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} (D_j f_i)(x)e_i$

$\qquad = \sum_{i=1}^{m} [\sum_{j=1}^{n} x_j (D_j f_i)(x)]\, e_i$

Definition 16.2.8: Gradient and Directional Derivative

Let γ: (a,b) $\subset \mathbb{R} \to$ open E $\subset \mathbb{R}^n$ and f: E $\subset \mathbb{R}^n \to \mathbb{R}$ both be differentiable. Then by theorem 16.2.4, g: $\mathbb{R} \to \mathbb{R}$ defined as g(t) = f($\gamma(t)$) is differentiable for any t $\in$ (a,b) such that:

g'(t) = f'($\gamma(t)$) $\gamma'(t)$

Since f($\gamma(t)$): E $\subset \mathbb{R}^n \to \mathbb{R}$, by theorem 16.2.6, then:

f'($\gamma(t)$)$e_j = (D_j f)(\gamma(t))$ for j = {1,...,n}

Since γ: (a,b) $\subset \mathbb{R} \to$ open E $\subset \mathbb{R}^n$, then:

$\gamma'(t) = \sum_{i=1}^{n} (D_1 \gamma_i)(t)e_i = \sum_{i=1}^{n} \gamma_i'(t)e_i$

Thus, g'(t) = $\sum_{i=1}^{n} (D_i f)(\gamma(t)) \gamma_i'(t)$.

For each x $\in$ E, let the gradient of f: E $\subset \mathbb{R}^n \to \mathbb{R}$ at x, $(\nabla f)(x)$:

$(\nabla f)(x) = \sum_{i=1}^{n} (D_i f)(x)e_i$

Since $e_i e_j = 1$ if i = j, but $e_i e_j = 0$ if i $\neq$ j, then:

$[f(\gamma(t))]$' = g'(t)

$= \sum_{i=1}^{n} (D_i f)(\gamma(t)) \gamma_i'(t)$

$= \sum_{i=1}^{n} [(D_i f)(\gamma(t))e_i \cdot \gamma_i'(t)e_i]$

$= [\sum_{i=1}^{n} (D_i f)(\gamma(t))e_i] \cdot [\sum_{i=1}^{n} \gamma_i'(t)e_i] = (\nabla f)(\gamma(t)) \cdot \gamma'(t)$

For t $\in (-\infty, \infty)$, let $\gamma(t) = x + tu$ where x $\in$ E and unit vector u $\in \mathbb{R}^n$. Then:

$(D_u f)(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t} = \lim_{t \to 0} \frac{g(t) - g(0)}{t}$ = g'(x)

$= (\nabla f)(\gamma(x)) \cdot \gamma'(x) = (\nabla f)(x) \cdot u$

Let $(D_u f)(x)$ be the directional derivative of f at x in direction of u.

For u = $u_1 e_1 + ... + u_n e_i$:

$(D_u f)(x) = (\nabla f)(x) \cdot u = \sum_{i=1}^{n} (D_i f)(x)e_i \cdot \sum_{i=1}^{n} u_i e_i = \sum_{i=1}^{n} (D_i f)(x)u_i$

Also, for a fixed f and x, $(D_u f)(x)$ is maximized when u = $\lambda(\nabla f)(x)$ for $\lambda > 1$ since $x \cdot y = |x||y|\cos(\theta)$ where θ is the angle between x and y.

Theorem 16.2.9: A bounded derivative over a convex space have bounded range

For differentiable f: convex open E $\subset \mathbb{R}^n \to \mathbb{R}^m$, there is a M $\in \mathbb{R}$ such that $||f'(x)|| \leq$ M for every x $\in$ E. Then for all a,b $\in$ E:

$|f(b) - f(a)| \leq$ M$|b - a|$

Proof

For fixed a,b $\in$ E, let $\gamma(t)$ = (1-t)a + tb. Since E is convex, for t $\in$ [0,1], then $\gamma(t) \in$ E. Let g(t) = f($\gamma(t)$). Then g'(t) = f'($\gamma(t)$)$\gamma'(t)$ = f'($\gamma(t)$)(b-a). Thus, for t $\in$ [0,1]:

$|g'(t)| = |f'(\gamma(t))(b - a)| \leq ||f'(\gamma(t))|| \ |b - a| \leq$ M$|b - a|$

Since g(0) = f($\gamma(0)$) = f(a) and g(1) = f($\gamma(1)$) = f(b), then by the Mean Value Theorem, for x $\in$ (0,1)

$|f(b) - f(a)| = |g(1) - g(0)| \leq (1 - 0)|g'(x)| \leq$ M$|b - a|$

Corollary 16.2.10: If the derivative is 0, the function is constant

For differentiable f: convex open E $\subset \mathbb{R}^n \to \mathbb{R}^m$, f'(x) = 0 for all x $\in$ E. Then, f is constant.

Proof

Since $||f'(x)|| = 0$ for all x $\in$ E, then by theorem 7.2.9, for all a,b $\in$ E:

$0 \leq |f(b) - f(a)| \leq 0(b - a) = 0$

Thus, f(b) = f(a) for all a,b $\in$ E so f is constant.

Definition 16.2.11: Continuously Differentiable

A differentiable f: open $E \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable in E if:

f': $E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous

For $\epsilon > 0$, there is a $\delta > 0$ such that for every x,y $\in$ E where $|x - y| < \delta$, then:

$||f'(y) - f'(x)|| < \epsilon$

If f is continuous differentiable, then f $\in \mathscr{C}'(E)$.

Theorem 16.2.12: Continuously differentiable imply continuous partial derivatives

Let f: open $E \subset \mathbb{R}^n \to \mathbb{R}^m$. Then f $\in \mathscr{C}'(E)$ if and only if each partial derivative $D_j f_i$ exist and are continuous on E.

Proof

If f $\in \mathscr{C}'(E)$, then f is differentiable. Thus, by theorem 16.2.6, partial derivative $D_j f_i$ where j = {1,...,n} exists for any x $\in$ E such that:

$$f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i \qquad \Rightarrow \qquad (D_j f_i)(x) = f'(x)e_j \cdot e_i$$

Thus, since f $\in \mathscr{C}'(E)$, then for $|x - y| < \delta$:

$$|(D_j f_i)(y) - (D_j f_i)(x)| = |f'(y)e_j \cdot e_i - f'(x)e_j \cdot e_i| = |[f'(y) - f'(x)]e_j \cdot e_i|$$
$$\leq |[f'(y) - f'(x)]e_j|\, |e_i| \leq ||f'(y) - f'(x)||\, |e_j|\, |e_i|$$
$$= ||f'(y) - f'(x)|| < \epsilon$$

Thus, each $D_j f_i$ is continuous.

Since each $D_j f_i$ is continuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for $|y - x| < \delta$, then for all j $\in$ {1,...,n} and i $\in$ {1,...,m}, then $|D_j f_i(y) - D_j f_i(x)| < \epsilon$.

Then for h = $h_1 e_1 + ... + h_n e_n$ where $|x - h| < \delta$:

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|}$$

$$= \lim_{h \to 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + ... + h_n e_n) - f_i(x)]e_i - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|}$$

$$= \lim_{h \to 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + ... + h_n e_n) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|}$$

$$= \lim_{h \to 0} \frac{\left|\sum_{i=1}^m \left[\begin{array}{c} f_i(x+\sum_{k=1}^n h_k e_k) - f_i(x+\sum_{k=1}^{n-1} h_k e_k) \\ +f_i(x+\sum_{k=1}^{n-1} h_k e_k) - f_i(x+\sum_{k=1}^{n-2} h_k e_k) \\ +... + f_i(x+h_1) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x)h_j \end{array} \right] e_i \right|}{|h|}$$

Since each $D_j f_i$ exist, then by the Mean Value Theorem, for each j = {1,...,n}, there is a $t_j \in (0, h_j)$ such that:

$$f_i(x + \sum_{k=1}^j h_k e_k) - f_i(x + \sum_{k=1}^{j-1} h_k e_k) = D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j)h_j$$

Thus:

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|}$$

$$= \lim_{h \to 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j)h_j - \sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|}$$

$$< \lim_{h \to 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n [\epsilon h_j]]e_i|}{|h|} \leq \lim_{h \to 0} \frac{|\sum_{i=1}^m [n\epsilon |h|]e_i|}{|h|} = \lim_{h \to 0} \frac{\sqrt{m}n\epsilon |h|}{|h|} = \sqrt{m}n\epsilon$$

Thus, f(x) is differentiable where:

$$f'(x) = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & ... & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & ... & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & ... & (D_n f_m)(x) \end{bmatrix}$$

Thus, for $|y - x| < \delta$:

$$||f'(y) - f'(x)|| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n [(D_j f_i)(y) - (D_j f_i)(x)]^2} < \sqrt{\sum_{i=1}^m \sum_{j=1}^n \epsilon^2} = \sqrt{m}n\epsilon$$

Thus, f $\in \mathscr{C}'(E)$.

16.3   The Contraction Principle

Definition 16.3.1: Contraction
>   For metric space X with metric d, then ϕ: X $\to$ X is a contraction if there is
>   c $\in$ (0,1) such that for all x,y $\in$ X:
>   d(ϕ(x),ϕ(y)) $\leq$ c d(x,y)

Theorem 16.3.2: Banach's Fixed Point Theorem
>   If X is a complete metric space and ϕ is a contraction of X into X, then there
>   is a unique x $\in$ X such that ϕ(x) = x

Proof

Let ϕ(x) = x and ϕ(y) = y. Since ϕ is a contraction, then d(x,y) = d(ϕ(x),ϕ(y)) $\leq$ c
d(x,y) would hold true only if d(x,y) = 0 so x = y. Thus, such a ϕ(x) = x is unique.
For a fixed $x_0 \in$ X, let $\{x_n\}$ have $x_{n+1} = \phi(x_n)$. Thus, for some c $\in$ (0,1):
$\quad$ d(x_{n+1},x_n) = d($\phi(x_n)$,$\phi(x_{n-1})$) $\leq$ c d(x_n, x_{n-1})
$\qquad\qquad$ = c d($\phi(x_{n-1}), \phi(x_{n-2})$) = ... = c^n d(x_1, x_0)
Thus, for $\epsilon > 0$, choose N such that $d(x_1, x_0)\frac{c^N}{(1-c)} < \epsilon$. Then for m > n $\geq$ N:
$\quad$ d(x_m,x_n) $\leq \sum_{i=n}^{m-1}$ d(x_{i+1},x_i) $\leq \sum_{i=n}^{m-1} c^i$ d(x_1, x_0)
$\qquad\qquad \leq$ d(x_1, x_0) $\frac{c^n}{1-c} \leq$ d(x_1, x_0) $\frac{c^N}{1-c} < \epsilon$
Thus, $\{x_n\}$ is a Cauchy Sequence and since X is complete, then $\{x_n\}$ converges to a
x $\in$ X. Note a contraction is uniformly continuous so:
$\quad \phi(x) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} x_{n+1} =$ x

Example

>   For y' = y where y(0) = 1, show y(x) = e^x for x near 0.

Take the metric space of continuous functions, C[a,b], with the sup metric as defined
in definition 14.3.4 where 0 $\in$ [a,b]. By theorem 14.3.5, C[a,b] is complete.
Then for each f $\in$ C[a.b], let Tf(x) = 1 + $\int_0^x$ f(t) dt for x $\in$ [a,b].
$\quad |Tf(x) - Tg(x)| = |\int_0^x f(t) - g(t)dt| \leq \int_{\min(0,x)}^{\max(0,x)} |f(t) - g(t)|dt$
$\qquad\qquad \leq |x - 0|$ d(f,g) $\leq$ (b-a) d(f,g)
Thus, d(Tf(x),Tg(x)) $\leq$ (b-a) d(f,g) so for b-a < 1, then T is a contraction. By
theorem 16.3.2, there is a unique y where y(x) = 1 + $\int_0^x$ y(t) dt. To determine y, use
the process defined in theorem 16.3.2's proof referred as the Picard iteration. Using
any continuous f(x), let's take f(x) = 1. Then:
$\quad$ T(1) = 1 + $\int_0^x$ 1 dt = 1 + x
$\quad$ T(T(1)) = 1 + $\int_0^x$ 1+t dt = 1 + x + $\frac{1}{2}x^2$
$\quad$ T(T(T(1))) = 1 + $\int_0^x$ 1+t+$\frac{1}{2}t^2$ dt = 1 + x + $\frac{1}{2}x^2$ + $\frac{1}{6}x^3$
Thus, by definition 15.2.1, y(x) = $\lim_{n\to\infty}$ T^n(1) = $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

16.4   Inverse Function Theorem

Theorem 16.4.1: Inverse Function Theorem

> Let f $\in \mathscr{C}'(E)$: open E $\subset \mathbb{R}^n \to \mathbb{R}^n$ where Df(a) is invertible for some (a,b).
> (a) There are open U,V $\subset \mathbb{R}^n$ such that f: a $\in$ U $\to$ b $\in$ V is invertible
> (b) If g = f^{-1}: V $\to$ U where g(f(x)) = x, then for y = f(x):
> g $\in \mathscr{C}'(V)$ where Dg(y) = $[Df(g(y))]^{-1}$

Proof

Since Df(a) is invertible for a $\in$ E, then choose λ such that $||[Df(a)]^{-1}|| = \frac{1}{2\lambda}$
Since Df(a) is continuous at a, there is a $B_r(a) \subset$ E such that for x $\in$ U:
$\quad ||Df(x) - Df(a)|| < \lambda$
For each y $\in \mathbb{R}^n$, let $\phi(x) = $ x $+ [Df(a)]^{-1}(y - f(x))$ for x $\in$ E. Then f(x) = y if and only if $\phi(x) = $ x. Since:
$\quad \phi'(x) = $ I $- [Df(a)]^{-1}Df(x) = [Df(a)]^{-1}(Df(a) - Df(x)) < \frac{1}{2\lambda}\lambda = \frac{1}{2}$
Then by theorem 16.2.9, for all $x_1, x_2 \in B_r(a)$, then $|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|$.
Thus, ϕ is a contraction so on $\overline{B_r(a)}$ which is complete, then there is a unique x $\in \overline{B_r(a)}$ such that $\phi(x) = $ x. Thus for each y, then f(x) = y for a unique x so f is 1-1.
Let U = $B_r(a)$ and V = $f(B_r(a))$ so f maps U onto V. Thus, f is invertible on U.
Then for each $y_0 \in$ V, then $y_0 = $ f(x_0) for a unique $x_0 \in$ U. Choose t for $\overline{B_t(x_0)}$ such that $\overline{B_t(x_0)} \subset$ U = $B_r(a)$. Then for y $\in$ V where $|y - y_0| < \lambda t$ and x $\in \overline{B_t(x_0)}$:
$\quad |\phi(x_0) - x_0| = |[Df(a)]^{-1}(y - f(x_0))| \leq \frac{1}{2\lambda}\lambda t = \frac{t}{2}$
$\quad |\phi(x) - x_0| \leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| < \frac{1}{2}|x - x_0| + \frac{t}{2} \leq \frac{t}{2} + \frac{t}{2} = $ t
Thus, $\phi(x) \in B_t(x_0)$. Since $|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|$ for $x_1, x_2 \in \overline{B_t(x_0)}$, then ϕ is a contraction on $\overline{B_t(x_0)}$ so there is a unique x $\in \overline{B_t(x_0)}$ such that $\phi(x) = $ x so for y where $|y - y_0| < \lambda t$, then f(x) = y. Thus, y $\in$ f($\overline{B_t(x_0)}) \subset$ f(U) = V so V is open.

For y,y+k $\in$ V, there are x,x+h $\in$ U such that f(x) = y and f(x+h) = y+k.
$\quad \phi(x + h)$ - $\phi(x) = $ h $+ [Df(a)]^{-1}(f(x) - f(x + h)) = $ h $+ [Df(a)]^{-1}k$
Since $|\phi(x + h) - \phi(x)| < \frac{1}{2}|h|$, then $[Df(a)]^{-1}k \in [\frac{1}{2}|h|, \frac{3}{2}|h|]$.
$\quad |h| \leq 2[Df(a)]^{-1}k \leq 2||[Df(a)]^{-1}||\ |k| \leq \frac{|k|}{\lambda}$
Since $||Df(x) - Df(a)||\ ||[Df(a)]^{-1}|| < \lambda\frac{1}{2\lambda} = \frac{1}{2} < 1$, then by theorem 16.1.9a, then Df(x) is invertible and thus, have an inverse T. Since:
$\quad$ g(y+k) - g(y) - Tk = h - Tk = -T(f(x+h) - f(x) - Df(x)h)
then $\frac{|g(y+k)-g(y)-Tk|}{|k|} \leq \frac{||T||}{\lambda}\frac{|f(x+h)-f(x)-Df(x)h|}{|h|}$.
As k $\to$ 0, then h $\to$ 0. Since f is differentiable, then $\lim_{h \to 0}\frac{|f(x+h)-f(x)-Df(x)h|}{|h|} \to 0$ so $\lim_{k \to 0}\frac{|g(y+k)-g(y)-Tk|}{|k|} = 0$. Thus, Dg(y) = T where T is the inverse of Df(x).
$\quad$ Df(x)Dg(y) = Df(x)T = $I_{n \times n}$ $\qquad \to \qquad$ Dg(y) = $[Df(x)]^{-1} = [Df(g(y))]^{-1}$
Since g is differentiable and thus, continuous and $Df(x)$ is continuous, then by theorem 16.1.9b, $[Df(g(y))]^{-1}$ is continuous.

Corollary 16.4.2: f with continuous, invertible Df(x) at all x is an open mapping

> If f $\in \mathscr{C}'(E)$: open E $\subset \mathbb{R}^n \to \mathbb{R}^n$ where Df(x) is invertible for every x $\in$ E,
> then open f(W) $\subset \mathbb{R}^n$ for every open W $\subset$ E.

Proof

From theorem 16.4.1a, let U = W contain x. Then, V = f(U) = f(W) is open.

Example

$xe^{xy} - \sin(y) = $ a

$x^9 y^{10} + 3\cos(xy) = $ b

Prove there is a unique solution for all (a,b) close to $(e - \sin(1), 1 + 3\cos(1))$

Let f(x,y) = $(xe^{xy} - \sin(y), x^9 y^{10} + 3\cos(xy))$.

Since each component is differentiable at all x,y, then f(x,y) is differentiable where:

$$\text{Df(x,y)} = \begin{bmatrix} e^{xy} + xye^{xy} & x^2 e^{xy} - \cos(y) \\ 9x^8 y^{10} - 3y\sin(xy) & 10x^9 y^9 - 3x\sin(xy) \end{bmatrix}$$

Since Df(1,1) = $\begin{bmatrix} 2e & e - \cos(1) \\ 9 - 3\sin(1) & 10 - 3\sin(1) \end{bmatrix}$ so $\det(\text{Df(1,1)}) \neq 0$.

Then by the Inverse Function Theorem, f is invertible and thus 1-1. So, there is a unique solution (x,y) near (1,1) for all (a,b) close enough to $(e - \sin(1), 1 + 3\cos(1))$.

16.5   Implicit Function Theorem

Definition 16.5.1: Matrix Components

For x = $(x_1, ..., x_n) \in \mathbb{R}^n$ and y = $(y_1, ..., y_m) \in \mathbb{R}^m$:

(x,y) = $(x_1, ..., x_n, y_1, ..., y_m) \in \mathbb{R}^{n+m}$.

For A $\in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ where h $\in \mathbb{R}^n$ and k $\in \mathbb{R}^m$, let:

$A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$ $A_x h = $ A(h,0)

$A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ $A_y k = $ A(0,k)

Thus, A(h,k) = $A_x h + A_y k$.

$$A = n\{ \overbrace{\begin{bmatrix} A_x & A_y \end{bmatrix}}^{n+m} \begin{bmatrix} h \\ k \end{bmatrix} \}n+m$$

$$= \begin{bmatrix} a_{x_{11}} & \cdots & a_{x_{1n}} & a_{y_{11}} & \cdots & a_{y_{1m}} \\ a_{x_{21}} & \cdots & a_{x_{2n}} & a_{y_{21}} & \cdots & a_{y_{2m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{x_{n1}} & \cdots & a_{x_{nn}} & a_{y_{n1}} & \cdots & a_{y_{nm}} \end{bmatrix} \begin{bmatrix} h_1 \\ \cdots \\ h_n \\ k_1 \\ \cdots \\ k_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{x_{11}} h_1 & \cdots & a_{x_{1n}} h_n & a_{y_{11}} k_1 & \cdots & a_{y_{1m}} k_m \\ a_{x_{21}} h_1 & \cdots & a_{x_{2n}} h_n & a_{y_{21}} k_1 & \cdots & a_{y_{2m}} k_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{x_{n1}} h_1 & \cdots & a_{x_{nn}} h_n & a_{y_{n1}} k_1 & \cdots & a_{y_{nm}} k_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{x_{11}} h_1 & \cdots & a_{x_{1n}} h_n \\ a_{x_{21}} h_1 & \cdots & a_{x_{2n}} h_n \\ \vdots & \ddots & \vdots \\ a_{x_{n1}} h_1 & \cdots & a_{x_{nn}} h_n \end{bmatrix} + \begin{bmatrix} a_{y_{11}} k_1 & \cdots & a_{y_{1m}} k_m \\ a_{y_{21}} k_1 & \cdots & a_{y_{2m}} k_m \\ \vdots & \ddots & \vdots \\ a_{y_{n1}} k_1 & \cdots & a_{y_{nm}} k_m \end{bmatrix} = A_x h + A_y k$$

Theorem 16.5.2: Every k has a unique h such that A(h,k) = 0

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then for every $k \in \mathbb{R}^m$, there is a unique $h \in \mathbb{R}^n$ such that A(h,k) = 0. Then:

$\qquad$ h $= -(A_x)^{-1}A_y k$

Proof

Since $0 = A(h,k) = A_x h + A_y k$ and A_x is invertible and thus, $(A_x)^{-1}$ exist, then:
$\quad (A_x)^{-1}0 = (A_x)^{-1}A_x h + (A_x)^{-1}A_y k \qquad \rightarrow \qquad 0 = \text{h} + (A_x)^{-1}A_y k$
Thus, h $= -(A_x)^{-1}A_y k$ is unique.

Theorem 16.5.3: Implicit Function Theorem

Let $f \in \mathscr{C}'(E)$: open $E \subset \mathbb{R}^{n+m} \to \mathbb{R}^n$ such that f(a,b) = 0 for some (a,b) $\in$ E.
Let A = Df(a,b) where A_x is invertible.
Then there are open $U \in \mathbb{R}^{n+m}$, $W \in \mathbb{R}^m$ where (a,b) $\in$ U, b $\in$ W such that:
$\qquad$ For every y $\in$ W, there is a unique x such that (x,y) $\in$ U where f(x,y) = 0
If x = g(y), then $g \in \mathscr{C}'(W)$: W $\to \mathbb{R}^n$ where:
$\qquad$ g(b) = a $\qquad$ f(g(y),y) = 0 for y $\in$ W $\qquad$ g'(b) $= -(A_x)^{-1}A_y$

Proof

Let F(x,y) = (f(x,y),y) for (x,y) $\in$ E. Then F(x,y) $\in \mathscr{C}'(E)$: E $\to \mathbb{R}^{n+m}$.

Since DF(x,y) $= \begin{bmatrix} D_x f(x,y) & D_y f(x,y) \\ D_x y & D_y y \end{bmatrix} = \begin{bmatrix} D_x f(x,y) & D_y f(x,y) \\ 0_{m \times n} & I_{m \times m} \end{bmatrix}$, then

$\det(DF(x,y)) = \det(D_x f(x,y))\det(I_{m \times m}) = \det(D_x f(x,y))$.
Since $A_x = D_x f(a,b)$ is invertible so $\det(A_x) \neq 0$, then $\det(DF(a,b)) = \det(D_x f(a,b))$ $\neq 0$ and thus, DF(a,b) is invertible. Then by theorem 16.4.1a, there are open U,V $\in$ $\mathbb{R}^{n+m}$ such that F: (a,b) $\in$ U $\to$ (f(a,b),b) = (0,b) $\in$ V is invertible. Let W be the set of all y $\in \mathbb{R}^m$ such that (0,y) $\in$ V so b $\in$ W where W is open since V is open.
Since F is invertible on U so F is 1-1 on U, then for every y $\in$ W so (0,y) $\in$ V, there is a unique (x,y) $\in$ U such that F(x,y) = (f(x,y),y) = (0,y) so f(x,y) = 0.
For y $\in$ W, let g be (x,y) = (g(y),y) $\in$ U and f(g(y),y) = 0.
Thus, F(g(y),y) = (0,y) so f(g(y),y) = 0 for y $\in$ W.
Let G be the inverse of F. Then by theorem 16.4.1b, then G $\in \mathscr{C}'(V)$.
$\quad$ (g(y),y) = G(F(g(y),y)) = G(0,y)
Thus, $g \in \mathscr{C}'(W)$: W $\to \mathbb{R}^n$ where b $\in$ W so g(b) = a.
Let (g(y),y) $= \phi(y)$ so $\phi'(y)k = (\text{g'(y)}k,k)$ for k $\in \mathbb{R}^m$.
Since $f(\phi(y)) = $ f(g(y),y) = 0 for y $\in$ W, then f'$(\phi(y))\phi'(y) = 0$.
For y = b $\in$ W, then $\phi(b)$ = (g(b),b) = (a,b) so $D f(\phi(b))$ = Df(a,b) = A.
$\quad 0 = 0k = \text{f'}(\phi(b))\phi'(b)k = A\phi'(b)k = A(\text{g'(b)}k,k) = A_x \text{g'(b)}k + A_y k$
Since A_x is invertible so $(A_x)^{-1}$ exist, then g'(b)k $= (A_x)^{-1}A_x \text{g'(b)}k = -(A_x)^{-1}A_y k$.

Example

$\qquad xu^2 + yv^2 + xy = 11 \qquad xv^2 + yu^2 - xy$ = -1
$\qquad$ Show (u,v,x,y) close enough to (1,1,2,3) satisfy the system of equations.

Let $F(u,v,x,y) = (xu^2 + yv^2 + xy - 11, xv^2 + yu^2 - xy + 1)$.

Then $DF_{u,v} = \begin{bmatrix} 2xu & 2yv \\ 2yu & 2xv \end{bmatrix}$ so $DF_{u,v}(1,1,2,3) = \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ is invertible.

Then by the Implicit Function Theorem, there is an open W where (2,3) $\in$ W with g(2,3) = (1,1) so (u,v,x,y) = (g(x,y),x,y) near (1,1,2,3) satisfy the equations.

17   Lebesgue Measure

17.1   Regulated Integral

Definition 17.1.1: Basic Properties of the Integral

Let $\mathcal{V}$ be a vector space of real-valued functions on closed interval I.

If f,g $\in \mathcal{V}$ and c $\in \mathbb{R}$, then $f + g, cf \in \mathcal{V}$

For each f $\in \mathcal{V}$, the integral of f on [a,b] $\subset$ I, $\int_a^b f(x)dx$ should satisfy:

(a) Linearity: For f,g $\in \mathcal{V}$ and $c_1, c_2 \in \mathbb{R}$:
$\int_a^b c_1 f(x) + c_2 g(x)dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx$

(b) Monotonicity: For f,g $\in \mathcal{V}$ where g(x) $\leq$ f(x):
$\int_a^b g(x)dx \leq \int_a^b f(x)dx$

(c) Additivity: For f $\in \mathcal{V}$ and c $\in$ [a,b]:
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(d) Constant: For f(x) = C:
$\int_a^b Cdx = C(b-a)$

(e) Finite Sets: For f,g $\in \mathcal{V}$ where f(x) = g(x) for all, but finitely many x:
$\int_a^b f(x)dx = \int_a^b g(x)dx$

It should be noted that all integrals need not satisfy properties 3, 4, and 5. However, all integrals consider henceforth will satisfy them.

Theorem 17.1.2: Absolute Value

If f,$|f| \in \mathcal{V}$, then if a $\leq$ b:
$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$

Proof

Since f(x) $\leq |f(x)|$, then by definition 17.1.1b, $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$.
Also, since -f(x) $\leq |f(x)|$, then $\int_a^b -f(x)dx \leq \int_a^b |f(x)|dx$.
Since $|\int_a^b f(x)dx|$ is either equal to $\int_a^b f(x)dx$ or $-\int_a^b f(x)dx$, then:
$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

Definition 17.1.3: Step Function

Function f: [a,b] $\rightarrow \mathbb{R}$ is a step function if there is a partition $\{x_0, ..., x_n\}$:
a = $x_0 < x_1 < ... < x_{n-1} < x_n$ = b
such that f(x) = c_i on (x_{i-1}, x_i) for constant c_i

Theorem 17.1.4: Integral of a Step Function

If step function f with partition $\{x_0, ..., x_n\}$ of [a,b] is f(x) = c_i for x $\in (x_{i-1}, x_i)$:
$\int_a^b f(x)dx = \sum_{i=1}^n c_i(x_i - x_{i-1})$

Proof

By definition 17.1.1b, $\int_a^b f(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx$
Since f(x) = c_i, but finitely many x on $[x_{i-1}, x_i]$ (i.e. endpoints):
$\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} c_i dx = \sum_{i=1}^n c_i(x_i - x_{i-1})$

Theorem 17.1.5: Step Functions form a Vector Space

The collection of all step functions on [a,b] form a vector space

Proof

> Let f,g be step functions with values c_i and d_j on partitions $\{x_0, ..., x_n\}$ and $\{y_0, ..., y_m\}$ respectively. Let $k_1, k_2 \in \mathbb{R}$. Let partition Z = $\{x_0, ..., x_n\} \cup \{y_0, ..., y_m\}$.
> Then each $[z_{k-1}, z_k] \subset [x_{i-1}, x_i]$ and $[z_{k-1}, z_k] \subset [y_{j-1}, y_j]$ for some i and j.
> Then $k_1 f + k_2 g$ have value $k_1 c_i + k_2 d_j$ on (z_{k-1}, z_k) so $k_1 f + k_2 g$ is a step function.

Theorem 17.1.6: Integral of Step Functions are independent of Partition

Let step function f have value c_i on partition $\{x_0, ..., x_n\}$ and value d_j on partition $\{y_0, ..., y_m\}$. Then:

$$\sum_{i=1}^{n} c_i(x_i - x_{i-1}) = \sum_{j=m}^{n} d_j(y_j - y_{j-1})$$

Proof

> Let partition Z = $\{x_0, ..., x_n\} \cup \{y_0, ..., y_m\}$.
> Then each $[z_{k-1}, z_k] \subset [x_{i-1}, x_i]$ and $[z_{k-1}, z_k] \subset [y_{j-1}, y_j]$ for some i and j.
> Let $\{z_t^*\}$ be the set of z_k where $[z_{t-1}^*, z_t^*] = [z_{k-1}, z_k] \cup ... \cup [z_{k+t^*-1}, z_{k+t^*}] = [x_{i-1}, x_i]$.
> $\quad \sum_i c_i(x_i - x_{i-1}) = \sum_t c_i(z_t^* - z_{t-1}^*)$
> $\quad = \sum_k v_k(z_k - z_{k-1})$ where $v_k = c_i$ where $[z_{k-1}, z_k] \subset [x_{i-1}, x_i]$
> Let $\{z_t^{**}\}$ be the set of z_k where $[z_{t-1}^{**}, z_t^{**}] = [z_{k-1}, z_k] \cup ... \cup [z_{k+t^{**}-1}, z_{k+t^{**}}] = [y_{j-1}, y_j]$.
> $\quad \sum_j d_j(y_j - y_{j-1}) = \sum_t d_i(z_t^{**} - z_{t-1}^{**})$
> $\quad = \sum_k v_k(z_k - z_{k-1})$ where $v_k = d_j$ where $[z_{k-1}, z_k] \subset [y_{j-1}, y_j]$
> Thus, $\sum_i c_i(x_i - x_{i-1}) = \sum_k v_k(z_k - z_{k-1}) = \sum_j d_j(y_j - y_{j-1})$.

Definition 17.1.7: Regulated Function

Function f: [a,b] $\to \mathbb{R}$ is regulated if:

There is a sequence of step functions $\{f_n\}$ that converge uniformly to f

Theorem 17.1.8: Regulated Integral

Suppose step functions $\{f_n\}$ on [a,b] converge uniformly to f. Then $\{\int_a^b f_n(x)dx\}$ converges. If step functions $\{g_n\}$ also converge uniformly to f:

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \lim_{n\to\infty} \int_a^b g_n(x)dx$$

Then, the regulated integral of f on [a,b] can be defined:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

Proof

> Let $z_n = \int_a^b f_n(x)dx$.
> Since $\{f_n\}$ converges uniformly to f, there is a N where for m,n $\geq$ N and all x $\in$ [a,b]:
> $\quad |f_m(x) - f_n(x)| < \frac{\epsilon}{b-a}$
> Thus:
> $\quad |z_m - z_n| = |\int_a^b f_m(x)dx - \int_a^b f_n(x)dx| \leq \int_a^b |f_m(x) - f_n(x)|dx < \int_a^b \frac{\epsilon}{b-a}dx = \epsilon$
> Since $\{z_n\}$ is Cauchy on $\mathbb{R}$, then $\{z_n\}$ converges.
> If $\{g_n\}$ converges uniformly to f, then there is a M where for n $\geq$ M and all x $\in$ [a,b]:
> $\quad |f_n(x) - f| < \frac{\epsilon}{2(b-a)}$ $|g_n(x) - f| < \frac{\epsilon}{2(b-a)}$
> $\quad |f_n(x) - g_n(x)| \leq |f_n(x) - f| + |f - g_n(x)| < \frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{b-a}$
> Thus
> $\quad |\int_a^b f(x)dx - \int_a^b f_n(x)dx| \leq \int_a^b |f(x) - f_n(x)|dx < \int_a^b \frac{\epsilon}{b-a}dx = \epsilon$

Theorem 17.1.9: Continuous functions are regulated

Every continuous function f: [a,b] $\to \mathbb{R}$ is a regulated function

Proof

Since f is continuous on compact [a,b], then f is uniformly continuous on [a,b].
Thus for any $\epsilon_n = \frac{1}{2^n}$, there is a δ_n where for $|x - y| < \delta_n$, then $|f(x) - f(y)| < \epsilon_n$.
For a fixed n, choose a partition $\{x_0, ..., x_m\}$ such that each $\Delta x_i = \frac{b-a}{m} < \delta_n$.
Let step function $f_n(x) = f(x_i)$ for x $\in [x_{i-1}, x_i)$ for i = {1,...,m}. For x $\in$ [a,b], there
is an i such that x $\in [x_{i-1}, x_i)$ so $|f(x) - f_n(x)| = |f(x) - f(x_i)| < \epsilon_n$.
Thus, $\{f_n\}$ converges uniformly to f, then f is regulated.

Theorem 17.1.10: Lower and Upper Riemann Limit Redefined

Let f be a bounded on [a,b]. Let:
  $\mathcal{U}(f)$ = { u(x) | f(x) $\leq$ u(x) for all x , u(x) is a step function }.
  $\mathcal{L}(f)$ = { v(x) | f(x) $\geq$ v(x) for all x , v(x) is a step function }.
Then, $\sup\limits_{v \in \mathcal{L}(f)} (\int_a^b v(x)dx) \leq \inf\limits_{u \in \mathcal{U}(f)} (\int_a^b u(x)dx)$.

Proof

Since v(x) $\leq$ f(x) $\leq$ u(x), then $\int_a^b v(x)dx \leq \int_a^b u(x)dx$.
Since $\int_a^b v(x)dx \leq \int_a^b u(x)dx$ holds for any u(x) $\geq$ v(x), then:
  $\int_a^b v(x)dx \leq \inf(\int_a^b u(x)dx)$
Also, since $\int_a^b v(x)dx \leq \inf(\int_a^b u(x)dx)$ holds for any v(x) $\leq$ u(x), then:
  $\sup(\int_a^b v(x)dx) \leq \inf(\int_a^b u(x)dx)$

Definition 17.1.11: Riemann Integral Redefined

Let f be a bounded on [a,b]. Let:
  $\mathcal{U}(f)$ = { u(x) | f(x) $\leq$ u(x) for all x , u(x) is a step function }.
  $\mathcal{L}(f)$ = { v(x) | f(x) $\geq$ v(x) for all x , v(x) is a step function }.
Then f is Riemann integrable if:
  $\sup\limits_{v \in \mathcal{L}(f)} (\int_a^b v(x)dx) = \inf\limits_{u \in \mathcal{U}(f)} (\int_a^b u(x)dx)$

Theorem 17.1.12: Riemann-Integrability ϵ Definition Redefined

A bounded f on [a,b] is Riemann integrable if and only if:
  For $\epsilon > 0$, there are step functions $v(x), u(x)$ where v(x) $\leq$ f(x) $\leq$ u(x):
    $\int_a^b u(x)dx$ - $\int_a^b v(x)dx < \epsilon$

Proof

If f is Riemann integrable, then for $\epsilon > 0$, there are step functions u(x),v(x):
  $|\int_a^b f(x)dx - \int_a^b u(x)dx| < \frac{\epsilon}{2}$ $|\int_a^b f(x)dx - \int_a^b v(x)dx| < \frac{\epsilon}{2}$
Thus:
  $|\int_a^b u(x)dx - \int_a^b v(x)dx| \leq |\int_a^b u(x)dx - \int_a^b f(x)dx| + |\int_a^b f(x)dx - \int_a^b v(x)dx| < \epsilon$

If for $\epsilon > 0$, there are step functions $v(x), u(x)$ where v(x) $\leq$ f(x) $\leq$ u(x):
  $\int_a^b u(x)dx$ - $\int_a^b v(x)dx < \epsilon$
Since $\sup(\int_a^b v(x)dx) \geq \int_a^b u(x)dx$ and $\inf(\int_a^b u(x)dx) \leq \int_a^b v(x)dx$, then:
  $\inf(\int_a^b u(x)dx)$ - $\sup(\int_a^b v(x)dx) \leq \int_a^b u(x)dx$ - $\int_a^b v(x)dx < \epsilon$
Thus, $\sup(\int_a^b v(x)dx) = \inf(\int_a^b u(x)dx)$ so f is Riemann integrable.

Theorem 17.1.13: Regulated functions are Riemann Integrable
  Every regulated function is Riemann integrable where the regulated integral
  is equal to the Riemann integral
Proof

Since f is regulated, then for $\epsilon_n = \frac{1}{2^n}$, there is a step function f_n such that for all x $\in$ [a,b] so $|f(x) - f_n(x)| < \epsilon_n$. Thus, $\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$.
Let step functions $u_n(x) = f_n(x) + \frac{1}{2^n}$ and $v_n(x) = f_n(x) - \frac{1}{2^n}$ so $v_n(x) <$ f(x) $< u_n(x)$ for all x $\in$ [a,b]. Then:
  $|\int_a^b u_n(x)dx - \int_a^b v_n(x)dx| \le \int_a^b |u_n(x) - v_n(x)|dx = \int_a^b \frac{1}{2^{n-1}}dx = \frac{b-a}{2^{n-1}}$
Thus, by theorem 17.1.12, f is Riemann integrable. Since:
  $\lim_{n \to \infty} \int_a^b u_n(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx + \lim_{n \to \infty} \int_a^b \frac{1}{2^n}dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$
  $\lim_{n \to \infty} \int_a^b v_n(x)dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$ - $\lim_{n \to \infty} \int_a^b \frac{1}{2^n}dx = \lim_{n \to \infty} \int_a^b f_n(x)dx$
Thus, the Riemann integral of f is $\lim_{n \to \infty} \int_a^b f_n(x)dx$ so the regulated integral is equal to the Riemann integral.

Theorem 17.1.14: Riemann Intergrable functions form a vector space
  The set $\mathcal{R}$ of bounded Riemann integrable functions on [a,b] is a vector space
  that contains the vector space of regulated functions
Proof

By theorem 17.1.13, every regulated function is Riemann integrable so $\mathcal{R}$ contain the set of regulated functions. Let f,g $\in \mathcal{R}$ and $c_1, c_2 \in \mathbb{R}$.
Then for $\epsilon > 0$, there are step functions v_f, u_f where $v_f \le$ f $\le u_f$ such that:
  $\int_a^b u_f(x)dx - \int_a^b v_f(x)dx < \frac{\epsilon}{2c_1}$
Also, there are step functions v_g, u_g where $v_g \le$ g $\le u_g$ such that:
  $\int_a^b u_g(x)dx - \int_a^b v_g(x)dx < \frac{\epsilon}{2c_2}$
Since $c_1 v_f + c_2 v_g \le c_1 f + c_2 g \le c_1 u_f + c_2 u_g$ where $c_1 v_f + c_2 v_g, c_1 u_f + c_2 u_g$ are step functions such that:
  $\int_a^b (c_1 u_f(x) + c_2 u_g(x))dx - \int_a^b (c_1 v_f(x) + c_2 v_g(x))dx$
  $= \int_a^b c_1(u_f(x) - v_f(x))dx + \int_a^b c_2(u_g(x)) - v_g(x))dx < c_1 \frac{\epsilon}{2c_1} + c_2 \frac{\epsilon}{2c_2} = \epsilon$
then $c_1 f + c_2 g$ is Riemann integrable so $c_1 f + c_2 g \in \mathcal{R}$.

17.2   Outer Measure

Definition 17.2.1: Basic Properties of the Length / Measure of a Set
  For bounded A,B $\subset \mathbb{R}$, there is an associated non-negative real number $\mu(A)$:
  (a) Length: If A = (a,b) or A = [a,b], then:
      $\mu(A) = \text{len}(A) =$ b-a
  (b) Translation Invariance: If c $\in \mathbb{R}$, then:
      $\mu(A + c) = \mu(A)$
  (c) Countable Additivity: If $\{A_n\}_{n=1}^{\infty}$ is countable, then:
      $\mu(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$
      If each A_n are pairwise disjoint, then:
      $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$
  (d) Monotonicity: If A $\subset$ B, then:
      $\mu(A) \le \mu(B)$

Definition 17.2.2: Null Set
 X $\subset \mathbb{R}$ is a null set if for $\epsilon > 0$:
  There is a collection of open set $\{U_n\}_{n=1}^{\infty}$ where X $\subset \cup_{n=1}^{n} U_n$:
   $\sum_{n=1}^{\infty} \text{len}(U_n) < \epsilon$
 If X is a null set, then X^c has full measure.

Definition 17.2.3: Outer Measure
 Let A $\subset \mathbb{R}$. Let open intervals $\{I_n\}_{n=1}^{\infty}$ be such that A $\subset U_{n=1}^{\infty} I_n$.
 Then the outer measure $\mu^*(A)$:
  $\mu^*(A) = \inf(\sum_{n=1}^{\infty} \text{len}(I_n))$

Theorem 17.2.4: Null set A $\rightleftharpoons \mu^*(A) = 0$
 Let A $\subset \mathbb{R}$. Then, A is a null set if and only if $\mu^*(A) = 0$.
Proof

If A is a null set, then for $\epsilon > 0$, there are open intervals $\{I_n\}_{n=1}^{\infty}$ where A $\subset \cup_{n=1}^{\infty} I_n$:
 $\sum_{n=1}^{n} \text{len}(I_n) < \epsilon$
Then, $\mu^*(A) = \inf(\sum_{n=1}^{n} \text{len}(I_n)) \leq \sum_{n=1}^{n} \text{len}(I_n) = \epsilon$ so $\mu^*(A) < 0$.

If $\mu^*(A) = 0$, then for open intervals $\{I_n\}_{n=1}^{\infty}$ where A $\subset \cup_{n=1}^{\infty} I_n$:
 $0 = \mu^*(A) = \inf(\sum_{n=1}^{n} \text{len}(I_n))$
Thus, for $\epsilon > 0$, there is a $\{I_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{n} \text{len}(I_n) < \epsilon$ so A is a null set.

Theorem 17.2.5: Outer Measure: Length Property
 $\mu^*([a,b]) = \mu^*((a,b))$ = b - a

Let $I_n = (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$. Then:
 $\mu^*([a,b]) \leq \text{len}(I_n) = b - a + \epsilon$ $\rightarrow$ $\mu^*([a,b]) \leq b - a$
Since [a,b] is compact, then for any $\{I_i\}_{i=1}^{\infty}$ where [a,b] $\subset \cup_{i=1}^{\infty} I_i$, there is a M such
that [a,b] $\subset \cup_{i=1}^{M} I_i$. Let n be the number of elements in [a,b].
If n = 1, then a = b so $0 = \mu^*([a,b]) \geq$ b-a = b-b = 0 holds true.
If n > 1, then there is at least two intervals I_{n_1}, I_{n_2} that intersect since if c $\in$ (a,b),
then only (a,c),(c,b) will not contain c. Let $V_{n-1} = I_{n-1} \cup I_{n-2}$. Then, let $V_i = I_i$ for
the I_i where i $\neq n_1, n_2$ and i $< \max(n_1, n_2)$ and $V_i = I_{i-1}$ for the I_i where i $\neq n_1, n_2$
and i $> \max(n_1, n_2)$. Thus:
 $\sum_{i=1}^{M} \text{len}(I_i) > \sum_{i=1}^{M_1} \text{len}(V_i) \geq$ b - a $\rightarrow$ $\mu^*([a,b]) \geq b - a$

Since (a,b) $\subset$ (a,b), then $\mu^*((a,b)) \leq \text{len}((a,b))$ = b - a.
Since $\{I_i\}_{i=1}^{\infty}$ where (a,b) $\subset \cup_{i=1}^{\infty} I_i$ have $[a + \epsilon, b - \epsilon] \subset \cup_{i=1}^{\infty} I_i$, then by process above:
 $\sum_{i=1}^{\infty} \text{len}(I_i) \geq b - a - 2\epsilon$ $\rightarrow$ $\mu^*((a,b)) \geq b - a$

Theorem 17.2.6: Outer Measure: Monotonicity Property
 If A,B $\subset \mathbb{R}$ where A $\subset$ B, then $\mu^*(A) \leq \mu^*(B)$
Proof

Since A $\subset$ B, then every open intervals $\{I_i\}_{i=1}^{\infty}$ where B $\subset \cup_{i=1}^{\infty} I_i$ is A $\subset \cup_{i=1}^{\infty} I_i$. Thus:
 $\mu^*(A) = \inf_A(\sum_{i=1}^{\infty} \text{len}(I_i)) \leq \inf_B(\sum_{i=1}^{\infty} \text{len}(I_i)) = \mu^*(B)$

Theorem 17.2.7: Outer Measure: Countable Subadditivity Property

For $\{A_n\}_{n=1}^{\infty}$ where each $A_n \subset \mathbb{R}$:

$\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Proof

For each A_n, there are open intervals $\{I_i^n\}_{i=1}^{\infty}$ where $A_n \subset \cup_{i=1}^{\infty} I_i^n$ such that for $\epsilon > 0$:

$\sum_{i=1}^{\infty} \text{len}(I_i^n) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$

Since $\{\{I_i^n\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ have $\cup_{n=1}^{\infty} A_n \subset \cup_{n=1}^{\infty} \cup_{i=1}^{\infty} I_i^n$, then:

$\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \text{len}(I_i^n) \leq \sum_{n=1}^{\infty} [\mu^*(A_n) + \frac{\epsilon}{2^n}] = \sum_{n=1}^{\infty} \mu^*(A_n) + \frac{\epsilon}{2}$

Thus, $\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Corollary 17.2.8: Countable A $\rightleftharpoons \mu^*(A) = 0$

If A is countable, then $\mu^*(A) = 0$.

Thus, intervals are uncountable.

Proof

Since A is countable, let $A = \{x_1, x_2, ...\}$.

Since $\mu^*(\{x_n\}) = 0$, then:

$\mu^*(A) = \mu^*(\{x_1, x_2, ...\}) \leq \sum_{n=1}^{\infty} \mu^*(\{x_n\}) = 0$

Thus, $\mu^*(A) = 0$.

Since $\mu^*([a,b]) = $ b - a $\neq 0$, then A is uncountable.

Theorem 17.2.9: Outer Measure: Translation Invariance Property

If $A \subset \mathbb{R}$ and c $\in \mathbb{R}$, then $\mu^*(A + c) = \mu^*(A)$

Proof

There are open intervals $\{I_i\}_{i=1}^{\infty}$ where A+c $\subset \cup_{i=1}^{\infty} I_i$ such that:

$|\sum_{i=1}^{\infty} \text{len}(I_i) - \mu^*(A + c)| \leq \frac{\epsilon}{2}$

Let open intervals $\{I_i^*\}_{i=1}^{\infty}$ be $I_i^* = I_i - c$ so $A \subset \cup_{i=1}^{\infty} I_i^*$ where:

$|\sum_{i=1}^{\infty} \text{len}(I_i^*) - \mu^*(A)| \leq \frac{\epsilon}{2}$

Since $\text{len}(I_i^*) = \text{len}(I_i - c) = \text{len}(I_i)$, then:

$|\mu^*(A + c) - \mu^*(A)|$

$\leq |\mu^*(A + c) - \sum_{i=1}^{\infty} \text{len}(I_i)| + |\sum_{i=1}^{\infty} \text{len}(I_i) - \sum_{i=1}^{\infty} \text{len}(I_i^*)|$

$\quad + |\sum_{i=1}^{\infty} \text{len}(I_i^*) - \mu^*(A)|$

$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Thus, $\mu^*(A + c) = \mu^*(A)$.

Theorem 17.2.10: Outer Measure: Regularity Property

If $A \subset \mathbb{R}$ and $\mu^*(A)$ is finite, then for any $\epsilon > 0$, there is an open set V where $A \subset V$ such that $\mu^*(V) < \mu^*(A) + \epsilon$. Thus:

$\mu^*(A) = \inf(\mu^*(U) \mid U$ is open , $A \subset U)$

Proof

There are open intervals $\{I_i\}_{i=1}^{\infty}$ where $A \subset \cup_{i=1}^{\infty} I_i$ such that for $\epsilon > 0$:

$\sum_{i=1}^{\infty} \text{len}(I_i) < \mu^*(A) + \epsilon$

Let $V = \cup_{i=1}^{\infty} I_i$. Then:

$\mu^*(V) = \mu^*(\cup_{i=1}^{\infty} I_i) \leq \sum_{i=1}^{\infty} \text{len}(I_i) < \mu^*(A) + \epsilon$

Thus, $\inf(\mu^*(U) \mid U$ is open , $A \subset U) \leq \mu^*(A) + \epsilon$ so:

$\inf(\mu^*(U) \mid U$ is open , $A \subset U) \leq \mu^*(A)$.

Since $A \subset \cup_{i=1}^{\infty} I_i = V$, then $\mu^*(A) \leq \mu^*(V) = \inf(\mu^*(U) \mid U$ is open , $A \subset U)$.

Thus, $\mu^*(A) = \inf(\mu^*(U) \mid U$ is open , $A \subset U)$.

17.3   Lebesgue Measure

Definition 17.3.1: Sigma Algebra and Borel Sets

Let $\mathcal{A}$ be a collection of subsets of X.

Then, $\mathcal{A}$ is a σ-algebra of subsets of X if for A $\in \mathcal{A}$:

  (a) X $\in \mathcal{A}$

  (b) $A^c \in \mathcal{A}$ in respect to X

  (c) $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$

Some examples of σ-algebra of subsets of X are:

    $\mathcal{A} = \{$X,$\emptyset\}$ $\mathcal{A} = $ P(X) (i.e. all subsets of X ($2^{\mathbb{R}}$))

If C is a collection of subsets of $\mathbb{R}$ and $\mathcal{A}$ is the smallest σ-algebra of subsets of $\mathbb{R}$ that contains C, then $\mathcal{A}$ is a σ-algebra generated by C.

Let $\mathcal{B}$ be σ-algebra of subsets of $\mathbb{R}$ generated by the collection of all open intervals. Then, $\mathcal{B}$ is a Borel σ-algebra and any B $\in \mathcal{B}$ is a Borel set.

Definition 17.3.2: Lebesgue Measurable

Let the σ-algebra of subsets of $\mathbb{R}$ generated by the collection of all open intervals and null sets be $\mathcal{M}$. Then, sets in $\mathcal{M}$ are Lebesgue measurable.

If I is a closed interval, then $\mathcal{M}(I)$ is a σ-algebra of subsets of I.

Theorem 17.3.3: Lebesgue Measure

There is a unique μ, the Lebesgue measure, from A,B $\in \mathcal{M}(I)$ to $\mathbb{R}_+$:

  (a) Length: If A = (a,b), then:

    $\mu(A) = $ len(A) = b-a

  (b) Translation Invariance: If c $\in \mathbb{R}$ and A+c $\subset$ I, then A+c $\in \mathcal{M}(I)$ where:

    $\mu(A + c) = \mu(A)$

  (c) Countable Additivity: If $\{A_n\}_{n=1}^{\infty}$ is countable, then:

    $\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$

    If each A_n are pairwise disjoint, then:

    $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

  (d) Monotonicity: If A $\subset$ B, then:

    $\mu(A) \leq \mu(B)$

  (e) Null Sets: If A $\subset$ I is a null set, then A $\in \mathcal{M}(I)$ where:

    $\mu(A) = 0$

    Also, if A $\in \mathcal{M}(I)$ where $\mu(A) = 0$, then A is a null set.

  (f) Regularity

    $\mu(A) = \inf(\mu(U) \mid$ U is open , A $\subset$ U)

Theorem 17.3.4: The Lebesgue measure of set differences

If A,B $\in \mathcal{M}(I)$, then A\B $\in \mathcal{M}(I)$ where:

$\mu(A \cup B) = \mu(A\backslash B) + \mu(B)$

Thus, if I = [0,1], then $\mu(I) = 1$ so $\mu(A^c) = $ 1 - $\mu(A)$.

Proof

Since A\B = A $\cap B^c$ where A,$B^c \in \mathcal{M}(I)$, then A\B $\in \mathcal{M}(I)$.

Since A\B and B are disjoint where $A\backslash B \cup B = A \cup B$, then:

  $\mu(A \cup B) = \mu(A\backslash B \cup B) = \mu(A\backslash B) + \mu(B)$

  $\mu(I\backslash A) + \mu(A) = \mu(A^c) + \mu(A) = \mu(A^c \cup A) = \mu(I) = 1$

Theorem 17.3.5: Alternative Definition for Lebesgue Measurable

A $\subset$ I is Lebesgue Measurable if for all X $\subset$ I:
$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X)$$

Proof

> Since X $= (A \cap X) \cup (A^c \cap X)$, then by theorem 17.2.7, then:
> $$\mu^*(X) \leq \mu^*(A \cap X) + \mu^*(A^c \cap X)$$

Theorem 17.3.6: Lebesgue Measure's Regularity ϵ Definition

If A $\in \mathcal{M}(I)$, then for $\epsilon > 0$:
There is a closed C $\subset$ A such that:
$$\mu(C) > \mu(A) - \epsilon$$
There is a countable union of pairwise disjoint open intervals U $= \cup \, U_n$
where A $\subset$ U such that:
$$\mu(U) < \mu(A) + \epsilon$$

Proof

> Since A $\in \mathcal{M}(I)$, then $A^c \in \mathcal{M}(I)$. Thus for $\epsilon > 0$, there is an open set V such that $A^c \subset$ V where $\mu(V) < \mu(A^c) + \epsilon$. Let C $= V^c$ so C is closed and C $\subset$ A. Then:
> $$\mu(C) = \mu(V^c) = 1 \text{ - } \mu(V) > 1 \text{ - } \mu(A^c) \text{ - } \epsilon = \mu(A) \text{ - } \epsilon$$
>
> ---
>
> Since A $\in \mathcal{M}(I)$, then for $\epsilon > 0$, there is a open set U such that A $\subset$ U where:
> $$\mu(U) < \mu(A) + \epsilon$$

Theorem 17.3.7: Monotonic Measurable Sets

If $A_n \subset A_{n+1}$ are Lebesgue measurable subsets of I, then:
$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$$
If $B_{n+1} \subset B_n$ are Lebesgue measurable subsets of I, then:
$$\mu(\cap_{n=1}^{\infty} B_n) = \lim_{n \to \infty} \mu(B_n)$$

Proof

> Since A_n is Lebesgue measurable, then $\cup A_n$ is Lebesgue measurable.
> Let $F_n = A_n \backslash A_{n-1}$, then $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$ where each F_n is pairwsie disjoint.
> $$\mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} F_n) = \lim_{n \to \infty} \sum_{i=1}^{n} \mu(F_i) = \lim_{n \to \infty} \mu(A_n)$$
>
> ---
>
> Since B_n is Lebesgue measurable, then $\cap B_n$ is Lebesgue measurable.
> Let $E_n = B_n^c$. Since $(\cap B_n)^c = \cup E_n$ where each $E_n \subset E_{n+1}$, then:
> $$\mu(\cap_{n=1}^{\infty} B_n) = 1 \text{ - } \mu(\cup_{n=1}^{\infty} E_n) = \lim_{n \to \infty}(1 - \mu(E_n)) = \lim_{n \to \infty} \mu(B_n)$$

17.4   Lebesgue Integral

Definition 17.4.1: Indicator Function

For A $\subset$ [0,1], the indicator function:
$$\mathfrak{X}_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition 17.4.2: Measurable Partition

A finite measurable partition of [0,1] is a collection $\{A_i\}_{i=1}^n$ of measurable subsets which are pairwise disjoint where $\cup A_i = [0,1]$.

Definition 17.4.3: Simple Function

f: $[0,1] \to \mathbb{R}$ is simple if there exists a finite measurable partition, $\{A_i\}_{i=1}^n$ and $r_i \in \mathbb{R}$ such that $f(x) = \sum_{i=1}^n r_i \mathfrak{X}_{A_i}$.

Then the Lebesgue integral of a simple function:
$$\int f \, d\mu = \sum_{i=1}^n r_i \mu(A_i)$$

Theorem 17.4.4: Properties of Simple Functions

The set of simple functions is a vector space where:

(a) Linearity: If f,g are simple functions and $c_1, c_2 \in \mathbb{R}$:
$$\int c_1 f + c_2 g \, d\mu = c_1 \int f \, d\mu + c_2 \int g \, d\mu$$

(b) Monotonicity: If f,g are simple where $f(x) \leq g(x)$:
$$\int f \, d\mu \leq \int g \, d\mu$$

(c) Absolute Value: If f is simple, then $|f|$ is simple:
$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proof

Since f is simple, then there is a measurable partition $\cup_{i=1}^n A_i = [0,1]$ where A_i is disjoint so $f(x) = \sum_{i=1}^n r_i \mathfrak{X}_{A_i}$. Then, $c_1 f$ is simple since $c_1 f(x) = \sum_{i=1}^n c_1 r_i \mathfrak{X}_{A_i}$.
Since g is simple, then there is a measurable partition $\cup_{j=1}^m B_j = [0,1]$ where B_j is disjoint so $g(x) = \sum_{j=1}^m s_j \mathfrak{X}_{B_j}$.
Then for $c_1 f + c_2 g$, take the measurable partition $\cup_{i=1}^n \cup_{j=1}^m C_{i,j}$ where $C_{ij} = A_i \cap B_j$.
$$\begin{aligned} c_1 f(x) + c_2 g(x) &= \sum_{i=1}^n c_1 r_i \mathfrak{X}_{A_i} + \sum_{j=1}^m c_2 s_j \mathfrak{X}_{B_j} \\ &= \sum_{i=1}^n c_1 r_i \sum_{j=1}^m \mathfrak{X}_{C_{ij}} + \sum_{j=1}^m c_2 s_j \sum_{i=1}^n \mathfrak{X}_{C_{ij}} \\ &= \sum_{i=1}^n \sum_{j=1}^m (c_1 r_i + c_2 s_j) \mathfrak{X}_{C_{ij}} \end{aligned}$$
Thus, the simple functions form a vector space.
$$\begin{aligned} \int c_1 f + c_2 g \, d\mu &= \sum_{i=1}^n \sum_{j=1}^m (c_1 r_i + c_2 s_j) \mu(C_{ij}) \\ &= \sum_{i=1}^n c_1 r_i \sum_{j=1}^m \mu(C_{ij}) + \sum_{j=1}^m c_2 s_j \sum_{i=1}^n \mu(C_{ij}) \\ &= \sum_{i=1}^n c_1 r_i \mu(A_i) + \sum_{j=1}^m c_2 s_j \mu(B_j) = c_1 \int f \, d\mu + c_2 \int g \, d\mu \end{aligned}$$
$\int g \, d\mu - \int f \, d\mu = \int (f\text{-}g) \, d\mu \geq 0$
$\left| \int f \, d\mu \right| = \left| \sum_{i=1}^n r_i \mu(A_i) \right| \leq \sum_{i=1}^n |r_i| \mu(A_i) = \int |f| \, d\mu$

Theorem 17.4.5: Measurable Functions
If f: $X \subset \mathbb{R} \to [-\infty, \infty]$, then the following are equivalent:
- For any $a \in \mathbb{R}$, $f^{-1}([-\infty, a])$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}([a, \infty])$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is Lebesgue measurable

Then f is Lebesgue measurable.

Proof

Suppose for any $a \in \mathbb{R}$, $f^{-1}([-\infty, a])$ is Lebesgue measurable.
$f^{-1}([-\infty, a)) = \cup_{n=1}^{\infty} f^{-1}([-\infty, a - \frac{1}{2^n}])$ is measurable since it's countable measurables.
$f^{-1}([a, \infty]) = f^{-1}([-\infty, a)^c) = (f^{-1}([-\infty, a)))^c$ is measurable since it's the complement of a measurable.
$f^{-1}((a, \infty]) = \cup_{n=1}^{\infty} f^{-1}([a + \frac{1}{2^n}, \infty])$ is measurable since it's countable measurables.
$f^{-1}([-\infty, a]) = f^{-1}((a, \infty]^c) = (f^{-1}((a, \infty]))^c$ is measurable since it's the complement of a measurable.

Theorem 17.4.6: Measurable Functions and Null Sets
Let f,g: [a,b] $\to \mathbb{R}$.
(a) If there is a null set $A \subset$ [a,b] where f(x) = 0 if $x \notin A$, then f is measurable
(b) If f = g except on null set A, then f is measurable if and only if g is measurable

Proof

Since f(x) = 0 if $x \notin A$, then $f^{-1}([-\infty, 0)) \cup f^{-1}((0, \infty]) \subset$ A.
If a < 0, then $f^{-1}([-\infty, a]) \subset f^{-1}([-\infty, 0)) \subset$ A so $f^{-1}([-\infty, a])$ is a null set and thus, measurable. For $a \geq 0$, then $f^{-1}([-\infty, a]) = (f^{-1}((a, \infty]))^c \subset (f^{-1}((0, \infty)))^c$ so $f^{-1}([-\infty, a])$ is a complement of a null set and thus, measurable.

Suppose f is measurable. Let $a \in \mathbb{R}$.
$g^{-1}([a, \infty]) = (g^{-1}([a, \infty]) \cap A) \cup (g^{-1}([a, \infty]) \cap A^c)$
Since f = g on A^c, then $(g^{-1}([a, \infty]) \cap A^c) = (f^{-1}([a, \infty]) \cap A^c)$ which is measurable.
Since $(g^{-1}([a, \infty]) \cap A) \subset$ A, then $(g^{-1}([a, \infty]) \cap A)$ is a null set and thus, measurable.
Proof is analogous for g.

Theorem 17.4.7: Measurable Functions and Sequences
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Then:
$g_1(x) = \sup(f_n(x))$ $\qquad\qquad$ $g_2(x) = \inf(f_n(x))$
$g_3(x) = \lim_{n \to \infty} \sup(f_n(x))$ $\qquad$ $g_4(x) = \lim_{n \to \infty} \inf(f_n(x))$
are measurable.

Proof

For $a \in \mathbb{R}$, $\{$x $\mid g_1(x) > a\ \} = \cup_{i=1}^{n} \{$x $\mid f_n(x) > a\}$ which are measurable sets so countable implies measurable and thus, g_1 is measurable.
For $a \in \mathbb{R}$, $\{$x $\mid g_2(x) < a\ \} = \cup_{i=1}^{n} \{$x $\mid f_n(x) < a\}$ which are measurable sets so countable implies measurable and thus, g_2 is measurable.
Since $g_3(x) = \lim_{n \to \infty} \sup(f_n(x)) = \inf(\sup(f_n(x)))$ where $\sup(f_n(x))$ are measurable so g_3 is measurable.
Since $g_4(x) = \lim_{n \to \infty} \inf(f_n(x)) = \sup(\inf(f_n(x)))$ where $\inf(f_n(x))$ are measurable so g_4 is measurable.

Theorem 17.4.8: Lebesgue measurable functions form a Vector Space
      The set of Lebesgue measurable functions from [0,1] to $\mathbb{R}$ is a vector space
      with a vector subspace of bounded Lebesgue measurable functions. Thus, if
      f,g are measurable, then fg is measurable.

Proof

Let f,g be Lebesgue measurable functions.

For a $\in \mathbb{R}$, let set $U_a = \{$ f(x) + g(x) > a $\}$.

Since $\mathbb{Q} = \{r_m\}$ is countable and dense, there is a r_m such that:

    f(x) > r_m > a - g(x)

Let $V_m = \{$x $\mid$ f(x) > $r_m\} \cap \{$x $\mid$ g(x) > $a - r_m\}$

If x $\in U_a$, then x $\in V_m$ for some m since there is a r_m where f(x) > r_m > a - g(x). If x $\in V_m$, then f(x) + g(x) > a so x $\in U_a$. Thus, $U_a = \cup_{m=1}^{\infty} V_m$ which is countable and thus, measurable since f,g are measurable. Thus, f+g is measurable.

Note for c $\in \mathbb{R}$, $\{$x $\mid$ cf(x) > a$\}$ is measurable since $\{$x $\mid$ f(x) > v$\}$ for any v $\in \mathbb{R}$ is measurable including $\frac{a}{c}$. Thus, the measurable functions form a vector space.

If f,g are bounded and measurable, then $c_1 f + c_2 g$ is bounded which is measurable as proved above so bounded measurable functions is a vector subspace of measurable functions.

17.5   Lebesgue Integral of Bounded Functions

Theorem 17.5.1: Lebesgue Integral of a Bounded Function

If f: $[0,1] \to \mathbb{R}$ is bounded, then the following are equivalent:
- f is Lebesgue measurable
- There are simple functions $\{f_n\}$ which converge uniformly to f
- If simple functions u(x),v(x) where $v(x) \le f(x) \le u(x)$, then:
  $\sup(\int v \ d\mu) = \inf(\int u \ d\mu)$

Then, $\int f \ d\mu = \sup(\int v \ d\mu) = \inf(\int u \ d\mu)$

Proof

Suppose f is Lebesgue measurable.
Since f is bounded, there are m,M such that $m \le f(x) \le M$ for all $x \in [0,1]$. For $\epsilon_n > 0$, take a large enough n such that $\frac{M-m}{n} \le \epsilon_n$. For $\{c_0, ..., c_n\}$, let $c_k = m + k\epsilon_n$.
Let $f_n(x) = \sum_{i=1}^{n} c_{i-1}\mathfrak{X}_{f^{-1}([c_{i-1},c_i))}$ which is simple.
Then for any $x \in [0,1]$, there is a $[c_{i-1}, c_i)$ where $x \in [c_{i-1}, c_i)$ so $|f(x) - f_n(x)| \le \epsilon_n$.

Suppose simple functions $\{f_n\}$ converge uniformly to f.
Let $\delta_n = \sup(|f(x) - f_n(x)|)$ so $\lim_{n\to\infty} \delta_n = 0$. Let simple functions $v_n(x) = f_n(x) - \delta_n$ and $u_n(x) = f_n(x) + \delta_n$ so $v_n(x) \le f(x) \le u_n(x)$.
  $\inf(\int u \ d\mu) \le \lim_{n\to\infty} \inf(\int u_n(x) \ d\mu) = \lim_{n\to\infty} \inf(\int f_n(x) + \delta_n \ d\mu)$
  $= \lim_{n\to\infty} \inf(\int f_n(x) \ d\mu) \le \lim_{n\to\infty} \sup(\int f_n(x) \ d\mu)$
  $= \lim_{n\to\infty} \sup(\int f_n(x) - \delta_n \ d\mu) = \lim_{n\to\infty} \sup(\int v_n(x) \ d\mu) \le \sup(\int v \ d\mu)$
Since $\sup(\int v \ d\mu) \le \inf(\int u \ d\mu)$, then $\sup(\int v \ d\mu) = \inf(\int u \ d\mu)$.

For n, there are simple functions $v_n(x), u(x)$ where $v_n(x) \le f(x) \le u_n(x)$ such that:
  $\int u_n(x) \ d\mu$ - $\int v_n(x) \ d\mu < \frac{1}{2^n}$
Since $u_n(x)$ and $v_n(x)$ are simple and thus, measurable, then $g_1(x) = \sup(v_n(x))$ and $g_2(x) = \inf(u_n(x))$ are measurable. Let B = $\{x \mid g_1(x) < g_2(x)\}$. Suppose $\mu(B) > 0$.
If $B_m = \{x \mid g_1(x) < g_2(x) - \frac{1}{m}\}$, then B = $\int_{m=1}^{\infty} B_m$ so $\mu(B_m) > 0$ for some m.
Thus, for $x \in B_m$:
  $v_n(x) \le g_1(x) < g_2(x) - \frac{1}{m} \le u_n(x) - \frac{1}{m}$
  $\int u_n \ d\mu$ - $\int v_n \ d\mu = \int u_n - v_n \ d\mu \ge \int \frac{1}{m}\mathfrak{X}_{B_m} \ d\mu = \frac{1}{m}\mu(B_m)$
which contradicts $\int u_n(x) \ d\mu$ - $\int v_n(x) \ d\mu < \frac{1}{2^n}$ and thus, $\mu(B) = 0$ so $g_1(x) = g_2(x)$ except on a null set. Since $g_1(x) \le f(x) \le g_2(x)$, then f(x) - $g_1(x) = 0$ except on a null set and thus, by theorem 17.4.6, f(x) - $g_1(x)$ is measurable so f(x) is measurable.

Theorem 17.5.2: Uniform Convergence of Simple Functions are Lebesgue Integrable

If simple functions $\{f_n\}$ converge uniformly to bounded measurable f:
  $\int f \ d\mu = \lim_{n\to\infty} \int f_n \ d\mu$

Proof

Let $\delta_n = \sup(|f(x) - f_n(x)|)$. Since $\{f_n\}$ converge uniformly f, then $\lim_{n\to\infty} \delta_n = 0$:
  $f_n(x) - \delta_n \le f(x) \le f_n(x) + \delta_n$
Thus, by theorem 17.5.1:
  $\int f \ d\mu = \inf(\int u \ d\mu) \le \lim_{n\to\infty} \inf(\int f_n(x) + \delta_n \ d\mu)$
  $\le \lim_{n\to\infty} \inf(\int f_n(x) \ d\mu) \le \lim_{n\to\infty} \sup(\int f_n(x) \ d\mu)$
  $\le \lim_{n\to\infty} \sup(\int f_n(x) - \delta_n \ d\mu) \le \sup(\int v \ d\mu) = \int f \ d\mu$
Since $\lim_{n\to\infty} \inf(\int f_n(x) \ d\mu) \le \lim_{n\to\infty} \int f_n(x) \ d\mu \le \lim_{n\to\infty} \sup(\int f_n(x) \ d\mu)$, then:
  $\int f \ d\mu = \lim_{n\to\infty} \int f_n(x) \ d\mu$

Theorem 17.5.3: Properties of Bounded Measurable Functions

If f,g are bounded Lebesgue measurable functions. Then:
(a) Linearity: If $c_1, c_2 \in \mathbb{R}$:
$$\int c_1 f + c_2 g \ d\mu = c_1 \int \text{f} \ d\mu + c_2 \int \text{g} \ d\mu$$
(b) Monotonicity: If f(x) $\leq$ g(x):
$$\int \text{f} \ d\mu \leq \int \text{g} \ d\mu$$
(c) Absolute Value: $|f|$ is measurable where:
$$|\int \text{f} \ d\mu| \leq \int |f| \ d\mu$$
(d) Null Sets: If f(x) = g(x) except on a set of measure zero:
$$\int \text{f} \ d\mu = \int \text{g} \ d\mu$$

Proof

Since f and g are measurable, then there are simple functions $\{f_n\}, \{g_n\}$ where converge uniformly to f and g respectively. Thus, $\{c_1 f_n + c_2 g_n\}$ converge to $c_1 f + c_2 g$ uniformly.
$$\int c_1 f + c_2 g \ d\mu = \lim_{n \to \infty} \int c_1 f_n + c_2 g_n \ d\mu$$
$$= c_1 \lim_{n \to \infty} \int f_n \ d\mu + c_2 \lim_{n \to \infty} \int g_n \ d\mu = c_1 \int \text{f} \ d\mu + c_2 \int \text{g} \ d\mu$$

If f(x) $\leq$ g(x), then since f,g are measurable, then are simple functions v_f, u_g where $v_f \leq \text{f} \leq \text{g} \leq u_g$ such that:
$$\int \text{f} \ d\mu = \sup(\int v_f \ d\mu) \leq \inf(\int u_g \ d\mu) = \int \text{g} \ d\mu$$

Since $|[a, \infty)| = (-\infty, -a] \cup [a, \infty)$, then $|f|^{-1}([a, \infty)) = f^{-1}((-\infty, -a]) \cup f^{-1}([a, \infty))$ which are measurable since f is measurable, then $|f|$ is measurable. Also, there are simple functions $\{f_n\}$ that converge uniformly to f. Then by theorem 17.4.4:
$$|\int f d\mu| = \lim_{n \to \infty} |\int f_n d\mu| \leq \lim_{n \to \infty} \int |f_n| d\mu = \int |f| d\mu$$

Let h(x) = f(x) - g(x) = 0 except on a null set E and is bounded so $|h(x)| \leq M \mathfrak{X}_E$.
$$|\int f d\mu - \int g d\mu| = |\int h d\mu| \leq \int |h| d\mu \leq \int M \mathfrak{X}_E d\mu = M\mu(E) = 0$$

Definition 17.5.4: Bounded Lebesgue integral over a Measurable set

If E $\subset$ [0,1] is a measurable set and f is a bounded measurable function, the Lebesgue integral of f over E:
$$\int_E \text{f} \ d\mu = \int f\mathfrak{X}_E \ d\mu$$

Theorem 17.5.5: Additivity Property

If $\{E_n\}_{n=1}^N$ are pairwise disjoint measurable sets with E = $\cup E_n$ and f is a bounded measurable function:
$$\int_E \text{f} \ d\mu = \sum_{n=1}^N \int_{E_n} \text{f} \ d\mu$$

Proof

Since $\mathfrak{X}_E = \sum_{n=1}^N \mathfrak{X}_{E_n}$, then $f\mathfrak{X}_E = \sum_{n=1}^N f\mathfrak{X}_{E_n}$.
$$\int_E \text{f} \ d\mu = \int f\mathfrak{X}_E \ d\mu = \int \sum_{n=1}^N f\mathfrak{X}_{E_n} \ d\mu = \sum_{n=1}^N \int f\mathfrak{X}_{E_n} \ d\mu = \sum_{n=1}^N \int_{E_n} \text{f} \ d\mu$$

Theorem 17.5.6: Riemann Integrability implies Lebesgue Integrability

   Every bounded Riemann integrable f: $[0,1] \to \mathbb{R}$ is measurable and thus, Lebesgue integrable. The Riemann integral is equal to the Lebesgue integral.

Proof

Since the set of step functions $\mathcal{L}(f)$ less than f is a subset of the set of simple functions $\mathcal{L}_\mu(f)$ less than f and the set of step functions $\mathcal{U}(f)$ greater than f is a subset of the set of simple functions $\mathcal{U}_\mu(f)$ greater than f, then:

$$\sup_{v \in \mathcal{L}(f)} (\textstyle\int_0^1 v(t)dt) \leq \sup_{v \in \mathcal{L}_\mu(f)} (\textstyle\int_0^1 v d\mu) \leq \inf_{u \in \mathcal{U}_\mu(f)} (\textstyle\int_0^1 u d\mu) \leq \inf_{u \in \mathcal{U}(f)} (\textstyle\int_0^1 u(t)dt)$$

Thus, if f is Riemann integrable, then $\sup\limits_{v \in \mathcal{L}(f)} (\int_0^1 v(t)dt) = \inf\limits_{u \in \mathcal{U}(f)} (\int_0^1 u(t)dt)$ so $\sup\limits_{v \in \mathcal{L}_\mu(f)} (\int_0^1 v d\mu) = \inf\limits_{u \in \mathcal{U}_\mu(f)} (\int_0^1 u d\mu)$ and thus, f is Lebesgue measurable and the Riemann integral is equal to the Lebesgue integral since:

$$\sup_{v \in \mathcal{L}(f)} (\textstyle\int_0^1 v(t)dt) = \sup_{v \in \mathcal{L}_\mu(f)} (\textstyle\int_0^1 v d\mu) = \inf_{u \in \mathcal{U}_\mu(f)} (\textstyle\int_0^1 u d\mu) = \inf_{u \in \mathcal{U}(f)} (\textstyle\int_0^1 u(t)dt)$$

17.6   Bounded Convergence Theorem

Theorem 17.6.1: Bounded Convergence Theorem

   Suppose $\{f_n\}$ on [0,1] converge pointwise to f where each $|f_n(x)| \leq M$.
   Then, f is a bounded measurable function where:
   $\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$

Proof

Since $\lim_{n \to \infty} f_n = f$ pointwise, then for any $x \in [0,1]$, then is a N_x where for $n \geq N_x$:
   $|f(x) - f_n(x)| < \epsilon$
   $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \epsilon + M \qquad \Rightarrow \qquad |f(x)| \leq M$
Thus, f is bounded. Since $\lim_{n \to \infty} f_n = f$, then by theorem 17.4.7, f is measurable. Let set $E_n = \{\, x \in [0,1] \mid |f_n(x) - f(x)| < \frac{\epsilon}{2} \,\}$. Since $\lim_{n \to \infty} f_n = f$ pointwise, then $\cup_{n=1}^\infty E_n = [0,1]$. Since $E_n \subset E_{n+1}$, then $\lim_{n \to} \mu(E_n) = \mu([0,1]) = 1$. Then, there is a N where $\mu(E_N) > 1 - \frac{\epsilon}{4M}$ so $\mu(E_N^c) < \frac{\epsilon}{4M}$. Thus:
$$\lim_{n \to \infty} | \textstyle\int f_n d\mu - \int f d\mu | = \lim_{n \to \infty} | \int f_n - f d\mu | \leq \lim_{n \to \infty} \int |f_n - f| d\mu$$
$$= \textstyle\int_{E_N} |f_n - f| d\mu + \int_{E_N^c} |f_n - f| d\mu$$
$$< \tfrac{\epsilon}{2} \mu(E_N) + 2M \mu(E_N^c) < \tfrac{\epsilon}{2} + 2M \tfrac{\epsilon}{4M} = \epsilon$$

Definition 17.6.2: Almost Everywhere

   If a property holds for all x except for a null set, then it holds almost everywhere

Theorem 17.6.3: Bounded Convergence Theorem for Almost Everywhere

Suppose bounded $\{f_n\}$ on [0,1] are measurable and f is bounded such that $\lim_{n\to\infty} f_n = f$ for almost all x. If $|f_n(x)| \leq M$ almost everywhere, then f is measurable where:

$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$

<u>Proof</u>

Let A = { x | $\lim_{n\to\infty} f_n(x) \neq f(x)$ } so $\mu(A) = 0$. Let D_n = { x | $|f_n(x)| > M$ } so $\mu(D_n) = 0$. Let E = A $\cup_{n=1}^{\infty} D_n$. Thus:

$\mu(E) \leq \mu(A) + \sum_{i=1}^{\infty} \mu(D_n) = 0$ $\Rightarrow$ $\mu(E) = 0$

Let $g_n(x) = f_n(x)\mathfrak{X}_{E^c}(x)$ which is measurable since $f_n(x),\mathfrak{X}_{E^c}(x)$ are measurable. Then, $|g_n(x)| \leq M$. Let $g(x) = f(x)\mathfrak{X}_{E^c}(x)$ so $\lim_{n\to\infty} g_n(x) = g(x)$ and $g(x) \leq M$. Since $\lim_{n\to\infty} g_n(x) = g(x)$, then by theorem 17.4.7, g(x) is measurable. Since g(x) = f(x) almost everywhere, then by theorem 17.4.6b, f(x) is measurable.

$\int g d\mu = \int f d\mu$ $\int g_n d\mu = \int f_n d\mu$

By theorem 17.6.1, $\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu$. Thus:

$\lim_{n\to\infty} \int f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu = \int f \, d\mu$

17.7   Integral of Unbounded Functions

Definition 17.7.1: Integrable Function

If f: [0,1] $\to$ [0,∞] is Lebesgue measurable, let $f_n(x) = \min(f(x),n)$. Then f_n is a bounded measurable function and let:

$\int f \, d\mu = \lim_{n\to\infty} f_n(x) \, d\mu$

If $\int f \, d\mu < \infty$, then f is integrable.

Theorem 17.7.2: Unbounded sets of Integrable functions have measure 0

If f is a non-negative integrable function and A = { x | f(x) = ∞ }, then:

$\mu(A) = 0$

<u>Proof</u>

If $x \in A$, then $f_n(x) = n \geq n\mathfrak{X}_A(x)$. Thus, $\int f_n d\mu \geq \int n\mathfrak{X}_A d\mu = n\mu(A)$. If $\mu(A) > 0$, then:

$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu \geq \lim_{n\to\infty} \int n\mathfrak{X}_A d\mu = \lim_{n\to\infty} n\mu(A) = \infty$

Thus, if f is integrable, then $\mu(A) = 0$.

Theorem 17.7.3: Integrable functions for Almost Everywhere

Suppose f,g are non-negative measurable functions with $g(x) \leq f(x)$ for almost all x. If x is integrable, then g is integrable where:

$\int g \, d\mu \leq \int f \, d\mu$

If g = 0 almost everywhere, then $\int g \, d\mu = 0$.

<u>Proof</u>

If $f_n(x) = \min(f(x),n)$ and $g_n(x) = \min(g(x),n)$, then f_n, g_n are bounded measurable functions where $g_n(x) \leq f_n(x)$ almost everywhere. If f is integrable, then:

$\int g_n d\mu \leq \int f_n d\mu \leq \int f d\mu$

Since $\{g_n\}$ is increasing and bounded above by $\int f d\mu$, then $\int g d\mu$ is finite and thus, exist. If $0 \leq g(x) \leq 0$ almost everywhere, for almost all x so $\int g d\mu = \int 0 d\mu = 0$.

Corollary 17.7.4: If integrable $f \geq 0$, then $\int f \, d\mu \, 0 \rightleftharpoons f(x) = 0$ almost everywhere

If f: $[0,1] \to [0,\infty]$ is a non-negative integrable function and $\int f d\mu = 0$, then
f(x) = 0 almost everywhere

<u>Proof</u>

Let $E_n = \{ \, x \mid f(x) \geq \frac{1}{n} \, \}$. Then, f(x) $\geq \frac{1}{n}\mathfrak{X}_{E_n}(x)$ where:
$\frac{1}{n}\mu(E_n) = \int \frac{1}{n}\mathfrak{X}_{E_n}d\mu \leq \int f d\mu = 0$
Thus, $\mu(E_n) = 0$. Let E = $\{ \, x \mid f(x) > 0 \, \}$ so E = $\cup_{n=1}^{\infty}E_n$ where $E_n \subset E_{n+1}$ so:
$\mu(E) = \mu(\cup_{n=1}^{\infty}E_n) = \lim_{n\to\infty}\mu(E_n) = 0.$

Theorem 17.7.5: Absolute Continuity

If f is a non-negative integrable function, then for $\epsilon > 0$, there is a $\delta > 0$ where
for every measurable A $\subset [0,1]$ with $\mu(A) < \delta$, then $\int_A f \, d\mu < \epsilon$

<u>Proof</u>

Let $E_n = \{ \, x \mid f(x) \geq n \, \}$ so $f_n(x) = \begin{cases} f(x) & x \in E_n^c \\ n & x \in E_n \end{cases}$. Thus:

f(x) - $f_n(x) = \begin{cases} 0 & x \in E_n^c \\ f(x) - n & x \in E_n \end{cases}$

$\int f d\mu$ - $\int f_n d\mu = \int f - f_n d\mu = \int_{E_n} f(x) - n d\mu$
Since f is integrable, then $\lim_{n\to\infty}\int f d\mu$ - $\int f_n d\mu = 0$. Thus:
$\lim_{n\to\infty}\int_{E_n} f(x) - n d\mu = 0$
Thus, there is a N where $\int_{E_n} f(x) - n d\mu < \frac{\epsilon}{2}$. Then for $\delta < \frac{\epsilon}{2N}$, if $\mu(A) < \delta$:
$\int_A f d\mu = \int_{A\cap E_N} f d\mu + \int_{A\cap E_N^c} f d\mu$
$\leq \int_{A\cap E_N}(f - N)d\mu + \int_{A\cap E_N} N d\mu + \int_{A\cap E_N^c} N d\mu$
$\leq \int_{A\cap E_N}(f - N)d\mu + \int_A N d\mu < \frac{\epsilon}{2} + N\mu(A) < \frac{\epsilon}{2} + N\delta < \frac{\epsilon}{2} + N\frac{\epsilon}{2N} < \epsilon$

Corollary 17.7.6: Uniform Continuity of the Integral

If f: $[0,1] \to [0,\infty]$ is a non-negative integrable function where F(x) = $\int_{[0,x]} f$
$d\mu$, then F(x) is continuous

<u>Proof</u>

By theorem 17.7.5, for $\epsilon > 0$, there is a $\delta > 0$ where for $\mu([x,y]) < \delta$, then:
$\int_{[x,y]} f \, d\mu < \epsilon$
$|F(y) - F(x)| = |\int_{[0,y]} f d\mu - \int_{[0,x]} f d\mu| = |\int_{[x,y]} f d\mu| < \epsilon$
Thus, F(x) is uniformly continuous.

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