

Fall Real Analysis

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1 The Real Number System

1.1 Number Systems

Natural : $\mathbb{N} = \{1, 2, 3, \dots\}$

Integer : $\mathbb{Z} = \{-2, -1, 0, 1, 2, \dots\}$

Rational : $\mathbb{Q} = \frac{p}{q}$ where $p, q \in \mathbb{N}$

*** \mathbb{Q} is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.

Let set $A = \{x \in \mathbb{Q} \text{ for } x^2 < 2\}$ where $A \neq \emptyset$ and 2 is an upper bound for A .

A has no least upper bound in \mathbb{Q} , but A has a least upper bound in \mathbb{R} .

1.2 Real Number System

\mathbb{R} is the unique ordered field with the least upper bound property.

\mathbb{R} exists and unique.

Definition 1.2.1

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- **Trichotomy**: For all $x, y \in S$, only one holds true:

$$- x < y$$

$$- x = y$$

$$- x > y$$

- **Transitivity**: If $x < y$ and $y < z$, then $x < z$.

Definition 1.2.2

An ordered set is a set with an order.

Definition 1.2.3

Let S be an ordered set. Let $E \subset S$.

An upper bound of E is a $\beta \in S$ if $x \leq \beta$ for all $x \in E$.

If such a β exists, then E is bounded from above.

Definition 1.2.4

Let S be an ordered set. Let $E \subset S$ be bounded from above.
Then, there exists a least upper bound α if:

- α is an upper bound for E
- If $\gamma < \alpha$, then γ is not an upper bound for E .

Then $\alpha = \sup(E)$.

*** Greatest Lower Bound: $\inf(E)$ ***

Example 1.2.5

Let $S = (1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from \mathbb{R} . For subsets E of S :

- $E = (1, 2)$ is bounded above and $\sup(E) = 2$
- $E = (5, 6)$ is not bounded above so $\sup(E) = \text{DNE}$
- $E = [3, 4)$ is bounded below $\inf(E) = 3$ and $\sup(E) = \text{DNE}$

Observations on the Least Upper Bound

If $\sup E$ exists, it may or may not exist at E .

If α exists, then α is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1

An ordered set of S has a least upper bound property if:

For every nonempty subset $E \subset S$ that is bounded from above:
 $\sup(E)$ exists in S .

Example 1.3.2

\mathbb{Q} doesn't have a least upper bound property. For example, $z = \sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If $y^2 - 2 < 0$, then y is not an upper bound for E .
- If $y^2 - 2 > 0$, y is an upper bound for E , but not the $\sup(E)$.

Thus, E has no least upper bound in \mathbb{Q} .

However in \mathbb{R} , $\sqrt{2}$ is in E .

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound implies Greatest Upper Bound

Let S be an ordered set with the least upper bound property.

Let non-empty $B \subset S$ be bounded below.

Let L be the set of all lower bounds of B .

Then $\alpha = \sup(L)$ exists in S and $\alpha \in B$.

Proof

L is non-empty since B is bounded from below.

Thus, by the least upper bound property of S , $\alpha = \sup(L)$ exists in S .

We claim that $\alpha = \inf(B)$.

If $\gamma < \alpha$, then γ is not an upper bound for L so $\gamma \notin B$.

Thus, for every $x \in B$, $\alpha \leq x$.

If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in B$. Thus, $\inf(B) = \alpha$.

2.2 Fields

Addition Axioms

- If $x, y \in F$, then $x+y \in F$
- $x+y = y+x$ for all $x, y \in F$
- $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$
- There exists $0 \in F$ such that $0+x = x$ for all $x \in F$
- For every $x \in F$, there is $-x \in F$ where $x+(-x) = 0$

Multiplicative axioms

- If $x, y \in F$, then $xy \in F$
- $yx = xy$ for all $x, y \in F$
- $(xy)z = x(yz)$ for all $x, y, z \in F$
- There exists $1 \neq 0 \in F$ such that $1x = x$ for all $x \in F$
- If $x \neq 0 \in F$, there is $\frac{1}{x} \in F$ where $x(\frac{1}{x}) = 1$

Distributive Law

$x(y+z) = xy + xz$ hold for all $x, y, z \in F$.

Definition 2.2.1

- (a) If $x+y = x+z$, then $y = z$

Proof

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If
- $x+y = x$
- , then
- $y = 0$

ProofFrom (a), let $z = 0$.

- (c) If
- $x+y = 0$
- , then
- $y = -x$

ProofFrom (a), let $z = -x$.

- (d)
- $-(-x) = x$

ProofFrom (c), let $x = -x$ and $y = x$.

- (e) If
- $x \neq 0$
- and
- $xy = xz$
- , then
- $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If
- $x \neq 0$
- and
- $xy = x$
- , then
- $y = 1$

ProofFrom (e), let $z = 1$.

- (g) If
- $x \neq 0$
- and
- $xy = 1$
- , then
- $y = \frac{1}{x}$

ProofFrom (e), let $z = \frac{1}{x}$.

- (h) If
- $x \neq 0$
- , then
- $\frac{1}{1/x} = x$

ProofFrom (g), let $x = \frac{1}{x}$ and $y = x$.

- (i)
- $0x = 0$

ProofSince $0x + 0x = (0+0)x = 0x$, then $0x = 0$.

- (j) If
- $x, y \neq 0$
- , then
- $xy \neq 0$

ProofSuppose $xy = 0$, then $\frac{1}{y}\frac{1}{x}xy = \frac{1}{y}1y = \frac{1}{y}y = 1$. $xy = 0 = 1$ is a contradiction.

- (k)
- $(-x)y = -(xy) = x(-y)$

Proof

$$xy + (-x)y = (x+(-x))y = 0y = 0.$$

Then by part (c), $(-x)y = -(xy)$.

$$\text{Similarly, } xy + x(-y) = x(y+(-y)) = x0 = 0.$$

Then by part (c), $x(-y) = -(xy)$.

$$(l) \quad (-x)(-y) = xy$$

Proof

By part (k), then $(-x)(-y) = -[x(-y)] = -[-(xy)]$.

By part (d), $-[-(xy)] = xy$.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all $x, y, z \in F$.

- If $y < z$, then $y+x < z+x$
- If $x, y > 0$, then $xy > 0$

Definition 2.3.1: \mathbb{Q} and \mathbb{R} are ordered fields

\mathbb{Q} , \mathbb{R} are ordered fields, but \mathbb{C} is not an ordered field.

Definition 2.3.2

Let F be an ordered field. For all $x, y, z \in F$.

- If $x > 0$, $-x < 0$ and vice versa
- If $x > 0$ and $y < z$, then $xy < xz$
- If $x < 0$ and $y < z$, then $xy > xz$
- If $x \neq 0$, $x^2 > 0$
- If $0 < x < y$, then $0 < 1/y < 1/x$

Theorem 2.3.3: \mathbb{R} is a ordered field with $<$

There exists a unique ordered field \mathbb{R} with the least upper bound property.
Also, $\mathbb{Q} \subset \mathbb{R}$.

Theorem 2.3.4

For all $x, y \in \mathbb{R}$:

- **Archimedean Property:** If $x > 0$, there is $n \in \mathbb{Z}$ such that $nx > y$.

Proof

Fix $x > 0$. Suppose there is a y such that the property fails.

Let $A = \{ nx : n = 1, 2, 3, \dots \}$.

Then, A is nonempty and bounded from above by y .

Then by the least upper bound property by \mathbb{R} , $\alpha = \sup(A)$ exists in \mathbb{R} .

Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$.

So $\alpha - x$ is not an upper bound of A .

So there is a $mx \in A$ such that $mx > \alpha - x$

But then $\alpha < (m+1)x$ where $(m+1)x \in A$ which contradicts α is an upper bound for A .

- **\mathbb{Q} is dense in \mathbb{R} :** If $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof

Since $x < y$, then $y - x > 0$. Then by the Archimedean Property, there exists a $n \in \mathbb{Z}$ such that $n(y-x) > 1$. Thus, $ny > nx+1 > nx$

By the well-ordering principle, there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$.

Then, $m > nx \geq m-1$ so $nx+1 \geq m > nx$.

Since $ny > nx+1 \geq m > ny$, then $y > m/n > x$.

3 Temp

3.1 nth Root

If $0 < t < 1$, then $t^n \leq t$.

If $t > 1$, $t^n \geq t$.

If $0 < s < t$, $s^n < t^n$.

Theorem 3.1.1

Fix n . For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Proof

Uniqueness: y is unique since if $y_1 < y_2$, then $y_1^n < y_2^n$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ and $< x$.

Thus, $0 < t_1^n \leq t_1 < x$ so $t_1 \in A$.

A is bounded above since $t \geq x+1$.

Then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

Thus, $x+1$ is an upper bound of A .

By the least upper bound property, then $y = \sup(A)$.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

*** (Not an upper bound of A and y is not a least upper bound of A)

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n < (\beta - \alpha)n\beta^{n-1} \quad \text{Insert Proof}$$

Suppose $y^n < x$. Pick $0 < h < 1$ which contradicts the previous statement.

From inequality, let $\beta = y+h$ and $\alpha = y$. Insert Proof

Thus, $(y+h)^n < x$, thus $(y+h) \in A$ and thus, not an upper bound of A .

Suppose $y^n > x$. Pick $k = \frac{y^n - x}{ny^{n-1}} < y$. Insert Proof

Consider $t \geq y-k$, then:

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$$

So $t \notin A$.

Thus, $y-k$ is not an upper bound of A contradicting y is the least upper bound of A .

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

3.2 Decimals and Extended Reals

3.3 Complex Numbers

Definition 3.3.1

A complex number is an ordered pair (a,b) where $a,b \in \mathbb{R}$. For $x,y \in \mathbb{C}$

- $(a,b) + (c,d) = (a + c, b + d)$
- $(a,b) * (c,d) = (ac - bd, ad + bc)$

Thus, the axioms form a field where $(0,0) = 0$ and $(1,0) = 1$ and $(0,1) = i$.

Definition 3.3.2

Let $i = (0,1)$.

Proof

$i^2 = -1$. **Insert Proof**

Definition 3.3.3

$$(a, b) = a + bi$$

Definition 3.3.4

Let conjugate: $\bar{z} = a - bi$

- $(z + \bar{w}) = \bar{z} + w$
- Product
- Real
- $z\bar{z} \geq 0$

Definition 3.3.5

The $\|z\| = \sqrt{z\bar{z}}$.

Insert property for real numbers conjugate and absolute value.

Theorem 3.36

- If $x \neq 0$ and $\|x\| > 0$, then conjugate
- product conjugate
- Real part < Actual for abs
- $\|z + w\| \leq \|z\| + \|w\|$

References