

Real Analysis

Azure

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1 Ordered Sets and Fields

1.1 Ordered Sets and Bounds

Definition 1.1.1: Ordered Set

An order is:

- **Trichotomy**: For all $x, y \in S$, only one holds true:
 - $x < y$
 - $x = y$
 - $x > y$
- **Transitivity**: If $x < y$ and $y < z$, then $x < z$.

An ordered set is a set with an order.

Definition 1.1.2: Bounds

Let S be an ordered set and $E \subset S$.

An upper bound of E is a $\beta \in S$ such that for $x \leq \beta$ for all $x \in E$.

If such a β exists, then E is bounded from above.

A lower bound of E is a $\alpha \in S$ such that for $x \geq \alpha$ for all $x \in E$.

If such a α exists, then E is bounded from below.

Definition 1.1.3: Infimum & Supremum

Let S be an ordered set.

Let $E \subset S$ be bounded from above. Least upper bound $\beta \in S$ exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E .

Then $\beta = \sup(E)$.

Let $E \subset S$ be bounded from below. Greatest lower bound $\alpha \in S$ exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E .

Then $\alpha = \inf(E)$.

Even if $\sup(E)$ exists, it may or may not exist at S .

If $\sup(E)$ exists, then $\sup(E)$ is unique. Statement also holds true for $\inf(E)$.

Example

Let $S = (1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from \mathbb{R} . For subsets E of S :

- $E = (1, 2)$ is bounded above with $\sup(E) = 3$ and not bounded below.
- $E = (5, 6)$ is not bounded above or below so $\inf(E), \sup(E) = \text{DNE}$.
- $E = [3, 4)$ is bounded below with $\inf(E) = 3$, but $\sup(E) = \text{DNE}$.

1.2 Least Upper Bound Property

Theorem 1.2.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:

For every nonempty subset $E \subset S$ that is bounded from above:

$\sup(E)$ exists in S .

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If $y^2 - 2 < 0$, then $z > y$ where $z \in A$. So, y is not an upper bound.
Since for any y , there is $z > y$ where $z \in A$, then $\sup(A)$ doesn't exist in \mathbb{Q} .
- If $y^2 - 2 > 0$, then $z < y$ where $z \in B$. So, y is an upper bound, but not $\sup(E)$.
Since for any y , there is $z < y$ where $z \in B$, then $\inf(B)$ doesn't exist in \mathbb{Q} .

Thus, \mathbb{Q} doesn't have the least upper bound or greatest lower bound property.

Example

\mathbb{Q} doesn't have a least upper bound property. Take for example, $\sqrt{2}$.

Let $x^2 = 2$. If x was rational, there is a rational $\frac{p}{q}$ where $x = \frac{p}{q}$ where both p and q are not even.

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow \quad p^2 = 2q^2$$

Since $2q^2$ is even, then p^2 is even so p is even. Thus, p is divisible by 2 so p^2 is divisible by 4 so q^2 is divisible by 2 so q is even. Thus, both p and q must be even which is a contradiction so $x = \sqrt{2}$ cannot be rational.

So if $\sqrt{2} < \frac{a}{b}$ for some rational $\frac{a}{b}$, there is always another rational $\frac{p}{q}$:

$$\sqrt{2} < \frac{p}{q} < \frac{a}{b}$$

and there will never be a rational $\frac{p}{q}$ such that $\sqrt{2} = \frac{p}{q}$ since $\sqrt{2}$ is not rational.

Theorem 1.2.2: Least Upper Bound + Lower Bound implies Greatest Lower Bound

Let S be an ordered set with the least upper bound property.

Let non-empty $B \subset S$ be bounded below.

Let L be the set of all lower bounds of B .

Then $\alpha = \sup(L)$ exists in S .

Proof

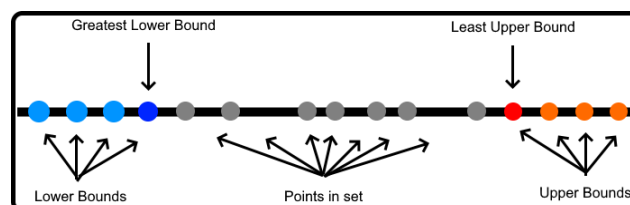
L is non-empty since B is bounded from below.

Thus, by the least upper bound property of S , $\alpha = \sup(L)$ exists in S .

We claim that $\alpha = \inf(B)$.

If $\gamma < \alpha$, then γ is not an upper bound for L so $\gamma \notin B$ since all upper bounds for L are in B . Thus, for every $x \in B$, $\alpha \leq x$.

If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in B$. Thus, $\inf(B) = \alpha$.



1.3 Fields

Definition 1.3.1: Fields Axioms

- (a) Addition Axioms
- If $x, y \in F$, then $x+y \in F$
 - $x+y = y+x$ for all $x, y \in F$
 - $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$
 - There exists $0 \in F$ such that $0+x = x$ for all $x \in F$
 - For every $x \in F$, there is $-x \in F$ where $x+(-x) = 0$
- (b) Multiplicative Axioms
- If $x, y \in F$, then $xy \in F$
 - $yx = xy$ for all $x, y \in F$
 - $(xy)z = x(yz)$ for all $x, y, z \in F$
 - There exists $1 \neq 0 \in F$ such that $1x = x$ for all $x \in F$
 - If $x \neq 0 \in F$, there is $\frac{1}{x} \in F$ where $x(\frac{1}{x}) = 1$
- (c) Distributive Law
- $x(y+z) = xy + xz$ hold for all $x, y, z \in F$

Theorem 1.3.2: Consequences of the Field Axioms

- (a) If $x+y = x+z$, then $y = z$

Proof

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If $x+y = x$, then $y = 0$

Proof

$$\text{From (a), let } z = 0$$

- (c) If $x+y = 0$, then $y = -x$

Proof

$$\text{From (a), let } z = -x$$

- (d) $-(-x) = x$

Proof

$$\text{From (c), let } x = -x \text{ and } y = x$$

- (e) If $x \neq 0$ and $xy = xz$, then $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If $x \neq 0$ and $xy = x$, then $y = 1$

Proof

$$\text{From (e), let } z = 1$$

- (g) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$

Proof

$$\text{From (e), let } z = \frac{1}{x}$$

- (h) If
- $x \neq 0$
- , then
- $\frac{1}{1/x} = x$

Proof

From (g), let $x = \frac{1}{x}$ and $y = x$

- (i)
- $0x = 0$

Proof

Since $0x + 0x = (0+0)x = 0x = 0x + 0$, then $0x = 0$
--

- (j) If
- $x, y \neq 0$
- , then
- $xy \neq 0$

Proof

Suppose $xy = 0$, then $1 = \frac{1}{y} \frac{1}{x} xy = \frac{1}{y} \frac{1}{x} 0 = 0$. $0 = 1$ is a contradiction.

- (k)
- $(-x)y = -(xy) = x(-y)$

Proof

$xy + (-x)y = (x+(-x))y = 0y = 0$. Then by part (c), $(-x)y = -(xy)$. $xy + x(-y) = x(y+(-y)) = x0 = 0$. Then by part (c), $x(-y) = -(xy)$.
--

- (l)
- $(-x)(-y) = xy$

Proof

By part (k), then $(-x)(-y) = -[x(-y)] = -[-(xy)]$. By part (d), $-[-(xy)] = xy$.

1.4 Ordered Fields

Definition 1.4.1: Ordered Field

An ordered field F is a field F which is also an ordered set for all $x, y, z \in F$.

- If $y < z$, then $y+x < z+x$
- If $x, y > 0$, then $xy > 0$

\mathbb{Q}, \mathbb{R} are ordered fields, but \mathbb{C} is not an ordered field since $i^2 = -1 \not> 0$.

Theorem 1.4.2: Properties of the Ordered Field

- (a) If
- $x > 0$
- , then
- $-x < 0$
- and vice versa

Proof

$-x = -x + 0 < -x + x = 0$

- (b) If
- $x > 0$
- and
- $y < z$
- , then
- $xy < xz$

Proof

Since $z-y > 0$, then $0 < x(z-y) = xz - xy$

- (c) If
- $x < 0$
- and
- $y < z$
- , then
- $xy > xz$

Proof

Since $-x > 0$ and $z-y > 0$, then $0 < -x(z-y) = xy - xz$

- (d) If
- $x \neq 0$
- ,
- $x^2 > 0$

Proof

If $x > 0 \Rightarrow x^2 = x \cdot x > 0$. If $x < 0 \Rightarrow (-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 > 0$
--

- (e) If
- $0 < x < y$
- , then
- $0 < 1/y < 1/x$

Proof

$(\frac{1}{y})y = 1 > 0$ so $\frac{1}{y} > 0$. Since $x < y$, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$.
--

Theorem 1.4.3: \mathbb{R} is an ordered field

There exists a unique ordered field \mathbb{R} with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$ so \mathbb{Q} is also an ordered field.

Proof

The proof in Day 5 is a construction of \mathbb{R} by defining a specific order $<$.

Theorem 1.4.4: \mathbb{Q} is dense in \mathbb{R}

- (a) **Archimedean Property:** For $x, y \in \mathbb{R}$, if $x > 0$, there is $n \in \mathbb{Z}$ where $nx > y$.

Proof

Fix $x > 0$. Let $A = \{ nx : n = 1, 2, \dots \}$. Suppose there is a y where $nx \leq y$. Then, A is nonempty and bounded from above by y . By the least upper bound property of \mathbb{R} , $\alpha = \sup(A)$ exists in \mathbb{R} . Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$. So $\alpha - x$ is not an upper bound of A . So there is a $mx \in A$ such that $mx > \alpha - x$. Then $\alpha < (m+1)x$, but $(m+1)x \in A$ contradicting α is an upper bound for A .

- (b) **\mathbb{Q} is dense in \mathbb{R} :** For $x, y \in \mathbb{R}$, if $x < y$, there is a $p \in \mathbb{Q}$ where $x < p < y$.

Proof

Since $x < y$, then $y - x > 0$. Then by the Archimedean Property, there exists $n \in \mathbb{Z}$ such that $n(y - x) > 1$. Thus, $ny > nx + 1 > nx$. Since there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$, then $m > nx \geq m-1$ so $nx + 1 \geq m > nx$. Since $ny > nx + 1 \geq m > nx$, then $y > m/n > x$.

2 Roots, Complex Field, & Euclidean Spaces

2.1 nth Root

Theorem 2.1.1: nth Root

- (a) If $0 < t \leq 1$, then $t^n \leq t$.

Proof

Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
 Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
 Applying the process n times, then $t^n \leq t$.

- (b) If $t \geq 1$, then $t^n \geq t$.

Proof

Since $0 < 1 \leq t$, then $t \leq t^2$.
 Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
 Applying the process n times, $t \leq t^n$.

- (c) If $0 < s < t$, then $s^n < t^n$.

Proof

$$\underbrace{s \cdot s \cdot \dots \cdot s}_n < t \cdot s \cdot \dots \cdot s < t \cdot t \cdot \dots \cdot s < \dots < \underbrace{t \cdot \dots \cdot t}_n$$

Theorem 2.1.2: $y^n = x$ has a unique y

Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = y_1^n < y_2^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$.

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < t_1^n < t_1 < x$ so $t_1 \in A$.

A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

So $x+1$ is an upper bound of A .

Thus by the least upper bound property, $y = \sup(A)$ exists.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

*** (Not an upper bound of A if $<$ and not a least upper bound of A if $>$)***

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\beta^{n-1} + \beta^{n-2}\alpha^1 + \dots + \alpha^{n-1})}_{\substack{\beta^{n-1} < \beta^{n-1} < \beta^{n-1}}} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$.

From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.

$$\text{Consider } t \geq y-k, \text{ then: } y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$$

Thus, $t^n > x$ so $t \notin A$.

Thus, $y-k$ is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 2.1.3: n-th root of product = product of n-th root

If $a, b > 0$ and $n \in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

Proof

Let $A = a^{\frac{1}{n}}$, $B = b^{\frac{1}{n}}$. By **theorem 2.1.2**, since A is a root for $y_1^n = a$, then $A^n = a$. Similarly, B is a solution of $y_2^n = b$ so $B^n = b$. Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 \dots A_n B_1 B_2 \dots B_n \\ &= A_1 A_2 \dots B_1 A_n B_2 \dots B_n = \dots = A_1 B_1 A_2 \dots A_{n-1} A_n B_2 \dots B_n \\ &= \dots = A_1 B_1 A_2 B_2 \dots A_n B_n = (AB)^n \end{aligned}$$

Then again by **theorem 2.1.2**, there is a unique $(ab)^{\frac{1}{n}} = AB = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

2.2 Decimals**Definition 2.2.1: Decimals**

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.

Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x$

Let E be the set of d_k for $k = 0, 1, \dots, \infty$. Then, $x = \sup(E)$.

2.3 Extended Reals**Definition 2.3.1: Extended Reals**

The extended real number system consist of \mathbb{R} and $\pm\infty$ such that:

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}$$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If $x > 0$, then $x(\pm\infty) = \pm\infty$. If $x < 0$, then $x(\pm\infty) = \mp\infty$

2.4 Complex Numbers**Definition 2.4.1: Complex Number**

A complex number is an ordered pair (a, b) where $a, b \in \mathbb{R}$. For $x, y \in \mathbb{C}$

- $x + y = (a, b) + (c, d) = (a + c, b + d)$
- $xy = (a, b)(c, d) = (ac - bd, ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)^{-1}(a, -b)$

Thus, the axioms form a field where $(0, 0) = 0$ and $(1, 0) = 1$ and $(0, 1) = i$.

Theorem 2.4.2: Imaginary i and Form a + bi

Let $i = (0, 1)$. Then, $i^2 = -1$.

Then, $(a, b) = a + bi$

Proof

$$\begin{aligned} i^2 &= (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1 \\ (a, b) &= (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi \end{aligned}$$

Definition 2.4.3: Conjugate

Let conjugate: $\bar{z} = a - bi$ where $\text{Re}(z) = a$, $\text{Im}(z) = b$.

Let $z = (a, b)$ and $w = (c, d)$:

(a) $\overline{z + w} = \bar{z} + \bar{w}$

Proof

$$\overline{z + w} = \overline{(a + c, b + d)} = (a + c, -b - d) = (a, -b) + (c, -d) = \bar{z} + \bar{w}$$

(b) $\overline{zw} = \bar{z} \bar{w}$

Proof

$$\overline{zw} = \overline{(ac - bd, ad + bc)} = (ac - bd, -ad - bc) = (a, -b)(c, -d) = \bar{z} \bar{w}$$

(c) $z + \bar{z} = 2 \text{Re}(z)$ $z - \bar{z} = 2i \text{Im}(z)$

Proof

$$\begin{aligned} z + \bar{z} &= (a, b) + (a, -b) = (2a, 0) = 2 \text{Re}(z) \\ z - \bar{z} &= (a, b) - (a, -b) = (0, 2b) = (0, 2)b = 2i \text{Im}(z) \end{aligned}$$

(d) $z\bar{z} \geq 0$

Proof

$$z\bar{z} = (a, b)(a, -b) = (a^2 + b^2, -ab + ab) = (a^2 + b^2, 0) = a^2 + b^2 \geq 0$$

Definition 2.4.4: Absolute Value

Let absolute value: $|z| = \sqrt{z\bar{z}}$

Let $z = (a, b)$ and $w = (c, d)$:

(a) If $z \neq 0$, then $|z| > 0$.

Proof

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0 \text{ where } |z| = 0 \text{ only if } a, b = 0 \text{ so only if } z = (0, 0).$$

(b) $|\bar{z}| = |z|$

Proof

$$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

(c) $|zw| = |z| |w|$

Proof

$$\begin{aligned} |zw| &= |(ac - bd, ad + bc)| = \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z| |w| \end{aligned}$$

(d) $|\text{Re}(z)| \leq |z|$

Proof

$$|\text{Re}(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

(e) $|z + w| \leq |z| + |w|$

Proof

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + |w|^2 + 2 \text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

2.5 Euclidean Spaces

Definition 2.5.1: Euclidean Spaces

For each positive integer k , let \mathbb{R}^k be the set of all ordered k -tuples:

$$x = (x_1, \dots, x_k) \quad \text{for each } x_i \in \mathbb{R}$$

with the properties:

- $x+y = (x_1 + y_1, \dots, x_k + y_k) \in \mathbb{R}^k$
- $cx = (cx_1, \dots, cx_k) \in \mathbb{R}^k$

So, \mathbb{R}^n has a vector space structure. Similarly, for \mathbb{C}^n .

Definition 2.5.2: Inner Product for \mathbb{R}^k

$$x \cdot y = x_1 y_1 + \dots + x_k y_k \in \mathbb{R}$$

Definition 2.5.3: Norm

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^k x_i^2}$$

Definition 2.5.4: Extension to \mathbb{C}^k

For $z, w \in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + \dots + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + \dots + z_k \overline{z_k} = |z_1|^2 + \dots + |z_k|^2 = |z|^2$

2.6 Cauchy-Schwarz

Theorem 2.6.1: Cauchy-Schwarz

If $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $b_1, \dots, b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^n \alpha_j \overline{b_j}|^2 \leq \sum_{j=1}^n |\alpha_j|^2 \sum_{j=1}^n |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j \overline{b_j}$.

If $B = 0$, then $b_1 = \dots = b_n = 0$. Thus, $0 \leq A(0)$ holds true.

Suppose $B > 0$. Then:

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) \\ &= \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B \overline{C} a_j \overline{b_j} - B C \overline{a_j} b_j + C \overline{C} b_j \overline{b_j} \\ &= B^2 \sum |a_j|^2 - B \overline{C} \sum a_j \overline{b_j} - B C \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B \overline{C} C - B C \overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

Since $|Ba_j - Cb_j|^2 \geq 0$, then $B(AB - |C|^2) \geq 0$.

Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Corollary 2.6.2: $|z \cdot w| \leq |z||w|$

Since $|z_i|^2 = z_i \overline{z_i}$, then $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$$|z \cdot w|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2 |w|^2$$

Thus, $|z \cdot w| \leq |z||w|$.

Theorem 2.6.3: Properties of \mathbb{R}^k

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

- (a) $|x| \geq 0$ where $|x| = 0$ only if $x = 0$

Proof

$$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = \dots = x_k = 0$$

- (b) $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2} \sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$$

- (c) $|x + y| \leq |x| + |y|$

Proof

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

- (d) $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

3 Construction of \mathbb{R}

There exists an ordered field \mathbb{R} which has the least upper bound property.
Also, \mathbb{R} contains \mathbb{Q} as a subfield.

Definition 5.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$
- If $p \in \alpha$ and $q \in \mathbb{Q} < p$, then $q \in \alpha$
- If $p \in \alpha$, then $p < r \in \mathbb{Q}$ for some $r \in \alpha$

Proposition 5.2: Order of $\mathbb{R} \rightarrow$ ordered set \mathbb{R}

Define $\alpha < \beta$ if α is a proper subset of β .

- If $\alpha \not\subseteq \beta$, then β is not a subset of α .
Then there is a $p \in \beta$ such that $p \notin \alpha$.
Then for any $q \in \alpha$, $q < p$ and thus, $q \in \beta$.
Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.
- If $\alpha \not\subseteq \beta$ and $\alpha \not\supset \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
If $\alpha \neq \beta$, there are p, q such that $p \in \alpha$, but $p \notin \beta$ and $q \in \beta$, but $q \notin \alpha$.
But if $p \notin \beta$, then for any $b \in \beta$, $b < p$. Thus, $q < p$.
Similarly, if $q \notin \alpha$, then for any $a \in \alpha$, $a < q$. Thus, $p < q$.
Thus, there is a contradiction since $p > q$ and $p < q$ so $\alpha = \beta$.
- If $\alpha \not\subseteq \beta$, then α is not a subset of β .
Then there is a $p \in \alpha$ such that $p \notin \beta$.
Then for any $q \in \beta$, $q < p$ and thus, $q \in \alpha$.
Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.
- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ , then α is a proper subset of γ . Thus, $\alpha < \gamma$.

Thus, \mathbb{R} is an ordered set with such an order $<$.

Proposition 5.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let $A \subset \mathbb{R}$ and β be an upper bound for A . Let γ be the union of all $\alpha \in A$.
Thus, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$.

γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in A$ where α_0 is nonempty.
Since α_0 is nonempty, then γ is nonempty.
Since every $\alpha \in A$ is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.
- If $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$ so $q \in A$.
- If $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$. Thus, there is a $r \in \alpha_1$ such that $r > p$ so $r \in \gamma$. Thus, there is a $r \in \gamma$ where $r > p$.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in A \subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A .

Suppose $\delta < \gamma$. Then there is a $s \in \gamma$ such that $s \notin \delta$. Since $s \in \gamma$, then there is a $\alpha \in A$ such that $s \in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A .

Thus, $\gamma = \sup(A)$.

Proposition 5.4: \mathbb{R} is a field

If $\alpha, \beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Also, let 0^* be the set of all negative rational numbers which is a cut since:

- 0^* is nonempty and $0^* \neq \mathbb{Q}$
- If $p \in 0^*$, then any $q \in \mathbb{Q} < p$ is a negative rational and thus, $q \in 0^*$.
- Since \mathbb{Q} is dense in \mathbb{R} , then for any $p \in 0^*$, there is a $r \in \mathbb{Q}$ where $p < r < 0$ so r is a negative rational so $r \in 0^*$.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take $r' \notin \alpha, s' \notin \beta$, then $r' + s' > r + s$ for $r \in \alpha, s \in \beta$. Thus, $r' + s' \notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If $p \in \alpha + \beta$, then $p = r + s$ where $r \in \alpha$ and $s \in \beta$.
If $q < p$, then $q - s < p - s = (r + s) - s = r$ so $q - s \in \alpha$.
Since $q - s \in \alpha$ and $s \in \beta$, then $(q - s) + s = q \in \alpha + \beta$.
- If $r \in \alpha$, then there is a $t \in \alpha$ such that $t > r$. Let $s \in \beta$.
Thus, for any $p = r + s \in \alpha + \beta$, there is a $q = t + s \in \alpha + \beta$ such that $p = r + s < t + s = q$.

$\alpha + \beta = \beta + \alpha$

If $p = r + s \in \alpha + \beta$ where $r \in \alpha, s \in \beta$, then $s + r = r + s = p \in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If $r \in \alpha, s \in \beta, t \in \gamma$, then $r + s + t = (r + s) + t \in (\alpha + \beta) + \gamma$ and $r + s + t = r + (s + t) \in \alpha + (\beta + \gamma)$.

$\alpha + 0^* = \alpha$

If $r \in \alpha, s \in 0^*$, then $r + s < r$. Thus, $r + s \in \alpha$. Thus, $\alpha + 0^* \subset \alpha$.

If $p \in \alpha$, there is a $r \in \alpha$ where $r > p$. Thus, $p - r \in 0^*$.

Since $p = r + (p - r) \in \alpha + 0^*$, then $\alpha \subset \alpha + 0^*$. Thus, $\alpha + 0^* = \alpha$.

There is a $-\alpha$ such that $\alpha + (-\alpha) = 0^*$

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is $r > 0$ such that $-p - r \notin \alpha$.

$\beta \in \mathbb{R}$ since β is a cut:

- If $s \notin \alpha$ and $p = -s - 1$, then $-p - 1 \notin \alpha$. Thus, $p \in \beta$ so β is nonempty. If $q \in \alpha$, then $-q \notin \beta$ so $\beta \neq \mathbb{R}$.
- If $p \in \beta$, let $r > 0$ so $-p - r \notin \alpha$. If $q < p$, then $-q - r > -p - r$ and thus, $-q - r \notin \alpha$ so $q \in \beta$.
- If $p \in \beta$, let $t = p + (r/2)$. Then $-t - (r/2) = -p - r \notin \alpha$ and thus, $t \in \beta$ where $p < t$.

If $r \in \alpha, s \in \beta$, then $s \notin \alpha$. Thus, $r < -s$ so $r + s < 0$. Thus, $\alpha + \beta \subset 0^*$.

Let $v \in 0^*$ and let $w = -v/2$ so $w > 0$.

Thus, by the Archimedean property, there is an integer n such that $nw \in \alpha$, but $(n+1)w \notin \alpha$. Let $p = -(n+2)w$ so $-p - w = (n+1)w \notin \alpha$ so $p \in \beta$.

Then, $v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p \in \alpha + \beta$.

Since $v \in 0^*$, then $0^* \subset \alpha + \beta$. Thus, $\alpha + \beta = 0^*$. Then, let $-\alpha = \beta$.

Thus, if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Thus, if $\alpha > 0^*$, then $-\alpha = -\alpha + 0^* < -\alpha + \alpha = 0^*$ so $-\alpha < 0^*$.

If $\alpha, \beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that $p \leq rs$ for $r \in \alpha, s \in \beta$.

Define 1^* as the set of all $q < 1$. Then all multiplication axioms holds with similar proofs as addition. Also, note since $\alpha, \beta > 0^*$, then $\alpha\beta > 0^*$.

Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were $\alpha, \beta, \gamma >, < 0^*$.

4 Cardinality

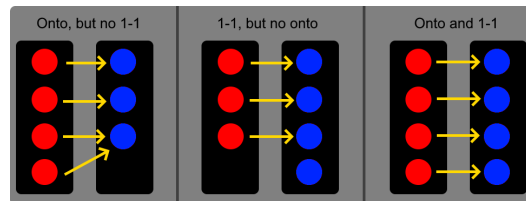
4.1 Cardinality

Definition 4.1.1: Onto and 1-1 Mapping

Suppose for every $x \in A$, there is an associated $f(x) \in B$.

Then f maps A into $B = f: A \rightarrow B$.

- If $f(A) = B$, then f maps A onto B .
- If for each $y \in B$, $f^{-1}(y)$ consist of at most one $x \in A$ where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B .



Definition 4.1.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function $f: A \rightarrow B$. (1-1 correspondence between A and B) Then, $A \sim B$.

If $f: A \rightarrow B$ is 1-1 and onto, then there is a $f^{-1}: B \rightarrow A$ that is 1-1 and onto.

Definition 4.1.3: Countability

- A is **finite** if $A \sim J_n = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$
- A is **infinite** if A is not finite
- A is **countably infinite** if $A \sim J = \mathbb{Z}_+$
- A is **uncountable** if A is not finite or countably infinite
- A is **at most countable** if A is finite or countably infinite

Example

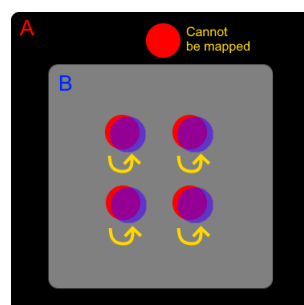
\mathbb{Z} is countably infinite

Proof

Let $f(n): \mathbb{Z}_+ \rightarrow \mathbb{Z} = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n-1}{2} & n \text{ is odd} \end{cases}$
 So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto -1$, $4 \mapsto 2$, $5 \mapsto -2$, etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 4.1.4: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A .



Theorem 4.1.5: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite

Proof

Let $E \subset A$ be an infinite subset. For every distinct $x_i \in A$, let $\{x_1, x_2, \dots\} \in A$.
 Let n_1 be smallest integer such that $x_{n_1} \in E$.
 Then let n_2 be the smallest integer where $n_2 > n_1$ such that $x_{n_2} \in E$.
 Repeat the process to create sequence $f(k) = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$.
 Thus, there is a 1-1 correspondence between E and \mathbb{Z}_+ so E is countably infinite.

**4.2 Set of Sets****Definition 4.2.1: Union and Intersection**

Let sets Ω, B be such that for each $x \in \Omega$, there is an associated $E_x \subset B$.

- $E = \cup_{x=1}^n E_x$ only if for every $x \in E$, $x \in E_x$ for at least one $x \in \Omega$.
- $P = \cap_{x=1}^n E_x$ only if for every $x \in P$, $x \in E_x$ for all $x \in \Omega$.

with properties:

- | | |
|---|---|
| (a) $A \cup B = B \cup A$ | (a) $A \cap B = B \cap A$ |
| (b) $(A \cup B) \cup C = A \cup (B \cup C)$ | (b) $(A \cap B) \cap C = A \cap (B \cap C)$ |
| (c) $A \subset A \cup B$ | (c) $(A \cap B) \subset A$ |
| (d) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$ | |

Proof

If $x \in A \cup B$, then $x \in A$ or/and $x \in B$.

- If $x \in A$, since $A \subset B$, then $x \in B$. Then, $(A \cup B) \subset B$.
- If $x \in B$, then immediately $(A \cup B) \subset B$.

If $x \in B$, then $x \in A \cup B$ so $B \subset (A \cup B)$. Thus, $A \cup B = B$.

If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, $(A \cap B) \subset A$.

If $x \in A$, since $A \subset B$, then $x \in B$ so $x \in A \cap B$. Thus, $A \subset (A \cap B)$.

Thus, $A \cap B = A$.

- (e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof

If $x \in A \cap (B \cup C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

- If $x \in B$, then $x \in (A \cap B)$ so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then $x \in (A \cap C)$ so $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

If $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$$(f) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

If $x \in A \cup (B \cap C)$, then $x \in A$ or/and $(x \in B \text{ and } x \in C)$.

- If $x \in A$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
- If $x \in B, C$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

If $x \in (A \cup B) \cap (A \cup C)$, then $x \in A$ or/and $(x \in B \text{ and } x \in C)$.

Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 4.2.2: Union of countably infinite sets is countably infinite

If E_1, E_2, \dots are countably infinite sets, then $S = \bigcup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

For each E_n , there is a sequence $\{x_{n1}, x_{n2}, \dots\}$. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence $S^* = \{x_{11}; x_{21}, x_{12}; x_{31}, x_{32}, x_{23}; \dots\}$.

Since $S^* \sim S$ so S is at most countable and S is infinite since E_1, E_2, \dots are infinite, then S cannot be finite and thus, countably infinite.

Alternative Proof

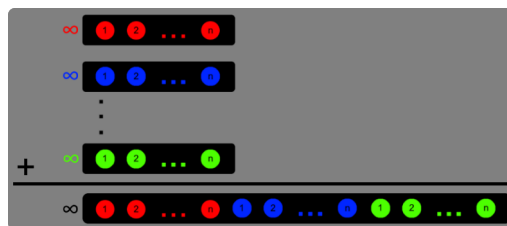
For each E_n , let set $\widetilde{E}_n = E_n - \bigcup_{m=1}^{\infty} E_m$ where $m \neq n$. Thus, $S = \bigcup_{n=1}^{\infty} \widetilde{E}_n$.

Since each E_n is countably infinite, there exists a 1-1 mapping $\delta_n: E_n \rightarrow \mathbb{Z}_+$.

Thus, for each \widetilde{E}_n , there is a 1-1 mapping $\delta_n: \widetilde{E}_n \rightarrow A \subset \mathbb{Z}_+$.

Let p_1, p_2, \dots be distinct primes. Since for $s \in S$, there exists a unique \widetilde{E}_i such that $s \in \widetilde{E}_i$, then let $f(s) = p_1^{\delta_1(s)} p_2^{\delta_2(s)} \dots$ where $p_k^{\delta_k(s)} = 1$ if $k \neq i$.

Then, by the Fundamental theorem of arithmetic, f maps s to a unique $z \in \mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable. Since any $E_n \subset S$ is countably infinite, then S cannot be finite and thus, S is countably infinite.



Theorem 4.2.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B_n be the set of all n -tuples (a_1, \dots, a_n) where $a_k \in A$. Then B_n is countably infinite.

Proof

The base case B_1 is countably infinite since $B_1 = A$.

Suppose B_{n-1} is countably infinite. Then for every $x \in B$:

$$x = (b, a) \quad b \in B_{n-1} \text{ and } a \in A$$

Since for every fixed b , $(b, a) \sim A$ and thus, countably infinite.

Since B is a set of countably infinite sets, then B_n is countably infinite.

Theorem 4.2.4: \mathbb{Q} is countable

The set of rational numbers, \mathbb{Q} , is countably infinite

Proof

Since elements of \mathbb{Q} are of form $\frac{a}{b}$ which is a 2-tuple, then by the **theorem 4.2.3**, \mathbb{Q} is countably infinite.

Alternative Proof

For every $x \in \mathbb{Q}$, let $x = (-1)^i \frac{p}{q}$ where $p, q \in \mathbb{Z}_+$.
 Let $f(x) = 2^i 3^p 5^q$. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to $E \subset \mathbb{Z}_+$.
 Thus, \mathbb{Q} is at most countable, but since $p, q \in \mathbb{Z}_+$, then \mathbb{Q} cannot be finite and thus, is countably infinite.

Example

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences s_1, s_2, \dots .
 Then construct a sequence s as follows:
 If the n -th digit in s_n is 1, then let the n -th digit of s be 0 and vice versa.
 Thus, s differs from every $s_n \in E$ so $s \notin E$.
 But, $s \in A$ so E is a proper subset of A .
 Thus, every countably infinite subset of A is a proper subset of A .
 If A is countably infinite, then A is a proper subset of A which is a contradiction.

5 Metric Spaces & Closed/Open

5.1 Metric Spaces

Definition 5.1.1: Metric Spaces

A set X is a metric space if for any $p, q \in X$, there is an associated $d(p, q) \in \mathbb{R}$ such that:

- $d(p, q) > 0$ if $p \neq q$
- $d(p, q) = 0$ if and only if $p = q$
- **Symmetry**: $d(p, q) = d(q, p)$
- **Triangle Inequality**: $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

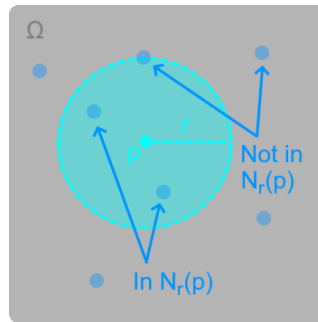
For euclidean spaces \mathbb{R}^k , $d(x, y) = |x - y|$ where $x, y \in \mathbb{R}^k$.

Definition 5.1.2: Types of Points and Sets

For metric space X and set $E \subset X$:

(a) Neighborhood

For $p \in X$ and $r > 0$, $N_r(p)$ is the set of all $q \in X$ where $d(q, p) < r$



(b) Limit Points and Closed Sets

Closed set E contain all $p \in X$ where every $N_r(p)$ contain a $q \neq p \in E$

• Limit Points

For point $p \in X$, every $N_r(p)$ contains a $q \neq p \in E$

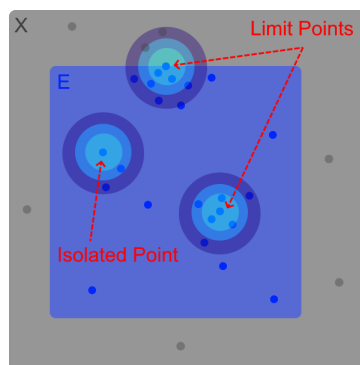
The set of all limit points of $E = E'$

• Isolated Points

If $p \in E$ is not a limit point of E

• Closed

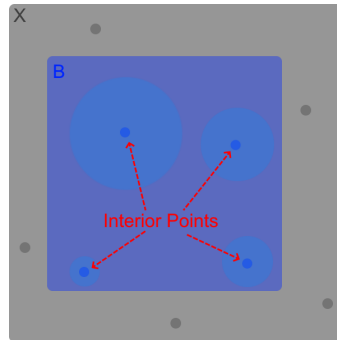
If every limit point p of E is a $p \in E$



(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset E$

- Interior Point
For $p \in X$, there is a $N_r(p) \subset E$
The set of all interior points = E°
- Open
If every $p \in E$ is an interior point of E



(d) More about Sets

- Bounded
If there is $M \in \mathbb{R}$, $q \in X$ such that $d(p, q) < M$ for all $p \in E$
- Complement
From E , E^c is the set of all $p \in X$ such that $p \notin E$
- Perfect
If E is closed and if every $p \in E$ is a limit point of E
- Dense
If every $p \in X$ is a limit point of E or/and $p \in E$
- Boundary Point
For $p \in X$, if every $N_r(p)$ contains a $x \in E$ and $y \in E^c$
The set of all boundary points = ∂E

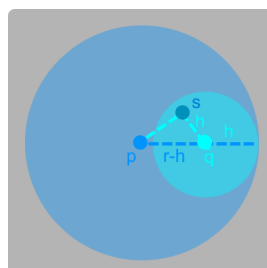
For a metric space X , $\{X, \emptyset\}$ are both open and closed.

Theorem 5.1.3: $N_r(p)$ is open

Every neighborhood is an open set

Proof

Let $q \in N_r(p)$. Then there is a $h > 0 \in \mathbb{R}$ such that $d(q, p) = r - h$.
Then for any $s \in N_h(q)$, $d(s, p) \leq d(s, q) + d(q, p) = h + (r - h) = r$.
Thus, for any $q \in N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

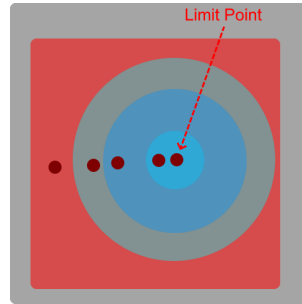


Theorem 5.1.4: If a set has a limit point, there are infinite $q \in E$ in $N_r(p)$

If p is a limit point of set E , then every $N_r(p)$ contains infinitely many $q \in E$.

Proof

Suppose there is $N_{r_1}(p)$ which contains finitely many $q = \{q_1, \dots, q_n\}$.
 Let $r = \min_{m \in [1, n]} d(p, q_m)$. Then $N_r(p)$ contains no $q \in E$ such that $q \neq p$.
 So, p is not a limit point of E which is a contradiction since p is a limit point of E .



Corollary 5.1.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since $\emptyset \in E$, all finite set must be closed.

Proof

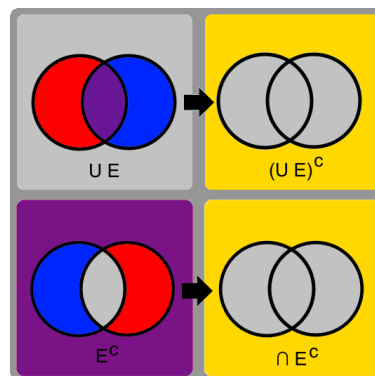
Let p be a limit point of finite set E . By **theorem 5.1.4**, then any $N_r(p)$ contain infinite $q \in E$ so E is an infinite set which is a contradiction since E is finite.
 So p cannot be limit point of E and thus, E has no limit points. Since finite set E contains all its limit points because there are no limit points, then E is closed.

Theorem 5.1.6: De Morgan's Laws

Let E_1, E_2, \dots be a collection of sets. Then, $(\cup E_x)^c = \cap (E_x^c)$.

Proof

If $p \in (\cup E_x)^c$, then $p \notin (\cup E_x)$.
 Thus, $p \notin E_x$ for any x so $p \in E_x^c$ for all x . Thus, $p \in \cap (E_x^c)$ so $(\cup E_x)^c \subset \cap (E_x^c)$.
 If $p \in \cap (E_x^c)$, then $p \in E_x^c$ for all x .
 Thus, $p \notin E_x$ for any x so $p \notin \cup E_x$. Thus, $p \in (\cup E_x)^c$ so $\cap (E_x^c) \subset (\cup E_x)^c$.
 Thus, $(\cup E_x)^c = \cap (E_x^c)$.



Theorem 5.1.7: Open set \rightarrow Closed complement

A set E is open if and only if E^c is closed

Proof

Suppose E is open. Let x be a limit point of E^c .

Then for every $r > 0$, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.

Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.

Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.

Since $x \notin E^c$, x is not a limit point of E^c . Then there exists a $r > 0$ such that any $p \in N_r(x)$ is not in E^c . Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E . Since any $x \in E$ is an interior point of E , then E is open.

Corollary 5.1.8: Closed set \rightarrow Open complement

A set F is closed if and only if F^c is open.

Proof

From **theorem 5.1.7**, let $E = F^c$

Theorem 5.1.9: Union open \rightarrow open and Intersection closed \rightarrow closed

- (a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup G_x$ is open.

Proof

If $p \in \cup G_x$, then $p \in G_x$ for at least one x . Let \bar{x} be such an x .

Since $G_{\bar{x}}$ is open, then p is an interior point of $G_{\bar{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\bar{x}} \subset \cup G_x$. So p is an interior point of $\cup G_x$.

Since any $p \in \cup G_x$ is an interior point, then $\cup G_x$ is open.

- (b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap F_x$ is closed.

Proof

By **theorem 5.1.7**, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup F_x^c$ is open.

Thus, again by **theorem 5.1.7**, $(\cup F_x^c)^c$ is closed.

By **theorem 5.1.6**, $(\cup F_x^c)^c = \cap (F_x^c)^c = \cap F_x$.

- (c) If G_1, \dots, G_n is a finite collection of open sets, then $\cap_{x=1}^n G_x$ is open.

Proof

If $p \in \cap_{x=1}^n G_x$, then $p \in G_x$ for all G_x for $x = \{1, 2, \dots, n\}$.

Since each G_x is open, then for any G_x , there is a $N_{r_x}(p) \subset G_x$.

Let $r = \min(r_1, r_2, \dots, r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x .

So, $N_r(p) \subset \cap_{x=1}^n G_x$ and thus, p is an interior point of $\cap_{x=1}^n G_x$ so $\cap_{x=1}^n G_x$ is open.

Infinite + Closed: $G_i = (-1/i, 1/i)$

Infinite + Open: $G_i = (-i, i)$

- (d) If F_1, \dots, F_n is a finite collection of closed sets, then $\cup_{x=1}^n F_x$ is closed.

Proof

By **theorem 5.1.7**, any F_x^c is open. Since F_1^c, \dots, F_n^c is a finite collection of open set, then by part (c), $\cap_{x=1}^n F_x^c$ is open.

Thus, again by **theorem 5.1.7**, $(\cap_{x=1}^n F_x^c)^c$ is closed.

By **theorem 5.1.6**, $(\cap_{x=1}^n F_x^c)^c = \cup_{x=1}^n (F_x^c)^c = \cup_{x=1}^n F_x$.

Infinite + Closed: $F_i = [-1/i, 1/i]$

Infinite + Open: $F_i = [1/i, \infty)$

Theorem 5.1.10: E' is closed

Let $E \subset X$. Then, $(E')' \subset E'$. Thus, E' is closed.

Proof

If $x \in (E')'$, then for every $N_{r_1}(x)$, there is a $y \neq x$ where $y \in E'$.
 Since $y \in E'$, then for every $N_{r_2}(y)$, there is a $z \neq y$ where $z \in E$.
 Let $r = r_1 + r_2$.
 Then for every $N_r(x)$, there exists a $z \neq x$ where $z \in E$. Thus, $x \in E'$ so $(E')' \subset E'$.

Theorem 5.1.11: E° is open

Let $E \subset X$. Then, E° is open.

Proof

If $p \in E^\circ$, there is a $r > 0$ such that $N_r(p) \subset E$.
 Then for $0 < s < r$, $N_s(p) \subset N_r(p)$ so any $q \in N_s(p)$ is $q \in E^\circ$.
 Since any $p \in E^\circ$ have a $N_s(p) \subset E^\circ$, then E° is open.

5.2 Intervals and Balls

Definition 5.2.1: Segments and Intervals

In \mathbb{R} , a **segment** is an open interval $(a,b) = \{ x \in \mathbb{R} : a < x < b \}$

In \mathbb{R} , a **interval** is a closed interval $[a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$

Definition 5.2.2: Open Balls

In \mathbb{R}^k , an **open ball** of radius $r > 0$ centered at p is:

$$N_r(p) = \{ x \in \mathbb{R}^k : |x - p| < r \} = \{ x \in \mathbb{R}^k : d(x,p) < r \}$$

A **closed ball** has $d(x,p) \leq r$.

Definition 5.2.3: Convex

$E \subset \mathbb{R}^k$ is **convex** if for all $x, y \in E$ and $t \in [0,1]$, $tx + (1-t)y \in E$.

Example

Balls in \mathbb{R}^k are convex

Let $x, y \in$ open ball $N_r(p)$. Let $z = tx + (1-t)y$ for $t \in [0,1]$.
 Since $|x - p| < r$ and $|y - p| < r$:

$$\begin{aligned} |z - p| &= |tx + (1-t)y - p| = |tx + (1-t)y - tp + (t-1)p| \\ &= |t(x-p) + (1-t)(y-p)| \leq t|x-p| + (1-t)|y-p| \\ &< tr + (1-t)r = r \end{aligned}$$

 Thus, $z \in N_r(p)$ so balls are convex. Same proof applies to closed balls.

Definition 5.2.4: Dense

$E \subset X$ is dense if every $x \in X$ is either in E or a limit point of E .

Example

Let $X = \mathbb{R}$. Then, $E = \mathbb{Q}$ is dense in \mathbb{R} .

Fix $x \in \mathbb{R}$ and $r > 0$. There is a $q \in \mathbb{Q}$ such that $x-r < q < x$. So for any $r > 0$ and $q \in \mathbb{Q}$, $q \neq x$ and $q \in N_r(x)$. Thus, every $x \in \mathbb{R}$ is a limit point of \mathbb{Q} .

6 Closure, Open Relative, & Compact

6.1 Closure

Definition 6.1.1: Closure

Let $E \subset$ metric space X and E' be the set of all limit points of E in X .

Then the closure of E : $\overline{E} = E \cup E'$

with the properties:

- (a) \overline{E} is closed

Proof

Suppose $x \in X$, but $x \notin \overline{E}$. Thus, $x \in \overline{E}^c$.

Thus, there is a $N_r(x) \subset \overline{E}^c$ since else there is always a $p \in N_r(x)$ where $p \in \overline{E}$ so x is a limit point of \overline{E} so $x \in \overline{E}$. Thus, \overline{E}^c is open so \overline{E} is closed by [theorem 5.1.7](#).

- (b) $E = \overline{E}$ if and only if E is closed

Proof

If $E = \overline{E}$, then by part (a), E is closed.

If E is closed, then $E' \subset E$ so $E = E \cup E' = \overline{E}$.

- (c) $\overline{E} \subset F$ for every closed $F \subset X$ such that $E \subset F$

Proof

If closed set F , then $F' \subset F$. Since $E \subset F$, then $E' \subset F' \subset F$ so $\overline{E} \subset F$.

Theorem 6.1.2: $\sup(E) \in \overline{E}$

Let non-empty set of real numbers, E , be bounded above. Let $y = \sup(E)$.

Then, $y \in \overline{E}$. Thus, $y \in E$ if E is closed and $y \notin E$ if E is open in \mathbb{R} .

Proof

If $y \in E$, then $y \in \overline{E}$. Suppose $y \notin E$.

For every $h > 0$, there exists a $x \in E$ such that $y-h < x < y$ otherwise $y-h$ is an upper bound for E which is a contradiction since $y = \sup(E)$.

Thus, y is a limit point of E so $y \in E'$.

If E is closed, then $y \in E$ since $y \in E'$. Also, $y \in \overline{E}$.

If E is open, then any $N_r(y) \not\subset E$ since $N_r(y)$ in \mathbb{R} must contain a $\gamma > y$ so $y \notin E^\circ$.

6.2 Open Relative

Definition 6.2.1: Open Relative

Suppose $E \subset Y \subset$ metric space X .

Then E is open relative to Y if for each $p \in E$:

There is an $r > 0$ such that for any $q \in Y$ where $d(q,p) < r$, then $q \in E$.

Theorem 6.2.2: E is open relative to $Y \subset X$ if $E = Y \cap G$ and G is open in X

Suppose $E \subset Y \subset X$.

E is open relative to Y if and only if $E = Y \cap G$ for some open $G \subset X$.

Proof

Suppose E is open relative to Y .

Then for each $p \in E$, there is a $r_p > 0$ such that for any $q \in Y$ where $d(p, q) < r_p$, then $q \in E$.

Since $Y \subset X$, let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$ and define $G = \bigcup_{p \in E} V_p$. Since V_p is open by **theorem 5.1.3**, then by **theorem 5.1.9a**, open $G \subset X$.

Since $p \in V_p$ for all $p \in E$, then $E \subset G \cap Y$. Also, by construction, then $V_p \cap Y \subset E$ so $G \cap Y \subset E$. Thus, $E = Y \cap G$.

If G is open in X and $E = G \cap Y$, then every $p \in E$ has a $V_p \subset G$.

Then, $V_p \cap Y \subset G \cap Y = E$ so E is open relative to Y .

6.3 Compact Sets

Definition 6.3.1: Open Cover

An open cover of set $E \subset X$ is a collection of open $G_1, G_2, \dots \subset X$ such that $E \subset \bigcup G_i$.

Definition 6.3.2: Compact

$K \subset X$ is compact if every open cover of K contains a finite subcover.

If G_1, G_2, \dots is an open cover of K , then $K \subset \bigcup_{i=1}^n G_i$ for some n .

Theorem 6.3.3: A compact set is compact in every metric space

Suppose $K \subset Y \subset X$.

Then K is compact relative to X if and only if K is compact relative to Y .

Proof

Suppose K is compact relative to X .

Let V_1, V_2, \dots be sets open relative to Y such that $K \subset \bigcup V_x$. Then by **theorem 6.2.2** for each V_x , there is a G_x open relative to X where $V_x = Y \cap G_x$.

Since K is compact relative to X , then there is a n such that $K \subset G_{x_1} \cup \dots \cup G_{x_n}$.

Thus, $K = K \cap Y \subset (\bigcup_{i=1}^n G_{x_i}) \cap Y = (\bigcup_{i=1}^n (G_{x_i} \cap Y)) = \bigcup_{i=1}^n V_{x_i}$.

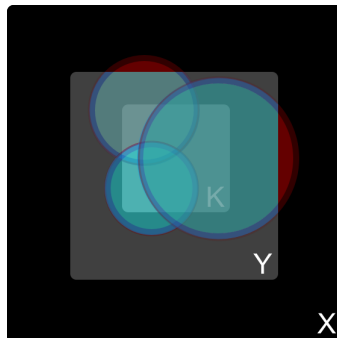
Since there are open V_{x_1}, \dots, V_{x_n} where $K \subset \bigcup_{i=1}^n V_{x_i}$ so K is compact relative to Y .

Suppose K is compact relative to Y .

Let open $G_1, G_2, \dots \subset X$ such that $X \subset \bigcup G_x$. For each G_x , let $V_x = Y \cap G_x \subset Y$.

Since K is compact relative to Y , there is a n such that $K \subset \bigcup_{i=1}^n V_{x_i}$.

Thus, $K \subset \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (Y \cap G_{x_i}) \subset \bigcup_{i=1}^n G_{x_i}$ so K is compact relative to X .

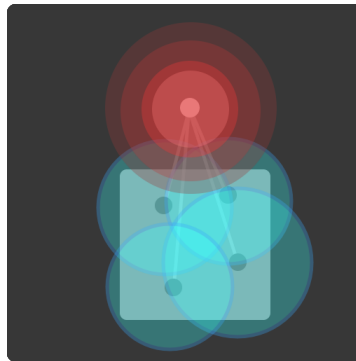


Theorem 6.3.4: A compact set is closed

Compact subsets of metric spaces are closed

Proof

Let compact $K \subset X$. Suppose $p \in X$, but $p \notin K$ so $p \in K^c$.
 If $q \in K$, let W_q be a neighborhood of q with $r < \frac{1}{2}d(p,q)$. Let $V_{p,q}$ be a neighborhood of p with $r < \frac{1}{2}d(p,q)$. Since K is compact, then there are finite points q_1, \dots, q_n such that $K \subset W$ where $W = W_{q_1} \cup \dots \cup W_{q_n}$.
 Let $V = V_{p,q_1} \cap \dots \cap V_{p,q_n}$, then $K \cap V \subset W \cap V = \emptyset$ so $V \subset K^c$.
 Since there is a neighborhood V for $p \in K^c$ where $V \subset K^c$, then every $p \in K^c$ is an interior point so K^c is open. Then by [theorem 5.1.7](#), K is closed.

**Theorem 6.3.5: If closed $E \subset$ compact set K , E is compact**

Closed subsets of compact sets are compact

Proof

Suppose $F \subset K \subset X$ where F is closed relative to X and K is compact.
 Let V_1, V_2, \dots be an open cover for F . Let open set F^c be all $k \in K$ where $k \notin F$.
 $K = F \cup F^c \subset V_1 \cup V_2 \cup \dots \cup F^c$
 Thus, $V_1 \cup V_2 \cup \dots \cup F^c$ is an open cover for K .
 Since K is compact, there is a finite subcover Ω that covers K and thus, finite subcover Ω covers $F \cup F^c$.
 Remove F^c from Ω . Since finite subcover $\Omega - F^c$ covers F , then F is compact.

Corollary 6.3.6: Closed $F \cap$ compact $K =$ compact

If F is closed and K is compact, then $F \cap K$ is compact

Proof

Since K is compact, then K is closed by [theorem 6.3.4](#).
 Then, by [5.1.9b](#), $F \cap K$ is closed.
 Since $F \cap K \subset K$, then by [theorem 6.3.5](#), $F \cap K$ is compact.

Theorem 6.3.7: Nonempty $\cap_{i=1}^n K_i \rightarrow$ nonempty $\cap K_i$

For compact sets $K_1, K_2, \dots \subset X$ where any intersection of finite K_i is nonempty, then $\cap K_i$ is nonempty

Proof

Fix K_1 . If there is a $k \in K_1$ where $k \in K_i$ for all i , then $k \in \cap K_i$ so $\cap K_i \neq \emptyset$.
 Suppose for every $k \in K_1$, $k \notin K_i$ for some i .
 Then for every $k \in K_1$, there is a K_i such that $k \notin K_i$ so $k \in K_i^c$.
 Thus, K_2^c, K_3^c, \dots form an open cover for K_1 . Since K_1 is compact, there is a n where $K_1 \subset K_{i_1}^c \cup \dots \cup K_{i_n}^c$. But then, $K_1 \cap K_{i_1} \cap \dots \cap K_{i_n} = \emptyset$ which is a contradiction.

Corollary 6.3.8: Nonempty K_i where $K_{i+1} \subset K_i \rightarrow$ nonempty $\cap K_i$

If K_1, K_2, \dots is a sequence of nonempty compact sets such that $K_{i+1} \subset K_i$, then $\cap K_i$ is nonempty

Proof

Since each K_i is nonempty and if $i_1 < \dots < i_n$, then $K_{i_1} \cap \dots \cap K_{i_n} = K_{i_n}$ is nonempty, then by **theorem 6.3.7**, $\cap K_i$ is nonempty.

Theorem 6.3.9: Nonempty intervals I_n where $I_{n+1} \subset I_n \rightarrow$ nonempty $\cap I_n$

If I_1, I_2, \dots is a sequence of intervals in \mathbb{R}^1 such that $I_{n+1} \subset I_n$, then $\cap I_n$ is nonempty.

Proof

Let $I_n = [a_n, b_n]$ and thus, each I_n is nonempty. If $n_1 < \dots < n_m$, then $I_{n_1} \cap \dots \cap I_{n_m} = [a_{n_m}, b_{n_m}]$ is nonempty. Thus, by **theorem 6.3.7**, $\cap I_n$ is nonempty.

Theorem 6.3.10: $p \in E'$ exists if infinite $E \subset$ compact K

If E is an infinite subset of compact set K , then E has a limit point in K

Proof

If no $p \in K$ is a $p \in E'$, then each p would have a neighborhood V_p contains at most $p \in E$ if $p \in E$. Thus, there is no finite subcover that covers E and thus, there is no finite subcover that covers K since $E \subset K$ which contradicts K is compact.

Definition 6.3.11: K-cells

The set of all $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$

Theorem 6.3.12: K-cells are compact

Every k -cell is compact

Proof

Let k -cell I consists of all $x = (x_1, \dots, x_k)$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.
 Let $\delta = \sqrt{\sum_{i=1}^k (b_i - a_i)^2}$. Thus, $|x - y| \leq \delta$ for $x, y \in I$.
 Suppose there exists an open cover G_1, G_2, \dots of I which contain no finite subcover.
 Let $c_i = \frac{a_i + b_i}{2}$. Then each interval splits into $[a_i, c_i]$ and $[c_i, b_i]$ for $i \in [1, k]$ so there now exists 2^k k -cells Q_i whose union is I .
 At least one Q_i cannot be covered else I would be covered. Then subdivide Q_i as before and repeating the process so $Q_{i+1} \subset Q_i$ and each are not covered.
 However, there is a point $x^* \in Q_{i_j}$ for all j such that $N_r(x^*) \subset G$ so Q_{i_1} is covered which is a contradiction.

Theorem 6.3.13: Heine-Borel Theorem

If a set $E \subset \mathbb{R}^k$ has one of the three properties, then it has the other two:

- (a) E is closed and bounded
- (b) E is compact
- (c) Every infinite subset of E has a limit point in E

Proof

Suppose E is closed and bounded.

Then there exists a $M \in \mathbb{R}$ and $q \in \mathbb{R}^k$ such that $d(p, q) < M$ for all $p \in E$.

Thus, there is a k -cell $K = [-M + q_1, q_1 + M] \times \dots \times [-M + q_k, q_k + M]$ such that $E \subset K$. Then by **theorem 6.3.12**, K is compact and thus by **theorem 6.3.5**, E is compact so (a) \rightarrow (b).

Then by **theorem 6.3.10**, any infinite subset of E has a limit point in E so (b) \rightarrow (c). Suppose E is not bounded.

Then there exists $p \in E$ such that $d(p, q) > M$ for any $M \in \mathbb{R}$ and $q \in \mathbb{R}^k$.

Let $S \subset E$ be such points p .

Then S is infinite else there is a maximal p and thus, p is bounded. Thus, S is infinite and contains no limit points in E since any $d(p_1, p_2) > M$ which contradicts that every infinite subset of E has a limit point in E . Thus, E is bounded.

Suppose E is not closed.

Then there exists a $p \in E'$, but $p \notin E$. Since p is a limit point, then there is a $q \in E$ such that $\frac{1}{n+1} < d(q, p) < \frac{1}{n}$ for $n = \{1, 2, \dots\}$.

Let $S \subset E$ be such points q .

Thus, p is the only limit point of S since for $r < \frac{1}{n}$, any $N_r(q_i)$ contains no points of S other than q_i since $d(q_i, q_j) > \frac{1}{n}$ for any $q_1, q_2 \in S$.

Thus, S is infinite, but the only $p \in S'$ is $p \notin E$ which contradicts that every infinite subset of E has a limit point in E . Thus, E is closed. So, (c) \rightarrow (a).

Theorem 6.3.14: Weierstrass Theorem

Every bounded infinite set $E \subset \mathbb{R}^k$ has a limit point in \mathbb{R}^k .

Proof

Since E is bounded, then there exists a k -cell K such that $E \subset K$. Since K is compact, then by **theorem 6.3.10**, E has a limit point in K and thus, in \mathbb{R}^k .

7 Perfect and Connected Sets

7.1 Perfect Sets

Definition 7.1.1: Perfect Set

$E \subset X$ is perfect if E is closed and if every $p \in E$ is $p \in E'$

Theorem 7.1.2: Perfect sets are uncountable

Let P be a nonempty perfect set in \mathbb{R}^k . Then, P is uncountable.

Proof

Since P has limit points, then by [theorem 5.1.4](#), P is infinite.

Suppose P is countable. Then let $x_1, x_2, \dots \in P$.

Let V_i be a neighborhood of x_i where $y \in V_i$ for any $y \in \mathbb{R}^k$ such that $|y - x_i| < r$.

Thus, the $\overline{V_i}$ is the set of all $y \in \mathbb{R}^k$ such that $|y - x_i| \leq r$.

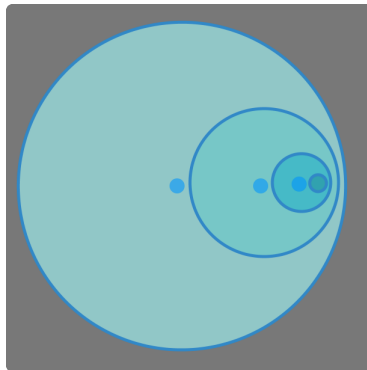
Since every x_i are limit points, then any $V_i \cap P$ is not empty where there is a V_{i+1}

(a) $\overline{V_{i+1}} \subset V_i$

(b) $x_i \notin \overline{V_{i+1}}$

(c) $V_{i+1} \cap P$ is nonempty

Let $K_i = \overline{V_i} \cap P$. Since $\overline{V_i}$ is closed and bounded, then by [theorem 6.3.11](#), $\overline{V_i}$ is compact. Since $x_i \notin K_{i+1}$, then no $x_i \in P$ is $x_i \in \cap K_i$. Since $K_n \subset P$, then $\cap K_i$ is empty which contradicts [corollary 6.3.8](#) since each K_i is nonempty and $K_{i+1} \subset K_i$.



Corollary 7.1.3: \mathbb{R} is not countable

Every interval $[a, b]$ is uncountable. Thus, \mathbb{R} is uncountable.

Proof

Since $[a, b]$ is closed and every $p \in [a, b]$ is a limit point, then nonempty set $[a, b]$ is perfect. Thus, by [theorem 7.1.2](#), $[a, b]$ is uncountable.

Definition 7.1.4: Cantor Set

There exists perfect segments in \mathbb{R}^1 which contain no segment.

Let $E_0 = [0,1]$.

For E_1 , remove $(\frac{1}{3}, \frac{2}{3})$. Thus, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

For E_2 , remove $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. Thus, $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

Continuing such a sequence, the set of compact sets E_n are such that:

$$(a) \ E_{n+1} \subset E_n$$

$$(b) \ E_n \text{ is the union of } 2^n \text{ intervals each of length } 3^{-n}.$$

$P = \cap E_n$ is called the Cantor set. P is compact and nonempty.

Thus, any segment of form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ where $k, m \in \mathbb{Z}_+$ has no points in common with P . Since any segment (a, b) contain a segment of such a form since $3^{-m} < \frac{b-a}{6}$, then P contains no segment.

Let $x \in P$ and segment S contain x . Let I_n be an interval of E_n containing x . Then choose a large enough n so $I_n \subset S$.

Let x_n be an endpoint of I_n where $x_n \neq x$ and thus, x is a limit point. Since P is closed and every $p \in P$ is $p \in P'$, then P is perfect.

7.2 Connected Sets**Definition 7.2.1: Connected Set**

$A, B \subset X$ are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.

$E \subset X$ is connected if E is not the union of two nonempty separated sets.

Separated sets are disjoint, but disjoint sets need not be separated.

Theorem 7.2.2: All points between points in connected sets exists

$E \subset \mathbb{R}^1$ is connected if and only if:

If $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof

If there exists $x, y \in E$ and $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where $A_z = E \cap (-\infty, z)$ and $B_z = E \cap (z, \infty)$.

Since $x \in A_z$ and $y \in B_z$, then A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, then A_z and B_z are separated. Thus, E is not connected.

Suppose E is not connected. Then, there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$ where $x < y$. Let $z = \sup(A \cap [x, y])$.

Since, $z \in \overline{A}$ so $z \notin B$, then $x \leq z < y$. If $z \notin A$, then $x < z < y$ so $z \notin E$.

If $z \in A$, then $z \notin \overline{B}$ and thus, there exists a z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then, $x < z_1 < y$ so $z_1 \notin E$.

8 Convergent and Cauchy Sequences

8.1 Convergent Sequences

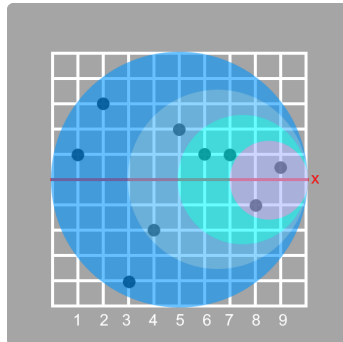
Definition 8.1.1: Convergent Sequence

A sequence $\{x_n\}$ in metric space X converge if there is a $x \in X$ such that:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$

Then, $\{x_n\}$ converges to x : $\lim_{n \rightarrow \infty} x_n = x$

If $\{x_n\}$ does not converge, then it diverges.



Example

- (a) Let $x_n = \frac{1}{n}$ in \mathbb{R}^2 . Then, $\lim_{n \rightarrow \infty} x_n = 0$

Proof

For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$. Then:

$$d(x_n, 0) = |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

- (b) Let $x_n = (-1)^n + \frac{1}{n}$ in \mathbb{R}^2 . Then, $\{x_n\}$ diverges.

Proof

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (-1)^n + \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} (-1)^n$$

Since $(-1)^n$ alternates between -1 and 1, then $\{x_n\}$ diverges.

Theorem 8.1.2: A convergent sequence is unique and bounded

- (a) $\{p_n\}$ converges to $p \in X$ if and only if every $N_r(p)$ contains p_n for all, but finitely many n .

Proof

Suppose $p_n \rightarrow p$. Then for $N_\epsilon(p)$, any $q \in X$ such that $d(q, p) < \epsilon$ is $q \in N_\epsilon(p)$. Since $p_n \rightarrow p$, there is a N such that for $n \geq N$, $d(p_n, p) < \epsilon$.

Thus, for $n \geq N$, $p_n \in N_\epsilon(p)$.

Suppose every $N_r(p)$ contains p_n for all, but finitely many n .

For $\epsilon > 0$, let $N_\epsilon(p)$ be the set of all $q \in X$ such that $d(p, q) < \epsilon$. Thus, there exists a N such that $p_n \in N_\epsilon(p)$ if $n \geq N$.

Thus, $d(p_n, p) < \epsilon$ so $p_n \rightarrow p$.

- (b) If $p, p' \in X$ and $\{p_n\}$ converges to p and p' , then $p = p'$.

Proof

For $\epsilon > 0$, there exists N, N' such that:

$$d(p_n, p) < \frac{\epsilon}{2} \text{ for } n \geq N \quad d(p_n, p') < \frac{\epsilon}{2} \text{ for } n \geq N'$$

Then for $n \geq \max(N, N')$, $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$.

Thus, $p = p'$.

- (c) If
- $\{p_n\}$
- converges, then
- $\{p_n\}$
- is bounded.

Proof

If $\{p_n\} \rightarrow p$, there is a N such that for $n > N$, $d(p_n, p) < 1$.
 Let $r = \max(d(p_1, p), \dots, d(p_N, p), 1)$. Thus for all n , $d(p_n, p) \leq r$.

- (d) If
- $E \subset X$
- and
- $p \in E'$
- , there is a
- $\{p_n\}$
- in
- E
- such that
- $p = \lim_{n \rightarrow \infty} p_n$
- .

Proof

Since $p \in E'$, then for each $n \in \mathbb{Z}_+$, there is a $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$.
 For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$ so for $n \geq N$, $d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$.
 Thus, $p = \lim_{n \rightarrow \infty} p_n$.

Theorem 8.1.3: Arithmetic Operations for sequences

Suppose $\{s_n\}, \{t_n\} \in \mathbb{C}$ where $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} t_n = t$.

- (a)
- $\lim_{n \rightarrow \infty} s_n + t_n = s + t$

Proof

For $\epsilon > 0$, there exists N_1, N_2 such that
 $|s_n - s| < \frac{\epsilon}{2}$ for $n \geq N_1$ $|t_n - t| < \frac{\epsilon}{2}$ for $n \geq N_2$
 If $N = \max(N_1, N_2)$, then for $n \geq N$:
 $|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$

- (b)
- $\lim_{n \rightarrow \infty} cs_n = cs$
- and
- $\lim_{n \rightarrow \infty} c + s_n = c + s$

Proof

For $\epsilon > 0$, there exists a N such that
 $|s_n - s| < \frac{\epsilon}{|c|}$ for $n \geq N$
 $|cs_n - cs| \leq |c| \cdot |s_n - s| < \epsilon$

- (c)
- $\lim_{n \rightarrow \infty} s_n t_n = st$

Proof

Note $s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$.
 For $\epsilon > 0$, there exists N_1, N_2 such that
 $|s_n - s| < \sqrt{\epsilon}$ for $n \geq N_1$ $|t_n - t| < \sqrt{\epsilon}$ for $n \geq N_2$
 If $N = \max(N_1, N_2)$, then for $n \geq N$, $|(s_n - s)(t_n - t)| < \epsilon$.
 Thus, $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$.

$$\lim_{n \rightarrow \infty} (s_n t_n - st) = \lim_{n \rightarrow \infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$$

$$= 0 + t \cdot 0 + s \cdot 0 = 0$$

- (d)
- $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$
- where
- $s_n, s \neq 0$

Proof

Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$ so $|s_n| > \frac{1}{2}|s|$ for $n \geq m$.
 For $\epsilon > 0$, there is a $N > m$ such that for $n \geq N$, $|s_n - s| < \frac{1}{2}|s|^2\epsilon$.
 Thus, for $n \geq N$, $|\frac{1}{s_n} - \frac{1}{s}| = \frac{|s_n - s|}{|s_n s|} < \frac{2}{|s|^2}|s_n - s| < \epsilon$.

Theorem 8.1.4: Extension to \mathbb{R}^k

- (a) Suppose $x_n \in \mathbb{R}^k$ and $x_n = (\alpha_{n_1}, \dots, \alpha_{n_k})$. Then $\{x_n\}$ converges to $x = (\alpha_1, \dots, \alpha_k)$ if and only if $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ for $i \in [1, k]$.

Proof

Suppose $\{x_n\}$ converges to $x = (\alpha_1, \dots, \alpha_k)$.

Since for any $i \in [1, k]$:

$$|\alpha_{n_i} - \alpha_i| \leq \sqrt{|\alpha_{n_1} - \alpha_1|^2 + \dots + |\alpha_{n_k} - \alpha_k|^2} = |x_n - x| < \epsilon.$$

Then, $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$.

Suppose $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ for $i \in [1, k]$.

Then for $\epsilon > 0$, there is an N such that for $n \geq N$:

$$|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}} \text{ for } i \in [1, k]$$

$$|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \epsilon$$

- (b) Suppose $\{x_n\}, \{y_n\} \in \mathbb{R}^k$ and $\{\beta_n\} \in \mathbb{R}$ and $x_n \rightarrow x, y_n \rightarrow y, \beta_n \rightarrow \beta$.
 $\lim_{n \rightarrow \infty} x_n + y_n = x + y \quad \lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y \quad \lim_{n \rightarrow \infty} \beta_n x_n = \beta x$

Proof

By part a, then $\lim_{n \rightarrow \infty} x_{n_i} + y_{n_i} = x_i + y_i$ so $\{x_n + y_n\} \rightarrow x + y$.

Also, $\lim_{n \rightarrow \infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$ so $\{x_n \cdot y_n\} \rightarrow x \cdot y$.

Also, $\lim_{n \rightarrow \infty} \beta_i x_{n_i} = \beta_i x_i$ so $\{\beta_n x_n\} \rightarrow \beta x$.

8.2 Subsequences

Definition 8.2.1: Subsequence

For sequence $\{p_n\}$, let $\{n_k\} \in \mathbb{Z}_+$ where $n_k < n_{k+1}$.

Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.

If $\{p_{n_k}\}$ converges, then its limit is called a subsequential limit.

Theorem 8.2.2: $\{p_n\} \rightarrow p \Leftrightarrow \text{Every } \{p_{n_k}\} \rightarrow p$

$\{p_n\}$ converges to p if and only if every subsequence converges to p

Proof

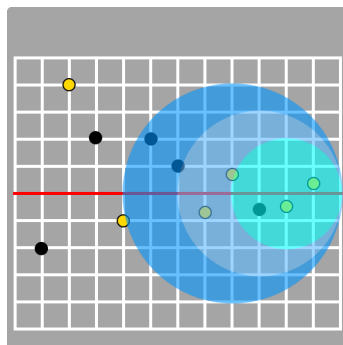
Suppose $\{p_n\}$ converges to p .

Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $d(p_n, p) < \epsilon$.

Let $\{p_{n_k}\} \subset \{p_n\}$. Then for $n_k \geq N$, $|p_{n_k} - p| < \epsilon$. Thus, $\{p_{n_k}\} \rightarrow p$.

Suppose every subsequence converges to p .

Since $\{p_n\}$ is a subsequence of itself, then $\{p_n\}$ converges to p .



Theorem 8.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \rightarrow p$

- (a) If $\{p_n\}$ is a sequence in a compact metric space X , then some subsequence converges to $p \in X$.

Proof

Let E be the range of $\{p_n\}$.

If E is finite, there is a $p \in E$ and sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $p_{n_1} = p_{n_2} = \dots = p$. Thus, $\{p_{n_k}\} \rightarrow p$.

If E is infinite, then by **theorem 6.3.10**, then there exists a $p \in E'$.

Then there are n_k such that $d(p_{n_k}, p) < \frac{1}{k}$. Thus, $\{p_{n_k}\} \rightarrow p$.

- (b) Every bounded sequence in \mathbb{R}^k contains a convergent subsequence.

Proof

Let E be a bounded sequence in \mathbb{R}^k . Since $E \cup E'$ is bounded and closed, then by **theorem 6.3.13**, $E \cup E'$ is compact.

Thus by part a, E contains a convergent subsequence.

Theorem 8.2.4: The set of subsequential limits is closed

The subsequential limits of $\{p_n\}$ in metric space X form a closed subset of X

Proof

Let E be the range of the set of all subsequential limits of $\{p_n\}$.

If E is empty, then E is closed. If E is finite, then E' is empty so E is closed.

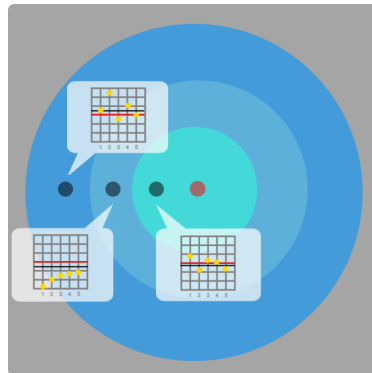
Suppose E is infinite. Then, let $q \in E'$.

Since $q \in E'$, there is a $x \in E$ where $d(x, q) < \frac{\epsilon}{2}$.

Since $x \in E$, there is a $\{p_{n_k}\} \rightarrow x$ so there is a N such that for $n \geq N$, $d(p_{n_k}, x) < \frac{\epsilon}{2}$.

Thus, $d(p_{n_k}, q) \leq d(p_{n_k}, x) + d(x, q) < \epsilon$ so q is a subsequential limit of $\{p_n\}$.

Thus, $q \in E$ so E is closed.



8.3 Cauchy Sequences

Definition 8.3.1: Metric Spaces

Sequence $\{p_n\} \in X$ is a Cauchy sequence if:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$

Let nonempty $E \subset X$ and $S \subset \mathbb{R}$ of $d(p, q)$ where $p, q \in E$.

Let $\sup(S) = \text{diam}(E)$. If $\{p_n\} \in X$, and $p_N, p_{N+1}, \dots \in E_N$, then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N \rightarrow \infty} \text{diam}(E_N) = 0$.

Theorem 8.3.2: Cauchy sequences and its closure have the same diam

- (a) If
- $\overline{E} \subset X$
- , then
- $\text{diam}(\overline{E}) = \text{diam}(E)$
- .

Proof

Since $E \subset \overline{E}$, then $\text{diam}(E) \leq \text{diam}(\overline{E})$.

For $\epsilon > 0$, let $p, q \in E$.

Thus, there are $p', q' \in E$ such that $d(p', p) < \epsilon$ and $d(q', q) < \epsilon$. Thus:
 $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \leq 2\epsilon + \text{diam}(E)$.

Thus, $\text{diam}(\overline{E}) \leq 2\epsilon + \text{diam}(E)$ so $\text{diam}(\overline{E}) = \text{diam}(E)$.

- (b) If
- K_n
- is a sequence of compact sets of
- X
- such that
- $K_{n+1} \subset K_n$
- and
- $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$
- , then
- $\bigcap K_n$
- consist of only one point.

Proof

Let $K = \bigcap K_n$. Since K_n is a sequence of compact sets, then by [corollary 6.3.8](#), K is nonempty.

If K contains more than one point, then $\text{diam}(K) > 0$. But since $K \subset K_n$, then $\text{diam}(K) \leq \text{diam}(K_n)$ which contradicts that $\text{diam}(K_n) \rightarrow 0$.

Theorem 8.3.3: Convergent sequences are cauchy sequences

- (a) Every convergent sequence is a Cauchy sequence.

Proof

If $p_n \rightarrow p$ and $\epsilon > 0$, there is a N such that for all $n \geq N$, $d(p, p_n) < \frac{\epsilon}{2}$. Thus, for $m, n \geq N$:

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon.$$

Thus, $\{p_n\}$ is a Cauchy sequence.

- (b) If
- $\{p_n\}$
- is a Cauchy sequence in compact metric space
- X
- , then
- $\{p_n\}$
- converges to some
- $p \in X$
- .

Proof

Let $\{p_n\}$ be a Cauchy sequence in compact space X .

Let $p_N, p_{N+1}, \dots \in E_N$.

Since $\{p_n\}$ is a Cauchy sequence, then $\lim_{N \rightarrow \infty} \text{diam}(\overline{E_N}) = 0$. Since $\overline{E_N}$ is closed in compact X , then by [theorem 6.3.5](#), $\overline{E_N}$ is compact.

Since $E_{N+1} \subset E_N$, then $\overline{E_{N+1}} \subset \overline{E_N}$ and thus, by [theorem 8.3.2b](#), then there is a unique $p \in \overline{E_N}$ for every N .

Since $p \in \overline{E_N}$, then $d(p, q) < \epsilon$ for every $q \in \overline{E_N}$ so every $q \in E_N$.

Then for $\epsilon > 0$, there is a N_0 such that for $N \geq N_0$, $\text{diam}(\overline{E_N}) < \epsilon$.

Thus, $d(p_n, p) < \epsilon$ for $n \geq N_0$ so $\{p_n\} \rightarrow p$.

- (c) In
- \mathbb{R}^k
- , every Cauchy sequence converges.

Proof

Let $\{x_n\}$ be a Cauchy sequence in \mathbb{R}^k . Let $x_N, x_{N+1}, \dots \in E_N$.

Then for some N , $\text{diam}(E_N) < 1$. Thus, the range of $\{x_n\} = E_N \cup \{x_1, \dots, x_{N-1}\}$. Thus, $\{x_n\}$ is bounded.

Thus, the $\{x_n\}$ is closed and bounded so by [theorem 6.3.13](#), $\overline{\{x_n\}}$ is compact.

Thus, by part b, $\{x_n\}$ converges to some $p \in \mathbb{R}^k$.

Definition 8.3.4: Complete

A metric space where every Cauchy sequence converges is complete.

Thus, by **theorem 8.3.3**, all compact and Euclidean spaces are complete.

Definition 8.3.5: Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is:

- (a) monotonically increasing if $s_n \leq s_{n+1}$
- (b) monotonically decreasing if $s_n \geq s_{n+1}$

Theorem 8.3.6: Monotonic sequences converge if bounded

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded

Proof

Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$.

Suppose $\{s_n\}$ is bounded.

Let $s = \sup(E)$ so $s_n \leq s$. For every $\epsilon > 0$, there is a N such that $s - \epsilon < s_N \leq s$ else $s - \epsilon$ would be an upper bound of E which contradicts $s = \sup(E)$.

Since $\{s_n\}$ increases, then for $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s$ so $\{s_n\} \rightarrow s$.

Suppose $\{s_n\}$ converges to s .

Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s$.

Thus, $\{s_n\}$ is bounded from above.

Suppose $s_n \geq s_{n+1}$. Let E be the range of $\{s_n\}$.

Suppose $\{s_n\}$ is bounded.

Let $s = \inf(E)$ so $s_n \geq s$. For every $\epsilon > 0$, there is a N such that $s \leq s_N < s + \epsilon$ else $s + \epsilon$ would be a lower bound of E which contradicts $s = \inf(E)$.

Since $\{s_n\}$ decreases, then for $n \geq N$, $s \leq s_n \leq s_N < s + \epsilon$ so $\{s_n\} \rightarrow s$.

Suppose $\{s_n\}$ converges to s .

Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $s \leq s_n \leq s_N < s + \epsilon$.

Thus, $\{s_n\}$ is bounded from below.

9 Limits and Special Sequences

9.1 Upper and Lower Limits

Definition 9.1.1: Infinite limits

Let $\{s_n\}$ be a sequence of real numbers such that:

For every real M , there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \geq M$.

Then, $s_n \rightarrow +\infty$.

For every real M , there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \leq M$.

Then, $s_n \rightarrow -\infty$.

Definition 9.1.2: Upper and Lower Limits

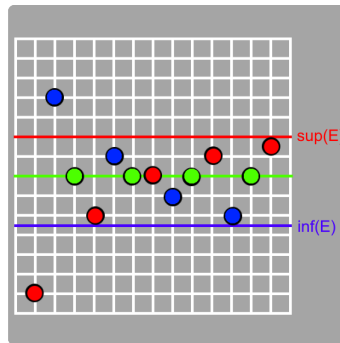
Let $\{s_n\} \subset \mathbb{R}$ and E contain all subsequential limits of $\{s_n\}$ plus possibly $\pm\infty$.

Then, the upper limit of $\{s_n\}$:

$$s^* = \sup(E) \quad \lim_{n \rightarrow \infty} \sup(s_n) = s^*$$

Then, the lower limit of $\{s_n\}$:

$$s_* = \inf(E) \quad \lim_{n \rightarrow \infty} \inf(s_n) = s_*$$



Theorem 9.1.3: Upper and Lower limits are unique

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and s^* be the upper limit of $\{s_n\}$. Then:

(a) $s^* \in E$

Proof

If $s^* = +\infty$, then there is a $\{s_{n_k}\} \rightarrow +\infty$ so E is not bounded above.

If $s^* \in \mathbb{R}$, then E is bounded above so $s^* \in E'$.

Then by [theorem 8.2.4](#), $s^* \in E$.

If $s^* = -\infty$, then there are no subsequential limits in E . Thus, for every M , there is a N such that for $n \geq N$, $s_n \leq M$ so $-\infty \in E$.

(b) If $x > s^*$, there is a N such that for $n \geq N$, $s_n < x$

Proof

Suppose there is a $x > s^*$ such that $s_n \geq x$ for infinitely many n .

Then, there is a $y \in E$ where $y \geq x > s^*$ which contradicts $s^* = \sup(E)$.

(c) s^* is the only number that satisfies (a) and (b)

Proof

Suppose p, q satisfy part a and b where $p < q$. Choose x where $p < x < q$. Since p satisfies b, then $s_n < x$ for $n \geq N$. Thus, x is an upper bound for E so $q \notin E$ since $q > x$ contradicting that q satisfies part a.

The same properties are analogous for s_* .

Theorem 9.1.4: Inf & Sup of $s_n \leq t_n$

If $s_t \leq t_n$ for $n \geq$ fixed N , then

$$\lim_{n \rightarrow \infty} \inf(s_n) \leq \lim_{n \rightarrow \infty} \inf(t_n)$$

$$\lim_{n \rightarrow \infty} \sup(s_n) \leq \lim_{n \rightarrow \infty} \sup(t_n)$$

Proof

Let E_1 be the set of extended reals x such that $\{s_{n_k}\} \rightarrow x$ for some $\{s_{n_k}\}$.

Let E_2 be the set of extended reals y such that $\{t_{n_k}\} \rightarrow y$ for some $\{s_{n_k}\}$.

Let $s^* = \sup(E_1)$, $s_* = \inf(E_1)$, $t^* = \sup(E_2)$, and $t_* = \inf(E_2)$.

Since there is a N such that $s_n \leq t_n$ for $n \geq N$, then:

$$x \leftarrow \{s_N, s_{N+1}, \dots\} \leq \{t_N, t_{N+1}, \dots\} \rightarrow y$$

Thus, for $n \geq N$, $\inf(s_n) \leq \inf(t_n)$ and $\sup(s_n) \leq \sup(t_n)$.

9.2 Special Sequences**Theorem 9.2.1: Special sequences**

- (a) If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

Proof

For $\epsilon > 0$, let $N > \sqrt[p]{\frac{1}{\epsilon}}$. Then for $n \geq N$, $\lim_{n \rightarrow \infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}} = \epsilon$

- (b) If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$

Proof

If $p > 1$, then let $x_n = \sqrt[p]{p} - 1 > 0$.

$$p = (x_n + 1)^n = x_n^n + nx_n^{n-1} + \dots + nx_n + 1 \geq nx_n + 1$$

Thus, $0 < x_n \leq \frac{p-1}{n}$ so $\{x_n\} \rightarrow 0$ and thus, $\{\sqrt[p]{p}\} \rightarrow 1$.

If $p = 1$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} 1 = 1$.

If $0 < p < 1$, then $\frac{1}{p} > 1$. From the proof above for $p > 1$, $\{\sqrt[\frac{1}{p}]{\frac{1}{p}}\} \rightarrow 1$.

Thus, $\{\frac{1}{\sqrt[p]{p}}\} \rightarrow 1$ so $\{\sqrt[p]{p}\} \rightarrow 1$.

- (c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

Proof

Let $x_n = \sqrt[n]{n} - 1 \geq 0$. Then, $n = (x_n + 1)^n \geq \frac{n(n-1)}{2} x_n^2$.

Thus, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ so $\{x_n\} \rightarrow 0$ and thus, $\{\sqrt[n]{n}\} \rightarrow 1$.

- (d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$

Proof

Let $k \in \mathbb{Z}$ such that $k > \alpha$ and $k > 0$. For $n > 2k$:

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

Thus, $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$.

Since $\alpha - k < 0$, then $\{n^{\alpha-k}\} \rightarrow 0$ so $\{\frac{n^\alpha}{(1+p)^n}\} \rightarrow 0$.

- (e) If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$

Proof

From part d, let $\alpha = 0$.

Thus, $\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = 0$ and since $p > 0$, then $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$.

Also, $-\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = \lim_{n \rightarrow \infty} \frac{-1}{(1+p)^n} = 0$ so $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n > -1$.

10 Series and Convergence Tests

10.1 Series

Definition 10.1.1: Series

For sequence $\{a_n\}$, define $\sum_{n=p}^q a_n = a_p + a_{p+1} + \dots + a_q$.

Then associate $\{a_n\}$ with a sequence $\{s_n\}$ such that $s_n = \sum_{k=1}^n a_k$.

Then $\{s_n\}$ is a series with partial sums s_n .

If $\{s_n\} \rightarrow s$, then $\sum_{n=1}^{\infty} a_n = s$ is the sum of the convergent series.

Note $a_1 = s_1$ and $a_n = s_n - s_{n-1}$.

Theorem 10.1.2: Cauchy Criterion for series

$\sum a_n$ converges if and only if:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$

Proof

Suppose $\sum_{k=1}^n a_k$ converges.

Then by [theorem 8.3.3a](#), $\sum_{k=1}^n a_k$ is a Cauchy sequence.

Then for $\epsilon > 0$, there is a N such that for $m \geq n \geq N$:

$$d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = |\sum_{k=n}^m a_k| \leq \epsilon$$

Suppose for every $\epsilon > 0$, there is a N such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$.

$$|\sum_{k=n}^m a_k| = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) \leq \epsilon$$

Thus, $\sum_{k=1}^n a_k$ is a Cauchy sequence and thus, convergent.

Theorem 10.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \rightarrow 0$

If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof

Since $\sum a_n$ converges, then by [theorem 10.1.2](#), for $\epsilon > 0$, there is a N such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$. Then if $m = n \geq N$, $|\sum_{k=n}^m a_k| = |a_n| \leq \epsilon$ so $\{a_n\} \rightarrow 0$.

Example

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \dots + \frac{1}{8}) + (\frac{1}{9} + \dots) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Thus, $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$ which is unbounded and thus, not convergent.

Theorem 10.1.4: Convergent series \Leftrightarrow Bounded sequence

A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.

Proof

Suppose $\sum a_n$ converges where $a_n \geq 0$.

Since $a_n \geq 0$, then $\{s_n\}$ is monotonic so by [theorem 8.3.6](#), $\{s_n\}$ is bounded above.

Suppose $\{s_n\}$ is bounded where $a_n \geq 0$.

Since $\{s_n\}$ is monotonic and bounded, then by [theorem 8.3.6](#), $\{s_n\}$ converges.

Theorem 10.1.5: Comparison Test

- (a) If $|a_n| \leq c_n$ for $n \geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.

Proof

For $\epsilon > 0$, there exists a $N \geq N_0$ such that for $m \geq n \geq N$, $\sum_{k=n}^m c_k \leq \epsilon$.

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \epsilon$$
Thus, $\sum a_n$ converges.

- (b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof

Suppose $\sum a_n$ converges.
Then from part a, $\sum d_n$ converges which contradicts that $\sum a_n$ diverges.
Thus, $\sum a_n$ diverges.

10.2 Series of Nonnegative Terms**Theorem 10.2.1: Infinite Geometric Series**

If $x \in [0, 1)$, then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geq 1$, the series diverges.

Proof

If $x \neq 1$, then using the geometric series $s_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$. Let $n \rightarrow \infty$.
If $x \in [0, 1)$, then by **theorem 9.2.1e**, $s_n = \frac{1}{1-x} (1 - x^{n+1}) = \frac{1}{1-x} (1 - 0) = \frac{1}{1-x}$.
Also, by **theorem 9.2.1e**, if $x \geq 1$, then the series diverges.

Theorem 10.2.2: Cauchy's Convergence Criterion

Suppose $0 \leq a_{i+1} \leq a_i$.

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots \text{ converges.}$$

Proof

Let $s_n = a_1 + a_2 + \dots + a_n$ and $t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$. For $n < 2^k$:

$$\begin{aligned} s_n &\leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \dots + 2^k a_{2^k} = t_k \end{aligned}$$

By **comparison test**, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. For $n > 2^k$:

$$\begin{aligned} s_n &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k \end{aligned}$$

By **comparison test**, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem 10.2.3: P-series

$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Proof

If $p \leq 0$, then by **theorem 10.1.3**, $\sum \frac{1}{n^p}$ diverges.
If $p > 0$, then by **theorem 10.2.2**, $\sum \frac{1}{n^p}$ converges only if $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges.
Since $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$, then by **theorem 10.2.1**, $\sum_{k=0}^{\infty} 2^{k(1-p)}$ converges if $2^{1-p} < 1$ so if $1-p < 0$ so $p > 1$.

Theorem 10.2.4: Log P-series

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof

Since $\frac{1}{n(\log(n))^p}$ decreases, then by **theorem 10.2.2**,
 $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges.
 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k}$
 Then by **theorem 10.2.3**, $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$ converges if $p > 1$ and diverges if $p \leq 1$.
 Thus, $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $p > 1$ and diverges and $p \leq 1$.

Corollary 10.2.5: Log P-series extended

$\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges if $p > 1$ and diverges if $p \leq 1$

Proof

From **theorem 10.2.4**, replace $n = \log(n)$ and multiplying by $\frac{1}{n} \rightarrow \frac{1}{n \log(n)(\log(\log(n)))^p}$.
 Since $\frac{1}{n \log(n)(\log(\log(n)))^p}$ decreases, by **theorem 10.2.2** $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)(\log(\log(2^k)))^p}$.
 $\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k \log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$
 Since $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$ converges by **theorem 10.2.4**, $\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges.

10.3 The Number e**Definition 10.3.1: Summation equivalence to e**

$$s_m = \sum_{n=0}^m \frac{1}{n!} = 1 + \sum_{n=1}^m \frac{1}{n!} < 1 + \sum_{n=1}^m \frac{1}{2^{n-1}} < 3$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Theorem 10.3.2: Limit equivalence to e

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Proof

Let $s_n = \sum_{k=0}^n \frac{1}{k!}$ and $t_n = \left(1 + \frac{1}{n}\right)^n$. Using the binomial theorem:
 $t_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)$
 Thus, $t_n \leq s_n$ so $\lim_{n \rightarrow \infty} \sup(t_n) \leq e$.
 If $n \geq m$, then $t_n \geq \sum_{k=0}^m \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)$.
 As $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \inf(t_n) \geq \sum_{k=0}^m \frac{1}{k!} = s_m$. As $m \rightarrow \infty$, $\lim_{n \rightarrow \infty} \inf(t_n) \geq e$.

Theorem 10.3.3: Rapidity of convergence of e

$$0 < e - s_n < \frac{1}{n!n}$$

Proof

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) = \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!n}$$

Theorem 10.3.4: e is irrational

e is irrational

Proof

Suppose r is rational. Then let $e = \frac{p}{q}$ for $p, q \in \mathbb{Z}_+$.
 Thus, by **theorem 10.3.3**, $0 < e - s_q < \frac{1}{q!q}$ so $0 < q!(e - s_q) < \frac{1}{q}$.
 Since $e = \frac{p}{q}$, then $q!e$ is an integer and $q!s_q = q!(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!})$ is an integer.
 Thus, $q!(e - s_q)$ is an integer which is between 0 and $\frac{1}{q}$ and thus, a contradiction.

10.4 Root and Ratio Tests**Theorem 10.4.1: Root Test**

- For $\sum a_n$, let $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|})$.
- (a) If $\alpha < 1$, $\sum a_n$ converges
 - (b) If $\alpha > 1$, $\sum a_n$ diverges
 - (c) If $\alpha = 1$, unclear

Proof

If $\alpha < 1$, choose β such that $\beta \in (\alpha, 1)$ and $N \in \mathbb{Z}$ such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$.
 Since $\beta \in (0, 1)$, then by **theorem 10.2.1**, $\sum \beta^n$ converges. Then by the **comparison test**, $\sum a_n$ converges.
 If $\alpha > 1$, then there is a a_{n_k} such that $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$.
 Thus, $|a_n| > 1$ for infinitely many n so by **theorem 10.1.3**, $\sum a_n$ doesn't converge.
 $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$ have $\alpha = 1$, but $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges by **theorem 10.2.3**.

Theorem 10.4.2: Ratio Test

- (a) $\sum a_n$ converges if $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$
- (b) $\sum a_n$ diverges if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq n_0$ for $n_0 \in \mathbb{Z}$

Proof

If $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$, there is a $\beta < 1$ and N such that for $n \geq N$, $|\frac{a_{n+1}}{a_n}| < \beta$.
 Then $|a_{N+1}| < \beta|a_N|$ so $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$.
 Thus, $|a_{N+p}| < \beta^p|a_N|$ so $|a_n| < |a_N|\beta^{-N}\beta^n$.
 Thus, by the **comparison test**, $\sum a_n$ converges.
 If $|a_{n+1}| \geq |a_n| > 0$ for $n \geq n_0$, then by **theorem 10.1.3**, $\sum a_n$ diverges.

Theorem 10.4.3: Ratio convergence \rightarrow Root convergence

$$\lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$$

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$$

Proof

Let $\alpha = \lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n})$. If $\alpha = -\infty$, then $-\infty \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$ holds true.
 If α is finite, there is a $\beta \leq \alpha$ and N such that for $n \geq N$, $\frac{c_{n+1}}{c_n} \geq \beta$ so $c_{N+p} \geq \beta^p c_N$.
 Then, $c_n \geq c_N \beta^{-N} \beta^n$ so $\sqrt[n]{c_n} \geq \sqrt[n]{c_N \beta^{-N} \beta^n} = \sqrt[n]{c_N} \beta^{-N/n} \beta$. Thus, $\lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$.
 Let $\alpha = \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$. If $\alpha = \infty$, then $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \infty$ holds true.
 If α is finite, there is a $\beta \geq \alpha$ and N such that for $n \geq N$, $\frac{c_{n+1}}{c_n} \leq \beta$ so $c_{N+p} \leq \beta^p c_N$.
 Then, $c_n \leq c_N \beta^{-N} \beta^n$ so $\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N} \beta^n} = \sqrt[n]{c_N} \beta^{-N/n} \beta$. Thus, $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$.

10.5 Power Series

Definition 10.5.1: Power series

For a sequence $\{c_n\} \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is a power series.
 c_n are the coefficients and $z \in \mathbb{C}$.

Theorem 10.5.2: Radius of Convergence

For power series $\sum c_n z^n$, let $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$ and $R = \frac{1}{\alpha}$.
 Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Proof

Let $a_n = c_n z^n$. Using the **root test**,

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|}) = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n z^n|})$$

$$= |z| \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R}$$

Thus, $\sum c_n z^n$ converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$

10.6 Summation By Parts

Theorem 10.6.1: Summation by parts

For sequences $\{a_n\}$, $\{b_n\}$, let $A_n = \sum_{k=0}^n a_k$. Then for $0 \leq p \leq q$:

$$\sum_{n=p}^q a_n b_n = (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Proof

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p}^q A_{n-1} b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p \\ &= (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p \end{aligned}$$

Theorem 10.6.2: Conditions for convergent $\sum a_n b_n$

Suppose for $\{a_n\}$, $\{b_n\}$:

- partial sums A_n of $\sum a_n$ form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n \rightarrow \infty} b_n = 0$

Then $\sum a_n b_n$ converges.

Proof

Since $\{A_n\}$ is bounded, $|A_n| \leq M$ for all n .
 Since $\{b_n\}$ is monotonically decreasing and $\lim_{n \rightarrow \infty} b_n = 0$, then for $\epsilon > 0$, there is a N such that $b_N \leq \frac{\epsilon}{2M}$. Then for $N \leq p \leq q$:

$$\begin{aligned} |\sum_{n=p}^q a_n b_n| &= |(\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p| \\ &\leq M |\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p| = 2M b_p \leq 2M b_N \leq \epsilon \end{aligned}$$

Corollary 10.6.3: Convergent series of Alternating Sequences

Suppose for $\{c_n\}$:

- $|c_i| \geq |c_{i+1}|$
- $c_{2i-1} \geq 0$ and $c_{2i} \leq 0$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then $\sum c_n$ converges.

Proof

From **theorem 10.6.2**, let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

Corollary 10.6.4: Convergent power series at radius of convergence

Suppose for $\{c_n\}$:

- Radius of convergence of $\sum c_n z^n$ is 1
- $c_i \geq c_{i+1}$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then $\sum c_n z^n$ converges at every point where $|z| = 1$ except possibly $z = 1$.

Proof

From **theorem 10.6.2**, let $a_n = z^n$ and $b_n = c_n$.
 A_n of $\sum a_n$ form a bounded sequence since $|A_n| = |\sum_0^n z^n| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|}$.

10.7 Absolute Convergence**Definition 10.7.1: Absolute convergence**

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ converges non-absolutely.

Theorem 10.7.2: Absolute convergence \rightarrow convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges

Proof

Since $\sum a_n$ converges absolutely, then for every $\epsilon > 0$, there is an integer N such that for $m \geq n \geq N$, $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$.
 Thus, $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$ so $\sum a_n$ converges.

10.8 Addition & Multiplication of Series**Theorem 10.8.1: Addition and Scalar Multiplication**

If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$ and $\sum ca_n = cA$.

Proof

Let $A_n = \sum_{k=0}^n a_k$ and $B_n = \sum_{k=0}^n b_k$.
 Then $A_n + B_n = \sum_{k=0}^n a_k + b_k$ so $\lim_{n \rightarrow \infty} A_n + B_n = A + B$.
 Then $\lim_{n \rightarrow \infty} cA_n = \underbrace{A + \dots + A}_c = cA$

Definition 10.8.2: Cauchy Product

For $\sum a_n$ and $\sum b_n$, let $c_n = \sum_{k=0}^n a_k b_{n-k}$ and the product as $\sum c_n$.

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n) (b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots\end{aligned}$$

Theorem 10.8.3: Conditions $\sum c_n = AB$

Suppose

- $\sum_{n=0}^{\infty} a_n$ converges absolutely
- $\sum_{n=0}^{\infty} a_n = A$
- $\sum_{n=0}^{\infty} b_n = B$
- $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then $\sum_{n=0}^{\infty} c_n = AB$.

Proof

Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$, and $\beta_n = B_n - B$.

$$\begin{aligned}C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \dots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0\end{aligned}$$

Let $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$ so $C_n = A_n B + \gamma_n$.

Since a_n converges absolutely, then $\sum_{n=0}^{\infty} |a_n| = \alpha$.

Since $\sum_{n=0}^{\infty} b_n = B$, then $\beta_n \rightarrow 0$.

Then for $\epsilon > 0$, there is a N such that $|\beta_n| \leq \frac{\epsilon}{\alpha}$ for $n \geq N$.

$$\begin{aligned}|\gamma_n| &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \dots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |a_{n-N-1} + \dots + a_0| \frac{\epsilon}{\alpha} \\ &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + \alpha \frac{\epsilon}{\alpha}\end{aligned}$$

Thus, with a fixed N , since $a_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} |\gamma_n| \leq \epsilon$ so $\lim_{n \rightarrow \infty} \gamma_n = 0$.

Thus, $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} A_n B + \gamma_n = AB$.

Theorem 10.8.4: By Cauchy Product, $\sum c_n = C$ implies $C = AB$

If $\sum a_n = A$, $\sum b_n = B$, $\sum c_n = C$ where $c_n = a_0 b_n + \dots + a_n b_0$, then $C = AB$.

11 Continuity

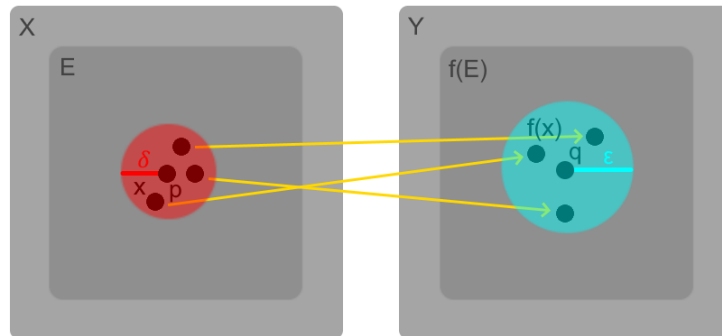
11.1 Limits of Functions

Definition 11.1.1: Limits of functions

For metric spaces X, Y , let $E \subset X$, f maps E into Y , and $p \in E'$.

Then $\lim_{x \rightarrow p} f(x) = q$ if there is a $q \in Y$ such that:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in E$ where $d_X(x, p) < \delta$, then $d_Y(f(x), q) < \epsilon$



Theorem 11.1.2: Sequence definition of $\lim_{x \rightarrow p} f(x) = q$

$\lim_{x \rightarrow p} f(x) = q$ if and only if $\lim_{n \rightarrow \infty} f(p_n) = q$ for every sequence $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Proof

Suppose $\lim_{x \rightarrow p} f(x) = q$.

For $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $d_X(x, p) < \delta$.

Choose $\{p_n\} \in E$ such that $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Then for $\delta > 0$, there is N such that for $n > N$, then $d_X(p_n, p) < \delta$ so $d_Y(f(p_n), q) < \epsilon$.

Suppose $\lim_{x \rightarrow p} f(x) \neq q$. Then there is a $\epsilon > 0$ such that for every $\delta > 0$, there is a $x \in E$ where $d_Y(f(x), q) \geq \epsilon$, but $d_X(x, p) < \delta$. Let $\delta_n = \frac{1}{n}$ and thus, there is a $\{p_n\}$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$, but $\lim_{n \rightarrow \infty} f(p_n) \neq q$.

Corollary 11.1.3: A limit of a function is unique

If f has a limit at p , this limit is unique.

Proof

If $\lim_{x \rightarrow p} f(x) = q$, then by **theorem 11.1.2**, $\lim_{n \rightarrow \infty} f(p_n) = q$ for every $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Thus, if there exists $\lim_{x \rightarrow p} f(x) = q'$, then there is a $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$, but $\lim_{n \rightarrow \infty} f(p_n) = q'$ which is a contradiction.

Theorem 11.1.4: Arithmetic operations on functions of limits

Let $E \subset X$, $p \in E'$, and $f(x), g(x) \in \mathbb{C}$ so $\lim_{x \rightarrow p} f(x) = A$, $\lim_{x \rightarrow p} g(x) = B$.

(a) $\lim_{x \rightarrow p} (f + g)(x) = A + B$

(b) $\lim_{x \rightarrow p} (fg)(x) = AB$

(c) $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$

11.2 Continuous Functions

Definition 11.2.1: Continuous functions on a set

Suppose X, Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y .

f is continuous at p if:

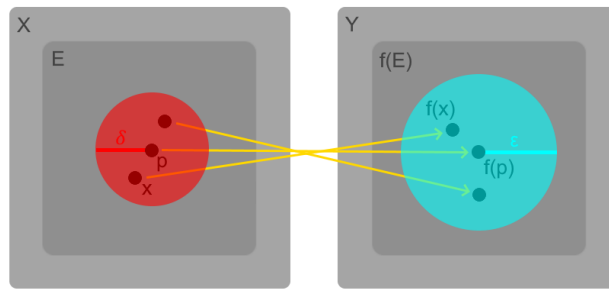
For every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x \in E$

where $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$

$f(p)$ have to be defined to be continuous.

If f is continuous at every $p \in E$, then f is continuous on E .

f is continuous at isolated points since regardless of ϵ , there is a $\delta > 0$ such that $d_X(x, p) < \delta$ is $x = p$ so $d_Y(f(x), f(p)) = 0 < \epsilon$.



Theorem 11.2.2: Continuity at $p \Leftrightarrow \lim_{x \rightarrow p} f(x) = f(p)$

Suppose $E \subset X$, $p \in E$, and f maps E into Y . Let $p \in E$.

Then f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.

Proof

If f is continuous at p , then for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all $x \in E$ where $d_X(x, p) < \delta$. Thus, $\lim_{x \rightarrow p} f(x) = f(p)$.

If $\lim_{x \rightarrow p} f(x) = f(p)$, then for every $\epsilon > 0$, there is a $\delta > 0$ where $d_Y(f(x), f(p)) < \epsilon$ for all $x \in E$ where $d_X(x, p) < \delta$. Thus, f is continuous at p .

Theorem 11.2.3: Continuity Chain Rule

Suppose $E \subset X$, $f: E \rightarrow Y$, $g: f(E) \rightarrow Z$, and $h: E \rightarrow Z$ where $h(x) = g(f(x))$.

If f is continuous at p and g is continuous at $f(p)$, then h is continuous at p .

Proof

Since g is continuous at $f(p)$, then for $\epsilon > 0$, there is a δ_1 such that:

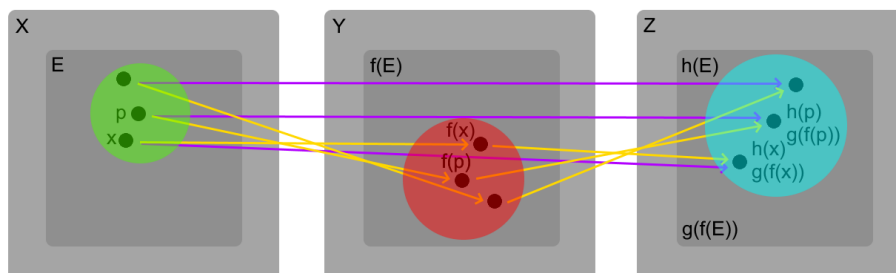
$d_Z(g(y), g(f(p))) < \epsilon$ for $d_Y(y, f(p)) < \delta_1$ where $y \in f(E)$

Since f is continuous at p , there is a $\delta_2 > 0$ such that:

$d_Y(f(x), f(p)) < \delta_1$ for $d_X(x, p) < \delta_2$ where $x \in E$

Thus, $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$ for $d_X(x, p) < \delta_2$ where $x \in E$.

Thus, h is continuous at p .



Theorem 11.2.4: Continuous functions map open sets to open sets

$f: X \rightarrow Y$ is continuous on X if and only if:

$f^{-1}(V)$ is open in X for every open set V in Y .

Proof

Suppose f is continuous on X and V is an open set in Y .

Suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\epsilon > 0$ such that $y \in V$ if $d_Y(y, f(p)) < \epsilon$. Since f is continuous at p , there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for $d_X(x, p) < \delta$. Thus, $x \in f^{-1}(V)$ for $d_X(x, p) < \delta$.

Suppose $f^{-1}(V)$ is open in X for every open V in Y .

Fix $p \in X$ and $\epsilon > 0$. Let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \epsilon$ so V is open and thus, $f^{-1}(V)$ is open. Thus, there exists $\delta > 0$ such that $x \in f^{-1}(V)$ for $d_X(x, p) < \delta$. Since $x \in f^{-1}(V)$, then $f(x) \in V$ so $d_Y(f(x), f(p)) < \epsilon$.

Corollary 11.2.5: Continuous functions map closed sets to closed sets

$f: X \rightarrow Y$ is continuous on X if and only if:

$f^{-1}(C)$ is closed in X for every closed set C in Y .

Proof

By **theorem 11.2.4**, f is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y . Let $C = V^c$. Since V is open, then C is closed.

Since $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$, then $f^{-1}(C)$ is closed since $f^{-1}(V)$ is open.

Theorem 11.2.6: Continuous functions

Let f, g be complex continuous functions on X .

Then $f+g$, fg , and $\frac{f}{g}$ where $g \neq 0$ for all $x \in X$ are continuous on X .

Proof

If x is an isolated point, $f+g$, fg , and $\frac{f}{g}$ are continuous by definition. If x is a limit point, then by **theorems 11.1.4 and 11.2.2**, $f+g$, fg , and $\frac{f}{g}$ are continuous since

- $\lim_{x \rightarrow p} (f + g)(x) = \lim_{x \rightarrow p} f(x) + \lim_{x \rightarrow p} g(x) = f(p) + g(p)$
- $\lim_{x \rightarrow p} (fg)(x) = \lim_{x \rightarrow p} f(x) \lim_{x \rightarrow p} g(x) = f(p)g(p)$
- $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \rightarrow p} f(x)}{\lim_{x \rightarrow p} g(x)} = \frac{f(p)}{g(p)}$

Theorem 11.2.7: Continuous functions on \mathbb{R}^k

(a) Let $f_1, \dots, f_k: X \rightarrow \mathbb{R}$ and $f: X \rightarrow \mathbb{R}^k$ where $f(x) = (f_1(x), \dots, f_k(x))$.

Then f is continuous if and only if f_1, \dots, f_k are continuous.

(b) If f and g are continuous mappings of X into \mathbb{R}^k , then $f + g$ and $f \cdot g$ are continuous on X .

Proof

Since $|f_i(x) - f_i(y)| \leq \sqrt{\sum_{i=1}^k |f_i(x) - f_i(y)|^2} = |f(x) - f(y)|$, then if f is continuous, then each f_i is continuous and vice versa.

Since f, g are continuous, then by part a, each f_i, g_i are continuous. Then by **theorem 11.2.6**, each $f_i + g_i$ and $f_i g_i$ are continuous so by part a, $f + g$ and $f \cdot g$ are continuous.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are $x_1 \cdot \dots \cdot x_k$ where each x_i is continuous, rationals are polynomials divided by polynomials, and $||x| - |y|| \leq |x - y|$ implies $|x|$ is continuous.

11.3 Continuity and Compactness

Definition 11.3.1: Bounded Functions

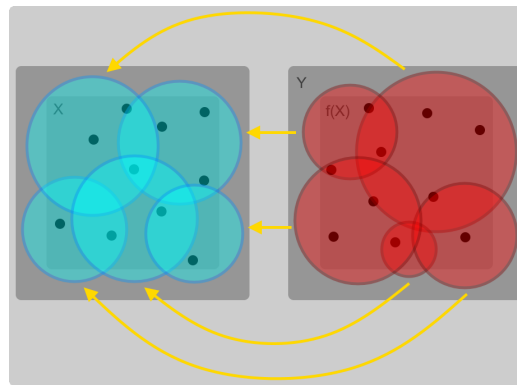
$f: E \rightarrow \mathbb{R}^k$ is bounded if there is a $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in E$.

Theorem 11.3.2: Continuous functions from compact spaces are compact

Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Proof

Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, then by [theorem 11.2.4](#), each $f^{-1}(V_\alpha)$ is open. Since X is compact, there is n where $X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})$. Thus, $f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ so $f(X)$ is compact.



Theorem 11.3.3: Continuous functions from compact to \mathbb{R}^k are bounded

If f is a continuous mapping of a compact metric space X into \mathbb{R}^k , then $f(X)$ is closed and bounded.

Proof

By [theorem 11.2.2](#), $f(X)$ is compact. By [theorem 6.3.13](#), $f(X)$ is closed and bounded.

Theorem 11.3.4: Generalized extreme value theorem

Suppose f is a continuous real function of a compact metric space X such that $M = \sup_{x \in X} f(x)$ and $m = \inf_{x \in X} f(x)$.

Then there exists $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proof

By [theorem 11.3.3](#), $f(X)$ is closed and bounded. Let $M = \sup_{x \in X} f(x)$, $m = \inf_{x \in X} f(x)$. Since $f(X)$ is bounded, then $M, m \in (f(X))'$ and since $f(X)$ is closed, then $M, m \in f(X)$. Thus, there exists $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Theorem 11.3.5: If f is continuous 1-1, then f^{-1} is continuous

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y . Then f^{-1} is a continuous mapping of Y onto X .

Proof

Let V be an open set in X . Since V^c is closed and $V^c \subset$ compact set X , then by [theorem 6.3.5](#), V^c is compact. Thus by [theorem 11.3.2](#), $f(V^c)$ is a compact subset of Y so $f(V^c)$ is closed. Since f is 1-1 and onto, $f(V^c) = (f(V))^c$ so $f(V)$ is open. Since from any open set V in X , $f(V)$ is open in Y , then by [theorem 11.2.4](#), f^{-1} is continuous.

Definition 11.3.6: Uniformly Continuous

Let $f: X \rightarrow Y$. Then f is uniformly continuous on X if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that for all $p, q \in X$ where $d_X(p, q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

Theorem 11.3.7: Continuous functions on compact are uniformly continuous

Let f be a continuous mapping of a compact metric space X into metric space Y . Then f is uniformly continuous on X .

Proof

For $\epsilon > 0$, since f is continuous, then for each $p \in X$, there is a $\phi(p)$ such that for all $q \in X$ where $d_X(q, p) < \phi(p)$, $d_Y(f(q), f(p)) < \frac{\epsilon}{2}$.

Let $J(p)$ be the set of all $q \in X$ where $d_X(q, p) < \frac{1}{2}\phi(p)$.

Since the set of all $J(p)$ is an open cover of X and since X is compact, then there is a n such that $X \subset J(p_1) \cup \dots \cup J(p_n)$. Let $\delta = \frac{1}{2} \min(\phi(p_1), \dots, \phi(p_n)) > 0$.

Then for $p, q \in X$ where $d_X(p, q) < \delta$, there is a m where $1 \leq m \leq n$ such that $p \in J(p_m)$ so $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. Thus:

$$\begin{aligned} d_X(q, p_m) &\leq d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \leq \phi(p_m) \\ d_Y(f(p), f(q)) &\leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Theorem 11.3.8: Continuous functions from noncompact \nrightarrow uniformly continuous

Let E be a noncompact set in \mathbb{R}^1 .

- (a) There exists a continuous function which is not bounded.
- (b) There exists a continuous, bounded function which has no maximum.
- (c) If E is bounded, there exists a continuous function which is not uniformly continuous.

Proof

Suppose E is bounded so there is a $x_0 \in E'$, but $x_0 \notin E$.

Consider $f(x) = \frac{1}{x-x_0}$ which is continuous on E , but unbounded.

For $\epsilon > 0$ and $\delta > 0$, there is a $x \in E$ such that $|x - x_0| < \delta$. Take t close enough to x_0 so $|f(t) - f(x_0)| > \epsilon$, but $|t - x| < \delta$. Thus, f is not uniformly continuous.

Consider $g(x) = \frac{1}{1+(x-x_0)^2}$ which is continuous on E and bounded since $g(x) \in (0,1)$. Since $\sup_{x \in E} g(x) = 1$, but $g(x) < 1$ for all $x \in E$, then g has no maximum on E .

11.4 Continuity and Connectedness**Theorem 11.4.1: Continuous functions map connected to connected**

If f is a continuous mapping of X into Y and E is a connected subset of X , then $f(E)$ is connected.

Proof

Suppose $f(E) = A \cup B$ where A and B are nonempty separated subsets of Y .

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Then $E = G \cup H$.

Since $A \subset \bar{A}$, $G \subset f^{-1}(\bar{A})$. Since f is continuous, then $f^{-1}(\bar{A})$ is closed so $\bar{G} \subset f^{-1}(\bar{A})$. Thus, $f(\bar{G}) \subset \bar{A}$.

Since $f(H) = B$ and $\bar{A} \cap B$ is empty, $\bar{G} \cap H$ is empty. Similarly, $G \cap \bar{H}$ is empty so G and H are separated which contradicts that $E = G \cup H$ is connected.

Theorem 11.4.2: Generalized Intermediate Value Theorem

Let f be a continuous real function on $[a, b]$. If $f(a) < c < f(b)$, then there exists $x \in (a, b)$ such that $f(x) = c$.

Proof

Since $[a, b]$ is connected, then by [theorem 11.4.1](#), $f([a, b])$ is a connected subset of \mathbb{R}^1 . Thus, by [theorem 7.2.2](#), any c where $f(a) < c < f(b)$ is $c \in f(x)$ for some $x \in [a, b]$.

11.5 Discontinuities**Definition 11.5.1: Right and Left Limits**

Let f be defined on (a, b) .

Then for any x where $x \in [a, b)$, $f(x+) = q$ if:

$f(t_n) \rightarrow q$ as $n \rightarrow \infty$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \rightarrow x$.

Then for any x where $x \in (a, b]$, $f(x-) = q$ if:

$f(t_n) \rightarrow q$ as $n \rightarrow \infty$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \rightarrow x$.

Then $\lim_{t \rightarrow x} f(t)$ exists if and only if $f(x-) = f(x+) = \lim_{t \rightarrow x} f(t)$.

Definition 11.5.2: Types of discontinuities

If f is discontinuous at x , but $f(x+)$ and $f(x-)$ exists, then f have a simple discontinuity of the first kind else it is a discontinuity of the second kind.

Thus, a simple discontinuity is either:

- $f(x-) \neq f(x+)$
- $f(x-) = f(x+) \neq f(x)$

11.6 Monotonic Functions**Definition 11.6.1: Monotonic**

$f: (a, b) \rightarrow \mathbb{R}$ is monotonically increasing if $f(x) \leq f(y)$ for $a < x < y < b$.

$f: (a, b) \rightarrow \mathbb{R}$ is monotonically decreasing if $f(x) \geq f(y)$ for $a < x < y < b$.

Theorem 11.6.2: Right and Left Limits of monotonic functions on (a, b)

Let f be monotonically increasing on (a, b) .

Then $f(x+)$ and $f(x-)$ exists at every $x \in (a, b)$ where:

$$\sup_{t \in (a, x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x, b)} f(t)$$

Furthermore, for $a < x < y < b$, $f(x+) \leq f(y-)$.

Properties analogous for monotonically decreasing functions.

Proof

Since f is monotonically increasing, then for $t \in (a, x)$, $f(t)$ is bounded above by $f(x)$ and thus, by the least upper bounded property, $\sup_{t \in (a, x)} f(t)$ exists.

For $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_{t \in (a, x)} f(t) - \epsilon < f(x - \delta) \leq \sup_{t \in (a, x)} f(t)$ for $a < x - \delta < x$. Since $f(x - \delta) \leq f(t) \leq \sup_{t \in (a, x)} f(t)$ for $t \in (x - \delta, x)$, then $|f(t) - \sup_{t \in (a, x)} f(t)| < \epsilon$ for $t \in (x - \delta, x)$ so $f(x-) = \sup_{t \in (a, x)} f(t)$.

For $\epsilon > 0$, there exists a $\delta > 0$ such that $\inf_{t \in (x, b)} f(t) < f(x + \delta) \leq \inf_{t \in (x, b)} f(t) + \epsilon$ for $x < x + \delta < b$. Since $f(x + \delta) \geq f(t) \geq \inf_{t \in (x, b)} f(t)$ for $t \in (x, x + \delta)$, then $|f(t) - \inf_{t \in (x, b)} f(t)| < \epsilon$ for $t \in (x, x + \delta)$ so $f(x+) = \inf_{t \in (x, b)} f(t)$.

Thus, $\sup_{t \in (a, x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x, b)} f(t)$.

If $a < x < y < b$, then:

$$f(x+) = \inf_{t \in (x, b)} f(t) = \inf_{t \in (x, y)} f(t) \leq \sup_{t \in (x, y)} f(t) = \sup_{t \in (a, y)} f(t) = f(y-)$$

Corollary 11.6.3: Monotonics can only have simple discontinuities

Monotonic functions have no discontinuities of the second kind

Proof

By **theorem 11.6.2**, $f(x_-)$ and $f(x_+)$ exists and thus, f can only have simple discontinuities and not discontinuities of the second kind.

Theorem 11.6.4: Discontinuities of monotonics is at most countable

Let f be monotonic on (a,b) .

Then the set of points of (a,b) where f is discontinuous is at most countable.

Proof

Suppose f is increasing. Let E be the set of points where f is discontinuous. Then for $x \in E$, there is a rational $r(x)$ where $f(x_-) < r(x) < f(x_+)$.
 Then for $x_1 < x_2$, by **theorem 11.6.2**, $f(x_1+) \leq f(x_2-)$. Then:

$$f(x_{1-}) < r(x_1) < f(x_{1+}) \leq f(x_{2-}) < r(x_2) < f(x_{2+})$$

 Thus, $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.
 Since there is a 1-1 correspondence between E and a subset of rational numbers which is countable, then E is at most countable.
 If f is decreasing, proof is analogous.

11.7 Infinite Limits / Limits at Infinity**Definition 11.7.1: Neighborhoods in extended reals**

For any real c , a neighborhood of $+\infty = (c, +\infty)$.

For any real c , a neighborhood of $-\infty = (-\infty, c)$.

Definition 11.7.2: Infinite Limits

Let real function f be defined on $E \subset \mathbb{R}$.

Then $f(t) \rightarrow A$ as $t \rightarrow x$ where A and x are extended reals if:

For every neighborhood U of A , there is a neighborhood V of x such that $V \cap E \neq \emptyset$ and $f(t) \in U$ for all $t \in V \cap E$ where $t \neq x$.

Theorem 11.7.3: Arithmetic operations on functions of infinite limits

Let f, g be defined on $E \subset \mathbb{R}$ where $f(t) \rightarrow A$ and $g(t) \rightarrow B$ as $t \rightarrow x$.

(a) If $f(t) \rightarrow A'$, then $A' = A$.

(b) $(f+g)(t) \rightarrow A + B$

(c) $(fg)(t) \rightarrow AB$

(d) $\frac{f}{g}(t) \rightarrow \frac{A}{B}$

12 Differentiation

12.1 Derivative of a function

Definition 12.1.1: Derivative

Let f be defined on any $x \in [a, b]$.

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \text{ for } t \neq x$$

The derivative of f at x :

$$f'(x) = \lim_{t \rightarrow x} \phi(t)$$

if the limit exist as defined by [definition 11.1.1](#).

If f' is defined at x , then f is differentiable at x .

Theorem 12.1.2: Differentiability \rightarrow Continuity

Let f be defined on $[a, b]$.

If f is differentiable at $x \in [a, b]$, then f is continuous at x .

Proof

As $t \rightarrow x$:

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0$$

Theorem 12.1.3: Arithmetic operations on differentiation

Suppose f, g are defined on $[a, b]$ and differentiable on $x \in [a, b]$. Then $f+g$, fg , and $\frac{f}{g}$ are differentiable at x :

(a) $(f+g)'(x) = f'(x) + g'(x)$

Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{(f+g)(t) - (f+g)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{f(t) - f(x) + g(t) - g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} + \lim_{t \rightarrow x} \frac{g(t) - g(x)}{t - x} = f'(x) + g'(x) \end{aligned}$$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{(fg)(t) - (fg)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{[f(t) - f(x)]g(t)}{t - x} + \lim_{t \rightarrow x} \frac{f(x)[g(t) - g(x)]}{t - x} \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

(c) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{\left(\frac{f}{g}\right)(t) - \left(\frac{f}{g}\right)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t - x} = \lim_{t \rightarrow x} \frac{\frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)(t - x)}}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)(t - x)} \\ &= \lim_{t \rightarrow x} \frac{[f(t) - f(x)]g(x)}{g(t)g(x)(t - x)} + \lim_{t \rightarrow x} \frac{f(x)[g(x) - g(t)]}{g(t)g(x)(t - x)} \\ &= \frac{f'(x)g(x)}{g^2(x)} + \frac{f(x)[-g'(x)]}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \end{aligned}$$

Theorem 12.1.4: Differentiation Chain Rule

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at $x \in [a,b]$, g is defined on interval I containing $f([a,b])$, and g is differentiable at $f(x)$.

If $h(t) = g(f(t))$, then h is differentiable at x and $h'(x) = g'(f(x)) \cdot f'(x)$

Proof

Since f is differentiable at x and g is differentiable at $f(x)$, then:

$$\begin{aligned} f(t) - f(x) &= (t-x) [f'(x) + u(t)] && \text{for } t \in [a,b] \text{ and } \lim_{t \rightarrow x} u(t) = 0 \\ g(s) - g(f(x)) &= (s-f(x)) [g'(f(x)) + v(s)] && \text{for } s \in I \text{ and } \lim_{s \rightarrow f(x)} v(s) = 0 \end{aligned}$$

Thus:

$$\begin{aligned} \lim_{t \rightarrow x} \frac{h(t) - h(x)}{t - x} &= \lim_{t \rightarrow x} \frac{g(f(t)) - g(f(x))}{t - x} \\ &= \lim_{t \rightarrow x} \frac{(f(t) - f(x)) [g'(f(x)) + v(f(t))]}{t - x} \\ &= \lim_{t \rightarrow x} \frac{(t - x) [f'(x) + u(t)] [g'(f(x)) + v(f(t))]}{t - x} \\ &= g'(f(x)) \cdot f'(x) + f'(x) \cdot 0 + g'(f(x)) \cdot 0 + 0 \cdot 0 = g'(f(x)) \cdot f'(x) \end{aligned}$$

12.2 Mean Value Theorems**Definition 12.2.1: Local Extrema**

Let real-valued $f \in X$.

Then f has a local maximum at $p \in X$ if:

There is $\delta > 0$ such that for all $q \in X$ where $d_X(q, p) < \delta$, $f(q) \leq f(p)$.

Then f has a local minimum at $p \in X$ if:

There is $\delta > 0$ such that for all $q \in X$ where $d_X(q, p) < \delta$, $f(q) \geq f(p)$.

Theorem 12.2.2: Derivative at local extrema is 0

Let f be defined on $[a,b]$.

If f has a local maximum at $x \in (a,b)$ and $f'(x)$ exists, then $f'(x) = 0$.

If f has a local minimum at $x \in (a,b)$ and $f'(x)$ exists, then $f'(x) = 0$.

Proof

Suppose x is a local maximum.

Then there is a $\delta > 0$ such that for all $t \in (a,b)$ where $|t - x| < \delta$, then $f(t) \leq f(x)$.

Then for $t < x$, $\frac{f(t) - f(x)}{t - x} \geq 0$. Thus, $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = f'(x) \geq 0$.

For $t > x$, $\frac{f(t) - f(x)}{t - x} \leq 0$. Thus, $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = f'(x) \leq 0$.

Since $f'(x)$ exists, then $f'(x) = 0$.

Proof is analogous for local minimum.

Theorem 12.2.3: Generalized Mean Value Theorem

If f, g are continuous real functions on $[a,b]$ and differentiable on (a,b) , then there is a $x \in (a,b)$ such that $[f(b) - f(a)] \cdot g'(x) = [g(b) - g(a)] \cdot f'(x)$.

Proof

Let $h(t) = [f(b) - f(a)] \cdot g(t) - [g(b) - g(a)] \cdot f(t)$ for $t \in [a,b]$.

Since f, g are continuous on $[a,b]$ and differentiable on (a,b) , then h is continuous on $[a,b]$ and differentiable on (a,b) . Also, $h(a) = f(b)g(a) - f(a)g(b) = h(b)$.

If h is constant, then $h'(x) = 0$ and thus, theorem holds true for every $x \in (a,b)$.

If $h(t) > h(a)$ for some $t \in (a,b)$, let $x \in [a,b]$ where h attains a local maximum. If $h(t) < h(a)$ for some $t \in (a,b)$, let $x \in [a,b]$ where h attains a local minimum. Then by **theorem 12.2.2**, $h'(x) = 0$ and thus, theorem holds true at local extrema.

Theorem 12.2.4: Mean Value Theorem

If f is a real continuous function on $[a,b]$ and differentiable on (a,b) , then there is a $x \in (a,b)$ such that $f(b) - f(a) = (b-a) f'(x)$.

Proof

From **theorem 12.2.3**, let $g(x) = x$.

Theorem 12.2.5: Sign of derivative determines increasing/decreasing

Suppose f is differentiable on (a,b) .

- (a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotonically increasing.
- (b) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.
- (c) If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is monotonically decreasing

Proof

From **theorem 12.2.4**, $f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$ for $x \in (x_1, x_2) \subset (a,b)$.

If $f'(x) \geq 0$ for all $x \in (a,b)$, then $f(x_2) - f(x_1) \geq 0$. Since $f(x_2) \geq f(x_1)$ for $x_2 > x_1$, then f is monotonically increasing.

If $f'(x) = 0$ for all $x \in (a,b)$, then $f(x_2) - f(x_1) = 0$. Since $f(x_2) = f(x_1)$ for $x_2 > x_1$, then f is constant.

If $f'(x) \leq 0$ for all $x \in (a,b)$, then $f(x_2) - f(x_1) \leq 0$. Since $f(x_2) \leq f(x_1)$ for $x_2 > x_1$, then f is monotonically decreasing.

12.3 Continuity of Derivatives

Theorem 12.3.1: Intermediate values of derivatives exists

Suppose f is a real differentiable function on $[a,b]$ and $f'(a) < \lambda < f'(b)$.

Then there is a $x \in (a,b)$ such that $f'(x) = \lambda$.

Statement holds true if $f'(a) > f'(b)$.

Proof

Suppose $f'(a) < \lambda < f'(b)$. Let $g(t) = f(t) - \lambda t$.

Since $f(t), t$ are differentiable on $[a,b]$, then $g(t)$ is differentiable on $[a,b]$.

Then $g'(a) = f'(a) - \lambda < 0$ so $g(t_1) < g(a)$ for some $t_1 \in (a,b)$.

Also, $g'(b) = f'(b) - \lambda > 0$ so $g(t_2) < g(b)$ for some $t_2 \in (a,b)$.

Thus, there is a x where $g(x)$ is a local minimum so $g'(x) = 0$ and thus, $f'(x) = \lambda$.

Corollary 12.3.2: Differentiable functions have no simple discontinuities

If f is differentiable on $[a,b]$, then f' cannot have simple discontinuities on $[a,b]$.

Proof

By **theorem 12.3.1**, $f'(x)$ exists for any $x \in [a,b]$.

12.4 L'Hospital's Rule

Theorem 12.4.1: L'Hospital's Rule

Suppose f, g are real and differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$.

Suppose $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \rightarrow A$. If either:

- $\lim_{x \rightarrow a} f(x) \rightarrow 0$ and $\lim_{x \rightarrow a} g(x) \rightarrow 0$
- $\lim_{x \rightarrow a} g(x) \rightarrow \infty$ or $\lim_{x \rightarrow a} g(x) \rightarrow -\infty$

Then, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow A$.

Statement holds true if $x \rightarrow b$.

Proof

Consider the case $-\infty \leq A < \infty$.

Choose q such that $A < q$ and r such that $A < r < q$.

Thus, there is a $c \in (a, b)$ such that $a < x < c$ for $\frac{f'(x)}{g'(x)} < r$.

For $a < x < y < c$, then by [theorem 12.2.3](#), there is a $t \in (x, y)$ such that:

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$$

If $\lim_{x \rightarrow a} f(x) \rightarrow 0$ and $\lim_{x \rightarrow a} g(x) \rightarrow 0$, then as $x \rightarrow a$, $\frac{f(y)}{g(y)} \leq r < q$ for $y \in (a, c)$.

If $\lim_{x \rightarrow a} g(x) \rightarrow \infty$, then keeping y fixed, choose $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Thus:

$$\begin{aligned} \frac{g(x)-g(y)}{g(x)} \cdot \frac{f(x)-f(y)}{g(x)-g(y)} &< \frac{g(x)-g(y)}{g(x)} \cdot r \text{ for } x \in (a, c_1) \\ \frac{f(x)}{g(x)} &< r - r \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \end{aligned}$$

Thus as $x \rightarrow a$, there is a $c_2 \in (a, c_1)$ such that $\frac{f(x)}{g(x)} < r < q$ for $x \in (a, c_2)$.

Proof is analogous if $\lim_{x \rightarrow a} g(x) \rightarrow -\infty$.

Thus, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow A$.

12.5 Derivative of Higher Order

Definition 12.5.1: Derivative of Higher Order

If f has a derivative f' on an interval and f' is differentiable, then the derivative of f' is f'' , the second derivative of f . Then, $f^{(n)}$ is the n th derivative of f .

For $f^{(n)}(x)$ to exist at x , $f^{(n-1)}(t)$ must exist in a neighborhood of x and $f^{(n-1)}$ must be differentiable at x .

If $f^{(n-1)}$ exist in a neighborhood of x , then $f^{(n-2)}$ must be differentiable in that neighborhood and so on until f is differentiable on that neighborhood.

12.6 Taylor's Theorem

Theorem 12.6.1: Taylor's Theorem

Suppose f is a real function on $[a,b]$, $n \in \mathbb{Z}_+$, $f^{(n-1)}$ is continuous on $[a,b]$, $f^n(t)$ exists at every $t \in (a,b)$.

Let $\alpha, \beta \in [a,b]$ be distinct and $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k$.

Then there exists a x between α and β such that $f(\beta) = P(\beta) + \frac{f^n(x)}{n!} (\beta - \alpha)^n$

Proof

Let M be the number defined by $f(\beta) = P(\beta) + M(\beta - \alpha)^n$.
 Let $g(t) = f(t) - P(t) - M(t - \alpha)^n$ for $t \in [\alpha, \beta]$. Thus, $g^{(n)}(t) = f^{(n)}(t) - n!M$.
 Also since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = [0, n-1]$, then $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$.
 Since the choice of M gives $g(\beta) = 0$, then by the Mean Value Theorem, $g'(x_1) = 0$ for some x_1 between α and β .
 Since $g'(\alpha) = 0$, then $g''(x_2) = 0$ for some x_2 between α and x_1 .
 Thus, $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1} so x_n is between α and β .
 Thus, there exists an $x_n \in (\alpha, \beta)$ such that:

$$0 = g^{(n)}(x_n) = f^{(n)}(x_n) - n!M$$

$$M = \frac{f^n(x_n)}{n!}$$

12.7 Differentiation of Vector-Valued Functions

Definition 12.7.1: Extending derivative to Vector-Valued Functions

For vector-valued function $f: t \in [a,b] \rightarrow \mathbb{R}^k$, the derivative of f at x :

$$f'(x) = \lim_{t \rightarrow x} \left| \frac{f(t) - f(x)}{t - x} \right|$$

if the limit exist as defined by [definition 14.1.1](#).

If $f = (f_1, \dots, f_k)$, then $f' = (f'_1, \dots, f'_k)$ and f is differentiable at x if and only if f_1, \dots, f_k are differentiable at x .

Thus, by [theorem 11.2.7](#), these theorems hold true for vector-valued functions:

- [12.1.2](#): If f is differentiable at x , then f is continuous at x .
- [12.1.3a](#): If f, g are differentiable at x , then $f+g, f \cdot g$ are differentiable at x .

However, [theorem 12.2.4: Mean Value Theorem](#) and [theorem 12.4.1: L'Hospital's Rule](#) does not always hold true since [theorem 12.1.3b/c](#), multiplying/dividing vectors by vectors, is not defined for vector-valued functions.

Theorem 12.7.2: Mean Value Theorem for \mathbb{R}^k

Suppose f is a continuous mapping of $[a,b]$ into \mathbb{R}^k and f is differentiable on (a,b) . Then there is a $x \in (a,b)$ such that $|f(b) - f(a)| \leq (b-a) |f'(x)|$

Proof

Let $z = f(b) - f(a)$ and define $\phi(t) = z \cdot f(t)$ for $t \in [a,b]$.
 Then $\phi(t)$ is real-valued continuous on $[a,b]$ and differentiable on (a,b) .
 Then by the Mean Value Theorem, for some $x \in (a,b)$:

$$\phi(b) - \phi(a) = (b-a) \phi'(x) = (b-a) z \cdot f'(x)$$
 Since $\phi(b) - \phi(a) = z \cdot f(b) - z \cdot f(a) = z \cdot z = |z|^2$, then by the Schwarz Inequality:

$$|z|^2 = (b-a) |z \cdot f'(x)| \leq (b-a) |z| |f'(x)|$$

$$|z| \leq (b-a) |f'(x)|$$

$$|f(b) - f(a)| \leq (b-a) |f'(x)|$$

13 Riemann-Stieltjes Integral

13.1 Riemann-Stieltjes Integral

Definition 13.1.1: Riemann Integral

For $[a,b]$, let $a = x_0 \leq x_1 \leq \dots \leq x_n = b$ and $\Delta x_i = x_i - x_{i-1}$.

Suppose real f is bounded. Then for each partition P , $\{x_0, \dots, x_n\}$,

let $m_i = \inf f([x_{i-1}, x_i])$ and $M_i = \sup f([x_{i-1}, x_i])$.

Then let $L(P,f) = \sum_{i=1}^n m_i \Delta x_i$ and $U(P,f) = \sum_{i=1}^n M_i \Delta x_i$. Thus, over all P :

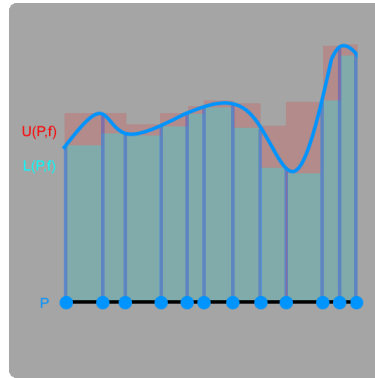
Lower Riemann Integral: $\int_a^b f(x) dx = \sup L(P,f)$

Upper Riemann Integral: $\int_a^b f(x) dx = \inf U(P,f)$

If $\int_a^b f(x) dx = \overline{\int}_a^b f(x) dx$, then f is Riemann-integrable ($f \in \mathcal{R}$) and $\int_a^b f(x) dx$.

Since f is bounded, there are m, M such that $m \leq f(x) \leq M$.

Thus, $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$.



Definition 13.1.2: Riemann-Stieltjes Integral

Let α be monotonically increasing on $[a,b]$.

Then for each partition P , let $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

For real bounded f , let $L(P,f,\alpha) = \sum_{i=1}^n m_i \Delta \alpha_i$ and $U(P,f,\alpha) = \sum_{i=1}^n M_i \Delta \alpha_i$.

Thus, $\int_a^b f(x) d\alpha(x) = \sup L(P,f,\alpha)$ and $\overline{\int}_a^b f(x) d\alpha(x) = \inf U(P,f,\alpha)$.

If $\int_a^b f(x) d\alpha(x) = \overline{\int}_a^b f(x) d\alpha(x)$, then $f \in \mathcal{R}(\alpha)$ with value $\int_a^b f(x) d\alpha(x)$.

Definition 13.1.3: Refinement

Partition Q is a refinement of P if $P \subset Q$.

For partitions P_1, P_2 , then $Q = P_1 \cup P_2$ is the common refinement.

Theorem 13.1.4: Refinements monotonically increase $L(P,f)$ & decrease $U(P,f)$

If Q is a refinement of P , then:

$$L(P,f,\alpha) \leq L(Q,f,\alpha) \leq U(Q,f,\alpha) \leq U(P,f,\alpha)$$

Proof

Since Q is a refinement of P , then $P \subset Q$.

Suppose $Q = P \cup \{x^*\}$ where $P = \{x_0, \dots, x_n\}$ and $Q = \{x_0, \dots, x_{k-1}, x^*, x_k, \dots, x_n\}$.

Regardless of anymore points, the process below will be analogous.

$$L(P,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta \alpha_i + m_{[x_{k-1}, x_k]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + m_{[x_{k-1}, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n m_i \Delta \alpha_i$$

$$L(Q,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta \alpha_i + m_{[x_{k-1}, x^*]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + m_{[x^*, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n m_i \Delta \alpha_i$$

Since $[x_{k-1}, x^*], [x^*, x_k] \subset [x_{k-1}, x_k]$, then $m_{[x_{k-1}, x_k]} \leq m_{[x_{k-1}, x^*]}, m_{[x^*, x_k]}$. Thus:

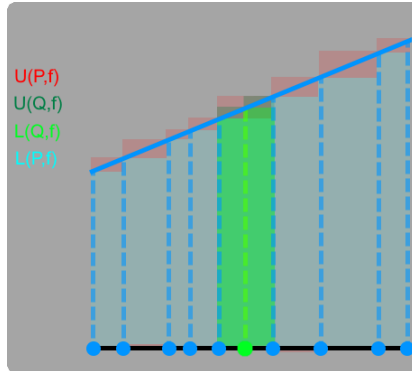
$$L(Q,f,\alpha) - L(P,f,\alpha) = (m_{[x_{k-1}, x^*]} - m_{[x_{k-1}, x_k]}) [\alpha(x^*) - \alpha(x_{k-1})] \\ + (m_{[x^*, x_k]} - m_{[x_{k-1}, x_k]}) [\alpha(x_k) - \alpha(x^*)] \geq 0.$$

$$U(P,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta \alpha_i + M_{[x_{k-1}, x_k]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + M_{[x_{k-1}, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n M_i \Delta \alpha_i$$

$$U(Q,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta \alpha_i + M_{[x_{k-1}, x^*]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + M_{[x^*, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n M_i \Delta \alpha_i$$

Since $[x_{k-1}, x^*], [x^*, x_k] \subset [x_{k-1}, x_k]$, then $M_{[x_{k-1}, x_k]} \geq M_{[x_{k-1}, x^*]}, M_{[x^*, x_k]}$. Thus:

$$U(Q,f,\alpha) - U(P,f,\alpha) = (M_{[x_{k-1}, x^*]} - M_{[x_{k-1}, x_k]}) [\alpha(x^*) - \alpha(x_{k-1})] \\ + (M_{[x^*, x_k]} - M_{[x_{k-1}, x_k]}) [\alpha(x_k) - \alpha(x^*)] \leq 0.$$

**Theorem 13.1.5: Lower Riemann Integral \leq Upper Riemann Integral**

$$\int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$$

Proof

For partitions P_1, P_2 , let $L(P_1, f, \alpha)$ and $U(P_2, f, \alpha)$. Let $P = P_1 \cup P_2$. Thus:

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$$

Thus, over all partitions for P_1 , $\int_a^b f d\alpha \leq U(P_2, f, \alpha)$

Thus, over all partitions for P_2 , $\int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$

Theorem 13.1.6: Riemann-Integrability ϵ Definition

$f \in \mathcal{R}(\alpha)$ if and only if for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

Proof

If $f \in \mathcal{R}(\alpha)$, then $\int_a^f d\alpha = \overline{\int}_a^b f d\alpha = \int_a^b f d\alpha$. For $\epsilon > 0$, there exists partitions P_1, P_2 :

$$\int_a^b f d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2} \quad U(P_2, f, \alpha) - \int_a^b f d\alpha < \frac{\epsilon}{2}$$

Then for partition $P = P_1 \cup P_2$, then:

$$\int_a^b f d\alpha - L(P, f, \alpha) \leq \int_a^b f d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2}$$

$$U(P, f, \alpha) - \int_a^b f d\alpha \leq U(P_2, f, \alpha) - \int_a^b f d\alpha < \frac{\epsilon}{2}$$

Thus, $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

For $\epsilon > 0$, there is a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Since $L(P, f, \alpha) \leq \int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha \leq U(P, f, \alpha)$, then $\overline{\int}_a^b f d\alpha - \int_a^b f d\alpha < \epsilon$.

Theorem 13.1.7: Properties of Riemann-Integrability

(a) If $f \in \mathcal{R}(\alpha)$, then $U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon$ for every refinement of P , Q

Proof

By **theorem 13.1.6**, for $\epsilon > 0$, there is a P such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Then by **theorem 13.1.4**, for any refinement of P , Q , then:

$$U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon.$$

(b) If $f \in \mathcal{R}(\alpha)$ where $P = \{x_0, \dots, x_n\}$ and $s_i, t_i \in [x_{i-1}, x_i]$, then:

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

Proof

By **theorem 13.1.6**, for $\epsilon > 0$, there is a P such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\sum_{i=1}^n M_i \Delta\alpha_i - \sum_{i=1}^n m_i \Delta\alpha_i < \epsilon$$

Since $s_i, t_i \in [x_{i-1}, x_i]$, then $m_i \leq f(s_i), f(t_i) \leq M_i$.

Thus, $|f(s_i) - f(t_i)| \leq M_i - m_i$.

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^n M_i - m_i \Delta\alpha_i \leq \epsilon$$

(c) If $f \in \mathcal{R}(\alpha)$ where $P = \{x_0, \dots, x_n\}$ and $t_i \in [x_{i-1}, x_i]$, then:

$$|\sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha| < \epsilon$$

Proof

Since $\sup L(P, f, \alpha) = \int_a^b f d\alpha = \int_a^b f d\alpha = \overline{\int}_a^b f d\alpha = \inf U(P, f, \alpha)$, then:

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha)$$

Since $t_i \in [x_{i-1}, x_i]$, then $m_i \leq f(t_i) \leq M_i$. Thus:

$$\begin{aligned} L(P, f, \alpha) &= \sum_{i=1}^n m_i \Delta\alpha_i \leq \sum_{i=1}^n f(t_i) \Delta\alpha_i \\ &\leq \sum_{i=1}^n M_i \Delta\alpha_i = U(P, f, \alpha) \end{aligned}$$

Thus, $|\sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha| \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

13.2 Riemann-Integrable Functions

Theorem 13.2.1: Continuous functions are Riemann-Integrable

If f is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$

Proof

For $\epsilon > 0$, choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$. Since f is continuous and $[a, b]$ is compact, then f is uniformly continuous. Thus, for $\eta > 0$, there is a $\delta > 0$ such that for all $x, t \in [a, b]$ where $|x - t| < \delta$, then $|f(x) - f(t)| < \eta$. For partition P of $[a, b]$ such that $\Delta x_i < \delta$ for all $i = \{1, \dots, n\}$, then $M_i - m_i \leq \eta$ for each i . Thus:

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \leq \sum_{i=1}^n \eta \Delta \alpha_i = \eta [\alpha(b) - \alpha(a)] < \epsilon$$

Theorem 13.2.2: Monotonic functions are Riemann-Integrable

If f is monotonic on $[a, b]$ and α is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$

Proof

Since α is continuous on $[a, b]$, let $\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$ where $n \in \mathbb{Z}_+$. Let partition $P = \{\alpha(x_0), \dots, \alpha(x_n)\}$. Suppose f is monotonically increasing. Thus:

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n (M_i - m_i) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] = \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] \end{aligned}$$

For $\epsilon > 0$, there exists a n such that $\frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \epsilon$ so $f \in \mathcal{R}(\alpha)$.

If f is monotonically decreasing, then $\sum_{i=1}^n (M_i - m_i) = \sum_{i=1}^n [f(x_{i-1}) - f(x_i)]$.

Theorem 13.2.3: Bounded functions with finite discontinuities are Riemann-Integrable

If f is bounded on $[a, b]$ with finitely many discontinuities and α is continuous at every discontinuity, then $f \in \mathcal{R}(\alpha)$

Proof

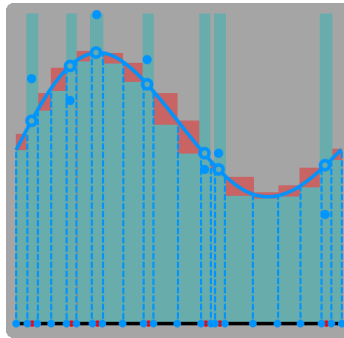
Since f is bounded, let $M = \sup |f(x)|$ and E be the set of discontinuities of f .

Since E is finite and α is continuous over E , then for $\epsilon > 0$, there are finitely many disjoint $[u_j, v_j]$ where $\sum [\alpha(v_j) - \alpha(u_j)] < \epsilon$ which cover E .

Let $K = [a, b] \setminus \bigcup (u_j, v_j)$ which is compact. Since f is continuous over compact K , then f is uniformly continuous over K . Thus, for $\epsilon > 0$, there is a $\delta > 0$ such that for $s, t \in K$ where $|s - t| < \delta$, then $|f(s) - f(t)| < \epsilon$.

Let partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ where each $\Delta x_i < \delta$ and if $x \in (u_j, v_j) \notin P$, but $u_j, v_j \in P$. Thus, $M_i - m_i \leq 2M$ for each i and $M_i - m_i \leq \epsilon$ unless x_{i-1} is a u_j , then:

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i = \sum_K (M_i - m_i) \Delta \alpha_i + \sum_{K^c} (M_i - m_i) \Delta \alpha_i \\ &\leq \epsilon \sum_K \Delta \alpha_i + 2M \sum_{K^c} \Delta \alpha_i \leq [\alpha(b) - \alpha(a)]\epsilon + 2M\epsilon \end{aligned}$$



Theorem 13.2.4: Composite of continuous-integrable functions are Riemann-Integrable

If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ where $f \in [m, M]$ and ϕ is continuous on $[m, M]$ such that $h(x) = \phi(f(x))$, then $h \in \mathcal{R}(\alpha)$

Proof

Since ϕ is continuous and $[m, M]$ is compact, then ϕ is uniformly continuous. Thus, for $\epsilon > 0$, there is a $0 < \delta < \epsilon$ such that for all $s, t \in [m, M]$ where $|s - t| \leq \delta$, then $|\phi(s) - \phi(t)| < \epsilon$.

Since $f \in \mathcal{R}(\alpha)$, there is a partition $P = \{x_0, \dots, x_n\}$ such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$$

For each $i = \{1, \dots, n\}$, let $i \in A$ if $M_i - m_i < \delta$ and $i \in B$ if $M_i - m_i \geq \delta$.

Let $m_i^* = \inf \phi(f([x_{i-1}, x_i]))$ and $M_i^* = \sup \phi(f([x_{i-1}, x_i]))$.

For A , since $M_i - m_i < \delta$, then $M_i^* - m_i^* \leq \epsilon$.

For B , $M_i^* - m_i^* \leq 2K$ where $K = \sup_{[m, M]} |\phi|$.

$$\delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2$$

$$\sum_{i \in B} \Delta \alpha_i \leq \delta < \epsilon$$

Thus:

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\ &\leq \epsilon \sum_{i \in A} \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i \\ &\leq \epsilon [\alpha(b) - \alpha(a)] + 2K\epsilon < \epsilon [\alpha(b) - \alpha(a) + 2K] \end{aligned}$$

13.3 Integral Properties**Theorem 13.3.1: Integral Additive Properties**

(a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$ and constant c , then $f_1 + f_2, cf_1 \in \mathcal{R}(\alpha)$ and

$$\int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$\int_a^b cf_1 \, d\alpha = c \int_a^b f_1 \, d\alpha$$

Proof

Since $f_1, f_2 \in \mathcal{R}(\alpha)$, then there are partitions P_1, P_2 such that for $\epsilon > 0$:

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2} \quad U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2}$$

Thus for partition $P = P_1 \cup P_2$:

$$U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) - L(P, f_2, \alpha) < \epsilon$$

$$U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) < \epsilon$$

For any partition Q :

$$\begin{aligned} L(Q, f_1, \alpha) + L(Q, f_2, \alpha) &\leq L(Q, f_1 + f_2, \alpha) \leq U(Q, f_1 + f_2, \alpha) \\ &\leq U(Q, f_1, \alpha) + U(Q, f_2, \alpha) \end{aligned}$$

Thus, $f_1 + f_2 \in \mathcal{R}(\alpha)$ where:

$$\begin{aligned} \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha &= \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \leq \int_a^b f_1 + f_2 \, d\alpha \\ &= \int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 + f_2 \, d\alpha \\ &\leq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \end{aligned}$$

Proof for cf_1 is analogous by replacing $\frac{\epsilon}{2}$ with $\frac{\epsilon}{c}$.

(b) If $f_1, f_2 \in \mathcal{R}(\alpha)$ and $f_1(x) \leq f_2(x)$ on $[a, b]$, then $\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$

Proof

Since $f_1, f_2 \in \mathcal{R}(\alpha)$, then by part a, $0 \leq \int_a^b f_2 - f_1 \, d\alpha = \int_a^b f_2 \, d\alpha - \int_a^b f_1 \, d\alpha$.

- (c) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $c \in (a, b)$, then $f \in \mathcal{R}(\alpha)$ on $[a, c], [c, b]$ and
- $$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

Proof

Since $f \in \mathcal{R}(\alpha)$ on $[a, b]$, there is a partition P of $[a, b]$ such that for $\epsilon > 0$:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

For partition P of $[a, b]$, let refinement of P , $Q = P \cup \{c\}$. Thus:

$$L(P, f, \alpha) \leq L(Q, f, \alpha) \leq U(Q, f, \alpha) \leq U(P, f, \alpha)$$

Thus, let $A = (P < c) \cup c \in [a, c]$ and $B = c \cup (c < P) \in (c, b)$:

$$\begin{aligned} L(Q, f, \alpha) &= \sum_Q m_q \Delta \alpha_q \\ &\leq \sum_A m_a \Delta \alpha_a + \sum_B m_b \Delta \alpha_b = L(A, f, \alpha) + L(B, f, \alpha) \end{aligned}$$

$$\begin{aligned} U(Q, f, \alpha) &= \sum_Q M_q \Delta \alpha_q \\ &\geq \sum_A M_a \Delta \alpha_a + \sum_B M_b \Delta \alpha_b = U(A, f, \alpha) + U(B, f, \alpha) \end{aligned}$$

Since Q is a refinement of P , then $U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon$. Thus:

$$0 \leq U(A, f, \alpha) + U(B, f, \alpha) - L(A, f, \alpha) - L(B, f, \alpha) < \epsilon$$

$$U(A, f, \alpha) - L(A, f, \alpha) < \epsilon \quad U(B, f, \alpha) - L(B, f, \alpha) < \epsilon$$

Thus, $f \in \mathcal{R}(\alpha)$ on $[a, c], [c, b]$ where:

$$\begin{aligned} \int_a^b f d\alpha &\leq \int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha \\ &= \int_a^c f d\alpha + \int_c^b f d\alpha \leq \int_a^b f d\alpha \end{aligned}$$

Since $\int_a^b f d\alpha, \int_a^c f d\alpha = \int_a^b f d\alpha$, then $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.

- (d) If $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$ and constant c , then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$, $f \in \mathcal{R}(c\alpha_1)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\int_a^b f d(c\alpha_1) = c \int_a^b f d\alpha_1$$

Proof

Since $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$, then there are partitions P_1, P_2 where for $\epsilon > 0$:

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\epsilon}{2} \quad U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\epsilon}{2}$$

Thus, for partition $P = P_1 \cup P_2$:

$$\sum_{i=1}^n (M_i - m_i) \Delta \alpha_{1i} < \frac{\epsilon}{2} \quad \sum_{i=1}^n (M_i - m_i) \Delta \alpha_{2i} < \frac{\epsilon}{2}$$

$$\sum_{i=1}^n (M_i - m_i) (\Delta \alpha_{1i} + \Delta \alpha_{2i}) < \epsilon$$

$$U(P, f, \alpha_1 + \alpha_2) - L(P, f, \alpha_1 + \alpha_2) < \epsilon$$

For any partition Q :

$$\begin{aligned} L(Q, f, \alpha_1) + L(Q, f, \alpha_2) &\leq L(Q, f, \alpha_1 + \alpha_2) \\ &\leq U(Q, f, \alpha_1 + \alpha_2) \\ &\leq U(Q, f, \alpha_1) + U(Q, f, \alpha_2) \end{aligned}$$

Thus, $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ where:

$$\begin{aligned} \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \leq \int_a^b f d(\alpha_1 + \alpha_2) \\ &= \int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d(\alpha_1 + \alpha_2) \\ &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \end{aligned}$$

Proof for $c\alpha_1$ is analogous by replacing $\frac{\epsilon}{2}$ with $\frac{\epsilon}{c}$.

Theorem 13.3.2: Integral Multiplicative Properties(a) If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, then $fg \in \mathcal{R}(\alpha)$ **Proof**Since $f, g \in \mathcal{R}(\alpha)$, then $f+g, f-g \in \mathcal{R}(\alpha)$.By **theorem 13.2.4**, let $\phi(t) = t^2$ which is continuous so $\phi(f+g) = (f+g)^2, \phi(f-g) = (f-g)^2 \in \mathcal{R}(\alpha)$.Thus, $4fg = (f+g)^2 - (f-g)^2 \in \mathcal{R}(\alpha)$.(b) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $|f| \in \mathcal{R}(\alpha)$ where $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$ **Proof**By **theorem 13.2.4**, let $\phi(t) = |t|$ which is continuous so $|f| \in \mathcal{R}(\alpha)$.Then choose $c = \pm 1$ such that $c \int f d\alpha \geq 0$. Then:

$$|\int f d\alpha| = c \int f d\alpha = \int c f d\alpha \leq \int |f| d\alpha$$

13.4 Change of Variable**Definition 13.4.1: Unit Step Function**

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Theorem 13.4.2: Integrating f over I centered at s If f is bounded on $[a, b]$ and continuous at $s \in (a, b)$ where $\alpha(x) = I(x-s)$, then:

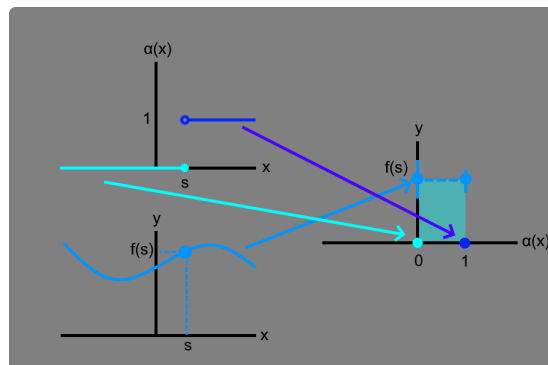
$$\int_a^b f d\alpha = f(s)$$

IntuitionIf $x < s < y$, then $\Delta I = I(y-s) - I(x-s) = 1 - 0 = 1$ else $\Delta I = 0$.So, $f(x)d\alpha(x) \approx f(x)\Delta I$ have only $f(s)\Delta I = f(s)$ since the others $\Delta I = 0$.**Proof**For partition $P = \{x_0, x_1, x_2, x_3\}$ where $x_1 = s$:

$$L(P, f, \alpha) = m_2 \quad U(P, f, \alpha) = M_2$$

Since f is continuous at s , then for $\epsilon > 0$, there is a $\delta > 0$ where for all $x \in [s, s+\delta]$, then $|f(x) - f(s)| < \frac{\epsilon}{2}$. Thus, $M_2 - m_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ so $\int f d\alpha$ exist where:

$$f(s) - m_2 < \frac{\epsilon}{2} \text{ so } \underline{\int} f d\alpha = f(s) \quad M_2 - f(s) < \frac{\epsilon}{2} \text{ so } \overline{\int} f d\alpha = f(s)$$



Theorem 13.4.3: Integrating f over a step function

If $c_n \geq 0$ where $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b) , and $\alpha(x) = \sum c_n I(x - s_n)$. Then for continuous f on $[a,b]$:

$$\int_a^b f d\alpha = \sum c_n f(s_n)$$

Intuition

Similar to **theorem 13.4.2**, but over a step function. The $\{s_n\}$ determines where the steps are and the $\{\sum c_n\}$ determines the value at each step.

Thus, $f(x)d\alpha(x)$ have only:

$$f(s_n) \cdot (\text{value}_{\text{current step}} - \text{value}_{\text{previous step}}) = f(s_n) \cdot (\sum c_n - \sum c_{n-1}) = f(s_n) \cdot c_n$$

Proof

Since $\alpha(x) = \sum c_n I(x - s_n) \leq \sum c_n$, then by the comparison test, $\alpha(x)$ converges.

Since $c_n, I(x - s_n) \geq 0$, then $\alpha(x)$ is monotonic.

Since $a < s_n$ for any n , then $\alpha(a) = \sum c_n I(a - s_n) = \sum c_n 0 = 0$.

Since $b > s_n$ for any n , then $\alpha(b) = \sum c_n I(b - s_n) = \sum c_n 1 = \sum c_n$.

Since $\sum c_n$ converges, then for $\epsilon > 0$, there is a N such that $\sum_{n=N+1}^{\infty} c_n < \epsilon$.

Let $\alpha_1(x) = \sum_{n=1}^N c_n I(x - s_n)$ and $\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$. By **theorem 13.4.2**:

$$\int_a^b f d\alpha_1 = \int_a^b f d(\sum_{n=1}^N c_n I(x - s_n)) = \sum_{n=1}^N c_n f(s_n)$$

$$|\int_a^b f d\alpha_2| = \sum_{n=N+1}^{\infty} c_n f(s_n) \leq \sum_{n=N+1}^{\infty} c_n \sup(|f(x)|) = \sup(|f(x)|)\epsilon$$

Thus, $\int f d\alpha = \int f d(\alpha_1 + \alpha_2) = \int f d\alpha_1 + \int f d\alpha_2 = \sum_{n=1}^N c_n f(s_n) + \sup(|f(x)|)\epsilon$

Theorem 13.4.4: $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$

If $\alpha' \in \mathcal{R}$ on $[a,b]$ and f is real, bounded on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. Then:

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$$

Intuition

If α is differentiable on $[x,y]$, then by the Mean Value Theorem, there is a $t \in [x,y]$:

$$\alpha(x) - \alpha(y) = \alpha'(t) \cdot (x - y)$$

Since $d\alpha \approx \Delta\alpha(x) = \alpha'(t)\Delta x \approx \alpha'(x) dx$, then $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$.

Proof

Since $\alpha' \in \mathcal{R}$, then $\epsilon > 0$, there is a partition $P = \{x_0, \dots, x_n\}$ such that:

$$U(P, \alpha') - L(P, \alpha') < \epsilon$$

By the Mean Value Theorem, there are $t_i \in [x_{i-1}, x_i]$ such that $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$.

Then for $s_i \in [x_{i-1}, x_i]$:

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i \leq U(P, \alpha') - L(P, \alpha') < \epsilon$$

Let $M = \sup(|f(x)|)$. Since $\sum_{i=1}^n f(s_i)\Delta\alpha_i = \sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i$, then:

$$\begin{aligned} & |\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i| \\ &= |\sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i| \\ &\leq M |\sum_{i=1}^n \alpha'(t_i)\Delta x_i - \sum_{i=1}^n \alpha'(s_i)\Delta x_i| = M\epsilon \end{aligned}$$

Thus:

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\epsilon \quad \sum_{i=1}^n f(s_i)\Delta\alpha_i \geq L(P, f\alpha') + M\epsilon$$

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\epsilon \quad L(P, f, \alpha) \geq L(P, f\alpha') + M\epsilon$$

$$|\int f d\alpha - \int f\alpha' dx| < M\epsilon \quad |\int f d\alpha - \int f\alpha' dx| < M\epsilon$$

Thus, $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$.

Theorem 13.4.5: Integral Change of Variable: $\int_a^b f(x) dx = \int_A^B f(\phi(y))\phi'(y) dy$

Let strictly increasing continuous $\phi: [A,B] \rightarrow [a,b]$ and $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

Let $\beta(y) = \alpha(\phi(y))$ and $g(y) = f(\phi(y))$ for $y \in [A,B]$. Then $g \in \mathcal{R}(\beta)$ where:

$$\int_A^B g d\beta = \int_a^b f d\alpha$$

Intuition

Partition of $[a,b] = \{x_0, \dots, x_n\} \sim$ partition of $[A,B] = \{y_0, \dots, y_n\}$ where $x_i = \phi(y_i)$.
Thus, $g(y)d\beta(y) \approx f(\phi(y))\Delta\alpha(\phi(y)) = f(x)\Delta\alpha(x) \approx f(x)d\alpha$.

Proof

Since $f \in \mathcal{R}(\alpha)$, then for $\epsilon > 0$, there is a partition P such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

For partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$, there is a partition $Q = \{y_0, \dots, y_n\}$ of $[A,B]$ where $x_i = \phi(y_i)$. Thus:

$$L(Q, g, \beta) = L(Q, f(\phi(y)), \alpha(\phi(y))) = L(P, f(x), \alpha(x)) = L(P, f, \alpha)$$

$$U(Q, g, \beta) = U(Q, f(\phi(y)), \alpha(\phi(y))) = U(P, f(x), \alpha(x)) = U(P, f, \alpha)$$

Thus, $U(Q, g, \beta) - L(Q, g, \beta) = U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ so $g \in \mathcal{R}(\beta)$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Let $\alpha(x) = x$. Then $\beta(y) = \phi(y)$. If $\beta' \in \mathcal{R}$ on $[A,B]$, then by **theorem 13.4.5**:

$$\int_a^b f(x) dx = \int_a^b f d\alpha = \int_A^B g d\beta = \int_A^B g(y)\beta'(y) dy = \int_A^B f(\phi(y))\phi'(y) dy$$

13.5 Fundamental Theorem of Calculus

Theorem 13.5.1: If $F(x) = \int f(x)dx$, then $F'(x) = f(x)$

Let $f \in \mathcal{R}$ on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t) dt$.

Then F is continuous on $[a,b]$ and if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 where $F'(x_0) = f(x_0)$.

Intuition

If f is integrable, then $|F(x) - F(y)| = |\int_x^y f(t)dt| < \epsilon$ if x and y are close enough.
If f is continuous at $x_0 \in [t, y]$, then for close enough t, y :

$$|\frac{F(y)-F(t)}{y-t} - f(x_0)| = |\frac{1}{y-t} \int_t^y [f(x) - f(x_0)]| < \epsilon$$

Proof

Since $f \in \mathcal{R}$, then f is bounded. Let $|f(t)| \leq M$ for any $t \in [a,b]$. Then for $\epsilon > 0$, there is a $\frac{\epsilon}{M} > \delta > 0$ such that for all $x, y \in [a,b]$ where $|y - x| < \delta$, then:

$$|F(y) - F(x)| = |\int_a^y f(t)dt - \int_a^x f(t)dt| = |\int_x^y f(t)dt| \leq M|y - x| < M\delta < \epsilon$$

Thus, F is uniformly continuous on $[a,b]$.

Suppose f is continuous at x_0 . Then for $\epsilon > 0$, there is a $\delta > 0$ such that for all $t \in [a,b]$ where $|t - x_0| < \delta$, then $|f(t) - f(x_0)| < \epsilon$.

Thus, for $s, t \in [x_0 - \delta, x_0 + \delta]$ where $s < x_0 < t$:

$$\begin{aligned} |\frac{F(t)-F(s)}{t-s} - f(x_0)| &= |\frac{1}{t-s} \int_s^t f(x)dx - f(x_0)| \\ &= |\frac{1}{t-s} \int_s^t f(x)dx - \frac{1}{t-s}(t-s)f(x_0)| \\ &= |\frac{1}{t-s} \int_s^t f(x)dx - \frac{1}{t-s} \int_s^t f(x_0)dx| \\ &= |\frac{1}{t-s} \int_s^t [f(x) - f(x_0)]dx| < |\frac{1}{t-s}(t-s)\epsilon| = \epsilon \end{aligned}$$

Thus, $F'(x_0) = f(x_0)$.

Theorem 13.5.2: Fundamental Theorem of Calculus: $\int_a^b f(x) dx = F(b) - F(a)$

If $f \in \mathcal{R}$ on $[a,b]$ and there is a differentiable F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Intuition

Since F is differentiable, then by the Mean Value Theorem, there is a $t \in [x,y]$

$$F(y) - F(x) = (y - x) \cdot F'(t) = (y - x) \cdot f(t)$$

Thus, $\int_a^b f(x) dx \approx \sum f(t) \Delta x = \sum [F(x_i) - F(x_{i-1})] = F(b) - F(a)$

Proof

Since $f \in \mathcal{R}$, then for $\epsilon > 0$, there is a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ such that:

$$U(P,f) - L(P,f) < \epsilon$$

Since there is a differentiable F on $[a,b]$, then F is differentiable over any $[x_{i-1}, x_i]$.

Then by the Mean Value Theorem, there are $t_i \in (x_{i-1}, x_i)$ such that:

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) F'(t_i) = \Delta x_i f(t_i)$$

Thus, $\sum_{i=1}^n f(t_i) \Delta x_i = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(b) - F(a)$.

Since $\sum_{i=1}^n f(t_i) \Delta x_i \leq \sum_{i=1}^n \sup(f([x_{i-1}, x_i])) \Delta x_i = U(P,f)$, then:

$$|F(b) - F(a) - \int_a^b f(x) dx| = |\sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f(x) dx| \leq U(P,f) - L(P,f) < \epsilon$$

Theorem 13.5.3: Integration by Parts

Suppose F, G are differentiable on $[a,b]$ and $F' = f, G' = g \in \mathcal{R}$. Then:

$$\int_a^b F(x)g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x) dx$$

Intuition

By the derivative product rule, $(HG)' = H'G + HG'$. Then:

$$\int H'G dx = \int (HG)' - HG' dx = [HG]_a^b - \int HG' dx$$

Proof

Let $H(x) = F(x)G(x)$ where $H'(x) = f(x)G(x) + F(x)g(x)$.

Since F, G are differentiable and thus, continuous, then $F, G \in \mathcal{R}$.

Thus, $H' \in \mathcal{R}$. Then by **theorem 13.5.2**:

$$\int_a^b H'(x) dx = H(b) - H(a)$$

$$\int_a^b f(x)G(x) + F(x)g(x) dx = H(b) - H(a)$$

$$\int_a^b F(x)g(x) dx = H(b) - H(a) - \int_a^b f(x)G(x) dx$$

13.6 Integration of Vector-Valued Functions

Definition 13.6.1: Integration of Vector-Valued Functions

Let real f_1, \dots, f_k be defined on $[a, b]$ where $f = (f_1, \dots, f_k)$.

Then, let $f \in \mathcal{R}(\alpha)$ if each $f_i \in \mathcal{R}(\alpha)$ where $\int_a^b f \, d\alpha = (\int_a^b f_1 \, d\alpha, \dots, \int_a^b f_k \, d\alpha)$.

Thus, all these theorems hold true for vector-valued functions:

(a) **Theorem 13.3.1a**

If $f_1, f_2 \in \mathcal{R}(\alpha)$ and constant c , then:

$$f_1 + f_2 \in \mathcal{R}(\alpha) \text{ with } \int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$cf_1 \in \mathcal{R}(\alpha) \text{ with } \int_a^b cf_1 \, d\alpha = c \int_a^b f_1 \, d\alpha$$

(b) **Theorem 13.3.1c**

If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ where $c \in (a, b)$, then $f \in \mathcal{R}(\alpha)$ on $[a, c], [c, b]$ where:

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha$$

(c) **Theorem 13.3.1e**

If $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$ and constant c , then:

$$f \in \mathcal{R}(\alpha_1 + \alpha_2) \text{ with } \int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$$

$$f \in \mathcal{R}(c\alpha_1) \text{ with } \int_a^b f \, d(c\alpha_1) = c \int_a^b f \, d\alpha_1$$

(d) **Theorem 13.4.4**

If $\alpha' \in \mathcal{R}$ on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$.

$$\int_a^b f(x) \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$$

(e) **Theorem 13.5.2**

If $f \in \mathcal{R}$ and there is a differentiable F on $[a, b]$ such that $F' = f$, then:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Theorem 13.6.2: $|\int f \, d\alpha| \leq \int |f| \, d\alpha$

If $f: [a, b] \rightarrow \mathbb{R}^k$ where $f \in \mathcal{R}(\alpha)$, then $|f| \in \mathcal{R}(\alpha)$ where:

$$|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, d\alpha$$

Proof

For $f = (f_1, \dots, f_k)$, then $|f| = (f_1^2 + \dots + f_k^2)^{\frac{1}{2}}$.

Since $f \in \mathcal{R}(\alpha)$, then each $f_i \in \mathcal{R}(\alpha)$ so $f_1^2 + \dots + f_k^2 \in \mathcal{R}(\alpha)$.

Since $x^{\frac{1}{2}}$ is continuous on $[0, \infty)$, then by **theorem 13.2.4**, $|f| = (f_1^2 + \dots + f_k^2)^{\frac{1}{2}} \in \mathcal{R}(\alpha)$.

Let $y = (y_1, \dots, y_k)$ where each $y_i = \int f_i \, d\alpha$. Thus, $y = \int f \, d\alpha$ where:

$$|y|^2 = \sum_1^k y_i^2 = \sum_1^k (y_i \int f_i \, d\alpha) = \int (\sum y_i f_i) \, d\alpha$$

By the Schwarz inequality, $\sum y_i f_i(t) \leq |y||f(t)|$. Thus:

$$|y|^2 = \int (\sum y_i f_i) \, d\alpha \leq \int |y||f| \, d\alpha$$

$$|\int_a^b f \, d\alpha| = |y| \leq \int |f| \, d\alpha$$

13.7 Line Integrals

Definition 13.7.1: Rectifiable Curves

A curve in \mathbb{R}^k is a continuous $\gamma: [a, b] \rightarrow \mathbb{R}^k$.

If γ is 1-1, then γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is a closed curve.

For partition $P = \{x_0, \dots, x_n\}$ and curve γ on $[a, b]$, let:

$$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$$

Then the length of γ is defined:

$$\Lambda(\gamma) = \sup(\Lambda(P, \gamma))$$

If $\Lambda(\gamma) < \infty$, then γ is rectifiable.

Theorem 13.7.2: Line Integral of $\gamma = \int_a^b |\gamma'(x)| dx$

If γ' is continuous on $[a, b]$, then γ is rectifiable where

$$\Lambda(\gamma) = \int_a^b |\gamma'(x)| dx$$

Proof

Since γ is differentiable, then by [theorem 13.5.2](#), for $a \leq x_{i-1} < x_i \leq b$:

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(x) dx \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(x)| dx$$

Thus, for any partition $P = \{x_0, \dots, x_n\}$:

$$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^n \left(\int_{x_{i-1}}^{x_i} |\gamma'(x)| dx \right) = \int_a^b |\gamma'(x)| dx$$

$$\Lambda(\gamma) \leq \int_a^b |\gamma'(x)| dx$$

Since γ' is continuous on compact $[a, b]$, then γ' is uniformly continuous. Thus, for $\epsilon > 0$, there is a $\delta > 0$ such that for all $s, t \in [a, b]$ where $|s - t| < \delta$, then $|\gamma'(s) - \gamma'(t)| < \epsilon$.

Then for partition P where each $\Delta x_i < \delta$ and $x \in [x_{i-1}, x_i]$:

$$|\gamma'(x)| \leq |\gamma'(x_i)| + \epsilon$$

Then:

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |\gamma'(x)| dx &\leq (|\gamma'(x_i)| + \epsilon) \Delta x_i = |\gamma'(x_i)| \Delta x_i + \epsilon \Delta x_i \\ &= \left| \int_{x_{i-1}}^{x_i} [\gamma'(x) + \gamma'(x_i) - \gamma'(x)] dx \right| + \epsilon \Delta x_i \\ &\leq \left| \int_{x_{i-1}}^{x_i} \gamma'(x) dx \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(x)] dx \right| + \epsilon \Delta x_i \\ &\leq |\gamma(x_i) - \gamma(x_{i-1})| + \epsilon \Delta x_i + \epsilon \Delta x_i \end{aligned}$$

Thus:

$$\begin{aligned} \int_a^b |\gamma'(x)| dx &= \int_{x_0}^{x_1} |\gamma'(x)| dx + \dots + \int_{x_{n-1}}^{x_n} |\gamma'(x)| dx \\ &\leq \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon(b-a) = \Lambda(P, \gamma) + 2\epsilon(b-a) \end{aligned}$$

Since $\int_a^b |\gamma'(x)| dx \leq \Lambda(\gamma) + 2\epsilon(b-a) \leq \int_a^b |\gamma'(x)| dx + 2\epsilon(b-a)$, then:

$$\Lambda(\gamma) = \int_a^b |\gamma'(x)| dx.$$

14 Sequences and Series of Functions

14.1 Pointwise Convergence of Functions

Definition 14.1.1: Sequences and Series of Functions

Suppose $\{f_n\}$ is a sequence of functions defined on set E .

If $\{f_n(x)\}$ converges for any $x \in E$, then:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for } x \in E$$

So for $x \in E$ and $\epsilon > 0$, there is a N_x such that for $n \geq N_x$:

$$|f_n(x) - f(x)| < \epsilon$$

If $\sum f_n(x)$ converges for every $x \in E$, then:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ for } x \in E$$

14.2 Uniform Convergence of Functions

Definition 14.2.1: Uniform Convergence

$\{f_n\}$ converges uniformly on E to a function f if for all $x \in E$:

For $\epsilon > 0$, there is a $N \in \mathbb{Z}$ where for $n \geq N$, then $|f_n(x) - f(x)| \leq \epsilon$

$\sum f_n(X)$ converges uniformly if $\{s_n\}$ converges uniformly on E

where $\sum_{i=1}^n f_i(x) = s_n(x)$:

For $\epsilon > 0$, there is a $N \in \mathbb{Z}$ where for $m \geq n \geq N$, then $|\sum_{i=n}^m f_i(x)| \leq \epsilon$

Theorem 14.2.2: Cauchy Criterion for sequence of functions

$\{f_n\}$ converges uniformly on E if and only if:

For $\epsilon > 0$, there is a $N \in \mathbb{Z}$ where for $n, m \geq N$ and every $x \in E$, then:

$$|f_n(x) - f_m(x)| \leq \epsilon$$

Intuition

Convergent sequences are Cauchy and Cauchy sequences in \mathbb{R} are convergent.

Proof

If $\{f_n\}$ converges uniformly on E , then for $\epsilon > 0$, there is a N where for $n, m \geq N$:

$$\begin{aligned} |f_n(x) - f(x)| &\leq \frac{\epsilon}{2} & |f_m(x) - f(x)| &\leq \frac{\epsilon}{2} \\ |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

If for $\epsilon > 0$, there is a $N \in \mathbb{Z}$ where for $n, m \geq N$ and every $x \in E$ so

$|f_n(x) - f_m(x)| \leq \epsilon$, then $\{f_n\}$ is a Cauchy sequence in \mathbb{R}^k and thus, converges.

Then there is a $f(x)$ where $f(x) = \lim_{m \rightarrow \infty} f_m(x)$. Thus:

$$|f_n(x) - f(x)| \leq |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| \leq \epsilon$$

Theorem 14.2.3: Connection between Convergence and Uniform Convergence

Suppose for $x \in E$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$. Let $M_n = \sup_{x \in E} (|f_n(x) - f(x)|)$.

Then $\{f_n\}$ converges uniformly to f on E if and only if $\lim_{n \rightarrow \infty} M_n = 0$.

Intuition

Pointwise convergence implies for any particular x_0 and $\epsilon > 0$ so $|f_n(x_0) - f(x_0)| < \epsilon$.
Uniform convergence implies for every x and $\epsilon > 0$ so $|f_n(x) - f(x)| < \epsilon$.

Thus, uniform convergence implies pointwise convergence, but pointwise convergence might not imply uniform convergence since for $n \geq N_1$, $|f_n(x_0) - f(x_0)| < \epsilon$, but there might always exist $x_1 \neq x_0$ where $|f_n(x_1) - f(x_1)| \not< \epsilon$ until $N_2 > N_1$.

If $\sup_{x \in E} (|f_n(x) - f(x)|) \rightarrow 0$, then x_1 cannot exist and thus, pointwise implies uniform.

Proof

If $\{f_n\}$ converges uniformly to f on E , then for $\epsilon > 0$, there is a N where for $n \geq N$:

$$|f_n(x) - f(x)| \leq \epsilon \quad \text{for all } x \in E$$

Thus, $M_n = \sup_{x \in E} (|f_n(x) - f(x)|) \leq \epsilon$ so $\lim_{n \rightarrow \infty} M_n \leq \epsilon$.

If $\lim_{n \rightarrow \infty} M_n = 0$, then for $\epsilon > 0$, there is a N where for $n \geq N$ so $\lim_{n \rightarrow \infty} M_n \leq \epsilon$.
Since $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for $x \in E$, there is a N_x for each x where for $n \geq N_x$:

$$|f_n(x) - f(x)| \leq \epsilon$$

Since there is a N such that for $n \geq N$ so $M_n = \sup_{x \in E} (|f_n(x) - f(x)|) \leq \epsilon$, then there

is $\sup_{x \in E} (\{N_x\}) = N$ such that for all $x \in E$ where $n \geq N$:

$$|f_n(x) - f(x)| \leq \sup_{x \in E} (|f_n(x) - f(x)|) = M_n \leq \epsilon$$

Theorem 14.2.4: Condition for Uniform Convergence for Series

For $\{f_n\}$ defined on E , suppose $|f_n(x)| \leq M_n$ for any $x \in E$.

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on E .

Proof

If $\sum M_n$ converges, then for $\epsilon > 0$, there is a N where for $m \geq n \geq N$:

$$|\sum_{i=n}^m f_i(x)| \leq \sum_{i=n}^m |f_i(x)| \leq \sum_{i=n}^m M_i \leq \epsilon$$

14.3 Uniform Convergence and Continuity

Theorem 14.3.1: $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$

Suppose $\{f_n\}$ converges uniformly to f on a set E .

Let $x \in E$ where $\lim_{t \rightarrow x} f_n(t) = A_n$.

Then $\{A_n\}$ converges where $\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} A_n$.

Intuition

Since $\{f_n\}$ converges uniformly so for any t , then $\lim_{n \rightarrow \infty} f_n(t) = f(t)$.

For t near x , then $\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{t \rightarrow x} f(t)$.

Note uniform convergence is essential since $f_n \rightarrow f$ and $f_n(t) \rightarrow f(t)$ for any t including t near x . Since pointwise convergence possibly $f_n(t) \not\rightarrow f(t)$ for some t near x , then continuity possibly might not hold.

Proof

Since $\{f_n\}$ converges uniformly, then for $\epsilon > 0$, there is a N where for $m, n \geq N$ and every $t \in E$, then $|f_n(t) - f_m(t)| \leq \epsilon$. Then for $t \rightarrow x$:

$$|A_n - A_m| = |\lim_{t \rightarrow x} f_n(t) - \lim_{t \rightarrow x} f_m(t)| \leq \epsilon$$

Thus, $\{A_n\}$ is a Cauchy Sequence in \mathbb{R}^k so $\{A_n\}$ converges to $A = \lim_{n \rightarrow \infty} A_n$.

Since $\{A_n\}$ converges to A , then for $\epsilon > 0$, there is a N_1 where for $n \geq N_1$:

$$|A - A_n| \leq \frac{\epsilon}{3}$$

Since $\{f_n\}$ converges uniformly to f , then for $\epsilon > 0$, there is a N_2 where for $n \geq N_2$:

$$|f(t) - f_n(t)| \leq \frac{\epsilon}{3}.$$

Since there is a r such that for $t \in N_r(x)$, then:

$$|f_n(t) - \lim_{t \rightarrow x} f_n(t)| = |f_n(t) - A_n| \leq \frac{\epsilon}{3}$$

Thus, for $t \rightarrow x$, $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \leq \epsilon$.

Thus, $\lim_{t \rightarrow x} f(t) = A = \lim_{n \rightarrow \infty} A_n$.

Theorem 14.3.2: Uniform Convergence preserve Continuity

If $\{f_n\}$ converges uniformly to f on E where each f_n is continuous on E , then f is continuous on E .

Intuition

If each f_n is continuous:

$$\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

Proof

Since $\{f_n\}$ converges uniformly to f , then by **theorem 14.3.1**, for any $x \in E$:

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$$

Since each f_n is continuous, then:

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{t \rightarrow x} f(t)$$

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

Theorem 14.3.3: Decreasing, continuous sequence over compact converges uniformly

Suppose K is compact and

- (a) $\{f_n\}$ is a sequence of continuous functions on K
- (b) $\{f_n\}$ converges pointwise to a continuous f on K
- (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$

Then f_n converges uniformly to f on K .

Proof

Let $g_n = f_n - f$ so g_n is continuous where $g_n \geq g_{n+1}$.
 Thus, $\lim_{n \rightarrow \infty} g_n(x) = 0$ pointwise. For $\epsilon > 0$, let $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
 Since g_n is continuous and the set of $g_n(x) \geq \epsilon$ is closed, then K_n is closed. Since closed $K_n \subset$ compact K , then K_n is compact.
 Since $g_n \geq g_{n+1}$, then $K_{n+1} \subset K_n$. For any $x \in K$, $\lim_{n \rightarrow \infty} g_n(x) = 0$ so there is a N_x such that $x \notin K_n$ if $n > N_x$. Thus, any $x \notin \bigcap_{n=1}^{\infty} K_n$ so $\bigcap_{n=1}^{\infty} K_n = \emptyset$.
 Since $\bigcap_{n=1}^{\infty} K_n = \emptyset$, then K_n is empty for some N .
 Thus, $0 \leq g_n(x) < \epsilon$ for all $x \in K$ where $n \geq N$.

Definition 14.3.4: Supremum Norm

$\mathcal{C}(X)$ is the set of all complex, continuous, bounded functions in metric X .

If X is compact, then bounded is not needed

Then for each $f \in \mathcal{C}(X)$, associate a supremum norm:

$$\|f\| = \sup_{x \in X} |f(x)| < \infty$$

- (a) $\|f(x)\| = 0$ if and only if $f(x) = 0$ for every $x \in X$
- (b) Since $|f + g| \leq |f| + |g| \leq \|f\| + \|g\|$, then $\|f + g\| \leq \|f\| + \|g\|$

Then for $f, g \in \mathcal{C}(X)$, let distance $\|f - g\|$ and thus, $\mathcal{C}(X)$ is a metric space.

By **theorem 14.2.3**, $\{f_n\} \rightarrow f$ on $\mathcal{C}(X)$ if and only if $\{f_n\} \rightarrow f$ uniformly on X .

Theorem 14.3.5: $\mathcal{C}(X)$ is a complete metric space

$\mathcal{C}(X)$ is a complete metric space

Intuition

A Cauchy sequence $\{f_n\}$ is uniformly convergent to f .
 Since $\mathcal{C}(X)$ contain continuous functions, then f is continuous.
 Since functions in $\mathcal{C}(X)$ are bounded, then f is bounded.

Proof

Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$.
 Since $\{f_n\} \in \mathcal{C}(X)$, then each f_n is continuous and bounded.
 Then for $\epsilon > 0$, there is a N such that for $n, m \geq N$, then:
 $|f_n - f_m| \leq \|f_n - f_m\| \leq \epsilon$
 Then by **theorem 14.2.2**, $\{f_n\}$ converges uniformly to f .
 Since each f_n is continuous and $\{f_n\}$ converges uniformly to f , then by **theorem 14.3.2**, f is continuous on $\mathcal{C}(X)$.
 Since $\{f_n\}$ converges uniformly to f , there is a N where for $n \geq N$:
 $|f - f_n(x)| \leq \epsilon$
 Since each f_n is bounded, then f is bounded. Since f is continuous and bounded, then $f \in \mathcal{C}(X)$. Thus, every Cauchy sequence $\{f_n\}$ converges to $f \in \mathcal{C}(X)$.

14.4 Uniform Convergence and Integration

Theorem 14.4.1: Uniform Convergence preserves Integrability

If $\{f_n\} \in \mathcal{R}(\alpha)$ converges uniformly to f on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$ where:

$$\int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha$$

Intuition

Since f_n is integrable, then $\int_a^b f_n \, d\alpha$ exist and since $\{f_n\}$ uniformly converges, then for $\epsilon > 0$, $|f - f_n| < \epsilon$. Thus, for a large enough n , $\int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha$.

Proof

Since $\{f_n\}$ converges uniformly to f , then for $\epsilon > 0$:

$$|f - f_n| < \epsilon \quad \rightarrow \quad f_n - \epsilon < f < f_n + \epsilon$$

Then:

$$\int_a^b f_n - \epsilon \, d\alpha < \int_a^b f \, d\alpha \leq \int_a^b f_n \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$$

Thus,

$$\int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha - \int_a^b f_n - \epsilon \, d\alpha = 2\epsilon[\alpha(b) - \alpha(a)]$$

So, $\int_a^b f \, d\alpha$ exists and since $f_n \in \mathcal{R}(\alpha)$ where $\int_a^b f_n - \epsilon \, d\alpha < \int_a^b f_n \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$:

$$\int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha$$

Theorem 14.4.2: Uniform Convergence preserves Integrability for series

If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then:

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$$

Proof

Since $f_n \in \mathcal{R}(\alpha)$, then $f(x) \in \mathcal{R}(\alpha)$. Since $f(x)$ converges uniformly, then by [theorem 14.4.1](#), then $\int_a^b f \, d\alpha = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_a^b f_n \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$.

14.5 Uniform Convergence and Differentiation

Theorem 14.5.1: Uniform Convergence of Derivatives preserve Differentiability

Suppose $\{f_n\}$ are differentiable on $[a,b]$ such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f'_n\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly to f on $[a,b]$ where:

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad \text{for } x \in [a,b]$$

Intuition

Since $\{f'_n\}$ converges uniformly, for t near x , then by the Mean Value Theorem:

$$\frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$$

Since $\{f'_n\}$ converges uniformly, by the Mean Value Theorem, there is a $t \in [x_1, x_2]$:

$$|[f_n(x_2) - f_m(x_2)] - [f_n(x_1) - f_m(x_1)]| = (x_2 - x_1)|f'_n(t) - f'_m(t)| < \epsilon$$

Thus, $\{f_n - f_m\}$ converges uniformly so if $\{f_n\}$ converges for some x_0 :

$$|f_n(x) - f_m(x)| = |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] + [f_n(x_0) - f_m(x_0)]| \leq \epsilon$$

Thus, $\{f_n\}$ converges uniformly which preserves continuity so for t near x as $n \rightarrow \infty$:

$$f'(x) = \frac{f(t) - f(x)}{t - x} = \frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$$

Note uniform convergence of $\{f'_n\}$ gives $\frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x}$. Then uniform convergence of $\{f'_n\}$ with convergent $f_n(x_0)$ leads to uniform convergence of $\{f_n\}$ which gives $\frac{f(t) - f(x)}{t - x} = \frac{f_n(t) - f_n(x)}{t - x}$.

Proof

Since $f_n(x_0)$ converges for some $x_0 \in [a,b]$, then for $\epsilon > 0$, there is a N_1 such that for $n_1, m_1 \geq N_1$:

$$|f_{n_1}(x_0) - f_{m_1}(x_0)| < \frac{\epsilon}{2}$$

Since f'_n converges uniformly, then there is a N_2 such that for $n_2, m_2 \geq N_2$:

$$|f'_{n_2}(t) - f'_{m_2}(t)| < \frac{\epsilon}{2(b-a)}$$

Let $N = \max(N_1, N_2)$. Then for $n, m \geq N$:

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \quad |f'_n(t) - f'_m(t)| < \frac{\epsilon}{2(b-a)}$$

Since f_n is differentiable, then $f_n - f_m$ is differentiable. Then by the Mean Value Theorem, there is a $x \in (a,b)$ such that:

$$|[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)]| \leq |x - t||f'_n(t) - f'_m(t)| < |x - t|\frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2}$$

Thus, for $n, m \geq N$:

$$|f_n(x) - f_m(x)| \leq |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| + |f_n(x_0) - f_m(x_0)| < \epsilon$$

Thus, $\{f_n\}$ converges uniformly to $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ where:

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \quad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

Since $\lim_{t \rightarrow x} |\phi_n(t) - \phi_m(t)| < \frac{\epsilon}{2(b-a)}$, then:

$$\lim_{n \rightarrow \infty} \phi_n(t) = \frac{f(t) - f(x)}{t - x} = \phi(t)$$

Since $\{\phi_n(t)\}$ converges uniformly to $\phi(t)$, then by **theorem 14.3.1**:

$$\lim_{t \rightarrow x} \phi(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \phi_n(t) = \lim_{n \rightarrow \infty} f'_n(x)$$

Theorem 14.5.2: Continuous functions can be non-differentiable

There exists a real continuous function on \mathbb{R} which is nowhere differentiable.

Proof

Let $\phi(x) = |x|$ for $x \in [-1, 1]$. Then to extend to all real x , let $\phi(x+2) = \phi(x)$. Then ϕ is continuous on \mathbb{R} where for $s, t \in \mathbb{R}$, $|\phi(s) - \phi(t)| \leq |s - t|$.
 Let $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$.
 Since $f(x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n$, then $f(x)$ converges uniformly and since $\phi(x)$ is continuous, then $f(x)$ is continuous. Then for a fixed x and positive integer m , choose $\delta_m = \pm \frac{1}{2} 4^{-m}$ such that no integer lies in $(4^m x, 4^m(x + \delta_m))$. Let $\gamma_n = \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}$.
 For $n > m$, $4^n \delta_m$ is even so $\gamma_n = 0$. For $n \in [0, m]$, $|\gamma_n| \leq \frac{|4^n \delta_m|}{\delta_m} = 4^m < 4^n$.
 Since $|\gamma_m| = 4^m$, then:

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^m (\frac{3}{4})^n \gamma_n \right| + \left| \sum_{n=m+1}^{\infty} (\frac{3}{4})^n \gamma_n \right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1)$$

 As $m \rightarrow \infty$, then $\delta_m \rightarrow 0$, but $\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \rightarrow \infty$ so f is not differentiable at any x .

14.6 Equicontinuous Families of Functions**Definition 14.6.1: Boundedness**

Let $\{f_n\}$ be defined on set E .

$\{f_n\}$ is pointwise bounded on E if for $x \in E$ and every n , there is a ϕ where:

$$|f_n(x)| < \phi(x)$$

$\{f_n\}$ is uniformly bounded on E if for every n and $x \in E$, there is a M where:

$$|f_n(x)| < M$$

Definition 14.6.2: Equicontinuous

A family of complex functions, \mathcal{F} , defined on set $E \subset X$ is equicontinuous if for every $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in E$ and $f \in \mathcal{F}$ where $d(x, y) < \delta$, then $|f(x) - f(y)| < \epsilon$.

Theorem 14.6.3: Pointwise bounded $\{f_n\}$ over countable sets have convergent $\{f_{n_k}\}$

If $\{f_n\}$ are pointwise bounded, complex functions on countable set E , then

$\{f_n\}$ has subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Intuition

Any $\{f_{n_k}\} \subset \{f_n\}$ is pointwise bounded so there is a convergent subsequence for a particular x . Let $\{f_{n_{k_1}}\}$ be a convergent subsequence for x_1 . Then find a subsequence $\{f_{n_{k_2}}\} \subset \{f_{n_{k_1}}\}$ which converges for x_2 . Continue the process until every x .

Proof

For each $x_i \in E$, let $\{x_i\}$. For x_1 , $\{f_n(x_1)\}$ is piecewise bounded so there exists a subsequence $\{f_{1,k}(x_1)\}$ which converges as $k \rightarrow \infty$.

Since $\{f_{1,k}\}$ is piecewise bounded since $\{f_{1,k}\} \subset \{f_n\}$, then there is a subsequence $\{f_{2,k}\} \subset \{f_{1,k}\}$ such that $\{f_{2,k}(x_2)\}$ converges as $k \rightarrow \infty$. Then continuing the pattern:

$S_1:$	$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	\dots
$S_2:$	$f_{2,1}$	$f_{2,2}$	$f_{2,3}$	\dots
$S_3:$	$f_{3,1}$	$f_{3,2}$	$f_{3,3}$	\dots
\dots				

Thus, $\{f_{n,n}(x_i)\}$ converges as $n \rightarrow \infty$ for every $x_i \in E$.

Theorem 14.6.4: Uniform convergent $\{f_n\}$ where $f_n \in \mathcal{C}(K)$ is equicontinuous

If K is a compact metric space where $f_n \in \mathcal{C}(K)$ and $\{f_n\}$ converges uniformly on K , then $\{f_n\}$ is equicontinuous on K .

Intuition

Since $\{f_n\}$ converges uniformly, then there is a N where for $n > N$, then $|f_n - f_N| < \epsilon$. Since $\{f_n\}$ is continuous over compact K , then $\{f_n\}$ is uniformly continuous. So for $d(x,y) < \delta$, then:

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\epsilon$$

Proof

Since $\{f_n\}$ converges uniformly, then for $\epsilon > 0$, there is a N such that for $n > N$:

$$||f_n - f_N|| < \frac{\epsilon}{3}$$

Since f_i for $i \in [1, N]$ is continuous over compact K , then f_i is uniformly continuous so there is a $\delta > 0$ such that for all x, y where $d(x, y) < \delta$, then $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$.

Then for $n > N$ and $d(x, y) < \delta$:

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \epsilon$$

Thus, for $\epsilon > 0$, there is a $\delta > 0$ such that for all f_n and $x, y \in K$ where $d(x, y) < \delta$, $|f_n(x) - f_n(y)| < \epsilon$. So, $\{f_n\}$ is equicontinuous.

Theorem 14.6.5: Pointwise bounded and equicontinuous $\{f_n\}$ over compact K is uniformly bounded and have uniformly convergent $\{f_{n_k}\}$

If K is compact where $\{f_n\} \in \mathcal{C}(K)$ is pointwise bounded and equicontinuous:

- (a) $\{f_n\}$ is uniformly bounded on K
- (b) $\{f_n\}$ contains a uniformly convergent subsequence

Intuition

Since $\{f_n\}$ is equicontinuous, for $d(x, y) < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$.

Since $\{f_n\}$ is pointwise bounded on compact K , there are finite x_0, \dots, x_n such that $d(x, x_i) < \delta$ so $|f_n(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i)| < \epsilon + M$.

For a countable dense subset of K , the countability gives a convergent subsequence $\{g_n\}$ and the dense gives $d(x, x_i) < \delta$ for finite x_1, \dots, x_m so:

$$|g_n(x) - g_m(y)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)| < \epsilon.$$

Proof

Since f_n is equicontinuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for $x, y \in K$ where $d(x, y) < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$.

Since K is compact, there are finite $p_1, \dots, p_r \in K$ so for any $x \in K$, there is at least one p_i so $d(x, p_i) < \delta$. Since $\{f_i\}$ is pointwise bounded, there is a M_i so $|f_n(p_i)| < M_i$. Let $M = \max(M_1, \dots, M_r)$. So, $|f_n(x)| < |f_n(x) - f_n(p_i)| + |f_n(p_i)| < \epsilon + M_i < \epsilon + M$. Thus, $\{f_n\}$ is uniformly bounded on K .

Let countable dense $E \subset K$. By **theorem 14.6.3**, $\{f_n\}$ has a convergent subsequence $\{f_{n_i}(x)\}$ for every $x \in E$. Let $V(x, \delta) = \{y \in K : d(x, y) < \delta\}$ so $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Since E is dense in compact K , there are finitely many $x_1, \dots, x_m \in E$ such that:

$$K \subset V(x_1, \delta) \cup \dots \cup V(x_m, \delta).$$

Since $\{f_{n_i}(x)\}$ converges for every $x \in E$, there is a N where for $n_i, n_j \geq N$, $s \in [1, m]$:

$$|f_{n_i}(x_s) - f_{n_j}(x_s)| < \frac{\epsilon}{3}$$

Thus, for any $x \in K$, there is a $x_s \in E$ such that:

$$|f_{n_i}(x) - f_{n_j}(x)| \leq |f_{n_i}(x) - f_{n_i}(x_s)| + |f_{n_i}(x_s) - f_{n_j}(x_s)| + |f_{n_j}(x_s) - f_{n_j}(x)| < \epsilon$$

Thus, $\{f_n\}$ contains a subsequence that uniformly converges.

14.7 Stone-Weierstrass Theorem

Theorem 14.7.1: There are polynomials that converge uniformly to continuous f

For complex continuous f on $[a,b]$, there is a sequence of polynomials $\{P_n\}$ that converges uniformly to $f(x)$.

Proof

Let $[a,b] = [0,1]$ where $f(0) = f(1) = 0$ and $f(x) = 0$ if $x \notin [0,1]$.

Thus, f is uniformly continuous over \mathbb{R} .

Let $Q_n(x) = c_n(1 - x^2)^n$ where c_n is chosen so $\int_{-1}^1 Q_n(x) dx = 1$. Since:

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= 2 \int_0^1 (1 - x^2)^n dx \geq 2 \int_0^{\frac{1}{\sqrt{n}}} (1 - x^2)^n dx \geq 2 \int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2 dx \\ &= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}} \end{aligned}$$

so $c_n < \sqrt{n}$. Thus for $\delta > 0$, $Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n$ so $Q_n \rightarrow 0$ on $|x| \in [\delta, 1]$.

Let $P_n(x) = \int_{-1}^1 f(x+t)Q_n(t) dt$ for $x \in [0,1]$. Since $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t) dt = \int_0^1 f(t)Q_n(t-x) dt$ which is a polynomial so $\{P_n\}$ is a sequence of polynomials.

Since f is uniformly continuous, for $\epsilon > 0$, there is a $\delta > 0$ such that for $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $M = \sup(|f(x)|)$. Then:

$$\begin{aligned} |P_n(x) - f(x)| &\leq \int_{-1}^1 |f(x+t) - f(x)|Q_n(t) dt \\ &\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 Q_n(t) dt \\ &\leq 4M\sqrt{n}(1 - \delta^2)^n + \frac{\epsilon}{2} < \epsilon \quad \text{for a large enough } n \end{aligned}$$

Corollary 14.7.2: There are polynomials that converges uniformly to $|x|$

For $[-a,a]$, there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and $P_n(x)$ converges uniformly to $|x|$.

Proof

By **Theorem 14.7.1**, there is a $\{P_n^*\}$ of real polynomials that converges uniformly to $|x|$. Since $P_n^*(0) \rightarrow |0| = 0$, let $P_n(x) = P_n^*(x) - P_n^*(0)$.

Definition 14.7.3: Algebra, Uniformly Closed, and Uniform Closure

A family of complex functions on E , \mathcal{A} , is an algebra if for $f, g \in \mathcal{A}$, then:

- (a) $f+g \in \mathcal{A}$
- (b) $fg \in \mathcal{A}$
- (c) $cf \in \mathcal{A}$ for complex constant c

\mathcal{A} is uniformly closed if:

For any $f_n \in \mathcal{A}$ where f_n uniformly converges to f , then $f \in \mathcal{A}$

Let the uniform closure, \mathcal{B} , be the set of all uniformly convergent f from \mathcal{A} .

Theorem 14.7.4: Bounded algebra implies Uniformly closed uniform closure

For algebra \mathcal{A} of bounded functions, \mathcal{B} is a uniformly closed algebra.

Proof

If $f, g \in \mathcal{B}$, there are uniformly convergent $\{f_n\}, \{g_n\}$ where $f_n \rightarrow f, g_n \rightarrow g$ and $f_n, g_n \in \mathcal{A}$. Since f_n, g_n are bounded and \mathcal{A} is an algebra, then uniformly convergent:

$$f_n + g_n \rightarrow f+g \quad f_n g_n \rightarrow fg \quad c f_n \rightarrow cf$$

Thus, $f + g, fg, cf \in \mathcal{B}$ so \mathcal{B} is a uniformly closed algebra.

Definition 14.7.5: Separate Points

For family of functions, \mathcal{A} , separate points on E :

If for every pair of distinct $x_1, x_2 \in E$, there is a $f \in \mathcal{A}$ where $f(x_1) \neq f(x_2)$.

\mathcal{A} vanishes at no point of E :

If for each $x \in E$, there is a $g \in \mathcal{A}$ such that $g(x) \neq 0$

Theorem 14.7.6: Non-vashing, separate algebra contain all values

Suppose algebra \mathcal{A} separates points and vanishes at no points on E . If x_1, x_2 are distinct points, then for constants c_1, c_2 , there is a $f \in \mathcal{A}$ where:

$$f(x_1) = c_1 \text{ and } f(x_2) = c_2.$$

Proof

Since \mathcal{A} separates points and vanishes at no points on E , then there are $g, h, k \in \mathcal{A}$:

$$g(x_1) \neq g(x_2) \quad h(x_1) \neq 0 \quad k(x_2) \neq 0$$

Let $u = k(g - g(x_1))$ and $v = h(g - g(x_2))$ so $u, v \in \mathcal{A}$ where $u(x_1) = v(x_2) = 0$ and $u(x_2), v(x_1) \neq 0$. Then, $f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$ have $f(x_1) = c_1$ and $f(x_2) = c_2$.

Theorem 14.7.7: Stone-Weierstrass Theorem

If algebra of real continuous functions on compact K , \mathcal{A} , separates points and vanishes at no points on K , then \mathcal{B} consist of all real continuous functions.

Proof

Claim: If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Let $a = \sup(|f(x)|)$. By **Corollary 14.7.2**, for $\epsilon > 0$, there are c_1, \dots, c_n such that:

$$|\sum_{i=1}^n c_i y^i - |y|| < \epsilon \quad \text{for } y \in [-a, a]$$

Since \mathcal{B} is an algebra, then $g = \sum_{i=1}^n c_i f^i \in \mathcal{B}$. Thus:

$$|g(x) - |f(x)|| < \epsilon \quad \text{for } x \in K$$

Since \mathcal{B} is uniformly closed, then $|f(x)| \in \mathcal{B}$.

Claim: If $f, g \in \mathcal{B}$, then $\min(f, g), \max(f, g) \in \mathcal{B}$.

Since:

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2} \quad \min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$

then $\max(f, g), \min(f, g) \in \mathcal{B}$.

Claim: For real, continuous f on K and $\epsilon > 0$, there exist $g_x \in \mathcal{B}$ where $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$ for $t \in K$.

Since $\mathcal{A} \subset \mathcal{B}$ where \mathcal{A} separates points and vanishes at no points on E , then \mathcal{B} is the same. Then by **theorem 14.7.6**, for $y \in K$, there is a $h_y \in \mathcal{B}$ where:

$$h_y(x) = f(x) \quad h_y(y) = f(y)$$

Since h_y is continuous, there is an open set J_y such that $h_y(t) > f(t) - \epsilon$ for $t \in J_y$.

Since K is compact, there are finite y_1, \dots, y_n such that $K \subset J_{y_1} \cup \dots \cup J_{y_n}$.

Let $g_x = \max(h_{y_1}, \dots, h_{y_n})$ so $g_x \in \mathcal{B}$ where $g_x(t) > f(t) - \epsilon$ for $t \in K$.

Claim: For real, continuous f on K and $\epsilon > 0$, there is a $h \in \mathcal{B}$ where $|h(x) - f(x)| < \epsilon$.

Since g_x is continuous, there is an open set V_x where $g_x(t) < f(t) + \epsilon$ for $t \in V_x$.

Since K is compact, there are finite x_1, \dots, x_m such that $K \subset V_{x_1} \cup \dots \cup V_{x_m}$.

Let $h = \min(g_{x_1}, \dots, g_{x_m})$ so $h \in \mathcal{B}$ where $h(t) > f(t) - \epsilon$. But, $h(t) < f(t) + \epsilon$ so $|h(x) - f(x)| < \epsilon$. Since \mathcal{B} is uniformly closed, then the theorem holds true.

Definition 14.7.8: Self-Adjoint

\mathcal{A} is self-adjoint if for every $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$

Theorem 14.7.9: Stone-Weierstrass for complex functions

If self-adjoint algebra of complex continuous functions on compact K , \mathcal{A} , separates points and vanishes at no points on K , then \mathcal{B} consist of all complex continuous functions on K . In other words, \mathcal{A} is dense in $\mathcal{C}(K)$.

Proof

Let \mathcal{A}_R be the set of all real functions on K in \mathcal{A} .

If $f \in \mathcal{A}$ and $f = u + iv$ for real u, v then $2u = f + \overline{f} \in \mathcal{A}_R$.

If $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) = 1$ and $f(x_2) = 0$ so $u(x_1) \neq u(x_2)$ so \mathcal{A}_R separates points.

If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathcal{A}$ and there is a complex λ such that $\lambda g(x) > 0$.

If $f = \lambda g$, then $u(x) > 0$ so \mathcal{A}_R vanishes at no point of K .

Then by [theorem 14.7.7](#), every real continuous function on K lies in $\mathcal{B}_{\mathcal{A}_R}$ and since $\mathcal{B}_{\mathcal{A}_R} \subset \mathcal{B}$, then every real continuous function lies in \mathcal{B} . If f is complex continuous where $f = u + iv$, then $f \in \mathcal{B}$ since $u, v \in \mathcal{B}$.

15 Special Functions

15.1 Power Series

Definition 15.1.1: Analytic Functions

Power series, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

If $f(x)$ converges for $|x-a| < R$ for some R , then f is expanded in a power series about $x = a$.

Theorem 15.1.2: Convergent Power Series are differentiable

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$, then $f(x)$ converges uniformly on $[-R+\epsilon, R-\epsilon]$ for any $\epsilon > 0$.

Then, f is continuous and differentiable in $(-R, R)$ where:

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Proof

For $\epsilon > 0$ and $|x| \leq R - \epsilon$:

$$|c_n x^n| \leq |c_n (R - \epsilon)^n|$$

Since $\sum c_n (R - \epsilon)^n$ converges absolutely in $[-R + \epsilon, R - \epsilon]$, then $f(x)$ uniformly converges on $[-R + \epsilon, R - \epsilon]$.

Since $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$, then:

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{n} |c_n|) = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$$

so $f(x)$ and $f'(x)$ have the same interval of convergence so $f'(x)$ uniformly converges on $[-R + \epsilon, R - \epsilon]$. Since $f'(x)$ exists, then f is differentiable and thus, continuous.

Corollary 15.1.3: Power Series have infinite derivatives

On $(-R, R)$, f has derivatives of all orders:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)c_n x^{n-k}$$

$$f^{(k)}(0) = k! c_k$$

Proof

By [theorem 15.1.2](#), apply derivative k times.

Theorem 15.1.4: Continuity of Power Series at endpoints

Suppose $\sum c_n$ converges where $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$.

Then $\lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} c_n$.

Proof

Let $s_n = c_0 + \dots + c_n$.

$$\begin{aligned} \sum_{n=0}^m c_n x^n &= \sum_{n=0}^m (s_n - s_{n-1}) x^n = \sum_{n=0}^m s_n x^n - \sum_{n=0}^m s_{n-1} x^n \\ &= \sum_{n=0}^m s_n x^n - \sum_{n=0}^{m-1} s_n x^{n+1} = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m \end{aligned}$$

Since $|x| < 1$, then as $m \rightarrow \infty$, then $s_m x^m \rightarrow 0$. Let $s = \lim_{n \rightarrow \infty} s_n$.

Thus, for $\epsilon > 0$, there is a N such that for $n > N$, then $|s - s_n| < \frac{\epsilon}{2}$.

Since $(1-x) \sum_{n=0}^{\infty} x^n = (1-x) \frac{1}{1-x} = 1$, then:

$$|f(x) - s| = |(1-x) \sum_{n=0}^{\infty} (s_n - s) x^n| \leq (1-x) \sum_{n=0}^N |s_n - s| |x|^n + \frac{\epsilon}{2}$$

Then choose $\delta > 0$ such that $(1-x) \sum_{n=0}^N |s_n - s| < \frac{\epsilon}{2}$ for $x > 1 - \delta$. Thus:

$$|\lim_{x \rightarrow 1} f(x) - s| < \epsilon$$

Corollary 15.1.5: Cauchy Product

If $\sum a_n \rightarrow A$, $\sum b_n \rightarrow B$, and $\sum c_n \rightarrow C$ where $c_n = \sum_{k=0}^n a_k b_{n-k}$, then:
 $C = AB$

Proof

For $x \in (0,1)$, let:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad h(x) = \sum_{n=0}^{\infty} c_n x^n$$

Then f, g, h absolutely converges. Note $fg = h$.

By **theorem 15.1.4**, then $AB = \lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} h(x) = C$.

Theorem 15.1.6: $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$

Suppose $\sum_{j=1}^{\infty} |a_{i,j}| = b_i$ where $\sum b_i$ converges, then:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$$

Proof

Let countable set E contain points x_n where $x_n \rightarrow x_0$. Let:

$$f_i(x_n) = \sum_{j=1}^n a_{i,j} \quad f_i(x_0) = \sum_{j=1}^{\infty} a_{i,j} \quad g(x) = \sum_{i=1}^{\infty} f_i(x)$$

Thus, each f_i is continuous at x_0 . Since $|f_i(x)| \leq b_i$, then $g(x)$ converges uniformly so g is continuous at x_0 . Thus:

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} &= \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} f_i(x_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{\infty} a_{i,j} = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \sum_{i=1}^n a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j} \end{aligned}$$

Theorem 15.1.7: Extension to Taylor's Theorem

If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ where $a \in (-R, R)$, then f is expanded in a power series about $x = a$ which converges in $|x - a| < R - |a|$ where:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

Proof

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n [(x - a) + a]^n = \sum_{n=0}^{\infty} c_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x - a)^m \\ &= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} \right] (x - a)^m \end{aligned}$$

Then by **corollary 15.1.3**, $\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} = \frac{f^{(m)}(a)}{m!}$ so $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$.

Theorem 15.1.8: Equivalent Power Series have the same coefficients

If $\sum a_n x^n, \sum b_n x^n$ converge in $S = (-R, R)$, let E be the set of all $x \in S$ where $\sum a_n x^n = \sum b_n x^n$. If E has a limit point in S , then $a_n = b_n$ for all n .

Proof

Let $c_n = a_n - b_n$ and $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $f(x) = 0$ on E .

Let $A = E'$ and $B = S \setminus E'$. Thus, B is open. If $x_0 \in A$, then:

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n \quad |x - x_0| < R - |x_0|$$

Suppose $d_n \neq 0$ for some n . Let k be the smallest integer where $d_k \neq 0$. Then:

$$f(x) = (x - x_0)^k g(x) \quad |x - x_0| < R - |x_0| \text{ and } g(x) = \sum_{m=0}^{\infty} d_{k+m} (x - x_0)^m$$

Since g is continuous at x_0 and $g(x_0) = d_k \neq 0$, there is a $\delta > 0$ such that $g(x) \neq 0$ for $|x - x_0| < \delta$.

Thus, $f(x) \neq 0$ if $|x - x_0| < \delta$ which contradicts that x_0 is a limit point of E . Thus, $d_n = 0$ for all n so $f(x) = 0$ for all x so A is open. Thus, A and B are disjoint and thus, are separated. Since $S = A \cup B$ and S is connected, then either A or B is empty. Since A cannot be empty, then B is empty so $A = S$. Since f is continuous in S , then $A \subset S$ so $E = S$ so $c_n = 0$ for all n .

15.2 Exponential and Logarithmic Functions

Definition 15.2.1: Exponential Function

Define $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{C}$.

By the ratio test:

$$\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) = \lim_{n \rightarrow \infty} \sup(|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}|) = \lim_{n \rightarrow \infty} \sup(|\frac{z}{n+1}|) = 0 < 1$$

Thus, $E(x)$ converges. Then by [corollary 15.1.5](#):

$$\begin{aligned} E(x)E(y) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^k y^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y) \end{aligned}$$

As a result, $E(x)E(-x) = E(0) = 1$. As a consequence:

- (a) $E(x) \neq 0$ for all x
- (b) If $x > 0$, then $E(x) > 0$ and thus, $E(x) > 0$ for all $x \in \mathbb{R}$
- (c) $\lim_{x \rightarrow \infty} E(x) \rightarrow \infty$ so $\lim_{x \rightarrow -\infty} E(x) \rightarrow 0$ for $x \in \mathbb{R}$
- (d) For $0 < x < y$, $E(x) < E(y)$ so $E(-y) = \frac{1}{E(y)} < \frac{1}{E(x)} = E(-x)$ so $E(x)$ is strictly increasing on \mathbb{R}
- (e) $E'(x) = \lim_{h \rightarrow 0} \frac{E(x+h) - E(x)}{h} = \lim_{h \rightarrow 0} \frac{E(x)E(h) - E(x)}{h}$
 $= E(x) \lim_{h \rightarrow 0} \frac{E(h) - 1}{h} = E(x) (\lim_{h \rightarrow 0} \frac{E(h)}{h} - \lim_{h \rightarrow 0} \frac{1}{h})$
 $= E(x) (\lim_{h \rightarrow 0} \frac{1}{h} + 1 - \lim_{h \rightarrow 0} \frac{1}{h}) = E(x)$
- (f) For $n > 0 \in \mathbb{Z}$:

$$E(n) = \underbrace{E(1) \dots E(1)}_n = e^n$$

For $p = \frac{n}{m} > 0 \in \mathbb{Q}$:

$$[E(p)]^m = E(mp) = E(n) = e^n \text{ so } E(p) = e^{n/m} = e^p$$

Since $E(-p) = \frac{1}{E(p)} = e^{-p}$, then $E(p) = e^p$ hold for all $p \in \mathbb{Q}$.

For $x \in \mathbb{R}$, let $e^x = \sup_{x > p} (e^p)$ for $p \in \mathbb{Q}$. Since $E(x)$ is continuous and

monotonically increasing, for every $\epsilon > 0$, there is a $\delta > 0$ where $|x - p| < \delta$, then $|\sup_{x > p} (e^p) - e^p| < \epsilon$. Thus:

$$e^x = \sup_{x > p} (e^p) = \lim_{p \rightarrow x} E(p) = E(x).$$

Theorem 15.2.2: Properties of e^x

- (a) e^x is continuous and differentiable for all $x \in \mathbb{R}$
- (b) $(e^x)' = e^x$
- (c) e^x is strictly increasing where $e^x > 0$
- (d) $e^{x+y} = e^x e^y$
- (e) $\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow -\infty} e^x = 0$
- (f) $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$ for every $n > 0$

Proof

Part (a) is proved by convergent power series while parts (c) to (e) is proved by properties of $E(x)$ above. Since $e^x > \frac{x^{n+1}}{(n+1)!}$ for $x > 0$ and every $n \in \mathbb{Z}_+$, then:

$$0 \leq \lim_{x \rightarrow \infty} x^n e^{-x} < \lim_{x \rightarrow \infty} \frac{(n+1)!}{x} = 0$$

Thus, $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$ for every $n \in \mathbb{Z}_+$. Since $x^n e^{-x}$ is continuous and $n \in \mathbb{Z}_+$ is dense in \mathbb{R}_+ , then $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$ for every $n > 0$.

Definition 15.2.3: Logarithmic Function

Since $y = E(x)$ is strictly increasing on \mathbb{R} , then $E(x)$ is injective and thus, there exist an inverse function $L(y)$ which is also strictly increasing. Since $E(x)$ is differentiable, then $L(y)$ is also differentiable. Then:

$$\begin{aligned} E(L(y)) &= y & \text{where } y > 0 \\ L(E(x)) &= x & \text{where } x \in \mathbb{R} \end{aligned}$$

Then:

$$L'(E(x))E'(x) = L'(y)E'(x) = L'(y)y = 1 \quad \Rightarrow \quad L'(y) = \frac{1}{y}$$

Since for $x = 0$ have $y = E(0) = 1$, then $L(1) = 0$. Thus:

$$L(y) = \int_1^y L'(t) dt = \int_1^y \frac{1}{t} dt$$

As a consequence:

(a) Let $y_1 = E(x_1)$ and $y_2 = E(x_2)$, then:

$$L(y_1 y_2) = L(E(x_1)E(x_2)) = L(E(x_1 + x_2)) = x_1 + x_2 = L(y_1) + L(y_2)$$

(b) Let $\log(y) = L(y)$. Then:

$$\text{Since } \lim_{x \rightarrow \infty} E(x) = \infty, \text{ then } \lim_{y \rightarrow \infty} L(y) = \infty.$$

$$\text{Since } \lim_{x \rightarrow -\infty} E(x) = 0, \text{ then } \lim_{y \rightarrow 0} L(y) = -\infty.$$

(c) For $n \in \mathbb{Z}$:

$$\text{If } n \geq 0, E(nL(y)) = E(\underbrace{L(y) + \dots + L(y)}_n) = E(L(y^n)) = y^n$$

$$\text{If } n < 0, E(nL(y)) = E(\underbrace{-(L(y) + \dots + L(y))}_{-n}) = [E(L(y^{-n}))]^{-1} = y^n$$

For $p = \frac{a}{b} \in \mathbb{Q}$ where $b > 0$, let $t^b = y$:

$$\begin{aligned} E(pL(y)) &= \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(y))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(t^b))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}bL(t))^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(aL(t))^n}{n!} = \sum_{n=0}^{\infty} \frac{(L(t^a))^n}{n!} = t^a = y^{\frac{a}{b}} = y^p \end{aligned}$$

For $c \in \mathbb{R}$, let $y^c = \sup_{c > p} E(pL(y))$. Since $E(x), L(y)$ are continuous and

monotonically increasing, then for every $\epsilon > 0$, there is a $\delta > 0$ where $|c - p| < \delta$, then $|\sup_{c > p} (E(pL(y)) - E(pL(y)))| < \epsilon$. Thus:

$$y^c = \sup_{c > p} E(pL(y)) = \lim_{p \rightarrow c} E(pL(y)) = E(cL(y))$$

(d) For $y \in \mathbb{C}$ and $c \neq 0 \in \mathbb{R}$:

$$(y^c)' = E'(cL(y))cL'(y) = E(cL(y))c\frac{1}{y} = y^c c\frac{1}{y} = cy^{c-1}$$

Thus:

$$\text{If } c \neq -1, \text{ then } \int y^c dy = \int \frac{1}{c+1}(y^{c+1})' dy = \frac{1}{c+1}y^{c+1}$$

$$\text{If } c = -1, \text{ then } \int y^{-1} dy = \int L'(y) dy = L(y) = \log(y)$$

(e) $\lim_{y \rightarrow \infty} y^{-c} \log(y) = 0$ for every $c > 0$

For $\epsilon \in (0, c)$ and $y > 1$:

$$y^{-c} \log(y) = y^{-c} \int_1^y t^{-1} dt < y^{-c} \int_1^y t^{\epsilon-1} dt = y^{-c} \frac{y^{\epsilon}-1}{\epsilon} < \frac{1}{y^{c-\epsilon}}$$

$$0 \leq \lim_{y \rightarrow \infty} y^{-c} \log(y) < \lim_{y \rightarrow \infty} \frac{1}{y^{c-\epsilon}} = 0$$

15.3 Trigonometric Function

Definition 15.3.1: Trigonometric Function

Define for $x \in \mathbb{C}$:

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \quad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

Since $E(\bar{x}) = \sum_{n=0}^{\infty} \frac{\bar{x}^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{x^n}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = \overline{E(x)}$, then for $x \in \mathbb{R}$:

$$C(x), S(x) \in \mathbb{R}$$

Also, $E(ix) = C(x) + iS(x)$. Then:

$$(a) |E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(0) = 1 \text{ so } |E(ix)| = 1$$

$$(b) C(0) = \frac{1}{2}[E(0) + E(0)] = 1$$

$$S(0) = \frac{1}{2i}[E(0) - E(0)] = 0$$

$$(c) C'(x) = \frac{1}{2}[E'(ix)i + E'(-ix)(-i)] = \frac{1}{2}[E(ix)i - E(-ix)i] = -S(x)$$

$$S'(x) = \frac{1}{2i}[E'(ix)i - E'(-ix)(-i)] = \frac{1}{2i}[E(ix)i + E(-ix)i] = C(x)$$

$$(d) \text{ There exists positive numbers such that } C(x) = 0.$$

If the claim is false, since C is continuous and $C(0) = 1$, then $S'(x) = C(x) > 0$. Then $S(x)$ is strictly increasing and since $S(0) = 0$, then $S(x) > 0$ for $x > 0$. Then for $0 < x < y$:

$$\begin{aligned} S(x)(y-x) &< \int_x^y S(t) dt = \int_x^y -C'(t) dt = C(x) - C(y) \\ &\leq |C(x) - C(y)| \leq |C(x)| + |C(y)| = 2 \end{aligned}$$

But if $S(x) > 0$, then $S(x)(y-x) \not\leq 2$ for a large enough y for any $S(x)$. Thus by contradiction, there are positive numbers where $C(x) = 0$.

Since the set of zeros of a continuous function is closed, there exists a smallest positive number x_0 such that $C(x_0) = 0$. Let $\pi = 2x_0$.

Then, $C(\frac{\pi}{2}) = C(x_0) = 0$ and since $|E(ix)| = |C(x) + iS(x)| = 1$, then $S(\frac{\pi}{2}) = \pm 1$. Since $C(x)$ is continuous where $C(0) = 1$ and $C(\frac{\pi}{2}) = 0$, then $S'(x) = C(x) > 0$ for $x \in (0, \frac{\pi}{2})$ where $S(0) = 0$ so $S(\frac{\pi}{2}) = 1$.

Thus, $E(\frac{\pi}{2}i) = C(\frac{\pi}{2}) + iS(\frac{\pi}{2}) = 0 + i1 = i$. Then:

$$-1 = i^2 = E(\frac{\pi}{2}i)E(\frac{\pi}{2}i) = E(\frac{\pi}{2}i + \frac{\pi}{2}i) = E(\pi i)$$

$$1 = (-1)^2 = E(\pi i)E(\pi i) = E(\pi i + \pi i) = E(2\pi i)$$

$$E(z) = E(z)1 = E(z)E(2\pi i) = E(z + 2\pi i)$$

Theorem 15.3.2: Properties of $C(x)$ and $S(x)$

$$(a) E \text{ is periodic with period } 2\pi i$$

Proof

$$\text{Since } E(z) = E(z + 2\pi i), E \text{ has period } 2\pi i.$$

$$(b) C(x) \text{ and } S(x) \text{ are periodic with period } 2\pi$$

Proof

$$\text{Since } C(x) = \frac{1}{2}[E(ix) + E(-ix)] \text{ and } S(x) = \frac{1}{2i}[E(ix) - E(-ix)] \text{ where } E(x) \text{ have period } 2\pi i \text{ so } C(x) \text{ and } S(x) \text{ have period } 2\pi.$$

$$(c) \text{ If } t \in (0, 2\pi), \text{ then } E(it) \neq 1$$

Proof

$$\text{If } t \in (0, \frac{\pi}{2}) \text{ where } E(it) = x + iy, \text{ then } x, y \in (0, 1).$$

$$\text{Note } E(4it) = [E(it)]^4 = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

$$\text{If } E(4it) \text{ is real, then } x^2 - y^2 = 0. \text{ Thus, since } |E(ix)| = 1, \text{ then } x^2 + y^2 = 1 \text{ so } x^2 = y^2 = \frac{1}{2} \text{ and thus, } E(4it) = -1 \neq 1.$$

- (d) For $z \in \mathbb{C}$ where $|z| = 1$, there is a unique $t \in [0, 2\pi)$ such that $E(it) = z$

Proof

By part (c), for $0 \leq t_1 < t_2 < 2\pi$:

$$E(it_2)[E(it_1)]^{-1} = E(it_2)[E(-it_1)] = E(it_2 - it_1) \neq 1$$

Thus, $t \in [0, 2\pi)$ must be unique. Let fixed $z = x + iy$ where $|z| = 1$.

For $x, y \geq 0$, since $C(x)$ decreases from 1 to 0 on $[0, \frac{\pi}{2}]$, then $C(t) = x$ for some $t \in [0, \frac{\pi}{2}]$. Since $|E(it)| = C(t)^2 + S(t)^2 = 1$ and $x^2 + y^2 = 1$, then $S(t) = y$ so $E(it) = x + yi = z$.

If $x < 0$, $y \geq 0$, fix $-iz$ instead of z and thus, $E(it) = -iz$ for some $t \in [0, \frac{\pi}{2}]$. Since $E(\frac{\pi}{2}i) = i$, then $z = -iz(i) = E(it)E(\frac{\pi}{2}i) = E(i(t + \frac{\pi}{2}))$.

If $x, y < 0$, fix $-z$ instead of z and thus, $E(it) = -z$ for some $t \in [0, \frac{\pi}{2}]$. Since $E(\pi i) = -1$, then $z = -z(-1) = E(it)E(\pi i) = E(i(t + \pi))$.

If $x \geq 0$, $y < 0$, fix iz instead of z and thus, $E(it) = iz$ for some $t \in [0, \frac{\pi}{2}]$. Then $z = iz(-1)(i) = E(it)E(\pi i)E(\frac{\pi}{2}i) = E(i(t + \frac{3\pi}{2}))$.

Definition 15.3.3: Unit Curve

Let $\gamma(t) = E(it)$ for $t \in [0, 2\pi]$.

By **theorem 15.3.2(d)** and $E(z) = E(z + 2\pi i)$, then $\gamma(t)$ is a simple closed curve whose range is the unit circle. Since $\gamma'(t) = iE'(it) = iE(it)$, the length of γ :

$$\Lambda(\gamma) = \int_0^{2\pi} |\gamma'(t)| dt = 2\pi$$

Thus, $\pi = 2x_0$ defined earlier have the same geometric significance as π . Then consider the triangle with vertices at:

$$z_1 = 0 \quad z_2 = C(t_0) \quad z_3 = \gamma(t_0) = (C(t_0), S(t_0))$$

Thus, $C(t) = \cos(t)$ and $S(t) = \sin(t)$.

15.4 Algebraic Completeness of the Complex Field

Theorem 15.4.1: Every complex polynomial has a complex root

For $a_0, \dots, a_n \neq 0 \in \mathbb{C}$ where $n \geq 1$, let $P(z) = \sum_{k=0}^n a_k z^k$.

Then $P(z) = 0$ for some $z \in \mathbb{C}$.

Proof

Assume $a_n = 1$. Let $\mu = \inf(|P(z)|)$. If $|z| = R$, then:

$$|P(z)| \geq R^n(1 - |a_{n-1}|R^{-1} - \dots - |a_0|R^{-n})$$

Thus, $\lim_{R \rightarrow \infty} |P(z)| = \infty$ so there is a R_0 such that $|R(z)| > \mu$ if $|z| > R_0$.

Since $|P|$ is continuous, then for a closed $N_{R_0}(0)$, by the Extreme Value Theorem:

$$|P(z_0)| = \mu \quad \text{for some } z_0$$

Suppose $\mu \neq 0$. Let polynomial $Q(z) = \frac{P(z+z_0)}{P(z_0)}$ where $Q(0) = 1$, $Q(z) \geq 1$ for all z .

Then there is a smallest integer $k \leq n$ so $b_k \neq 0$ so $Q(z) = 1 + b_k z^k + \dots + b_n z^n$.

By **theorem 15.3.2(d)**, there is a $\theta \in \mathbb{R}$ such that $e^{ik\theta} b_k = -|b_k|$.

If $r > 0$ and $r^k |b_k| < 1$, then $|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|$. Thus:

$$\begin{aligned} |Q(re^{i\theta})| &= |1 + b_k r^k e^{ik\theta} + b_{k+1} r^{k+1} e^{i\theta(k+1)} + \dots + b_n r^n e^{i\theta n}| \\ &\leq |1 + b_k r^k e^{ik\theta}| + |b_{k+1} r^{k+1} e^{i\theta(k+1)}| + \dots + |b_n r^n e^{i\theta n}| \\ &= 1 - r^k |b_k| + r^{k+1} |b_{k+1}| + \dots + r^n |b_n| = 1 - r^k (|b_k| - r |b_{k+1}| - \dots - r^{n-k} |b_n|) \end{aligned}$$

Thus, for a sufficiently small r , $|Q(re^{i\theta})| < 1$ which contradicts $Q(z) \geq 1$ for all z .

Thus, $\mu = 0$ so there is a z_0 such that $|P(z_0)| = \mu = 0$ so $P(z_0) = 0$.

15.5 Fourier Series

Definition 15.5.1: Trigonometric Polynomial

A trigonometric polynomial is a finite sum where for $x \in \mathbb{R}$:

$$f(x) = a_0 + \sum_{n=1}^N [a_n \cos(nx) + b_n \sin(nx)] = \sum_{n=-N}^N c_n e^{inx}$$

A trigonometric series is then:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Thus:

(a) $f(x)$ has period of 2π

(b) Since $(\frac{1}{in} e^{inx})' = e^{inx}$ where $\frac{1}{in} e^{inx}$ have period of 2π , then for $n \in \mathbb{Z}$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\frac{1}{in} e^{inx})' dx = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \dots \end{cases}$$

(c) For $m \in \{-N, -N+1, \dots, N\}$, then:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sum_{n=-N}^N c_n e^{inx} e^{-imx}] dx \\ &= \sum_{n=-N}^N [\frac{1}{2\pi} \int_{-\pi}^{\pi} c_n e^{inx} e^{-imx} dx] = c_m \end{aligned}$$

(d) If $f(x)$ is real, then:

$$\overline{c_m} = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} e^{imx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} dx = c_{-m}$$

Thus, $f(x)$ is real if and only if $c_{-n} = \overline{c_n}$ for $n = \{0, 1, \dots, N\}$.

If $f(x)$ is integrable on $[-\pi, \pi]$, then c_m are called the Fourier coefficients and $f(x)$ is a Fourier series of f .

Definition 15.5.2: Orthogonal System of Functions

Let $\{\phi_n\}$ be a sequence of complex functions on $[a, b]$ such that:

$$\int_a^b \phi_n(x) \overline{\phi_m(x)} dx = 0 \quad \text{for } m \neq n$$

Then, $\{\phi_n\}$ is an orthogonal system of functions on $[a, b]$. Additionally, if:

$$\int_a^b \phi_n(x) \overline{\phi_n(x)} dx = \int_a^b |\phi_n(x)|^2 dx = 1$$

for all n , then $\{\phi_n\}$ is orthonormal.

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and $c_n = \int_a^b f(t) \overline{\phi_n(t)} dt$ for $n = \{1, 2, \dots\}$, then c_n is the n -th Fourier coefficient of f relative to $\{\phi_n\}$. Then:

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

Theorem 15.5.3: Fourier Series of f is the best approximation to f

For orthonormal $\{\phi_n\}$ on $[a,b]$, let n-th partial sum of the Fourier series of f, $\sum_{m=1}^n c_m \phi_m(x) = s_n(x)$. Suppose $f \in \mathcal{R}$ and $t_n(x) = \sum_{m=1}^n \gamma_m \phi_m(x)$. Then:

$$\int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx$$

where

$$\int_a^b |f - s_n|^2 dx = \int_a^b |f - t_n|^2 dx$$

if and only if $\gamma_m = c_m$ for every $m = \{1, \dots, n\}$.

Also, $\int |s_n(x)|^2 dx \leq \int |f(x)|^2 dx$.

Proof

$$\int f(x) \overline{t_n(x)} dx = \int f(x) \sum [\overline{\gamma_m \phi_m(x)}] dx = \sum [\int f(x) \overline{\gamma_m \phi_m(x)} dx] = \sum c_m \overline{\gamma_m}$$

Since $\{\phi_n\}$ is orthonormal, then:

$$\begin{aligned} \int |t_n(x)|^2 dx &= \int t_n(x) \overline{t_n(x)} dx = \int [\sum_m \gamma_m \phi_m(x)] [\sum_k \overline{\gamma_k \phi_k(x)}] dx \\ &= \sum_m \sum_k [\int \gamma_m \phi_m(x) \overline{\gamma_k \phi_k(x)} dx] = \sum |\gamma_m|^2 \end{aligned}$$

Thus:

$$\begin{aligned} \int |f(x) - t_n(x)|^2 dx &= \int |f(x)|^2 dx - \int f(x) \overline{t_n(x)} dx - \int \overline{f(x)} t_n(x) dx + \int |t_n(x)|^2 dx \\ &= \int |f(x)|^2 dx - \sum c_m \overline{\gamma_m} - \sum \overline{c_m} \gamma_m + \sum |\gamma_m|^2 \\ &= \int |f(x)|^2 dx - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2 \end{aligned}$$

Thus, $\int |f(x) - t_n(x)|^2 dx$ is minimized if and only if $\gamma_m = c_m$ for every $m = \{1, \dots, n\}$.

Let $\gamma_m = c_m$ and since $\int |f(x) - s_n(x)|^2 dx \geq 0$, then:

$$\begin{aligned} \int |f(x) - s_n(x)|^2 dx &= \int |f(x)|^2 dx - \sum |c_m|^2 \\ \int |s_n(x)|^2 dx &= \sum |c_m|^2 \leq \int |f(x)|^2 dx \end{aligned}$$

Theorem 15.5.4: Bessel Inequality

For $\{\phi_n\}$ is orthonormal on $[a,b]$ and $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$, if $f \in \mathcal{R}$, then:

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n = 0$$

Proof

Since $\{\phi_n\}$ is orthonormal on $[a,b]$ and $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$, then by **theorem 15.5.3**, for any integer $n > 1$:

$$\sum_{m=1}^n |c_m|^2 \leq \int_a^b |f(x)|^2 dx$$

Thus, as $n \rightarrow \infty$, then $\sum_{m=1}^{\infty} |c_m|^2 \leq \int_a^b |f(x)|^2 dx$.

Since $\sum_{m=1}^{\infty} |c_m|^2$ is monotonically increasing and bounded above, then $\sum_{m=1}^{\infty} |c_m|^2$ converges and thus, $\lim_{n \rightarrow \infty} c_n = 0$.

Definition 15.5.5: Trigonometric Series

Consider functions $f \in \mathcal{R}$ on $[-\pi, \pi]$ with period 2π . Let $\phi_n(x) = e^{inx}$ which is orthogonal and orthonormal when $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$.

Thus, the N -th partial sum of the Fourier series of f is:

$s_N(f; x) = \sum_{n=-N}^N c_n e^{inx}$
where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$. Then by **theorem 15.5.3**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(f; x)|^2 dx = \sum_{n=-N}^N |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

From the Dirichlet kernel, $D_N(x) = \sum_{n=-N}^N e^{inx}$.

$$(e^{ix} - 1)D_N(x) = \sum_{n=-N}^N [e^{i(n+1)x} - e^{inx}] = e^{i(N+1)x} - e^{-iNx}$$

$$\begin{aligned} D_N(x) &= \frac{e^{-\frac{1}{2}ix}(e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x})}{e^{-\frac{1}{2}ix}(e^{ix} - 1)} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \\ &= \frac{2i \sin((N+\frac{1}{2})x)}{2i \sin(\frac{1}{2}x)} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} \end{aligned}$$

Since e^{inx} is periodic for 2π for each $n \in [-N, N]$, then $D_N(x)$ is periodic for 2π .

Thus, since f is also periodic for 2π , then:

$$\begin{aligned} s_N(f; x) &= \sum_{n=-N}^N c_n e^{inx} = \sum_{n=-N}^N \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right] e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[\sum_{n=-N}^N e^{in(x-t)} \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt \end{aligned}$$

Theorem 15.5.6: If f is continuous at some x , then Fourier Series of f converges to f

If for some x , there are $\delta > 0$ and M such that $|f(x+t) - f(x)| \leq M|t|$ for all $t \in (-\delta, \delta)$, then:

$$\lim_{N \rightarrow \infty} s_N(f; x) = f(x)$$

Proof

Let $g(t) = \frac{f(x-t) - f(x)}{\sin(\frac{1}{2}t)}$ for $t \in [-\pi, \pi]$ where $g(0) = 0$. Then by **definition 15.5.1(b)**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{n=-N}^N e^{inx} \right] dx = 1$$

Thus:

$$\begin{aligned} s_N(f; x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt - f(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt - f(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)] D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin((N+\frac{1}{2})t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) [\sin(Nt) \cos(\frac{1}{2}t) + \sin(\frac{1}{2}t) \cos(Nt)] dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \cos(\frac{1}{2}t)] \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \sin(\frac{1}{2}t)] \cos(Nt) dt \end{aligned}$$

Since $g(t)$ and $\cos(\frac{1}{2}t), \sin(\frac{1}{2}t)$ are bounded on $[-\pi, \pi]$, then $g(t) \cos(\frac{1}{2}t)$ and $g(t) \sin(\frac{1}{2}t)$ are bounded on $[-\pi, \pi]$. As $N \rightarrow \infty$, then $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \cos(\frac{1}{2}t)] \sin(Nt) dt = 0$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \sin(\frac{1}{2}t)] \cos(Nt) dt = 0$ so $\lim_{N \rightarrow \infty} s_N(f; x) = f(x)$.

Corollary 15.5.7: Localization Theorem

If $f(x) = 0$ for all x in some segment J , then for every $x \in J$:

$$\lim_{N \rightarrow \infty} s_N(f; x) = 0$$

Proof

Let $J = (a, b)$. Then for $x \in J$, choose δ such that $(x - \delta, x + \delta) \subset J$.

Thus for any $t \in (-\delta, \delta)$, then $|f(x+t) - f(x)| = |0 - 0| = 0$.

Then by **theorem 15.5.6**, for every $x \in J$, $\lim_{N \rightarrow \infty} s_N(f; x) = f(x) = 0$.

Corollary 15.5.8: Equivalent functions on (a,b) have similar Fourier Series on (a,b)

If $f(t) = g(t)$ for all t in some neighborhood of x , then:

$$\lim_{N \rightarrow \infty} [s_N(f; x) - s_N(g; x)] = 0$$

Proof

Since $f(t) - g(t) = 0$ for all $t \in (x - \delta, x + \delta)$, then by **corollary 15.5.7**, then:

$$\lim_{N \rightarrow \infty} s_N(f - g; x) = 0$$

The Fourier series for $f - g$:

$$s_N(f - g; x) = \sum_{n=-N}^N c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f - g)(t) e^{-int} dt$$

The Fourier series for f and g :

$$s_N(f; x) = \sum_{n=-N}^N a_n e^{inx} \quad \text{where } a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

$$s_N(g; x) = \sum_{n=-N}^N b_n e^{inx} \quad \text{where } b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt$$

Then $s_N(f - g; x) = s_N(f; x) - s_N(g; x)$ and thus:

$$\lim_{N \rightarrow \infty} [s_N(f; x) - s_N(g; x)] = \lim_{N \rightarrow \infty} s_N(f - g; x) = 0$$

Theorem 15.5.9: There are Fourier Series that converge uniformly to continuous f

If f is continuous with period 2π , then for $\epsilon > 0$, there is a trigonometric polynomial P such that for all $x \in \mathbb{R}$:

$$|P(x) - f(x)| < \epsilon$$

Proof

Since $f(x)$ has a period of 2π , then for a fixed $x \in \mathbb{R}$, $f(x)$ on \mathbb{R} can be defined on compact $[x, x+2\pi]$ which is the complex unit circle T by a mapping of $x \rightarrow e^{ix}$.

The set of trigonometric polynomials, $P(x) = \sum_{n=-N}^N c_n e^{inx}$ for constants $c_n \in \mathbb{C}$ and integer $N \geq 0$, is an algebra \mathcal{A} since for $P_1(x) = \sum_{n=-N_1}^{N_1} a_n e^{inx}$ and $P_2(x) = \sum_{n=-N_2}^{N_2} b_n e^{inx}$, let $N = \max(N_1, N_2)$ and $a_n, b_n = 0$ if $n \geq N_1, N_2$ respectively:

$$P_1(x) + P_2(x) = \sum_{n=-N}^N (a_n + b_n) e^{inx} \text{ so } P_1(x) + P_2(x) \in \mathcal{A}$$

$$P_1(x)P_2(x) = \sum_{n=-2N}^{2N} d_n e^{inx} \text{ where } d_n = \sum_{k=-N}^N a_k b_{n-k} \text{ so } P_1(x)P_2(x) \in \mathcal{A}$$

$$cP_1(x) = \sum_{n=-N_1}^{N_1} (ca_n) e^{inx} \text{ where } ca_n \in \mathbb{C} \text{ so } cP_1(x) \in \mathcal{A}$$

Also, \mathcal{A} is self-adjoint since:

$$\overline{P_1(x)} = \sum_{n=-N_1}^{N_1} \overline{a_n} e^{-inx} = \sum_{n=-N_1}^{N_1} \overline{a_{-n}} e^{inx} \text{ where } \overline{a_{-n}} \in \mathbb{C} \text{ so } \overline{P_1(x)} \in \mathcal{A}$$

Also, \mathcal{A} separates points on T since any two points on T are distinct and \mathcal{A} vanishes at no point of T since $(0,0)$ does not exist on the complex unit circle. For $\pi > \epsilon > 0$, since the mapping $x \rightarrow e^{ix}$ is 1-1 from $[x+\epsilon, x+2\pi-\epsilon]$, then \mathcal{A} separates points and vanishes at no point on $[x+\epsilon, x+2\pi-\epsilon]$.

Thus, by **theorem 14.7.9**, then \mathcal{B} , the set of all uniformly convergent $P(x)$ from \mathcal{A} , consist of all complex continuous f on $[x+\epsilon, x+2\pi-\epsilon]$.

So there is a $P(x)$ such that $P(x)$ converges uniformly to f so for all $t \in [x, x+2\pi]$, then $|P(t) - f(t)| < \epsilon$. Since f has a period of 2π , then for all $x \in \mathbb{R}$, then $|P(t) - f(t)| < \epsilon$.

Definition 15.5.10: L^p Space

For $p \geq 1$, let $L^p = \{ f: [a,b] \rightarrow \mathbb{C} \mid \|f\|_p = [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} < \infty \}$.

For complex $f, g \in \mathcal{R}$:

- (a) **Holder's Inequality**: If $\frac{1}{p} + \frac{1}{q} = 1$ where $p, q \geq 1$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$

Proof

Claim: If $a, b \geq 0$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ and equality only if $a^p = b^q$.
 Take $y = f(x) = x^{p-1}$ for $x \in [0, a]$ and $x = f^{-1}(y) = \sqrt[p-1]{y}$ for $y \in [0, b]$.
 The total area is $\int_0^a x^{p-1} dx + \int_0^b y^{\frac{1}{p-1}} dy = \frac{a^p}{p} + \frac{p-1}{p} b^{\frac{p}{p-1}} = \frac{a^p}{p} + \frac{b^q}{q}$.
 Graphing each function on their respective axes, it is shown that regardless if $a^{p-1} > b$ or $a^{p-1} < b$, the total area is greater than ab and equality holds only if $a^{p-1} = b$ so $b^q = a^{(p-1)q} = a^{(p-1)\frac{p}{p-1}} = a^p$.

$$\begin{aligned} \frac{1}{\|f\|_p \|g\|_q} \|fg\|_1 &= \frac{1}{\|f\|_p \|g\|_q} \int |fg| dx = \frac{1}{\|f\|_p \|g\|_q} \int |f| |g| dx \\ &= \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} dx \leq \int \frac{|f|^p}{\|f\|_p^p} + \frac{|g|^q}{\|g\|_q^q} dx \\ &= \frac{1}{\|f\|_p^p} \int |f|^p dx + \frac{1}{\|g\|_q^q} \int |g|^q dx \\ &= \frac{1}{\|f\|_p^p} \|f\|_p^p + \frac{1}{\|g\|_q^q} \|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Since $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, then equality holds only if $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$.

- (b) **Minkowski's Inequality**: $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

Proof

Since $f, g \in \mathcal{R}$, then $|f + g|^p \in \mathcal{R}$. By Holder's Inequality:

$$\begin{aligned} \|f + g\|_p^p &= \int_a^b |f(x) + g(x)|^p dx = \int_a^b |f(x) + g(x)| |f(x) + g(x)|^{p-1} dx \\ &\leq \int_a^b (|f(x)| + |g(x)|) |f(x) + g(x)|^{p-1} dx \\ &\leq \int_a^b |f(x)| |f(x) + g(x)|^{p-1} dx + \int_a^b |g(x)| |f(x) + g(x)|^{p-1} dx \\ &\leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} + (\int_a^b |g(x)|^p dx)^{\frac{1}{p}} (\int_a^b |f(x) + g(x)|^{p-1(\frac{p}{p-1})} dx)^{1-\frac{1}{p}} \\ &= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1} \end{aligned}$$

Theorem 15.5.11: For integrable f , there are continuous g where $f, g \in L^2$

Let $f \in \mathcal{R}$ on $[a, b]$. Then for $\epsilon > 0$, there is a continuous g where:

$$g(a) = f(a) \quad g(b) = f(b) \quad \|f(x) - g(x)\|_2 < \epsilon$$

Proof

Since $f \in \mathcal{R}$, then $|f(x)| < M$. For $\epsilon > 0$, there is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$:

$$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \Delta x_i < \frac{\epsilon^2}{2M}$$

Let $g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$ for $t \in [x_{i-1}, x_i]$ which is continuous on $[a, b]$ since:

$$g(x_i+) = f(x_i) = g(x_i-) \Rightarrow g(x_i) = f(x_i) \text{ so } g(a) = f(a), g(b) = f(b)$$

Thus, for $t \in [x_{i-1}, x_i]$:

$$\begin{aligned} |f(t) - g(t)| &= |f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) - \frac{t - x_{i-1}}{\Delta x_i} f(x_i)| \\ &= |\frac{x_i - t}{\Delta x_i} [f(t) - f(x_{i-1})] + \frac{t - x_{i-1}}{\Delta x_i} [f(t) - f(x_i)]| \\ &\leq |\frac{x_i - t}{\Delta x_i}| |f(t) - f(x_{i-1})| + |\frac{t - x_{i-1}}{\Delta x_i}| |f(t) - f(x_i)| = M_i - m_i \end{aligned}$$

Since g is continuous, then $g \in \mathcal{R}$ and thus, $|f(x) - g(x)|^2 \in \mathcal{R}$. Thus:

$$\begin{aligned} \|f(x) - g(x)\|_2 &= [\int_a^b |f(x) - g(x)|^2 dx]^{\frac{1}{2}} = \lim_{n \rightarrow \infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t) - g(t)|^2 dt]^{\frac{1}{2}} \\ &\leq \lim_{n \rightarrow \infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i)^2 dt]^{\frac{1}{2}} \leq \lim_{n \rightarrow \infty} [\sum_{i=1}^n 2M \int_{x_{i-1}}^{x_i} (M_i - m_i) dt]^{\frac{1}{2}} \\ &= \lim_{n \rightarrow \infty} [2M \sum_{i=1}^n (M_i - m_i) \Delta x_i]^{\frac{1}{2}} < \lim_{n \rightarrow \infty} [2M \frac{\epsilon^2}{2M}]^{\frac{1}{2}} = \epsilon \end{aligned}$$

Theorem 15.5.12: Parseval's Theorem

For $f, g \in \mathcal{R}$ with period of 2π where:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$$

then:

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 dx = 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof

Since $f \in \mathcal{R}$ on $[x, x+2\pi]$ for a fixed $x \in \mathbb{R}$, where $f(x) = f(x+2\pi)$, then by **theorem 15.5.11**, for $\epsilon > 0$, there is a continuous h such that:

$$\|f(x) - h(x)\|_2 < \epsilon$$

Also, $h(x) = f(x)$ and $h(x+2\pi) = f(x+2\pi)$ for any $x \in \mathcal{R}$, and since $f(x) = f(x+2\pi)$, then h has a period of 2π . Then by **theorem 15.5.9**, there is a trigonometric polynomial $P(x)$ such that for all $x \in \mathbb{R}$:

$$|h(x) - P(x)| < \epsilon \quad \Rightarrow \quad \|h(x) - P(x)\|_2 = \left[\int_x^{x+2\pi} |h(x) - P(x)|^2 dx \right]^{\frac{1}{2}} < \sqrt{2\pi} \epsilon$$

Then by **theorem 15.5.3**:

$$\|h(x) - s_N(h; x)\|_2 \leq \|h(x) - P(x)\|_2 < \sqrt{2\pi} \epsilon$$

$$\|s_N(h; x) - s_N(f; x)\|_2 = \|s_N(h - f; x)\|_2 \leq \|h(x) - f(x)\|_2 < \epsilon$$

Thus:

$$\begin{aligned} \|f(x) - s_N(f; x)\|_2 &\leq \|f(x) - h(x)\|_2 + \|h(x) - s_N(h; x)\|_2 + \|s_N(h; x) - s_N(f; x)\|_2 \\ &< (2 + \sqrt{2\pi}) \epsilon \end{aligned}$$

Note $\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx = \sum_{n=-N}^N [c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \overline{g(x)} dx] = \sum_{n=-N}^N c_n \overline{\gamma_n}$.

By Holder's Inequality:

$$\begin{aligned} & \left| \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx - \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx \right| \\ & \leq \int_{-\pi}^{\pi} |f(x) - s_N(f; x)| |g(x)| dx \\ & \leq \left[\int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 dx \right]^{\frac{1}{2}} \left[\int_{-\pi}^{\pi} |g(x)|^2 dx \right]^{\frac{1}{2}} \\ & = \|f(x) - s_N(f; x)\|_2 \|g(x)\|_2 \end{aligned}$$

Since $g \in \mathcal{R}$, then $|g|^2 \in \mathcal{R}$ and thus, $\|g(x)\|_2$ is bounded.

Since $\lim_{N \rightarrow \infty} \|f(x) - s_N(f; x)\|_2 = 0$, then:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx = \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n \overline{\gamma_n} = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{f(x)} dx = \sum_{n=-\infty}^{\infty} c_n \overline{c_n} = \sum_{n=-\infty}^{\infty} |c_n|^2$$

16 Multivariable Functions

16.1 Linear Transformations

Definition 16.1.1: Vector Spaces

(a) **Vector Space**

A nonempty set $X \subset \mathbb{R}^n$ is a vector space if for all $x, y \in X$ and scalar c :

$$x+y \in X \quad cx \in X$$

Null vector 0 is also defined as $0 = (0, \dots, 0) \in \mathbb{R}^k$.

(b) **Linear Combinations and Span**

For scalars c_1, \dots, c_k , a linear combination of $x_1, \dots, x_k \in \mathbb{R}^n$:

$$c_1x_1 + \dots + c_kx_k$$

The span of x_1, \dots, x_k is the set of all linear combinations of x_1, \dots, x_k .

(c) **Independence and Dimension**

If $c_1x_1 + \dots + c_kx_k = 0$ only if $c_1 = \dots = c_k = 0$, then x_1, \dots, x_k are independent. Any independent set does not contain 0 since $c0 + c_1x_1 + \dots + c_kx_k = 0$ holds true for $c, 0, \dots, 0$ where c is any number, not just $0, 0, \dots, 0$.

If vector space X have r independent vectors, but not $r+1$ independent vectors, then $\dim(X) = r$. The set $\{0\}$ has dimension 0 .

(d) **Basis**

If $x_1, \dots, x_k \in X$ are independent and spans X , then x_1, \dots, x_k is a basis of X . Thus, for every $x \in X$:

Since x_1, \dots, x_k spans X , there exists c_1, \dots, c_k such that $x = c_1x_1 + \dots + c_kx_k$.

Since x_1, \dots, x_k are independent, then such c_1, \dots, c_k are unique else there are a_1, \dots, a_k where at least one $a_i \neq c_i$ such that:

$$x = a_1x_1 + \dots + a_kx_k \quad \Rightarrow \quad 0 = x - x = (a_1 - c_1)x_1 + \dots + (a_k - c_k)x_k$$

where at least one $(a_i - c_i) \neq 0$ contradicting x_1, \dots, x_k are independent.

The c_1, \dots, c_k are called the coordinates of x with respect to basis x_1, \dots, x_k .

(e) **Standard Basis of \mathbb{R}^k**

Let $e_i = (0, \dots, \underset{1}{0}, \dots, \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, \dots, \underset{k}{0}) \in \mathbb{R}^k$.

Thus, e_1, \dots, e_k is a basis for \mathbb{R}^k where any $x = (x_1, \dots, x_k) = x_1e_1 + \dots + x_ke_k$.

Theorem 16.1.2: $\dim(X) \leq (\# \text{ vectors that span } X)$

If vector space X is spanned by r vectors, then $\dim(X) \leq r$.

Proof

If $\dim(X) > r$, then there are at minimum $r+1$ independent vectors that spans X which contradicts that X is spanned by r vectors.

Let X be spanned by $x_1, \dots, x_r \neq 0$. If x_1, \dots, x_r are independent, then $\dim(X) = r$.

If x_1, \dots, x_r are not independent, then there is at least two $c_k \neq 0$ where:

$$0 = c_1x_1 + \dots + c_rx_r$$

since if only one $c_k \neq 0$, then $0 = c_1x_1 + \dots + c_rx_r = c_kx_k$ which implies $x_k = 0$ since $c_k \neq 0$ which is a contradiction. Thus, for $c_k, c_{i_1}, \dots, c_{i_n} \neq 0$:

$$0 = c_1x_1 + \dots + c_rx_r = c_kx_k + c_{i_1}x_{i_1} + \dots + c_{i_n}x_{i_n} \quad \Rightarrow \quad x_k = \frac{-c_{i_1}}{c_k}x_{i_1} + \dots + \frac{-c_{i_n}}{c_k}x_{i_n}$$

Remove x_k from x_1, \dots, x_r and repeat the process until all x_i are independent and thus, $\dim(X) = r - (\# x_i \text{ removed}) < r$.

Corollary 16.1.3: $\dim(X) = (\# \text{ vectors in a basis})$

If x_1, \dots, x_n is a basis for X , then $\dim(X) = n$.

Thus, $\dim(\mathbb{R}^n) = n$.

Proof

Since x_1, \dots, x_n is a basis for X , then x_1, \dots, x_n spans X and are independent.
 Since x_1, \dots, x_n span X , then by **theorem 16.1.2**, then $\dim(X) \leq n$. Since x_1, \dots, x_n are independent, then $\dim(X) \geq n$ since there might be another x_i independent to x_1, \dots, x_n and another and so on. Thus, $\dim(\mathbb{R}^n) = n$.
 Since e_1, \dots, e_n is a basis for \mathbb{R}^n , then $\dim(\mathbb{R}^n) = n$.

Theorem 16.1.4: Properties of Basis

For vector space X where $\dim(X) = n$:

- (a) n vectors span X if and only if the n vectors are independent
- (b) X has a basis where every basis have only n vectors
- (c) For independent x_1, \dots, x_r where $r \in \{1, \dots, n\}$, X has a basis with x_1, \dots, x_r

Intuition

x_1, \dots, x_m can span X , but not independent since there might be a x_i that is dependent on the other x_i (aka $x_i = a_i x_i + \dots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \dots + a_m x_m$).
 x_1, \dots, x_k can be independent, but not span X since there might be another x that is independent to each x_i (aka $x \neq b_1 x_1 + \dots + b_k x_k$ for any b_1, \dots, b_k).
 So to get a basis, either remove the dependent elements from x_1, \dots, x_m to get independent or add independent elements to x_1, \dots, x_k to get a span of X . Simply, a basis has a set amount of vectors, but x_1, \dots, x_m has too much while x_1, \dots, x_k has too few.

Proof

Let x_1, \dots, x_n span X . If x_1, \dots, x_n are not independent, then remove x_i until x_1, \dots, x_k are independent as performed in **theorem 16.1.2**. Thus, $\dim(X) = k < n$ which is a contradiction and thus, x_1, \dots, x_n are independent.

For independent x_1, \dots, x_n , add $y_1, \dots, y_k \in X$ so $x_1, \dots, x_n, y_1, \dots, y_k$ span X . Since $\dim(X) = n$, then $x_1, \dots, x_n, y_1, \dots, y_k$ are not independent. Since any non-independent set can remove elements in its span until it is independent and thus, preserves its span as performed in **theorem 16.1.2**, then each y_i can be removed to reach independent x_1, \dots, x_n which still spans X .

By part (a), any n independent vectors spans X so thus, forms a basis for X . For x_1, \dots, x_k where $k > n$, since $\dim(X) = n$, then x_1, \dots, x_k is non-independent and is thus, not a basis. For x_1, \dots, x_k where $k < n$, since $\dim(X) = n$, there is a $x \in X$ such that x_1, \dots, x_k, x are independent. Then $x \neq c_1 x_1 + \dots + c_k x_k$ for any c_1, \dots, c_k else

$$x = c_1 x_1 + \dots + c_k x_k \quad \Rightarrow \quad 0 = c_1 x_1 + \dots + c_k x_k + -x$$

so x_1, \dots, x_k, x are not independent. Thus, there is a $x \in X$ that is not in the span of x_1, \dots, x_k so x_1, \dots, x_k does not span X .

For independent x_1, \dots, x_r , since $\dim(X) = n$, there are x_{r+1}, \dots, x_n such that x_1, \dots, x_n are independent. By part (a), x_1, \dots, x_n spans X so x_1, \dots, x_n forms a basis which contain x_1, \dots, x_r .

Definition 16.1.5: Linear Transformation

A mapping A of vector space X into vector space Y is a linear transformation if for all $x_1, x_2 \in X$ and scalar c :

$$A(x_1 + x_2) = Ax_1 + Ax_2 \quad A(cx_1) = cAx_1$$

Since $A0 + A0 = A(0+0) = A0$, then $A0 = 0$.

If x_1, \dots, x_n is a basis for X , then for any $x \in X$, there is a unique set of c_1, \dots, c_n where $x = c_1x_1 + \dots + c_nx_n$ such that:

$$Ax = A(c_1x_1 + \dots + c_nx_n) = c_1Ax_1 + \dots + c_nAx_n$$

Linear transformation that maps X into X are **linear operators**.

Additionally, if A is 1-1 and maps X onto X , then A is **invertible**.

Thus, there is a A^{-1} such that:

$$A^{-1}(Ax) = x \quad \text{for all } x \in X$$

Since A maps X onto X , for any $x \in X$, then $Ax = y \in X$.

Thus, for all $y \in X$, then $x = A^{-1}(Ax) = A^{-1}y$. Thus:

$$A(A^{-1}y) = Ax = y$$

Also, for any $x_1, x_2 \in X$ and scalars c_1, c_2 where $Ax_1 = y_1$ and $Ax_2 = y_2$:

$$\begin{aligned} A^{-1}(c_1y_1 + c_2y_2) &= A^{-1}(c_1Ax_1 + c_2Ax_2) = A^{-1}(A(c_1x_1 + c_2x_2)) \\ &= c_1x_1 + c_2x_2 = c_1A^{-1}(y_1) + c_2A^{-1}(y_2) \end{aligned}$$

So, A^{-1} is a linear transformation.

Theorem 16.1.6: Linear Operators imply 1-1 \Leftrightarrow onto

Linear operator A preserves independence if and only if A is 1-1.

Thus, linear operator A is 1-1 if and only if $A(X) = X$.

Proof

Let x_1, \dots, x_n be a basis for X where each $Ax_i = y_i \in X$. So for any $y \in A(X)$, there is $x \in X$ where $x = c_1x_1 + \dots + c_nx_n$ for a unique set of c_1, \dots, c_n such that:

$$y = Ax = A(c_1x_1 + \dots + c_nx_n) = c_1Ax_1 + \dots + c_nAx_n = c_1y_1 + \dots + c_ny_n$$

If A is 1-1, then there is only one such x so in respect to y_1, \dots, y_n , then any

$y = k_1y_1 + \dots + k_ny_n$ must have $k_1 = c_1, \dots, k_n = c_n$. Thus, for $y = 0$, since $0 = A0$ and x_1, \dots, x_n are independent, then $c_1 = \dots = c_n = 0$ so y_1, \dots, y_n are independent.

If A is not 1-1, then there is y where there are at least two distinct such x so in respect to y_1, \dots, y_n , then $y = k_1y_1 + \dots + k_ny_n$ holds true for at least 2 distinct k_1, \dots, k_n so y_1, \dots, y_n are not independent. Thus, A is 1-1 if and only if y_1, \dots, y_n is independent. By **theorem 16.1.4a**, y_1, \dots, y_n span X so $A(X) = X$ if and only if y_1, \dots, y_n are independent.

Definition 16.1.7: Operations of Linear Transformatons

Let $L(X, Y)$ be the set of all linear transformation of X into Y .

Let Ω be the set of all invertible linear operators on \mathbb{R}^n .

- (a) If $A_1, A_2 \in L(X, Y)$ and c_1, c_2 are scalars, then for any $x \in X$, define:
 $(c_1 A_1 + c_2 A_2)x = c_1 A_1 x + c_2 A_2 x$
- (b) For vector space Z , if $A \in L(X, Y)$ and $B \in L(Y, Z)$, then for any $x \in X$, define:
 $(BA)x = B(Ax) \in L(X, Z)$
- (c) For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, define the norm:
 $\|A\| = \sup(|Ax| \mid x \in \mathbb{R}^n \text{ where } |x| \leq 1)$
- (d) $|Ax| = |A(|x| \frac{x}{|x|})| = |A(\frac{x}{|x|})| |x| \leq \sup(|A(\frac{x}{|x|})|) |x| = \|A\| |x|$
 If there is a λ such that $|Ax| \leq \lambda |x|$ for all $x \in \mathbb{R}^n$, then $\|A\| \leq \lambda |1| = \lambda$.
- (e) For $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$, the distance between A and B is defined $\|A - B\|$

Theorem 16.1.8: Operations of Norms of Linear Transformations

- (a) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then $\|A\| < \infty$. Thus, A is uniformly continuous.

Proof

For $|x| \leq 1$:

$$|Ax| = |A(x_1 e_1 + \dots + x_n e_n)| \leq |x_1| |Ae_1| + \dots + |x_n| |Ae_n|$$

$$\leq |Ae_1| + \dots + |Ae_n| = M$$

Thus, $\|Ax\| \leq |Ae_1| + \dots + |Ae_n| = M < \infty$.

Let $|x - y| < \epsilon$ and thus, $|Ax - Ay| = |A(x - y)| \leq \|A\| |x - y| < M\epsilon$ so A is uniformly continuous.

- (b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then:

$$\|A + B\| \leq \|A\| + \|B\| \quad \|cA\| = |c| \|A\|$$

Proof

For $|x| \leq 1$, $|(A + B)x| \leq |Ax + Bx| \leq |Ax| + |Bx| \leq \|A\| + \|B\|$.

Thus, $\|A + B\| \leq \|A\| + \|B\|$. Since $|cAx| = |c| |Ax|$, then $\|cA\| = |c| \|A\|$.

Also, for the distance between A and B , by part a:

$$\|A - B\| \leq \|A + B\| \leq \|A\| + \|B\| \leq M_1 + M_2$$

- (c) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then:

$$\|BA\| \leq \|B\| \|A\|$$

Proof

For $|x| \leq 1$, $|BAx| = |B(Ax)| \leq \|B\| |Ax| \leq \|B\| \|A\| |x| \leq \|B\| \|A\|$.

Thus, $\|BA\| \leq \|B\| \|A\|$.

Theorem 16.1.9: Operations of Norms of Invertible Linear Operators

- (a) If $A \in \Omega$ and $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ where $\|B - A\| \|A^{-1}\| < 1$, then $B \in \Omega$

Proof

$$\begin{aligned} \frac{1}{\|A^{-1}\|} |x| &= \frac{1}{\|A^{-1}\|} |A^{-1}Ax| \leq \frac{1}{\|A^{-1}\|} \|A^{-1}\| |Ax| \\ &= |Ax| \leq |(A - B)x| + |Bx| \leq \|A - B\| |x| + |Bx| \end{aligned}$$

Thus, $|Bx| \geq (\frac{1}{\|A^{-1}\|} - \|A - B\|) |x| \geq \frac{2}{\|A^{-1}\|} |x| \geq 0$ so $Bx \neq 0$ if $x \neq 0$ so B is 1-1. Then by **theorem 16.1.4a**, B spans \mathbb{R}^n so B is invertible so $B \in \Omega$.

- (b) $\Omega \subset L(\mathbb{R}^n, \mathbb{R}^n)$ is open and the mapping $T: A \rightarrow A^{-1}$ is continuous on Ω

Proof

Since $\|B - A\| < \frac{1}{\|A^{-1}\|}$ for any $B \in \Omega$, then for every $B \in \Omega$, there exist an open subset of $L(\mathbb{R}^n, \mathbb{R}^n)$ that contains B so Ω is open.

Since

$$\begin{aligned} |y| &= |BB^{-1}y| \geq (\frac{1}{\|A^{-1}\|} - \|A - B\|) |B^{-1}y| \\ &\geq (\frac{1}{\|A^{-1}\|} - \|A - B\|) \|B^{-1}\| |y| \end{aligned}$$

then $\frac{1}{\frac{1}{\|A^{-1}\|} - \|A - B\|} \geq \|B^{-1}\|$. Thus, by **theorem 16.1.8**:

$$\begin{aligned} \|B^{-1} - A^{-1}\| &= \|B^{-1}(A - B)A^{-1}\| \\ &\leq \|B^{-1}\| \|A - B\| \|A^{-1}\| \leq \frac{\|A - B\| \|A^{-1}\|}{\frac{1}{\|A^{-1}\|} - \|A - B\|} \end{aligned}$$

Since $\lim_{B \rightarrow A} \|A - B\| \rightarrow 0$ so $\|B^{-1} - A^{-1}\| \rightarrow 0$, then T is continuous on Ω .

Definition 16.1.10: Matrices

Let x_1, \dots, x_n be a basis for X and y_1, \dots, y_m be a basis for Y .

Then every $A \in L(X, Y)$ determines a set of numbers a_{ij} such that:

$$Ax_j = \sum_{i=1}^m a_{ij} y_i \quad \text{for } j = \{1, \dots, n\}$$

Thus, A can be represented by an m by n matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Since the a_{ij} of Ax_j are from the j -th column $[A]$, then Ax_j is called the column vector of $[A]$. Thus, the $\text{span}(A)$ is the span of the column vectors of $[A]$.

For any $x \in X$, there is a unique set of c_1, \dots, c_n such that $x = c_1 x_1 + \dots + c_n x_n$:

$$[Ax] = \begin{bmatrix} (y_1) \{ \overbrace{a_{11}}^{c_1} \overbrace{a_{12}}^{c_2} \dots \overbrace{a_{1n}}^{c_n} \\ (y_2) \{ a_{21} \quad a_{22} \quad \dots \quad a_{2n} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ (y_m) \{ a_{m1} \quad a_{m2} \quad \dots \quad a_{mn} \end{bmatrix}$$

$$\begin{aligned} Ax &= A(\sum_{j=1}^n c_j x_j) \\ &= \sum_{j=1}^n c_j Ax_j \\ &= \sum_{j=1}^n c_j \sum_{i=1}^m a_{ij} y_i \\ &= \sum_{j=1}^n \sum_{i=1}^m a_{ij} c_j y_i \\ &= \sum_{i=1}^m [\sum_{j=1}^n a_{ij} c_j] y_i \end{aligned}$$

So $[\sum_{j=1}^n a_{1j} c_j], \dots, [\sum_{j=1}^n a_{mj} c_j]$ are Ax 's coordinates in respect to y_1, \dots, y_m .

Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then, $BA \in L(X, Z)$.

Let z_1, \dots, z_p be a basis for Z where:

$$By_i = \sum_{k=1}^p b_{ki} z_k \quad (BA)x_j = \sum_{k=1}^p c_{kj} z_k$$

Thus, B as a p by m matrix and BA as a p by n matrix can be represented:

$$[B] = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \quad [BA] = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix}$$

$$\begin{aligned} (BA)x_j &= B(Ax_j) = B(\sum_{i=1}^m a_{ij} y_i) \\ &= \sum_{i=1}^m a_{ij} By_i \\ &= \sum_{i=1}^m a_{ij} \sum_{k=1}^p b_{ki} z_k \\ &= \sum_{i=1}^m \sum_{k=1}^p b_{ki} a_{ij} z_k \\ &= \sum_{k=1}^p [\sum_{i=1}^m b_{ki} a_{ij}] z_k \end{aligned}$$

Thus, $c_{kj} = \sum_{i=1}^m b_{ki} a_{ij}$ for $j = \{1, \dots, n\}$ and $k = \{1, \dots, p\}$.

So to get matrix $[BA]$ from $[B]$ and $[A]$:

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m b_{1i} a_{i1} & \sum_{i=1}^m b_{1i} a_{i2} & \dots & \sum_{i=1}^m b_{1i} a_{in} \\ \sum_{i=1}^m b_{2i} a_{i1} & \sum_{i=1}^m b_{2i} a_{i2} & \dots & \sum_{i=1}^m b_{2i} a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m b_{pi} a_{i1} & \sum_{i=1}^m b_{pi} a_{i2} & \dots & \sum_{i=1}^m b_{pi} a_{in} \end{bmatrix}$$

For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, since $Ax = \sum_{i=1}^m [\sum_{j=1}^n a_{ij}c_j] e_i$ where $x = \sum_{j=1}^n c_j e_j$, then by the Cauchy-Schwarz Inequality:

$$\begin{aligned} |Ax|^2 &= \sum_{i=1}^m [\sum_{j=1}^n a_{ij}c_j]^2 \\ &\leq \sum_{i=1}^m [(\sum_{j=1}^n a_{ij}^2) (\sum_{j=1}^n c_j^2)] \\ &= [\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2] (\sum_{j=1}^n c_j^2) \\ &= [\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2] |x|^2 \end{aligned}$$

Thus, for $|x| \leq 1$, then $\|A\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$.

Theorem 16.1.11: A linear transformation of continuous functions is continuous

If each a_{ij} is a continuous function on S and for each $p \in S$, then $A_p \in L(\mathbb{R}^n, \mathbb{R}^m)$ with entries $a_{ij}(p)$, then the mapping $T: S \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

Proof

Since each $a_{i,j}$ is continuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for $t, p \in S$ where $|t - p| < \delta$, then $|a_{i,j}(t) - a_{i,j}(p)| < \frac{\epsilon}{\sqrt{mn}}$. Thus, for $|t - p| < \delta$:

$$\|A_p - A_t\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n (a_{ij}(p) - a_{ij}(t))^2} < \sqrt{\sum_{i=1}^m \sum_{j=1}^n (\frac{\epsilon}{\sqrt{mn}})^2} = \epsilon$$

16.2 Differentiation

Definition 16.2.1: Derivative Extended to Higher Dimensions

First, let's redefine the derivative such that it can be extended to higher dimensions. For $f: (a,b) \subset \mathbb{R} \rightarrow \mathbb{R}^m$, let $f'(x) = y \in \mathbb{R}^m$ such that:

$$f(x+h) - f(x) = yh + r(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{r(h)}{h} = 0$$

Since $y: h \rightarrow yh$ is a linear transformation from \mathbb{R} to \mathbb{R}^m , then $f'(x) \in L(\mathbb{R}, \mathbb{R}^m)$.

Now for derivatives in higher dimensions.

Let $f: x \in \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$.

If there is an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that for any $h \in E$:

$$f(x+h) - f(x) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

then f is differentiable at x . Then differential of f at x , $f'(x) = A$.

If f is differentiable at every $x \in E$, then f is differentiable on E .

Theorem 16.2.2: The derivative of a function is unique

Let $f: x \in \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that for any $h \in E$:

$$f(x+h) - f(x) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

Suppose $A = A_1$ and $A = A_2$ satisfies such conditions. Then $A_1 = A_2$.

Proof

For any $h \in \mathbb{R}^n$:

$$\begin{aligned} |(A_2 - A_1)h| &= |[f(x+h) - f(x) - r_{A_1}(h)] - [f(x+h) - f(x) - r_{A_2}(h)]| \\ &= |r_{A_2}(h) - r_{A_1}(h)| \\ &\leq |r_{A_2}(h)| + |r_{A_1}(h)| \end{aligned}$$

Since $A_1, A_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$, for any t where h is fixed, then:

$$\begin{aligned} |(A_2 - A_1)(th)| &\leq |r_{A_2}(th)| + |r_{A_1}(th)| \\ |t| |(A_2 - A_1)h| &\leq |r_{A_2}(th)| + |r_{A_1}(th)| \\ |(A_2 - A_1)h| &\leq \frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|} \end{aligned}$$

So as $t \rightarrow 0$, then $\frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|} \rightarrow 0 + 0 = 0$. Thus, $A_1 = A_2$.

Theorem 16.2.3: Derivative of a Linear Transformation

If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $x \in \mathbb{R}^n$, then:

$$A'(x) = A$$

Proof

Since $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then let $f(x) = Ax$.

$$f(x+h) - f(x) = A(x+h) - Ax = Ax + Ah - Ax = Ah$$

Thus, $r_A(h) = 0$ so $\lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = \lim_{h \rightarrow 0} 0 = 0$. Thus, $A'(x) = f'(x) = A$.

Theorem 16.2.4: Chain Rule in Higher Dimensions

Let $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at $x_0 \in E$ and $g: f(E) \subset \text{open } H \subset \mathbb{R}^m \rightarrow \mathbb{R}^k$ be differentiable at $f(x_0)$.

Then $F: E \rightarrow \mathbb{R}^k$ where $F(x) = g(f(x))$ is differentiable at x_0 such that:

$$F'(x_0) = g'(f(x_0)) f'(x_0)$$

Proof

Since f is differentiable at x_0 and g is differentiable at $f(x_0)$, then there is a $A = f'(x_0)$ and $B = g'(f(x_0))$ such that:

$$f(x_0+h) - f(x_0) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

$$g(f(x_0)+k) - g(f(x_0)) = Bk + r_B(k) \quad \text{where } \lim_{k \rightarrow 0} \frac{|r_B(k)|}{|k|} = 0$$

Let $k = f(x_0+h) - f(x_0)$. Thus:

$$\begin{aligned} F(x_0+h) - F(x_0) - BAh &= g(f(x_0+h)) - g(f(x_0)) - BAh \\ &= g(f(x_0)+k) - g(f(x_0)) - BAh = Bk + r_B(k) - BAh \\ &= B(k - Ah) + r_B(k) = B(f(x_0+h) - f(x_0) - Ah) + r_B(k) \\ &= Br_A(h) + r_B(k) \end{aligned}$$

$$\frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} = \frac{|Br_A(h) + r_B(k)|}{|h|} \leq \frac{|Br_A(h)| + |r_B(k)|}{|h|} \leq \frac{\|B\| |r_A(h)| + |r_B(k)|}{|h|}$$

Since f is differentiable at x_0 , then f is continuous at x_0 and thus, $\lim_{h \rightarrow 0} k = 0$.

Since $\lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$ and $\lim_{k \rightarrow 0} \frac{|r_B(k)|}{|k|} = 0$, then:

$$\lim_{h \rightarrow 0} \frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} \leq \lim_{h \rightarrow 0} \|B\| \frac{|r_A(h)|}{|h|} + \lim_{h \rightarrow 0} \frac{|r_B(k)|}{|h|} = 0 + 0 = 0$$

Thus, $F'(x_0) = BA = g'(f(x_0)) f'(x_0)$.

Definition 16.2.5: Partial Derivatives: Derivatives along the standard basis

Let $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. The components of f are the $f_1, \dots, f_m \in \mathbb{R}$ such that for $x \in E$, then $f(x) = \sum_{i=1}^m f_i(x) e_i$.

Since $e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, then $f(x) \cdot e_i = [\sum_{i=1}^m f_i(x) e_i] \cdot e_i = f_i(x)$.

Then for $x \in E$ and $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, let the **partial derivative** $\frac{\partial f_i}{\partial x_j} = D_j f_i$ be the derivative of f_i with respect to x_j . Then for $t \in \mathbb{R}$:

$$f_i(x + te_j) - f_i(x) = D_j f_i(te_j) + r_{D_j f_i}(te_j) \quad \text{where } \lim_{t \rightarrow 0} \frac{|r_{D_j f_i}(te_j)|}{|t|} = 0$$

Theorem 16.2.6: Derivative of f is the sum of all partial derivatives

Let $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at $x \in E$. Then the partial derivatives $(D_j f_i)(x)$ exists such that for $j \in \{1, \dots, n\}$:

$$f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i$$

Proof

For a fixed j , since f is differentiable at x , then:

$$f(x+te_j) - f(x) = f'(x)(te_j) + r(te_j) \quad \text{where } \lim_{t \rightarrow 0} \frac{|r(te_j)|}{|t|} = 0$$

Then $f'(x)$ exist where:

$$\lim_{t \rightarrow 0} \frac{f(x+te_j) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{f'(x)(te_j)}{t} + \frac{r(te_j)}{t} = \lim_{t \rightarrow 0} t \frac{f'(x)e_j}{t} = f'(x)e_j$$

Since $f(x) = \sum_{i=1}^m f_i(x)e_i$, then:

$$\lim_{t \rightarrow 0} \frac{f(x+te_j) - f(x)}{t} = \lim_{t \rightarrow 0} \sum_{i=1}^m \frac{f_i(x+te_j) - f_i(x)}{t} e_i = f'(x)e_j$$

Since $f'(x)$ exist and $\lim_{t \rightarrow 0} \frac{f_i(x+te_j) - f_i(x)}{t} = D_j f_i(x)$, then each $D_j f_i(x)$ exists where:

$$f'(x)e_j = \sum_{i=1}^m \lim_{t \rightarrow 0} \frac{f_i(x+te_j) - f_i(x)}{t} e_i = \sum_{i=1}^m (D_j f_i)(x)e_i$$

Definition 16.2.7: Matrix of the Differential of f

By **theorem 16.2.6**, $f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i$ where $(D_j f_i)(x)$ is the derivative of the component f_i in respect to x_j for $j = \{1, \dots, n\}$.

Since $f'(x)e_j$ is the j -th column of $[f'(x)]$, then:

$$[f'(x)] = \begin{bmatrix} \sum_{i=1}^m (D_1 f_i)(x)e_i & \sum_{i=1}^m (D_2 f_i)(x)e_i & \dots & \sum_{i=1}^m (D_n f_i)(x)e_i \end{bmatrix}$$

where each $\sum_{i=1}^m (D_j f_i)(x)e_i$ is a column vector at the j -th column.

Since each $\sum_{i=1}^m (D_j f_i)(x)e_i$ has a coordinate of $(D_j f_i)(x)$ for e_i where each $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^m$, then:

$$[f'(x)] = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & \dots & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & \dots & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & \dots & (D_n f_m)(x) \end{bmatrix}$$

Thus, for $x \in \mathbb{R}^n$ where $x = x_1 e_1 + \dots + x_n e_n$, then:

$$\begin{aligned} f'(x)x &= f'(x) \left[\sum_{j=1}^n x_j e_j \right] \\ &= \sum_{j=1}^n x_j f'(x)e_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m (D_j f_i)(x)e_i \\ &= \sum_{i=1}^m \left[\sum_{j=1}^n x_j (D_j f_i)(x) \right] e_i \end{aligned}$$

Definition 16.2.8: Gradient and Directional Derivative

Let $\gamma: (a,b) \subset \mathbb{R} \rightarrow \text{open } E \subset \mathbb{R}^n$ and $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ both be differentiable. Then by **theorem 16.2.4**, $g: \mathbb{R} \rightarrow \mathbb{R}$ defined as $g(t) = f(\gamma(t))$ is differentiable for any $t \in (a,b)$ such that:

$$g'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

Since $f(\gamma(t)): E \subset \mathbb{R}^n \rightarrow \mathbb{R}$, by **theorem 16.2.6**, then:

$$f'(\gamma(t))e_j = (D_j f)(\gamma(t)) \text{ for } j = \{1, \dots, n\}$$

Since $\gamma: (a,b) \subset \mathbb{R} \rightarrow \text{open } E \subset \mathbb{R}^n$, then:

$$\gamma'(t) = \sum_{i=1}^n (D_i \gamma_i)(t)e_i = \sum_{i=1}^n \gamma'_i(t)e_i$$

Thus, $g'(t) = \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t)$.

For each $x \in E$, let the **gradient** of $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ at x , $(\nabla f)(x)$:

$$(\nabla f)(x) = \sum_{i=1}^n (D_i f)(x)e_i$$

Since $e_i e_j = 1$ if $i = j$, but $e_i e_j = 0$ if $i \neq j$, then:

$$\begin{aligned} [f(\gamma(t))]' &= g'(t) \\ &= \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t) \\ &= \sum_{i=1}^n [(D_i f)(\gamma(t))e_i \cdot \gamma'_i(t)e_i] \\ &= [\sum_{i=1}^n (D_i f)(\gamma(t))e_i] \cdot [\sum_{i=1}^n \gamma'_i(t)e_i] = (\nabla f)(\gamma(t)) \cdot \gamma'(t) \end{aligned}$$

For $t \in (-\infty, \infty)$, let $\gamma(t) = x + tu$ where $x \in E$ and unit vector $u \in \mathbb{R}^n$. Then:

$$\begin{aligned} (D_u f)(x) &= \lim_{t \rightarrow 0} \frac{f(x+tu) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} = g'(x) \\ &= (\nabla f)(\gamma(x)) \cdot \gamma'(x) = (\nabla f)(x) \cdot u \end{aligned}$$

Let $(D_u f)(x)$ be the **directional derivative** of f at x in direction of u .

For $u = u_1 e_1 + \dots + u_n e_n$:

$$(D_u f)(x) = (\nabla f)(x) \cdot u = \sum_{i=1}^n (D_i f)(x)e_i \cdot \sum_{i=1}^n u_i e_i = \sum_{i=1}^n (D_i f)(x)u_i$$

Also, for a fixed f and x , $(D_u f)(x)$ is maximized when $u = \lambda(\nabla f)(x)$ for $\lambda > 1$ since $x \cdot y = |x||y| \cos(\theta)$ where θ is the angle between x and y .

Theorem 16.2.9: A bounded derivative over a convex space have bounded range

For differentiable $f: \text{convex open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, there is a $M \in \mathbb{R}$ such that $\|f'(x)\| \leq M$ for every $x \in E$. Then for all $a, b \in E$:

$$|f(b) - f(a)| \leq M|b - a|$$

Proof

For fixed $a, b \in E$, let $\gamma(t) = (1-t)a + tb$. Since E is convex, for $t \in [0,1]$, then $\gamma(t) \in E$. Let $g(t) = f(\gamma(t))$. Then $g'(t) = f'(\gamma(t))\gamma'(t) = f'(\gamma(t))(b-a)$. Thus, for $t \in [0,1]$:

$$|g'(t)| = |f'(\gamma(t))(b-a)| \leq \|f'(\gamma(t))\| |b-a| \leq M|b-a|$$

Since $g(0) = f(\gamma(0)) = f(a)$ and $g(1) = f(\gamma(1)) = f(b)$, then by the Mean Value Theorem, for $x \in (0,1)$

$$|f(b) - f(a)| = |g(1) - g(0)| \leq (1-0)|g'(x)| \leq M|b-a|$$
Corollary 16.2.10: If the derivative is 0, the function is constant

For differentiable $f: \text{convex open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $f'(x) = 0$ for all $x \in E$.

Then, f is constant.

Proof

Since $\|f'(x)\| = 0$ for all $x \in E$, then by **theorem 7.2.9**, for all $a, b \in E$:

$$0 \leq |f(b) - f(a)| \leq 0(b-a) = 0$$

Thus, $f(b) = f(a)$ for all $a, b \in E$ so f is constant.

Definition 16.2.11: Continuously Differentiable

A differentiable f : open $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable in E if:

$f': E \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous

For $\epsilon > 0$, there is a $\delta > 0$ such that for every $x, y \in E$ where $|x - y| < \delta$, then:

$$\|f'(y) - f'(x)\| < \epsilon$$

If f is continuously differentiable, then $f \in \mathcal{C}'(E)$.

Theorem 16.2.12: Continuously differentiable imply continuous partial derivatives

Let f : open $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then $f \in \mathcal{C}'(E)$ if and only if each partial derivative

$D_j f_i$ exist and are continuous on E .

Proof

If $f \in \mathcal{C}'(E)$, then f is differentiable. Thus, by **theorem 16.2.6**, partial derivative $D_j f_i$ where $j = \{1, \dots, n\}$ exists for any $x \in E$ such that:

$$f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i \quad \Rightarrow \quad (D_j f_i)(x) = f'(x)e_j \cdot e_i$$

Thus, since $f \in \mathcal{C}'(E)$, then for $|x - y| < \delta$:

$$\begin{aligned} |(D_j f_i)(y) - (D_j f_i)(x)| &= |f'(y)e_j \cdot e_i - f'(x)e_j \cdot e_i| = |[f'(y) - f'(x)]e_j \cdot e_i| \\ &\leq |[f'(y) - f'(x)]e_j| |e_i| \leq \|f'(y) - f'(x)\| |e_j| |e_i| \\ &= \|f'(y) - f'(x)\| < \epsilon \end{aligned}$$

Thus, each $D_j f_i$ is continuous.

Since each $D_j f_i$ is continuous, then for $\epsilon > 0$, there is a $\delta > 0$ such that for $|y - x| < \delta$, then for all $j \in \{1, \dots, n\}$ and $i \in \{1, \dots, m\}$, then $|D_j f_i(y) - D_j f_i(x)| < \epsilon$.

Then for $h = h_1 e_1 + \dots + h_n e_n$ where $|x - h| < \delta$:

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + \dots + h_n e_n) - f_i(x)]e_i - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + \dots + h_n e_n) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{\left| \begin{array}{c} f_i(x + \sum_{k=1}^n h_k e_k) - f_i(x + \sum_{k=1}^{n-1} h_k e_k) \\ + f_i(x + \sum_{k=1}^{n-1} h_k e_k) - f_i(x + \sum_{k=1}^{n-2} h_k e_k) \\ + \dots + f_i(x + h_1) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x)h_j \end{array} \right|}{|h|} e_i| \end{aligned}$$

Since each $D_j f_i$ exist, then by the Mean Value Theorem, for each $j = \{1, \dots, n\}$, there is a $t_j \in (0, h_j)$ such that:

$$f_i(x + \sum_{k=1}^j h_k e_k) - f_i(x + \sum_{k=1}^{j-1} h_k e_k) = D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j)h_j$$

Thus:

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j)h_j - \sum_{j=1}^n (D_j f_i)(x)h_j]e_i|}{|h|} \\ &< \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n \epsilon h_j]e_i|}{|h|} \leq \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [n\epsilon|h|]e_i|}{|h|} = \lim_{h \rightarrow 0} \frac{\sqrt{mn}\epsilon|h|}{|h|} = \sqrt{mn}\epsilon \end{aligned}$$

Thus, $f(x)$ is differentiable where:

$$f'(x) = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & \dots & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & \dots & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & \dots & (D_n f_m)(x) \end{bmatrix}$$

Thus, for $|y - x| < \delta$:

$$\|f'(y) - f'(x)\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n [(D_j f_i)(y) - (D_j f_i)(x)]^2} < \sqrt{\sum_{i=1}^m \sum_{j=1}^n \epsilon^2} = \sqrt{mn}\epsilon$$

Thus, $f \in \mathcal{C}'(E)$.

16.3 The Contraction Principle

Definition 16.3.1: Contraction

For metric space X with metric d , then $\phi: X \rightarrow X$ is a **contraction** if there is $c \in (0,1)$ such that for all $x,y \in X$:

$$d(\phi(x), \phi(y)) \leq c d(x,y)$$

Theorem 16.3.2: Banach's Fixed Point Theorem

If X is a complete metric space and ϕ is a contraction of X into X , then there is a unique $x \in X$ such that $\phi(x) = x$

Proof

Let $\phi(x) = x$ and $\phi(y) = y$. Since ϕ is a contraction, then $d(x,y) = d(\phi(x), \phi(y)) \leq c d(x,y)$ would hold true only if $d(x,y) = 0$ so $x = y$. Thus, such a $\phi(x) = x$ is unique. For a fixed $x_0 \in X$, let $\{x_n\}$ have $x_{n+1} = \phi(x_n)$. Thus, for some $c \in (0,1)$:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\phi(x_n), \phi(x_{n-1})) \leq c d(x_n, x_{n-1}) \\ &= c d(\phi(x_{n-1}), \phi(x_{n-2})) = \dots = c^n d(x_1, x_0) \end{aligned}$$

Thus, for $\epsilon > 0$, choose N such that $d(x_1, x_0) \frac{c^N}{(1-c)} < \epsilon$. Then for $m > n \geq N$:

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} c^i d(x_1, x_0) \\ &\leq d(x_1, x_0) \frac{c^n}{1-c} \leq d(x_1, x_0) \frac{c^N}{1-c} < \epsilon \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy Sequence and since X is complete, then $\{x_n\}$ converges to a $x \in X$. Note a contraction is uniformly continuous so:

$$\phi(x) = \lim_{n \rightarrow \infty} \phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

Example

For $y' = y$ where $y(0) = 1$, show $y(x) = e^x$ for x near 0.

Take the metric space of continuous functions, $C[a,b]$, with the sup metric as defined in [definition 14.3.4](#) where $0 \in [a,b]$. By [theorem 14.3.5](#), $C[a,b]$ is complete.

Then for each $f \in C[a,b]$, let $Tf(x) = 1 + \int_0^x f(t) dt$ for $x \in [a,b]$.

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| \int_0^x f(t) - g(t) dt \right| \leq \int_{\min(0,x)}^{\max(0,x)} |f(t) - g(t)| dt \\ &\leq |x - 0| d(f,g) \leq (b-a) d(f,g) \end{aligned}$$

Thus, $d(Tf(x), Tg(x)) \leq (b-a) d(f,g)$ so for $b-a < 1$, then T is a contraction. By [theorem 16.3.2](#), there is a unique y where $y(x) = 1 + \int_0^x y(t) dt$. To determine y , use the process defined in [theorem 16.3.2](#)'s proof referred as the Picard iteration. Using any continuous $f(x)$, let's take $f(x) = 1$. Then:

$$T(1) = 1 + \int_0^x 1 dt = 1 + x$$

$$T(T(1)) = 1 + \int_0^x 1+t dt = 1 + x + \frac{1}{2}x^2$$

$$T(T(T(1))) = 1 + \int_0^x 1+t+\frac{1}{2}t^2 dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Thus, by [definition 15.2.1](#), $y(x) = \lim_{n \rightarrow \infty} T^n(1) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$.

16.4 Inverse Function Theorem

Theorem 16.4.1: Inverse Function Theorem

Let $f \in \mathcal{C}'(E)$: open $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $Df(a)$ is invertible for some (a, b) .

(a) There are open $U, V \subset \mathbb{R}^n$ such that $f: a \in U \rightarrow b \in V$ is invertible

(b) If $g = f^{-1}: V \rightarrow U$ where $g(f(x)) = x$, then for $y = f(x)$:

$$g \in \mathcal{C}'(V) \text{ where } Dg(y) = [Df(g(y))]^{-1}$$

Proof

Since $Df(a)$ is invertible for $a \in E$, then choose λ such that $\|[Df(a)]^{-1}\| = \frac{1}{2\lambda}$. Since $Df(a)$ is continuous at a , there is a $B_r(a) \subset E$ such that for $x \in U$:

$$\|Df(x) - Df(a)\| < \lambda$$

For each $y \in \mathbb{R}^n$, let $\phi(x) = x + [Df(a)]^{-1}(y - f(x))$ for $x \in E$. Then $f(x) = y$ if and only if $\phi(x) = x$. Since:

$$\phi'(x) = I - [Df(a)]^{-1}Df(x) = [Df(a)]^{-1}(Df(a) - Df(x)) < \frac{1}{2\lambda}\lambda = \frac{1}{2}$$

Then by [theorem 16.2.9](#), for all $x_1, x_2 \in B_r(a)$, then $|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|$.

Thus, ϕ is a contraction so on $\overline{B_r(a)}$ which is complete, then there is a unique $x \in \overline{B_r(a)}$ such that $\phi(x) = x$. Thus for each y , then $f(x) = y$ for a unique x so f is 1-1.

Let $U = B_r(a)$ and $V = f(B_r(a))$ so f maps U onto V . Thus, f is invertible on U .

Then for each $y_0 \in V$, then $y_0 = f(x_0)$ for a unique $x_0 \in U$. Choose t for $B_t(x_0)$ such that $\overline{B_t(x_0)} \subset U = B_r(a)$. Then for $y \in V$ where $|y - y_0| < \lambda t$ and $x \in \overline{B_t(x_0)}$:

$$|\phi(x_0) - x_0| = |[Df(a)]^{-1}(y - f(x_0))| \leq \frac{1}{2\lambda}\lambda t = \frac{t}{2}$$

$$|\phi(x) - x_0| \leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| < \frac{1}{2}|x - x_0| + \frac{t}{2} \leq \frac{t}{2} + \frac{t}{2} = t$$

Thus, $\phi(x) \in B_t(x_0)$. Since $|\phi(x_1) - \phi(x_2)| \leq \frac{1}{2}|x_1 - x_2|$ for $x_1, x_2 \in \overline{B_t(x_0)}$, then ϕ is a contraction on $\overline{B_t(x_0)}$ so there is a unique $x \in \overline{B_t(x_0)}$ such that $\phi(x) = x$ so for y where $|y - y_0| < \lambda t$, then $f(x) = y$. Thus, $y \in f(\overline{B_t(x_0)}) \subset f(U) = V$ so V is open.

For $y, y+k \in V$, there are $x, x+h \in U$ such that $f(x) = y$ and $f(x+h) = y+k$.

$$\phi(x+h) - \phi(x) = h + [Df(a)]^{-1}(f(x) - f(x+h)) = h + [Df(a)]^{-1}k$$

Since $|\phi(x+h) - \phi(x)| < \frac{1}{2}|h|$, then $[Df(a)]^{-1}k \in [\frac{1}{2}|h|, \frac{3}{2}|h|]$.

$$|h| \leq 2[Df(a)]^{-1}k \leq 2\|[Df(a)]^{-1}\| |k| \leq \frac{|k|}{\lambda}$$

Since $\|Df(x) - Df(a)\| \|[Df(a)]^{-1}\| < \lambda \frac{1}{2\lambda} = \frac{1}{2} < 1$, then by [theorem 16.1.9a](#), then $Df(x)$ is invertible and thus, have an inverse T . Since:

$$g(y+k) - g(y) - Tk = h - Tk = -T(f(x+h) - f(x) - Df(x)h)$$

$$\text{then } \frac{|g(y+k) - g(y) - Tk|}{|k|} \leq \frac{\|T\| |f(x+h) - f(x) - Df(x)h|}{|h|}.$$

As $k \rightarrow 0$, then $h \rightarrow 0$. Since f is differentiable, then $\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - Df(x)h|}{|h|} \rightarrow 0$

so $\lim_{k \rightarrow 0} \frac{|g(y+k) - g(y) - Tk|}{|k|} = 0$. Thus, $Dg(y) = T$ where T is the inverse of $Df(x)$.

$$Df(x)Dg(y) = Df(x)T = I_{n \times n} \rightarrow Dg(y) = [Df(x)]^{-1} = [Df(g(y))]^{-1}$$

Since g is differentiable and thus, continuous and $Df(x)$ is continuous, then by [theorem 16.1.9b](#), $[Df(g(y))]^{-1}$ is continuous.

Corollary 16.4.2: f with continuous, invertible $Df(x)$ at all x is an open mapping

If $f \in \mathcal{C}'(E)$: open $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $Df(x)$ is invertible for every $x \in E$, then open $f(W) \subset \mathbb{R}^n$ for every open $W \subset E$.

Proof

From [theorem 16.4.1a](#), let $U = W$ contain x . Then, $V = f(U) = f(W)$ is open.

Example

$$xe^{xy} - \sin(y) = a$$

$$x^9y^{10} + 3\cos(xy) = b$$

Prove there is a unique solution for all (a,b) close to $(e - \sin(1), 1 + 3\cos(1))$

Let $f(x,y) = (xe^{xy} - \sin(y), x^9y^{10} + 3\cos(xy))$.

Since each component is differentiable at all x,y, then $f(x,y)$ is differentiable where:

$$Df(x,y) = \begin{bmatrix} e^{xy} + xy e^{xy} & x^2 e^{xy} - \cos(y) \\ 9x^8y^{10} - 3y \sin(xy) & 10x^9y^9 - 3x \sin(xy) \end{bmatrix}$$

$$\text{Since } Df(1,1) = \begin{bmatrix} 2e & e - \cos(1) \\ 9 - 3\sin(1) & 10 - 3\sin(1) \end{bmatrix} \text{ so } \det(Df(1,1)) \neq 0.$$

Then by the Inverse Function Theorem, f is invertible and thus 1-1. So, there is a unique solution (x,y) near $(1,1)$ for all (a,b) close enough to $(e - \sin(1), 1 + 3\cos(1))$.

16.5 Implicit Function Theorem**Definition 16.5.1: Matrix Components**

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^m$:

$$(x,y) = (x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}.$$

For $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ where $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$, let:

$$A_x \in L(\mathbb{R}^n, \mathbb{R}^n) \quad A_x h = A(h, 0)$$

$$A_y \in L(\mathbb{R}^m, \mathbb{R}^n) \quad A_y k = A(0, k)$$

Thus, $A(h,k) = A_x h + A_y k$.

$$A = \overbrace{n\{ \begin{bmatrix} A_x & A_y \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \}}^{n+m} \quad \text{ } n+m$$

$$\begin{aligned} &= \begin{bmatrix} a_{x_{11}} & \dots & a_{x_{1n}} & a_{y_{11}} & \dots & a_{y_{1m}} \\ a_{x_{21}} & \dots & a_{x_{2n}} & a_{y_{21}} & \dots & a_{y_{2m}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{x_{n1}} & \dots & a_{x_{nn}} & a_{y_{n1}} & \dots & a_{y_{nm}} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \\ k_1 \\ \vdots \\ k_m \end{bmatrix} \\ &= \begin{bmatrix} a_{x_{11}}h_1 & \dots & a_{x_{1n}}h_n & a_{y_{11}}k_1 & \dots & a_{y_{1m}}k_m \\ a_{x_{21}}h_1 & \dots & a_{x_{2n}}h_n & a_{y_{21}}k_1 & \dots & a_{y_{2m}}k_m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{x_{n1}}h_1 & \dots & a_{x_{nn}}h_n & a_{y_{n1}}k_1 & \dots & a_{y_{nm}}k_m \end{bmatrix} \\ &= \begin{bmatrix} a_{x_{11}}h_1 & \dots & a_{x_{1n}}h_n \\ a_{x_{21}}h_1 & \dots & a_{x_{2n}}h_n \\ \vdots & \ddots & \vdots \\ a_{x_{n1}}h_1 & \dots & a_{x_{nn}}h_n \end{bmatrix} + \begin{bmatrix} a_{y_{11}}k_1 & \dots & a_{y_{1m}}k_m \\ a_{y_{21}}k_1 & \dots & a_{y_{2m}}k_m \\ \vdots & \ddots & \vdots \\ a_{y_{n1}}k_1 & \dots & a_{y_{nm}}k_m \end{bmatrix} = A_x h + A_y k \end{aligned}$$

Theorem 16.5.2: Every k has a unique h such that $A(h,k) = 0$

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and A_x is invertible, then for every $k \in \mathbb{R}^m$, there is a unique $h \in \mathbb{R}^n$ such that $A(h,k) = 0$. Then:

$$h = -(A_x)^{-1}A_y k$$

Proof

Since $0 = A(h,k) = A_x h + A_y k$ and A_x is invertible and thus, $(A_x)^{-1}$ exist, then:
 $(A_x)^{-1}0 = (A_x)^{-1}A_x h + (A_x)^{-1}A_y k \rightarrow 0 = h + (A_x)^{-1}A_y k$
 Thus, $h = -(A_x)^{-1}A_y k$ is unique.

Theorem 16.5.3: Implicit Function Theorem

Let $f \in \mathcal{C}'(E)$: open $E \subset \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ such that $f(a,b) = 0$ for some $(a,b) \in E$.

Let $A = Df(a,b)$ where A_x is invertible.

Then there are open $U \subset \mathbb{R}^{n+m}$, $W \subset \mathbb{R}^m$ where $(a,b) \in U$, $b \in W$ such that:

For every $y \in W$, there is a unique x such that $(x,y) \in U$ where $f(x,y) = 0$

If $x = g(y)$, then $g \in \mathcal{C}'(W)$: $W \rightarrow \mathbb{R}^n$ where:

$$g(b) = a \quad f(g(y),y) = 0 \text{ for } y \in W \quad g'(b) = -(A_x)^{-1}A_y$$

Proof

Let $F(x,y) = (f(x,y),y)$ for $(x,y) \in E$. Then $F(x,y) \in \mathcal{C}'(E)$: $E \rightarrow \mathbb{R}^{n+m}$.
 Since $DF(x,y) = \begin{bmatrix} D_x f(x,y) & D_y f(x,y) \\ D_x y & D_y y \end{bmatrix} = \begin{bmatrix} D_x f(x,y) & D_y f(x,y) \\ 0_{m \times n} & I_{m \times m} \end{bmatrix}$, then
 $\det(DF(x,y)) = \det(D_x f(x,y))\det(I_{m \times m}) = \det(D_x f(x,y))$.
 Since $A_x = D_x f(a,b)$ is invertible so $\det(A_x) \neq 0$, then $\det(DF(a,b)) = \det(D_x f(a,b)) \neq 0$ and thus, $DF(a,b)$ is invertible. Then by **theorem 16.4.1a**, there are open $U, V \subset \mathbb{R}^{n+m}$ such that $F: (a,b) \in U \rightarrow (f(a,b),b) = (0,b) \in V$ is invertible. Let W be the set of all $y \in \mathbb{R}^m$ such that $(0,y) \in V$ so $b \in W$ where W is open since V is open.
 Since F is invertible on U so F is 1-1 on U , then for every $y \in W$ so $(0,y) \in V$, there is a unique $(x,y) \in U$ such that $F(x,y) = (f(x,y),y) = (0,y)$ so $f(x,y) = 0$.
 For $y \in W$, let g be $(x,y) = (g(y),y) \in U$ and $f(g(y),y) = 0$.
 Thus, $F(g(y),y) = (0,y)$ so $f(g(y),y) = 0$ for $y \in W$.
 Let G be the inverse of F . Then by **theorem 16.4.1b**, then $G \in \mathcal{C}'(V)$.
 $(g(y),y) = G(F(g(y),y)) = G(0,y)$
 Thus, $g \in \mathcal{C}'(W)$: $W \rightarrow \mathbb{R}^n$ where $b \in W$ so $g(b) = a$.
 Let $(g(y),y) = \phi(y)$ so $\phi'(y)k = (g'(y)k,k)$ for $k \in \mathbb{R}^m$.
 Since $f(\phi(y)) = f(g(y),y) = 0$ for $y \in W$, then $f'(\phi(y))\phi'(y) = 0$.
 For $y = b \in W$, then $\phi(b) = (g(b),b) = (a,b)$ so $Df(\phi(b)) = Df(a,b) = A$.
 $0 = 0k = f'(\phi(b))\phi'(b)k = A\phi'(b)k = A(g'(b)k,k) = A_x g'(b)k + A_y k$
 Since A_x is invertible so $(A_x)^{-1}$ exist, then $g'(b)k = (A_x)^{-1}A_x g'(b)k = -(A_x)^{-1}A_y k$.

Example

$$xu^2 + yv^2 + xy = 11 \quad xv^2 + yu^2 - xy = -1$$

Show (u,v,x,y) close enough to $(1,1,2,3)$ satisfy the system of equations.

Let $F(u,v,x,y) = (xu^2 + yv^2 + xy - 11, xv^2 + yu^2 - xy + 1)$.
 Then $DF_{u,v} = \begin{bmatrix} 2xu & 2yv \\ 2yu & 2xv \end{bmatrix}$ so $DF_{u,v}(1,1,2,3) = \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ is invertible.
 Then by the Implicit Function Theorem, there is an open W where $(2,3) \in W$ with $g(2,3) = (1,1)$ so $(u,v,x,y) = (g(x,y),x,y)$ near $(1,1,2,3)$ satisfy the equations.

17 Lebesgue Integral

17.1 Regulated Integral

Definition 17.1.1: Basic Properties of the Integral

Let \mathcal{V} be a vector space of real-valued functions on closed interval I .

If $f, g \in \mathcal{V}$ and $c \in \mathbb{R}$, then $f + g, cf \in \mathcal{V}$

For each $f \in \mathcal{V}$, the integral of f on $[a, b] \subset I$, $\int_a^b f(x)dx$ should satisfy:

(a) **Linearity**: For $f, g \in \mathcal{V}$ and $c_1, c_2 \in \mathbb{R}$:

$$\int_a^b c_1 f(x) + c_2 g(x) dx = c_1 \int_a^b f(x) dx + c_2 \int_a^b g(x) dx$$

(b) **Monotonicity**: For $f, g \in \mathcal{V}$ where $g(x) \leq f(x)$:

$$\int_a^b g(x) dx \leq \int_a^b f(x) dx$$

(c) **Additivity**: For $f \in \mathcal{V}$ and $c \in [a, b]$:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

(d) **Constant**: For $f(x) = C$:

$$\int_a^b C dx = C(b - a)$$

(e) **Finite Sets**: For $f, g \in \mathcal{V}$ where $f(x) = g(x)$ for all, but finitely many x :

$$\int_a^b f(x) dx = \int_a^b g(x) dx$$

It should be noted that all integrals need not satisfy properties 3, 4, and 5.

However, all integrals considered henceforth will satisfy them.

Theorem 17.1.2: Absolute Value

If $f, |f| \in \mathcal{V}$, then if $a \leq b$:

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$$

Proof

Since $f(x) \leq |f(x)|$, then by [definition 17.1.1b](#), $\int_a^b f(x) dx \leq \int_a^b |f(x)| dx$.

Also, since $-f(x) \leq |f(x)|$, then $\int_a^b -f(x) dx \leq \int_a^b |f(x)| dx$.

Since $\left| \int_a^b f(x) dx \right|$ is either equal to $\int_a^b f(x) dx$ or $-\int_a^b f(x) dx$, then:

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

Definition 17.1.3: Step Function

Function $f: [a, b] \rightarrow \mathbb{R}$ is a **step function** if there is a partition $\{x_0, \dots, x_n\}$:

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

such that $f(x) = c_i$ on (x_{i-1}, x_i) for constant c_i

Theorem 17.1.4: Integral of a Step Function

If step function f with partition $\{x_0, \dots, x_n\}$ of $[a, b]$ is $f(x) = c_i$ for $x \in (x_{i-1}, x_i)$:

$$\int_a^b f(x) dx = \sum_{i=1}^n c_i (x_i - x_{i-1})$$

Proof

By [definition 17.1.1b](#), $\int_a^b f(x) dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) dx$

Since $f(x) = c_i$, but finitely many x on $[x_{i-1}, x_i]$ (i.e. endpoints):

$$\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} c_i dx = \sum_{i=1}^n c_i (x_i - x_{i-1})$$

Theorem 17.1.5: Step Functions form a Vector Space

The collection of all step functions on $[a,b]$ form a vector space

Proof

Let f, g be step functions with values c_i and d_j on partitions $\{x_0, \dots, x_n\}$ and $\{y_0, \dots, y_m\}$ respectively. Let $k_1, k_2 \in \mathbb{R}$. Let partition $Z = \{x_0, \dots, x_n\} \cup \{y_0, \dots, y_m\}$. Then each $[z_{k-1}, z_k] \subset [x_{i-1}, x_i]$ and $[z_{k-1}, z_k] \subset [y_{j-1}, y_j]$ for some i and j . Then $k_1 f + k_2 g$ have value $k_1 c_i + k_2 d_j$ on (z_{k-1}, z_k) so $k_1 f + k_2 g$ is a step function.

Theorem 17.1.6: Integral of Step Functions are independent of Partition

Let step function f have value c_i on partition $\{x_0, \dots, x_n\}$ and value d_j on partition $\{y_0, \dots, y_m\}$. Then:

$$\sum_{i=1}^n c_i (x_i - x_{i-1}) = \sum_{j=m}^n d_j (y_j - y_{j-1})$$

Proof

Let partition $Z = \{x_0, \dots, x_n\} \cup \{y_0, \dots, y_m\}$. Then each $[z_{k-1}, z_k] \subset [x_{i-1}, x_i]$ and $[z_{k-1}, z_k] \subset [y_{j-1}, y_j]$ for some i and j . Let $\{z_t^*\}$ be the set of z_k where $[z_{t-1}^*, z_t^*] = [z_{k-1}, z_k] \cup \dots \cup [z_{k+t^*-1}, z_{k+t^*}] = [x_{i-1}, x_i]$.

$$\sum_i c_i (x_i - x_{i-1}) = \sum_t c_i (z_t^* - z_{t-1}^*)$$

$$= \sum_k v_k (z_k - z_{k-1}) \quad \text{where } v_k = c_i \text{ where } [z_{k-1}, z_k] \subset [x_{i-1}, x_i]$$
Let $\{z_t^{**}\}$ be the set of z_k where $[z_{t-1}^{**}, z_t^{**}] = [z_{k-1}, z_k] \cup \dots \cup [z_{k+t^{**}-1}, z_{k+t^{**}}] = [y_{j-1}, y_j]$.

$$\sum_j d_j (y_j - y_{j-1}) = \sum_t d_i (z_t^{**} - z_{t-1}^{**})$$

$$= \sum_k v_k (z_k - z_{k-1}) \quad \text{where } v_k = d_j \text{ where } [z_{k-1}, z_k] \subset [y_{j-1}, y_j]$$
Thus, $\sum_i c_i (x_i - x_{i-1}) = \sum_k v_k (z_k - z_{k-1}) = \sum_j d_j (y_j - y_{j-1})$.

Definition 17.1.7: Regulated Function

Function $f: [a,b] \rightarrow \mathbb{R}$ is **regulated** if:

There is a sequence of step functions $\{f_n\}$ that converge uniformly to f

Theorem 17.1.8: Regulated Integral

Suppose step functions $\{f_n\}$ on $[a,b]$ converge uniformly to f . Then $\{\int_a^b f_n(x) dx\}$ converges. If step functions $\{g_n\}$ also converge uniformly to f :

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \lim_{n \rightarrow \infty} \int_a^b g_n(x) dx$$

Then, the regulated integral of f on $[a,b]$ can be defined:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx$$

Proof

Let $z_n = \int_a^b f_n(x) dx$. Since $\{f_n\}$ converges uniformly to f , there is a N where for $m, n \geq N$ and all $x \in [a,b]$:
 $|f_m(x) - f_n(x)| < \frac{\epsilon}{b-a}$
Thus:
 $|z_m - z_n| = |\int_a^b f_m(x) dx - \int_a^b f_n(x) dx| \leq \int_a^b |f_m(x) - f_n(x)| dx < \int_a^b \frac{\epsilon}{b-a} dx = \epsilon$
Since $\{z_n\}$ is Cauchy on \mathbb{R} , then $\{z_n\}$ converges.
If $\{g_n\}$ converges uniformly to f , then there is a M where for $n \geq M$ and all $x \in [a,b]$:
 $|f_n(x) - f| < \frac{\epsilon}{2(b-a)} \quad |g_n(x) - f| < \frac{\epsilon}{2(b-a)}$
 $|f_n(x) - g_n(x)| \leq |f_n(x) - f| + |f - g_n(x)| < \frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{b-a}$
Thus
 $|\int_a^b f(x) dx - \int_a^b f_n(x) dx| \leq \int_a^b |f(x) - f_n(x)| dx < \int_a^b \frac{\epsilon}{b-a} dx = \epsilon$

Theorem 17.1.9: Continuous functions are regulated

Every continuous function $f: [a, b] \rightarrow \mathbb{R}$ is a regulated function

Proof

Since f is continuous on compact $[a, b]$, then f is uniformly continuous on $[a, b]$.
 Thus for any $\epsilon_n = \frac{1}{2^n}$, there is a δ_n where for $|x - y| < \delta_n$, then $|f(x) - f(y)| < \epsilon_n$.
 For a fixed n , choose a partition $\{x_0, \dots, x_m\}$ such that each $\Delta x_i = \frac{b-a}{m} < \delta_n$.
 Let step function $f_n(x) = f(x_i)$ for $x \in [x_{i-1}, x_i]$ for $i = \{1, \dots, m\}$. For $x \in [a, b]$, there is an i such that $x \in [x_{i-1}, x_i]$ so $|f(x) - f_n(x)| = |f(x) - f(x_i)| < \epsilon_n$.
 Thus, $\{f_n\}$ converges uniformly to f , then f is regulated.

Theorem 17.1.10: Lower and Upper Riemann Limit Redefined

Let f be a bounded on $[a, b]$. Let:

$$\mathcal{U}(f) = \{ u(x) \mid f(x) \leq u(x) \text{ for all } x, u(x) \text{ is a step function} \}.$$

$$\mathcal{L}(f) = \{ v(x) \mid f(x) \geq v(x) \text{ for all } x, v(x) \text{ is a step function} \}.$$

$$\text{Then, } \sup_{v \in \mathcal{L}(f)} \left(\int_a^b v(x) dx \right) \leq \inf_{u \in \mathcal{U}(f)} \left(\int_a^b u(x) dx \right).$$

Proof

Since $v(x) \leq f(x) \leq u(x)$, then $\int_a^b v(x) dx \leq \int_a^b u(x) dx$.
 Since $\int_a^b v(x) dx \leq \int_a^b u(x) dx$ holds for any $u(x) \geq v(x)$, then:

$$\int_a^b v(x) dx \leq \inf \left(\int_a^b u(x) dx \right)$$

 Also, since $\int_a^b v(x) dx \leq \inf \left(\int_a^b u(x) dx \right)$ holds for any $v(x) \leq u(x)$, then:

$$\sup \left(\int_a^b v(x) dx \right) \leq \inf \left(\int_a^b u(x) dx \right)$$

Definition 17.1.11: Riemann Integral Redefined

Let f be a bounded on $[a, b]$. Let:

$$\mathcal{U}(f) = \{ u(x) \mid f(x) \leq u(x) \text{ for all } x, u(x) \text{ is a step function} \}.$$

$$\mathcal{L}(f) = \{ v(x) \mid f(x) \geq v(x) \text{ for all } x, v(x) \text{ is a step function} \}.$$

Then f is Riemann integrable if:

$$\sup_{v \in \mathcal{L}(f)} \left(\int_a^b v(x) dx \right) = \inf_{u \in \mathcal{U}(f)} \left(\int_a^b u(x) dx \right)$$

Theorem 17.1.12: Riemann-Integrability ϵ Definition Redefined

A bounded f on $[a, b]$ is Riemann integrable if and only if:

$$\text{For } \epsilon > 0, \text{ there are step functions } v(x), u(x) \text{ where } v(x) \leq f(x) \leq u(x):$$

$$\int_a^b u(x) dx - \int_a^b v(x) dx < \epsilon$$

Proof

If f is Riemann integrable, then for $\epsilon > 0$, there are step functions $u(x), v(x)$:

$$\left| \int_a^b f(x) dx - \int_a^b u(x) dx \right| < \frac{\epsilon}{2} \quad \left| \int_a^b f(x) dx - \int_a^b v(x) dx \right| < \frac{\epsilon}{2}$$

 Thus:

$$\left| \int_a^b u(x) dx - \int_a^b v(x) dx \right| \leq \left| \int_a^b u(x) dx - \int_a^b f(x) dx \right| + \left| \int_a^b f(x) dx - \int_a^b v(x) dx \right| < \epsilon$$

If for $\epsilon > 0$, there are step functions $v(x), u(x)$ where $v(x) \leq f(x) \leq u(x)$:

$$\int_a^b u(x) dx - \int_a^b v(x) dx < \epsilon$$

 Since $\sup \left(\int_a^b v(x) dx \right) \geq \int_a^b u(x) dx$ and $\inf \left(\int_a^b u(x) dx \right) \leq \int_a^b v(x) dx$, then:

$$\inf \left(\int_a^b u(x) dx \right) - \sup \left(\int_a^b v(x) dx \right) \leq \int_a^b u(x) dx - \int_a^b v(x) dx < \epsilon$$

 Thus, $\sup \left(\int_a^b v(x) dx \right) = \inf \left(\int_a^b u(x) dx \right)$ so f is Riemann integrable.

Theorem 17.1.13: Regulated functions are Riemann Integrable

Every regulated function is Riemann integrable where the regulated integral is equal to the Riemann integral

Proof

Since f is regulated, then for $\epsilon_n = \frac{1}{2^n}$, there is a step function f_n such that for all $x \in [a, b]$ so $|f(x) - f_n(x)| < \epsilon_n$. Thus, $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$.
 Let step functions $u_n(x) = f_n(x) + \frac{1}{2^n}$ and $v_n(x) = f_n(x) - \frac{1}{2^n}$ so $v_n(x) < f(x) < u_n(x)$ for all $x \in [a, b]$. Then:

$$|\int_a^b u_n(x)dx - \int_a^b v_n(x)dx| \leq \int_a^b |u_n(x) - v_n(x)|dx = \int_a^b \frac{1}{2^{n-1}}dx = \frac{b-a}{2^{n-1}}$$

 Thus, by **theorem 17.1.12**, f is Riemann integrable. Since:

$$\lim_{n \rightarrow \infty} \int_a^b u_n(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx + \lim_{n \rightarrow \infty} \int_a^b \frac{1}{2^n}dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$$

$$\lim_{n \rightarrow \infty} \int_a^b v_n(x)dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx - \lim_{n \rightarrow \infty} \int_a^b \frac{1}{2^n}dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$$

 Thus, the Riemann integral of f is $\lim_{n \rightarrow \infty} \int_a^b f_n(x)dx$ so the regulated integral is equal to the Riemann integral.

Theorem 17.1.14: Riemann Intergrable functions form a vector space

The set \mathcal{R} of bounded Riemann integrable functions on $[a, b]$ is a vector space that contains the vector space of regulated functions

Proof

By **theorem 17.1.13**, every regulated function is Riemann integrable so \mathcal{R} contain the set of regulated functions. Let $f, g \in \mathcal{R}$ and $c_1, c_2 \in \mathbb{R}$.
 Then for $\epsilon > 0$, there are step functions v_f, u_f where $v_f \leq f \leq u_f$ such that:

$$\int_a^b u_f(x)dx - \int_a^b v_f(x)dx < \frac{\epsilon}{2c_1}$$

 Also, there are step functions v_g, u_g where $v_g \leq g \leq u_g$ such that:

$$\int_a^b u_g(x)dx - \int_a^b v_g(x)dx < \frac{\epsilon}{2c_2}$$

 Since $c_1 v_f + c_2 v_g \leq c_1 f + c_2 g \leq c_1 u_f + c_2 u_g$ where $c_1 v_f + c_2 v_g, c_1 u_f + c_2 u_g$ are step functions such that:

$$\begin{aligned} & \int_a^b (c_1 u_f(x) + c_2 u_g(x))dx - \int_a^b (c_1 v_f(x) + c_2 v_g(x))dx \\ &= \int_a^b c_1 (u_f(x) - v_f(x))dx + \int_a^b c_2 (u_g(x) - v_g(x))dx < c_1 \frac{\epsilon}{2c_1} + c_2 \frac{\epsilon}{2c_2} = \epsilon \end{aligned}$$

 then $c_1 f + c_2 g$ is Riemann integrable so $c_1 f + c_2 g \in \mathcal{R}$.

17.2 Outer Measure

Definition 17.2.1: Basic Properties of the Length / Measure of a Set

For bounded $A, B \subset \mathbb{R}$, there is an associated non-negative real number $\mu(A)$:

- (a) **Length**: If $A = (a, b)$ or $A = [a, b]$, then:

$$\mu(A) = \text{len}(A) = b - a$$

- (b) **Translation Invariance**: If $c \in \mathbb{R}$, then:

$$\mu(A + c) = \mu(A)$$

- (c) **Countable Additivity**: If $\{A_n\}_{n=1}^{\infty}$ is countable, then:

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

If each A_n are pairwise disjoint, then:

$$\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

- (d) **Monotonicity**: If $A \subset B$, then:

$$\mu(A) \leq \mu(B)$$

Definition 17.2.2: Null Set

$X \subset \mathbb{R}$ is a **null set** if for $\epsilon > 0$:

There is a collection of open set $\{U_n\}_{n=1}^{\infty}$ where $X \subset \bigcup_{n=1}^{\infty} U_n$:

$$\sum_{n=1}^{\infty} \text{len}(U_n) < \epsilon$$

If X is a null set, then X^c has full measure.

Definition 17.2.3: Outer Measure

Let $A \subset \mathbb{R}$. Let open intervals $\{I_n\}_{n=1}^{\infty}$ be such that $A \subset \bigcup_{n=1}^{\infty} I_n$.

Then the **outer measure** $\mu^*(A)$:

$$\mu^*(A) = \inf(\sum_{n=1}^{\infty} \text{len}(I_n))$$

Theorem 17.2.4: Null set $A \Rightarrow \mu^*(A) = 0$

Let $A \subset \mathbb{R}$. Then, A is a null set if and only if $\mu^*(A) = 0$.

Proof

If A is a null set, then for $\epsilon > 0$, there are open intervals $\{I_n\}_{n=1}^{\infty}$ where $A \subset \bigcup_{n=1}^{\infty} I_n$:

$$\sum_{n=1}^{\infty} \text{len}(I_n) < \epsilon$$

Then, $\mu^*(A) = \inf(\sum_{n=1}^{\infty} \text{len}(I_n)) \leq \sum_{n=1}^{\infty} \text{len}(I_n) = \epsilon$ so $\mu^*(A) < 0$.

If $\mu^*(A) = 0$, then for open intervals $\{I_n\}_{n=1}^{\infty}$ where $A \subset \bigcup_{n=1}^{\infty} I_n$:

$$0 = \mu^*(A) = \inf(\sum_{n=1}^{\infty} \text{len}(I_n))$$

Thus, for $\epsilon > 0$, there is a $\{I_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \text{len}(I_n) < \epsilon$ so A is a null set.

Theorem 17.2.5: Outer Measure: Length Property

$$\mu^*([a, b]) = \mu^*((a, b)) = b - a$$

Let $I_n = (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$. Then:

$$\mu^*([a, b]) \leq \text{len}(I_n) = b - a + \epsilon \quad \rightarrow \quad \mu^*([a, b]) \leq b - a$$

Since $[a, b]$ is compact, then for any $\{I_i\}_{i=1}^{\infty}$ where $[a, b] \subset \bigcup_{i=1}^{\infty} I_i$, there is a M such that $[a, b] \subset \bigcup_{i=1}^M I_i$. Let n be the number of elements in $[a, b]$.

If $n = 1$, then $a = b$ so $0 = \mu^*([a, b]) \geq b - a = b - b = 0$ holds true.

If $n > 1$, then there is at least two intervals I_{n_1}, I_{n_2} that intersect since if $c \in (a, b)$, then only $(a, c), (c, b)$ will not contain c . Let $V_{n-1} = I_{n-1} \cup I_{n-2}$. Then, let $V_i = I_i$ for the I_i where $i \neq n_1, n_2$ and $i < \max(n_1, n_2)$ and $V_i = I_{i-1}$ for the I_i where $i \neq n_1, n_2$ and $i > \max(n_1, n_2)$. Thus:

$$\sum_{i=1}^M \text{len}(I_i) > \sum_{i=1}^{M_1} \text{len}(V_i) \geq b - a \quad \rightarrow \quad \mu^*([a, b]) \geq b - a$$

Since $(a, b) \subset [a, b]$, then $\mu^*((a, b)) \leq \text{len}([a, b]) = b - a$.

Since $\{I_i\}_{i=1}^{\infty}$ where $(a, b) \subset \bigcup_{i=1}^{\infty} I_i$ have $[a + \epsilon, b - \epsilon] \subset \bigcup_{i=1}^{\infty} I_i$, then by process above:

$$\sum_{i=1}^{\infty} \text{len}(I_i) \geq b - a - 2\epsilon \quad \rightarrow \quad \mu^*((a, b)) \geq b - a$$

Theorem 17.2.6: Outer Measure: Monotonicity Property

If $A, B \subset \mathbb{R}$ where $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$

Proof

Since $A \subset B$, then every open intervals $\{I_i\}_{i=1}^{\infty}$ where $B \subset \bigcup_{i=1}^{\infty} I_i$ is $A \subset \bigcup_{i=1}^{\infty} I_i$. Thus:

$$\mu^*(A) = \inf_A(\sum_{i=1}^{\infty} \text{len}(I_i)) \leq \inf_B(\sum_{i=1}^{\infty} \text{len}(I_i)) = \mu^*(B)$$

Theorem 17.2.7: Outer Measure: Countable Subadditivity Property

For $\{A_n\}_{n=1}^{\infty}$ where each $A_n \subset \mathbb{R}$:

$$\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Proof

For each A_n , there are open intervals $\{I_i^n\}_{i=1}^{\infty}$ where $A_n \subset \cup_{i=1}^{\infty} I_i^n$ such that for $\epsilon > 0$:

$$\sum_{i=1}^{\infty} \text{len}(I_i^n) \leq \mu^*(A_n) + \frac{\epsilon}{2^n}$$

Since $\{\{I_i^n\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ have $\cup_{n=1}^{\infty} A_n \subset \cup_{n=1}^{\infty} \cup_{i=1}^{\infty} I_i^n$, then:

$$\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \text{len}(I_i^n) \leq \sum_{n=1}^{\infty} [\mu^*(A_n) + \frac{\epsilon}{2^n}] = \sum_{n=1}^{\infty} \mu^*(A_n) + \frac{\epsilon}{2}$$

Thus, $\mu^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

Corollary 17.2.8: Countable A $\Rightarrow \mu^*(A) = 0$

If A is countable, then $\mu^*(A) = 0$.

Thus, intervals are uncountable.

Proof

Since A is countable, let $A = \{x_1, x_2, \dots\}$.

Since $\mu^*(\{x_n\}) = 0$, then:

$$\mu^*(A) = \mu^*(\{x_1, x_2, \dots\}) \leq \sum_{n=1}^{\infty} \mu^*(\{x_n\}) = 0$$

Thus, $\mu^*(A) = 0$.

Since $\mu^*([a, b]) = b - a \neq 0$, then A is uncountable.

Theorem 17.2.9: Outer Measure: Translation Invariance Property

If $A \subset \mathbb{R}$ and $c \in \mathbb{R}$, then $\mu^*(A + c) = \mu^*(A)$

Proof

There are open intervals $\{I_i\}_{i=1}^{\infty}$ where $A+c \subset \cup_{i=1}^{\infty} I_i$ such that:

$$|\sum_{i=1}^{\infty} \text{len}(I_i) - \mu^*(A+c)| \leq \frac{\epsilon}{2}$$

Let open intervals $\{I_i^*\}_{i=1}^{\infty}$ be $I_i^* = I_i - c$ so $A \subset \cup_{i=1}^{\infty} I_i^*$ where:

$$|\sum_{i=1}^{\infty} \text{len}(I_i^*) - \mu^*(A)| \leq \frac{\epsilon}{2}$$

Since $\text{len}(I_i^*) = \text{len}(I_i - c) = \text{len}(I_i)$, then:

$$\begin{aligned} & |\mu^*(A+c) - \mu^*(A)| \\ & \leq |\mu^*(A+c) - \sum_{i=1}^{\infty} \text{len}(I_i)| + |\sum_{i=1}^{\infty} \text{len}(I_i) - \sum_{i=1}^{\infty} \text{len}(I_i^*)| \\ & \quad + |\sum_{i=1}^{\infty} \text{len}(I_i^*) - \mu^*(A)| \\ & \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, $\mu^*(A+c) = \mu^*(A)$.

Theorem 17.2.10: Outer Measure: Regularity Property

If $A \subset \mathbb{R}$ and $\mu^*(A)$ is finite, then for any $\epsilon > 0$, there is an open set V where

$A \subset V$ such that $\mu^*(V) < \mu^*(A) + \epsilon$. Thus:

$$\mu^*(A) = \inf(\mu^*(U) \mid U \text{ is open, } A \subset U)$$

Proof

There are open intervals $\{I_i\}_{i=1}^{\infty}$ where $A \subset \cup_{i=1}^{\infty} I_i$ such that for $\epsilon > 0$:

$$\sum_{i=1}^{\infty} \text{len}(I_i) < \mu^*(A) + \epsilon$$

Let $V = \cup_{i=1}^{\infty} I_i$. Then:

$$\mu^*(V) = \mu^*(\cup_{i=1}^{\infty} I_i) \leq \sum_{i=1}^{\infty} \text{len}(I_i) < \mu^*(A) + \epsilon$$

Thus, $\inf(\mu^*(U) \mid U \text{ is open, } A \subset U) \leq \mu^*(A) + \epsilon$ so:

$$\inf(\mu^*(U) \mid U \text{ is open, } A \subset U) \leq \mu^*(A).$$

Since $A \subset \cup_{i=1}^{\infty} I_i = V$, then $\mu^*(A) \leq \mu^*(V) = \inf(\mu^*(U) \mid U \text{ is open, } A \subset U)$.

Thus, $\mu^*(A) = \inf(\mu^*(U) \mid U \text{ is open, } A \subset U)$.

17.3 Lebesgue Measure

Definition 17.3.1: Sigma Algebra and Borel Sets

Let \mathcal{A} be a collection of subsets of X .

Then, \mathcal{A} is a σ -algebra of subsets of X if for $A \in \mathcal{A}$:

- (a) $X \in \mathcal{A}$
- (b) $A^c \in \mathcal{A}$ in respect to X
- (c) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Some examples of σ -algebra of subsets of X are:

$$\mathcal{A} = \{X, \emptyset\} \quad \mathcal{A} = P(X) \text{ (i.e. all subsets of } X \text{ (} 2^{\mathbb{R}} \text{))}$$

If C is a collection of subsets of \mathbb{R} and \mathcal{A} is the smallest σ -algebra of subsets of \mathbb{R} that contains C , then \mathcal{A} is a σ -algebra generated by C .

Let \mathcal{B} be σ -algebra of subsets of \mathbb{R} generated by the collection of all open intervals. Then, \mathcal{B} is a **Borel σ -algebra** and any $B \in \mathcal{B}$ is a Borel set.

Definition 17.3.2: Lebesgue Measurable

Let the σ -algebra of subsets of \mathbb{R} generated by the collection of all open intervals and null sets be \mathcal{M} . Then, sets in \mathcal{M} are **Lebesgue measurable**.

If I is a closed interval, then $\mathcal{M}(I)$ is a σ -algebra of subsets of I .

Theorem 17.3.3: Lebesgue Measure

There is a unique μ , the **Lebesgue measure**, from $A, B \in \mathcal{M}(I)$ to \mathbb{R}_+ :

- (a) **Length**: If $A = (a, b)$, then:

$$\mu(A) = \text{len}(A) = b - a$$
- (b) **Translation Invariance**: If $c \in \mathbb{R}$ and $A + c \subset I$, then $A + c \in \mathcal{M}(I)$ where:

$$\mu(A + c) = \mu(A)$$
- (c) **Countable Additivity**: If $\{A_n\}_{n=1}^{\infty}$ is countable, then:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$
 If each A_n are pairwise disjoint, then:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$
- (d) **Monotonicity**: If $A \subset B$, then:

$$\mu(A) \leq \mu(B)$$
- (e) **Null Sets**: If $A \subset I$ is a null set, then $A \in \mathcal{M}(I)$ where:

$$\mu(A) = 0$$
 Also, if $A \in \mathcal{M}(I)$ where $\mu(A) = 0$, then A is a null set.
- (f) **Regularity**

$$\mu(A) = \inf\{\mu(U) \mid U \text{ is open, } A \subset U\}$$

Theorem 17.3.4: The Lebesgue measure of set differences

If $A, B \in \mathcal{M}(I)$, then $A \setminus B \in \mathcal{M}(I)$ where:

$$\mu(A \cup B) = \mu(A \setminus B) + \mu(B)$$

Thus, if $I = [0, 1]$, then $\mu(I) = 1$ so $\mu(A^c) = 1 - \mu(A)$.

Proof

Since $A \setminus B = A \cap B^c$ where $A, B^c \in \mathcal{M}(I)$, then $A \setminus B \in \mathcal{M}(I)$.

Since $A \setminus B$ and B are disjoint where $A \setminus B \cup B = A \cup B$, then:

$$\mu(A \cup B) = \mu(A \setminus B \cup B) = \mu(A \setminus B) + \mu(B)$$

$$\mu(I \setminus A) + \mu(A) = \mu(A^c) + \mu(A) = \mu(A^c \cup A) = \mu(I) = 1$$

Theorem 17.3.5: Alternative Definition for Lebesgue Measurable

$A \subset I$ is Lebesgue Measurable if for all $X \subset I$:

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = \mu^*(X)$$

Proof

Since $X = (A \cap X) \cup (A^c \cap X)$, then by **theorem 17.2.7**, then:

$$\mu^*(X) \leq \mu^*(A \cap X) + \mu^*(A^c \cap X)$$

Theorem 17.3.6: Lebesgue Measure's Regularity ϵ Definition

If $A \in \mathcal{M}(I)$, then for $\epsilon > 0$:

There is a closed $C \subset A$ such that:

$$\mu(C) > \mu(A) - \epsilon$$

There is a countable union of pairwise disjoint open intervals $U = \cup U_n$

where $A \subset U$ such that:

$$\mu(U) < \mu(A) + \epsilon$$

Proof

Since $A \in \mathcal{M}(I)$, then $A^c \in \mathcal{M}(I)$. Thus for $\epsilon > 0$, there is an open set V such that $A^c \subset V$ where $\mu(V) < \mu(A^c) + \epsilon$. Let $C = V^c$ so C is closed and $C \subset A$. Then:

$$\mu(C) = \mu(V^c) = 1 - \mu(V) > 1 - \mu(A^c) - \epsilon = \mu(A) - \epsilon$$

Since $A \in \mathcal{M}(I)$, then for $\epsilon > 0$, there is a open set U such that $A \subset U$ where:

$$\mu(U) < \mu(A) + \epsilon$$

Theorem 17.3.7: Monotonic Measurable Sets

If $A_n \subset A_{n+1}$ are Lebesgue measurable subsets of I , then:

$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$$

If $B_{n+1} \subset B_n$ are Lebesgue measurable subsets of I , then:

$$\mu(\cap_{n=1}^{\infty} B_n) = \lim_{n \rightarrow \infty} \mu(B_n)$$

Proof

Since A_n is Lebesgue measurable, then $\cup A_n$ is Lebesgue measurable.

Let $F_n = A_n \setminus A_{n-1}$, then $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} F_n$ where each F_n is pairwise disjoint.

$$\mu(\cup_{n=1}^{\infty} A_n) = \mu(\cup_{n=1}^{\infty} F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(F_i) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Since B_n is Lebesgue measurable, then $\cap B_n$ is Lebesgue measurable.

Let $E_n = B_n^c$. Since $(\cap B_n)^c = \cup E_n$ where each $E_n \subset E_{n+1}$, then:

$$\mu(\cap_{n=1}^{\infty} B_n) = 1 - \mu(\cup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} (1 - \mu(E_n)) = \lim_{n \rightarrow \infty} \mu(B_n)$$

17.4 Lebesgue Integral

Definition 17.4.1: Indicator Function

For $A \subset [0,1]$, the indicator function:

$$\mathfrak{X}_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}$$

Definition 17.4.2: Measurable Partition

A finite measurable partition of $[0,1]$ is a collection $\{A_i\}_{i=1}^n$ of measurable subsets which are pairwise disjoint where $\cup A_i = [0,1]$.

Definition 17.4.3: Simple Function

$f: [0,1] \rightarrow \mathbb{R}$ is simple if there exists a finite measurable partition, $\{A_i\}_{i=1}^n$ and $r_i \in \mathbb{R}$ such that $f(x) = \sum_{i=1}^n r_i \mathfrak{X}_{A_i}$.

Then the Lebesgue integral of a simple function:

$$\int f \, d\mu = \sum_{i=1}^n r_i \mu(A_i)$$

Theorem 17.4.4: Properties of Simple Functions

The set of simple functions is a vector space where:

(a) **Linearity**: If f, g are simple functions and $c_1, c_2 \in \mathbb{R}$:

$$\int c_1 f + c_2 g \, d\mu = c_1 \int f \, d\mu + c_2 \int g \, d\mu$$

(b) **Monotonicity**: If f, g are simple where $f(x) \leq g(x)$:

$$\int f \, d\mu \leq \int g \, d\mu$$

(c) **Absolute Value**: If f is simple, then $|f|$ is simple:

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proof

Since f is simple, then there is a measurable partition $\cup_{i=1}^n A_i = [0,1]$ where A_i is disjoint so $f(x) = \sum_{i=1}^n r_i \mathfrak{X}_{A_i}$. Then, $c_1 f$ is simple since $c_1 f(x) = \sum_{i=1}^n c_1 r_i \mathfrak{X}_{A_i}$. Since g is simple, then there is a measurable partition $\cup_{j=1}^m B_j = [0,1]$ where B_j is disjoint so $g(x) = \sum_{j=1}^m s_j \mathfrak{X}_{B_j}$.

Then for $c_1 f + c_2 g$, take the measurable partition $\cup_{i=1}^n \cup_{j=1}^m C_{ij}$ where $C_{ij} = A_i \cap B_j$.

$$\begin{aligned} c_1 f(x) + c_2 g(x) &= \sum_{i=1}^n c_1 r_i \mathfrak{X}_{A_i} + \sum_{j=1}^m c_2 s_j \mathfrak{X}_{B_j} \\ &= \sum_{i=1}^n c_1 r_i \sum_{j=1}^m \mathfrak{X}_{C_{ij}} + \sum_{j=1}^m c_2 s_j \sum_{i=1}^n \mathfrak{X}_{C_{ij}} \\ &= \sum_{i=1}^n \sum_{j=1}^m (c_1 r_i + c_2 s_j) \mathfrak{X}_{C_{ij}} \end{aligned}$$

Thus, the simple functions form a vector space.

$$\begin{aligned} \int c_1 f + c_2 g \, d\mu &= \sum_{i=1}^n \sum_{j=1}^m (c_1 r_i + c_2 s_j) \mu(C_{ij}) \\ &= \sum_{i=1}^n c_1 r_i \sum_{j=1}^m \mu(C_{ij}) + \sum_{j=1}^m c_2 s_j \sum_{i=1}^n \mu(C_{ij}) \\ &= \sum_{i=1}^n c_1 r_i \mu(A_i) + \sum_{j=1}^m c_2 s_j \mu(B_j) = c_1 \int f \, d\mu + c_2 \int g \, d\mu \end{aligned}$$

$$\int g \, d\mu - \int f \, d\mu = \int (g-f) \, d\mu \geq 0$$

$$\left| \int f \, d\mu \right| = \left| \sum_{i=1}^n r_i \mu(A_i) \right| \leq \sum_{i=1}^n |r_i| \mu(A_i) = \int |f| \, d\mu$$

Theorem 17.4.5: Measurable Functions

If $f: X \subset \mathbb{R} \rightarrow [-\infty, \infty]$, then the following are equivalent:

- For any $a \in \mathbb{R}$, $f^{-1}([-\infty, a])$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}([a, \infty])$ is Lebesgue measurable
- For any $a \in \mathbb{R}$, $f^{-1}((a, \infty])$ is Lebesgue measurable

Then f is **Lebesgue measurable**.

Proof

Suppose for any $a \in \mathbb{R}$, $f^{-1}([-\infty, a])$ is Lebesgue measurable.
 $f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, a - \frac{1}{2^n}])$ is measurable since it's countable measurables.
 $f^{-1}([a, \infty]) = f^{-1}([-\infty, a)^c) = (f^{-1}([-\infty, a)))^c$ is measurable since it's the complement of a measurable.
 $f^{-1}((a, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}([a + \frac{1}{2^n}, \infty])$ is measurable since it's countable measurables.
 $f^{-1}([-\infty, a]) = f^{-1}((a, \infty]^c) = (f^{-1}((a, \infty]))^c$ is measurable since it's the complement of a measurable.

Theorem 17.4.6: Measurable Functions and Null Sets

Let $f, g: [a, b] \rightarrow \mathbb{R}$.

- (a) If there is a null set $A \subset [a, b]$ where $f(x) = 0$ if $x \notin A$, then f is measurable
- (b) If $f = g$ except on null set A , then f is measurable if and only if g is measurable

Proof

Since $f(x) = 0$ if $x \notin A$, then $f^{-1}([-\infty, 0)) \cup f^{-1}((0, \infty]) \subset A$.
 If $a < 0$, then $f^{-1}([-\infty, a]) \subset f^{-1}([-\infty, 0)) \subset A$ so $f^{-1}([-\infty, a])$ is a null set and thus, measurable. For $a \geq 0$, then $f^{-1}([-\infty, a]) = (f^{-1}((a, \infty]))^c \subset (f^{-1}((0, \infty)))^c$ so $f^{-1}([-\infty, a])$ is a complement of a null set and thus, measurable.

Suppose f is measurable. Let $a \in \mathbb{R}$.

$$g^{-1}([a, \infty]) = (g^{-1}([a, \infty]) \cap A) \cup (g^{-1}([a, \infty]) \cap A^c)$$

Since $f = g$ on A^c , then $(g^{-1}([a, \infty]) \cap A^c) = (f^{-1}([a, \infty]) \cap A^c)$ which is measurable. Since $(g^{-1}([a, \infty]) \cap A) \subset A$, then $(g^{-1}([a, \infty]) \cap A)$ is a null set and thus, measurable. Proof is analogous for g .

Theorem 17.4.7: Measurable Functions and Sequences

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Then:

$$\begin{aligned} g_1(x) &= \sup(f_n(x)) & g_2(x) &= \inf(f_n(x)) \\ g_3(x) &= \lim_{n \rightarrow \infty} \sup(f_n(x)) & g_4(x) &= \lim_{n \rightarrow \infty} \inf(f_n(x)) \end{aligned}$$

are measurable.

Proof

For $a \in \mathbb{R}$, $\{x \mid g_1(x) > a\} = \bigcup_{i=1}^{\infty} \{x \mid f_n(x) > a\}$ which are measurable sets so countable implies measurable and thus, g_1 is measurable.
 For $a \in \mathbb{R}$, $\{x \mid g_2(x) < a\} = \bigcup_{i=1}^{\infty} \{x \mid f_n(x) < a\}$ which are measurable sets so countable implies measurable and thus, g_2 is measurable.
 Since $g_3(x) = \lim_{n \rightarrow \infty} \sup(f_n(x)) = \inf(\sup(f_n(x)))$ where $\sup(f_n(x))$ are measurable so g_3 is measurable.
 Since $g_4(x) = \lim_{n \rightarrow \infty} \inf(f_n(x)) = \sup(\inf(f_n(x)))$ where $\inf(f_n(x))$ are measurable so g_4 is measurable.

Theorem 17.4.8: Lebesgue measurable functions form a Vector Space

The set of Lebesgue measurable functions from $[0,1]$ to \mathbb{R} is a vector space with a vector subspace of bounded Lebesgue measurable functions. Thus, if f, g are measurable, then fg is measurable.

Proof

Let f, g be Lebesgue measurable functions.

For $a \in \mathbb{R}$, let set $U_a = \{ f(x) + g(x) > a \}$.

Since $\mathbb{Q} = \{r_m\}$ is countable and dense, there is a r_m such that:

$$f(x) > r_m > a - g(x)$$

Let $V_m = \{x \mid f(x) > r_m\} \cap \{x \mid g(x) > a - r_m\}$

If $x \in U_a$, then $x \in V_m$ for some m since there is a r_m where $f(x) > r_m > a - g(x)$. If $x \in V_m$, then $f(x) + g(x) > a$ so $x \in U_a$. Thus, $U_a = \cup_{m=1}^{\infty} V_m$ which is countable and thus, measurable since f, g are measurable. Thus, $f+g$ is measurable.

Note for $c \in \mathbb{R}$, $\{x \mid cf(x) > a\}$ is measurable since $\{x \mid f(x) > v\}$ for any $v \in \mathbb{R}$ is measurable including $\frac{a}{c}$. Thus, the measurable functions form a vector space.

If f, g are bounded and measurable, then $c_1f + c_2g$ is bounded which is measurable as proved above so bounded measurable functions is a vector subspace of measurable functions.

17.5 Lebesgue Integral of Bounded Functions

Theorem 17.5.1: Lebesgue Integral of a Bounded Function

If $f: [0,1] \rightarrow \mathbb{R}$ is bounded, then the following are equivalent:

- f is Lebesgue measurable
- There are simple functions $\{f_n\}$ which converge uniformly to f
- If simple functions $u(x), v(x)$ where $v(x) \leq f(x) \leq u(x)$, then:

$$\sup(\int v \, d\mu) = \inf(\int u \, d\mu)$$

Then, $\int f \, d\mu = \sup(\int v \, d\mu) = \inf(\int u \, d\mu)$

Proof

Suppose f is Lebesgue measurable.

Since f is bounded, there are m, M such that $m \leq f(x) \leq M$ for all $x \in [0,1]$. For $\epsilon_n > 0$, take a large enough n such that $\frac{M-m}{n} \leq \epsilon_n$. For $\{c_0, \dots, c_n\}$, let $c_k = m + k\epsilon_n$. Let $f_n(x) = \sum_{i=1}^n c_{i-1} \chi_{f^{-1}([c_{i-1}, c_i])}$ which is simple.

Then for any $x \in [0,1]$, there is a $[c_{i-1}, c_i)$ where $x \in [c_{i-1}, c_i)$ so $|f(x) - f_n(x)| \leq \epsilon_n$.

Suppose simple functions $\{f_n\}$ converge uniformly to f .

Let $\delta_n = \sup(|f(x) - f_n(x)|)$ so $\lim_{n \rightarrow \infty} \delta_n = 0$. Let simple functions $v_n(x) = f_n(x) - \delta_n$ and $u_n(x) = f_n(x) + \delta_n$ so $v_n(x) \leq f(x) \leq u_n(x)$.

$$\begin{aligned} \inf(\int u \, d\mu) &\leq \lim_{n \rightarrow \infty} \inf(\int u_n(x) \, d\mu) = \lim_{n \rightarrow \infty} \inf(\int f_n(x) + \delta_n \, d\mu) \\ &= \lim_{n \rightarrow \infty} \inf(\int f_n(x) \, d\mu) \leq \lim_{n \rightarrow \infty} \sup(\int f_n(x) \, d\mu) \\ &= \lim_{n \rightarrow \infty} \sup(\int f_n(x) - \delta_n \, d\mu) = \lim_{n \rightarrow \infty} \sup(\int v_n(x) \, d\mu) \leq \sup(\int v \, d\mu) \end{aligned}$$

Since $\sup(\int v \, d\mu) \leq \inf(\int u \, d\mu)$, then $\sup(\int v \, d\mu) = \inf(\int u \, d\mu)$.

For n , there are simple functions $v_n(x), u_n(x)$ where $v_n(x) \leq f(x) \leq u_n(x)$ such that:

$$\int u_n(x) \, d\mu - \int v_n(x) \, d\mu < \frac{1}{2^n}$$

Since $u_n(x)$ and $v_n(x)$ are simple and thus, measurable, then $g_1(x) = \sup(v_n(x))$ and $g_2(x) = \inf(u_n(x))$ are measurable. Let $B = \{x \mid g_1(x) < g_2(x)\}$. Suppose $\mu(B) > 0$. If $B_m = \{x \mid g_1(x) < g_2(x) - \frac{1}{m}\}$, then $B = \bigcup_{m=1}^{\infty} B_m$ so $\mu(B_m) > 0$ for some m .

Thus, for $x \in B_m$:

$$\begin{aligned} v_n(x) &\leq g_1(x) < g_2(x) - \frac{1}{m} \leq u_n(x) - \frac{1}{m} \\ \int u_n \, d\mu - \int v_n \, d\mu &= \int u_n - v_n \, d\mu \geq \int \frac{1}{m} \chi_{B_m} \, d\mu = \frac{1}{m} \mu(B_m) \end{aligned}$$

which contradicts $\int u_n(x) \, d\mu - \int v_n(x) \, d\mu < \frac{1}{2^n}$ and thus, $\mu(B) = 0$ so $g_1(x) = g_2(x)$ except on a null set. Since $g_1(x) \leq f(x) \leq g_2(x)$, then $f(x) - g_1(x) = 0$ except on a null set and thus, by [theorem 17.4.6](#), $f(x) - g_1(x)$ is measurable so $f(x)$ is measurable.

Theorem 17.5.2: Uniform Convergence of Simple Functions are Lebesgue Integrable

If simple functions $\{f_n\}$ converge uniformly to bounded measurable f :

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Proof

Let $\delta_n = \sup(|f(x) - f_n(x)|)$. Since $\{f_n\}$ converge uniformly f , then $\lim_{n \rightarrow \infty} \delta_n = 0$:
 $f_n(x) - \delta_n \leq f(x) \leq f_n(x) + \delta_n$

Thus, by [theorem 17.5.1](#):

$$\begin{aligned} \int f \, d\mu &= \inf(\int u \, d\mu) \leq \lim_{n \rightarrow \infty} \inf(\int f_n(x) + \delta_n \, d\mu) \\ &\leq \lim_{n \rightarrow \infty} \inf(\int f_n(x) \, d\mu) \leq \lim_{n \rightarrow \infty} \sup(\int f_n(x) \, d\mu) \\ &\leq \lim_{n \rightarrow \infty} \sup(\int f_n(x) - \delta_n \, d\mu) \leq \sup(\int v \, d\mu) = \int f \, d\mu \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \inf(\int f_n(x) \, d\mu) \leq \lim_{n \rightarrow \infty} \int f_n(x) \, d\mu \leq \lim_{n \rightarrow \infty} \sup(\int f_n(x) \, d\mu)$, then:
 $\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n(x) \, d\mu$

Theorem 17.5.3: Properties of Bounded Measurable Functions

If f, g are bounded Lebesgue measurable functions. Then:

(a) **Linearity**: If $c_1, c_2 \in \mathbb{R}$:

$$\int c_1 f + c_2 g \, d\mu = c_1 \int f \, d\mu + c_2 \int g \, d\mu$$

(b) **Monotonicity**: If $f(x) \leq g(x)$:

$$\int f \, d\mu \leq \int g \, d\mu$$

(c) **Absolute Value**: $|f|$ is measurable where:

$$|\int f \, d\mu| \leq \int |f| \, d\mu$$

(d) **Null Sets**: If $f(x) = g(x)$ except on a set of measure zero:

$$\int f \, d\mu = \int g \, d\mu$$

Proof

Since f and g are measurable, then there are simple functions $\{f_n\}, \{g_n\}$ where converge uniformly to f and g respectively. Thus, $\{c_1 f_n + c_2 g_n\}$ converge to $c_1 f + c_2 g$ uniformly.

$$\begin{aligned} \int c_1 f + c_2 g \, d\mu &= \lim_{n \rightarrow \infty} \int c_1 f_n + c_2 g_n \, d\mu \\ &= c_1 \lim_{n \rightarrow \infty} \int f_n \, d\mu + c_2 \lim_{n \rightarrow \infty} \int g_n \, d\mu = c_1 \int f \, d\mu + c_2 \int g \, d\mu \end{aligned}$$

If $f(x) \leq g(x)$, then since f, g are measurable, then are simple functions v_f, u_g where $v_f \leq f \leq g \leq u_g$ such that:

$$\int f \, d\mu = \sup(\int v_f \, d\mu) \leq \inf(\int u_g \, d\mu) = \int g \, d\mu$$

Since $|[a, \infty)| = (-\infty, -a] \cup [a, \infty)$, then $|f|^{-1}([a, \infty)) = f^{-1}((-\infty, -a]) \cup f^{-1}([a, \infty))$ which are measurable since f is measurable, then $|f|$ is measurable. Also, there are simple functions $\{f_n\}$ that converge uniformly to f . Then by **theorem 17.4.4**:

$$|\int f \, d\mu| = \lim_{n \rightarrow \infty} |\int f_n \, d\mu| \leq \lim_{n \rightarrow \infty} \int |f_n| \, d\mu = \int |f| \, d\mu$$

Let $h(x) = f(x) - g(x) = 0$ except on a null set E and is bounded so $|h(x)| \leq M \chi_E$.

$$|\int f \, d\mu - \int g \, d\mu| = |\int h \, d\mu| \leq \int |h| \, d\mu \leq \int M \chi_E \, d\mu = M \mu(E) = 0$$

Definition 17.5.4: Bounded Lebesgue integral over a Measurable set

If $E \subset [0, 1]$ is a measurable set and f is a bounded measurable function, the **Lebesgue integral of f over E** :

$$\int_E f \, d\mu = \int f \chi_E \, d\mu$$

Theorem 17.5.5: Additivity Property

If $\{E_n\}_{n=1}^N$ are pairwise disjoint measurable sets with $E = \cup E_n$ and f is a bounded measurable function:

$$\int_E f \, d\mu = \sum_{n=1}^N \int_{E_n} f \, d\mu$$

Proof

Since $\chi_E = \sum_{n=1}^N \chi_{E_n}$, then $f \chi_E = \sum_{n=1}^N f \chi_{E_n}$.

$$\int_E f \, d\mu = \int f \chi_E \, d\mu = \int \sum_{n=1}^N f \chi_{E_n} \, d\mu = \sum_{n=1}^N \int f \chi_{E_n} \, d\mu = \sum_{n=1}^N \int_{E_n} f \, d\mu$$

Theorem 17.5.6: Riemann Integrability implies Lebesgue Integrability

Every bounded Riemann integrable $f: [0,1] \rightarrow \mathbb{R}$ is measurable and thus, Lebesgue integrable. The Riemann integral is equal to the Lebesgue integral.

Proof

Since the set of step functions $\mathcal{L}(f)$ less than f is a subset of the set of simple functions $\mathcal{L}_\mu(f)$ less than f and the set of step functions $\mathcal{U}(f)$ greater than f is a subset of the set of simple functions $\mathcal{U}_\mu(f)$ greater than f , then:

$$\sup_{v \in \mathcal{L}(f)} \left(\int_0^1 v(t) dt \right) \leq \sup_{v \in \mathcal{L}_\mu(f)} \left(\int_0^1 v d\mu \right) \leq \inf_{u \in \mathcal{U}_\mu(f)} \left(\int_0^1 u d\mu \right) \leq \inf_{u \in \mathcal{U}(f)} \left(\int_0^1 u(t) dt \right)$$

Thus, if f is Riemann integrable, then $\sup_{v \in \mathcal{L}(f)} \left(\int_0^1 v(t) dt \right) = \inf_{u \in \mathcal{U}(f)} \left(\int_0^1 u(t) dt \right)$ so

$\sup_{v \in \mathcal{L}_\mu(f)} \left(\int_0^1 v d\mu \right) = \inf_{u \in \mathcal{U}_\mu(f)} \left(\int_0^1 u d\mu \right)$ and thus, f is Lebesgue measurable and the Riemann integral is equal to the Lebesgue integral since:

$$\sup_{v \in \mathcal{L}(f)} \left(\int_0^1 v(t) dt \right) = \sup_{v \in \mathcal{L}_\mu(f)} \left(\int_0^1 v d\mu \right) = \inf_{u \in \mathcal{U}_\mu(f)} \left(\int_0^1 u d\mu \right) = \inf_{u \in \mathcal{U}(f)} \left(\int_0^1 u(t) dt \right)$$

18 Lebesgue Convergence Theorems

18.1 Bounded Convergence Theorem

Theorem 18.1.1: Bounded Convergence Theorem

Suppose measurable $\{f_n\}$ on $[0,1]$ converge pointwise to f where $|f_n(x)| \leq M$.

Then, f is a bounded measurable function where:

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$$

Proof

Since $\lim_{n \rightarrow \infty} f_n = f$ pointwise, then for any $x \in [0,1]$, then is a N_x where for $n \geq N_x$:

$$|f(x) - f_n(x)| < \epsilon$$

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \epsilon + M \Rightarrow |f(x)| \leq M$$

Thus, f is bounded. Since $\lim_{n \rightarrow \infty} f_n = f$, then by [theorem 17.4.7](#), f is measurable.

Let set $E_n = \{x \in [0,1] \mid |f_n(x) - f(x)| < \frac{\epsilon}{2}\}$. Since $\lim_{n \rightarrow \infty} f_n = f$ pointwise, then $\cup_{n=1}^{\infty} E_n = [0,1]$. Since $E_n \subset E_{n+1}$, then $\lim_{n \rightarrow \infty} \mu(E_n) = \mu([0,1]) = 1$.

Then, there is a N where $\mu(E_N) > 1 - \frac{\epsilon}{4M}$ so $\mu(E_N^c) < \frac{\epsilon}{4M}$. Thus:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \int f_n d\mu - \int f d\mu \right| &= \lim_{n \rightarrow \infty} \left| \int f_n - f d\mu \right| \leq \lim_{n \rightarrow \infty} \int |f_n - f| d\mu \\ &= \int_{E_N} |f_n - f| d\mu + \int_{E_N^c} |f_n - f| d\mu \\ &< \frac{\epsilon}{2} \mu(E_N) + 2M \mu(E_N^c) < \frac{\epsilon}{2} + 2M \frac{\epsilon}{4M} = \epsilon \end{aligned}$$

Definition 18.1.2: Almost Everywhere

If a property holds for all x except for a null set, then it holds [almost everywhere](#)

Theorem 18.1.3: Bounded Convergence Theorem for Almost Everywhere

Suppose bounded $\{f_n\}$ on $[0,1]$ are measurable and f is bounded such that $\lim_{n \rightarrow \infty} f_n = f$ for almost all x . If $|f_n(x)| \leq M$ almost everywhere, then f is measurable where:

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu$$

Proof

Let $A = \{x \mid \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}$ so $\mu(A) = 0$. Let $D_n = \{x \mid |f_n(x)| > M\}$ so $\mu(D_n) = 0$. Let $E = A \cup \cup_{n=1}^{\infty} D_n$. Thus:

$$\mu(E) \leq \mu(A) + \sum_{i=1}^{\infty} \mu(D_n) = 0 \Rightarrow \mu(E) = 0$$

Let $g_n(x) = f_n(x) \chi_{E^c}(x)$ which is measurable since $f_n(x), \chi_{E^c}(x)$ are measurable.

Then, $|g_n(x)| \leq M$. Let $g(x) = f(x) \chi_{E^c}(x)$ so $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ and $g(x) \leq M$.

Since $\lim_{n \rightarrow \infty} g_n(x) = g(x)$, then by [theorem 17.4.7](#), $g(x)$ is measurable.

Since $g(x) = f(x)$ almost everywhere, then by [theorem 17.4.6b](#), $f(x)$ is measurable.

$$\int g d\mu = \int f d\mu \quad \int g_n d\mu = \int f_n d\mu$$

By [theorem 18.1.1](#), $\lim_{n \rightarrow \infty} \int g_n d\mu = \int g d\mu$. Thus:

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu = \int g d\mu = \int f d\mu$$

18.2 Integral of Unbounded Functions

Definition 18.2.1: Integrable Function

If $f: [0,1] \rightarrow [0,\infty]$ is Lebesgue measurable, let $f_n(x) = \min(f(x), n)$.

Then f_n is a bounded measurable function and let:

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n(x) d\mu$$

If $\int f d\mu < \infty$, then f is **integrable**.

Theorem 18.2.2: Unbounded sets of Integrable functions have measure 0

If f is a non-negative integrable function and $A = \{x \mid f(x) = \infty\}$, then:
 $\mu(A) = 0$

Proof

If $x \in A$, then $f_n(x) = n \geq n\chi_A(x)$. Thus, $\int f_n d\mu \geq \int n\chi_A d\mu = n\mu(A)$.

If $\mu(A) > 0$, then:

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu \geq \lim_{n \rightarrow \infty} \int n\chi_A d\mu = \lim_{n \rightarrow \infty} n\mu(A) = \infty$$

Thus, if f is integrable, then $\mu(A) = 0$.

Theorem 18.2.3: Integrable functions for Almost Everywhere

Suppose f, g are non-negative measurable functions with $g(x) \leq f(x)$ for almost all x . If f is integrable, then g is integrable where:

$$\int g d\mu \leq \int f d\mu$$

If $g = 0$ almost everywhere, then $\int g d\mu = 0$.

Proof

If $f_n(x) = \min(f(x), n)$ and $g_n(x) = \min(g(x), n)$, then f_n, g_n are bounded measurable functions where $g_n(x) \leq f_n(x)$ almost everywhere. If f is integrable, then:

$$\int g_n d\mu \leq \int f_n d\mu \leq \int f d\mu$$

Since $\{g_n\}$ is increasing and bounded above by $\int f d\mu$, then $\int g d\mu$ is finite and thus, exist. If $0 \leq g(x) \leq 0$ almost everywhere, for almost all x so $\int g d\mu = \int 0 d\mu = 0$.

Corollary 18.2.4: If integrable $f \geq 0$, then $\int f d\mu = 0 \iff f(x) = 0$ almost everywhere

If $f: [0,1] \rightarrow [0,\infty]$ is a non-negative integrable function and $\int f d\mu = 0$, then
 $f(x) = 0$ almost everywhere

Proof

Let $E_n = \{x \mid f(x) \geq \frac{1}{n}\}$. Then, $f(x) \geq \frac{1}{n}\chi_{E_n}(x)$ where:

$$\frac{1}{n}\mu(E_n) = \int \frac{1}{n}\chi_{E_n} d\mu \leq \int f d\mu = 0$$

Thus, $\mu(E_n) = 0$. Let $E = \{x \mid f(x) > 0\}$ so $E = \cup_{n=1}^{\infty} E_n$ where $E_n \subset E_{n+1}$ so:

$$\mu(E) = \mu(\cup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \mu(E_n) = 0.$$

Theorem 18.2.5: Absolute Continuity

If f is a non-negative integrable function, then for $\epsilon > 0$, there is a $\delta > 0$ where for every measurable $A \subset [0,1]$ with $\mu(A) < \delta$, then $\int_A f d\mu < \epsilon$

Proof

Let $E_n = \{x \mid f(x) \geq n\}$ so $f_n(x) = \begin{cases} f(x) & x \in E_n^c \\ n & x \in E_n \end{cases}$. Thus:

$$f(x) - f_n(x) = \begin{cases} 0 & x \in E_n^c \\ f(x) - n & x \in E_n \end{cases}$$

$$\int f d\mu - \int f_n d\mu = \int f - f_n d\mu = \int_{E_n} f(x) - n d\mu$$

Since f is integrable, then $\lim_{n \rightarrow \infty} \int f d\mu - \int f_n d\mu = 0$. Thus:

$$\lim_{n \rightarrow \infty} \int_{E_n} f(x) - n d\mu = 0$$

Thus, there is a N where $\int_{E_N} f(x) - n d\mu < \frac{\epsilon}{2}$. Then for $\delta < \frac{\epsilon}{2N}$, if $\mu(A) < \delta$:

$$\begin{aligned} \int_A f d\mu &= \int_{A \cap E_N} f d\mu + \int_{A \cap E_N^c} f d\mu \leq \int_{A \cap E_N} (f - N) d\mu + \int_{A \cap E_N} N d\mu + \int_{A \cap E_N^c} N d\mu \\ &\leq \int_{A \cap E_N} (f - N) d\mu + \int_A N d\mu < \frac{\epsilon}{2} + N\mu(A) < \frac{\epsilon}{2} + N\delta < \frac{\epsilon}{2} + N\frac{\epsilon}{2N} < \epsilon \end{aligned}$$
Corollary 18.2.6: Uniform Continuity of the Integral

If $f: [0,1] \rightarrow [0,\infty]$ is an integrable function where $F(x) = \int_{[0,x]} f d\mu$, then $F(x)$ is continuous

Proof

By **theorem 17.7.5**, for $\epsilon > 0$, there is a $\delta > 0$ where for $\mu([x,y]) < \delta$, then $\int_{[x,y]} f d\mu < \epsilon$.

$$|F(y) - F(x)| = \left| \int_{[0,y]} f d\mu - \int_{[0,x]} f d\mu \right| = \left| \int_{[x,y]} f d\mu \right| < \epsilon$$

Thus, $F(x)$ is uniformly continuous.

18.3 Dominated Convergence Theorems**Theorem 18.3.1: Dominated Convergence Theorem**

Suppose non-negative measurable $\{f_n\}$ on $[0,1]$ converge pointwise to f for almost all x . If there is a non-negative integrable g where $f_n(x) \leq g(x)$ for almost all x , then f is integrable where:

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu$$

Proof

Let $h_n = f_n \chi_E$ and $h = f \chi_E$ where $E = \{x \mid \lim_{n \rightarrow \infty} f_n(x) = f(x)\}$ so $\lim_{n \rightarrow \infty} h_n(x) = h(x)$ for all x . Since $h_n(x) = f_n \chi_E \leq g(x)$ for almost all x and g is integrable, then $h(x) \leq g(x)$ for almost all x so by **theorem 18.2.3**, h is integrable.

For $\epsilon > 0$, let $E_n = \{x \mid |h_m(x) - h(x)| < \frac{\epsilon}{2} \text{ for all } m \geq n\}$. By **theorem 18.2.5**, there is a $\delta > 0$ where for each measurable $A \subset [0,1]$ with $\mu(A) < \delta$, then $\int_A g d\mu < \frac{\epsilon}{4}$.

Since $\lim_{n \rightarrow \infty} h_n(x) = h(x)$ for all $x \in [0,1]$, then any $x \in E_n$ in some n so $\cup_{n=1}^{\infty} E_n = [0,1]$. Since $E_n \subset E_{n+1}$, then $\lim_{n \rightarrow \infty} \mu(E_n) = \mu([0,1]) = 1$. Thus, there is a n where $\mu(E_n) > 1 - \delta$ so $\mu(E_n^c) < \delta$. Note $|h_n(x) - h(x)| \leq |h_n(x)| + |h(x)| \leq 2g(x)$ for almost all x . Thus, for any $m > n$:

$$\begin{aligned} \left| \int h_m d\mu - \int h d\mu \right| &\leq \int |h_m - h| d\mu = \int_{E_n} |h_m - h| d\mu + \int_{E_n^c} |h_m - h| d\mu \\ &< \frac{\epsilon}{2} \mu(E_n) + 2 \int_{E_n^c} g d\mu < \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon \end{aligned}$$

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \lim_{n \rightarrow \infty} \int h_n d\mu = \int h d\mu = \int f d\mu$$

Theorem 18.3.2: Fatou's Lemma

If non-negative measurable $\{g_n\}$ on $[0,1]$ converge pointwise to $g(x)$ for almost all x , then:

$$\int g \, d\mu \leq \liminf_{n \rightarrow \infty} \int g_n \, d\mu$$

Thus, if $\liminf_{n \rightarrow \infty} \int g_n \, d\mu < \infty$, then g is integrable.

Proof

Since g_n is measurable and $\lim_{n \rightarrow \infty} g_n = g$ for almost all x , then g is measurable. Let bounded, measurable h be $h(x) \leq g(x)$ for all x . Let $h_n(x) = \min(h(x), g_n(x))$ so h_n is bounded and measurable where $\lim_{n \rightarrow \infty} h_n = h$. Then by [theorem 18.1.1](#):

$$\int h \, d\mu = \lim_{n \rightarrow \infty} \int h_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int g_n \, d\mu$$

Since the inequality holds for any bounded, measurable h where $h(x) \leq g(x)$, then let $h(x) = g_m(x) = \min(g_n(x), m)$. Thus, for any m :

$$\int g_m \, d\mu \leq \liminf_{n \rightarrow \infty} \int g_n \, d\mu$$

$$\int g \, d\mu = \lim_{m \rightarrow \infty} \int g_m \, d\mu \leq \liminf_{n \rightarrow \infty} \int g_n \, d\mu$$

Theorem 18.3.3: Monotone Convergence Theorem

If non-negative measurable $\{g_n\}$ on $[0,1]$ converge pointwise to $g(x)$ for almost all x where $g_n(x) \leq g_{n+1}(x)$, then:

$$\int g \, d\mu = \lim_{n \rightarrow \infty} \int g_n \, d\mu$$

Thus, g is integrable if and only if $\lim_{n \rightarrow \infty} \int g_n \, d\mu < \infty$.

Proof

Since g_n is measurable and $\lim_{n \rightarrow \infty} g_n = g$ for almost all x , then g is measurable.

If f is integrable, then by [theorem 18.3.1](#), then:

$$\int g \, d\mu = \lim_{n \rightarrow \infty} \int g_n \, d\mu.$$

If $\lim_{n \rightarrow \infty} \int f \, d\mu = \infty$, then by [theorem 18.3.2](#):

$$\lim_{n \rightarrow \infty} \int g_n \, d\mu = \infty \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \int g_n \, d\mu = \infty$$

Corollary 18.3.4: Integral of Infinite Series

For non-negative measurable $u_n(x)$ and non-negative f , let $\sum_{n=1}^{\infty} u_n(x) = f(x)$ for almost all x . Then:

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int u_n \, d\mu$$

Proof

Let $f_N(x) = \sum_{n=1}^N u_n(x)$ so $\lim_{N \rightarrow \infty} f_N(x) = \sum_{n=1}^{\infty} u_n(x) = f(x)$ for almost all x . Since $u_n(x)$ is non-negative, then $f_N(x) \leq f_{N+1}(x)$. Then by [theorem 18.3.3](#):

$$\begin{aligned} \int f \, d\mu &= \lim_{N \rightarrow \infty} \int f_N \, d\mu = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N u_n(x) \, d\mu \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int u_n(x) \, d\mu = \sum_{n=1}^{\infty} \int u_n \, d\mu \end{aligned}$$

Corollary 18.3.5: Lebesgue Integral: Countable Additivity

Suppose $\{E_n\}$ are pairwise disjoint measurable subsets of I and f is a non-negative integrable function. If $E = \cup_{n=1}^{\infty} E_n$, then:

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$$

Proof

Let $u_n(x) = f \chi_{E_n}$. Since $\chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$, then $f \chi_E = f \sum_{n=1}^{\infty} \chi_{E_n} = \sum_{n=1}^{\infty} u_n(x)$. Thus, by [corollary 18.3.4](#):

$$\int_E f \, d\mu = \int f \chi_E \, d\mu = \sum_{n=1}^{\infty} \int u_n \, d\mu = \sum_{n=1}^{\infty} \int f \chi_{E_n} \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$$

18.4 General Lebesgue Integral

Definition 18.4.1: Measurable Function Redefined

For measurable function $f: [0,1] \rightarrow [-\infty, \infty]$, let:

$$f^+(x) = \max(f(x), 0) \quad f^-(x) = -\min(f(x), 0)$$

Thus, $f^+(x)$ and $f^-(x)$ are non-negative measurable functions where:

$$f(x) = f^+(x) - f^-(x)$$

Then f is Lebesgue integrable if $f^+(x)$ and $f^-(x)$ are integrable. Thus:

$$\int f \, d\mu = \int f^+(x) \, d\mu - \int f^-(x) \, d\mu$$

Theorem 18.4.2: For $f = g$ almost everywhere, then $\int f \, d\mu = \int g \, d\mu$

Suppose f, g are measurable functions on $[0,1]$ where $f = g$ almost everywhere.

Then if f is integrable, then g is integrable where $\int f \, d\mu = \int g \, d\mu$.

Proof

If f and g are measurable functions where $f = g$ almost everywhere, then $f^+ = g^+$ and $f^- = g^-$ almost everywhere. Then if f is integrable, then f^+ and f^- are integrable so by **theorem 18.2.3**, g^+ and g^- are integrable where:

$$\begin{aligned} \int f^+ \, d\mu &= \int g^+ \, d\mu & \int f^- \, d\mu &= \int g^- \, d\mu \\ \int f \, d\mu &= \int f^+(x) \, d\mu - \int f^-(x) \, d\mu &= \int g^+(x) \, d\mu - \int g^-(x) \, d\mu &= \int g \, d\mu \end{aligned}$$

Theorem 18.4.3: Integrable $f \Leftrightarrow$ Integrable $|f|$

Measurable $f: [0,1] \rightarrow [-\infty, \infty]$ is integrable if and only if $|f|$ is integrable

Proof

If f is integrable, then f^+, f^- are integrable. Since $|f| = f^+ + f^-$, then $|f|$ is integrable. If $|f|$ is integrable, then since $f^+, f^- \leq |f|$, by **theorem 18.2.3**, f^+, f^- are integrable so f is integrable.

Theorem 18.4.4: Lebesgue Convergence Theorem

Let measurable $\{f_n\}$ on $[0,1]$ converge pointwise to f for almost all x . If there is a integrable g where $|f_n(x)| \leq g(x)$ for almost all x , then f is integrable where:

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Proof

Let $f_n^+(x) = \max(f_n(x), 0)$ and $f_n^-(x) = -\min(f_n(x), 0)$. Thus, $\lim_{n \rightarrow \infty} f_n^+(x) = f^+(x)$ and $\lim_{n \rightarrow \infty} f_n^-(x) = f^-(x)$ for almost all x . Since $|f_n(x)| \leq g(x)$, then $f_n^+(x), f_n^-(x) \leq g(x)$ for almost all x . Then by **theorem 18.3.1**, f^+, f^- are integrable where:

$$\int f^+ \, d\mu = \lim_{n \rightarrow \infty} \int f_n^+ \, d\mu \quad \int f^- \, d\mu = \lim_{n \rightarrow \infty} \int f_n^- \, d\mu$$

Thus, $f = f^+ - f^-$ is integrable where:

$$\int f \, d\mu = \int f^+ - f^- \, d\mu = \lim_{n \rightarrow \infty} \int f_n^+ - f_n^- \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu$$

Theorem 18.4.5: Integrable f can be approximated by a step function

For integrable $f: [0,1] \rightarrow [-\infty, \infty]$ and $\epsilon > 0$, there is a step function g and measurable $A \subset [0,1]$ such that:

$$\mu(A) < \epsilon \quad |f(x) - g(x)| < \epsilon \text{ for all } x \notin A$$

If $|f(x)| \leq M$ for all x , then there is a step function g where $|g(x)| \leq M$.

Proof

Suppose $f(x) = \chi_E$ for some measurable set E .

Let $E \subset \bigcup_{i=1}^{\infty} U_i$ for open intervals $\{U_i\}$ such that:

$$\mu(E) \leq \mu(\bigcup_{i=1}^{\infty} U_i) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \mu(E) + \frac{\epsilon}{2} \Rightarrow \mu((\bigcup_{i=1}^{\infty} U_i) \cap E^c) < \frac{\epsilon}{2}$$

Then choose an N such that for $V_N = \bigcup_{i=1}^N U_i$, then $\mu(\bigcup_{i=1}^N U_i) \leq \sum_{i=1}^N \mu(U_i) < \frac{\epsilon}{2}$. Let $g(x) = \chi_{V_N}$ so g is a step function since V_N is finite. Let $A = \{x \mid f(x) \neq g(x)\}$.

$$A \subset (V_N \cap E^c) \cup (E \cap V_N^c) \subset ((\bigcup_{i=1}^N U_i) \cap E^c) \cup (\bigcup_{i=N+1}^{\infty} U_i)$$

$$\mu(A) \leq \mu((\bigcup_{i=1}^N U_i) \cap E^c) + \mu(\bigcup_{i=N+1}^{\infty} U_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Suppose simple function $f(x) = \sum_{i=1}^n r_i \chi_{E_i}$.

Proof is analogous to proof above except change $\frac{\epsilon}{2}$ into $\frac{\epsilon}{2n}$. For $j = \{1, \dots, n\}$, let step function $g_j(x) = \chi_{V_{N_j}}$ where $V_{N_j} = \bigcup_{i=1}^{N_j} U_{ji}$ where $E_i \subset \bigcup_{j=1}^{\infty} U_{ji}$ open intervals. Thus for $A_j = \{x \mid f(x) \neq r_j g_j(x)\}$, then $\mu(A_j) < \frac{\epsilon}{n}$ so $\mu(\bigcup_{j=1}^n A_j) \leq \sum_{j=1}^n \mu(A_j) < \epsilon$.

Suppose $f(x)$ is a bounded measurable function.

Then by [theorem 17.5.1](#), there is a simple function $h(x)$ where $|f(x) - h(x)| < \frac{\epsilon}{2}$ for all x . As shown above, there is a step function $g(x)$ such that $|h(x) - g(x)| < \frac{\epsilon}{2}$ for all $x \notin A$ for some measurable $A \subset [0,1]$ where $\mu(A) < \epsilon$. Thus, for all $x \notin A$:

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Suppose f is a non-negative integrable function.

Let $A_n = \{x \mid f(x) > n\}$. Then:

$$n\mu(A_n) = \int n \chi_{A_n} d\mu \leq \int f d\mu < \infty \Rightarrow \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \frac{1}{n} \int f d\mu = 0$$

Thus, there is a N where $\mu(A_N) < \frac{\epsilon}{2}$. Let $f_N = \min(f, N)$ so f is a bounded measurable function. As shown above, there is a step function g where $|f_N(x) - g(x)| < \frac{\epsilon}{2}$ for all $x \notin B$ for some measurable B where $\mu(B) < \frac{\epsilon}{2}$. Let $A = A_N \cup B$ so $\mu(A) \leq \mu(A_N) + \mu(B) < \epsilon$. Note if $x \notin A$, then $x \notin B$ so $f(x) = f_N(x)$. Thus, for all $x \notin A$:

$$|f(x) - g(x)| \leq |f(x) - f_N(x)| + |f_N(x) - g(x)| < \frac{\epsilon}{2} < \epsilon$$

Suppose f is a integrable function.

Since $f = f^+ - f^-$ where f^+, f^- are non-negative integrable functions, then as shown above, there are step functions g^+, g^- where $\mu(A^+), \mu(A^-) < \frac{\epsilon}{2}$ and $|f^+(x) - g^+(x)|, |f^-(x) - g^-(x)| < \frac{\epsilon}{2}$ for all $x \notin A^+, A^-$ respectively.

Let $A = A^+ \cup A^-$ and $g(x) = g^+ + g^-$. Thus, for any $x \notin A$:

$$|f(x) - g(x)| \leq |f^+(x) - g^+(x)| + |f^-(x) - g^-(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

If $|f(x)| \leq M$, take g from before and let $g_1(x) = \begin{cases} M & g(x) > M \\ g(x) & g(x) \in [-M, M] \\ -M & g(x) < -M \end{cases}$.

Thus, step function g_1 is $|g_1| \leq M$ where $g_1(x) = g(x)$ for $|g(x)| \leq M$. For $x \notin A$:

$$\text{If } g(x) > M: \quad f(x) \leq M = g_1(x) < g(x) \Rightarrow |f(x) - g_1(x)| < |f(x) - g(x)| < \epsilon$$

$$\text{If } g(x) < -M: \quad f(x) \geq -M = g_1(x) > g(x) \Rightarrow |f(x) - g_1(x)| < |f(x) - g(x)| < \epsilon$$

Theorem 18.4.6: Properties of the Lebesgue Integral

If f, g are Lebesgue integrable functions. Then:

- (a) **Linearity**: If $c_1, c_2 \in \mathbb{R}$:

$$\int c_1 f + c_2 g \, d\mu = c_1 \int f \, d\mu + c_2 \int g \, d\mu$$

- (b) **Monotonicity**: If $f(x) \leq g(x)$:

$$\int f \, d\mu \leq \int g \, d\mu$$

- (c) **Absolute Value**: $|f|$ is integrable where:

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

- (d) **Null Sets**: If $f(x) = g(x)$ except on a set of measure zero, then if f is integrable, then g is integrable where:

$$\int f \, d\mu = \int g \, d\mu$$

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