Fall Real Analysis

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Contents

1 The Real Number System

$\mathbb{N} = 1, 2, 3, ...$
$Z = -2, -1, 0, 1, 2$
$\mathbb{Q} = p/q$ where Q is countable and fails to have the least upper bound property

Ex 1.1
$\alpha \in \mathbb{R}$ where $\alpha^2 = 2$
Let $\alpha = p/q$ where p and q cannot both be even

$LetsetA = \{x \in \mathbb{Q}\ forx^2 < 2whereAisnonemptyand2isanupperboundforA$

A has no least upper bound in R
A has a least upper bound in Q

$\mathbb{R}$ is the unique ordered field withthe least upper bound property.
Theorem: $\mathbb{R}$ exists and unique.

Def 1.5
Let S be a set. An order on S is a relation ¡ satisfying 2 axioms:

- Trichotomy: For all x,y $\in S$:
  - x < y
  - x = y
  - x > y

Transitivity: If x < y and y < z, then x < z.

Def 1.5
An ordered set is a set with an order.

Def 1.7
Let S be an ordered set. Let E $\subset S$.
$AnupperboundofEisa$ $\beta \in S$ where x = $\beta forallx \in E$.
$Ifsucha\beta exists, thenEisboundedfromabove.$

Def 1.8
Let S be an ordered set. Let E $\subset Sbeboundedfromabove.$
$hereexistsan\alpha where$:
$\alpha isanupperboundforEifx\alpha, thenxisnotanupperboundforE$
Then $\alpha = sup(E)$.

*** Lower Bound in textbook known as an inf(E) ***
Example 1.9
Let S = $(1,2) \cup [3,4) \cup (5,6)$ with an order $<$ from $\mathbb{R}$.
For subsets E of S

- E = (1,2) is bound above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations
If sup E exists, it may or may not exists at E.
If $\alpha exists, then \alpha is unique. If$ $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$

Def 1.10
A is an ordered set of S.
A has a least upper bound property if:

- For every nonempty subset E $\subset Sthatisboundedfromabove, sup(E) existsinS.$

Example 1.1
$\mathbb{Q}$ doesn't have a least upper bound property for example $\sqrt(2)$.
Proof is at textbook.
$z = y - \frac{y^2-2}{y+2} then take z^2 - 2 = 2\frac{2(y^2-2)}{(y+2)^2}$
Then compare if $y^2 < 2$ and $y^2 > 2$.

- y$^2 < 2 is not an upper bound for E y^2 > 2 is an upper bound for E, but not a sup(E).$
ThusE has no least upper bound in Q.
However, since E R then $\sqrt(2) is in E.$

References