Fall Real Analysis

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Contents

1 The Real Number System

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** $\mathbb{Q}$ is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A = $\{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
But, A has no least upper bound in $\mathbb{Q}$, but A has a least upper bound in $\mathbb{R}$.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
Also, $\mathbb{R}$ exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x $<$ y
  - x $=$ y
  - x $>$ y
- Transitivity: If x $<$ y and y $<$ z, then x $<$ z.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.
A lower bound of E is a $\alpha \in$ S if x $\geq \alpha$ for all x $\in$ E.
If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.
Let E $\subset$ S be bounded from above. Least upper bound $\beta \in$ S exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E.

Then β = sup(E).

Let E $\subset$ S be bounded from below. Greatest lower bound $\alpha \in$ S exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E.

Then α = inf(E).

Example 1.2.5

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup(E) exists, it may or may not exists at S.
If sup(E) exists, then sup(E) is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:
For every nonempty subset E $\subset$ S that is bounded from above:
sup(E) exists in S.

Example 1.3.2

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then z > y where z $\in$ A. So, y is not an upper bound.
  Since for any y, there is z > y where z $\in$ A, then sup(A) doesn't exists in $\mathbb{Q}$.
- If $y^2 - 2 > 0$, then z < y where z $\in$ B. So, y is an upper bound, but not sup(E).
  Since for any y, there is z < y where z $\in$ B, then inf(B) doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound or greatest lower bound property.

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be a ordered set with the least upper bound property.

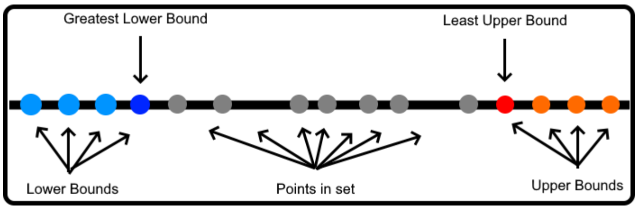
Let non-empty B $\subset$ S be bounded below.

Let L be the set of all lower bounds of B.

Then α = sup(L) exists in S.

Proof

L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, α = sup(L) exists in S.
We claim that α = inf(B).
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B since all upper bounds for L are in B. Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, inf(B) = α.



2.2 Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F
- x+y = y+x for all x,y $\in$ F
- (x+y)+z = x+(y+z) for all x,y,z $\in$ F
- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F
- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative xioms

- If x,y $\in$ F, then xy $\in$ F
- yx = xy for all x,y $\in$ F
- (xy)z = x(yx) for all x,y,z $\in$ F
- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F
- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x} \in$ F where x($\frac{1}{x}$) = 1

Distributive Law

x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Propositions 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0
Proof
From (a), let z = 0.

(c) If x+y = 0, then y = -x
Proof
From (a), let z = -x.

(d) -(-x) = x
Proof
From (c), let x = -x and y = x.

(e) If x $\neq$ 0 and xy = xz, then y = z
Proof
y = 1y = $\frac{1}{x}$xy = $\frac{1}{x}$xz = 1z = z

(f) If x $\neq$ 0 and xy = x, then y = 1
Proof
From (e), let z = 1.

(g) If x $\neq$ 0 and xy = 1, then y = $\frac{1}{x}$
Proof
From (e), let z = $\frac{1}{x}$.

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x
Proof
From (g), let x = $\frac{1}{x}$ and y = x.

(i) 0x = 0
Proof
Since 0x + 0x = (0+0)x = 0x = 0x + 0, then 0x = 0.

(j) If x,y $\neq$ 0, then xy $\neq$ 0
Proof
Suppose xy = 0, then 1 = $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}\frac{1}{x}$0 = 0.
0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)
Proof
xy + (-x)y = (x+(-x))y = 0y = 0.
Then by part (c), (-x)y = -(xy).
Similarly, xy + x(-y) = x(y+(-y)) = x0 = 0.
Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy
Proof
By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].
By part (d), -[-(xy)] = xy.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x
- If x,y > 0, then xy > 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field since i^2 = -1 $\not> 1$.

Propositions 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

(a) If x > 0, then -x < 0 and vice versa

Proof

-x = -x + 0 < -x + x = 0

(b) If x > 0 and y < z, then xy < xz

Proof

Since z-y > 0, then 0 < x(z-y) = xz - xy

(c) If x < 0 and y < z, then xy > xz

Proof

Since -x > 0 and z-y > 0, then 0 < -x(z-y) = xy - xz

(d) If x $\neq$ 0, $x^2 > 0$

Proof

If x > 0, then $x^2 = x \cdot x > 0$

If x < 0, then $(-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 > 0$

(e) If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

Since $(\frac{1}{y})y = 1 > 0$, then $(\frac{1}{y}) > 0$

Since x < y, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$

Theorem 2.3.3: $\mathbb{R}$ is an ordered field with <

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.
Also, $\mathbb{Q} \subset \mathbb{R}$ so $\mathbb{Q}$ is also an ordered field.

Theorem 2.3.4

For all x,y $\in \mathbb{R}$:

- Archimedean Property: If x > 0, there is n $\in \mathbb{Z}$ such that nx > y.

  Proof

  Fix x > 0. Suppose there is a y such that the property fails.
  Let A = { nx: n = 1, 2, 3, ... }.
  Then, A is nonempty and bounded from above by y.
  Then by the least upper bound property of $\mathbb{R}$, $\alpha = \sup(A)$ exists in $\mathbb{R}$.
  Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$.
  So $\alpha - x$ is not an upper bound of A.
  So there is a mx $\in$ A such that mx > $\alpha - x$.
  Then α < (m+1)x, but (m+1)x $\in$ A contradicting α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If x < y, there is a p $\in \mathbb{Q}$ such that x < p < y.

  Proof

  Since x < y, then y-x > 0. Then by the Archimedean Property, there exists a n $\in$ Z such that n(y-x) > 1. Thus, ny > nx+1 > nx
  By the well-ordering principle, there is a smallest m $\in \mathbb{Z}_+$ such that m > nx.
  Then, m > nx $\geq$ m-1 so nx+1 $\geq$ m > nx.
  Since ny > nx+1 $\geq$ m > nx, then y > m/n > x.

3 Roots & Complex Field

3.1 nth Root

(a) If $0 < t \leq 1$, then $t^n \leq t$.
Proof
Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
Applying the process n times, then $t^n \leq t$.

(b) If $t \geq 1$, $t^n \geq t$.
Proof
Since $0 < 1 \leq t$, then $t \leq t^2$.
Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
Applying the process n times, $t \leq t^n$.

(c) If $0 < s < t$, then $s^n < t^n$.
Proof
$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$

Theorem 3.1.1: $y^n = x$ has a unique y
Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.
Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof
Uniqueness:
y is unique since if $y_1 < y_2$, then $x = {y_1}^n < {y_2}^n \neq x$.
Existence:
Let set $A = \{ t > 0 : t^n < x \}$.
$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < {t_1}^n < t_1 < x$ so $t_1 \in A$.
A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.
So x+1 is an upper bound of A.
Thus by the least upper bound property, $y = \sup(A)$ exists.
For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.
(Not an upper bound of A if < and not a least upper bound of A if >)
For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha)\,(\underbrace{\beta^{n-1} + \beta^{n-2}\alpha^1 + ... + \alpha^{n-1}}_{\beta^{n-1} \quad <\beta^{n-1} \quad <\beta^{n-1}}) < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.
From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.
Suppose $y^n > x$. Pick $0 < k = \frac{y^n-x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.
Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$
Thus, $t^n > x$ so $t \notin A$.
Thus, y-k is an upper bound of A which is a contradiction since $y = \sup(A)$.
Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 3.1.2: n-th root of product = product of n-th root

If a,b > 0 and n $\in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}$.

Proof

Let A = $a^{\frac{1}{n}}$ and B = $b^{\frac{1}{n}}$.
Then by theorem 3.1.1, since A is a solution to y_1^n = a, then A^n = a.
Similarly, B is a solution of y_2^n = b so B^n = b. Thus:

$$\begin{aligned} ab &= A^nB^n = A_1A_2...A_nB_1B_2...B_n \\ &= A_1A_2...B_1A_nB_2...B_n = ... = A_1B_1A_2...A_{n-1}A_nB_3...B_n \\ &= ... = A_1B_1A_2B_2...A_nB_n = (AB)^n \end{aligned}$$

Then again by theorem 3.1.1, there is a unique $(ab)^{\frac{1}{n}}$ = AB = $a^{\frac{1}{n}}b^{\frac{1}{n}}$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for x > 0 $\in \mathbb{R}$.
Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + ... + \frac{n_k}{10^k} \leq x$
Let E be the set of d_k for k = 0, 1, ... ∞. Then, x = sup(E).

3.3 Extended Reals

The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:

$-\infty < x < \infty$ for every x $\in \mathbb{R}$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If x > 0, then $x(\pm\infty) = \pm\infty$
- If x < 0, then $x(\pm\infty) = \mp\infty$

3.4 Complex Numbers

Definition 3.4.1: Complex

A complex number is an ordered pair (a,b) where a,b $\in \mathbb{R}$. For x,y $\in \mathbb{C}$

- x + y = (a,b) + (c,d) = (a + c , b + d)
- xy = (a,b) (c,d) = (ac - bd , ad + bc)
- $\frac{1}{x} = (a^2 + b^2)(a,-b)$

Thus, the axioms form a field where (0,0) = 0 and (1,0) = 1 and (0,1) = i.

Definition 3.4.2: Imaginary i

Let i = (0,1). Then, i^2 = -1.

Proof

i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1

Definition 3.4.3: Form a + bi

(a,b) = a + bi

Proof

(a,b) = (a,0) +(0,b) = (a,0) + (b,0)(0,1) = a + bi

Definition 3.4.4: Conjugate

Let conjugate: $\bar{z}$ = a - bi where Re(z) = a , Im(z) = b

Let z = (a,b) and w = (c,d):

(a) $\overline{z+w} = \overline{z} + \overline{w}$

Proof

$\overline{z+w} = \overline{(a+c, b+d)}$ = (a+c,-b-d) = (a,-b) + (c,-d) = $\overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\ \overline{w}$

Proof

$\overline{zw} = \overline{(ac-bd, ad+bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\overline{z}\ \overline{w}$

(c) z + $\overline{z}$ = 2 Re(z) z - $\overline{z}$ = 2i Im(z)

Proof

z + $\overline{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)

z - $\overline{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

(d) z$\overline{z} \geq 0$

Proof

z$\overline{z}$ = (a,b)(a,-b) = (a^2 + b^2 , -ab+ab) = a^2 + b^2 ≥ 0

Definition 3.4.5: Absolute Value

Let absolute value: | z | = $\sqrt{z\overline{z}}$

Let z = (a,b) and w = (c,d):

(a) If z $\neq$ 0, then | z | > 0.

Proof

$\sqrt{z\overline{z}} = \sqrt{a^2+b^2} \geq 0$ where | z | = 0 only if a,b = 0 so only if z = (0,0).

(b) | $\overline{z}$ | = | z |

Proof

| $\overline{z}$ | = $\sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$ = | z |

(c) | zw | = | z | | w |

Proof

| zw | = | (ac-bd,ad+bc) | = $\sqrt{(ac-bd)^2+(ad+bc)^2}$

= $\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$

= $\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$ = | z | | w |

(d) | Re(z) | $\leq$ | z |

Proof

| Re(z) | = | a | = $\sqrt{a^2} \leq \sqrt{a^2+b^2}$ = | z |

(e) | z+w | $\leq$ | z | + | w |

Proof

$|z+w|^2$ = (z+w)$\overline{(z+w)}$ = (z+w)$(\overline{z}+\overline{w})$ = z$\overline{z}$ + z$\overline{w}$ + w$\overline{z}$ + w$\overline{w}$

= $|z|^2$ + $|w|^2$ + 2 Re(z$\overline{w}$) $\leq$ $|z|^2$ + $|w|^2$ + 2 $|z\overline{w}|$

= $|z|^2$ + $|w|^2$ + $2|z||w|$ = $(|z| + |w|)^2$

4 Euclidean Spaces

4.1 Euclidean Spaces

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples:
x $= (x_1, ..., x_k)$ for each $x_i \in \mathbb{R}$
with the properties:

- x+y $= (x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- cx $= (cx_1, ..., cx_k) \in \mathbb{R}^k$

So, $\mathbb{R}^n$ has a vector space structure. Similarly, for $\mathbb{C}^n$.

Definition 4.1.1: Inner Product for $\mathbb{R}^k$

$x \cdot y = x_1y_1 + ... + x_ky_k \in \mathbb{R}$

Definition 4.1.2: Norm

$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$

Definition 4.1.3: Extension to $\mathbb{C}^k$

For z,w $\in \mathbb{C}^n$

- $z \cdot w = z_1\overline{w_1} + ... + z_k\overline{w_k}$
- $z \cdot z = z_1\overline{z_1} + ... + z_k\overline{z_k} = |z_1|^2 + ... + |z_n|^2 = |z|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $b_1, ..., b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^{n} a_j(\overline{b_j})|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j(\overline{b_j})$.
If $B = 0$, then $b_1 = ... = b_n = 0$. Thus, $0 \leq A(0)$ holds true.
Suppose $B > 0$. Then:

$$\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j)(\overline{B}\ \overline{a_j} - \overline{C}\ \overline{b_j})$$
$$= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\ \overline{b_j}) = \sum B^2 a_j\overline{a_j} - B\overline{C}a_j\overline{b_j} - BC\overline{a_j}b_j + C\overline{C}b_j\overline{b_j}$$
$$= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j\overline{b_j} - BC \sum \overline{a_j}b_j + |C|^2 \sum |b_j|^2$$
$$= B^2A - B\overline{C}C - BC\overline{C} + |C|^2B = B^2A - 2|C|^2B + |C|^2B = B^2A - |C|^2B$$
$$= B(AB - |C|^2)$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.
Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i\overline{z_i}$, then $\sum z_i\overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$$|z \cdot w|^2 = |\sum z_i\overline{w_i}\ |^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2\ |w|^2$$

Thus, $|z \cdot w| \leq |z||w|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

(a) $|x| \geq 0$ where $|x| = 0$ only if x = 0

Proof

$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0$ where $|x| = 0$ only if $x_1 = ... = x_k = 0$

(b) $|\alpha x| = |\alpha||x|$

Proof

$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$

(c) $|x + y| \leq |x| + |y|$

Proof

$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2$
$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$

(d) $|x - y| \leq |x - z| + |y - z|$

Proof

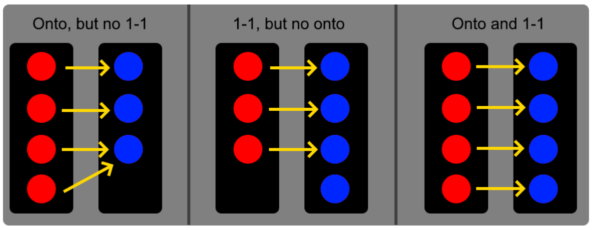
$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$

4.3 Cardinality

Definition 4.3.1: Onto and 1-1 Mapping

Suppose for every x ∈ A, there is an associated f(x) ∈ B.
Then f maps A into B = f: A → B.

- If f(A) = B, then f maps A onto B.
- If for each y ∈ B, $f^{-1}(y)$ consist of at most one x ∈ A where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B.



Definition 4.3.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function f: A → B. (1-1 correspondence between A and B) Then:

A ∼ B

If f: A → B is 1-1 and onto, then there is a f^{-1}: B → A that is 1-1 and onto.

Definition 4.3.3: Countability

- A is finite if A ∼ J_n = {0, 1, ..., n} for some n ∈ ℕ
- A is infinite if A is not finite
- A is countably infinite if A ∼ $\mathbb{Z}_+$ = J
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite.

Example 4.3.4

$\mathbb{Z}$ is countably infinite

Proof

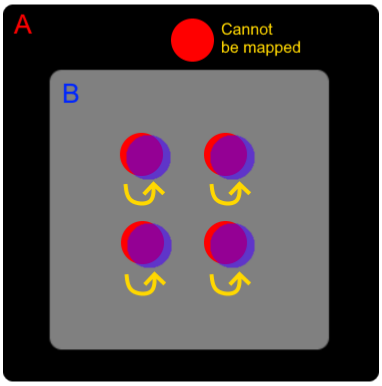
Let f: $\mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n-1}{2} & \text{n is odd} \end{cases}$$

So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto$ -1, $4 \mapsto 2$, $5 \mapsto$ -2 , etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 4.3.5: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A.

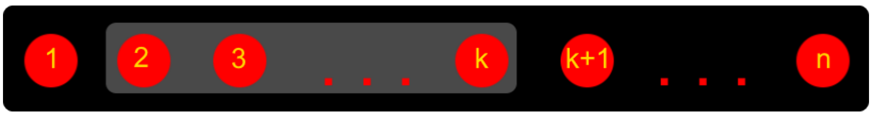


Theorem 4.3.6: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let E $\subset$ A be an infinite subset. For every distinct $x_i \in$ A, let x = { $x_1, x_2, ...$ }.
Let n_1 be smallest integer such that x$_{n_1} \in$ E.
Then let n_2 be the smallest integer where $n_2 > n_1$ such that x$_{n_2} \in$ E.
Repeat the process to create sequence f(k) = { $x_{n_1}, x_{n_2}, ..., x_{n_k}, ...$ }.
Thus, there is a 1-1 correspondence between E and $\mathbb{Z}_+$ so E is countably infinite.



5 Metric Spaces

5.1 Set of Sets

Definition 5.1.1: Union and Intersection

Let sets Ω,B be such that for each x $\in \Omega$, there is an associated $E_x \subset B$.

- $E = \cup_{x=1}^{n} E_x$ only if for every $x \in E$, $x \in E_x$ for at least one $x \in \Omega$.
- $P = \cap_{x=1}^{n} E_x$ only if for every $x \in P$, $x \in E_x$ for all $x \in \Omega$.

with properties:

(a) $A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$ $\qquad$ $(A \cap B) \cap C = A \cap (B \cap C)$

(c) $A \subset A \cup B$ $\qquad$ $(A \cap B) \subset A$

(d) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$

Proof

If $x \in A \cup B$, then $x \in A$ or/and $x \in B$.

- If $x \in A$, since $A \subset B$, then $x \in B$. Then, $(A \cup B) \subset B$.
- If $x \in B$, then immediately $(A \cup B) \subset B$.

If $x \in B$, then $x \in A \cup B$ so $B \subset (A \cup B)$. Thus, $A \cup B = B$.

If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, $(A \cap B) \subset A$.
If $x \in A$, since $A \subset B$, then $x \in B$ so $x \in A \cap B$. Thus, $A \subset (A \cap B)$.
Thus, $A \cap B = A$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

If $x \in A \cap (B \cup C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

- If $x \in B$, then $x \in (A \cap B)$ so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then $x \in (A \cap C)$ so $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.
If $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).
Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.
Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof

If $x \in A \cup (B \cap C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).

- If $x \in A$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
- If $x \in B,C$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

If $x \in (A \cup B) \cap (A \cup C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).
Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 5.1.2: Union of countably infinite sets is countably infinite

If $E_1, E_2, \ldots$ are countably infinite sets, then S $= \cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

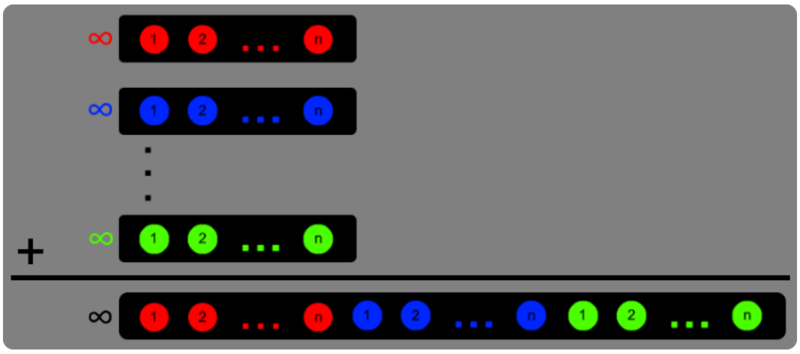
For each E_n, there is a sequence $\{ x_{n1}, x_{n2}, \ldots \}$. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence S$^* = \{ x_{11} \ ; \ x_{21}, x_{12} \ ; \ x_{31}, x_{32}, x_{33} \ ; \ \ldots \}$.
Since S$^* \sim$ S so S is at most countable and S is infinite since $E_1, E_2, \ldots$ are infinite, then S cannot be finite and thus, countably infinite.

Alternative Proof

For each E_n, let set $\widetilde{E_n} = E_n$ - $\cup_{m=1}^{\infty} E_m$ where m $\neq$ n. Thus, S $= \cup_{n=1}^{\infty} \widetilde{E_n}$.
Since each E_n is countably infinite, there exists a 1-1 mapping δ_n: $E_n \to \mathbb{Z}_+$.
Thus, for each $\widetilde{E_n}$, there is a 1-1 mapping δ_n: $\widetilde{E_n} \to$ A $\subset \mathbb{Z}_+$.
Let $p_1, p_2, \ldots$ be distinct primes.
Since for s $\in$ S, there exists a unique $\widetilde{E_i}$ such that s $\in \widetilde{E_i}$, then let f(s) $= p_1^{\delta_1(s)} p_2^{\delta_2(s)} \ldots$ where $p_k^{\delta_k(s)} = 1$ if k $\neq$ i.
Then, by the Fundamental theorem of arithmetic, f maps s to a unique z $\in \mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable.
Since any $E_n \subset$ S is countably infinite, then S cannot be finite and thus, S is countably infinite.



Theorem 5.1.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B$_n$ be the set of all n-tuples (a_1,...,a_n) where $a_k \in$ A. Then B_n is countably infinite.

Proof

The base case B_1 is countably infinite since $B_1 =$ A.
Suppose B_{n-1} is countably infinite. Then for every x $\in$ B:

x = (b,a) b $\in B_{n-1}$ and a $\in$ A

Since for every fixed b, (b,a) $\sim$ A and thus, countably infinite.
Since B is a set of countably infinite sets, then B_n is countably infinite.

Definition 5.1.4: $\mathbb{Q}$ is countably infinite

The set of rational numbers, $\mathbb{Q}$, is countably infinite.

Proof

Since elements of $\mathbb{Q}$ are of form $\frac{a}{b}$ which is a 2-tuple, then by the theorem 5.1.3, $\mathbb{Q}$ is countably infinite.

Alternative Proof

For every x $\in \mathbb{Q}$, let x = $(-1)^i \frac{p}{q}$ where p,q $\in \mathbb{Z}_+$.

Let f(x) = 2^i 3^p 5^q. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to E $\subset \mathbb{Z}_+$.

Thus, $\mathbb{Q}$ is at most countable, but since p,q $\in \mathbb{Z}_+$, then $\mathbb{Q}$ cannot be finite and thus, is countably infinite.

Example 5.1.5: Sequences of 0 and 1 are uncountable

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences $s_1, s_2, \ldots$.
Then construct a sequence s as follows:

If the n-th digit in s_n is 1, then let the n-th digit of s be 0 and vice versa.

Thus. s differs from every $s_n \in$ E so s $\notin$ E.
But, s $\in$ A so E is a proper subset of A.
Thus, every countably infinite subset of A is a proper subset of A.
If A is countably infinite, then A is a proper subset of A which is a contradiction.

5.2 Metric Spaces

Definition 5.2.1: Metric Spaces

A set X is a metric space if for ant p,q $\in$ X, there is an associated d(p,q) $\in \mathbb{R}$ such that:

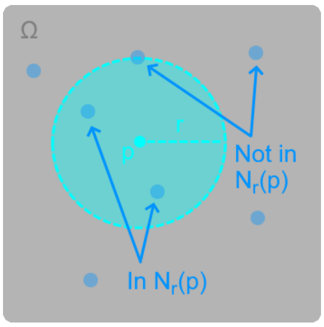
- d(p,q) > 0 if p $\neq$ q
- d(p,q) = 0 if and only if p = q
- Symmetry: d(p,q) = d(q,p)
- Triangle Inequality: d(p,q) $\leq$ d(p,r) + d(r,q) for any r $\in \mathbb{R}$.

For euclidean spaces $\mathbb{R}^k$, d(x,y) = $|x - y|$ where x,y $\in \mathbb{R}^k$.

Definition 5.2.2: Types of points and sets

(a) Neighborhood

For p $\in \mathbb{R}^k$ and r > 0, N$_r(p)$ is the set of all q such that d(q,p) < r



(b) Limit Points and Closed Sets

Closed set E contains all p where every N$_r(p)$ contains a q $\neq$ p $\in$ E

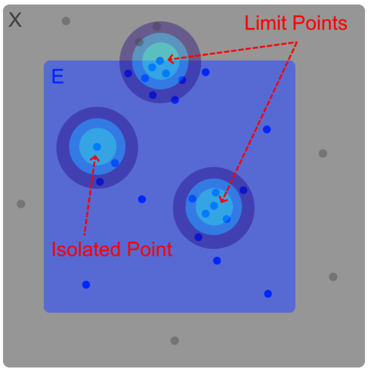
- Limit Points

For point p $\in$ X, every N$_r(p)$ contains a q $\neq$ p $\in$ E

- Isolated Points

  If p $\in$ E is not a limit point of E
- Closed

  If every limit point p of E is a p $\in$ E



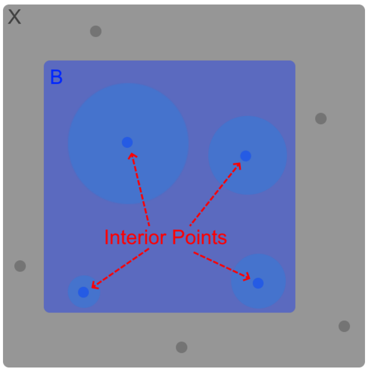
(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset$ E

- Interior Point

  For p $\in$ X, there is a $N_r(p) \subset$ E
- Open

  If every p $\in$ E is an interior point of E



(d) More about Sets

- Bounded

  If there is M $\in \mathbb{R}$, q $\in$ X such that d(p,q) $<$ M for all p $\in$ E
- Complement

  From E, E^c is the set of all p $\in$ X such that p $\notin$ E
- Perfect

  If E is closed and if every p $\in$ E is a limit point of E
- Dense

  If every p $\in$ X is a limit point of E or/and p $\in$ E

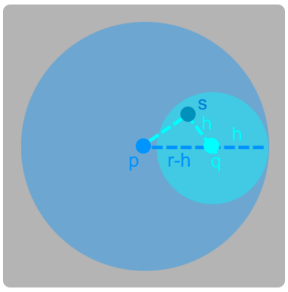
For a metric space X, X and $\emptyset$ are both open and closed. ‘

Theorem 5.2.3: $N_r(p)$ is open

Every neighborhood is an open set.

Proof

Let q $\in N_r(p)$. Then there is a h $> 0 \in \mathbb{R}$ such that d(q,p) = r - h.
Then for any s $\in N_h(q)$:
d(s,p) $\leq$ d(s,q) + d(q,p) = h + (r - h) = r
Thus, for any q $\in N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

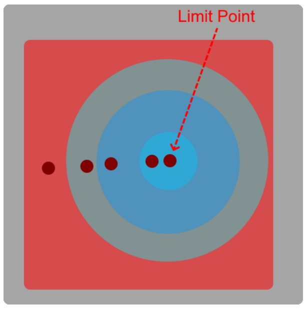


Theorem 5.2.4: If a set has a limit point, there are infinite q $\in$ E in $N_r(p)$

If p is a limit point of set E, then every $N_r(p)$ contains infinitely many q $\in$ E.

Proof

Suppose there is $N_{r_1}(p)$ which contains finitely many q = { q_1, ... , q_n }.
Let r = $\min_{m\in[1,n]}$ d(p,q_m). Then $N_r(p)$ contains no q $\in$ E such that q $\neq$ p.
So, p is not a limit point of E which is a contradiction since p is a limit point of E.



Corollary 5.2.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since $\emptyset \in$ A, all finite set must be closed.

Proof

Let p be a limit point of finite set E. By theorem 5.2.4, then any $N_r(p)$ contain infinite q $\in$ E so E is an infinite set which is a contradiction since E is finite.
So p cannot be limit point of E and thus, E has no limit points.

Theorem 5.2.6: De Morgan's Laws

Let $E_1, E_2, ...$ be a collection of sets. Then, $(\cup\ E_x)^c = \cap\ (E_x{}^c)$.

Proof

If p $\in (\cup\ E_x)^c$, then p $\notin (\cup\ E_x)$.
Thus, p $\notin E_x$ for any x so p $\in E_x^c$ for all x. Thus, p $\in \cap\ (E_x^c)$ so $(\cup\ E_x)^c \subset \cap\ (E_x^c)$.
If p $\in \cap\ (E_x^c)$, then p $\in E_x^c$ for all x.
Thus, p $\notin E_x$ for any x so p $\notin \cup\ E_x$. Thus, p $\in (\cup\ E_x)^c$ so $\cap\ (E_x^c) \subset (\cup\ E_x)^c$.
Thus, $(\cup\ E_x)^c = \cap\ (E_x^c)$.

Theorem 5.2.7: Open set $\rightarrow$ Closed complement

A set E is open if and only if E^c is closed.

Proof

Suppose E is open. Let x be a limit point of E^c.
Then for every $r > 0$, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.
Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.
Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.
Since $x \notin E$, x is not a limit point of E.
Then there exists a $r > 0$ such that any $p \in N_r(x)$ is not in E.
Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E.
Since any $x \in E$ is an interior point of E, then E is open.

Corollary 5.2.8: Closed set $\rightarrow$ Open complement

A set F is closed if only only if F^c is open.

Proof

From theorem 5.2.7, let $E = F^c$.

Theorem 5.2.9: Union open $\rightarrow$ open and Intersection closed $\rightarrow$ closed

(a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup\, G_x$ is open.
Proof
If $p \in \cup\, G_x$, then $p \in G_x$ for at least one x. Let $\overline{x}$ be such an x.
Since $G_{\overline{x}}$ is open, then p is an interior point of $G_{\overline{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\overline{x}} \subset \cup\, G_x$. So p is an interior point of $\cup\, G_x$.
Since any $p \in \cup\, G_x$ is an interior point, then $\cup\, G_x$ is open.

(b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap\, F_x$ is closed.
Proof
By Corollary 5.2.8, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup\, F_x^c$ is open.
Thus, again by Corollary 5.2.8, $(\cup\, F_x^c)^c$ is closed.
By theorem 5.2.6, $(\cup\, F_x^c)^c = \cap\, (F_x^c)^c = \cap\, F_x$.

(c) If $G_1, ..., G_n$ is a finite collection of open sets, then $\cap_{x=1}^{n} G_x$ is open.
Proof
If $p \in \cap_{x=1}^{n} G_x$, then $p \in G_x$ for all G_x for x = {1, 2, ... , n}.
Since each G_x is open, then for any G_x, there is a $N_{r_x}(p) \subset G_x$.
Let $r = \min(r_1, r_2, ..., r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x.
So, $N_r(p) \subset \cap_{x=1}^{n} G_x$ and thus, p is an interior point of $\cap_{x=1}^{n} G_x$ so $\cap_{x=1}^{n} G_x$ is open.

(d) If $F_1, ..., F_n$ is a finite collection of closed sets, then $\cup_{x=1}^{n} F_x$ is closed.
Proof
By Corollary 5.2.8, any F_x^c is open. Since $F_1^c, ..., F_n^c$ is a finite collection of open set, then by part (c), $\cap_{x=1}^{n} F_x^c$ is open.
Thus, again by Corollary 5.2.8, $(\cap_{x=1}^{n} F_x^c)^c$ is closed.
By theorem 5.2.6, $(\cap_{x=1}^{n} F_x^c)^c = \cup_{x=1}^{n} (F_x^c)^c = \cup_{x=1}^{n} F_x$.

6 Existence of $\mathbb{R}$: Theorem 2.3.3

There exists an ordered field $\mathbb{R}$ which has the least upper bound property.
Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Definition 6.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$
- If p $\in \alpha$ and q $\in \mathbb{Q} <$ p, then q $\in \alpha$
- If p $\in \alpha$, then p $<$ r $\in \mathbb{Q}$ for some r $\in \alpha$

Proposition 6.2: Order of $\mathbb{R} \rightarrow$ ordered set $\mathbb{R}$

Define $\alpha < \beta$ if α is a proper subset of β.

- If $\alpha \ngtr \beta$, then β is not a subset of α.
  Then there is a p $\in \beta$ such that p $\notin \alpha$.
  Then for any q $\in \alpha$, q $<$ p and thus, q $\in \beta$.
  Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.
- If $\alpha \nless \beta$ and $\alpha \ngtr \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
  If $\alpha \neq \beta$, there are p,q such that p $\in \alpha$, but p $\notin \beta$ and q $\in \beta$, but q $\notin \alpha$.
  But if p $\notin \beta$, then for any b $\in \beta$, b $<$ p. Thus, q $<$ p.
  Similarly, if q $\notin \alpha$, then for any a $\in \alpha$, a $<$ q. Thus, p $<$ q.
  Thus, there is a contradiction since p $>$ q and p $<$ q so $\alpha = \beta$.
- If $\alpha \nless \beta$, then α is not a subset of β.
  Then there is a p $\in \alpha$ such that p $\notin \beta$.
  Then for any q $\in \beta$, q $<$ p and thus, q $\in \alpha$.
  Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.
- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ, then α is a proper subset of γ. Thus, $\alpha < \gamma$.

Thus, $\mathbb{R}$ is an ordered set with such an order $<$.

Proposition 6.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let A $\subset \mathbb{R}$ and β be an upper bound for A. Let γ be the union of all $\alpha \in$ A.
Thus, p $\in \gamma$ if and only if p $\in \alpha$ for some $\alpha \in$ A.
γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in$ A where α_0 is nonempty.
  Since α_0 is nonempty, then γ is nonempty.
  Since every $\alpha \in$ A is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. If q $<$ p, then q $\in \alpha_1$ so q $\in$ A.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. Thus, there is a r $\in \alpha_1$ such that r $>$ p so r $\in \gamma$. Thus, there is a r $\in \gamma$ where r $>$ p.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in$ A $\subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A.
Suppose $\delta < \gamma$. Then there is a s $\in \gamma$ such that s $\notin \delta$. Since s $\in \gamma$, then there is a $\alpha \in$ A such that s $\in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A.
Thus, $\gamma = \sup(\text{A})$.

Proposition 6.4: $\mathbb{R}$ is a field

If α, $\beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums r + s where r $\in \alpha$ and s $\in \beta$. Also, let 0* be the set of all negative rational numbers which is a cut since:

- 0* is nonempty and 0* $\neq \mathbb{Q}$
- If p $\in$ 0*, then any q $\in \mathbb{Q}$ < p is a negative rational and thus, q $\in$ 0*.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then for any p $\in$ 0*, there is a r $\in \mathbb{Q}$ where p < r < 0 so r is a negative rational so r $\in$ 0*.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take r' $\notin \alpha$, s' $\notin \beta$, then r' + s' > r + s for r $\in \alpha$, s $\in \beta$. Thus, r' + s' $\notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If p $\in \alpha + \beta$, then p = r + s where r $\in \alpha$ and s $\in \beta$.
  If q < p, then q - s < p - s = (r + s) - s = r so q - s $\in \alpha$.
  Since q - s $\in \alpha$ and s $\in \beta$, then (q - s) + s = q $\in \alpha + \beta$.
- If r $\in \alpha$, then there is a t $\in \alpha$ such that t > r. Let s $\in \beta$.
  Thus, for any p = r + s $\in \alpha + \beta$, there is a q = t + s $\in \alpha + \beta$ such that p = r + s < t + s = q.

$\alpha + \beta = \beta + \alpha$

If p = r + s $\in \alpha + \beta$ where r $\in \alpha$, s $\in \beta$, then s + r = r + s = p $\in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If r $\in \alpha$, s $\in \beta$, t $\in \gamma$, then r + s + t = (r + s) + t $\in (\alpha + \beta) + \gamma$ and r + s + t = r + (s + t) $\in \alpha + (\beta + \gamma)$.

α + 0* = α

If r $\in \alpha$, s $\in$ 0*, then r + s < r. Thus, r + s $\in \alpha$. Thus, α + 0* $\subset \alpha$.
If p $\in \alpha$, there is a r $\in \alpha$ where r > p. Thus, p - r $\in$ 0*.
Since p = r + (p - r) $\in \alpha$ + 0*, then $\alpha \subset \alpha$ + 0*. Thus, α + 0* = α.

There is a -α such that α + -α = 0*

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is r > 0 such that -p - r $\notin \alpha$.
$\beta \in \mathbb{R}$ since β is a cut:

- If s $\notin \alpha$ and p = -s - 1, then -p - 1 $\notin \alpha$. Thus, p $\in \beta$ so β is nonempty. If q $\in \alpha$, then -q $\notin \beta$ so $\beta \neq \mathbb{R}$.
- If p $\in \beta$, let r > 0 so -p - r $\notin \alpha$. If q < p, then -q - r > -p - r and thus, -q - r $\notin \alpha$ so q $\in \beta$.
- If p $\in \beta$, let t = p + (r/2). Then -t - (r/2) = -p - r $\notin \alpha$ and thus, t $\in \beta$ where p < t.

If r $\in \alpha$, s $\in \beta$, then s $\notin \alpha$. Thus, r < -s so r + s < 0. Thus, $\alpha + \beta \subset$ 0*.
Let v $\in$ 0* and let w = -v/2 so w > 0.
Thus, by the Achimedean property, there is an integer n such that nw $\in \alpha$, but (n+1)w $\notin \alpha$. Let p = -(n+2)w so -p - w = (n+1)w $\notin \alpha$ so p $\in \beta$.
Then, v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p $\in \alpha + \beta$.
Since v $\in$ 0*, then 0* $\subset \alpha + \beta$. Thus, $\alpha + \beta$ = 0*. Then, let -α = β.

Thus, if α, β, $\gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
Thus, if α > 0*, then -α = -α + 0* < -α + α = 0* so -α < 0*.
If α, $\beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that p $\leq$ rs for r $\in \alpha$, s $\in \beta$. Define 1* as the set of all q < 1. Then all multiplication axioms holds with similar proofs as addition. Also, note since α, β > 0*, then $\alpha\beta$ > 0*.
Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were α, β, γ >,< 0*.

7 Balls and Convex

7.1 Intervals and Balls

Definition 7.1.1: Intervals

In $\mathbb{R}$, a segement is an open interval (a,b) = { x $\in \mathbb{R}$: a < x < b }
In $\mathbb{R}$, a interval is a closed interval [a,b] = { x $\in \mathbb{R}$: a $\leq$ x < b }

Definition 7.1.2: Open Balls

In $\mathbb{R}^k$, an open ball of radius r > 0 centered at p is:
N$_r(p)$ = { x $\in \mathbb{R}^k$: $|x-p|$ < r } = { x $\in \mathbb{R}^k$: d(x,p) < r }
A closed ball has d(y,p) $\leq$ r.

Definition 7.1.3: Convex

E $\subset \mathbb{R}^k$ is convex if for all x,y $\in$ E and t $\in$ [0,1]:
tx + (1-t)y $\in$ E
Balls in $\mathbb{R}^k$ are convex.
***Insert Proof later

Definition 7.1.4: Dense

E $\subset$ X is dense if every x $\in$ X is either in E or a limit point of E.

Example 7.1.5: $\mathbb{Q}$ is dense in $\mathbb{R}$

Let X = $\mathbb{R}$. Then, E = $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

Fix x $\in \mathbb{R}$ and r > 0.
There is a q $\in \mathbb{Q}$ such that x-r < q < x.
So for any r > 0 and q $\in \mathbb{Q}$, q $\neq$ x and q $\in$ N$_r(x)$.
Thus, every x $\in \mathbb{R}$ is a limit point of $\mathbb{Q}$.

8 Closure, Open Relative, & Compact

8.1 Closure

Definition 8.1.1: Closure

Let E $\subset$ metric space X and E' be the set of all limit points of E in X.

Then the closure of E: cl(E) = $\overline{E}$ = E $\cup$ E'

with the properties:

- $\overline{E}$ is closed
- E = $\overline{E}$ if and only if E is closed
- $\overline{E}$ $\subset$ F for every closed F $\subset$ X such that E $\subset$ F

Theorem 8.1.2: sup(E) $\in \overline{E}$ if E is bounded

Let non-empty set of real numbers, E, be bounded above.

Let y = sup(E). Then, y $\in \overline{E}$. Thus, y $\in$ E if E is closed.

Proof

If y $\in$ E, then y $\in \overline{E}$. Suppose y $\notin$ E.

For every h > 0, there exists a x $\in$ E such that y-h < x < y otherwise y-h is an upper bound for E which is a contradiction since y = sup(E).

Thus, y is a limit point of E so y $\in \overline{E}$.

8.2 Open Relative

Definition 8.2.1: Open Relative

Suppose E $\subset$ Y $\subset$ metric space X.

Then E is open relative to Y if for each p $\in$ E, there is an r > 0 such that q $\in$ E if d(q,p) < r and q $\in$ Y.

Theorem 8.2.2: E is open relative to Y $\subset$ X if E = Y $\cap$ G and G is open in X

Suppose E $\subset$ Y $\subset$ X. E is open relative to Y if and only if E = Y $\cap$ G for some open G $\subset$ X.

8.3 Compact Sets

Definition 8.3.1: Open Cover

An open cover of set E $\subset$ X is a collection $G_1, G_2, ...$ of open subsets of X such that E $\subset \cup\, G_i$.

Definition 8.3.2: Compact

K $\subset$ metric space X is compact if every open cover of K contains a finite subcover.

If $G_1, G_2, ...$ is an open cover of K, then K $\subset \cup_{i=1}^{n} G_i$ for some n.

Theorem 8.3.3: A compact set is compact in every metric space

Suppose $K \subset Y \subset X$.

Then, K is compact relative to X if and only if K is compact relative to Y.

Proof

***Insert Proof later

Theorem 8.3.4: A compact set is closed

Compact subsets of metric spaces are closed

Proof

***Insert Proof later

Theorem 8.3.5: If closed $E \subset$ compact set K, E is compact

Closed subsets of compact sets are compact

Proof

***Insert Proof later

Corollary 8.3.6:

If F is closed and K is compact, then $F \cap K$ is compact.

Proof

***Insert proof later

8.4

References