Fall Real Analysis

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Contents

1 The Real Number System

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** $\mathbb{Q}$ is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A $= \{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
But, A has no least upper bound in $\mathbb{Q}$, but A has a least upper bound in $\mathbb{R}$.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
Also, $\mathbb{R}$ exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x $<$ y
  - x $=$ y
  - x $>$ y
- Transitivity: If x $<$ y and y $<$ z, then x $<$ z.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.
A lower bound of E is a $\alpha \in$ S if x $\geq \alpha$ for all x $\in$ E.
If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.
Let E $\subset$ S be bounded from above. Least upper bound $\beta \in$ S exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E.

Then β = sup(E).

Let E $\subset$ S be bounded from below. Greatest lower bound $\alpha \in$ S exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E.

Then α = inf(E).

Example 1.2.5

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup(E) exists, it may or may not exists at S.
If sup(E) exists, then sup(E) is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:
For every nonempty subset E $\subset$ S that is bounded from above:
sup(E) exists in S.

Example 1.3.2

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then z > y where z $\in$ A. So, y is not a upper bound.

  Since for any y, there is z > y where z $\in$ A, then sup(A) doesn't exists in $\mathbb{Q}$.
- If $y^2 - 2 > 0$, then z < y where z $\in$ B. So, y is an upper bound, but not sup(E).

  Since for any y, there is z < y where z $\in$ B, then inf(B) doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound.

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be a ordered set with the least upper bound property.
Let non-empty B $\subset$ S be bounded below.
Let L be the set of all lower bounds of B.
Then α = sup(L) exists in S.

Proof

L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, α = sup(L) exists in S.
We claim that α = inf(B).
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B.
Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, inf(B) = α.

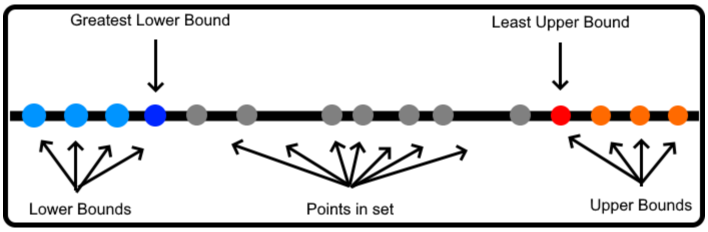

Figure 1: Infimum, Supremum, & Bounds

2.2 Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F
- x+y = y+x for all x,y $\in$ F
- (x+y)+z = x+(y+z) for all x,y,z $\in$ F
- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F
- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative xioms

- If x,y $\in$ F, then xy $\in$ F
- yx = xy for all x,y $\in$ F
- (xy)z = x(yx) for all x,y,z $\in$ F
- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F
- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x}$ $\in$ F where x($\frac{1}{x}$) = 1

Distributive Law

x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Propositions 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0

Proof

From (a), let z = 0.

(c) If x+y = 0, then y = -x

Proof

From (a), let z = -x.

(d) -(-x) = x

Proof

From (c), let x = -x and y = x.

(e) If x $\neq$ 0 and xy = xz, then y = z

Proof

y = 1y = $\frac{1}{x}$xy = $\frac{1}{x}$xz = 1z = z

(f) If x $\neq$ 0 and xy = x, then y = 1

Proof

From (e), let z = 1.

(g) If x $\neq$ 0 and xy = 1, then y = $\frac{1}{x}$

Proof

From (e), let z = $\frac{1}{x}$.

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x

Proof

From (g), let x = $\frac{1}{x}$ and y = x.

(i) 0x = 0

Proof

Since 0x + 0x = (0+0)x = 0x, then 0x = 0.

(j) If x,y $\neq$ 0, then xy $\neq$ 0

Proof

Suppose xy = 0, then $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}$1y = $\frac{1}{y}$y = 1.

xy = 0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)

Proof

xy + (-x)y = (x+(-x))y = 0y = 0.

Then by part (c), (-x)y = -(xy).

Similarly, xy + x(-y) = x(y+(-y)) = x0 = 0.

Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy

Proof

By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].

By part (d), -[-(xy)] = xy.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y $<$ z, then y+x $<$ z+x
- If x,y $>$ 0, then xy $>$ 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field since i^2 = -1 $\not>$ 1.

Propositions 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

(a) If x $>$ 0, -x $<$ 0 and vice versa

Proof

-x = (-x) + 0 $<$ (-x) + x = 0

(b) If x $>$ 0 and y $<$ z, then xy $<$ xz

Proof

Since z-y $>$ 0, then 0 $<$ x(z-y) = xz - xy

(c) If x $<$ 0 and y $<$ z, then xy $>$ xz

Proof

Since -x $>$ 0 and z-y $>$ 0, then 0 $<$ -x(z-y) = xy - xz

(d) If x $\neq$ 0, $x^2 > 0$

Proof

If x $>$ 0, then x^2 = x $\cdot$ x $>$ 0

If x $<$ 0, then x^2 = (-x) $\cdot$ (-x) $>$ 0

(e) If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

Since $(\frac{1}{y})y = 1 > 0$, then $(\frac{1}{y}) > 0$

Since $x < y$, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$

Theorem 2.3.3: $\mathbb{R}$ is a ordered field with $<$

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property. Also, $\mathbb{Q} \subset \mathbb{R}$ so $\mathbb{Q}$ is also an ordered field.

Theorem 2.3.4

For all x,y $\in \mathbb{R}$:

- Archimedean Property: If $x > 0$, there is $n \in \mathbb{Z}$ such that $nx > y$.

  Proof

  Fix $x > 0$. Suppose there is a y such that the property fails.

  Let $A = \{ nx: n = 1, 2, 3, ... \}$.

  Then, A is nonempty and bounded from above by y.

  Then by the least upper bound property by $\mathbb{R}$, $\alpha = \sup(A)$ exists in $\mathbb{R}$.

  Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$.

  So $\alpha - x$ is not an upper bound of A.

  So there is a $mx \in A$ such that $mx > \alpha - x$

  But then $\alpha < (m+1)x$ where $(m+1)x \in A$ which contradicts α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

  Proof

  Since $x < y$, then $y-x > 0$. Then by the Archimedean Property, there exists a $n \in Z$ such that $n(y-x) > 1$. Thus, $ny > nx+1 > nx$

  By the well-ordering principle, there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$.

  Then, $m > nx \geq m-1$ so $nx+1 \geq m > nx$.

  Since $ny > nx+1 \geq m > nx$, then $y > m/n > x$.

3 Roots & Complex Field

3.1 nth Root

(a) If $0 < t \leq 1$, then $t^n \leq t$.

Proof

Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.

Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.

Applying the process n times, then $t^n \leq t$.

(b) If $t \geq 1$, $t^n \geq t$.

Proof

Since $0 < 1 \leq t$, then $t \leq t^2$.

Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.

Applying the process n times, $t \leq t^n$.

(c) If $0 < s < t$, $s^n < t^n$.

Proof

$$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix n. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = {y_1}^n < {y_2}^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ and $< x$ and thus, $0 < {t_1}^n < t_1 < x$ so $t_1 \in A$.

A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

So x+1 is an upper bound of A.

Thus by the least upper bound property, $y = \sup(A)$ exists.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

(Not an upper bound of A if < and not a least upper bound of A if >)

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha)\,\underbrace{(\underbrace{\beta^{n-1}}_{\beta^{n-1}} + \underbrace{\beta^{n-2}\alpha^1}_{<\beta^{n-1}} + ... + \underbrace{\alpha^{n-1}}_{<\beta^{n-1}})} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.

From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n-x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.

Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus, $t^n > x$ so $t \notin A$.

Thus, y-k is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.
Then let n_k be the largest integer such that:
$d_k = n_0 + \frac{n_1}{10} + \ldots + \frac{n_k}{10^k} \leq x$
Let E be the set of d_k for $k = 0, 1, \ldots \infty$. Then, $x = \sup(E)$.

3.3 Extended Reals

The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:
$-\infty < x < \infty$ for every $x \in \mathbb{R}$
with the properties:

- $x \pm \infty = \pm\infty$
- $x \,/\, \pm\infty = 0$
- If $x > 0$, then $x(\pm\infty) = \pm\infty$
- If $x < 0$, then $x(\pm\infty) = \mp\infty$

3.4 Complex Numbers

Definition 3.3.1: Complex
A complex number is an ordered pair (a,b) where $a,b \in \mathbb{R}$. For $x,y \in \mathbb{C}$

- $x + y = (a,b) + (c,d) = (a + c , b + d)$
- $xy = (a,b)(c,d) = (ac - bd , ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)(a,-b)$

Thus, the axioms form a field where (0,0) = 0 and (1,0) = 1 and (0,1) = i.

Definition 3.3.2: Imaginary i
Let i = (0,1). Then, $i^2 = -1$.
Proof
$i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1$

Definition 3.3.3: Form a + bi
(a,b) = a + bi
Proof
(a,b) = (a,0) +(0,b) = (a,0) + (b,0)(0,1) = a + bi

Definition 3.3.4: Conjugate
Let conjugate: $\bar{z}$ = a - bi where Re(z) = a , Im(z) = b

Let z = (a,b) and w = (c,d):

(a) $\overline{z+w} = \bar{z} + \overline{w}$

Proof
$\overline{z+w} = \overline{(a+c, b+d)} = (a+c,-b-d) = (a,-b) + (c,-d) = \bar{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\ \overline{w}$

Proof

$\overline{zw} = \overline{(ac-bd, ad+bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\overline{z}\ \overline{w}$

(c) z + $\overline{z}$ = 2 Re(z)   z - $\overline{z}$ = 2i Im(z)

Proof

z + $\overline{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)

z - $\overline{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

(d) z$\overline{z} \geq 0$

Proof

z$\overline{z}$ = (a,b)(a,-b) = (a^2 + b^2 , -ab+ab) = a^2 + b$^2 \geq 0$

Definition 3.3.5: Absolute Value

Let absolute value: | z | = $\sqrt{z\overline{z}}$

Let z = (a,b) and w = (c,d):

(a) If z $\neq$ 0, then | z | > 0.

Proof

$\sqrt{z\overline{z}} = \sqrt{a^2+b^2} \geq 0$ where | z | = 0 only if a,b = 0 so only if z = (0,0).

(b) | $\overline{z}$ | = | z |

Proof

| $\overline{z}$ | = $\sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$ = | z |

(c) | zw | = | z | | w |

Proof

| zw | = | (ac-bd,ad+bc) | = $\sqrt{(ac-bd)^2+(ad+bc)^2}$

= $\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$

= $\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$ = | z | | w |

(d) | Re(z) | $\leq$ | z |

Proof

| Re(z) | = | a | = $\sqrt{a^2} \leq \sqrt{a^2+b^2}$ = | z |

(e) | z+w | $\leq$ | z | + | w |

Proof

$|z+w|^2$ = (z+w)$\overline{(z+w)}$ = (z+w)($\overline{z}+\overline{w}$) = z$\overline{z}$ + z$\overline{w}$ + w$\overline{z}$ + w$\overline{w}$

= $|z|^2 + |w|^2$ + 2 Re(z$\overline{w}$) $\leq |z|^2 + |w|^2 + 2\ |z\overline{w}| = |z|^2 + |w|^2 + 2|z||w|$

= $(|z| + |w|)^2$

4 Euclidean Spaces

4.1 Euclidean Spaces

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples:

$\mathrm{x} = (x_1, ..., x_k)$ for each $x_i \in \mathbb{R}$

with the properties:

- $\mathrm{x+y} = (x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- $\mathrm{cx} = (cx_1, ..., cx_k) \in \mathbb{R}^k$

So, $\mathbb{R}^n$ has a vector space structure. Similarly, for $\mathbb{C}^n$.

Definition 4.1.1: Inner Product for $\mathbb{R}^k$

$x \cdot y = x_1 y_1 + ... + x_k y_k \in \mathbb{R}$

Definition 4.1.2: Norm

$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$

Definition 4.1.3: Extension to $\mathbb{C}^k$

For z,w $\in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + ... + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + ... + z_k \overline{z_k} = |z_1|^2 + ... + |z_n|^2 = |z|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $b_1, ..., b_n \in \mathbb{C}$, then:

$|\sum_{j=1}^{n} a_j (\overline{b_j})|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j (\overline{b_j})$.
If $B = 0$, then $b_1 = ... = b_n = 0$. Thus, $0 \leq A(0)$ holds true.
Suppose $B > 0$. Then:

$$\begin{aligned}
&\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j)(\overline{B}\ \overline{a_j} - \overline{C}\ \overline{b_j}) \\
&= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\ \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B\overline{C} a_j \overline{b_j} - BC\overline{a_j} b_j + C\overline{C} b_j \overline{b_j} \\
&= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j \overline{b_j} - BC \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\
&= B^2 A - B\overline{C} C - BC\overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\
&= B(AB - |C|^2)
\end{aligned}$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.
Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i \overline{z_i}$, then $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$|z \cdot w|^2 = |\sum z_i \overline{w_i}\ |^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2\ |w|^2$

Thus, $|z \cdot w| \leq |z||w|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

(a) $|x| \geq 0$ where $|x| = 0$ only if x = 0

Proof

$|x| = \sqrt{\sum_{i=1}^{k} x_i^2} \geq 0$ where $|x| = 0$ only if $x_1 = ... = x_k = 0$

(b) $|\alpha x| = |\alpha||x|$

Proof

$|\alpha x| = \sqrt{\sum_{i=1}^{k} (\alpha x_i)^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^{k} x_i^2} = |\alpha||x|$

(c) $|x + y| \leq |x| + |y|$

Proof

$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2$
$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$

(d) $|x - y| \leq |x - z| + |y - z|$

Proof

$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$

4.3 Cardinality

Definition 4.3.1: Onto and 1-1 Mapping

Suppose for every x $\in$ A, there is an associated f(x) $\in$ B.
Then f maps A into B = f: A $\rightarrow$ B.

- If f(A) = B, then f maps A onto B.
- If for each y $\in$ B, $f^{-1}(y)$ consist of at most one x $\in$ A where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B.

Definition 4.3.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function f: A $\rightarrow$ B. (1-1 correspondence between A and B) Then:
A $\sim$ B
If f: A $\rightarrow$ B is 1-1 and onto, then there is a f^{-1}: B $\rightarrow$ A that is 1-1 and onto.

Definition 4.3.3: Countability

- A is finite if A $\sim J_n = \{0, 1, ..., n\}$ for some n $\in \mathbb{N}$
- A is infinite if A is not finite
- A is countably infinite if A $\sim \mathbb{Z}_+ = J$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite.

Example 4.3.4

$\mathbb{Z}$ is countably infinite

Proof

Let f: $\mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n-1}{2} & \text{n is odd} \end{cases}$$

So 1 $\mapsto$ 0 , 2 $\mapsto$ 1, 3 $\mapsto$ -1, 4 $\mapsto$ 2, 5 $\mapsto$ -2 , etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 4.3.4: Pigeohole Principle

If A is finite, A is not equivalent to any proper set of A.

Theorem 4.3.6: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let E $\subset$ A be an infinite subset. For every distinct $x_i \in$ A, let x = { $x_1, x_2, ...$ }.
Let n_1 be smallest integer such that $\mathrm{x}_{n_1} \in$ E.
Then let n_2 be the smallest integer where $n_2 > n_1$ such that $\mathrm{x}_{n_2} \in$ E.
Repeat the process to create sequence f(k) = { $x_{n_1}, x_{n_2}, ..., x_{n_k}, ...$ }.
Thus, there is a 1-1 correspondence between E and $\mathbb{Z}_+$ so E is countably infinite.

5 Metric Spaces

5.1 Set of Sets

Definition 5.1.1: Union and Intersection

Let sets A,B be such that for each $x \in A$, there is an associated $E_x \subset B$.

- $E = \cup_{x=1}^{n} E_x$ only if for every $x \in E$, $x \in E_x$ for at least one $x \in A$.
- $P = \cap_{x=1}^{n} E_x$ only if for every $x \in P$, $x \in E_x$ for all $x \in A$.

with properties:

(a) $A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$

(b) $(A \cup B) \cup C = A \cup (B \cup C)$ $\qquad$ $(A \cap B) \cap C = A \cap (B \cap C)$

(c) $A \subset A \cup B$ $\qquad$ $(A \cap B) \subset A$

(d) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$

Proof

If $x \in A \cup B$, then $x \in A$ or/and $x \in B$.

- If $x \in A$, since $A \subset B$, then $x \in B$. Then, $(A \cup B) \subset B$.
- If $x \in B$, then immediately $(A \cup B) \subset B$.

If $x \in B$, then $x \in A \cup B$ so $B \subset (A \cup B)$. Thus, $A \cup B = B$.

If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, $(A \cap B) \subset A$.

If $x \in A$, since $A \subset B$, then $x \in B$ so $x \in A \cap B$. Thus, $A \subset (A \cap B)$.

Thus, $A \cap B = A$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

If $x \in A \cap (B \cup C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

- If $x \in B$, then $x \in (A \cap B)$ so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then $x \in (A \cap C)$ so $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

If $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(f) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Proof

If x ∈ A ∪ (B ∩ C), then x ∈ A or/and (x ∈ B and x ∈ C).

- If x ∈ A, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).
- If x ∈ B,C, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).

If x ∈ (A ∪ B) ∩ (A ∪ C), then x ∈ A or/and (x ∈ B and x ∈ C).

Thus, (A ∪ B) ∩ (A ∪ C) ⊂ A ∪ (B ∩ C).

Thus, A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

Theorem 5.1.2: Union of countably infinite sets is countably infinite

If $E_1, E_2, ...$ are countably infinite sets, then S = $\cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

For each E_n, there is a sequence { x_{n1}, x_{n2}, ... }. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence S* = { x_{11} ; x_{21}, x_{12} ; x_{31}, x_{32}, x_{33} ; ... }.
Since S* ∼ S and S is infinite since $E_1, E_2, ...$ are infinite, then S is countably infinite.

Theorem 5.1.3: The set of countable n-tuples are countable

Let A be a countable set and B_n be the set of all n-tuples (a_1,...,a_n) where a_k ∈ A. Then B_n is countable.

Proof

The base case B_1 is countable since B_1 = A.
Suppose B_{n-1} is countable. Then for every x ∈ B:

x = (b,a) b ∈ B_{n-1} and a ∈ A

Since for every fixed b, (b,a) ∼ A and thus, countably infinite.
Since B is a set of countably infinite sets, then B_n is countably infinite.

Definition 5.1.4: ℚ is countably infinite

Since ℚ are of form $\frac{a}{b}$ which is a 2-tuple and thus countable, then ℚ is countable by the previous theorem.

Example 5.1.5: Sequences of 0 and 1 are uncountable

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof

Let set E be a countably infinite subset of A which consist of sequences $s_1, s_2, ...$.
Then construct a sequence s as follows:

If the n-th digit in s_n is 1, then let the n-th digit of s be 0 and vice versa.

Thus. s differs from every s_n ∈ E so s ∉ E.
But, s ∈ A so E is a proper subset of A.
Thus, every countably infinite subset of A is a proper subset of A.
If A is countably infinite, then A is a proper subset of A which is a contradiction.

5.2 Metric Spaces

Definition 5.2.1: Metric Spaces

A set X is a metric space if for ant p,q $\in$ X, there is an associated d(p,q) $\in \mathbb{R}$ such that:

- d(p,q) > 0 if p $\neq$ q and d(p,p) = 0
- d(p,q) = d(q,p)
- d(p,q) $\leq$ d(p,r) + d(r,q) for any r $\in \mathbb{R}$.

For euclidean spaces $\mathbb{R}^k$, d(x,y) = $|x - y|$ where x,y $\in \mathbb{R}^k$.

Definition 5.2.2: Types of points and sets

(a) Neighborhood

For p $\in \mathbb{R}^k$ and r > 0, N$_r(p)$ is the set of all q such that d(q,p) < r

(b) Limit Points and Closed Sets

Closed set E contains all p where every N$_r(p)$ contains a q $\neq$ p $\in$ E

- Limit Points

  For point p $\in$ E $\subset$ X, every N$_r(p)$ contains a q $\neq$ p $\in$ E
- Isolated Points

  If p $\in$ E is not a limit point of E
- Closed

  If every limit point p of E is a p $\in$ E

(c) Interior Points and Open Sets

Open set E contains all its p which has a N$_r(p)$ $\subset$ E

- Interior Point

  For p $\in$ E, there is a N$_r(p)$ $\subset$ E
- Open

  If every p $\in$ E is an interior point of E

(d) More about Sets

- Bounded

  If there is M $\in \mathbb{R}$, q $\in$ X such that d(p,q) < M for all p $\in$ E
- Complement

  From E, E^c is the set of all p $\in$ X such that p $\notin$ E
- Perfect

  If E is closed and if every p $\in$ E is a limit point of E
- Dense

  If every p $\in$ X is a limit point of E or/and p $\in$ E

Definition 5.2.3: $N_r(p)$ is open

Every neighborhood is an open set.

Proof

Let q $\in$ $N_r(p)$. Then there is a h $> 0 \in \mathbb{R}$ such that:

d(q,p) = r - h

Then for any s $\in$ $N_h(q)$:

d(s,p) $\leq$ d(s,q) + d(q,p) = h + (r - h) = r

Thus, for any q $\in$ $N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

5.3

References