Fall Real Analysis

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Fall 2021

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1 Least Upper Bound Property

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** Q is countable, but fails to have the least upper bound property ***

Example 1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A = $\{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
A has no least upper bound in Q, but A has a least upper bound in R.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
$\mathbb{R}$ exists and unique.

Definition 1.5

Let S be a set. An order on S is a relation < satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x < y
  - x = y
  - x > y
- Transitivity: If x < y and y < z, then x < z.

Definition 1.6

An ordered set is a set with an order.

Definition 1.7

Let S be an ordered set. Let E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.

Definition 1.8

Let S be an ordered set. Let E $\subset$ S be bounded from above.
Then, there exists a least upper bound α where:

- α is an upper bound for E
- If $\gamma < \alpha$, then γ is not an upper bound for E.

Then α = sup(E).
*** Lower Bound: inf(E) ***

Example 1.9

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order < from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup E exists, it may or may not exists at E.
If α exists, then α is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Definition 1.10

An ordered set of S has a least upper bound property if:
For every nonempty subset E $\subset$ S that is bounded from above:
sup(E) exists in S.

Example 1.1

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then y is not an upper bound for E.
- If $y^2 - 2 > 0$, y is an upper bound for E, but not the sup(E).

Thus, E has no least upper bound in $\mathbb{Q}$.
However in $\mathbb{R}$, $\sqrt{2}$ is in E.

References