Fall Real Analysis

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Contents

1 The Real Number System

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** Q is countable, but fails to have the least upper bound property ***

Example 1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A $= \{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
A has no least upper bound in Q, but A has a least upper bound in R.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
$\mathbb{R}$ exists and unique.

Definition 1.5

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x $<$ y
  - x $=$ y
  - x $>$ y
- Transitivity: If x $<$ y and y $<$ z, then x $<$ z.

Definition 1.6

An ordered set is a set with an order.

Definition 1.7

Let S be an ordered set. Let E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.

Definition 1.8

Let S be an ordered set. Let E $\subset$ S be bounded from above.
Then, there exists a least upper bound α where:

- α is an upper bound for E
- If $\gamma < \alpha$, then γ is not an upper bound for E.

Then α = sup(E).
*** Lower Bound: inf(E) ***

Example 1.9

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup E exists, it may or may not exists at E.
If α exists, then α is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Definition 1.10

An ordered set of S has a least upper bound property if:
For every nonempty subset E $\subset$ S that is bounded from above:
sup(E) exists in S.

Example 1.1

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then y is not an upper bound for E.
- If $y^2 - 2 > 0$, y is an upper bound for E, but not the sup(E).

Thus, E has no least upper bound in $\mathbb{Q}$.
However in $\mathbb{R}$, $\sqrt{2}$ is in E.

2 Day 2

2.1 Greatest Upper Bound Property

Theorem

If If an ordered set has the least upper bound property, then it has the greatest upper bound property.

Let S be a ordered set with the least upper bound property. Let non-empty B $\subset$ Sand bounded below. Let L be the set of all lower bounds of B. Then α = sup(L) exists and $\alpha \in$ B.

Proof

L is non-empty since B is bounded from below and $\gamma \in$ L.
Thus, by the least upper bound property of S, α = sup(L) exists.
We claim that α = inf(B).
For any x $< \alpha$, then x is not an upper bound for L so x $\in$ L.
For any x $> \alpha$, then x is an upper bound for L so x not in L and thus, x $\in$ B.

2.2 Fields

Addition Axioms

- x,y $\in$ F, then x+y $\in$ F
- Addition is commutative
- Addition is associative
- x+0 = x

Multiplicative xioms

- xy
- xy = yx
- (xy)z = x(yx)
- 1/x

Distributive Law

x(y+z) = xy + xz hold for all x,y,z $\in$ F.

*** Remember to insert all the propostitions

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y $<$ z, then y+x $<$ z+x
- If x,y $>$ 0, then xy $>$ 0

*** If x > 0, then x is positive ***

Definition 2.3.1

$\mathbb{Q}$,$\mathbb{R}$ are ordered fields. $\mathbb{C}$ is not a ordered field.

Definition 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

- If x > 0, -x < 0 and vice versa
- If x > 0 and y < z, then xy ¡ xz
- If x < 0 and y < z, then xy > xz
- If x $\neq$ 0, $x^2 > 0$
- If $0 < x < y$, then $0 < 1/y < 1/x$

  Proof for A

  If x > 0, then x+(-x) > 0+(-x) so 0 ¿ (-x)

Theorem 2.3.3: R is a ordered field with <

There exists a ordered field $\mathbb{R}$ with the least upper bound property.
Also, $\mathbb{Q} \subset \mathbb{R}$.
$\mathbb{R}$ is unique ordered field with least upper bound property.

Theorem 2.3.4

For all x,y $\in$ $\mathbb{R}$:

- Archimedean Property: If x > 0, there is n $\in$ $\mathbb{Z}$ such that nx > y.

  Proof

  Fix x > 0. Suppose there is a y such that the property fails.

  Let A = { nx: n = 1, 2, 3... }.

  Then, A is nonempty and bounded from above by y.

  Then by the least upper bound property by $\mathbb{R}$, then α = sup(A) exists in $\mathbb{R}$.

  Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$.

  So $\alpha - x$ is an upper bound of A. So there is a mx $\in$ A such that mx > $\alpha - x$

  But then $\alpha < (m + 1)x$ so (m+1)x $\in$ A which contradicts α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If x < y, there is a p $\in$ $\mathbb{Q}$ such that x < p < y.

  Proof

  n(y-x) > 1

  ny > nx+1 > nx

  1 > nx

By the well-ordering principle, there is a smallest m of positive integers such that m $>$ nx

Then, m $>$ nx $\geq$ m-1 and nx+1 $\geq$ m $>$ nx

By ny $>$ nx+1 $\geq$ m $>$ nx.

SO y $>$ m/n $>$ x.

References