Fall Real Analysis

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Contents

1 The Real Number System

$\mathbb{N} = 1, 2, 3, ...$
$Z = -2, -1, 0, 1, 2$
$\mathbb{Q} = p/q$ where Q is countable and fails to have the least upper bound property

Ex 1.1
$\alpha \in \mathbb{R}$ where $\alpha^2 = 2$
Let $\alpha = p/q$ where p and q cannot both be even

$Let set A = \{x \in \mathbb{Q}\ for x^2 < 2 where A is nonempty and 2 is an upper bound for A$

A has no least upper bound in R
A has a least upper bound in Q

$\mathbb{R}$ is the unique ordered field withthe least upper bound property.
Theorem: $\mathbb{R}$ exists and unique.

Def 1.5
Let S be a set. An order on S is a relation ¡ satisfying 2 axioms:

- Trichotomy: For all x,y $\in S$:
    - x < y
    - x = y
    - x > y

Transitivity: If x < y and y < z, then x < z.

Def 1.5
An ordered set is a set with an order.

Def 1.7
Let S be an ordered set. Let E $\subset S$.
$An upper bound of E is a$ $\beta \in S$ where x = $\beta for all x \in E$.
$If such a \beta exists, then E is bounded from above.$

Def 1.8
Let S be an ordered set. Let E $\subset S be bounded from above.$
$here exists an \alpha where$:
$\alpha is an upper bound for E if x \alpha, then x is not an upper bound for E$
Then $\alpha = sup(E)$.

*** Lower Bound in textbook known as an inf(E) ***
Example 1.9
Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with an order $<$ from $\mathbb{R}$.
For subsets E of S

- E = (1,2) is bound above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations
If sup E exists, it may or may not exists at E.
If $\alpha exists, then \alpha is unique. If$ $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$

Def 1.10
A is an ordered set of S.
A has a least upper bound property if:

- For every nonempty subset E $\subset S that is bounded from above, sup(E) exists in S$.

Example 1.1
$\mathbb{Q}$ doesn't have a least upper bound property for example $\sqrt(2)$.
Proof is at textbook.

References