

# Real Analysis

Azure

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# 1 Ordered Sets and Fields

## 1.1 Ordered Sets and Bounds

### Definition 1.1.1: Ordered Set

An order is:

- **Trichotomy**: For all  $x, y \in S$ , only one holds true:
  - $x < y$
  - $x = y$
  - $x > y$
- **Transitivity**: If  $x < y$  and  $y < z$ , then  $x < z$ .

An ordered set is a set with an order.

### Definition 1.1.2: Bounds

Let  $S$  be an ordered set and  $E \subset S$ .

An upper bound of  $E$  is a  $\beta \in S$  such that for  $x \leq \beta$  for all  $x \in E$ .

If such a  $\beta$  exists, then  $E$  is bounded from above.

A lower bound of  $E$  is a  $\alpha \in S$  such that for  $x \geq \alpha$  for all  $x \in E$ .

If such a  $\alpha$  exists, then  $E$  is bounded from below.

### Definition 1.1.3: Infimum & Supremum

Let  $S$  be an ordered set.

Let  $E \subset S$  be bounded from above. Least upper bound  $\beta \in S$  exists if:

- $\beta$  is an upper bound for  $E$
- If  $\gamma < \beta$ , then  $\gamma$  is not an upper bound for  $E$ .

Then  $\beta = \sup(E)$ .

Let  $E \subset S$  be bounded from below. Greatest lower bound  $\alpha \in S$  exists if:

- $\alpha$  is a lower bound for  $E$
- If  $\gamma > \alpha$ , then  $\gamma$  is not a lower bound for  $E$ .

Then  $\alpha = \inf(E)$ .

Even if  $\sup(E)$  exists, it may or may not exist at  $S$ .

If  $\sup(E)$  exists, then  $\sup(E)$  is unique. Statement also holds true for  $\inf(E)$ .

### Example

Let  $S = (1, 2) \cup [3, 4) \cup (5, 6)$  with the order  $<$  from  $\mathbb{R}$ . For subsets  $E$  of  $S$ :

- $E = (1, 2)$  is bounded above with  $\sup(E) = 2$  and not bounded below.
- $E = (5, 6)$  is not bounded above or below so  $\inf(E), \sup(E) = \text{DNE}$ .
- $E = [3, 4)$  is bounded below with  $\inf(E) = 3$ , but  $\sup(E) = \text{DNE}$ .

## 1.2 Least Upper Bound Property

### Theorem 1.2.1: Least Upper Bound Property

An ordered set  $S$  has a least upper bound property if:

For every nonempty subset  $E \subset S$  that is bounded from above:

$\sup(E)$  exists in  $S$ .

#### Proof

Let  $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$ , then take  $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$ .

Let set  $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$  and set  $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If  $y^2 - 2 < 0$ , then  $z > y$  where  $z \in A$ . So,  $y$  is not an upper bound.  
Since for any  $y$ , there is  $z > y$  where  $z \in A$ , then  $\sup(A)$  doesn't exist in  $\mathbb{Q}$ .
- If  $y^2 - 2 > 0$ , then  $z < y$  where  $z \in B$ . So,  $y$  is an upper bound, but not  $\sup(E)$ .  
Since for any  $y$ , there is  $z < y$  where  $z \in B$ , then  $\inf(B)$  doesn't exist in  $\mathbb{Q}$ .

Thus,  $\mathbb{Q}$  doesn't have the least upper bound or greatest lower bound property.

#### Example

$\mathbb{Q}$  doesn't have a least upper bound property. Take for example,  $\sqrt{2}$ .

Let  $x^2 = 2$ . If  $x$  was rational, there is a rational  $\frac{p}{q}$  where  $x = \frac{p}{q}$  where both  $p$  and  $q$  are not even.

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow \quad p^2 = 2q^2$$

Since  $2q^2$  is even, then  $p^2$  is even so  $p$  is even. Thus,  $p$  is divisible by 2 so  $p^2$  is divisible by 4 so  $q^2$  is divisible by 2 so  $q$  is even. Thus, both  $p$  and  $q$  must be even which is a contradiction so  $x = \sqrt{2}$  cannot be rational.

So if  $\sqrt{2} < \frac{a}{b}$  for some rational  $\frac{a}{b}$ , there is always another rational  $\frac{p}{q}$ :

$$\sqrt{2} < \frac{p}{q} < \frac{a}{b}$$

and there will never be a rational  $\frac{p}{q}$  such that  $\sqrt{2} = \frac{p}{q}$  since  $\sqrt{2}$  is not rational.

### Theorem 1.2.2: Least Upper Bound + Lower Bound implies Greatest Lower Bound

Let  $S$  be an ordered set with the least upper bound property.

Let non-empty  $B \subset S$  be bounded below.

Let  $L$  be the set of all lower bounds of  $B$ .

Then  $\alpha = \sup(L)$  exists in  $S$ .

#### Proof

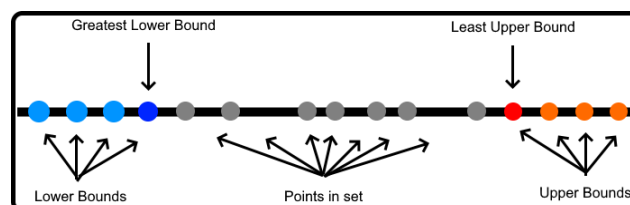
$L$  is non-empty since  $B$  is bounded from below.

Thus, by the least upper bound property of  $S$ ,  $\alpha = \sup(L)$  exists in  $S$ .

We claim that  $\alpha = \inf(B)$ .

If  $\gamma < \alpha$ , then  $\gamma$  is not an upper bound for  $L$  so  $\gamma \notin B$  since all upper bounds for  $L$  are in  $B$ . Thus, for every  $x \in B$ ,  $\alpha \leq x$ .

If  $\gamma \geq \alpha$ , then  $\gamma$  is an upper bound of  $L$  so  $\gamma \in B$ . Thus,  $\inf(B) = \alpha$ .



## 1.3 Fields

### Definition 1.3.1: Fields Axioms

- (a) Addition Axioms
- If  $x, y \in F$ , then  $x+y \in F$
  - $x+y = y+x$  for all  $x, y \in F$
  - $(x+y)+z = x+(y+z)$  for all  $x, y, z \in F$
  - There exists  $0 \in F$  such that  $0+x = x$  for all  $x \in F$
  - For every  $x \in F$ , there is  $-x \in F$  where  $x+(-x) = 0$
- (b) Multiplicative Axioms
- If  $x, y \in F$ , then  $xy \in F$
  - $yx = xy$  for all  $x, y \in F$
  - $(xy)z = x(yz)$  for all  $x, y, z \in F$
  - There exists  $1 \neq 0 \in F$  such that  $1x = x$  for all  $x \in F$
  - If  $x \neq 0 \in F$ , there is  $\frac{1}{x} \in F$  where  $x(\frac{1}{x}) = 1$
- (c) Distributive Law
- $x(y+z) = xy + xz$  hold for all  $x, y, z \in F$

### Theorem 1.3.2: Consequences of the Field Axioms

- (a) If  $x+y = x+z$ , then  $y = z$

Proof

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If  $x+y = x$ , then  $y = 0$

Proof

$$\text{From (a), let } z = 0$$

- (c) If  $x+y = 0$ , then  $y = -x$

Proof

$$\text{From (a), let } z = -x$$

- (d)  $-(-x) = x$

Proof

$$\text{From (c), let } x = -x \text{ and } y = x$$

- (e) If  $x \neq 0$  and  $xy = xz$ , then  $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If  $x \neq 0$  and  $xy = x$ , then  $y = 1$

Proof

$$\text{From (e), let } z = 1$$

- (g) If  $x \neq 0$  and  $xy = 1$ , then  $y = \frac{1}{x}$

Proof

$$\text{From (e), let } z = \frac{1}{x}$$

- (h) If
- $x \neq 0$
- , then
- $\frac{1}{1/x} = x$

Proof

From (g), let $x = \frac{1}{x}$ and $y = x$
---

- (i)
- $0x = 0$

Proof

Since $0x + 0x = (0+0)x = 0x = 0x + 0$ , then $0x = 0$
--

- (j) If
- $x, y \neq 0$
- , then
- $xy \neq 0$

Proof

Suppose $xy = 0$ , then $1 = \frac{1}{y} \frac{1}{x} xy = \frac{1}{y} \frac{1}{x} 0 = 0$ . $0 = 1$ is a contradiction.
---

- (k)
- $(-x)y = -(xy) = x(-y)$

Proof

$xy + (-x)y = (x+(-x))y = 0y = 0$ . Then by part (c), $(-x)y = -(xy)$ . $xy + x(-y) = x(y+(-y)) = x0 = 0$ . Then by part (c), $x(-y) = -(xy)$ .
--

- (l)
- $(-x)(-y) = xy$

Proof

By part (k), then $(-x)(-y) = -[x(-y)] = -[-(xy)]$ . By part (d), $-[-(xy)] = xy$ .
---

## 1.4 Ordered Fields

### Definition 1.4.1: Ordered Field

An ordered field  $F$  is a field  $F$  which is also an ordered set for all  $x, y, z \in F$ .

- If  $y < z$ , then  $y+x < z+x$
- If  $x, y > 0$ , then  $xy > 0$

$\mathbb{Q}, \mathbb{R}$  are ordered fields, but  $\mathbb{C}$  is not an ordered field since  $i^2 = -1 \not> 0$ .

### Theorem 1.4.2: Properties of the Ordered Field

- (a) If
- $x > 0$
- , then
- $-x < 0$
- and vice versa

Proof

$-x = -x + 0 < -x + x = 0$
----------------------------

- (b) If
- $x > 0$
- and
- $y < z$
- , then
- $xy < xz$

Proof

Since $z-y > 0$ , then $0 < x(z-y) = xz - xy$
---

- (c) If
- $x < 0$
- and
- $y < z$
- , then
- $xy > xz$

Proof

Since $-x > 0$ and $z-y > 0$ , then $0 < -x(z-y) = xy - xz$
---

- (d) If
- $x \neq 0$
- ,
- $x^2 > 0$

Proof

If $x > 0 \Rightarrow x^2 = x \cdot x > 0$ . If $x < 0 \Rightarrow (-x)^2 = (-x) \cdot (-x) = x \cdot x = x^2 > 0$
--

- (e) If
- $0 < x < y$
- , then
- $0 < 1/y < 1/x$

Proof

$(\frac{1}{y})y = 1 > 0$ so $\frac{1}{y} > 0$ . Since $x < y$ , then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$ .
--



**Theorem 1.4.3:  $\mathbb{R}$  is an ordered field**

There exists a unique ordered field  $\mathbb{R}$  with the least upper bound property.

Also,  $\mathbb{Q} \subset \mathbb{R}$  so  $\mathbb{Q}$  is also an ordered field.

**Proof**

The proof in Day 5 is a construction of  $\mathbb{R}$  by defining a specific order  $<$ .

**Theorem 1.4.4:  $\mathbb{Q}$  is dense in  $\mathbb{R}$** 

- (a) **Archimedean Property:** For  $x, y \in \mathbb{R}$ , if  $x > 0$ , there is  $n \in \mathbb{Z}$  where  $nx > y$ .

**Proof**

Fix  $x > 0$ . Let  $A = \{ nx : n = 1, 2, \dots \}$ . Suppose there is a  $y$  where  $nx \leq y$ . Then,  $A$  is nonempty and bounded from above by  $y$ . By the least upper bound property of  $\mathbb{R}$ ,  $\alpha = \sup(A)$  exists in  $\mathbb{R}$ . Since  $x > 0$ , then  $-x < 0$  so  $\alpha - x < \alpha - 0 = \alpha$ . So  $\alpha - x$  is not an upper bound of  $A$ . So there is a  $mx \in A$  such that  $mx > \alpha - x$ . Then  $\alpha < (m+1)x$ , but  $(m+1)x \in A$  contradicting  $\alpha$  is an upper bound for  $A$ .

- (b)  **$\mathbb{Q}$  is dense in  $\mathbb{R}$ :** For  $x, y \in \mathbb{R}$ , if  $x < y$ , there is a  $p \in \mathbb{Q}$  where  $x < p < y$ .

**Proof**

Since  $x < y$ , then  $y - x > 0$ . Then by the Archimedean Property, there exists  $n \in \mathbb{Z}$  such that  $n(y - x) > 1$ . Thus,  $ny > nx + 1 > nx$ . Since there is a smallest  $m \in \mathbb{Z}_+$  such that  $m > nx$ , then  $m > nx \geq m-1$  so  $nx + 1 \geq m > nx$ . Since  $ny > nx + 1 \geq m > nx$ , then  $y > m/n > x$ .

## 2 Roots, Complex Field, & Euclidean Spaces

### 2.1 nth Root

#### Theorem 2.1.1: nth Root

(a) If  $0 < t \leq 1$ , then  $t^n \leq t$ .

Proof

Since  $t > 0$  and  $t \leq 1$ , then  $t^2 \leq t$ .  
 Since  $t^2 \leq t$ , then  $t^3 \leq t^2$  so  $t^3 \leq t^2 \leq t$ .  
 Applying the process  $n$  times, then  $t^n \leq t$ .

(b) If  $t \geq 1$ ,  $t^n \geq t$ .

Proof

Since  $0 < 1 \leq t$ , then  $t \leq t^2$ .  
 Since  $t \leq t^2$ , then  $t^2 \leq t^3$  so  $t \leq t^2 \leq t^3$ .  
 Applying the process  $n$  times,  $t \leq t^n$ .

(c) If  $0 < s < t$ , then  $s^n < t^n$ .

Proof

$$\underbrace{s \cdot s \cdot \dots \cdot s}_n < t \cdot s \cdot \dots \cdot s < t \cdot t \cdot \dots \cdot s < \dots < \underbrace{t \cdot \dots \cdot t}_n$$

#### Theorem 2.1.2: $y^n = x$ has a unique $y$

Fix  $n \in \mathbb{Z}_+$ . For every  $x > 0$ , there exists a unique  $y \in \mathbb{R}$  such that  $y^n = x$ .

Also, such a  $y$  is written as  $y = \sqrt[n]{x} = x^{\frac{1}{n}}$ .

Proof

Uniqueness:

$y$  is unique since if  $y_1 < y_2$ , then  $x = y_1^n < y_2^n \neq x$ .

Existence:

Let set  $A = \{ t > 0 : t^n < x \}$ .

$A \neq \emptyset$  since let  $t_1 = \frac{x}{x+1} < 1$  so  $t_1 < x$  and thus,  $0 < t_1^n < t_1 < x$  so  $t_1 \in A$ .

$A$  is bounded above since if  $t \geq x+1$ , then  $t > 1$  so  $t^n \geq t \geq x+1 > x$  so  $t \notin A$ .  
 So  $x+1$  is an upper bound of  $A$ .

Thus by the least upper bound property,  $y = \sup(A)$  exists.

For  $y^n = x$ , show  $y^n < x$  and  $y^n > x$  cannot hold true.

\*\*\* (Not an upper bound of  $A$  if  $<$  and not a least upper bound of  $A$  if  $>$ )\*\*\*

For  $0 < \alpha < \beta$ :

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\beta^{n-1} + \beta^{n-2}\alpha^1 + \dots + \alpha^{n-1})}_{\substack{\beta^{n-1} < \beta^{n-1} < \beta^{n-1}}} < (\beta - \alpha)n\beta^{n-1}$$

Suppose  $y^n < x$ . Pick  $0 < h < 1$  and  $h < \frac{x - y^n}{n(y+1)^{n-1}}$ .

From inequality, let  $\beta = y+h$  and  $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus,  $(y+h)^n < x$  so  $y+h \in A$  and thus, not an upper bound of  $A$  which is a contradiction since  $y = \sup(A)$ .

Suppose  $y^n > x$ . Pick  $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$ .

$$\text{Consider } t \geq y-k, \text{ then: } y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$$

Thus,  $t^n > x$  so  $t \notin A$ .

Thus,  $y-k$  is an upper bound of  $A$  which is a contradiction since  $y = \sup(A)$ .

Since  $y^n < x$  and  $y^n > x$ , then  $y^n = x$ .

**Corollary 2.1.3: n-th root of product = product of n-th root**

If  $a, b > 0$  and  $n \in \mathbb{Z}_+$ , then  $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$

**Proof**

Let  $A = a^{\frac{1}{n}}$ ,  $B = b^{\frac{1}{n}}$ . By **theorem 2.1.2**, since  $A$  is a root for  $y_1^n = a$ , then  $A^n = a$ . Similarly,  $B$  is a solution of  $y_2^n = b$  so  $B^n = b$ . Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 \dots A_n B_1 B_2 \dots B_n \\ &= A_1 A_2 \dots B_1 A_n B_2 \dots B_n = \dots = A_1 B_1 A_2 \dots A_{n-1} A_n B_2 \dots B_n \\ &= \dots = A_1 B_1 A_2 B_2 \dots A_n B_n = (AB)^n \end{aligned}$$

Then again by **theorem 2.1.2**, there is a unique  $(ab)^{\frac{1}{n}} = AB = a^{\frac{1}{n}} b^{\frac{1}{n}}$ .

**2.2 Decimals****Definition 2.2.1: Decimals**

Let  $n_0$  be the largest integer such that  $n_0 \leq x$  for  $x > 0 \in \mathbb{R}$ .

Then let  $n_k$  be the largest integer such that  $d_k = n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x$

Let  $E$  be the set of  $d_k$  for  $k = 0, 1, \dots, \infty$ . Then,  $x = \sup(E)$ .

**2.3 Extended Reals****Definition 2.3.1: Extended Reals**

The extended real number system consist of  $\mathbb{R}$  and  $\pm\infty$  such that:

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}$$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If  $x > 0$ , then  $x(\pm\infty) = \pm\infty$ . If  $x < 0$ , then  $x(\pm\infty) = \mp\infty$

**2.4 Complex Numbers****Definition 2.4.1: Complex Number**

A complex number is an ordered pair  $(a, b)$  where  $a, b \in \mathbb{R}$ . For  $x, y \in \mathbb{C}$

- $x + y = (a, b) + (c, d) = (a + c, b + d)$
- $xy = (a, b)(c, d) = (ac - bd, ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)(a, -b)$

Thus, the axioms form a field where  $(0, 0) = 0$  and  $(1, 0) = 1$  and  $(0, 1) = i$ .

**Theorem 2.4.2: Imaginary i and Form a + bi**

Let  $i = (0, 1)$ . Then,  $i^2 = -1$ .

Then,  $(a, b) = a + bi$

**Proof**

$$\begin{aligned} i^2 &= (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1 \\ (a, b) &= (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi \end{aligned}$$

**Definition 2.4.3: Conjugate**

Let conjugate:  $\bar{z} = a - bi$  where  $\text{Re}(z) = a$ ,  $\text{Im}(z) = b$ .

Let  $z = (a, b)$  and  $w = (c, d)$ :

(a)  $\overline{z + w} = \bar{z} + \bar{w}$

**Proof**

$$\overline{z + w} = \overline{(a + c, b + d)} = (a + c, -b - d) = (a, -b) + (c, -d) = \bar{z} + \bar{w}$$

(b)  $\overline{zw} = \bar{z} \bar{w}$

**Proof**

$$\overline{zw} = \overline{(ac - bd, ad + bc)} = (ac - bd, -ad - bc) = (a, -b)(c, -d) = \bar{z} \bar{w}$$

(c)  $z + \bar{z} = 2 \text{Re}(z)$        $z - \bar{z} = 2i \text{Im}(z)$

**Proof**

$$\begin{aligned} z + \bar{z} &= (a, b) + (a, -b) = (2a, 0) = 2 \text{Re}(z) \\ z - \bar{z} &= (a, b) - (a, -b) = (0, 2b) = (0, 2)b = 2i \text{Im}(z) \end{aligned}$$

(d)  $z\bar{z} \geq 0$

**Proof**

$$z\bar{z} = (a, b)(a, -b) = (a^2 + b^2, -ab + ab) = (a^2 + b^2, 0) = a^2 + b^2 \geq 0$$

**Definition 2.4.4: Absolute Value**

Let absolute value:  $|z| = \sqrt{z\bar{z}}$

Let  $z = (a, b)$  and  $w = (c, d)$ :

(a) If  $z \neq 0$ , then  $|z| > 0$ .

**Proof**

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0 \text{ where } |z| = 0 \text{ only if } a, b = 0 \text{ so only if } z = (0, 0).$$

(b)  $|\bar{z}| = |z|$

**Proof**

$$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

(c)  $|zw| = |z| |w|$

**Proof**

$$\begin{aligned} |zw| &= |(ac - bd, ad + bc)| = \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z| |w| \end{aligned}$$

(d)  $|\text{Re}(z)| \leq |z|$

**Proof**

$$|\text{Re}(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

(e)  $|z + w| \leq |z| + |w|$

**Proof**

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + |w|^2 + 2 \text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

## 2.5 Euclidean Spaces

### Definition 2.5.1: Euclidean Spaces

For each positive integer  $k$ , let  $\mathbb{R}^k$  be the set of all ordered  $k$ -tuples:

$$x = (x_1, \dots, x_k) \quad \text{for each } x_i \in \mathbb{R}$$

with the properties:

- $x+y = (x_1 + y_1, \dots, x_k + y_k) \in \mathbb{R}^k$
- $cx = (cx_1, \dots, cx_k) \in \mathbb{R}^k$

So,  $\mathbb{R}^n$  has a vector space structure. Similarly, for  $\mathbb{C}^n$ .

### Definition 2.5.2: Inner Product for $\mathbb{R}^k$

$$x \cdot y = x_1 y_1 + \dots + x_k y_k \in \mathbb{R}$$

### Definition 2.5.3: Norm

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^k x_i^2}$$

### Definition 2.5.4: Extension to $\mathbb{C}^k$

For  $z, w \in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + \dots + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + \dots + z_k \overline{z_k} = |z_1|^2 + \dots + |z_k|^2 = |z|^2$

## 2.6 Cauchy-Schwarz

### Theorem 2.6.1: Cauchy-Schwarz

If  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$  and  $b_1, \dots, b_n \in \mathbb{C}$ , then:

$$|\sum_{j=1}^n \alpha_j (\overline{b_j})|^2 \leq \sum_{j=1}^n |\alpha_j|^2 \sum_{j=1}^n |b_j|^2$$

#### Proof

Let  $A = \sum |a_j|^2$  and  $B = \sum |b_j|^2$  and  $C = \sum a_j (\overline{b_j})$ .

If  $B = 0$ , then  $b_1 = \dots = b_n = 0$ . Thus,  $0 \leq A(0)$  holds true.

Suppose  $B > 0$ . Then:

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) \\ &= \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B \overline{C} a_j \overline{b_j} - B C \overline{a_j} b_j + C \overline{C} b_j \overline{b_j} \\ &= B^2 \sum |a_j|^2 - B \overline{C} \sum a_j \overline{b_j} - B C \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B \overline{C} C - B C \overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

Since  $|Ba_j - Cb_j|^2 \geq 0$ , then  $B(AB - |C|^2) \geq 0$ .

Since  $B > 0$ , then  $AB - |C|^2 \geq 0$  so  $AB \geq |C|^2$ .

### Corollary 2.6.2: $|z \cdot w| \leq |z||w|$

Since  $|z_i|^2 = z_i \overline{z_i}$ , then  $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$ . Thus:

$$|z \cdot w|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2 |w|^2$$

Thus,  $|z \cdot w| \leq |z||w|$ .

**Theorem 2.6.3: Properties of  $\mathbb{R}^k$** 

Let  $x, y, z \in \mathbb{R}^k$  where  $\alpha \in \mathbb{R}$ :

- (a)  $|x| \geq 0$  where  $|x| = 0$  only if  $x = 0$

Proof

$$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = \dots = x_k = 0$$

- (b)  $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2} \sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$$

- (c)  $|x + y| \leq |x| + |y|$

Proof

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

- (d)  $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

### 3 Construction of $\mathbb{R}$

There exists an ordered field  $\mathbb{R}$  which has the least upper bound property.  
Also,  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield.

#### Definition 5.1: Cuts

Define a cut as any set  $\alpha \subset \mathbb{Q}$  with the properties:

- $\alpha$  is not empty and  $\alpha \neq \mathbb{Q}$
- If  $p \in \alpha$  and  $q \in \mathbb{Q} < p$ , then  $q \in \alpha$
- If  $p \in \alpha$ , then  $p < r \in \mathbb{Q}$  for some  $r \in \alpha$

#### Proposition 5.2: Order of $\mathbb{R} \rightarrow$ ordered set $\mathbb{R}$

Define  $\alpha < \beta$  if  $\alpha$  is a proper subset of  $\beta$ .

- If  $\alpha \not\subset \beta$ , then  $\beta$  is not a subset of  $\alpha$ .  
Then there is a  $p \in \beta$  such that  $p \notin \alpha$ .  
Then for any  $q \in \alpha$ ,  $q < p$  and thus,  $q \in \beta$ .  
Thus,  $\alpha \subset \beta$  and since  $\alpha \neq \beta$ , then  $\alpha < \beta$ .
- If  $\alpha \not\subset \beta$  and  $\alpha \not\supset \beta$ , then either  $\alpha = \beta$  or  $\alpha \neq \beta$ .  
If  $\alpha \neq \beta$ , there are  $p, q$  such that  $p \in \alpha$ , but  $p \notin \beta$  and  $q \in \beta$ , but  $q \notin \alpha$ .  
But if  $p \notin \beta$ , then for any  $b \in \beta$ ,  $b < p$ . Thus,  $q < p$ .  
Similarly, if  $q \notin \alpha$ , then for any  $a \in \alpha$ ,  $a < q$ . Thus,  $p < q$ .  
Thus, there is a contradiction since  $p > q$  and  $p < q$  so  $\alpha = \beta$ .
- If  $\alpha \not\subset \beta$ , then  $\alpha$  is not a subset of  $\beta$ .  
Then there is a  $p \in \alpha$  such that  $p \notin \beta$ .  
Then for any  $q \in \beta$ ,  $q < p$  and thus,  $q \in \alpha$ .  
Thus,  $\beta \subset \alpha$  and since  $\alpha \neq \beta$ , then  $\beta < \alpha$ .
- If  $\alpha < \beta$  and  $\beta < \gamma$ , then since  $\alpha$  is a proper subset of  $\beta$  and  $\beta$  is a proper subset of  $\gamma$ , then  $\alpha$  is a proper subset of  $\gamma$ . Thus,  $\alpha < \gamma$ .

Thus,  $\mathbb{R}$  is an ordered set with such an order  $<$ .

#### Proposition 5.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let  $A \subset \mathbb{R}$  and  $\beta$  be an upper bound for  $A$ . Let  $\gamma$  be the union of all  $\alpha \in A$ .  
Thus,  $p \in \gamma$  if and only if  $p \in \alpha$  for some  $\alpha \in A$ .

$\gamma$  defines a cut since:

- Since  $A$  is nonempty, there exists a  $\alpha_0 \in A$  where  $\alpha_0$  is nonempty.  
Since  $\alpha_0$  is nonempty, then  $\gamma$  is nonempty.  
Since every  $\alpha \in A$  is  $\alpha < \beta$ , then  $\gamma < \beta$  so  $\gamma \subset \beta$  and thus,  $\gamma \neq \mathbb{Q}$ .
- If  $p \in \gamma$ , then  $p \in \alpha_1$  for some  $\alpha_1 \in A$ . If  $q < p$ , then  $q \in \alpha_1$  so  $q \in A$ .
- If  $p \in \gamma$ , then  $p \in \alpha_1$  for some  $\alpha_1 \in A$ . Thus, there is a  $r \in \alpha_1$  such that  $r > p$  so  $r \in \gamma$ . Thus, there is a  $r \in \gamma$  where  $r > p$ .

Since  $\gamma$  defines a cut, then  $\gamma \in \mathbb{R}$ . Since every  $\alpha \in A \subset \gamma$ , then  $\alpha \leq \gamma$  so  $\gamma$  is an upper bound for  $A$ .

Suppose  $\delta < \gamma$ . Then there is a  $s \in \gamma$  such that  $s \notin \delta$ . Since  $s \in \gamma$ , then there is a  $\alpha \in A$  such that  $s \in \alpha$ . Since  $\delta < \alpha$ , then  $\delta$  is not an upper bound of  $A$ .

Thus,  $\gamma = \sup(A)$ .

**Proposition 5.4:  $\mathbb{R}$  is a field**

If  $\alpha, \beta \in \mathbb{R}$ , define  $\alpha + \beta$  as the set of all sums  $r + s$  where  $r \in \alpha$  and  $s \in \beta$ . Also, let  $0^*$  be the set of all negative rational numbers which is a cut since:

- $0^*$  is nonempty and  $0^* \neq \mathbb{Q}$
- If  $p \in 0^*$ , then any  $q \in \mathbb{Q} < p$  is a negative rational and thus,  $q \in 0^*$ .
- Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , then for any  $p \in 0^*$ , there is a  $r \in \mathbb{Q}$  where  $p < r < 0$  so  $r$  is a negative rational so  $r \in 0^*$ .

$\alpha + \beta \in \mathbb{R}$  since  $\alpha + \beta$  is a cut:

- $\alpha + \beta$  is non-empty since  $\alpha, \beta$  are non-empty. Take  $r' \notin \alpha, s' \notin \beta$ , then  $r' + s' > r + s$  for  $r \in \alpha, s \in \beta$ . Thus,  $r' + s' \notin \alpha + \beta$  so  $\alpha + \beta \neq \mathbb{Q}$ .
- If  $p \in \alpha + \beta$ , then  $p = r + s$  where  $r \in \alpha$  and  $s \in \beta$ .  
If  $q < p$ , then  $q - s < p - s = (r + s) - s = r$  so  $q - s \in \alpha$ .  
Since  $q - s \in \alpha$  and  $s \in \beta$ , then  $(q - s) + s = q \in \alpha + \beta$ .
- If  $r \in \alpha$ , then there is a  $t \in \alpha$  such that  $t > r$ . Let  $s \in \beta$ .  
Thus, for any  $p = r + s \in \alpha + \beta$ , there is a  $q = t + s \in \alpha + \beta$  such that  $p = r + s < t + s = q$ .

$\alpha + \beta = \beta + \alpha$

If  $p = r + s \in \alpha + \beta$  where  $r \in \alpha, s \in \beta$ , then  $s + r = r + s = p \in \beta + \alpha$ .

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If  $r \in \alpha, s \in \beta, t \in \gamma$ , then  $r + s + t = (r + s) + t \in (\alpha + \beta) + \gamma$  and  $r + s + t = r + (s + t) \in \alpha + (\beta + \gamma)$ .

$\alpha + 0^* = \alpha$

If  $r \in \alpha, s \in 0^*$ , then  $r + s < r$ . Thus,  $r + s \in \alpha$ . Thus,  $\alpha + 0^* \subset \alpha$ .

If  $p \in \alpha$ , there is a  $r \in \alpha$  where  $r > p$ . Thus,  $p - r \in 0^*$ .

Since  $p = r + (p - r) \in \alpha + 0^*$ , then  $\alpha \subset \alpha + 0^*$ . Thus,  $\alpha + 0^* = \alpha$ .

There is a  $-\alpha$  such that  $\alpha + -\alpha = 0^*$

Fix  $\alpha \in \mathbb{R}$ . Let set  $\beta$  be all  $p$  where there is  $r > 0$  such that  $-p - r \notin \alpha$ .

$\beta \in \mathbb{R}$  since  $\beta$  is a cut:

- If  $s \notin \alpha$  and  $p = -s - 1$ , then  $-p - 1 \notin \alpha$ . Thus,  $p \in \beta$  so  $\beta$  is nonempty. If  $q \in \alpha$ , then  $-q \notin \beta$  so  $\beta \neq \mathbb{R}$ .
- If  $p \in \beta$ , let  $r > 0$  so  $-p - r \notin \alpha$ . If  $q < p$ , then  $-q - r > -p - r$  and thus,  $-q - r \notin \alpha$  so  $q \in \beta$ .
- If  $p \in \beta$ , let  $t = p + (r/2)$ . Then  $-t - (r/2) = -p - r \notin \alpha$  and thus,  $t \in \beta$  where  $p < t$ .

If  $r \in \alpha, s \in \beta$ , then  $s \notin \alpha$ . Thus,  $r < -s$  so  $r + s < 0$ . Thus,  $\alpha + \beta \subset 0^*$ .

Let  $v \in 0^*$  and let  $w = -v/2$  so  $w > 0$ .

Thus, by the Archimedean property, there is an integer  $n$  such that  $nw \in \alpha$ , but  $(n+1)w \notin \alpha$ . Let  $p = -(n+2)w$  so  $-p - w = (n+1)w \notin \alpha$  so  $p \in \beta$ .

Then,  $v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p \in \alpha + \beta$ .

Since  $v \in 0^*$ , then  $0^* \subset \alpha + \beta$ . Thus,  $\alpha + \beta = 0^*$ . Then, let  $-\alpha = \beta$ .

Thus, if  $\alpha, \beta, \gamma \in \mathbb{R}$  and  $\beta < \gamma$ , then  $\alpha + \beta < \alpha + \gamma$ .

Thus, if  $\alpha > 0^*$ , then  $-\alpha = -\alpha + 0^* < -\alpha + \alpha = 0^*$  so  $-\alpha < 0^*$ .

If  $\alpha, \beta \in \mathbb{R}_+$ , define  $\alpha\beta$  as the set of all  $p$  such that  $p \leq rs$  for  $r \in \alpha, s \in \beta$ .

Define  $1^*$  as the set of all  $q < 1$ . Then all multiplication axioms holds with similar proofs as addition. Also, note since  $\alpha, \beta > 0^*$ , then  $\alpha\beta > 0^*$ .

Also,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$  holds through cases were  $\alpha, \beta, \gamma >, < 0^*$ .



## 4 Cardinality

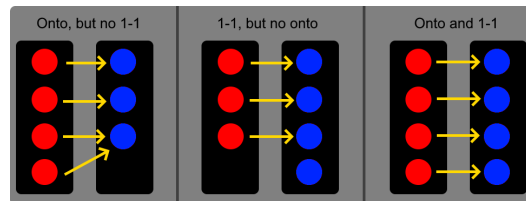
### 4.1 Cardinality

#### Definition 4.1.1: Onto and 1-1 Mapping

Suppose for every  $x \in A$ , there is an associated  $f(x) \in B$ .

Then  $f$  maps  $A$  into  $B = f: A \rightarrow B$ .

- If  $f(A) = B$ , then  $f$  maps  $A$  onto  $B$ .
- If for each  $y \in B$ ,  $f^{-1}(y)$  consist of at most one  $x \in A$  where  $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$  for  $y_1 \neq y_2$ , then  $f$  is a 1-1 mapping of  $A$  into  $B$ .



#### Definition 4.1.2: 1-1 Correspondence

Sets  $A$  and  $B$  are equivalent (have the same cardinality) if there is a 1-1 onto function  $f: A \rightarrow B$ . (1-1 correspondence between  $A$  and  $B$ ) Then,  $A \sim B$ .

If  $f: A \rightarrow B$  is 1-1 and onto, then there is a  $f^{-1}: B \rightarrow A$  that is 1-1 and onto.

#### Definition 4.1.3: Countability

- $A$  is **finite** if  $A \sim J_n = \{0, 1, \dots, n\}$  for some  $n \in \mathbb{N}$
- $A$  is **infinite** if  $A$  is not finite
- $A$  is **countably infinite** if  $A \sim J = \mathbb{Z}_+$
- $A$  is **uncountable** if  $A$  is not finite or countably infinite
- $A$  is **at most countable** if  $A$  is finite or countably infinite

#### Example

$\mathbb{Z}$  is countably infinite

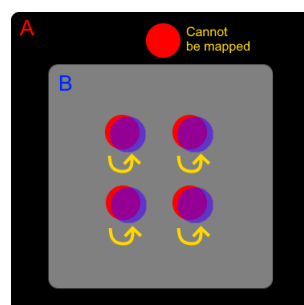
#### Proof

$$\text{Let } f(n): \mathbb{Z}_+ \rightarrow \mathbb{Z} = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n-1}{2} & n \text{ is odd} \end{cases}$$

So  $1 \mapsto 0$ ,  $2 \mapsto 1$ ,  $3 \mapsto -1$ ,  $4 \mapsto 2$ ,  $5 \mapsto -2$ , etc. Thus,  $\mathbb{Z} \sim \mathbb{Z}_+$ .

#### Definition 4.1.4: Pigeonhole Principle

If  $A$  is finite,  $A$  is not equivalent to any proper set of  $A$ .



**Theorem 4.1.5: Infinite subsets of countable sets are countable**

An infinite subset  $E$  of a countably infinite set  $A$  is countably infinite

**Proof**

Let  $E \subset A$  be an infinite subset. For every distinct  $x_i \in A$ , let  $\{x_1, x_2, \dots\} \in A$ .  
 Let  $n_1$  be smallest integer such that  $x_{n_1} \in E$ .  
 Then let  $n_2$  be the smallest integer where  $n_2 > n_1$  such that  $x_{n_2} \in E$ .  
 Repeat the process to create sequence  $f(k) = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$ .  
 Thus, there is a 1-1 correspondence between  $E$  and  $\mathbb{Z}_+$  so  $E$  is countably infinite.

**4.2 Set of Sets****Definition 4.2.1: Union and Intersection**

Let sets  $\Omega, B$  be such that for each  $x \in \Omega$ , there is an associated  $E_x \subset B$ .

- $E = \cup_{x=1}^n E_x$  only if for every  $x \in E$ ,  $x \in E_x$  for at least one  $x \in \Omega$ .
- $P = \cap_{x=1}^n E_x$  only if for every  $x \in P$ ,  $x \in E_x$  for all  $x \in \Omega$ .

with properties:

- |   |   |
|---|---|
| (a) $A \cup B = B \cup A$                                     | (a) $A \cap B = B \cap A$                   |
| (b) $(A \cup B) \cup C = A \cup (B \cup C)$                   | (b) $(A \cap B) \cap C = A \cap (B \cap C)$ |
| (c) $A \subset A \cup B$                                      | (c) $(A \cap B) \subset A$                  |
| (d) If $A \subset B$ , then $A \cup B = B$ and $A \cap B = A$ |   |

**Proof**

If  $x \in A \cup B$ , then  $x \in A$  or/and  $x \in B$ .

- If  $x \in A$ , since  $A \subset B$ , then  $x \in B$ . Then,  $(A \cup B) \subset B$ .
- If  $x \in B$ , then immediately  $(A \cup B) \subset B$ .

If  $x \in B$ , then  $x \in A \cup B$  so  $B \subset (A \cup B)$ . Thus,  $A \cup B = B$ .

If  $x \in A \cap B$ , then  $x \in A$  and  $x \in B$ . Thus,  $(A \cap B) \subset A$ .

If  $x \in A$ , since  $A \subset B$ , then  $x \in B$  so  $x \in A \cap B$ . Thus,  $A \subset (A \cap B)$ .

Thus,  $A \cap B = A$ .

- (e)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**Proof**

If  $x \in A \cap (B \cup C)$ , then  $x \in A$  and ( $x \in B$  or/and  $x \in C$ ).

- If  $x \in B$ , then  $x \in (A \cap B)$  so  $x \in (A \cap B) \cup (A \cap C)$ .
- If  $x \in C$ , then  $x \in (A \cap C)$  so  $x \in (A \cap B) \cup (A \cap C)$ .

Thus,  $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ .

If  $x \in (A \cap B) \cup (A \cap C)$ , then  $x \in A$  and ( $x \in B$  or/and  $x \in C$ ).

Thus,  $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ .

Thus,  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

$$(f) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

If  $x \in A \cup (B \cap C)$ , then  $x \in A$  or/and  $(x \in B \text{ and } x \in C)$ .

- If  $x \in A$ , then  $x \in (A \cup B)$  and  $x \in (A \cup C)$  so  $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ .
- If  $x \in B, C$ , then  $x \in (A \cup B)$  and  $x \in (A \cup C)$  so  $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ .

If  $x \in (A \cup B) \cap (A \cup C)$ , then  $x \in A$  or/and  $(x \in B \text{ and } x \in C)$ .

Thus,  $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ .

Thus,  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

### Theorem 4.2.2: Union of countably infinite sets is countably infinite

If  $E_1, E_2, \dots$  are countably infinite sets, then  $S = \bigcup_{n=1}^{\infty} E_n$  is countably infinite.

Proof

For each  $E_n$ , there is a sequence  $\{x_{n1}, x_{n2}, \dots\}$ . Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence  $S^* = \{x_{11}; x_{21}, x_{12}; x_{31}, x_{32}, x_{23}; \dots\}$ .

Since  $S^* \sim S$  so  $S$  is at most countable and  $S$  is infinite since  $E_1, E_2, \dots$  are infinite, then  $S$  cannot be finite and thus, countably infinite.

Alternative Proof

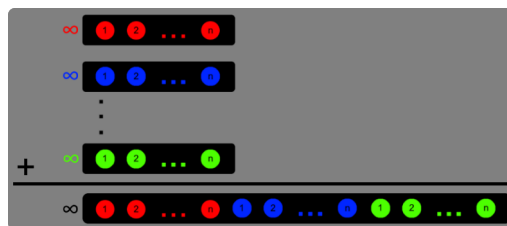
For each  $E_n$ , let set  $\widetilde{E}_n = E_n - \bigcup_{m=1}^{\infty} E_m$  where  $m \neq n$ . Thus,  $S = \bigcup_{n=1}^{\infty} \widetilde{E}_n$ .

Since each  $E_n$  is countably infinite, there exists a 1-1 mapping  $\delta_n: E_n \rightarrow \mathbb{Z}_+$ .

Thus, for each  $\widetilde{E}_n$ , there is a 1-1 mapping  $\delta_n: \widetilde{E}_n \rightarrow A \subset \mathbb{Z}_+$ .

Let  $p_1, p_2, \dots$  be distinct primes. Since for  $s \in S$ , there exists a unique  $\widetilde{E}_i$  such that  $s \in \widetilde{E}_i$ , then let  $f(s) = p_1^{\delta_1(s)} p_2^{\delta_2(s)} \dots$  where  $p_k^{\delta_k(s)} = 1$  if  $k \neq i$ .

Then, by the Fundamental theorem of arithmetic,  $f$  maps  $s$  to a unique  $z \in \mathbb{Z}_+$  and thus,  $f$  is a 1-1 function so  $S$  is at most countable. Since any  $E_n \subset S$  is countably infinite, then  $S$  cannot be finite and thus,  $S$  is countably infinite.



### Theorem 4.2.3: The set of countable n-tuples are countable

Let  $A$  be a countably infinite set and  $B_n$  be the set of all  $n$ -tuples  $(a_1, \dots, a_n)$  where  $a_k \in A$ . Then  $B_n$  is countably infinite.

Proof

The base case  $B_1$  is countably infinite since  $B_1 = A$ .

Suppose  $B_{n-1}$  is countably infinite. Then for every  $x \in B$ :

$$x = (b, a) \quad b \in B_{n-1} \text{ and } a \in A$$

Since for every fixed  $b$ ,  $(b, a) \sim A$  and thus, countably infinite.

Since  $B$  is a set of countably infinite sets, then  $B_n$  is countably infinite.

**Theorem 4.2.4:  $\mathbb{Q}$  is countable**

The set of rational numbers,  $\mathbb{Q}$ , is countably infinite

**Proof**

Since elements of  $\mathbb{Q}$  are of form  $\frac{a}{b}$  which is a 2-tuple, then by the **theorem 4.2.3**,  $\mathbb{Q}$  is countably infinite.

**Alternative Proof**

For every  $x \in \mathbb{Q}$ , let  $x = (-1)^i \frac{p}{q}$  where  $p, q \in \mathbb{Z}_+$ .  
 Let  $f(x) = 2^i 3^p 5^q$ . Then by the Fundamental theorem of arithmetic,  $f$  is a 1-1 mapping of  $x$  to  $E \subset \mathbb{Z}_+$ .  
 Thus,  $\mathbb{Q}$  is at most countable, but since  $p, q \in \mathbb{Z}_+$ , then  $\mathbb{Q}$  cannot be finite and thus, is countably infinite.

**Example**

Let  $A$  be the set of all sequences whose elements are digits 0 and 1. Then  $A$  is uncountable.

**Proof: Cantor's Diagonalization Proof**

Let set  $E$  be a countably infinite subset of  $A$  which consist of sequences  $s_1, s_2, \dots$ . Then construct a sequence  $s$  as follows:  
 If the  $n$ -th digit in  $s_n$  is 1, then let the  $n$ -th digit of  $s$  be 0 and vice versa.  
 Thus,  $s$  differs from every  $s_n \in E$  so  $s \notin E$ .  
 But,  $s \in A$  so  $E$  is a proper subset of  $A$ .  
 Thus, every countably infinite subset of  $A$  is a proper subset of  $A$ .  
 If  $A$  is countably infinite, then  $A$  is a proper subset of  $A$  which is a contradiction.

## 5 Metric Spaces & Closed/Open

### 5.1 Metric Spaces

#### Definition 5.1.1: Metric Spaces

A set  $X$  is a metric space if for any  $p, q \in X$ , there is an associated  $d(p, q) \in \mathbb{R}$  such that:

- $d(p, q) > 0$  if  $p \neq q$
- $d(p, q) = 0$  if and only if  $p = q$
- **Symmetry**:  $d(p, q) = d(q, p)$
- **Triangle Inequality**:  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ .

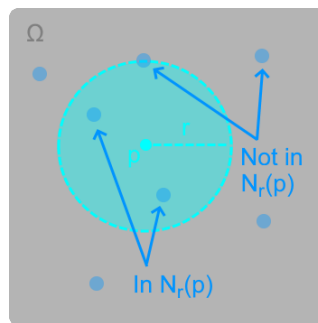
For euclidean spaces  $\mathbb{R}^k$ ,  $d(x, y) = |x - y|$  where  $x, y \in \mathbb{R}^k$ .

#### Definition 5.1.2: Types of Points and Sets

For metric space  $X$  and set  $E \subset X$ :

##### (a) Neighborhood

For  $p \in X$  and  $r > 0$ ,  $N_r(p)$  is the set of all  $q \in X$  where  $d(q, p) < r$



##### (b) Limit Points and Closed Sets

Closed set  $E$  contain all  $p \in X$  where every  $N_r(p)$  contain a  $q \neq p \in E$

##### • Limit Points

For point  $p \in X$ , every  $N_r(p)$  contains a  $q \neq p \in E$

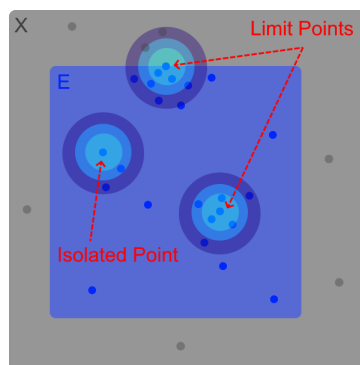
The set of all limit points of  $E = E'$

##### • Isolated Points

If  $p \in E$  is not a limit point of  $E$

##### • Closed

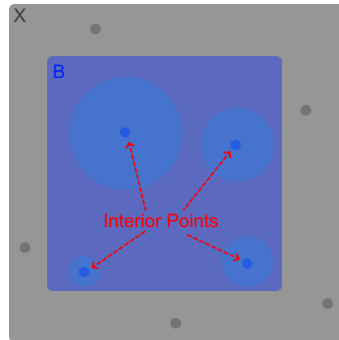
If every limit point  $p$  of  $E$  is a  $p \in E$



## (c) Interior Points and Open Sets

Open set  $E$  contains all its  $p$  which has a  $N_r(p) \subset E$

- Interior Point  
For  $p \in X$ , there is a  $N_r(p) \subset E$   
The set of all interior points =  $E^\circ$
- Open  
If every  $p \in E$  is an interior point of  $E$



## (d) More about Sets

- Bounded  
If there is  $M \in \mathbb{R}$ ,  $q \in X$  such that  $d(p, q) < M$  for all  $p \in E$
- Complement  
From  $E$ ,  $E^c$  is the set of all  $p \in X$  such that  $p \notin E$
- Perfect  
If  $E$  is closed and if every  $p \in E$  is a limit point of  $E$
- Dense  
If every  $p \in X$  is a limit point of  $E$  or/and  $p \in E$
- Boundary Point  
For  $p \in X$ , if every  $N_r(p)$  contains a  $x \in E$  and  $y \in E^c$   
The set of all boundary points =  $\partial E$

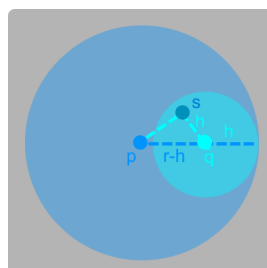
For a metric space  $X$ ,  $\{X, \emptyset\}$  are both open and closed.

**Theorem 5.1.3:  $N_r(p)$  is open**

Every neighborhood is an open set

**Proof**

Let  $q \in N_r(p)$ . Then there is a  $h > 0 \in \mathbb{R}$  such that  $d(q, p) = r - h$ .  
Then for any  $s \in N_h(q)$ ,  $d(s, p) \leq d(s, q) + d(q, p) = h + (r - h) = r$ .  
Thus, for any  $q \in N_r(p)$ , there exists a  $N_h(q) \subset N_r(p)$ .

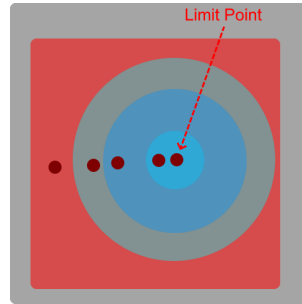


**Theorem 5.1.4:** If a set has a limit point, there are infinite  $q \in E$  in  $N_r(p)$

If  $p$  is a limit point of set  $E$ , then every  $N_r(p)$  contains infinitely many  $q \in E$ .

**Proof**

Suppose there is  $N_{r_1}(p)$  which contains finitely many  $q = \{q_1, \dots, q_n\}$ .  
 Let  $r = \min_{m \in [1, n]} d(p, q_m)$ . Then  $N_r(p)$  contains no  $q \in E$  such that  $q \neq p$ .  
 So,  $p$  is not a limit point of  $E$  which is a contradiction since  $p$  is a limit point of  $E$ .



**Corollary 5.1.5:** Limit points do not exist in finite sets

A finite set  $E$  has no limit points. Since  $\emptyset \in E$ , all finite set must be closed.

**Proof**

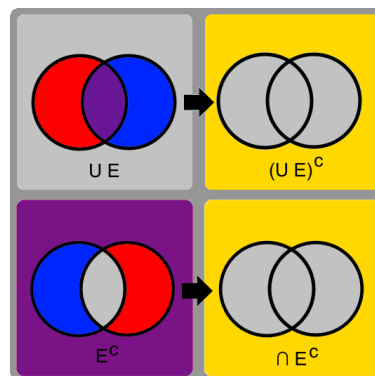
Let  $p$  be a limit point of finite set  $E$ . By **theorem 5.1.4**, then any  $N_r(p)$  contain infinite  $q \in E$  so  $E$  is an infinite set which is a contradiction since  $E$  is finite.  
 So  $p$  cannot be limit point of  $E$  and thus,  $E$  has no limit points. Since finite set  $E$  contains all its limit points because there are no limit points, then  $E$  is closed.

**Theorem 5.1.6:** De Morgan's Laws

Let  $E_1, E_2, \dots$  be a collection of sets. Then,  $(\cup E_x)^c = \cap (E_x^c)$ .

**Proof**

If  $p \in (\cup E_x)^c$ , then  $p \notin (\cup E_x)$ .  
 Thus,  $p \notin E_x$  for any  $x$  so  $p \in E_x^c$  for all  $x$ . Thus,  $p \in \cap (E_x^c)$  so  $(\cup E_x)^c \subset \cap (E_x^c)$ .  
 If  $p \in \cap (E_x^c)$ , then  $p \in E_x^c$  for all  $x$ .  
 Thus,  $p \notin E_x$  for any  $x$  so  $p \notin \cup E_x$ . Thus,  $p \in (\cup E_x)^c$  so  $\cap (E_x^c) \subset (\cup E_x)^c$ .  
 Thus,  $(\cup E_x)^c = \cap (E_x^c)$ .



**Theorem 5.1.7: Open set  $\rightarrow$  Closed complement**

A set  $E$  is open if and only if  $E^c$  is closed

**Proof**

Suppose  $E$  is open. Let  $x$  be a limit point of  $E^c$ .

Then for every  $r > 0$ ,  $N_r(x)$  must contain a  $p \in E^c$  such that  $p \neq x$ .

Then,  $N_r(x) \not\subset E$  so  $x$  is not an interior point of  $E$  and thus,  $x \notin E$  so  $x \in E^c$ .

Since any limit point  $x$  of  $E^c$  is a  $x \in E^c$ , then  $E^c$  is closed.

Suppose  $E^c$  is closed. Let  $x \in E$ .

Since  $x \notin E^c$ ,  $x$  is not a limit point of  $E^c$ . Then there exists a  $r > 0$  such that any  $p \in N_r(x)$  is not in  $E^c$ . Thus, every  $p \in N_r(x)$  is  $p \in E$  so  $N_r(x) \subset E$  and thus,  $x$  is an interior point of  $E$ . Since any  $x \in E$  is an interior point of  $E$ , then  $E$  is open.

**Corollary 5.1.8: Closed set  $\rightarrow$  Open complement**

A set  $F$  is closed if and only if  $F^c$  is open.

**Proof**

From **theorem 5.1.7**, let  $E = F^c$

**Theorem 5.1.9: Union open  $\rightarrow$  open and Intersection closed  $\rightarrow$  closed**

- (a) If  $\{G_x\}$  is a finite or infinite collection of open sets, then  $\cup G_x$  is open.

**Proof**

If  $p \in \cup G_x$ , then  $p \in G_x$  for at least one  $x$ . Let  $\bar{x}$  be such an  $x$ .

Since  $G_{\bar{x}}$  is open, then  $p$  is an interior point of  $G_{\bar{x}}$  and thus, there is a  $N_r(p)$  such that  $N_r(p) \subset G_{\bar{x}} \subset \cup G_x$ . So  $p$  is an interior point of  $\cup G_x$ .

Since any  $p \in \cup G_x$  is an interior point, then  $\cup G_x$  is open.

- (b) If  $\{F_x\}$  is a finite or infinite collection of closed sets, then  $\cap F_x$  is closed.

**Proof**

By **theorem 5.1.7**, any  $F_x^c$  is open. Since  $\{F_x^c\}$  is a finite or infinite collection of open set, then by part (a),  $\cup F_x^c$  is open.

Thus, again by **theorem 5.1.7**,  $(\cup F_x^c)^c$  is closed.

By **theorem 5.1.6**,  $(\cup F_x^c)^c = \cap (F_x^c)^c = \cap F_x$ .

- (c) If  $G_1, \dots, G_n$  is a finite collection of open sets, then  $\cap_{x=1}^n G_x$  is open.

**Proof**

If  $p \in \cap_{x=1}^n G_x$ , then  $p \in G_x$  for all  $G_x$  for  $x = \{1, 2, \dots, n\}$ .

Since each  $G_x$  is open, then for any  $G_x$ , there is a  $N_{r_x}(p) \subset G_x$ .

Let  $r = \min(r_1, r_2, \dots, r_n)$ . Thus,  $p \in N_r(p) \subset N_{r_x}(p)$  for all  $x$ .

So,  $N_r(p) \subset \cap_{x=1}^n G_x$  and thus,  $p$  is an interior point of  $\cap_{x=1}^n G_x$  so  $\cap_{x=1}^n G_x$  is open.

**Infinite + Closed:**  $G_i = (-1/i, 1/i)$

**Infinite + Open:**  $G_i = (-i, i)$

- (d) If  $F_1, \dots, F_n$  is a finite collection of closed sets, then  $\cup_{x=1}^n F_x$  is closed.

**Proof**

By **theorem 5.1.7**, any  $F_x^c$  is open. Since  $F_1^c, \dots, F_n^c$  is a finite collection of open set, then by part (c),  $\cap_{x=1}^n F_x^c$  is open.

Thus, again by **theorem 5.1.7**,  $(\cap_{x=1}^n F_x^c)^c$  is closed.

By **theorem 5.1.6**,  $(\cap_{x=1}^n F_x^c)^c = \cup_{x=1}^n (F_x^c)^c = \cup_{x=1}^n F_x$ .

**Infinite + Closed:**  $F_i = [-1/i, 1/i]$

**Infinite + Open:**  $F_i = [1/i, \infty)$



**Theorem 5.1.10:  $E'$  is closed**

Let  $E \subset X$ . Then,  $(E')' \subset E'$ . Thus,  $E'$  is closed.

**Proof**

If  $x \in (E')'$ , then for every  $N_{r_1}(x)$ , there is a  $y \neq x$  where  $y \in E'$ .  
 Since  $y \in E'$ , then for every  $N_{r_2}(y)$ , there is a  $z \neq y$  where  $z \in E$ .  
 Let  $r = r_1 + r_2$ .  
 Then for every  $N_r(x)$ , there exists a  $z \neq x$  where  $z \in E$ . Thus,  $x \in E'$  so  $(E')' \subset E'$ .

**Theorem 5.1.11:  $E^\circ$  is open**

Let  $E \subset X$ . Then,  $E^\circ$  is open.

**Proof**

If  $p \in E^\circ$ , there is a  $r > 0$  such that  $N_r(p) \subset E$ .  
 Then for  $0 < s < r$ ,  $N_s(p) \subset N_r(p)$  so any  $q \in N_s(p)$  is  $q \in E^\circ$ .  
 Since any  $p \in E^\circ$  have a  $N_s(p) \subset E^\circ$ , then  $E^\circ$  is open.

**5.2 Intervals and Balls****Definition 5.2.1: Segments and Intervals**

In  $\mathbb{R}$ , a **segment** is an open interval  $(a,b) = \{ x \in \mathbb{R} : a < x < b \}$   
 In  $\mathbb{R}$ , a **interval** is a closed interval  $[a,b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$

**Definition 5.2.2: Open Balls**

In  $\mathbb{R}^k$ , an **open ball** of radius  $r > 0$  centered at  $p$  is:

$$N_r(p) = \{ x \in \mathbb{R}^k : |x - p| < r \} = \{ x \in \mathbb{R}^k : d(x,p) < r \}$$

A **closed ball** has  $d(x,p) \leq r$ .

**Definition 5.2.3: Convex**

$E \subset \mathbb{R}^k$  is **convex** if for all  $x, y \in E$  and  $t \in [0,1]$ ,  $tx + (1-t)y \in E$ .

**Example**

Balls in  $\mathbb{R}^k$  are convex

Let  $x, y \in$  open ball  $N_r(p)$ . Let  $z = tx + (1-t)y$  for  $t \in [0,1]$ .  
 Since  $|x - p| < r$  and  $|y - p| < r$ :  

$$\begin{aligned} |z - p| &= |tx + (1-t)y - p| = |tx + (1-t)y - tp + (t-1)p| \\ &= |t(x-p) + (1-t)(y-p)| \leq t|x-p| + (1-t)|y-p| \\ &< tr + (1-t)r = r \end{aligned}$$
  
 Thus,  $z \in N_r(p)$  so balls are convex. Same proof applies to closed balls.

**Definition 5.2.4: Dense**

$E \subset X$  is dense if every  $x \in X$  is either in  $E$  or a limit point of  $E$ .

**Example**

Let  $X = \mathbb{R}$ . Then,  $E = \mathbb{Q}$  is dense in  $\mathbb{R}$ .

Fix  $x \in \mathbb{R}$  and  $r > 0$ . There is a  $q \in \mathbb{Q}$  such that  $x-r < q < x$ . So for any  $r > 0$  and  $q \in \mathbb{Q}$ ,  $q \neq x$  and  $q \in N_r(x)$ . Thus, every  $x \in \mathbb{R}$  is a limit point of  $\mathbb{Q}$ .

## 6 Closure, Open Relative, & Compact

### 6.1 Closure

#### Definition 6.1.1: Closure

Let  $E \subset$  metric space  $X$  and  $E'$  be the set of all limit points of  $E$  in  $X$ .

Then the closure of  $E$ :  $\overline{E} = E \cup E'$

with the properties:

- (a)  $\overline{E}$  is closed

#### Proof

Suppose  $x \in X$ , but  $x \notin \overline{E}$ . Thus,  $x \in \overline{E}^c$ .

Thus, there is a  $N_r(x) \subset \overline{E}^c$  since else there is always a  $p \in N_r(x)$  where  $p \in \overline{E}$  so  $x$  is a limit point of  $\overline{E}$  so  $x \in \overline{E}$ . Thus,  $\overline{E}^c$  is open so  $\overline{E}$  is closed by [theorem 5.1.7](#).

- (b)  $E = \overline{E}$  if and only if  $E$  is closed

#### Proof

If  $E = \overline{E}$ , then by part (a),  $E$  is closed.

If  $E$  is closed, then  $E' \subset E$  so  $E = E \cup E' = \overline{E}$ .

- (c)  $\overline{E} \subset F$  for every closed  $F \subset X$  such that  $E \subset F$

#### Proof

If closed set  $F$ , then  $F' \subset F$ . Since  $E \subset F$ , then  $E' \subset F' \subset F$  so  $\overline{E} \subset F$ .

#### Theorem 6.1.2: $\sup(E) \in \overline{E}$

Let non-empty set of real numbers,  $E$ , be bounded above. Let  $y = \sup(E)$ .

Then,  $y \in \overline{E}$ . Thus,  $y \in E$  if  $E$  is closed and  $y \notin E$  if  $E$  is open in  $\mathbb{R}$ .

#### Proof

If  $y \in E$ , then  $y \in \overline{E}$ . Suppose  $y \notin E$ .

For every  $h > 0$ , there exists a  $x \in E$  such that  $y-h < x < y$  otherwise  $y-h$  is an upper bound for  $E$  which is a contradiction since  $y = \sup(E)$ .

Thus,  $y$  is a limit point of  $E$  so  $y \in E'$ .

If  $E$  is closed, then  $y \in E$  since  $y \in E'$ . Also,  $y \in \overline{E}$ .

If  $E$  is open, then any  $N_r(y) \not\subset E$  since  $N_r(y)$  in  $\mathbb{R}$  must contain a  $\gamma > y$  so  $y \notin E^\circ$ .

### 6.2 Open Relative

#### Definition 6.2.1: Open Relative

Suppose  $E \subset Y \subset$  metric space  $X$ .

Then  $E$  is open relative to  $Y$  if for each  $p \in E$ :

There is an  $r > 0$  such that for any  $q \in Y$  where  $d(q,p) < r$ , then  $q \in E$ .

**Theorem 6.2.2:**  $E$  is open relative to  $Y \subset X$  if  $E = Y \cap G$  and  $G$  is open in  $X$

Suppose  $E \subset Y \subset X$ .

$E$  is open relative to  $Y$  if and only if  $E = Y \cap G$  for some open  $G \subset X$ .

**Proof**

Suppose  $E$  is open relative to  $Y$ .

Then for each  $p \in E$ , there is a  $r_p > 0$  such that for any  $q \in Y$  where  $d(p, q) < r_p$ , then  $q \in E$ .

Since  $Y \subset X$ , let  $V_p$  be the set of all  $q \in X$  such that  $d(p, q) < r_p$  and define  $G = \bigcup_{p \in E} V_p$ . Since  $V_p$  is open by **theorem 5.1.3**, then by **theorem 5.1.9a**, open  $G \subset X$ .

Since  $p \in V_p$  for all  $p \in E$ , then  $E \subset G \cap Y$ . Also, by construction, then  $V_p \cap Y \subset E$  so  $G \cap Y \subset E$ . Thus,  $E = Y \cap G$ .

If  $G$  is open in  $X$  and  $E = G \cap Y$ , then every  $p \in E$  has a  $V_p \subset G$ .

Then,  $V_p \cap Y \subset G \cap Y = E$  so  $E$  is open relative to  $Y$ .

## 6.3 Compact Sets

**Definition 6.3.1: Open Cover**

An open cover of set  $E \subset X$  is a collection of open  $G_1, G_2, \dots \subset X$  such that  $E \subset \bigcup G_i$ .

**Definition 6.3.2: Compact**

$K \subset X$  is compact if every open cover of  $K$  contains a finite subcover.

If  $G_1, G_2, \dots$  is an open cover of  $K$ , then  $K \subset \bigcup_{i=1}^n G_i$  for some  $n$ .

**Theorem 6.3.3: A compact set is compact in every metric space**

Suppose  $K \subset Y \subset X$ .

Then  $K$  is compact relative to  $X$  if and only if  $K$  is compact relative to  $Y$ .

**Proof**

Suppose  $K$  is compact relative to  $X$ .

Let  $V_1, V_2, \dots$  be sets open relative to  $Y$  such that  $K \subset \bigcup V_x$ . Then by **theorem 6.2.2** for each  $V_x$ , there is a  $G_x$  open relative to  $X$  where  $V_x = Y \cap G_x$ .

Since  $K$  is compact relative to  $X$ , then there is a  $n$  such that  $K \subset G_{x_1} \cup \dots \cup G_{x_n}$ .

Thus,  $K = K \cap Y \subset (\bigcup_{i=1}^n G_{x_i}) \cap Y = (\bigcup_{i=1}^n (G_{x_i} \cap Y)) = \bigcup_{i=1}^n V_{x_i}$ .

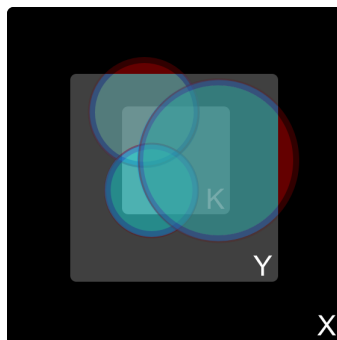
Since there are open  $V_{x_1}, \dots, V_{x_n}$  where  $K \subset \bigcup_{i=1}^n V_{x_i}$  so  $K$  is compact relative to  $Y$ .

Suppose  $K$  is compact relative to  $Y$ .

Let open  $G_1, G_2, \dots \subset X$  such that  $X \subset \bigcup G_x$ . For each  $G_x$ , let  $V_x = Y \cap G_x \subset Y$ .

Since  $K$  is compact relative to  $Y$ , there is a  $n$  such that  $K \subset \bigcup_{i=1}^n V_{x_i}$ .

Thus,  $K \subset \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (Y \cap G_{x_i}) \subset \bigcup_{i=1}^n G_{x_i}$  so  $K$  is compact relative to  $X$ .

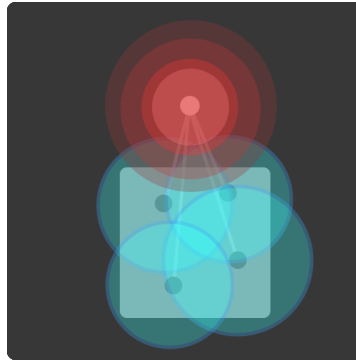


**Theorem 6.3.4: A compact set is closed**

Compact subsets of metric spaces are closed

**Proof**

Let compact  $K \subset X$ . Suppose  $p \in X$ , but  $p \notin K$  so  $p \in K^c$ .  
 If  $q \in K$ , let  $W_q$  be a neighborhood of  $q$  with  $r < \frac{1}{2}d(p,q)$ . Let  $V_{p,q}$  be a neighborhood of  $p$  with  $r < \frac{1}{2}d(p,q)$ . Since  $K$  is compact, then there are finite points  $q_1, \dots, q_n$  such that  $K \subset W$  where  $W = W_{q_1} \cup \dots \cup W_{q_n}$ .  
 Let  $V = V_{p,q_1} \cap \dots \cap V_{p,q_n}$ , then  $K \cap V \subset W \cap V = \emptyset$  so  $V \subset K^c$ .  
 Since there is a neighborhood  $V$  for  $p \in K^c$  where  $V \subset K^c$ , then every  $p \in K^c$  is an interior point so  $K^c$  is open. Then by [theorem 5.1.7](#),  $K$  is closed.

**Theorem 6.3.5: If closed  $E \subset$  compact set  $K$ ,  $E$  is compact**

Closed subsets of compact sets are compact

**Proof**

Suppose  $F \subset K \subset X$  where  $F$  is closed relative to  $X$  and  $K$  is compact.  
 Let  $V_1, V_2, \dots$  be an open cover for  $F$ . Let open set  $F^c$  be all  $k \in K$  where  $k \notin F$ .  
 $K = F \cup F^c \subset V_1 \cup V_2 \cup \dots \cup F^c$   
 Thus,  $V_1 \cup V_2 \cup \dots \cup F^c$  is an open cover for  $K$ .  
 Since  $K$  is compact, there is a finite subcover  $\Omega$  that covers  $K$  and thus, finite subcover  $\Omega$  covers  $F \cup F^c$ .  
 Remove  $F^c$  from  $\Omega$ . Since finite subcover  $\Omega - F^c$  covers  $F$ , then  $F$  is compact.

**Corollary 6.3.6: Closed  $F \cap$  compact  $K =$  compact**

If  $F$  is closed and  $K$  is compact, then  $F \cap K$  is compact

**Proof**

Since  $K$  is compact, then  $K$  is closed by [theorem 6.3.4](#).  
 Then, by [5.1.9b](#),  $F \cap K$  is closed.  
 Since  $F \cap K \subset K$ , then by [theorem 6.3.5](#),  $F \cap K$  is compact.

**Theorem 6.3.7: Nonempty  $\cap_{i=1}^n K_i \rightarrow$  nonempty  $\cap K_i$** 

For compact sets  $K_1, K_2, \dots \subset X$  where any intersection of finite  $K_i$  is nonempty, then  $\cap K_i$  is nonempty

**Proof**

Fix  $K_1$ . If there is a  $k \in K_1$  where  $k \in K_i$  for all  $i$ , then  $k \in \cap K_i$  so  $\cap K_i \neq \emptyset$ .  
 Suppose for every  $k \in K_1$ ,  $k \notin K_i$  for some  $i$ .  
 Then for every  $k \in K_1$ , there is a  $K_i$  such that  $k \notin K_i$  so  $k \in K_i^c$ .  
 Thus,  $K_2^c, K_3^c, \dots$  form an open cover for  $K_1$ . Since  $K_1$  is compact, there is a  $n$  where  $K_1 \subset K_{i_1}^c \cup \dots \cup K_{i_n}^c$ . But then,  $K_1 \cap K_{i_1} \cap \dots \cap K_{i_n} = \emptyset$  which is a contradiction.

**Corollary 6.3.8: Nonempty  $K_i$  where  $K_{i+1} \subset K_i \rightarrow$  nonempty  $\cap K_i$** 

If  $K_1, K_2, \dots$  is a sequence of nonempty compact sets such that  $K_{i+1} \subset K_i$ , then  $\cap K_i$  is nonempty

**Proof**

Since each  $K_i$  is nonempty and if  $i_1 < \dots < i_n$ , then  $K_{i_1} \cap \dots \cap K_{i_n} = K_{i_n}$  is nonempty, then by **theorem 6.3.7**,  $\cap K_i$  is nonempty.

**Theorem 6.3.9: Nonempty intervals  $I_n$  where  $I_{n+1} \subset I_n \rightarrow$  nonempty  $\cap I_n$** 

If  $I_1, I_2, \dots$  is a sequence of intervals in  $\mathbb{R}^1$  such that  $I_{n+1} \subset I_n$ , then  $\cap I_n$  is nonempty.

**Proof**

Let  $I_n = [a_n, b_n]$  and thus, each  $I_n$  is nonempty. If  $n_1 < \dots < n_m$ , then  $I_{n_1} \cap \dots \cap I_{n_m} = [a_{n_m}, b_{n_m}]$  is nonempty. Thus, by **theorem 6.3.7**,  $\cap I_n$  is nonempty.

**Theorem 6.3.10:  $p \in E'$  exists if infinite  $E \subset$  compact  $K$** 

If  $E$  is an infinite subset of compact set  $K$ , then  $E$  has a limit point in  $K$

**Proof**

If no  $p \in K$  is a  $p \in E'$ , then each  $p$  would have a neighborhood  $V_p$  contains at most  $p \in E$  if  $p \in E$ . Thus, there is no finite subcover that covers  $E$  and thus, there is no finite subcover that covers  $K$  since  $E \subset K$  which contradicts  $K$  is compact.

**Definition 6.3.11: K-cells**

The set of all  $x = (x_1, \dots, x_k) \in \mathbb{R}^k$  where  $x_i \in [a_i, b_i]$  for fixed  $a_i, b_i \in \mathbb{R}$

**Theorem 6.3.12: K-cells are compact**

Every k-cell is compact

**Proof**

Let k-cell  $I$  consists of all  $x = (x_1, \dots, x_k)$  where  $x_i \in [a_i, b_i]$  for fixed  $a_i, b_i \in \mathbb{R}$ .  
 Let  $\delta = \sqrt{\sum_{i=1}^k (b_i - a_i)^2}$ . Thus,  $|x - y| \leq \delta$  for  $x, y \in I$ .  
 Suppose there exists an open cover  $G_1, G_2, \dots$  of  $I$  which contain no finite subcover.  
 Let  $c_i = \frac{a_i + b_i}{2}$ . Then each interval splits into  $[a_i, c_i]$  and  $[c_i, b_i]$  for  $i \in [1, k]$  so there now exists  $2^k$  k-cells  $Q_i$  whose union is  $I$ .  
 At least one  $Q_i$  cannot be covered else  $I$  would be covered. Then subdivide  $Q_i$  as before and repeating the process so  $Q_{i+1} \subset Q_i$  and each are not covered.  
 However, there is a point  $x^* \in Q_{i_j}$  for all  $j$  such that  $N_r(x^*) \subset G$  so  $Q_{i_1}$  is covered which is a contradiction.

**Theorem 6.3.13: Heine-Borel Theorem**

If a set  $E \subset \mathbb{R}^k$  has one of the three properties, then it has the other two:

- (a)  $E$  is closed and bounded
- (b)  $E$  is compact
- (c) Every infinite subset of  $E$  has a limit point in  $E$

**Proof**

Suppose  $E$  is closed and bounded.

Then there exists a  $M \in \mathbb{R}$  and  $q \in \mathbb{R}^k$  such that  $d(p, q) < M$  for all  $p \in E$ .

Thus, there is a  $k$ -cell  $K = [-M + q_1, q_1 + M] \times \dots \times [-M + q_k, q_k + M]$  such that  $E \subset K$ . Then by [theorem 6.3.12](#),  $K$  is compact and thus by [theorem 6.3.5](#),  $E$  is compact so (a)  $\rightarrow$  (b).

Then by [theorem 6.3.10](#), any infinite subset of  $E$  has a limit point in  $E$  so (b)  $\rightarrow$  (c). Suppose  $E$  is not bounded.

Then there exists  $p \in E$  such that  $d(p, q) > M$  for any  $M \in \mathbb{R}$  and  $q \in \mathbb{R}^k$ .

Let  $S \subset E$  be such points  $p$ .

Then  $S$  is infinite else there is a maximal  $p$  and thus,  $p$  is bounded. Thus,  $S$  is infinite and contains no limit points in  $E$  since any  $d(p_1, p_2) > M$  which contradicts that every infinite subset of  $E$  has a limit point in  $E$ . Thus,  $E$  is bounded.

Suppose  $E$  is not closed.

Then there exists a  $p \in E'$ , but  $p \notin E$ . Since  $p$  is a limit point, then there is a  $q \in E$  such that  $\frac{1}{n+1} < d(q, p) < \frac{1}{n}$  for  $n = \{1, 2, \dots\}$ .

Let  $S \subset E$  be such points  $q$ .

Thus,  $p$  is the only limit point of  $S$  since for  $r < \frac{1}{n}$ , any  $N_r(q_i)$  contains no points of  $S$  other than  $q_i$  since  $d(q_i, q_j) > \frac{1}{n}$  for any  $q_1, q_2 \in S$ .

Thus,  $S$  is infinite, but the only  $p \in S'$  is  $p \notin E$  which contradicts that every infinite subset of  $E$  has a limit point in  $E$ . Thus,  $E$  is closed. So, (c)  $\rightarrow$  (a).

**Theorem 6.3.14: Weierstrass Theorem**

Every bounded infinite set  $E \subset \mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .

**Proof**

Since  $E$  is bounded, then there exists a  $k$ -cell  $K$  such that  $E \subset K$ . Since  $K$  is compact, then by [theorem 6.3.10](#),  $E$  has a limit point in  $K$  and thus, in  $\mathbb{R}^k$ .

## 7 Perfect and Connected Sets

### 7.1 Perfect Sets

#### Definition 7.1.1: Perfect Set

$E \subset X$  is perfect if  $E$  is closed and if every  $p \in E$  is  $p \in E'$

#### Theorem 7.1.2: Perfect sets are uncountable

Let  $P$  be a nonempty perfect set in  $\mathbb{R}^k$ . Then,  $P$  is uncountable.

#### Proof

Since  $P$  has limit points, then by [theorem 5.1.4](#),  $P$  is infinite.

Suppose  $P$  is countable. Then let  $x_1, x_2, \dots \in P$ .

Let  $V_i$  be a neighborhood of  $x_i$  where  $y \in V_i$  for any  $y \in \mathbb{R}^k$  such that  $|y - x_i| < r$ .

Thus, the  $\overline{V_i}$  is the set of all  $y \in \mathbb{R}^k$  such that  $|y - x_i| \leq r$ .

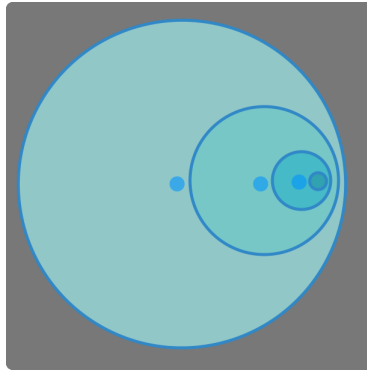
Since every  $x_i$  are limit points, then any  $V_i \cap P$  is not empty where there is a  $V_{i+1}$

(a)  $\overline{V_{i+1}} \subset V_i$

(b)  $x_i \notin \overline{V_{i+1}}$

(c)  $V_{i+1} \cap P$  is nonempty

Let  $K_i = \overline{V_i} \cap P$ . Since  $\overline{V_i}$  is closed and bounded, then by [theorem 6.3.11](#),  $\overline{V_i}$  is compact. Since  $x_i \notin K_{i+1}$ , then no  $x_i \in P$  is  $x_i \in \cap K_i$ . Since  $K_n \subset P$ , then  $\cap K_i$  is empty which contradicts [corollary 6.3.8](#) since each  $K_i$  is nonempty and  $K_{i+1} \subset K_i$ .



#### Corollary 7.1.3: $\mathbb{R}$ is not countable

Every interval  $[a, b]$  is uncountable. Thus,  $\mathbb{R}$  is uncountable.

#### Proof

Since  $[a, b]$  is closed and every  $p \in [a, b]$  is a limit point, then nonempty set  $[a, b]$  is perfect. Thus, by [theorem 7.1.2](#),  $[a, b]$  is uncountable.

**Definition 7.1.4: Cantor Set**

There exists perfect segments in  $\mathbb{R}^1$  which contain no segment.

Let  $E_0 = [0,1]$ .

For  $E_1$ , remove  $(\frac{1}{3}, \frac{2}{3})$ . Thus,  $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ .

For  $E_2$ , remove  $(\frac{1}{9}, \frac{2}{9})$  and  $(\frac{7}{9}, \frac{8}{9})$ . Thus,  $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ .

Continuing such a sequence, the set of compact sets  $E_n$  are such that:

(a)  $E_{n+1} \subset E_n$

(b)  $E_n$  is the union of  $2^n$  intervals each of length  $3^{-n}$ .

$P = \cap E_n$  is called the Cantor set.  $P$  is compact and nonempty.

Thus, any segment of form  $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$  where  $k, m \in \mathbb{Z}_+$  has no points in common with  $P$ . Since any segment  $(a, b)$  contain a segment of such a form since  $3^{-m} < \frac{b-a}{6}$ , then  $P$  contains no segment.

Let  $x \in P$  and segment  $S$  contain  $x$ . Let  $I_n$  be an interval of  $E_n$  containing  $x$ . Then choose a large enough  $n$  so  $I_n \subset S$ .

Let  $x_n$  be an endpoint of  $I_n$  where  $x_n \neq x$  and thus,  $x$  is a limit point. Since  $P$  is closed and every  $p \in P$  is  $p \in P'$ , then  $P$  is perfect.

**7.2 Connected Sets****Definition 7.2.1: Connected Set**

$A, B \subset X$  are separated if both  $A \cap \overline{B}$  and  $\overline{A} \cap B$  are empty.

$E \subset X$  is connected if  $E$  is not the union of two nonempty separated sets.

Separated sets are disjoint, but disjoint sets need not be separated.

**Theorem 7.2.2: All points between points in connected sets exists**

$E \subset \mathbb{R}^1$  is connected if and only if:

If  $x, y \in E$  and  $x < z < y$ , then  $z \in E$ .

**Proof**

If there exists  $x, y \in E$  and  $z \in (x, y)$  such that  $z \notin E$ , then  $E = A_z \cup B_z$  where  $A_z = E \cap (-\infty, z)$  and  $B_z = E \cap (z, \infty)$ .

Since  $x \in A_z$  and  $y \in B_z$ , then  $A$  and  $B$  are nonempty. Since  $A_z \subset (-\infty, z)$  and  $B_z \subset (z, \infty)$ , then  $A_z$  and  $B_z$  are separated. Thus,  $E$  is not connected.

Suppose  $E$  is not connected. Then, there are nonempty separated sets  $A$  and  $B$  such that  $A \cup B = E$ . Pick  $x \in A$ ,  $y \in B$  where  $x < y$ . Let  $z = \sup(A \cap [x, y])$ .

Since,  $z \in \overline{A}$  so  $z \notin B$ , then  $x \leq z < y$ . If  $z \notin A$ , then  $x < z < y$  so  $z \notin E$ .

If  $z \in A$ , then  $z \notin \overline{B}$  and thus, there exists a  $z_1$  such that  $z < z_1 < y$  and  $z_1 \notin B$ . Then,  $x < z_1 < y$  so  $z_1 \notin E$ .



## 8 Convergent and Cauchy Sequences

### 8.1 Convergent Sequences

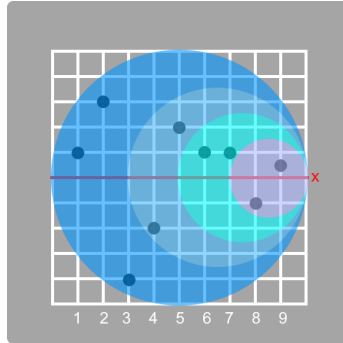
#### Definition 8.1.1: Convergent Sequence

A sequence  $\{x_n\}$  in metric space  $X$  converge if there is a  $x \in X$  such that:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for all  $n \geq N$ ,  $d(x_n, x) < \epsilon$

Then,  $\{x_n\}$  converges to  $x$ :  $\lim_{n \rightarrow \infty} x_n = x$

If  $\{x_n\}$  does not converge, then it diverges.



#### Example

- (a) Let  $x_n = \frac{1}{n}$  in  $\mathbb{R}^2$ . Then,  $\lim_{n \rightarrow \infty} x_n = 0$

#### Proof

For  $\epsilon > 0$ , there is a  $\frac{1}{N} < \epsilon$ . Then:

$$d(x_n, 0) = |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$$

- (b) Let  $x_n = (-1)^n + \frac{1}{n}$  in  $\mathbb{R}^2$ . Then,  $\{x_n\}$  diverges.

#### Proof

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (-1)^n + \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} (-1)^n$$

Since  $(-1)^n$  alternates between -1 and 1, then  $\{x_n\}$  diverges.

#### Theorem 8.1.2: A convergent sequence is unique and bounded

- (a)  $\{p_n\}$  converges to  $p \in X$  if and only if every  $N_r(p)$  contains  $p_n$  for all, but finitely many  $n$ .

#### Proof

Suppose  $p_n \rightarrow p$ . Then for  $N_\epsilon(p)$ , any  $q \in X$  such that  $d(q, p) < \epsilon$  is  $q \in N_\epsilon(p)$ . Since  $p_n \rightarrow p$ , there is a  $N$  such that for  $n \geq N$ ,  $d(p_n, p) < \epsilon$ .

Thus, for  $n \geq N$ ,  $p_n \in N_\epsilon(p)$ .

Suppose every  $N_r(p)$  contains  $p_n$  for all, but finitely many  $n$ .

For  $\epsilon > 0$ , let  $N_\epsilon(p)$  be the set of all  $q \in X$  such that  $d(p, q) < \epsilon$ . Thus, there exists a  $N$  such that  $p_n \in N_\epsilon(p)$  if  $n \geq N$ .

Thus,  $d(p_n, p) < \epsilon$  so  $p_n \rightarrow p$ .

- (b) If  $p, p' \in X$  and  $\{p_n\}$  converges to  $p$  and  $p'$ , then  $p = p'$ .

#### Proof

For  $\epsilon > 0$ , there exists  $N, N'$  such that:

$$d(p_n, p) < \frac{\epsilon}{2} \text{ for } n \geq N \quad d(p_n, p') < \frac{\epsilon}{2} \text{ for } n \geq N'$$

Then for  $n \geq \max(N, N')$ ,  $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$ .

Thus,  $p = p'$ .

- (c) If  $\{p_n\}$  converges, then  $\{p_n\}$  is bounded.

Proof

If  $\{p_n\} \rightarrow p$ , there is a  $N$  such that for  $n > N$ ,  $d(p_n, p) < 1$ .  
Let  $r = \max(d(p_1, p), \dots, d(p_N, p), 1)$ . Thus for all  $n$ ,  $d(p_n, p) \leq r$ .

- (d) If  $E \subset X$  and  $p \in E'$ , there is a  $\{p_n\}$  in  $E$  such that  $p = \lim_{n \rightarrow \infty} p_n$ .

Proof

Since  $p \in E'$ , then for each  $n \in \mathbb{Z}_+$ , there is a  $p_n \in E$  such that  $d(p_n, p) < \frac{1}{n}$ .  
For  $\epsilon > 0$ , there is a  $\frac{1}{N} < \epsilon$  so for  $n \geq N$ ,  $d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$ .  
Thus,  $p = \lim_{n \rightarrow \infty} p_n$ .

### Theorem 8.1.3: Arithmetic Operations for sequences

Suppose  $\{s_n\}, \{t_n\} \in \mathbb{C}$  where  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} t_n = t$ .

- (a)  $\lim_{n \rightarrow \infty} s_n + t_n = s + t$

Proof

For  $\epsilon > 0$ , there exists  $N_1, N_2$  such that  
 $|s_n - s| < \frac{\epsilon}{2}$  for  $n \geq N_1$        $|t_n - t| < \frac{\epsilon}{2}$  for  $n \geq N_2$   
 If  $N = \max(N_1, N_2)$ , then for  $n \geq N$ :  
 $|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$

- (b)  $\lim_{n \rightarrow \infty} cs_n = cs$  and  $\lim_{n \rightarrow \infty} c + s_n = c + s$

Proof

For  $\epsilon > 0$ , there exists a  $N$  such that  
 $|s_n - s| < \frac{\epsilon}{|c|}$  for  $n \geq N$   
 $|cs_n - cs| \leq |c| \cdot |s_n - s| < \epsilon$

- (c)  $\lim_{n \rightarrow \infty} s_n t_n = st$

Proof

Note  $s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$ .  
 For  $\epsilon > 0$ , there exists  $N_1, N_2$  such that  
 $|s_n - s| < \sqrt{\epsilon}$  for  $n \geq N_1$        $|t_n - t| < \sqrt{\epsilon}$  for  $n \geq N_2$   
 If  $N = \max(N_1, N_2)$ , then for  $n \geq N$ ,  $|(s_n - s)(t_n - t)| < \epsilon$ .  
 Thus,  $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$ .  
 $\lim_{n \rightarrow \infty} (s_n t_n - st) = \lim_{n \rightarrow \infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$   
 $= 0 + t \cdot 0 + s \cdot 0 = 0$

- (d)  $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$  where  $s_n, s \neq 0$

Proof

Choose  $m$  such that  $|s_n - s| < \frac{1}{2}|s|$  if  $n \geq m$  so  $|s_n| > \frac{1}{2}|s|$  for  $n \geq m$ .  
 For  $\epsilon > 0$ , there is a  $N > m$  such that for  $n \geq N$ ,  $|s_n - s| < \frac{1}{2}|s|^2\epsilon$ .  
 Thus, for  $n \geq N$ ,  $|\frac{1}{s_n} - \frac{1}{s}| = \frac{|s_n - s|}{|s_n s|} < \frac{2}{|s|^2}|s_n - s| < \epsilon$ .

**Theorem 8.1.4: Extension to  $\mathbb{R}^k$** 

- (a) Suppose  $x_n \in \mathbb{R}^k$  and  $x_n = (\alpha_{n_1}, \dots, \alpha_{n_k})$ . Then  $\{x_n\}$  converges to  $x = (\alpha_1, \dots, \alpha_k)$  if and only if  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$  for  $i \in [1, k]$ .

**Proof**

Suppose  $\{x_n\}$  converges to  $x = (\alpha_1, \dots, \alpha_k)$ .

Since for any  $i \in [1, k]$ :

$$|\alpha_{n_i} - \alpha_i| \leq \sqrt{|\alpha_{n_1} - \alpha_1|^2 + \dots + |\alpha_{n_k} - \alpha_k|^2} = |x_n - x| < \epsilon.$$

Then,  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ .

Suppose  $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$  for  $i \in [1, k]$ .

Then for  $\epsilon > 0$ , there is an  $N$  such that for  $n \geq N$ :

$$|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}} \text{ for } i \in [1, k]$$

$$|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot \left(\frac{\epsilon}{\sqrt{k}}\right)^2} = \epsilon$$

- (b) Suppose  $\{x_n\}, \{y_n\} \in \mathbb{R}^k$  and  $\{\beta_n\} \in \mathbb{R}$  and  $x_n \rightarrow x, y_n \rightarrow y, \beta_n \rightarrow \beta$ .  
 $\lim_{n \rightarrow \infty} x_n + y_n = x + y \quad \lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y \quad \lim_{n \rightarrow \infty} \beta_n x_n = \beta x$

**Proof**

By part a, then  $\lim_{n \rightarrow \infty} x_{n_i} + y_{n_i} = x_i + y_i$  so  $\{x_n + y_n\} \rightarrow x + y$ .

Also,  $\lim_{n \rightarrow \infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$  so  $\{x_n \cdot y_n\} \rightarrow x \cdot y$ .

Also,  $\lim_{n \rightarrow \infty} \beta_i x_{n_i} = \beta_i x_i$  so  $\{\beta_n x_n\} \rightarrow \beta x$ .

## 8.2 Subsequences

**Definition 8.2.1: Subsequence**

For sequence  $\{p_n\}$ , let  $\{n_k\} \in \mathbb{Z}_+$  where  $n_k < n_{k+1}$ .

Then  $\{p_{n_k}\}$  is a subsequence of  $\{p_n\}$ .

If  $\{p_{n_k}\}$  converges, then its limit is called a subsequential limit.

**Theorem 8.2.2:  $\{p_n\} \rightarrow p \Leftrightarrow$  Every  $\{p_{n_k}\} \rightarrow p$** 

$\{p_n\}$  converges to  $p$  if and only if every subsequence converges to  $p$

**Proof**

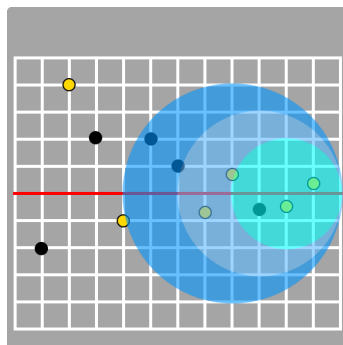
Suppose  $\{p_n\}$  converges to  $p$ .

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $d(p_n, p) < \epsilon$ .

Let  $\{p_{n_k}\} \subset \{p_n\}$ . Then for  $n_k \geq N$ ,  $|p_{n_k} - p| < \epsilon$ . Thus,  $\{p_{n_k}\} \rightarrow p$ .

Suppose every subsequence converges to  $p$ .

Since  $\{p_n\}$  is a subsequence of itself, then  $\{p_n\}$  converges to  $p$ .



**Theorem 8.2.3:**  $\{p_n\}$  in compact space have  $\{p_{n_k}\} \rightarrow p$

- (a) If  $\{p_n\}$  is a sequence in a compact metric space  $X$ , then some subsequence converges to  $p \in X$ .

**Proof**

Let  $E$  be the range of  $\{p_n\}$ .

If  $E$  is finite, there is a  $p \in E$  and sequence  $\{n_k\}$  with  $n_k < n_{k+1}$  such that  $p_{n_1} = p_{n_2} = \dots = p$ . Thus,  $\{p_{n_k}\} \rightarrow p$ .

If  $E$  is infinite, then by **theorem 6.3.10**, then there exists a  $p \in E'$ .

Then there are  $n_k$  such that  $d(p_{n_k}, p) < \frac{1}{k}$ . Thus,  $\{p_{n_k}\} \rightarrow p$ .

- (b) Every bounded sequence in  $\mathbb{R}^k$  contains a convergent subsequence.

**Proof**

Let  $E$  be a bounded sequence in  $\mathbb{R}^k$ . Since  $E \cup E'$  is bounded and closed, then by **theorem 6.3.13**,  $E \cup E'$  is compact.

Thus by part a,  $E$  contains a convergent subsequence.

**Theorem 8.2.4:** The set of subsequential limits is closed

The subsequential limits of  $\{p_n\}$  in metric space  $X$  form a closed subset of  $X$

**Proof**

Let  $E$  be the range of the set of all subsequential limits of  $\{p_n\}$ .

If  $E$  is empty, then  $E$  is closed. If  $E$  is finite, then  $E'$  is empty so  $E$  is closed.

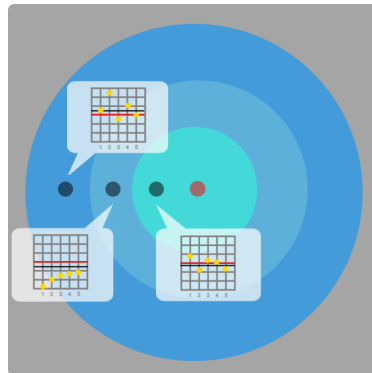
Suppose  $E$  is infinite. Then, let  $q \in E'$ .

Since  $q \in E'$ , there is a  $x \in E$  where  $d(x, q) < \frac{\epsilon}{2}$ .

Since  $x \in E$ , there is a  $\{p_{n_k}\} \rightarrow x$  so there is a  $N$  such that for  $n \geq N$ ,  $d(p_{n_k}, x) < \frac{\epsilon}{2}$ .

Thus,  $d(p_{n_k}, q) \leq d(p_{n_k}, x) + d(x, q) < \epsilon$  so  $q$  is a subsequential limit of  $\{p_n\}$ .

Thus,  $q \in E$  so  $E$  is closed.



## 8.3 Cauchy Sequences

### Definition 8.3.1: Metric Spaces

Sequence  $\{p_n\} \in X$  is a Cauchy sequence if:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for all  $n, m \geq N$ ,  $d(p_n, p_m) < \epsilon$

Let nonempty  $E \subset X$  and  $S \subset \mathbb{R}$  of  $d(p, q)$  where  $p, q \in E$ .

Let  $\sup(S) = \text{diam}(E)$ . If  $\{p_n\} \in X$ , and  $p_N, p_{N+1}, \dots \in E_N$ , then  $\{p_n\}$  is a Cauchy sequence if and only if  $\lim_{N \rightarrow \infty} \text{diam}(E_N) = 0$ .

**Theorem 8.3.2: Cauchy sequences and its closure have the same diam**

- (a) If
- $\overline{E} \subset X$
- , then
- $\text{diam}(\overline{E}) = \text{diam}(E)$
- .

**Proof**

Since  $E \subset \overline{E}$ , then  $\text{diam}(E) \leq \text{diam}(\overline{E})$ .

For  $\epsilon > 0$ , let  $p, q \in E$ .

Thus, there are  $p', q' \in E$  such that  $d(p', p) < \epsilon$  and  $d(q', q) < \epsilon$ . Thus:  
 $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \leq 2\epsilon + \text{diam}(E)$ .

Thus,  $\text{diam}(\overline{E}) \leq 2\epsilon + \text{diam}(E)$  so  $\text{diam}(\overline{E}) = \text{diam}(E)$ .

- (b) If
- $K_n$
- is a sequence of compact sets of
- $X$
- such that
- $K_{n+1} \subset K_n$
- and
- $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$
- , then
- $\bigcap K_n$
- consist of only one point.

**Proof**

Let  $K = \bigcap K_n$ . Since  $K_n$  is a sequence of compact sets, then by [corollary 6.3.8](#),  $K$  is nonempty.

If  $K$  contains more than one point, then  $\text{diam}(K) > 0$ . But since  $K \subset K_n$ , then  $\text{diam}(K) \leq \text{diam}(K_n)$  which contradicts that  $\text{diam}(K_n) \rightarrow 0$ .

**Theorem 8.3.3: Convergent sequences are cauchy sequences**

- (a) Every convergent sequence is a Cauchy sequence.

**Proof**

If  $p_n \rightarrow p$  and  $\epsilon > 0$ , there is a  $N$  such that for all  $n \geq N$ ,  $d(p, p_n) < \frac{\epsilon}{2}$ . Thus, for  $m, n \geq N$ :

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon.$$

Thus,  $\{p_n\}$  is a Cauchy sequence.

- (b) If
- $\{p_n\}$
- is a Cauchy sequence in compact metric space
- $X$
- , then
- $\{p_n\}$
- converges to some
- $p \in X$
- .

**Proof**

Let  $\{p_n\}$  be a Cauchy sequence in compact space  $X$ .

Let  $p_N, p_{N+1}, \dots \in E_N$ .

Since  $\{p_n\}$  is a Cauchy sequence, then  $\lim_{N \rightarrow \infty} \text{diam}(\overline{E_N}) = 0$ . Since  $\overline{E_N}$  is closed in compact  $X$ , then by [theorem 6.3.5](#),  $\overline{E_N}$  is compact.

Since  $E_{N+1} \subset E_N$ , then  $\overline{E_{N+1}} \subset \overline{E_N}$  and thus, by [theorem 8.3.2b](#), then there is a unique  $p \in \overline{E_N}$  for every  $N$ .

Since  $p \in \overline{E_N}$ , then  $d(p, q) < \epsilon$  for every  $q \in \overline{E_N}$  so every  $q \in E_N$ .

Then for  $\epsilon > 0$ , there is a  $N_0$  such that for  $N \geq N_0$ ,  $\text{diam}(\overline{E_N}) < \epsilon$ .

Thus,  $d(p_n, p) < \epsilon$  for  $n \geq N_0$  so  $\{p_n\} \rightarrow p$ .

- (c) In
- $\mathbb{R}^k$
- , every Cauchy sequence converges.

**Proof**

Let  $\{x_n\}$  be a Cauchy sequence in  $\mathbb{R}^k$ . Let  $x_N, x_{N+1}, \dots \in E_N$ .

Then for some  $N$ ,  $\text{diam}(E_N) < 1$ . Thus, the range of  $\{x_n\} = E_N \cup \{x_1, \dots, x_{N-1}\}$ . Thus,  $\{x_n\}$  is bounded.

Thus, the  $\{x_n\}$  is closed and bounded so by [theorem 6.3.13](#),  $\overline{\{x_n\}}$  is compact.

Thus, by part b,  $\{x_n\}$  converges to some  $p \in \mathbb{R}^k$ .

**Definition 8.3.4: Complete**

A metric space where every Cauchy sequence converges is complete.

Thus, by **theorem 8.3.3**, all compact and Euclidean spaces are complete.

**Definition 8.3.5: Monotonic Sequences**

A sequence  $\{s_n\}$  of real numbers is:

- (a) monotonically increasing if  $s_n \leq s_{n+1}$
- (b) monotonically decreasing if  $s_n \geq s_{n+1}$

**Theorem 8.3.6: Monotonic sequences converge if bounded**

Suppose  $\{s_n\}$  is monotonic. Then  $\{s_n\}$  converges if and only if it is bounded

**Proof**

Suppose  $s_n \leq s_{n+1}$ . Let  $E$  be the range of  $\{s_n\}$ .

Suppose  $\{s_n\}$  is bounded.

Let  $s = \sup(E)$  so  $s_n \leq s$ . For every  $\epsilon > 0$ , there is a  $N$  such that  $s - \epsilon < s_N \leq s$  else  $s - \epsilon$  would be an upper bound of  $E$  which contradicts  $s = \sup(E)$ .

Since  $\{s_n\}$  increases, then for  $n \geq N$ ,  $s - \epsilon < s_N \leq s_n \leq s$  so  $\{s_n\} \rightarrow s$ .

Suppose  $\{s_n\}$  converges to  $s$ .

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $s - \epsilon < s_N \leq s_n \leq s$ .

Thus,  $\{s_n\}$  is bounded from above.

Suppose  $s_n \geq s_{n+1}$ . Let  $E$  be the range of  $\{s_n\}$ .

Suppose  $\{s_n\}$  is bounded.

Let  $s = \inf(E)$  so  $s_n \geq s$ . For every  $\epsilon > 0$ , there is a  $N$  such that  $s \leq s_N < s + \epsilon$  else  $s + \epsilon$  would be a lower bound of  $E$  which contradicts  $s = \inf(E)$ .

Since  $\{s_n\}$  decreases, then for  $n \geq N$ ,  $s \leq s_n \leq s_N < s + \epsilon$  so  $\{s_n\} \rightarrow s$ .

Suppose  $\{s_n\}$  converges to  $s$ .

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n \geq N$ ,  $s \leq s_n \leq s_N < s + \epsilon$ .

Thus,  $\{s_n\}$  is bounded from below.

## 9 Limits and Special Sequences

### 9.1 Upper and Lower Limits

#### Definition 9.1.1: Infinite limits

Let  $\{s_n\}$  be a sequence of real numbers such that:

For every real  $M$ , there is a  $N \in \mathbb{Z}$  such that for  $n \geq N$ ,  $s_n \geq M$ .

Then,  $s_n \rightarrow +\infty$ .

For every real  $M$ , there is a  $N \in \mathbb{Z}$  such that for  $n \geq N$ ,  $s_n \leq M$ .

Then,  $s_n \rightarrow -\infty$ .

#### Definition 9.1.2: Upper and Lower Limits

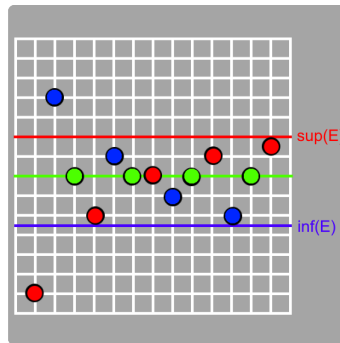
Let  $\{s_n\} \subset \mathbb{R}$  and  $E$  contain all subsequential limits of  $\{s_n\}$  plus possibly  $\pm\infty$ .

Then, the upper limit of  $\{s_n\}$ :

$$s^* = \sup(E) \quad \lim_{n \rightarrow \infty} \sup(s_n) = s^*$$

Then, the lower limit of  $\{s_n\}$ :

$$s_* = \inf(E) \quad \lim_{n \rightarrow \infty} \inf(s_n) = s_*$$



#### Theorem 9.1.3: Upper and Lower limits are unique

Let  $\{s_n\}$  be a sequence of real numbers. Let  $E$  be the set of subsequential limits and  $s^*$  be the upper limit of  $\{s_n\}$ . Then:

(a)  $s^* \in E$

##### Proof

If  $s^* = +\infty$ , then there is a  $\{s_{n_k}\} \rightarrow +\infty$  so  $E$  is not bounded above.

If  $s^* \in \mathbb{R}$ , then  $E$  is bounded above so  $s^* \in E'$ .

Then by [theorem 8.2.4](#),  $s^* \in E$ .

If  $s^* = -\infty$ , then there are no subsequential limits in  $E$ . Thus, for every  $M$ , there is a  $N$  such that for  $n \geq N$ ,  $s_n \leq M$  so  $-\infty \in E$ .

(b) If  $x > s^*$ , there is a  $N$  such that for  $n \geq N$ ,  $s_n < x$

##### Proof

Suppose there is a  $x > s^*$  such that  $s_n \geq x$  for infinitely many  $n$ .

Then, there is a  $y \in E$  where  $y \geq x > s^*$  which contradicts  $s^* = \sup(E)$ .

(c)  $s^*$  is the only number that satisfies (a) and (b)

##### Proof

Suppose  $p, q$  satisfy part a and b where  $p < q$ . Choose  $x$  where  $p < x < q$ . Since  $p$  satisfies b, then  $s_n < x$  for  $n \geq N$ . Thus,  $x$  is an upper bound for  $E$  so  $q \notin E$  since  $q > x$  contradicting that  $q$  satisfies part a.

The same properties are analogous for  $s_*$ .

**Theorem 9.1.4: Inf & Sup of  $s_n \leq t_n$** 

If  $s_n \leq t_n$  for  $n \geq$  fixed  $N$ , then

$$\lim_{n \rightarrow \infty} \inf(s_n) \leq \lim_{n \rightarrow \infty} \inf(t_n)$$

$$\lim_{n \rightarrow \infty} \sup(s_n) \leq \lim_{n \rightarrow \infty} \sup(t_n)$$

**Proof**

Let  $E_1$  be the set of extended reals  $x$  such that  $\{s_{n_k}\} \rightarrow x$  for some  $\{s_{n_k}\}$ .

Let  $E_2$  be the set of extended reals  $y$  such that  $\{t_{n_k}\} \rightarrow y$  for some  $\{s_{n_k}\}$ .

Let  $s^* = \sup(E_1)$ ,  $s_* = \inf(E_1)$ ,  $t^* = \sup(E_2)$ , and  $t_* = \inf(E_2)$ .

Since there is a  $N$  such that  $s_n \leq t_n$  for  $n \geq N$ , then:

$$x \leftarrow \{s_N, s_{N+1}, \dots\} \leq \{t_N, t_{N+1}, \dots\} \rightarrow y$$

Thus, for  $n \geq N$ ,  $\inf(s_n) \leq \inf(t_n)$  and  $\sup(s_n) \leq \sup(t_n)$ .

**9.2 Special Sequences****Theorem 9.2.1: Special sequences**

- (a) If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

**Proof**

For  $\epsilon > 0$ , let  $N > \sqrt[p]{\frac{1}{\epsilon}}$ . Then for  $n \geq N$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}} = \epsilon$

- (b) If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$

**Proof**

If  $p > 1$ , then let  $x_n = \sqrt[p]{p} - 1 > 0$ .

$$p = (x_n + 1)^n = x_n^n + nx_n^{n-1} + \dots + nx_n + 1 \geq nx_n + 1$$

Thus,  $0 < x_n \leq \frac{p-1}{n}$  so  $\{x_n\} \rightarrow 0$  and thus,  $\{\sqrt[p]{p}\} \rightarrow 1$ .

If  $p = 1$ , then  $\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} 1 = 1$ .

If  $0 < p < 1$ , then  $\frac{1}{p} > 1$ . From the proof above for  $p > 1$ ,  $\{\sqrt[\frac{1}{p}]{\frac{1}{p}}\} \rightarrow 1$ .

Thus,  $\{\frac{1}{\sqrt[p]{p}}\} \rightarrow 1$  so  $\{\sqrt[p]{p}\} \rightarrow 1$ .

- (c)  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

**Proof**

Let  $x_n = \sqrt[n]{n} - 1 \geq 0$ . Then,  $n = (x_n + 1)^n \geq \frac{n(n-1)}{2} x_n^2$ .

Thus,  $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$  so  $\{x_n\} \rightarrow 0$  and thus,  $\{\sqrt[n]{n}\} \rightarrow 1$ .

- (d) If  $p > 0$  and  $\alpha \in \mathbb{R}$ , then  $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$

**Proof**

Let  $k \in \mathbb{Z}$  such that  $k > \alpha$  and  $k > 0$ . For  $n > 2k$ :

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

Thus,  $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$ .

Since  $\alpha - k < 0$ , then  $\{n^{\alpha-k}\} \rightarrow 0$  so  $\{\frac{n^\alpha}{(1+p)^n}\} \rightarrow 0$ .

- (e) If  $|x| < 1$ , then  $\lim_{n \rightarrow \infty} x^n = 0$

**Proof**

From part d, let  $\alpha = 0$ .

Thus,  $\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = 0$  and since  $p > 0$ , then  $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$ .

Also,  $-\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = \lim_{n \rightarrow \infty} \frac{-1}{(1+p)^n} = 0$  so  $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n > -1$ .



## 10 Series and Convergence Tests

### 10.1 Series

#### Definition 10.1.1: Series

For sequence  $\{a_n\}$ , define  $\sum_{n=p}^q a_n = a_p + a_{p+1} + \dots + a_q$ .

Then associate  $\{a_n\}$  with a sequence  $\{s_n\}$  such that  $s_n = \sum_{k=1}^n a_k$ .

Then  $\{s_n\}$  is a series with partial sums  $s_n$ .

If  $\{s_n\} \rightarrow s$ , then  $\sum_{n=1}^{\infty} a_n = s$  is the sum of the convergent series.

Note  $a_1 = s_1$  and  $a_n = s_n - s_{n-1}$ .

#### Theorem 10.1.2: Cauchy Criterion for series

$\sum a_n$  converges if and only if:

For every  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$

#### Proof

Suppose  $\sum_{k=1}^n a_k$  converges.

Then by [theorem 8.3.3a](#),  $\sum_{k=1}^n a_k$  is a Cauchy sequence.

Then for  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ :

$$d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = |\sum_{k=n}^m a_k| \leq \epsilon$$

Suppose for every  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$ .

$$|\sum_{k=n}^m a_k| = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) \leq \epsilon$$

Thus,  $\sum_{k=1}^n a_k$  is a Cauchy sequence and thus, convergent.

#### Theorem 10.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \rightarrow 0$

If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

#### Proof

Since  $\sum a_n$  converges, then by [theorem 10.1.2](#), for  $\epsilon > 0$ , there is a  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m a_k| \leq \epsilon$ . Then if  $m = n \geq N$ ,  $|\sum_{k=n}^m a_k| = |a_n| \leq \epsilon$  so  $\{a_n\} \rightarrow 0$ .

#### Example

$\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \dots + \frac{1}{8}) + (\frac{1}{9} + \dots) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Thus,  $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$  which is unbounded and thus, not convergent.

#### Theorem 10.1.4: Convergent series $\Leftrightarrow$ Bounded sequence

A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.

#### Proof

Suppose  $\sum a_n$  converges where  $a_n \geq 0$ .

Since  $a_n \geq 0$ , then  $\{s_n\}$  is monotonic so by [theorem 8.3.6](#),  $\{s_n\}$  is bounded above.

Suppose  $\{s_n\}$  is bounded where  $a_n \geq 0$ .

Since  $\{s_n\}$  is monotonic and bounded, then by [theorem 8.3.6](#),  $\{s_n\}$  converges.

**Theorem 10.1.5: Comparison Test**

- (a) If
- $|a_n| \leq c_n$
- for
- $n \geq N_0$
- and
- $\sum c_n$
- converges, then
- $\sum a_n$
- converges.

**Proof**

For  $\epsilon > 0$ , there exists a  $N \geq N_0$  such that for  $m \geq n \geq N$ ,  $\sum_{k=n}^m c_k \leq \epsilon$ .  

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \epsilon$$
Thus,  $\sum a_n$  converges.

- (b) If
- $a_n \geq d_n \geq 0$
- for
- $n \geq N_0$
- and
- $\sum d_n$
- diverges, then
- $\sum a_n$
- diverges.

**Proof**

Suppose  $\sum a_n$  converges.  
Then from part a,  $\sum d_n$  converges which contradicts that  $\sum a_n$  diverges.  
Thus,  $\sum a_n$  diverges.

**10.2 Series of Nonnegative Terms****Theorem 10.2.1: Infinite Geometric Series**If  $x \in [0, 1)$ , then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If  $x \geq 1$ , the series diverges.**Proof**

If  $x \neq 1$ , then using the geometric series  $s_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$ . Let  $n \rightarrow \infty$ .  
If  $x \in [0, 1)$ , then by [theorem 9.2.1e](#),  $s_n = \frac{1}{1-x} (1 - x^{n+1}) = \frac{1}{1-x} (1 - 0) = \frac{1}{1-x}$ .  
Also, by [theorem 9.2.1e](#), if  $x \geq 1$ , then the series diverges.

**Theorem 10.2.2: Cauchy's Convergence Criterion**Suppose  $0 \leq a_{i+1} \leq a_i$ .Then the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots \text{ converges.}$$

**Proof**

Let  $s_n = a_1 + a_2 + \dots + a_n$  and  $t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$ . For  $n < 2^k$ :

$$\begin{aligned} s_n &\leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \dots + 2^k a_{2^k} = t_k \end{aligned}$$

By [comparison test](#), if  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges. For  $n > 2^k$ :

$$\begin{aligned} s_n &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k \end{aligned}$$

By [comparison test](#), if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  converges.

**Theorem 10.2.3: P-series** $\sum \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ **Proof**

If  $p \leq 0$ , then by [theorem 10.1.3](#),  $\sum \frac{1}{n^p}$  diverges.  
If  $p > 0$ , then by [theorem 10.2.2](#),  $\sum \frac{1}{n^p}$  converges only if  $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$  converges.  
Since  $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{(1-p)k}$ , then by [theorem 10.2.1](#),  $\sum_{k=0}^{\infty} 2^{k(1-p)}$  converges if  $2^{1-p} < 1$  so if  $1-p < 0$  so  $p > 1$ .

**Theorem 10.2.4: Log P-series**

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof**

Since  $\frac{1}{n(\log(n))^p}$  decreases, then by **theorem 10.2.2**,  
 $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$  converges.  
 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k}$   
 Then by **theorem 10.2.3**,  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .  
 Thus,  $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$  converges if  $p > 1$  and diverges and  $p \leq 1$ .

**Corollary 10.2.5: Log P-series extended**

$\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$

**Proof**

From **theorem 10.2.4**, replace  $n = \log(n)$  and multiplying by  $\frac{1}{n} \rightarrow \frac{1}{n \log(n)(\log(\log(n)))^p}$ .  
 Since  $\frac{1}{n \log(n)(\log(\log(n)))^p}$  decreases, by **theorem 10.2.2**  $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)(\log(\log(2^k)))^p}$ .  
 $\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k \log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$   
 Since  $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$  converges by **theorem 10.2.4**,  $\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$  converges.

**10.3 The Number e****Definition 10.3.1: Summation equivalence to e**

$$s_m = \sum_{n=0}^m \frac{1}{n!} = 1 + \sum_{n=1}^m \frac{1}{n!} < 1 + \sum_{n=1}^m \frac{1}{2^{n-1}} < 3$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

**Theorem 10.3.2: Limit equivalence to e**

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

**Proof**

Let  $s_n = \sum_{k=0}^n \frac{1}{k!}$  and  $t_n = \left(1 + \frac{1}{n}\right)^n$ . Using the binomial theorem:  
 $t_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)$   
 Thus,  $t_n \leq s_n$  so  $\lim_{n \rightarrow \infty} \sup(t_n) \leq e$ .  
 If  $n \geq m$ , then  $t_n \geq \sum_{k=0}^m \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)$ .  
 As  $n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \inf(t_n) \geq \sum_{k=0}^m \frac{1}{k!} = s_m$ . As  $m \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} \inf(t_n) \geq e$ .

**Theorem 10.3.3: Rapidity of convergence of e**

$$0 < e - s_n < \frac{1}{n!n}$$

**Proof**

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) = \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!n}$$

**Theorem 10.3.4: e is irrational**

e is irrational

**Proof**

Suppose r is rational. Then let  $e = \frac{p}{q}$  for  $p, q \in \mathbb{Z}_+$ .  
 Thus, by **theorem 10.3.3**,  $0 < e - s_q < \frac{1}{q!q}$  so  $0 < q!(e - s_q) < \frac{1}{q}$ .  
 Since  $e = \frac{p}{q}$ , then  $q!e$  is an integer and  $q!s_q = q!(1 + \frac{1}{2!} + \dots + \frac{1}{q!})$  is an integer.  
 Thus,  $q!(e - s_q)$  is an integer which is between 0 and  $\frac{1}{q}$  and thus, a contradiction.

**10.4 Root and Ratio Tests****Theorem 10.4.1: Root Test**

- For  $\sum a_n$ , let  $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|})$ .
- (a) If  $\alpha < 1$ ,  $\sum a_n$  converges
  - (b) If  $\alpha > 1$ ,  $\sum a_n$  diverges
  - (c) If  $\alpha = 1$ , unclear

**Proof**

If  $\alpha < 1$ , choose  $\beta$  such that  $\beta \in (\alpha, 1)$  and  $N \in \mathbb{Z}$  such that  $\sqrt[n]{|a_n|} < \beta$  for  $n \geq N$ .  
 Since  $\beta \in (0, 1)$ , then by **theorem 10.2.1**,  $\sum \beta^n$  converges. Then by the **comparison test**,  $\sum a_n$  converges.  
 If  $\alpha > 1$ , then there is a  $a_{n_k}$  such that  $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$ .  
 Thus,  $|a_n| > 1$  for infinitely many  $n$  so by **theorem 10.1.3**,  $\sum a_n$  doesn't converge.  
 $\sum \frac{1}{n}$ ,  $\sum \frac{1}{n^2}$  have  $\alpha = 1$ , but  $\sum \frac{1}{n}$  diverges and  $\sum \frac{1}{n^2}$  converges by **theorem 10.2.3**.

**Theorem 10.4.2: Ratio Test**

- (a)  $\sum a_n$  converges if  $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$
- (b)  $\sum a_n$  diverges if  $|\frac{a_{n+1}}{a_n}| \geq 1$  for all  $n \geq n_0$  for  $n_0 \in \mathbb{Z}$

**Proof**

If  $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$ , there is a  $\beta < 1$  and  $N$  such that for  $n \geq N$ ,  $|\frac{a_{n+1}}{a_n}| < \beta$ .  
 Then  $|a_{N+1}| < \beta|a_N|$  so  $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$ .  
 Thus,  $|a_{N+p}| < \beta^p|a_N|$  so  $|a_n| < |a_N|\beta^{-N}\beta^n$ .  
 Thus, by the **comparison test**,  $\sum a_n$  converges.  
 If  $|a_{n+1}| \geq |a_n| > 0$  for  $n \geq n_0$ , then by **theorem 10.1.3**,  $\sum a_n$  diverges.

**Theorem 10.4.3: Ratio convergence  $\rightarrow$  Root convergence**

$$\lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$$

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$$

**Proof**

Let  $\alpha = \lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n})$ . If  $\alpha = -\infty$ , then  $-\infty \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$  holds true.  
 If  $\alpha$  is finite, there is a  $\beta \leq \alpha$  and  $N$  such that for  $n \geq N$ ,  $\frac{c_{n+1}}{c_n} \geq \beta$  so  $c_{N+p} \geq \beta^p c_N$ .  
 Then,  $c_n \geq c_N \beta^{-N} \beta^n$  so  $\sqrt[n]{c_n} \geq \sqrt[n]{c_N \beta^{-N} \beta^n} = \sqrt[n]{c_N} \beta^{-N/n} \beta$ . Thus,  $\lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$ .  
 Let  $\alpha = \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$ . If  $\alpha = \infty$ , then  $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \infty$  holds true.  
 If  $\alpha$  is finite, there is a  $\beta \geq \alpha$  and  $N$  such that for  $n \geq N$ ,  $\frac{c_{n+1}}{c_n} \leq \beta$  so  $c_{N+p} \leq \beta^p c_N$ .  
 Then,  $c_n \leq c_N \beta^{-N} \beta^n$  so  $\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N} \beta^n} = \sqrt[n]{c_N} \beta^{-N/n} \beta$ . Thus,  $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$ .

## 10.5 Power Series

### Definition 10.5.1: Power series

For a sequence  $\{c_n\} \in \mathbb{C}$ , the series  $\sum_{n=0}^{\infty} c_n z^n$  is a power series.  $c_n$  are the coefficients and  $z \in \mathbb{C}$ .

### Theorem 10.5.2: Radius of Convergence

For power series  $\sum c_n z^n$ , let  $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$  and  $R = \frac{1}{\alpha}$ . Then  $\sum c_n z^n$  converges if  $|z| < R$  and diverges if  $|z| > R$ .

#### Proof

Let  $a_n = c_n z^n$ . Using the **root test**,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|}) &= \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n z^n|}) \\ &= |z| \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R} \end{aligned}$$

Thus,  $\sum c_n z^n$  converges if  $\frac{|z|}{R} < 1$  and diverges if  $\frac{|z|}{R} > 1$

## 10.6 Summation By Parts

### Theorem 10.6.1: Summation by parts

For sequences  $\{a_n\}$ ,  $\{b_n\}$ , let  $A_n = \sum_{k=0}^n a_k$ . Then for  $0 \leq p \leq q$ :

$$\sum_{n=p}^q a_n b_n = (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

#### Proof

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p}^q A_{n-1} b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p \\ &= (\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p \end{aligned}$$

### Theorem 10.6.2: Conditions for convergent $\sum a_n b_n$

Suppose for  $\{a_n\}$ ,  $\{b_n\}$ :

- partial sums  $A_n$  of  $\sum a_n$  form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n \rightarrow \infty} b_n = 0$

Then  $\sum a_n b_n$  converges.

#### Proof

Since  $\{A_n\}$  is bounded,  $|A_n| \leq M$  for all  $n$ .

Since  $\{b_n\}$  is monotonically decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ , then for  $\epsilon > 0$ , there is a  $N$  such that  $b_N \leq \frac{\epsilon}{2M}$ . Then for  $N \leq p \leq q$ :

$$\begin{aligned} |\sum_{n=p}^q a_n b_n| &= |(\sum_{n=p}^{q-1} A_n(b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p| \\ &\leq M |\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p| = 2M b_p \leq 2M b_N \leq \epsilon \end{aligned}$$

**Corollary 10.6.3: Convergent series of Alternating Sequences**

Suppose for  $\{c_n\}$ :

- $|c_i| \geq |c_{i+1}|$
- $c_{2i-1} \geq 0$  and  $c_{2i} \leq 0$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then  $\sum c_n$  converges.

**Proof**

From **theorem 10.6.2**, let  $a_n = (-1)^{n+1}$  and  $b_n = |c_n|$ .

**Corollary 10.6.4: Convergent power series at radius of convergence**

Suppose for  $\{c_n\}$ :

- Radius of convergence of  $\sum c_n z^n$  is 1
- $c_i \geq c_{i+1}$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then  $\sum c_n z^n$  converges at every point where  $|z| = 1$  except possibly  $z = 1$ .

**Proof**

From **theorem 10.6.2**, let  $a_n = z^n$  and  $b_n = c_n$ .  
 $A_n$  of  $\sum a_n$  form a bounded sequence since  $|A_n| = |\sum_0^n z^n| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|}$ .

**10.7 Absolute Convergence****Definition 10.7.1: Absolute convergence**

$\sum a_n$  converges absolutely if  $\sum |a_n|$  converges.

If  $\sum a_n$  converges, but  $\sum |a_n|$  diverges, then  $\sum a_n$  converges non-absolutely.

**Theorem 10.7.2: Absolute convergence  $\rightarrow$  convergence**

If  $\sum a_n$  converges absolutely, then  $\sum a_n$  converges

**Proof**

Since  $\sum a_n$  converges absolutely, then for every  $\epsilon > 0$ , there is an integer  $N$  such that for  $m \geq n \geq N$ ,  $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$ .  
 Thus,  $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$  so  $\sum a_n$  converges.

**10.8 Addition & Multiplication of Series****Theorem 10.8.1: Addition and Scalar Multiplication**

If  $\sum a_n = A$  and  $\sum b_n = B$ , then  $\sum (a_n + b_n) = A + B$  and  $\sum ca_n = cA$ .

**Proof**

Let  $A_n = \sum_{k=0}^n a_k$  and  $B_n = \sum_{k=0}^n b_k$ .  
 Then  $A_n + B_n = \sum_{k=0}^n a_k + b_k$  so  $\lim_{n \rightarrow \infty} A_n + B_n = A + B$ .  
 Then  $\lim_{n \rightarrow \infty} cA_n = \underbrace{A + \dots + A}_c = cA$

**Definition 10.8.2: Cauchy Product**

For  $\sum a_n$  and  $\sum b_n$ , let  $c_n = \sum_{k=0}^n a_k b_{n-k}$  and the product as  $\sum c_n$ .

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n) (b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots\end{aligned}$$

**Theorem 10.8.3: Conditions  $\sum c_n = AB$** 

Suppose

- $\sum_{n=0}^{\infty} a_n$  converges absolutely
- $\sum_{n=0}^{\infty} a_n = A$
- $\sum_{n=0}^{\infty} b_n = B$
- $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then  $\sum_{n=0}^{\infty} c_n = AB$ .

**Proof**

Let  $A_n = \sum_{k=0}^n a_k$ ,  $B_n = \sum_{k=0}^n b_k$ ,  $C_n = \sum_{k=0}^n c_k$ , and  $\beta_n = B_n - B$ .

$$\begin{aligned}C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0 \\ &= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \dots + a_n (B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0\end{aligned}$$

Let  $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$  so  $C_n = A_n B + \gamma_n$ .

Since  $a_n$  converges absolutely, then  $\sum_{n=0}^{\infty} |a_n| = \alpha$ .

Since  $\sum_{n=0}^{\infty} b_n = B$ , then  $\beta_n \rightarrow 0$ .

Then for  $\epsilon > 0$ , there is a  $N$  such that  $|\beta_n| \leq \frac{\epsilon}{\alpha}$  for  $n \geq N$ .

$$\begin{aligned}|\gamma_n| &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \dots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |a_{n-N-1} + \dots + a_0| \frac{\epsilon}{\alpha} \\ &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + \alpha \frac{\epsilon}{\alpha}\end{aligned}$$

Thus, with a fixed  $N$ , since  $a_n \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} |\gamma_n| \leq \epsilon$  so  $\lim_{n \rightarrow \infty} \gamma_n = 0$ .

Thus,  $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} A_n B + \gamma_n = AB$ .

**Theorem 10.8.4: By Cauchy Product,  $\sum c_n = C$  implies  $C = AB$** 

If  $\sum a_n = A$ ,  $\sum b_n = B$ ,  $\sum c_n = C$  where  $c_n = a_0 b_n + \dots + a_n b_0$ , then  $C = AB$ .

# 11 Continuity

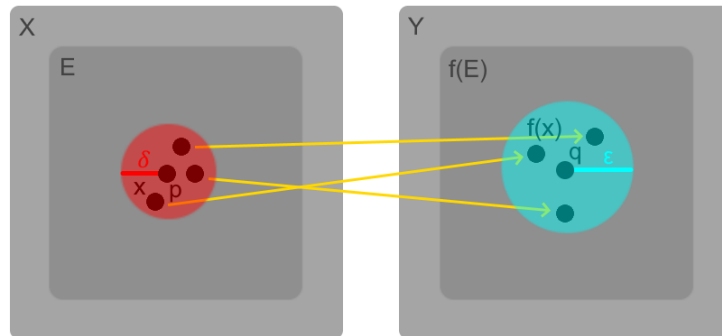
## 11.1 Limits of Functions

### Definition 11.1.1: Limits of functions

For metric spaces  $X, Y$ , let  $E \subset X$ ,  $f$  maps  $E$  into  $Y$ , and  $p \in E'$ .

Then  $\lim_{x \rightarrow p} f(x) = q$  if there is a  $q \in Y$  such that:

For every  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $x \in E$  where  $d_X(x, p) < \delta$ , then  $d_Y(f(x), q) < \epsilon$



### Theorem 11.1.2: Sequence definition of $\lim_{x \rightarrow p} f(x) = q$

$\lim_{x \rightarrow p} f(x) = q$  if and only if  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every sequence  $\{p_n\} \in E$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

#### Proof

Suppose  $\lim_{x \rightarrow p} f(x) = q$ .

For  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), q) < \epsilon$  if  $x \in E$  and  $d_X(x, p) < \delta$ .

Choose  $\{p_n\} \in E$  such that  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

Then for  $\delta > 0$ , there is  $N$  such that for  $n > N$ , then  $d_X(p_n, p) < \delta$  so  $d_Y(f(p_n), q) < \epsilon$ .

Suppose  $\lim_{x \rightarrow p} f(x) \neq q$ . Then there is a  $\epsilon > 0$  such that for every  $\delta > 0$ , there is a  $x \in E$  where  $d_Y(f(x), q) \geq \epsilon$ , but  $d_X(x, p) < \delta$ . Let  $\delta_n = \frac{1}{n}$  and thus, there is a  $\{p_n\}$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ , but  $\lim_{n \rightarrow \infty} f(p_n) \neq q$ .

### Corollary 11.1.3: A limit of a function is unique

If  $f$  has a limit at  $p$ , this limit is unique.

#### Proof

If  $\lim_{x \rightarrow p} f(x) = q$ , then by **theorem 11.1.2**,  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every  $\{p_n\} \in E$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ .

Thus, if there exists  $\lim_{x \rightarrow p} f(x) = q'$ , then there is a  $\{p_n\} \in E$  where  $p_n \neq p$  and  $\lim_{n \rightarrow \infty} p_n = p$ , but  $\lim_{n \rightarrow \infty} f(p_n) = q'$  which is a contradiction.

### Theorem 11.1.4: Arithmetic operations on functions of limits

Let  $E \subset X$ ,  $p \in E'$ , and  $f(x), g(x) \in \mathbb{C}$  so  $\lim_{x \rightarrow p} f(x) = A$ ,  $\lim_{x \rightarrow p} g(x) = B$ .

(a)  $\lim_{x \rightarrow p} (f + g)(x) = A + B$

(b)  $\lim_{x \rightarrow p} (fg)(x) = AB$

(c)  $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$



## 11.2 Continuous Functions

### Definition 11.2.1: Continuous functions on a set

Suppose  $X, Y$  are metric spaces,  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ .

$f$  is continuous at  $p$  if:

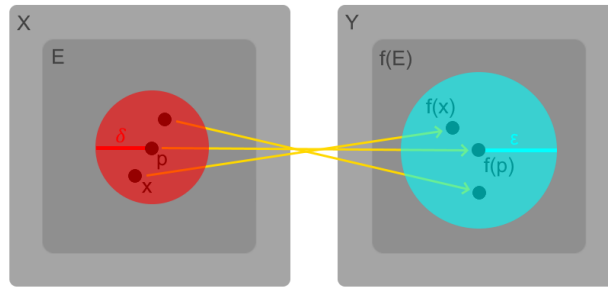
For every  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $x \in E$

where  $d_X(x, p) < \delta$ , then  $d_Y(f(x), f(p)) < \epsilon$

$f(p)$  have to be defined to be continuous.

If  $f$  is continuous at every  $p \in E$ , then  $f$  is continuous on  $E$ .

$f$  is continuous at isolated points since regardless of  $\epsilon$ , there is a  $\delta > 0$  such that  $d_X(x, p) < \delta$  is  $x = p$  so  $d_Y(f(x), f(p)) = 0 < \epsilon$ .



### Theorem 11.2.2: Continuity at $p \Leftrightarrow \lim_{x \rightarrow p} f(x) = f(p)$

Suppose  $E \subset X$ ,  $p \in E$ , and  $f$  maps  $E$  into  $Y$ . Let  $p \in E$ .

Then  $f$  is continuous at  $p$  if and only if  $\lim_{x \rightarrow p} f(x) = f(p)$ .

#### Proof

If  $f$  is continuous at  $p$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), f(p)) < \epsilon$  for all  $x \in E$  where  $d_X(x, p) < \delta$ . Thus,  $\lim_{x \rightarrow p} f(x) = f(p)$ .

If  $\lim_{x \rightarrow p} f(x) = f(p)$ , then for every  $\epsilon > 0$ , there is a  $\delta > 0$  where  $d_Y(f(x), f(p)) < \epsilon$  for all  $x \in E$  where  $d_X(x, p) < \delta$ . Thus,  $f$  is continuous at  $p$ .

### Theorem 11.2.3: Continuity Chain Rule

Suppose  $E \subset X$ ,  $f: E \rightarrow Y$ ,  $g: f(E) \rightarrow Z$ , and  $h: E \rightarrow Z$  where  $h(x) = g(f(x))$ .

If  $f$  is continuous at  $p$  and  $g$  is continuous at  $f(p)$ , then  $h$  is continuous at  $p$ .

#### Proof

Since  $g$  is continuous at  $f(p)$ , then for  $\epsilon > 0$ , there is a  $\delta_1$  such that:

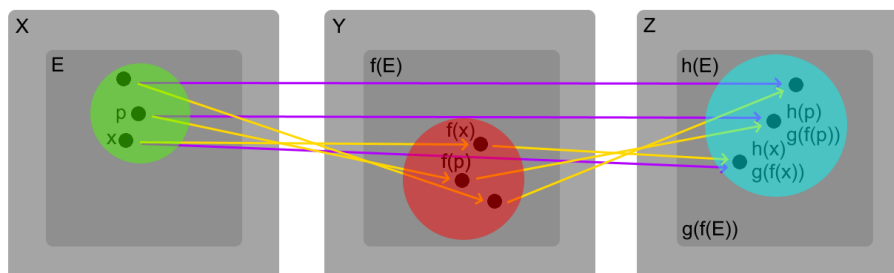
$d_Z(g(y), g(f(p))) < \epsilon$  for  $d_Y(y, f(p)) < \delta_1$  where  $y \in f(E)$

Since  $f$  is continuous at  $p$ , there is a  $\delta_2 > 0$  such that:

$d_Y(f(x), f(p)) < \delta_1$  for  $d_X(x, p) < \delta_2$  where  $x \in E$

Thus,  $d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon$  for  $d_X(x, p) < \delta_2$  where  $x \in E$ .

Thus,  $h$  is continuous at  $p$ .



**Theorem 11.2.4: Continuous functions map open sets to open sets**

$f: X \rightarrow Y$  is continuous on  $X$  if and only if:

$f^{-1}(V)$  is open in  $X$  for every open set  $V$  in  $Y$ .

**Proof**

Suppose  $f$  is continuous on  $X$  and  $V$  is an open set in  $Y$ .

Suppose  $p \in X$  and  $f(p) \in V$ . Since  $V$  is open, there exists  $\epsilon > 0$  such that  $y \in V$  if  $d_Y(y, f(p)) < \epsilon$ . Since  $f$  is continuous at  $p$ , there exists  $\delta > 0$  such that  $d_Y(f(x), f(p)) < \epsilon$  for  $d_X(x, p) < \delta$ . Thus,  $x \in f^{-1}(V)$  for  $d_X(x, p) < \delta$ .

Suppose  $f^{-1}(V)$  is open in  $X$  for every open  $V$  in  $Y$ .

Fix  $p \in X$  and  $\epsilon > 0$ . Let  $V$  be the set of all  $y \in Y$  such that  $d_Y(y, f(p)) < \epsilon$  so  $V$  is open and thus,  $f^{-1}(V)$  is open. Thus, there exists  $\delta > 0$  such that  $x \in f^{-1}(V)$  for  $d_X(x, p) < \delta$ . Since  $x \in f^{-1}(V)$ , then  $f(x) \in V$  so  $d_Y(f(x), f(p)) < \epsilon$ .

**Corollary 11.2.5: Continuous functions map closed sets to closed sets**

$f: X \rightarrow Y$  is continuous on  $X$  if and only if:

$f^{-1}(C)$  is closed in  $X$  for every closed set  $C$  in  $Y$ .

**Proof**

By **theorem 11.2.4**,  $f$  is continuous if and only if  $f^{-1}(V)$  is open in  $X$  for every open set  $V$  in  $Y$ . Let  $C = V^c$ . Since  $V$  is open, then  $C$  is closed.

Since  $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$ , then  $f^{-1}(C)$  is closed since  $f^{-1}(V)$  is open.

**Theorem 11.2.6: Continuous functions**

Let  $f, g$  be complex continuous functions on  $X$ .

Then  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  where  $g \neq 0$  for all  $x \in X$  are continuous on  $X$ .

**Proof**

If  $x$  is an isolated point,  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  are continuous by definition. If  $x$  is a limit point, then by **theorems 11.1.4 and 11.2.2**,  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  are continuous since

- $\lim_{x \rightarrow p} (f + g)(x) = \lim_{x \rightarrow p} f(x) + \lim_{x \rightarrow p} g(x) = f(p) + g(p)$
- $\lim_{x \rightarrow p} (fg)(x) = \lim_{x \rightarrow p} f(x) \lim_{x \rightarrow p} g(x) = f(p)g(p)$
- $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \rightarrow p} f(x)}{\lim_{x \rightarrow p} g(x)} = \frac{f(p)}{g(p)}$

**Theorem 11.2.7: Continuous functions on  $\mathbb{R}^k$** 

(a) Let  $f_1, \dots, f_k: X \rightarrow \mathbb{R}$  and  $f: X \rightarrow \mathbb{R}^k$  where  $f(x) = (f_1(x), \dots, f_k(x))$ .

Then  $f$  is continuous if and only if  $f_1, \dots, f_k$  are continuous.

(b) If  $f$  and  $g$  are continuous mappings of  $X$  into  $\mathbb{R}^k$ , then  $f + g$  and  $f \cdot g$  are continuous on  $X$ .

**Proof**

Since  $|f_i(x) - f_i(y)| \leq \sqrt{\sum_{i=1}^k |f_i(x) - f_i(y)|^2} = |f(x) - f(y)|$ , then if  $f$  is continuous, then each  $f_i$  is continuous and vice versa.

Since  $f, g$  are continuous, then by part a, each  $f_i, g_i$  are continuous. Then by **theorem 11.2.6**, each  $f_i + g_i$  and  $f_i g_i$  are continuous so by part a,  $f + g$  and  $f \cdot g$  are continuous.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are  $x_1 \cdot \dots \cdot x_k$  where each  $x_i$  is continuous, rationals are polynomials divided by polynomials, and  $||x| - |y|| \leq |x - y|$  implies  $|x|$  is continuous.

## 11.3 Continuity and Compactness

### Definition 11.3.1: Bounded Functions

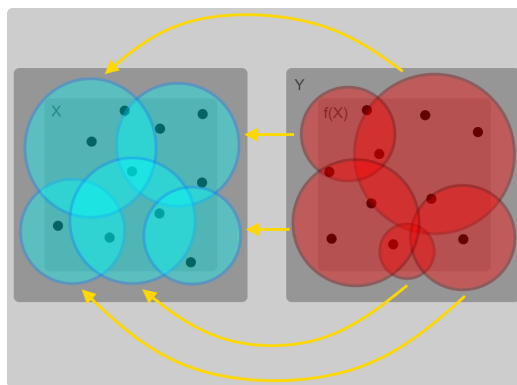
$f: E \rightarrow \mathbb{R}^k$  is bounded if there is a  $M \in \mathbb{R}$  such that  $f(x) \leq M$  for all  $x \in E$ .

### Theorem 11.3.2: Continuous functions from compact spaces are compact

Suppose  $f$  is a continuous mapping of a compact metric space  $X$  into a metric space  $Y$ . Then  $f(X)$  is compact.

#### Proof

Let  $\{V_\alpha\}$  be an open cover of  $f(X)$ . Since  $f$  is continuous, then by [theorem 11.2.4](#), each  $f^{-1}(V_\alpha)$  is open. Since  $X$  is compact, there is  $n$  where  $X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})$ . Thus,  $f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$  so  $f(X)$  is compact.



### Theorem 11.3.3: Continuous functions from compact to $\mathbb{R}^k$ are bounded

If  $f$  is a continuous mapping of a compact metric space  $X$  into  $\mathbb{R}^k$ , then  $f(X)$  is closed and bounded.

#### Proof

By [theorem 11.2.2](#),  $f(X)$  is compact. By [theorem 6.3.13](#),  $f(X)$  is closed and bounded.

### Theorem 11.3.4: Generalized extreme value theorem

Suppose  $f$  is a continuous real function of a compact metric space  $X$  such that  $M = \sup_{x \in X} f(x)$  and  $m = \inf_{x \in X} f(x)$ .

Then there exists  $p, q \in X$  such that  $f(p) = M$  and  $f(q) = m$ .

#### Proof

By [theorem 11.3.3](#),  $f(X)$  is closed and bounded. Let  $M = \sup_{x \in X} f(x)$ ,  $m = \inf_{x \in X} f(x)$ . Since  $f(X)$  is bounded, then  $M, m \in (f(X))'$  and since  $f(X)$  is closed, then  $M, m \in f(X)$ . Thus, there exists  $p, q \in X$  such that  $f(p) = M$  and  $f(q) = m$ .

### Theorem 11.3.5: If $f$ is continuous 1-1, then $f^{-1}$ is continuous

Suppose  $f$  is a continuous 1-1 mapping of a compact metric space  $X$  onto a metric space  $Y$ . Then  $f^{-1}$  is a continuous mapping of  $Y$  onto  $X$ .

#### Proof

Let  $V$  be an open set in  $X$ . Since  $V^c$  is closed and  $V^c \subset$  compact set  $X$ , then by [theorem 6.3.5](#),  $V^c$  is compact. Thus by [theorem 11.3.2](#),  $f(V^c)$  is a compact subset of  $Y$  so  $f(V^c)$  is closed. Since  $f$  is 1-1 and onto,  $f(V^c) = (f(V))^c$  so  $f(V)$  is open. Since from any open set  $V$  in  $X$ ,  $f(V)$  is open in  $Y$ , then by [theorem 11.2.4](#),  $f^{-1}$  is continuous.

**Definition 11.3.6: Uniformly Continuous**

Let  $f: X \rightarrow Y$ . Then  $f$  is uniformly continuous on  $X$  if:

For every  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $p, q \in X$  where  $d_X(p, q) < \delta$ , then  $d_Y(f(p), f(q)) < \epsilon$ .

**Theorem 11.3.7: Continuous functions on compact are uniformly continuous**

Let  $f$  be a continuous mapping of a compact metric space  $X$  into metric space  $Y$ . Then  $f$  is uniformly continuous on  $X$ .

**Proof**

For  $\epsilon > 0$ , since  $f$  is continuous, then for each  $p \in X$ , there is a  $\phi(p)$  such that for all  $q \in X$  where  $d_X(q, p) < \phi(p)$ ,  $d_Y(f(q), f(p)) < \frac{\epsilon}{2}$ .

Let  $J(p)$  be the set of all  $q \in X$  where  $d_X(q, p) < \frac{1}{2}\phi(p)$ .

Since the set of all  $J(p)$  is an open cover of  $X$  and since  $X$  is compact, then there is a  $n$  such that  $X \subset J(p_1) \cup \dots \cup J(p_n)$ . Let  $\delta = \frac{1}{2} \min(\phi(p_1), \dots, \phi(p_n)) > 0$ .

Then for  $p, q \in X$  where  $d_X(p, q) < \delta$ , there is a  $m$  where  $1 \leq m \leq n$  such that  $p \in J(p_m)$  so  $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$ . Thus:

$$\begin{aligned} d_X(q, p_m) &\leq d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \leq \phi(p_m) \\ d_Y(f(p), f(q)) &\leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

**Theorem 11.3.8: Continuous functions from noncompact  $\nrightarrow$  uniformly continuous**

Let  $E$  be a noncompact set in  $\mathbb{R}^1$ .

- (a) There exists a continuous function which is not bounded.
- (b) There exists a continuous, bounded function which has no maximum.
- (c) If  $E$  is bounded, there exists a continuous function which is not uniformly continuous.

**Proof**

Suppose  $E$  is bounded so there is a  $x_0 \in E'$ , but  $x_0 \notin E$ .

Consider  $f(x) = \frac{1}{x-x_0}$  which is continuous on  $E$ , but unbounded.

For  $\epsilon > 0$  and  $\delta > 0$ , there is a  $x \in E$  such that  $|x - x_0| < \delta$ . Take  $t$  close enough to  $x_0$  so  $|f(t) - f(x_0)| > \epsilon$ , but  $|t - x| < \delta$ . Thus,  $f$  is not uniformly continuous.

Consider  $g(x) = \frac{1}{1+(x-x_0)^2}$  which is continuous on  $E$  and bounded since  $g(x) \in (0, 1)$ . Since  $\sup_{x \in E} g(x) = 1$ , but  $g(x) < 1$  for all  $x \in E$ , then  $g$  has no maximum on  $E$ .

**11.4 Continuity and Connectedness****Theorem 11.4.1: Continuous functions map connected to connected**

If  $f$  is a continuous mapping of  $X$  into  $Y$  and  $E$  is a connected subset of  $X$ , then  $f(E)$  is connected.

**Proof**

Suppose  $f(E) = A \cup B$  where  $A$  and  $B$  are nonempty separated subsets of  $Y$ .

Let  $G = E \cap f^{-1}(A)$  and  $H = E \cap f^{-1}(B)$ . Then  $E = G \cup H$ .

Since  $A \subset \overline{A}$ ,  $G \subset f^{-1}(\overline{A})$ . Since  $f$  is continuous, then  $f^{-1}(\overline{A})$  is closed so  $\overline{G} \subset f^{-1}(\overline{A})$ . Thus,  $f(\overline{G}) \subset \overline{A}$ .

Since  $f(H) = B$  and  $\overline{A} \cap B$  is empty,  $\overline{G} \cap H$  is empty. Similarly,  $G \cap \overline{H}$  is empty so  $G$  and  $H$  are separated which contradicts that  $E = G \cup H$  is connected.

**Theorem 11.4.2: Generalized Intermediate Value Theorem**

Let  $f$  be a continuous real function on  $[a, b]$ . If  $f(a) < c < f(b)$ , then there exists  $x \in (a, b)$  such that  $f(x) = c$ .

**Proof**

Since  $[a, b]$  is connected, then by [theorem 11.4.1](#),  $f([a, b])$  is a connected subset of  $\mathbb{R}^1$ . Thus, by [theorem 7.2.2](#), any  $c$  where  $f(a) < c < f(b)$  is  $c \in f(x)$  for some  $x \in [a, b]$ .

**11.5 Discontinuities****Definition 11.5.1: Right and Left Limits**

Let  $f$  be defined on  $(a, b)$ .

Then for any  $x$  where  $x \in [a, b)$ ,  $f(x+) = q$  if:

$f(t_n) \rightarrow q$  as  $n \rightarrow \infty$  for all sequences  $\{t_n\}$  in  $(x, b)$  such that  $t_n \rightarrow x$ .

Then for any  $x$  where  $x \in (a, b]$ ,  $f(x-) = q$  if:

$f(t_n) \rightarrow q$  as  $n \rightarrow \infty$  for all sequences  $\{t_n\}$  in  $(a, x)$  such that  $t_n \rightarrow x$ .

Then  $\lim_{t \rightarrow x} f(t)$  exists if and only if  $f(x-) = f(x+) = \lim_{t \rightarrow x} f(t)$ .

**Definition 11.5.2: Types of discontinuities**

If  $f$  is discontinuous at  $x$ , but  $f(x+)$  and  $f(x-)$  exists, then  $f$  have a simple discontinuity of the first kind else it is a discontinuity of the second kind.

Thus, a simple discontinuity is either:

- $f(x-) \neq f(x+)$
- $f(x-) = f(x+) \neq f(x)$

**11.6 Monotonic Functions****Definition 11.6.1: Monotonic**

$f: (a, b) \rightarrow \mathbb{R}$  is monotonically increasing if  $f(x) \leq f(y)$  for  $a < x < y < b$ .

$f: (a, b) \rightarrow \mathbb{R}$  is monotonically decreasing if  $f(x) \geq f(y)$  for  $a < x < y < b$ .

**Theorem 11.6.2: Right and Left Limits of monotonic functions on  $(a, b)$** 

Let  $f$  be monotonically increasing on  $(a, b)$ .

Then  $f(x+)$  and  $f(x-)$  exists at every  $x \in (a, b)$  where:

$$\sup_{t \in (a, x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x, b)} f(t)$$

Furthermore, for  $a < x < y < b$ ,  $f(x+) \leq f(y-)$ .

Properties analogous for monotonically decreasing functions.

**Proof**

Since  $f$  is monotonically increasing, then for  $t \in (a, x)$ ,  $f(t)$  is bounded above by  $f(x)$  and thus, by the least upper bounded property,  $\sup_{t \in (a, x)} f(t)$  exists.

For  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\sup_{t \in (a, x)} f(t) - \epsilon < f(x - \delta) \leq \sup_{t \in (a, x)} f(t)$  for  $a < x - \delta < x$ . Since  $f(x - \delta) \leq f(t) \leq \sup_{t \in (a, x)} f(t)$  for  $t \in (x - \delta, x)$ , then  $|f(t) - \sup_{t \in (a, x)} f(t)| < \epsilon$  for  $t \in (x - \delta, x)$  so  $f(x-) = \sup_{t \in (a, x)} f(t)$ .

For  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\inf_{t \in (x, b)} f(t) < f(x + \delta) \leq \inf_{t \in (x, b)} f(t) + \epsilon$  for  $x < x + \delta < b$ . Since  $f(x + \delta) \geq f(t) \geq \inf_{t \in (x, b)} f(t)$  for  $t \in (x, x + \delta)$ , then  $|f(t) - \inf_{t \in (x, b)} f(t)| < \epsilon$  for  $t \in (x, x + \delta)$  so  $f(x+) = \inf_{t \in (x, b)} f(t)$ .

Thus,  $\sup_{t \in (a, x)} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{t \in (x, b)} f(t)$ .

If  $a < x < y < b$ , then:

$$f(x+) = \inf_{t \in (x, b)} f(t) = \inf_{t \in (x, y)} f(t) \leq \sup_{t \in (x, y)} f(t) = \sup_{t \in (a, y)} f(t) = f(y-)$$

**Corollary 11.6.3: Monotonics can only have simple discontinuities**

Monotonic functions have no discontinuities of the second kind

**Proof**

By **theorem 11.6.2**,  $f(x_-)$  and  $f(x_+)$  exists and thus,  $f$  can only have simple discontinuities and not discontinuities of the second kind.

**Theorem 11.6.4: Discontinuities of monotonics is at most countable**

Let  $f$  be monotonic on  $(a,b)$ .

Then the set of points of  $(a,b)$  where  $f$  is discontinuous is at most countable.

**Proof**

Suppose  $f$  is increasing. Let  $E$  be the set of points where  $f$  is discontinuous. Then for  $x \in E$ , there is a rational  $r(x)$  where  $f(x_-) < r(x) < f(x_+)$ .  
 Then for  $x_1 < x_2$ , by **theorem 11.6.2**,  $f(x_1+) \leq f(x_2-)$ . Then:  

$$f(x_{1-}) < r(x_1) < f(x_{1+}) \leq f(x_{2-}) < r(x_2) < f(x_{2+})$$
  
 Thus,  $r(x_1) \neq r(x_2)$  if  $x_1 \neq x_2$ .  
 Since there is a 1-1 correspondence between  $E$  and a subset of rational numbers which is countable, then  $E$  is at most countable.  
 If  $f$  is decreasing, proof is analogous.

**11.7 Infinite Limits / Limits at Infinity****Definition 11.7.1: Neighborhoods in extended reals**

For any real  $c$ , a neighborhood of  $+\infty = (c, +\infty)$ .

For any real  $c$ , a neighborhood of  $-\infty = (-\infty, c)$ .

**Definition 11.7.2: Infinite Limits**

Let real function  $f$  be defined on  $E \subset \mathbb{R}$ .

Then  $f(t) \rightarrow A$  as  $t \rightarrow x$  where  $A$  and  $x$  are extended reals if:

For every neighborhood  $U$  of  $A$ , there is a neighborhood  $V$  of  $x$  such that  $V \cap E \neq \emptyset$  and  $f(t) \in U$  for all  $t \in V \cap E$  where  $t \neq x$ .

**Theorem 11.7.3: Arithmetic operations on functions of infinite limits**

Let  $f, g$  be defined on  $E \subset \mathbb{R}$  where  $f(t) \rightarrow A$  and  $g(t) \rightarrow B$  as  $t \rightarrow x$ .

(a) If  $f(t) \rightarrow A'$ , then  $A' = A$ .

(b)  $(f+g)(t) \rightarrow A + B$

(c)  $(fg)(t) \rightarrow AB$

(d)  $\frac{f}{g}(t) \rightarrow \frac{A}{B}$

## 12 Differentiation

### 12.1 Derivative of a function

#### Definition 12.1.1: Derivative

Let  $f$  be defined on any  $x \in [a, b]$ .

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \text{ for } t \neq x$$

The derivative of  $f$  at  $x$ :

$$f'(x) = \lim_{t \rightarrow x} \phi(t)$$

if the limit exist as defined by [definition 11.1.1](#).

If  $f'$  is defined at  $x$ , then  $f$  is differentiable at  $x$ .

#### Theorem 12.1.2: Differentiability $\rightarrow$ Continuity

Let  $f$  be defined on  $[a, b]$ .

If  $f$  is differentiable at  $x \in [a, b]$ , then  $f$  is continuous at  $x$ .

#### Proof

As  $t \rightarrow x$ :

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0$$

#### Theorem 12.1.3: Arithmetic operations on differentiation

Suppose  $f, g$  are defined on  $[a, b]$  and differentiable on  $x \in [a, b]$ . Then  $f+g$ ,  $fg$ , and  $\frac{f}{g}$  are differentiable at  $x$ :

(a)  $(f+g)'(x) = f'(x) + g'(x)$

#### Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{(f+g)(t) - (f+g)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{f(t) - f(x) + g(t) - g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} + \lim_{t \rightarrow x} \frac{g(t) - g(x)}{t - x} = f'(x) + g'(x) \end{aligned}$$

(b)  $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

#### Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{(fg)(t) - (fg)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(t) - f(x)g(t) + f(x)g(t) - f(x)g(x)}{t - x} \\ &= \lim_{t \rightarrow x} \frac{[f(t) - f(x)]g(t)}{t - x} + \lim_{t \rightarrow x} \frac{f(x)[g(t) - g(x)]}{t - x} \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

(c)  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

#### Proof

$$\begin{aligned} \lim_{t \rightarrow x} \frac{\left(\frac{f}{g}\right)(t) - \left(\frac{f}{g}\right)(x)}{t - x} &= \lim_{t \rightarrow x} \frac{\frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}}{t - x} = \lim_{t \rightarrow x} \frac{\frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)(t - x)}}{t - x} \\ &= \lim_{t \rightarrow x} \frac{f(t)g(x) - f(x)g(t)}{g(t)g(x)(t - x)} \\ &= \lim_{t \rightarrow x} \frac{[f(t) - f(x)]g(x)}{g(t)g(x)(t - x)} + \lim_{t \rightarrow x} \frac{f(x)[g(x) - g(t)]}{g(t)g(x)(t - x)} \\ &= \frac{f'(x)g(x)}{g^2(x)} + \frac{f(x)[-g'(x)]}{g^2(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \end{aligned}$$

**Theorem 12.1.4: Differentiation Chain Rule**

Suppose  $f$  is continuous on  $[a,b]$ ,  $f'(x)$  exists at  $x \in [a,b]$ ,  $g$  is defined on interval  $I$  containing  $f([a,b])$ , and  $g$  is differentiable at  $f(x)$ .

If  $h(t) = g(f(t))$ , then  $h$  is differentiable at  $x$  and  $h'(x) = g'(f(x)) \cdot f'(x)$

**Proof**

Since  $f$  is differentiable at  $x$  and  $g$  is differentiable at  $f(x)$ , then:

$$\begin{aligned} f(t) - f(x) &= (t-x) [f'(x) + u(t)] && \text{for } t \in [a,b] \text{ and } \lim_{t \rightarrow x} u(t) = 0 \\ g(s) - g(f(x)) &= (s-f(x)) [g'(f(x)) + v(s)] && \text{for } s \in I \text{ and } \lim_{s \rightarrow f(x)} v(s) = 0 \end{aligned}$$

Thus:

$$\begin{aligned} \lim_{t \rightarrow x} \frac{h(t) - h(x)}{t - x} &= \lim_{t \rightarrow x} \frac{g(f(t)) - g(f(x))}{t - x} \\ &= \lim_{t \rightarrow x} \frac{(f(t) - f(x)) [g'(f(x)) + v(f(t))]}{t - x} \\ &= \lim_{t \rightarrow x} \frac{(t - x) [f'(x) + u(t)] [g'(f(x)) + v(f(t))]}{t - x} \\ &= g'(f(x)) \cdot f'(x) + f'(x) \cdot 0 + g'(f(x)) \cdot 0 + 0 \cdot 0 = g'(f(x)) \cdot f'(x) \end{aligned}$$

**12.2 Mean Value Theorems****Definition 12.2.1: Local Extrema**

Let real-valued  $f \in X$ .

Then  $f$  has a local maximum at  $p \in X$  if:

There is  $\delta > 0$  such that for all  $q \in X$  where  $d_X(q, p) < \delta$ ,  $f(q) \leq f(p)$ .

Then  $f$  has a local minimum at  $p \in X$  if:

There is  $\delta > 0$  such that for all  $q \in X$  where  $d_X(q, p) < \delta$ ,  $f(q) \geq f(p)$ .

**Theorem 12.2.2: Derivative at local extrema is 0**

Let  $f$  be defined on  $[a,b]$ .

If  $f$  has a local maximum at  $x \in (a,b)$  and  $f'(x)$  exists, then  $f'(x) = 0$ .

If  $f$  has a local minimum at  $x \in (a,b)$  and  $f'(x)$  exists, then  $f'(x) = 0$ .

**Proof**

Suppose  $x$  is a local maximum.

Then there is a  $\delta > 0$  such that for all  $t \in (a,b)$  where  $|t - x| < \delta$ , then  $f(t) \leq f(x)$ .

Then for  $t < x$ ,  $\frac{f(t) - f(x)}{t - x} \geq 0$ . Thus,  $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = f'(x) \geq 0$ .

For  $t > x$ ,  $\frac{f(t) - f(x)}{t - x} \leq 0$ . Thus,  $\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} = f'(x) \leq 0$ .

Since  $f'(x)$  exists, then  $f'(x) = 0$ .

Proof is analogous for local minimum.

**Theorem 12.2.3: Generalized Mean Value Theorem**

If  $f, g$  are continuous real functions on  $[a,b]$  and differentiable on  $(a,b)$ , then there is a  $x \in (a,b)$  such that  $[f(b) - f(a)] \cdot g'(x) = [g(b) - g(a)] \cdot f'(x)$ .

**Proof**

Let  $h(t) = [f(b) - f(a)] \cdot g(t) - [g(b) - g(a)] \cdot f(t)$  for  $t \in [a,b]$ .

Since  $f, g$  are continuous on  $[a,b]$  and differentiable on  $(a,b)$ , then  $h$  is continuous on  $[a,b]$  and differentiable on  $(a,b)$ . Also,  $h(a) = f(b)g(a) - f(a)g(b) = h(b)$ .

If  $h$  is constant, then  $h'(x) = 0$  and thus, theorem holds true for every  $x \in (a,b)$ .

If  $h(t) > h(a)$  for some  $t \in (a,b)$ , let  $x \in [a,b]$  where  $h$  attains a local maximum. If  $h(t) < h(a)$  for some  $t \in (a,b)$ , let  $x \in [a,b]$  where  $h$  attains a local minimum. Then by **theorem 12.2.2**,  $h'(x) = 0$  and thus, theorem holds true at local extrema.



**Theorem 12.2.4: Mean Value Theorem**

If  $f$  is a real continuous function on  $[a,b]$  and differentiable on  $(a,b)$ , then there is a  $x \in (a,b)$  such that  $f(b) - f(a) = (b-a) f'(x)$ .

**Proof**

From **theorem 12.2.3**, let  $g(x) = x$ .

**Theorem 12.2.5: Sign of derivative determines increasing/decreasing**

Suppose  $f$  is differentiable on  $(a,b)$ .

- (a) If  $f'(x) \geq 0$  for all  $x \in (a,b)$ , then  $f$  is monotonically increasing.
- (b) If  $f'(x) = 0$  for all  $x \in (a,b)$ , then  $f$  is constant.
- (c) If  $f'(x) \leq 0$  for all  $x \in (a,b)$ , then  $f$  is monotonically decreasing

**Proof**

From **theorem 12.2.4**,  $f(x_2) - f(x_1) = (x_2 - x_1) f'(x)$  for  $x \in (x_1, x_2) \subset (a,b)$ .  
 If  $f'(x) \geq 0$  for all  $x \in (a,b)$ , then  $f(x_2) - f(x_1) \geq 0$ . Since  $f(x_2) \geq f(x_1)$  for  $x_2 > x_1$ , then  $f$  is monotonically increasing.  
 If  $f'(x) = 0$  for all  $x \in (a,b)$ , then  $f(x_2) - f(x_1) = 0$ . Since  $f(x_2) = f(x_1)$  for  $x_2 > x_1$ , then  $f$  is constant.  
 If  $f'(x) \leq 0$  for all  $x \in (a,b)$ , then  $f(x_2) - f(x_1) \leq 0$ . Since  $f(x_2) \leq f(x_1)$  for  $x_2 > x_1$ , then  $f$  is monotonically decreasing.

**12.3 Continuity of Derivatives****Theorem 12.3.1: Intermediate values of derivatives exists**

Suppose  $f$  is a real differentiable function on  $[a,b]$  and  $f'(a) < \lambda < f'(b)$ .

Then there is a  $x \in (a,b)$  such that  $f'(x) = \lambda$ .

Statement holds true if  $f'(a) > f'(b)$ .

**Proof**

Suppose  $f'(a) < \lambda < f'(b)$ . Let  $g(t) = f(t) - \lambda t$ .  
 Since  $f(t), t$  are differentiable on  $[a,b]$ , then  $g(t)$  is differentiable on  $[a,b]$ .  
 Then  $g'(a) = f'(a) - \lambda < 0$  so  $g(t_1) < g(a)$  for some  $t_1 \in (a,b)$ .  
 Also,  $g'(b) = f'(b) - \lambda > 0$  so  $g(t_2) < g(b)$  for some  $t_2 \in (a,b)$ .  
 Thus, there is a  $x$  where  $g(x)$  is a local minimum so  $g'(x) = 0$  and thus,  $f'(x) = \lambda$ .

**Corollary 12.3.2: Differentiable functions have no simple discontinuities**

If  $f$  is differentiable on  $[a,b]$ , then  $f'$  cannot have simple discontinuities on  $[a,b]$ .

**Proof**

By **theorem 12.3.1**,  $f'(x)$  exists for any  $x \in [a,b]$ .

## 12.4 L'Hospital's Rule

### Theorem 12.4.1: L'Hospital's Rule

Suppose  $f, g$  are real and differentiable on  $(a, b)$  and  $g'(x) \neq 0$  for all  $x \in (a, b)$ .

Suppose  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \rightarrow A$ . If either:

- $\lim_{x \rightarrow a} f(x) \rightarrow 0$  and  $\lim_{x \rightarrow a} g(x) \rightarrow 0$
- $\lim_{x \rightarrow a} g(x) \rightarrow \infty$  or  $\lim_{x \rightarrow a} g(x) \rightarrow -\infty$

Then,  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow A$ .

Statement holds true if  $x \rightarrow b$ .

### Proof

Consider the case  $-\infty \leq A < \infty$ .

Choose  $q$  such that  $A < q$  and  $r$  such that  $A < r < q$ .

Thus, there is a  $c \in (a, b)$  such that  $a < x < c$  for  $\frac{f'(x)}{g'(x)} < r$ .

For  $a < x < y < c$ , then by [theorem 12.2.3](#), there is a  $t \in (x, y)$  such that:

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$$

If  $\lim_{x \rightarrow a} f(x) \rightarrow 0$  and  $\lim_{x \rightarrow a} g(x) \rightarrow 0$ , then as  $x \rightarrow a$ ,  $\frac{f(y)}{g(y)} \leq r < q$  for  $y \in (a, c)$ .

If  $\lim_{x \rightarrow a} g(x) \rightarrow \infty$ , then keeping  $y$  fixed, choose  $c_1 \in (a, y)$  such that  $g(x) > g(y)$  and  $g(x) > 0$  if  $a < x < c_1$ . Thus:

$$\begin{aligned} \frac{g(x)-g(y)}{g(x)} \cdot \frac{f(x)-f(y)}{g(x)-g(y)} &< \frac{g(x)-g(y)}{g(x)} \cdot r \text{ for } x \in (a, c_1) \\ \frac{f(x)}{g(x)} &< r - r \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \end{aligned}$$

Thus as  $x \rightarrow a$ , there is a  $c_2 \in (a, c_1)$  such that  $\frac{f(x)}{g(x)} < r < q$  for  $x \in (a, c_2)$ .

Proof is analogous if  $\lim_{x \rightarrow a} g(x) \rightarrow -\infty$ .

Thus,  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \rightarrow A$ .

## 12.5 Derivative of Higher Order

### Definition 12.5.1: Derivative of Higher Order

If  $f$  has a derivative  $f'$  on an interval and  $f'$  is differentiable, then the derivative of  $f'$  is  $f''$ , the second derivative of  $f$ . Then,  $f^{(n)}$  is the  $n$ th derivative of  $f$ .

For  $f^{(n)}(x)$  to exist at  $x$ ,  $f^{(n-1)}(t)$  must exist in a neighborhood of  $x$  and  $f^{(n-1)}$  must be differentiable at  $x$ .

If  $f^{(n-1)}$  exist in a neighborhood of  $x$ , then  $f^{(n-2)}$  must be differentiable in that neighborhood and so on until  $f$  is differentiable on that neighborhood.

## 12.6 Taylor's Theorem

### Theorem 12.6.1: Taylor's Theorem

Suppose  $f$  is a real function on  $[a,b]$ ,  $n \in \mathbb{Z}_+$ ,  $f^{(n-1)}$  is continuous on  $[a,b]$ ,  $f^n(t)$  exists at every  $t \in (a,b)$ .

Let  $\alpha, \beta \in [a,b]$  be distinct and  $P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k$ .

Then there exists a  $x$  between  $\alpha$  and  $\beta$  such that  $f(\beta) = P(\beta) + \frac{f^n(x)}{n!} (\beta - \alpha)^n$

#### Proof

Let  $M$  be the number defined by  $f(\beta) = P(\beta) + M(\beta - \alpha)^n$ .

Let  $g(t) = f(t) - P(t) - M(t - \alpha)^n$  for  $t \in [\alpha, \beta]$ . Thus,  $g^{(n)}(t) = f^{(n)}(t) - n!M$ .

Also since  $P^{(k)}(\alpha) = f^{(k)}(\alpha)$  for  $k = [0, n-1]$ , then  $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$ .

Since the choice of  $M$  gives  $g(\beta) = 0$ , then by the Mean Value Theorem,  $g'(x_1) = 0$  for some  $x_1$  between  $\alpha$  and  $\beta$ .

Since  $g'(\alpha) = 0$ , then  $g''(x_2) = 0$  for some  $x_2$  between  $\alpha$  and  $x_1$ .

Thus,  $g^{(n)}(x_n) = 0$  for some  $x_n$  between  $\alpha$  and  $x_{n-1}$  so  $x_n$  is between  $\alpha$  and  $\beta$ .

Thus, there exists an  $x_n \in (\alpha, \beta)$  such that:

$$0 = g^{(n)}(x_n) = f^{(n)}(x_n) - n!M$$

$$M = \frac{f^n(x_n)}{n!}$$

## 12.7 Differentiation of Vector-Valued Functions

### Definition 12.7.1: Extending derivative to Vector-Valued Functions

For vector-valued function  $f: t \in [a,b] \rightarrow \mathbb{R}^k$ , the derivative of  $f$  at  $x$ :

$$f'(x) = \lim_{t \rightarrow x} \left| \frac{f(t) - f(x)}{t - x} \right|$$

if the limit exist as defined by [definition 14.1.1](#).

If  $f = (f_1, \dots, f_k)$ , then  $f' = (f'_1, \dots, f'_k)$  and  $f$  is differentiable at  $x$  if and only if  $f_1, \dots, f_k$  are differentiable at  $x$ .

Thus, by [theorem 11.2.7](#), these theorems hold true for vector-valued functions:

- [12.1.2](#): If  $f$  is differentiable at  $x$ , then  $f$  is continuous at  $x$ .
- [12.1.3a](#): If  $f, g$  are differentiable at  $x$ , then  $f+g, f \cdot g$  are differentiable at  $x$ .

However, [theorem 12.2.4: Mean Value Theorem](#) and [theorem 12.4.1: L'Hospital's Rule](#) does not always hold true since [theorem 12.1.3b/c](#), multiplying/dividing vectors by vectors, is not defined for vector-valued functions.

### Theorem 12.7.2: Mean Value Theorem for $\mathbb{R}^k$

Suppose  $f$  is a continuous mapping of  $[a,b]$  into  $\mathbb{R}^k$  and  $f$  is differentiable on  $(a,b)$ . Then there is a  $x \in (a,b)$  such that  $|f(b) - f(a)| \leq (b-a) |f'(x)|$

#### Proof

Let  $z = f(b) - f(a)$  and define  $\phi(t) = z \cdot f(t)$  for  $t \in [a,b]$ .

Then  $\phi(t)$  is real-valued continuous on  $[a,b]$  and differentiable on  $(a,b)$ .

Then by the Mean Value Theorem, for some  $x \in (a,b)$ :

$$\phi(b) - \phi(a) = (b-a) \phi'(x) = (b-a) z \cdot f'(x)$$

Since  $\phi(b) - \phi(a) = z \cdot f(b) - z \cdot f(a) = z \cdot z = |z|^2$ , then by the Schwarz Inequality:

$$|z|^2 = (b-a) |z \cdot f'(x)| \leq (b-a) |z| |f'(x)|$$

$$|z| \leq (b-a) |f'(x)|$$

$$|f(b) - f(a)| \leq (b-a) |f'(x)|$$

## 13 Riemann-Stieltjes Integral

### 13.1 Riemann-Stieltjes Integral

#### Definition 13.1.1: Riemann Integral

For  $[a,b]$ , let  $a = x_0 \leq x_1 \leq \dots \leq x_n = b$  and  $\Delta x_i = x_i - x_{i-1}$ .

Suppose real  $f$  is bounded. Then for each partition  $P$ ,  $\{x_0, \dots, x_n\}$ ,

let  $m_i = \inf f([x_{i-1}, x_i])$  and  $M_i = \sup f([x_{i-1}, x_i])$ .

Then let  $L(P,f) = \sum_{i=1}^n m_i \Delta x_i$  and  $U(P,f) = \sum_{i=1}^n M_i \Delta x_i$ . Thus, over all  $P$ :

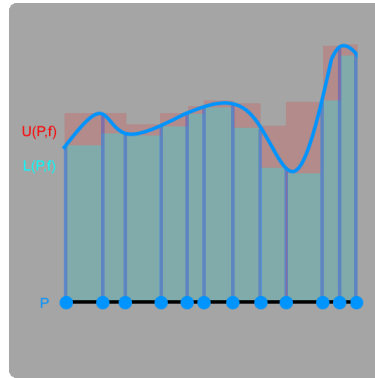
Lower Riemann Integral:  $\int_a^b f(x) dx = \sup L(P,f)$

Upper Riemann Integral:  $\int_a^b f(x) dx = \inf U(P,f)$

If  $\int_a^b f(x) dx = \overline{\int}_a^b f(x) dx$ , then  $f$  is Riemann-integrable ( $f \in \mathcal{R}$ ) and  $\int_a^b f(x) dx$ .

Since  $f$  is bounded, there are  $m, M$  such that  $m \leq f(x) \leq M$ .

Thus,  $m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$ .



#### Definition 13.1.2: Riemann-Stieltjes Integral

Let  $\alpha$  be monotonically increasing on  $[a,b]$ .

Then for each partition  $P$ , let  $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ .

For real bounded  $f$ , let  $L(P,f,\alpha) = \sum_{i=1}^n m_i \Delta \alpha_i$  and  $U(P,f,\alpha) = \sum_{i=1}^n M_i \Delta \alpha_i$ .

Thus,  $\int_a^b f(x) d\alpha(x) = \sup L(P,f,\alpha)$  and  $\overline{\int}_a^b f(x) d\alpha(x) = \inf U(P,f,\alpha)$ .

If  $\int_a^b f(x) d\alpha(x) = \overline{\int}_a^b f(x) d\alpha(x)$ , then  $f \in \mathcal{R}(\alpha)$  with value  $\int_a^b f(x) d\alpha(x)$ .

#### Definition 13.1.3: Refinement

Partition  $Q$  is a refinement of  $P$  if  $P \subset Q$ .

For partitions  $P_1, P_2$ , then  $Q = P_1 \cup P_2$  is the common refinement.

**Theorem 13.1.4: Refinements monotonically increase  $L(P,f)$  & decrease  $U(P,f)$** 

If  $Q$  is a refinement of  $P$ , then:

$$L(P,f,\alpha) \leq L(Q,f,\alpha) \leq U(Q,f,\alpha) \leq U(P,f,\alpha)$$

**Proof**

Since  $Q$  is a refinement of  $P$ , then  $P \subset Q$ .

Suppose  $Q = P \cup \{x^*\}$  where  $P = \{x_0, \dots, x_n\}$  and  $Q = \{x_0, \dots, x_{k-1}, x^*, x_k, \dots, x_n\}$ .

Regardless of anymore points, the process below will be analogous.

$$L(P,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta \alpha_i + m_{[x_{k-1}, x_k]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + m_{[x_{k-1}, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n m_i \Delta \alpha_i$$

$$L(Q,f,\alpha) = \sum_{i=1}^{k-1} m_i \Delta \alpha_i + m_{[x_{k-1}, x^*]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + m_{[x^*, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n m_i \Delta \alpha_i$$

Since  $[x_{k-1}, x^*], [x^*, x_k] \subset [x_{k-1}, x_k]$ , then  $m_{[x_{k-1}, x_k]} \leq m_{[x_{k-1}, x^*]}, m_{[x^*, x_k]}$ . Thus:

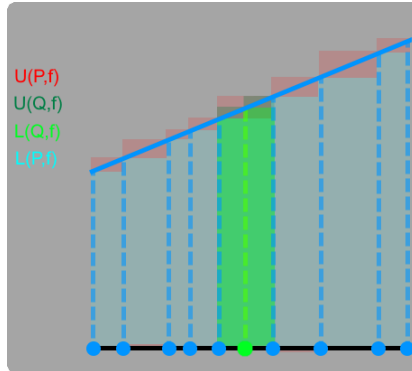
$$L(Q,f,\alpha) - L(P,f,\alpha) = (m_{[x_{k-1}, x^*]} - m_{[x_{k-1}, x_k]}) [\alpha(x^*) - \alpha(x_{k-1})] \\ + (m_{[x^*, x_k]} - m_{[x_{k-1}, x_k]}) [\alpha(x_k) - \alpha(x^*)] \geq 0.$$

$$U(P,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta \alpha_i + M_{[x_{k-1}, x_k]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + M_{[x_{k-1}, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n M_i \Delta \alpha_i$$

$$U(Q,f,\alpha) = \sum_{i=1}^{k-1} M_i \Delta \alpha_i + M_{[x_{k-1}, x^*]} [\alpha(x^*) - \alpha(x_{k-1})] \\ + M_{[x^*, x_k]} [\alpha(x_k) - \alpha(x^*)] + \sum_{i=k+1}^n M_i \Delta \alpha_i$$

Since  $[x_{k-1}, x^*], [x^*, x_k] \subset [x_{k-1}, x_k]$ , then  $M_{[x_{k-1}, x_k]} \geq M_{[x_{k-1}, x^*]}, M_{[x^*, x_k]}$ . Thus:

$$U(Q,f,\alpha) - U(P,f,\alpha) = (M_{[x_{k-1}, x^*]} - M_{[x_{k-1}, x_k]}) [\alpha(x^*) - \alpha(x_{k-1})] \\ + (M_{[x^*, x_k]} - M_{[x_{k-1}, x_k]}) [\alpha(x_k) - \alpha(x^*)] \leq 0.$$

**Theorem 13.1.5: Lower Riemann Integral  $\leq$  Upper Riemann Integral**

$$\int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$$

**Proof**

For partitions  $P_1, P_2$ , let  $L(P_1, f, \alpha)$  and  $U(P_2, f, \alpha)$ . Let  $P = P_1 \cup P_2$ . Thus:

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$$

Thus, over all partitions for  $P_1$ ,  $\int_a^b f d\alpha \leq U(P_2, f, \alpha)$

Thus, over all partitions for  $P_2$ ,  $\int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha$

**Theorem 13.1.6: Riemann-Integrability  $\epsilon$  Definition**

$f \in \mathcal{R}(\alpha)$  if and only if for every  $\epsilon > 0$ , there exists a partition  $P$  such that  $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

**Proof**

If  $f \in \mathcal{R}(\alpha)$ , then  $\int_a^f d\alpha = \overline{\int}_a^b f d\alpha = \int_a^b f d\alpha$ . For  $\epsilon > 0$ , there exists partitions  $P_1, P_2$ :  
 $\int_a^b f d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2}$        $U(P_2, f, \alpha) - \int_a^b f d\alpha < \frac{\epsilon}{2}$   
 Then for partition  $P = P_1 \cup P_2$ , then:  
 $\int_a^b f d\alpha - L(P, f, \alpha) \leq \int_a^b f d\alpha - L(P_1, f, \alpha) < \frac{\epsilon}{2}$   
 $U(P, f, \alpha) - \int_a^b f d\alpha \leq U(P_2, f, \alpha) - \int_a^b f d\alpha < \frac{\epsilon}{2}$   
 Thus,  $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ .

For  $\epsilon > 0$ , there is a partition  $P$  such that  $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ .

Since  $L(P, f, \alpha) \leq \int_a^b f d\alpha \leq \overline{\int}_a^b f d\alpha \leq U(P, f, \alpha)$ , then  $\overline{\int}_a^b f d\alpha - \int_a^b f d\alpha < \epsilon$ .

**Theorem 13.1.7: Properties of Riemann-Integrability**

- (a) If  $f \in \mathcal{R}(\alpha)$ , then  $U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon$  for every refinement of  $P, Q$

**Proof**

By **theorem 13.1.6**, for  $\epsilon > 0$ , there is a  $P$  such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Then by **theorem 13.1.4**, for any refinement of  $P, Q$ , then:

$$U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon.$$

- (b) If  $f \in \mathcal{R}(\alpha)$  where  $P = \{x_0, \dots, x_n\}$  and  $s_i, t_i \in [x_{i-1}, x_i]$ , then:

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

**Proof**

By **theorem 13.1.6**, for  $\epsilon > 0$ , there is a  $P$  such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\sum_{i=1}^n M_i \Delta\alpha_i - \sum_{i=1}^n m_i \Delta\alpha_i < \epsilon$$

Since  $s_i, t_i \in [x_{i-1}, x_i]$ , then  $m_i \leq f(s_i), f(t_i) \leq M_i$ .

Thus,  $|f(s_i) - f(t_i)| \leq M_i - m_i$ .

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i \leq \sum_{i=1}^n M_i - m_i \Delta\alpha_i \leq \epsilon$$

- (c) If  $f \in \mathcal{R}(\alpha)$  where  $P = \{x_0, \dots, x_n\}$  and  $t_i \in [x_{i-1}, x_i]$ , then:

$$|\sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha| < \epsilon$$

**Proof**

Since  $\sup L(P, f, \alpha) = \int_a^b f d\alpha = \int_a^b f d\alpha = \overline{\int}_a^b f d\alpha = \inf U(P, f, \alpha)$ , then:

$$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha)$$

Since  $t_i \in [x_{i-1}, x_i]$ , then  $m_i \leq f(t_i) \leq M_i$ . Thus:

$$\begin{aligned} L(P, f, \alpha) &= \sum_{i=1}^n m_i \Delta\alpha_i \leq \sum_{i=1}^n f(t_i) \Delta\alpha_i \\ &\leq \sum_{i=1}^n M_i \Delta\alpha_i = U(P, f, \alpha) \end{aligned}$$

Thus,  $|\sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha| \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ .

## 13.2 Riemann-Integrable Functions

### Theorem 13.2.1: Continuous functions are Riemann-Integrable

If  $f$  is continuous on  $[a, b]$ , then  $f \in \mathcal{R}(\alpha)$

#### Proof

For  $\epsilon > 0$ , choose  $\eta > 0$  such that  $[\alpha(b) - \alpha(a)]\eta < \epsilon$ . Since  $f$  is continuous and  $[a, b]$  is compact, then  $f$  is uniformly continuous. Thus, for  $\eta > 0$ , there is a  $\delta > 0$  such that for all  $x, t \in [a, b]$  where  $|x - t| < \delta$ , then  $|f(x) - f(t)| < \eta$ . For partition  $P$  of  $[a, b]$  such that  $\Delta x_i < \delta$  for all  $i = \{1, \dots, n\}$ , then  $M_i - m_i \leq \eta$  for each  $i$ . Thus:

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \leq \sum_{i=1}^n \eta \Delta \alpha_i = \eta [\alpha(b) - \alpha(a)] < \epsilon$$

### Theorem 13.2.2: Monotonic functions are Riemann-Integrable

If  $f$  is monotonic on  $[a, b]$  and  $\alpha$  is continuous on  $[a, b]$ , then  $f \in \mathcal{R}(\alpha)$

#### Proof

Since  $\alpha$  is continuous on  $[a, b]$ , let  $\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$  where  $n \in \mathbb{Z}_+$ . Let partition  $P = \{\alpha(x_0), \dots, \alpha(x_n)\}$ . Suppose  $f$  is monotonically increasing. Thus:

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n (M_i - m_i) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] = \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] \end{aligned}$$

For  $\epsilon > 0$ , there exists a  $n$  such that  $\frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \epsilon$  so  $f \in \mathcal{R}(\alpha)$ .

If  $f$  is monotonically decreasing, then  $\sum_{i=1}^n (M_i - m_i) = \sum_{i=1}^n [f(x_{i-1}) - f(x_i)]$ .

### Theorem 13.2.3: Bounded functions with finite discontinuities are Riemann-Integrable

If  $f$  is bounded on  $[a, b]$  with finitely many discontinuities and  $\alpha$  is continuous at every discontinuity, then  $f \in \mathcal{R}(\alpha)$

#### Proof

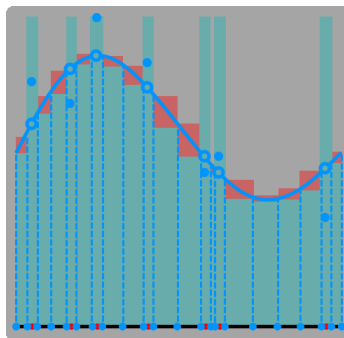
Since  $f$  is bounded, let  $M = \sup |f(x)|$  and  $E$  be the set of discontinuities of  $f$ .

Since  $E$  is finite and  $\alpha$  is continuous over  $E$ , then for  $\epsilon > 0$ , there are finitely many disjoint  $[u_j, v_j]$  where  $\sum [\alpha(v_j) - \alpha(u_j)] < \epsilon$  which cover  $E$ .

Let  $K = [a, b] \setminus \bigcup (u_j, v_j)$  which is compact. Since  $f$  is continuous over compact  $K$ , then  $f$  is uniformly continuous over  $K$ . Thus, for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for  $s, t \in K$  where  $|s - t| < \delta$ , then  $|f(s) - f(t)| < \epsilon$ .

Let partition  $P = \{x_0, \dots, x_n\}$  of  $[a, b]$  where each  $\Delta x_i < \delta$  and if  $x \in (u_j, v_j) \notin P$ , but  $u_j, v_j \in P$ . Thus,  $M_i - m_i \leq 2M$  for each  $i$  and  $M_i - m_i \leq \epsilon$  unless  $x_{i-1}$  is a  $u_j$ , then:

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i = \sum_K (M_i - m_i) \Delta \alpha_i + \sum_{K^c} (M_i - m_i) \Delta \alpha_i \\ &\leq \epsilon \sum_K \Delta \alpha_i + 2M \sum_{K^c} \Delta \alpha_i \leq [\alpha(b) - \alpha(a)]\epsilon + 2M\epsilon \end{aligned}$$



**Theorem 13.2.4: Composite of continuous-integrable functions are Riemann-Integrable**

If  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  where  $f \in [m, M]$  and  $\phi$  is continuous on  $[m, M]$  such that  $h(x) = \phi(f(x))$ , then  $h \in \mathcal{R}(\alpha)$

**Proof**

Since  $\phi$  is continuous and  $[m, M]$  is compact, then  $\phi$  is uniformly continuous. Thus, for  $\epsilon > 0$ , there is a  $0 < \delta < \epsilon$  such that for all  $s, t \in [m, M]$  where  $|s - t| \leq \delta$ , then  $|\phi(s) - \phi(t)| < \epsilon$ .

Since  $f \in \mathcal{R}(\alpha)$ , there is a partition  $P = \{x_0, \dots, x_n\}$  such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$$

For each  $i = \{1, \dots, n\}$ , let  $i \in A$  if  $M_i - m_i < \delta$  and  $i \in B$  if  $M_i - m_i \geq \delta$ .

Let  $m_i^* = \inf \phi(f([x_{i-1}, x_i]))$  and  $M_i^* = \sup \phi(f([x_{i-1}, x_i]))$ .

For  $A$ , since  $M_i - m_i < \delta$ , then  $M_i^* - m_i^* \leq \epsilon$ .

For  $B$ ,  $M_i^* - m_i^* \leq 2K$  where  $K = \sup_{[m, M]} |\phi|$ .

$$\delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2$$

$$\sum_{i \in B} \Delta \alpha_i \leq \delta < \epsilon$$

Thus:

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\ &\leq \epsilon \sum_{i \in A} \Delta \alpha_i + 2K \sum_{i \in B} \Delta \alpha_i \\ &\leq \epsilon [\alpha(b) - \alpha(a)] + 2K\epsilon < \epsilon [\alpha(b) - \alpha(a) + 2K] \end{aligned}$$

**13.3 Integral Properties****Theorem 13.3.1: Integral Additive Properties**

(a) If  $f_1, f_2 \in \mathcal{R}(\alpha)$  on  $[a, b]$  and constant  $c$ , then  $f_1 + f_2, cf_1 \in \mathcal{R}(\alpha)$  and

$$\int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$\int_a^b cf_1 \, d\alpha = c \int_a^b f_1 \, d\alpha$$

**Proof**

Since  $f_1, f_2 \in \mathcal{R}(\alpha)$ , then there are partitions  $P_1, P_2$  such that for  $\epsilon > 0$ :

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\epsilon}{2} \quad U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\epsilon}{2}$$

Thus for partition  $P = P_1 \cup P_2$ :

$$U(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_1, \alpha) - L(P, f_2, \alpha) < \epsilon$$

$$U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) < \epsilon$$

For any partition  $Q$ :

$$\begin{aligned} L(Q, f_1, \alpha) + L(Q, f_2, \alpha) &\leq L(Q, f_1 + f_2, \alpha) \leq U(Q, f_1 + f_2, \alpha) \\ &\leq U(Q, f_1, \alpha) + U(Q, f_2, \alpha) \end{aligned}$$

Thus,  $f_1 + f_2 \in \mathcal{R}(\alpha)$  where:

$$\begin{aligned} \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha &= \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \leq \int_a^b f_1 + f_2 \, d\alpha \\ &= \int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 + f_2 \, d\alpha \\ &\leq \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha \end{aligned}$$

Proof for  $cf_1$  is analogous by replacing  $\frac{\epsilon}{2}$  with  $\frac{\epsilon}{c}$ .

(b) If  $f_1, f_2 \in \mathcal{R}(\alpha)$  and  $f_1(x) \leq f_2(x)$  on  $[a, b]$ , then  $\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$

**Proof**

Since  $f_1, f_2 \in \mathcal{R}(\alpha)$ , then by part a,  $0 \leq \int_a^b f_2 - f_1 \, d\alpha = \int_a^b f_2 \, d\alpha - \int_a^b f_1 \, d\alpha$ .



- (c) If  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  and  $c \in (a, b)$ , then  $f \in \mathcal{R}(\alpha)$  on  $[a, c], [c, b]$  and
- $$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

**Proof**

Since  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$ , there is a partition  $P$  of  $[a, b]$  such that for  $\epsilon > 0$ :

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

For partition  $P$  of  $[a, b]$ , let refinement of  $P$ ,  $Q = P \cup \{c\}$ . Thus:

$$L(P, f, \alpha) \leq L(Q, f, \alpha) \leq U(Q, f, \alpha) \leq U(P, f, \alpha)$$

Thus, let  $A = (P < c) \cup c \in [a, c]$  and  $B = c \cup (c < P) \in (c, b)$ :

$$\begin{aligned} L(Q, f, \alpha) &= \sum_Q m_q \Delta \alpha_q \\ &\leq \sum_A m_a \Delta \alpha_a + \sum_B m_b \Delta \alpha_b = L(A, f, \alpha) + L(B, f, \alpha) \end{aligned}$$

$$\begin{aligned} U(Q, f, \alpha) &= \sum_Q M_q \Delta \alpha_q \\ &\geq \sum_A M_a \Delta \alpha_a + \sum_B M_b \Delta \alpha_b = U(A, f, \alpha) + U(B, f, \alpha) \end{aligned}$$

Since  $Q$  is a refinement of  $P$ , then  $U(Q, f, \alpha) - L(Q, f, \alpha) < \epsilon$ . Thus:

$$0 \leq U(A, f, \alpha) + U(B, f, \alpha) - L(A, f, \alpha) - L(B, f, \alpha) < \epsilon$$

$$U(A, f, \alpha) - L(A, f, \alpha) < \epsilon \quad U(B, f, \alpha) - L(B, f, \alpha) < \epsilon$$

Thus,  $f \in \mathcal{R}(\alpha)$  on  $[a, c], [c, b]$  where:

$$\begin{aligned} \int_a^b f d\alpha &\leq \int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha \\ &= \int_a^c f d\alpha + \int_c^b f d\alpha \leq \int_a^b f d\alpha \end{aligned}$$

Since  $\int_a^b f d\alpha, \int_a^c f d\alpha = \int_a^b f d\alpha$ , then  $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$ .

- (d) If  $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$  and constant  $c$ , then  $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ ,  $f \in \mathcal{R}(c\alpha_1)$  and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\int_a^b f d(c\alpha_1) = c \int_a^b f d\alpha_1$$

**Proof**

Since  $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$ , then there are partitions  $P_1, P_2$  where for  $\epsilon > 0$ :

$$U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \frac{\epsilon}{2} \quad U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \frac{\epsilon}{2}$$

Thus, for partition  $P = P_1 \cup P_2$ :

$$\begin{aligned} \sum_{i=1}^n (M_i - m_i) \Delta \alpha_{1i} &< \frac{\epsilon}{2} \quad \sum_{i=1}^n (M_i - m_i) \Delta \alpha_{2i} < \frac{\epsilon}{2} \\ \sum_{i=1}^n (M_i - m_i) (\Delta \alpha_{1i} + \Delta \alpha_{2i}) &< \epsilon \\ U(P, f, \alpha_1 + \alpha_2) - L(P, f, \alpha_1 + \alpha_2) &< \epsilon \end{aligned}$$

For any partition  $Q$ :

$$\begin{aligned} L(Q, f, \alpha_1) + L(Q, f, \alpha_2) &\leq L(Q, f, \alpha_1 + \alpha_2) \\ &\leq U(Q, f, \alpha_1 + \alpha_2) \\ &\leq U(Q, f, \alpha_1) + U(Q, f, \alpha_2) \end{aligned}$$

Thus,  $f \in \mathcal{R}(\alpha_1 + \alpha_2)$  where:

$$\begin{aligned} \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \leq \int_a^b f d(\alpha_1 + \alpha_2) \\ &= \int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d(\alpha_1 + \alpha_2) \\ &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \end{aligned}$$

Proof for  $c\alpha_1$  is analogous by replacing  $\frac{\epsilon}{2}$  with  $\frac{\epsilon}{c}$ .

**Theorem 13.3.2: Integral Multiplicative Properties**

(a) If  $f, g \in \mathcal{R}(\alpha)$  on  $[a, b]$ , then  $fg \in \mathcal{R}(\alpha)$

**Proof**

Since  $f, g \in \mathcal{R}(\alpha)$ , then  $f+g, f-g \in \mathcal{R}(\alpha)$ .

By **theorem 13.2.4**, let  $\phi(t) = t^2$  which is continuous so

$\phi(f+g) = (f+g)^2, \phi(f-g) = (f-g)^2 \in \mathcal{R}(\alpha)$ .

Thus,  $4fg = (f+g)^2 - (f-g)^2 \in \mathcal{R}(\alpha)$ .

(b) If  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$ , then  $|f| \in \mathcal{R}(\alpha)$  where  $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$

**Proof**

By **theorem 13.2.4**, let  $\phi(t) = |t|$  which is continuous so  $|f| \in \mathcal{R}(\alpha)$ .

Then choose  $c = \pm 1$  such that  $c \int f d\alpha \geq 0$ . Then:

$$|\int f d\alpha| = c \int f d\alpha = \int c f d\alpha \leq \int |f| d\alpha$$

**13.4 Change of Variable****Definition 13.4.1: Unit Step Function**

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

**Theorem 13.4.2: Integrating  $f$  over  $I$  centered at  $s$** 

If  $f$  is bounded on  $[a, b]$  and continuous at  $s \in (a, b)$  where  $\alpha(x) = I(x-s)$ , then:

$$\int_a^b f d\alpha = f(s)$$

**Intuition**

If  $x < s < y$ , then  $\Delta I = I(y-s) - I(x-s) = 1 - 0 = 1$  else  $\Delta I = 0$ .

So,  $f(x)d\alpha(x) \approx f(x)\Delta I$  have only  $f(s)\Delta I = f(s)$  since the others  $\Delta I = 0$ .

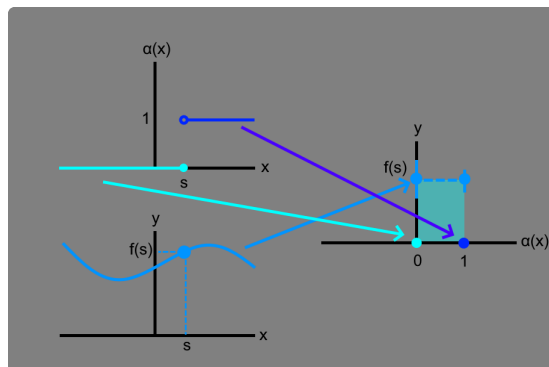
**Proof**

For partition  $P = \{x_0, x_1, x_2, x_3\}$  where  $x_1 = s$ :

$$L(P, f, \alpha) = m_2 \quad U(P, f, \alpha) = M_2$$

Since  $f$  is continuous at  $s$ , then for  $\epsilon > 0$ , there is a  $\delta > 0$  where for all  $x \in [s, s+\delta]$ , then  $|f(x) - f(s)| < \frac{\epsilon}{2}$ . Thus,  $M_2 - m_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$  so  $\int f d\alpha$  exist where:

$$f(s) - m_2 < \frac{\epsilon}{2} \text{ so } \underline{\int} f d\alpha = f(s) \quad M_2 - f(s) < \frac{\epsilon}{2} \text{ so } \overline{\int} f d\alpha = f(s)$$



**Theorem 13.4.3: Integrating f over a step function**

If  $c_n \geq 0$  where  $\sum c_n$  converges,  $\{s_n\}$  is a sequence of distinct points in  $(a,b)$ , and  $\alpha(x) = \sum c_n I(x - s_n)$ . Then for continuous  $f$  on  $[a,b]$ :

$$\int_a^b f d\alpha = \sum c_n f(s_n)$$

**Intuition**

Similar to **theorem 13.4.2**, but over a step function. The  $\{s_n\}$  determines where the steps are and the  $\{\sum c_n\}$  determines the value at each step.

Thus,  $f(x)d\alpha(x)$  have only:

$$f(s_n) \cdot (\text{value}_{\text{current step}} - \text{value}_{\text{previous step}}) = f(s_n) \cdot (\sum c_n - \sum c_{n-1}) = f(s_n) \cdot c_n$$

**Proof**

Since  $\alpha(x) = \sum c_n I(x - s_n) \leq \sum c_n$ , then by the comparison test,  $\alpha(x)$  converges.

Since  $c_n, I(x - s_n) \geq 0$ , then  $\alpha(x)$  is monotonic.

Since  $a < s_n$  for any  $n$ , then  $\alpha(a) = \sum c_n I(a - s_n) = \sum c_n 0 = 0$ .

Since  $b > s_n$  for any  $n$ , then  $\alpha(b) = \sum c_n I(b - s_n) = \sum c_n 1 = \sum c_n$ .

Since  $\sum c_n$  converges, then for  $\epsilon > 0$ , there is a  $N$  such that  $\sum_{n=N+1}^{\infty} c_n < \epsilon$ .

Let  $\alpha_1(x) = \sum_{n=1}^N c_n I(x - s_n)$  and  $\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$ . By **theorem 13.4.2**:

$$\int_a^b f d\alpha_1 = \int_a^b f d(\sum_{n=1}^N c_n I(x - s_n)) = \sum_{n=1}^N c_n f(s_n)$$

$$|\int_a^b f d\alpha_2| = \sum_{n=N+1}^{\infty} c_n f(s_n) \leq \sum_{n=N+1}^{\infty} c_n \sup(|f(x)|) = \sup(|f(x)|) \epsilon$$

Thus,  $\int f d\alpha = \int f d(\alpha_1 + \alpha_2) = \int f d\alpha_1 + \int f d\alpha_2 = \sum_{n=1}^N c_n f(s_n) + \sup(|f(x)|) \epsilon$

**Theorem 13.4.4:  $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$** 

If  $\alpha' \in \mathcal{R}$  on  $[a,b]$  and  $f$  is real, bounded on  $[a,b]$ , then  $f \in \mathcal{R}(\alpha)$  if and only if  $f\alpha' \in \mathcal{R}$ . Then:

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$$

**Intuition**

If  $\alpha$  is differentiable on  $[x,y]$ , then by the Mean Value Theorem, there is a  $t \in [x,y]$ :

$$\alpha(x) - \alpha(y) = \alpha'(t) \cdot (x - y)$$

Since  $d\alpha \approx \Delta\alpha(x) = \alpha'(t)\Delta x \approx \alpha'(x) dx$ , then  $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$ .

**Proof**

Since  $\alpha' \in \mathcal{R}$ , then  $\epsilon > 0$ , there is a partition  $P = \{x_0, \dots, x_n\}$  such that:

$$U(P, \alpha') - L(P, \alpha') < \epsilon$$

By the Mean Value Theorem, there are  $t_i \in [x_{i-1}, x_i]$  such that  $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$ .

Then for  $s_i \in [x_{i-1}, x_i]$ :

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq U(P, \alpha') - L(P, \alpha') < \epsilon$$

Let  $M = \sup(|f(x)|)$ . Since  $\sum_{i=1}^n f(s_i)\Delta\alpha_i = \sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i$ , then:

$$\begin{aligned} & |\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i| \\ &= |\sum_{i=1}^n f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^n f(s_i)\alpha'(s_i)\Delta x_i| \\ &\leq M |\sum_{i=1}^n \alpha'(t_i)\Delta x_i - \sum_{i=1}^n \alpha'(s_i)\Delta x_i| = M\epsilon \end{aligned}$$

Thus:

$$\sum_{i=1}^n f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\epsilon \quad \sum_{i=1}^n f(s_i)\Delta\alpha_i \geq L(P, f\alpha') + M\epsilon$$

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\epsilon \quad L(P, f, \alpha) \geq L(P, f\alpha') + M\epsilon$$

$$|\int f d\alpha - \int f\alpha' dx| < M\epsilon \quad |\int f d\alpha - \int f\alpha' dx| < M\epsilon$$

Thus,  $f \in \mathcal{R}(\alpha)$  if and only if  $f\alpha' \in \mathcal{R}$ .

**Theorem 13.4.5: Integral Change of Variable:**  $\int_a^b f(x) dx = \int_A^B f(\phi(y))\phi'(y) dy$

Let strictly increasing continuous  $\phi: [A,B] \rightarrow [a,b]$  and  $f \in \mathcal{R}(\alpha)$  on  $[a,b]$ .

Let  $\beta(y) = \alpha(\phi(y))$  and  $g(y) = f(\phi(y))$  for  $y \in [A,B]$ . Then  $g \in \mathcal{R}(\beta)$  where:

$$\int_A^B g d\beta = \int_a^b f d\alpha$$

### Intuition

Partition of  $[a,b] = \{x_0, \dots, x_n\} \sim$  partition of  $[A,B] = \{y_0, \dots, y_n\}$  where  $x_i = \phi(y_i)$ .  
Thus,  $g(y)d\beta(y) \approx f(\phi(y))\Delta\alpha(\phi(y)) = f(x)\Delta\alpha(x) \approx f(x)d\alpha$ .

### Proof

Since  $f \in \mathcal{R}(\alpha)$ , then for  $\epsilon > 0$ , there is a partition  $P$  such that:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

For partition  $P = \{x_0, \dots, x_n\}$  of  $[a,b]$ , there is a partition  $Q = \{y_0, \dots, y_n\}$  of  $[A,B]$  where  $x_i = \phi(y_i)$ . Thus:

$$L(Q, g, \beta) = L(Q, f(\phi(y)), \alpha(\phi(y))) = L(P, f(x), \alpha(x)) = L(P, f, \alpha)$$

$$U(Q, g, \beta) = U(Q, f(\phi(y)), \alpha(\phi(y))) = U(P, f(x), \alpha(x)) = U(P, f, \alpha)$$

Thus,  $U(Q, g, \beta) - L(Q, g, \beta) = U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$  so  $g \in \mathcal{R}(\beta)$  and

$$\int_A^B g d\beta = \int_a^b f d\alpha.$$

Let  $\alpha(x) = x$ . Then  $\beta(y) = \phi(y)$ . If  $\beta' \in \mathcal{R}$  on  $[A,B]$ , then by **theorem 13.4.5**:

$$\int_a^b f(x) dx = \int_a^b f d\alpha = \int_A^B g d\beta = \int_A^B g(y)\beta'(y) dy = \int_A^B f(\phi(y))\phi'(y) dy$$

## 13.5 Fundamental Theorem of Calculus

**Theorem 13.5.1:** If  $F(x) = \int f(x)dx$ , then  $F'(x) = f(x)$

Let  $f \in \mathcal{R}$  on  $[a,b]$ . For  $x \in [a,b]$ , let  $F(x) = \int_a^x f(t) dt$ .

Then  $F$  is continuous on  $[a,b]$  and if  $f$  is continuous at  $x_0 \in [a,b]$ , then  $F$  is differentiable at  $x_0$  where  $F'(x_0) = f(x_0)$ .

### Intuition

If  $f$  is integrable, then  $|F(x) - F(y)| = |\int_x^y f(t)dt| < \epsilon$  if  $x$  and  $y$  are close enough.  
If  $f$  is continuous at  $x_0 \in [t, y]$ , then for close enough  $t, y$ :

$$|\frac{F(y)-F(t)}{y-t} - f(x_0)| = |\frac{1}{y-t} \int_t^y [f(x) - f(x_0)]| < \epsilon$$

### Proof

Since  $f \in \mathcal{R}$ , then  $f$  is bounded. Let  $|f(t)| \leq M$  for any  $t \in [a,b]$ . Then for  $\epsilon > 0$ , there is a  $\frac{\epsilon}{M} > \delta > 0$  such that for all  $x, y \in [a,b]$  where  $|y - x| < \delta$ , then:

$$|F(y) - F(x)| = |\int_a^y f(t)dt - \int_a^x f(t)dt| = |\int_x^y f(t)dt| \leq M|y - x| < M\delta < \epsilon$$

Thus,  $F$  is uniformly continuous on  $[a,b]$ .

Suppose  $f$  is continuous at  $x_0$ . Then for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $t \in [a,b]$  where  $|t - x_0| < \delta$ , then  $|f(t) - f(x_0)| < \epsilon$ .

Thus, for  $s, t \in [x_0 - \delta, x_0 + \delta]$  where  $s < x_0 < t$ :

$$\begin{aligned} |\frac{F(t)-F(s)}{t-s} - f(x_0)| &= |\frac{1}{t-s} \int_s^t f(x)dx - f(x_0)| \\ &= |\frac{1}{t-s} \int_s^t f(x)dx - \frac{1}{t-s}(t-s)f(x_0)| \\ &= |\frac{1}{t-s} \int_s^t f(x)dx - \frac{1}{t-s} \int_s^t f(x_0)dx| \\ &= |\frac{1}{t-s} \int_s^t [f(x) - f(x_0)]dx| < |\frac{1}{t-s}(t-s)\epsilon| = \epsilon \end{aligned}$$

Thus,  $F'(x_0) = f(x_0)$ .

**Theorem 13.5.2: Fundamental Theorem of Calculus:**  $\int_a^b f(x) dx = F(b) - F(a)$ 

If  $f \in \mathcal{R}$  on  $[a,b]$  and there is a differentiable  $F$  on  $[a,b]$  such that  $F' = f$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

**Intuition**

Since  $F$  is differentiable, then by the Mean Value Theorem, there is a  $t \in [x,y]$

$$F(y) - F(x) = (y - x) \cdot F'(t) = (y - x) \cdot f(t)$$

Thus,  $\int_a^b f(x) dx \approx \sum f(t) \Delta x = \sum [F(x_i) - F(x_{i-1})] = F(b) - F(a)$

**Proof**

Since  $f \in \mathcal{R}$ , then for  $\epsilon > 0$ , there is a partition  $P = \{x_0, \dots, x_n\}$  of  $[a,b]$  such that:

$$U(P,f) - L(P,f) < \epsilon$$

Since there is a differentiable  $F$  on  $[a,b]$ , then  $F$  is differentiable over any  $[x_{i-1}, x_i]$ .

Then by the Mean Value Theorem, there are  $t_i \in (x_{i-1}, x_i)$  such that:

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) F'(t_i) = \Delta x_i f(t_i)$$

Thus,  $\sum_{i=1}^n f(t_i) \Delta x_i = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(b) - F(a)$ .

Since  $\sum_{i=1}^n f(t_i) \Delta x_i \leq \sum_{i=1}^n \sup(f([x_{i-1}, x_i])) \Delta x_i = U(P,f)$ , then:

$$|F(b) - F(a) - \int_a^b f(x) dx| = |\sum_{i=1}^n f(t_i) \Delta x_i - \int_a^b f(x) dx| \leq U(P,f) - L(P,f) < \epsilon$$

**Theorem 13.5.3: Integration by Parts**

Suppose  $F, G$  are differentiable on  $[a,b]$  and  $F' = f$ ,  $G' = g \in \mathcal{R}$ . Then:

$$\int_a^b F(x)g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x) dx$$

**Intuition**

By the derivative product rule,  $(HG)' = H'G + HG'$ . Then:

$$\int H'G dx = \int (HG)' - HG' dx = [HG]_a^b - \int HG' dx$$

**Proof**

Let  $H(x) = F(x)G(x)$  where  $H'(x) = f(x)G(x) + F(x)g(x)$ .

Since  $F, G$  are differentiable and thus, continuous, then  $F, G \in \mathcal{R}$ .

Thus,  $H' \in \mathcal{R}$ . Then by **theorem 13.5.2**:

$$\int_a^b H'(x) dx = H(b) - H(a)$$

$$\int_a^b f(x)G(x) + F(x)g(x) dx = H(b) - H(a)$$

$$\int_a^b F(x)g(x) dx = H(b) - H(a) - \int_a^b f(x)G(x) dx$$

## 13.6 Integration of Vector-Valued Functions

### Definition 13.6.1: Integration of Vector-Valued Functions

Let real  $f_1, \dots, f_k$  be defined on  $[a, b]$  where  $f = (f_1, \dots, f_k)$ .

Then, let  $f \in \mathcal{R}(\alpha)$  if each  $f_i \in \mathcal{R}(\alpha)$  where  $\int_a^b f \, d\alpha = (\int_a^b f_1 \, d\alpha, \dots, \int_a^b f_k \, d\alpha)$ .

Thus, all these theorems hold true for vector-valued functions:

(a) **Theorem 13.3.1a**

If  $f_1, f_2 \in \mathcal{R}(\alpha)$  and constant  $c$ , then:

$$f_1 + f_2 \in \mathcal{R}(\alpha) \text{ with } \int_a^b f_1 + f_2 \, d\alpha = \int_a^b f_1 \, d\alpha + \int_a^b f_2 \, d\alpha$$

$$cf_1 \in \mathcal{R}(\alpha) \text{ with } \int_a^b cf_1 \, d\alpha = c \int_a^b f_1 \, d\alpha$$

(b) **Theorem 13.3.1c**

If  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  where  $c \in (a, b)$ , then  $f \in \mathcal{R}(\alpha)$  on  $[a, c], [c, b]$  where:

$$\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha$$

(c) **Theorem 13.3.1e**

If  $f \in \mathcal{R}(\alpha_1), \mathcal{R}(\alpha_2)$  and constant  $c$ , then:

$$f \in \mathcal{R}(\alpha_1 + \alpha_2) \text{ with } \int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$$

$$f \in \mathcal{R}(c\alpha_1) \text{ with } \int_a^b f \, d(c\alpha_1) = c \int_a^b f \, d\alpha_1$$

(d) **Theorem 13.4.4**

If  $\alpha' \in \mathcal{R}$  on  $[a, b]$ , then  $f \in \mathcal{R}(\alpha)$  if and only if  $f\alpha' \in \mathcal{R}$ .

$$\int_a^b f(x) \, d\alpha = \int_a^b f(x)\alpha'(x) \, dx$$

(e) **Theorem 13.5.2**

If  $f \in \mathcal{R}$  and there is a differentiable  $F$  on  $[a, b]$  such that  $F' = f$ , then:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

### Theorem 13.6.2: $|\int f \, d\alpha| \leq \int |f| \, d\alpha$

If  $f: [a, b] \rightarrow \mathbb{R}^k$  where  $f \in \mathcal{R}(\alpha)$ , then  $|f| \in \mathcal{R}(\alpha)$  where:

$$|\int_a^b f \, d\alpha| \leq \int_a^b |f| \, d\alpha$$

#### Proof

For  $f = (f_1, \dots, f_k)$ , then  $|f| = (f_1^2 + \dots + f_k^2)^{\frac{1}{2}}$ .

Since  $f \in \mathcal{R}(\alpha)$ , then each  $f_i \in \mathcal{R}(\alpha)$  so  $f_1^2 + \dots + f_k^2 \in \mathcal{R}(\alpha)$ .

Since  $x^{\frac{1}{2}}$  is continuous on  $[0, \infty)$ , then by **theorem 13.2.4**,  $|f| = (f_1^2 + \dots + f_k^2)^{\frac{1}{2}} \in \mathcal{R}(\alpha)$ .

Let  $y = (y_1, \dots, y_k)$  where each  $y_i = \int f_i \, d\alpha$ . Thus,  $y = \int f \, d\alpha$  where:

$$|y|^2 = \sum_1^k y_i^2 = \sum_1^k (y_i \int f_i \, d\alpha) = \int (\sum y_i f_i) \, d\alpha$$

By the Schwarz inequality,  $\sum y_i f_i(t) \leq |y||f(t)|$ . Thus:

$$|y|^2 = \int (\sum y_i f_i) \, d\alpha \leq \int |y||f| \, d\alpha$$

$$|\int_a^b f \, d\alpha| = |y| \leq \int |f| \, d\alpha$$

## 13.7 Line Integrals

### Definition 13.7.1: Rectifiable Curves

A curve in  $\mathbb{R}^k$  is a continuous  $\gamma: [a, b] \rightarrow \mathbb{R}^k$ .

If  $\gamma$  is 1-1, then  $\gamma$  is called an arc.

If  $\gamma(a) = \gamma(b)$ ,  $\gamma$  is a closed curve.

For partition  $P = \{x_0, \dots, x_n\}$  and curve  $\gamma$  on  $[a, b]$ , let:

$$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$$

Then the length of  $\gamma$  is defined:

$$\Lambda(\gamma) = \sup(\Lambda(P, \gamma))$$

If  $\Lambda(\gamma) < \infty$ , then  $\gamma$  is rectifiable.

### Theorem 13.7.2: Line Integral of $\gamma = \int_a^b |\gamma'(x)| dx$

If  $\gamma'$  is continuous on  $[a, b]$ , then  $\gamma$  is rectifiable where

$$\Lambda(\gamma) = \int_a^b |\gamma'(x)| dx$$

### Proof

Since  $\gamma$  is differentiable, then by **theorem 13.5.2**, for  $a \leq x_{i-1} < x_i \leq b$ :

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(x) dx \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(x)| dx$$

Thus, for any partition  $P = \{x_0, \dots, x_n\}$ :

$$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^n \left( \int_{x_{i-1}}^{x_i} |\gamma'(x)| dx \right) = \int_a^b |\gamma'(x)| dx$$

$$\Lambda(\gamma) \leq \int_a^b |\gamma'(x)| dx$$

Since  $\gamma'$  is continuous on compact  $[a, b]$ , then  $\gamma'$  is uniformly continuous. Thus, for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $s, t \in [a, b]$  where  $|s - t| < \delta$ , then  $|\gamma'(s) - \gamma'(t)| < \epsilon$ .

Then for partition  $P$  where each  $\Delta x_i < \delta$  and  $x \in [x_{i-1}, x_i]$ :

$$|\gamma'(x)| \leq |\gamma'(x_i)| + \epsilon$$

Then:

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |\gamma'(x)| dx &\leq (|\gamma'(x_i)| + \epsilon) \Delta x_i = |\gamma'(x_i)| \Delta x_i + \epsilon \Delta x_i \\ &= \left| \int_{x_{i-1}}^{x_i} [\gamma'(x) + \gamma'(x_i) - \gamma'(x)] dx \right| + \epsilon \Delta x_i \\ &\leq \left| \int_{x_{i-1}}^{x_i} \gamma'(x) dx \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(x)] dx \right| + \epsilon \Delta x_i \\ &\leq |\gamma(x_i) - \gamma(x_{i-1})| + \epsilon \Delta x_i + \epsilon \Delta x_i \end{aligned}$$

Thus:

$$\begin{aligned} \int_a^b |\gamma'(x)| dx &= \int_{x_0}^{x_1} |\gamma'(x)| dx + \dots + \int_{x_{n-1}}^{x_n} |\gamma'(x)| dx \\ &\leq \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| + 2\epsilon(b-a) = \Lambda(P, \gamma) + 2\epsilon(b-a) \end{aligned}$$

Since  $\int_a^b |\gamma'(x)| dx \leq \Lambda(\gamma) + 2\epsilon(b-a) \leq \int_a^b |\gamma'(x)| dx + 2\epsilon(b-a)$ , then:

$$\Lambda(\gamma) = \int_a^b |\gamma'(x)| dx.$$

## 14 Sequences and Series of Functions

### 14.1 Pointwise Convergence of Functions

#### Definition 14.1.1: Sequences and Series of Functions

Suppose  $\{f_n\}$  is a sequence of functions defined on set  $E$ .

If  $\{f_n(x)\}$  converges for any  $x \in E$ , then:

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \text{ for } x \in E$$

So for  $x \in E$  and  $\epsilon > 0$ , there is a  $N_x$  such that for  $n \geq N_x$ :

$$|f_n(x) - f(x)| < \epsilon$$

If  $\sum f_n(x)$  converges for every  $x \in E$ , then:

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \text{ for } x \in E$$

### 14.2 Uniform Convergence of Functions

#### Definition 14.2.1: Uniform Convergence

$\{f_n\}$  converges uniformly on  $E$  to a function  $f$  if for all  $x \in E$ :

For  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  where for  $n \geq N$ , then  $|f_n(x) - f(x)| \leq \epsilon$

$\sum f_n(X)$  converges uniformly if  $\{s_n\}$  converges uniformly on  $E$

where  $\sum_{i=1}^n f_i(x) = s_n(x)$ :

For  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  where for  $m \geq n \geq N$ , then  $|\sum_{i=n}^m f_i(x)| \leq \epsilon$

#### Theorem 14.2.2: Cauchy Criterion for sequence of functions

$\{f_n\}$  converges uniformly on  $E$  if and only if:

For  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  where for  $n, m \geq N$  and every  $x \in E$ , then:

$$|f_n(x) - f_m(x)| \leq \epsilon$$

#### Intuition

Convergent sequences are Cauchy and Cauchy sequences in  $\mathbb{R}$  are convergent.

#### Proof

If  $\{f_n\}$  converges uniformly on  $E$ , then for  $\epsilon > 0$ , there is a  $N$  where for  $n, m \geq N$ :

$$\begin{aligned} |f_n(x) - f(x)| &\leq \frac{\epsilon}{2} & |f_m(x) - f(x)| &\leq \frac{\epsilon}{2} \\ |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

If for  $\epsilon > 0$ , there is a  $N \in \mathbb{Z}$  where for  $n, m \geq N$  and every  $x \in E$  so

$|f_n(x) - f_m(x)| \leq \epsilon$ , then  $\{f_n\}$  is a Cauchy sequence in  $\mathbb{R}^k$  and thus, converges.

Then there is a  $f(x)$  where  $f(x) = \lim_{m \rightarrow \infty} f_m(x)$ . Thus:

$$|f_n(x) - f(x)| \leq |f_n(x) - \lim_{m \rightarrow \infty} f_m(x)| \leq \epsilon$$



**Theorem 14.2.3: Connection between Convergence and Uniform Convergence**

Suppose for  $x \in E$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ . Let  $M_n = \sup_{x \in E} (|f_n(x) - f(x)|)$ .

Then  $\{f_n\}$  converges uniformly to  $f$  on  $E$  if and only if  $\lim_{n \rightarrow \infty} M_n = 0$ .

**Intuition**

Pointwise convergence implies for any particular  $x_0$  and  $\epsilon > 0$  so  $|f_n(x_0) - f(x_0)| < \epsilon$ .  
Uniform convergence implies for every  $x$  and  $\epsilon > 0$  so  $|f_n(x) - f(x)| < \epsilon$ .

Thus, uniform convergence implies pointwise convergence, but pointwise convergence might not imply uniform convergence since for  $n \geq N_1$ ,  $|f_n(x_0) - f(x_0)| < \epsilon$ , but there might always exist  $x_1 \neq x_0$  where  $|f_n(x_1) - f(x_1)| \not< \epsilon$  until  $N_2 > N_1$ .

If  $\sup_{x \in E} (|f_n(x) - f(x)|) \rightarrow 0$ , then  $x_1$  cannot exist and thus, pointwise implies uniform.

**Proof**

If  $\{f_n\}$  converges uniformly to  $f$  on  $E$ , then for  $\epsilon > 0$ , there is a  $N$  where for  $n \geq N$ :

$$|f_n(x) - f(x)| \leq \epsilon \quad \text{for all } x \in E$$

Thus,  $M_n = \sup_{x \in E} (|f_n(x) - f(x)|) \leq \epsilon$  so  $\lim_{n \rightarrow \infty} M_n \leq \epsilon$ .

If  $\lim_{n \rightarrow \infty} M_n = 0$ , then for  $\epsilon > 0$ , there is a  $N$  where for  $n \geq N$  so  $\lim_{n \rightarrow \infty} M_n \leq \epsilon$ .  
Since  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for  $x \in E$ , there is a  $N_x$  for each  $x$  where for  $n \geq N_x$ :

$$|f_n(x) - f(x)| \leq \epsilon$$

Since there is a  $N$  such that for  $n \geq N$  so  $M_n = \sup_{x \in E} (|f_n(x) - f(x)|) \leq \epsilon$ , then there

is  $\sup_{x \in E} (\{N_x\}) = N$  such that for all  $x \in E$  where  $n \geq N$ :

$$|f_n(x) - f(x)| \leq \sup_{x \in E} (|f_n(x) - f(x)|) = M_n \leq \epsilon$$

**Theorem 14.2.4: Condition for Uniform Convergence for Series**

For  $\{f_n\}$  defined on  $E$ , suppose  $|f_n(x)| \leq M_n$  for any  $x \in E$ .

If  $\sum M_n$  converges, then  $\sum f_n$  converges uniformly on  $E$ .

**Proof**

If  $\sum M_n$  converges, then for  $\epsilon > 0$ , there is a  $N$  where for  $m \geq n \geq N$ :

$$|\sum_{i=n}^m f_i(x)| \leq \sum_{i=n}^m |f_i(x)| \leq \sum_{i=n}^m M_i \leq \epsilon$$

### 14.3 Uniform Convergence and Continuity

**Theorem 14.3.1:**  $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$

Suppose  $\{f_n\}$  converges uniformly to  $f$  on a set  $E$ .

Let  $x \in E$  where  $\lim_{t \rightarrow x} f_n(t) = A_n$ .

Then  $\{A_n\}$  converges where  $\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} A_n$ .

#### Intuition

Since  $\{f_n\}$  converges uniformly so for any  $t$ , then  $\lim_{n \rightarrow \infty} f_n(t) = f(t)$ .

For  $t$  near  $x$ , then  $\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{t \rightarrow x} f(t)$ .

Note uniform convergence is essential since  $f_n \rightarrow f$  and  $f_n(t) \rightarrow f(t)$  for any  $t$  including  $t$  near  $x$ . Since pointwise convergence possibly  $f_n(t) \not\rightarrow f(t)$  for some  $t$  near  $x$ , then continuity possibly might not hold.

#### Proof

Since  $\{f_n\}$  converges uniformly, then for  $\epsilon > 0$ , there is a  $N$  where for  $m, n \geq N$  and every  $t \in E$ , then  $|f_n(t) - f_m(t)| \leq \epsilon$ . Then for  $t \rightarrow x$ :

$$|A_n - A_m| = |\lim_{t \rightarrow x} f_n(t) - \lim_{t \rightarrow x} f_m(t)| \leq \epsilon$$

Thus,  $\{A_n\}$  is a Cauchy Sequence in  $\mathbb{R}^k$  so  $\{A_n\}$  converges to  $A = \lim_{n \rightarrow \infty} A_n$ .

Since  $\{A_n\}$  converges to  $A$ , then for  $\epsilon > 0$ , there is a  $N_1$  where for  $n \geq N_1$ :

$$|A - A_n| \leq \frac{\epsilon}{3}$$

Since  $\{f_n\}$  converges uniformly to  $f$ , then for  $\epsilon > 0$ , there is a  $N_2$  where for  $n \geq N_2$ :

$$|f(t) - f_n(t)| \leq \frac{\epsilon}{3}.$$

Since there is a  $r$  such that for  $t \in N_r(x)$ , then:

$$|f_n(t) - \lim_{t \rightarrow x} f_n(t)| = |f_n(t) - A_n| \leq \frac{\epsilon}{3}$$

Thus, for  $t \rightarrow x$ ,  $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \leq \epsilon$ .

Thus,  $\lim_{t \rightarrow x} f(t) = A = \lim_{n \rightarrow \infty} A_n$ .

**Theorem 14.3.2: Uniform Convergence preserve Continuity**

If  $\{f_n\}$  converges uniformly to  $f$  on  $E$  where each  $f_n$  is continuous on  $E$ , then  $f$  is continuous on  $E$ .

#### Intuition

If each  $f_n$  is continuous:

$$\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

#### Proof

Since  $\{f_n\}$  converges uniformly to  $f$ , then by **theorem 14.3.1**, for any  $x \in E$ :

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$$

Since each  $f_n$  is continuous, then:

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{t \rightarrow x} f(t)$$

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{n \rightarrow \infty} f_n(x) = f(x)$$

**Theorem 14.3.3: Decreasing, continuous sequence over compact converges uniformly**

Suppose  $K$  is compact and

- (a)  $\{f_n\}$  is a sequence of continuous functions on  $K$
- (b)  $\{f_n\}$  converges pointwise to a continuous  $f$  on  $K$
- (c)  $f_n(x) \geq f_{n+1}(x)$  for all  $x \in K$

Then  $f_n$  converges uniformly to  $f$  on  $K$ .

**Proof**

Let  $g_n = f_n - f$  so  $g_n$  is continuous where  $g_n \geq g_{n+1}$ .  
 Thus,  $\lim_{n \rightarrow \infty} g_n(x) = 0$  pointwise. For  $\epsilon > 0$ , let  $K_n = \{x \in K : g_n(x) \geq \epsilon\}$ .  
 Since  $g_n$  is continuous and the set of  $g_n(x) \geq \epsilon$  is closed, then  $K_n$  is closed. Since closed  $K_n \subset$  compact  $K$ , then  $K_n$  is compact.  
 Since  $g_n \geq g_{n+1}$ , then  $K_{n+1} \subset K_n$ . For any  $x \in K$ ,  $\lim_{n \rightarrow \infty} g_n(x) = 0$  so there is a  $N_x$  such that  $x \notin K_n$  if  $n > N_x$ . Thus, any  $x \notin \bigcap_{n=1}^{\infty} K_n$  so  $\bigcap_{n=1}^{\infty} K_n = \emptyset$ .  
 Since  $\bigcap_{n=1}^{\infty} K_n = \emptyset$ , then  $K_n$  is empty for some  $N$ .  
 Thus,  $0 \leq g_n(x) < \epsilon$  for all  $x \in K$  where  $n \geq N$ .

**Definition 14.3.4: Supremum Norm**

$\mathcal{C}(X)$  is the set of all complex, continuous, bounded functions in metric  $X$ .

If  $X$  is compact, then bounded is not needed

Then for each  $f \in \mathcal{C}(X)$ , associate a supremum norm:

$$\|f\| = \sup_{x \in X} |f(x)| < \infty$$

- (a)  $\|f(x)\| = 0$  if and only if  $f(x) = 0$  for every  $x \in X$
- (b) Since  $|f + g| \leq |f| + |g| \leq \|f\| + \|g\|$ , then  $\|f + g\| \leq \|f\| + \|g\|$

Then for  $f, g \in \mathcal{C}(X)$ , let distance  $\|f - g\|$  and thus,  $\mathcal{C}(X)$  is a metric space.

By **theorem 14.2.3**,  $\{f_n\} \rightarrow f$  on  $\mathcal{C}(X)$  if and only if  $\{f_n\} \rightarrow f$  uniformly on  $X$ .

**Theorem 14.3.5:  $\mathcal{C}(X)$  is a complete metric space**

$\mathcal{C}(X)$  is a complete metric space

**Intuition**

A Cauchy sequence  $\{f_n\}$  is uniformly convergent to  $f$ .  
 Since  $\mathcal{C}(X)$  contain continuous functions, then  $f$  is continuous.  
 Since functions in  $\mathcal{C}(X)$  are bounded, then  $f$  is bounded.

**Proof**

Let  $\{f_n\}$  be a Cauchy sequence in  $\mathcal{C}(X)$ .  
 Since  $\{f_n\} \in \mathcal{C}(X)$ , then each  $f_n$  is continuous and bounded.  
 Then for  $\epsilon > 0$ , there is a  $N$  such that for  $n, m \geq N$ , then:  
 $|f_n - f_m| \leq \|f_n - f_m\| \leq \epsilon$   
 Then by **theorem 14.2.2**,  $\{f_n\}$  converges uniformly to  $f$ .  
 Since each  $f_n$  is continuous and  $\{f_n\}$  converges uniformly to  $f$ , then by **theorem 14.3.2**,  $f$  is continuous on  $\mathcal{C}(X)$ .  
 Since  $\{f_n\}$  converges uniformly to  $f$ , there is a  $N$  where for  $n \geq N$ :  
 $|f - f_n(x)| \leq \epsilon$   
 Since each  $f_n$  is bounded, then  $f$  is bounded. Since  $f$  is continuous and bounded, then  $f \in \mathcal{C}(X)$ . Thus, every Cauchy sequence  $\{f_n\}$  converges to  $f \in \mathcal{C}(X)$ .

## 14.4 Uniform Convergence and Integration

### Theorem 14.4.1: Uniform Convergence preserves Integrability

If  $\{f_n\} \in \mathcal{R}(\alpha)$  converges uniformly to  $f$  on  $[a, b]$ , then  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  where:

$$\int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha$$

#### Intuition

Since  $f_n$  is integrable, then  $\int_a^b f_n \, d\alpha$  exist and since  $\{f_n\}$  uniformly converges, then for  $\epsilon > 0$ ,  $|f - f_n| < \epsilon$ . Thus, for a large enough  $n$ ,  $\int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha$ .

#### Proof

Since  $\{f_n\}$  converges uniformly to  $f$ , then for  $\epsilon > 0$ :

$$|f - f_n| < \epsilon \quad \rightarrow \quad f_n - \epsilon < f < f_n + \epsilon$$

Then:

$$\int_a^b f_n - \epsilon \, d\alpha < \int_a^b f \, d\alpha \leq \int_a^b f \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$$

Thus,

$$\int_a^b f \, d\alpha - \int_a^b f \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha - \int_a^b f_n - \epsilon \, d\alpha = 2\epsilon[\alpha(b) - \alpha(a)]$$

So,  $\int_a^b f \, d\alpha$  exists and since  $f_n \in \mathcal{R}(\alpha)$  where  $\int_a^b f_n - \epsilon \, d\alpha < \int_a^b f \, d\alpha < \int_a^b f_n + \epsilon \, d\alpha$ :

$$\int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha$$

### Theorem 14.4.2: Uniform Convergence preserves Integrability for series

If  $f_n \in \mathcal{R}(\alpha)$  on  $[a, b]$  and  $f(x) = \sum_{n=1}^{\infty} f_n(x)$  converges uniformly, then:

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$$

#### Proof

Since  $f_n \in \mathcal{R}(\alpha)$ , then  $f(x) \in \mathcal{R}(\alpha)$ . Since  $f(x)$  converges uniformly, then by [theorem 14.4.1](#), then  $\int_a^b f \, d\alpha = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_a^b f_n \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha$ .

## 14.5 Uniform Convergence and Differentiation

### Theorem 14.5.1: Uniform Convergence of Derivatives preserve Differentiability

Suppose  $\{f_n\}$  are differentiable on  $[a,b]$  such that  $\{f_n(x_0)\}$  converges for some  $x_0 \in [a,b]$ . If  $\{f'_n\}$  converges uniformly on  $[a,b]$ , then  $\{f_n\}$  converges uniformly to  $f$  on  $[a,b]$  where:

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad \text{for } x \in [a,b]$$

#### Intuition

Since  $\{f'_n\}$  converges uniformly, for  $t$  near  $x$ , then by the Mean Value Theorem:

$$\frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$$

Since  $\{f'_n\}$  converges uniformly, by the Mean Value Theorem, there is a  $t \in [x_1, x_2]$ :

$$|[f_n(x_2) - f_m(x_2)] - [f_n(x_1) - f_m(x_1)]| = (x_2 - x_1)|f'_n(t) - f'_m(t)| < \epsilon$$

Thus,  $\{f_n - f_m\}$  converges uniformly so if  $\{f_n\}$  converges for some  $x_0$ :

$$|f_n(x) - f_m(x)| = |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] + [f_n(x_0) - f_m(x_0)]| \leq \epsilon$$

Thus,  $\{f_n\}$  converges uniformly which preserves continuity so for  $t$  near  $x$  as  $n \rightarrow \infty$ :

$$f'(x) = \frac{f(t) - f(x)}{t - x} = \frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x} = f'_n(x)$$

Note uniform convergence of  $\{f'_n\}$  gives  $\frac{f_n(t) - f_n(x)}{t - x} = \frac{(t-x)f'_n(x)}{t-x}$ . Then uniform convergence of  $\{f'_n\}$  with convergent  $f_n(x_0)$  leads to uniform convergence of  $\{f_n\}$  which gives  $\frac{f(t) - f(x)}{t - x} = \frac{f_n(t) - f_n(x)}{t - x}$ .

#### Proof

Since  $f_n(x_0)$  converges for some  $x_0 \in [a,b]$ , then for  $\epsilon > 0$ , there is a  $N_1$  such that for  $n_1, m_1 \geq N_1$ :

$$|f_{n_1}(x_0) - f_{m_1}(x_0)| < \frac{\epsilon}{2}$$

Since  $f'_n$  converges uniformly, then there is a  $N_2$  such that for  $n_2, m_2 \geq N_2$ :

$$|f'_{n_2}(t) - f'_{m_2}(t)| < \frac{\epsilon}{2(b-a)}$$

Let  $N = \max(N_1, N_2)$ . Then for  $n, m \geq N$ :

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \quad |f'_n(t) - f'_m(t)| < \frac{\epsilon}{2(b-a)}$$

Since  $f_n$  is differentiable, then  $f_n - f_m$  is differentiable. Then by the Mean Value Theorem, there is a  $x \in (a,b)$  such that:

$$|[f_n(x) - f_m(x)] - [f_n(t) - f_m(t)]| \leq |x - t||f'_n(t) - f'_m(t)| < |x - t|\frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2}$$

Thus, for  $n, m \geq N$ :

$$|f_n(x) - f_m(x)| \leq |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| + |f_n(x_0) - f_m(x_0)| < \epsilon$$

Thus,  $\{f_n\}$  converges uniformly to  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  where:

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \quad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

Since  $\lim_{t \rightarrow x} |\phi_n(t) - \phi_m(t)| < \frac{\epsilon}{2(b-a)}$ , then:

$$\lim_{n \rightarrow \infty} \phi_n(t) = \frac{f(t) - f(x)}{t - x} = \phi(t)$$

Since  $\{\phi_n(t)\}$  converges uniformly to  $\phi(t)$ , then by **theorem 14.3.1**:

$$\lim_{t \rightarrow x} \phi(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} \phi_n(t) = \lim_{n \rightarrow \infty} f'_n(x)$$

**Theorem 14.5.2: Continuous functions can be non-differentiable**

There exists a real continuous function on  $\mathbb{R}$  which is nowhere differentiable.

**Proof**

Let  $\phi(x) = |x|$  for  $x \in [-1, 1]$ . Then to extend to all real  $x$ , let  $\phi(x+2) = \phi(x)$ . Then  $\phi$  is continuous on  $\mathbb{R}$  where for  $s, t \in \mathbb{R}$ ,  $|\phi(s) - \phi(t)| \leq |s - t|$ .  
 Let  $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$ .  
 Since  $f(x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n$ , then  $f(x)$  converges uniformly and since  $\phi(x)$  is continuous, then  $f(x)$  is continuous. Then for a fixed  $x$  and positive integer  $m$ , choose  $\delta_m = \pm \frac{1}{2} 4^{-m}$  such that no integer lies in  $(4^m x, 4^m(x + \delta_m))$ . Let  $\gamma_n = \frac{\phi(4^n(x+\delta_m)) - \phi(4^n x)}{\delta_m}$ .  
 For  $n > m$ ,  $4^n \delta_m$  is even so  $\gamma_n = 0$ . For  $n \in [0, m]$ ,  $|\gamma_n| \leq \frac{|4^n \delta_m|}{\delta_m} = 4^m < 4^n$ .  
 Since  $|\gamma_m| = 4^m$ , then:  

$$|\frac{f(x+\delta_m) - f(x)}{\delta_m}| = |\sum_{n=0}^m (\frac{3}{4})^n \gamma_n| + |\sum_{n=m+1}^{\infty} (\frac{3}{4})^n \gamma_n| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1)$$
  
 As  $m \rightarrow \infty$ , then  $\delta_m \rightarrow 0$ , but  $|\frac{f(x+\delta_m) - f(x)}{\delta_m}| \rightarrow \infty$  so  $f$  is not differentiable at any  $x$ .

**14.6 Equicontinuous Families of Functions****Definition 14.6.1: Boundedness**

Let  $\{f_n\}$  be defined on set  $E$ .

$\{f_n\}$  is pointwise bounded on  $E$  if for  $x \in E$  and every  $n$ , there is a  $\phi$  where:

$$|f_n(x)| < \phi(x)$$

$\{f_n\}$  is uniformly bounded on  $E$  if for every  $n$  and  $x \in E$ , there is a  $M$  where:

$$|f_n(x)| < M$$

**Definition 14.6.2: Equicontinuous**

A family of complex functions,  $\mathcal{F}$ , defined on set  $E \subset X$  is equicontinuous if for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $x, y \in E$  and  $f \in \mathcal{F}$  where  $d(x, y) < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

**Theorem 14.6.3: Pointwise bounded  $\{f_n\}$  over countable sets have convergent  $\{f_{n_k}\}$** 

If  $\{f_n\}$  are pointwise bounded, complex functions on countable set  $E$ , then

$\{f_n\}$  has subsequence  $\{f_{n_k}\}$  such that  $\{f_{n_k}(x)\}$  converges for every  $x \in E$ .

**Intuition**

Any  $\{f_{n_k}\} \subset \{f_n\}$  is pointwise bounded so there is a convergent subsequence for a particular  $x$ . Let  $\{f_{n_{k_1}}\}$  be a convergent subsequence for  $x_1$ . Then find a subsequence  $\{f_{n_{k_2}}\} \subset \{f_{n_{k_1}}\}$  which converges for  $x_2$ . Continue the process until every  $x$ .

**Proof**

For each  $x_i \in E$ , let  $\{x_i\}$ . For  $x_1$ ,  $\{f_n(x_1)\}$  is piecewise bounded so there exists a subsequence  $\{f_{1,k}(x_1)\}$  which converges as  $k \rightarrow \infty$ .

Since  $\{f_{1,k}\}$  is piecewise bounded since  $\{f_{1,k}\} \subset \{f_n\}$ , then there is a subsequence  $\{f_{2,k}\} \subset \{f_{1,k}\}$  such that  $\{f_{2,k}(x_2)\}$  converges as  $k \rightarrow \infty$ . Then continuing the pattern:

$S_1:$	$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	$\dots$
$S_2:$	$f_{2,1}$	$f_{2,2}$	$f_{2,3}$	$\dots$
$S_3:$	$f_{3,1}$	$f_{3,2}$	$f_{3,3}$	$\dots$
$\dots$				

Thus,  $\{f_{n,n}(x_i)\}$  converges as  $n \rightarrow \infty$  for every  $x_i \in E$ .

**Theorem 14.6.4: Uniform convergent  $\{f_n\}$  where  $f_n \in \mathcal{C}(K)$  is equicontinuous**

If  $K$  is a compact metric space where  $f_n \in \mathcal{C}(K)$  and  $\{f_n\}$  converges uniformly on  $K$ , then  $\{f_n\}$  is equicontinuous on  $K$ .

**Intuition**

Since  $\{f_n\}$  converges uniformly, then there is a  $N$  where for  $n > N$ , then  $|f_n - f_N| < \epsilon$ . Since  $\{f_n\}$  is continuous over compact  $K$ , then  $\{f_n\}$  is uniformly continuous. So for  $d(x,y) < \delta$ , then:

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\epsilon$$

**Proof**

Since  $\{f_n\}$  converges uniformly, then for  $\epsilon > 0$ , there is a  $N$  such that for  $n > N$ :

$$||f_n - f_N|| < \frac{\epsilon}{3}$$

Since  $f_i$  for  $i \in [1, N]$  is continuous over compact  $K$ , then  $f_i$  is uniformly continuous so there is a  $\delta > 0$  such that for all  $x, y$  where  $d(x, y) < \delta$ , then  $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$ .

Then for  $n > N$  and  $d(x, y) < \delta$ :

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \epsilon$$

Thus, for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $f_n$  and  $x, y \in K$  where  $d(x, y) < \delta$ ,  $|f_n(x) - f_n(y)| < \epsilon$ . So,  $\{f_n\}$  is equicontinuous.

**Theorem 14.6.5: Pointwise bounded and equicontinuous  $\{f_n\}$  over compact  $K$  is uniformly bounded and have uniformly convergent  $\{f_{n_k}\}$** 

If  $K$  is compact where  $\{f_n\} \in \mathcal{C}(K)$  is pointwise bounded and equicontinuous:

- (a)  $\{f_n\}$  is uniformly bounded on  $K$
- (b)  $\{f_n\}$  contains a uniformly convergent subsequence

**Intuition**

Since  $\{f_n\}$  is equicontinuous, for  $d(x, y) < \delta$ , then  $|f_n(x) - f_n(y)| < \epsilon$ .

Since  $\{f_n\}$  is pointwise bounded on compact  $K$ , there are finite  $x_0, \dots, x_n$  such that  $d(x, x_i) < \delta$  so  $|f_n(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i)| < \epsilon + M$ .

For a countable dense subset of  $K$ , the countability gives a convergent subsequence  $\{g_n\}$  and the dense gives  $d(x, x_i) < \delta$  for finite  $x_1, \dots, x_m$  so:

$$|g_n(x) - g_m(y)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)| < \epsilon.$$

**Proof**

Since  $f_n$  is equicontinuous, then for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for  $x, y \in K$  where  $d(x, y) < \delta$ , then  $|f_n(x) - f_n(y)| < \epsilon$ .

Since  $K$  is compact, there are finite  $p_1, \dots, p_r \in K$  so for any  $x \in K$ , there is at least one  $p_i$  so  $d(x, p_i) < \delta$ . Since  $\{f_i\}$  is pointwise bounded, there is a  $M_i$  so  $|f_n(p_i)| < M_i$ . Let  $M = \max(M_1, \dots, M_r)$ . So,  $|f_n(x)| < |f_n(x) - f_n(p_i)| + |f_n(p_i)| < \epsilon + M_i < \epsilon + M$ . Thus,  $\{f_n\}$  is uniformly bounded on  $K$ .

Let countable dense  $E \subset K$ . By **theorem 14.6.3**,  $\{f_n\}$  has a convergent subsequence  $\{f_{n_i}(x)\}$  for every  $x \in E$ . Let  $V(x, \delta) = \{y \in K : d(x, y) < \delta\}$  so  $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ . Since  $E$  is dense in compact  $K$ , there are finitely many  $x_1, \dots, x_m \in E$  such that:

$$K \subset V(x_1, \delta) \cup \dots \cup V(x_m, \delta).$$

Since  $\{f_{n_i}(x)\}$  converges for every  $x \in E$ , there is a  $N$  where for  $n_i, n_j \geq N$ ,  $s \in [1, m]$ :

$$|f_{n_i}(x_s) - f_{n_j}(x_s)| < \frac{\epsilon}{3}$$

Thus, for any  $x \in K$ , there is a  $x_s \in E$  such that:

$$|f_{n_i}(x) - f_{n_j}(x)| \leq |f_{n_i}(x) - f_{n_i}(x_s)| + |f_{n_i}(x_s) - f_{n_j}(x_s)| + |f_{n_j}(x_s) - f_{n_j}(x)| < \epsilon$$

Thus,  $\{f_n\}$  contains a subsequence that uniformly converges.

## 14.7 Stone-Weierstrass Theorem

**Theorem 14.7.1:** There are polynomials that converge uniformly to continuous  $f$

For complex continuous  $f$  on  $[a,b]$ , there is a sequence of polynomials  $\{P_n\}$  that converges uniformly to  $f(x)$ .

**Proof**

Let  $[a,b] = [0,1]$  where  $f(0) = f(1) = 0$  and  $f(x) = 0$  if  $x \notin [0,1]$ .

Thus,  $f$  is uniformly continuous over  $\mathbb{R}$ .

Let  $Q_n(x) = c_n(1-x^2)^n$  where  $c_n$  is chosen so  $\int_{-1}^1 Q_n(x) dx = 1$ . Since:

$$\begin{aligned} \int_{-1}^1 (1-x^2)^n dx &= 2 \int_0^1 (1-x^2)^n dx \geq 2 \int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n dx \geq 2 \int_0^{\frac{1}{\sqrt{n}}} 1 - nx^2 dx \\ &= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}} \end{aligned}$$

so  $c_n < \sqrt{n}$ . Thus for  $\delta > 0$ ,  $Q_n(x) \leq \sqrt{n}(1-\delta^2)^n$  so  $Q_n \rightarrow 0$  on  $|x| \in [\delta, 1]$ .

Let  $P_n(x) = \int_{-1}^1 f(x+t)Q_n(t) dt$  for  $x \in [0,1]$ . Since  $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t) dt = \int_0^1 f(t)Q_n(t-x) dt$  which is a polynomial so  $\{P_n\}$  is a sequence of polynomials.

Since  $f$  is uniformly continuous, for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for  $|y-x| < \delta$ , then  $|f(y) - f(x)| < \frac{\epsilon}{2}$ . Let  $M = \sup(|f(x)|)$ . Then:

$$\begin{aligned} |P_n(x) - f(x)| &\leq \int_{-1}^1 |f(x+t) - f(x)|Q_n(t) dt \\ &\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 Q_n(t) dt \\ &\leq 4M\sqrt{n}(1-\delta^2)^n + \frac{\epsilon}{2} < \epsilon \quad \text{for a large enough } n \end{aligned}$$

**Corollary 14.7.2:** There are polynomials that converges uniformly to  $|x|$

For  $[-a,a]$ , there is a sequence of real polynomials  $P_n$  such that  $P_n(0) = 0$  and  $P_n(x)$  converges uniformly to  $|x|$ .

**Proof**

By **Theorem 14.7.1**, there is a  $\{P_n^*\}$  of real polynomials that converges uniformly to  $|x|$ . Since  $P_n^*(0) \rightarrow |0| = 0$ , let  $P_n(x) = P_n^*(x) - P_n^*(0)$ .

**Definition 14.7.3:** Algebra, Uniformly Closed, and Uniform Closure

A family of complex functions on  $E$ ,  $\mathcal{A}$ , is an algebra if for  $f, g \in \mathcal{A}$ , then:

- (a)  $f+g \in \mathcal{A}$
- (b)  $fg \in \mathcal{A}$
- (c)  $cf \in \mathcal{A}$  for complex constant  $c$

$\mathcal{A}$  is uniformly closed if:

For any  $f_n \in \mathcal{A}$  where  $f_n$  uniformly converges to  $f$ , then  $f \in \mathcal{A}$

Let the uniform closure,  $\mathcal{B}$ , be the set of all uniformly convergent  $f$  from  $\mathcal{A}$ .

**Theorem 14.7.4:** Bounded algebra implies Uniformly closed uniform closure

For algebra  $\mathcal{A}$  of bounded functions,  $\mathcal{B}$  is a uniformly closed algebra.

**Proof**

If  $f, g \in \mathcal{B}$ , there are uniformly convergent  $\{f_n\}, \{g_n\}$  where  $f_n \rightarrow f, g_n \rightarrow g$  and  $f_n, g_n \in \mathcal{A}$ . Since  $f_n, g_n$  are bounded and  $\mathcal{A}$  is an algebra, then uniformly convergent:

$$f_n + g_n \rightarrow f+g \quad f_n g_n \rightarrow fg \quad c f_n \rightarrow cf$$

Thus,  $f + g, fg, cf \in \mathcal{B}$  so  $\mathcal{B}$  is a uniformly closed algebra.



**Definition 14.7.5: Separate Points**

For family of functions,  $\mathcal{A}$ , separate points on  $E$ :

If for every pair of distinct  $x_1, x_2 \in E$ , there is a  $f \in \mathcal{A}$  where  $f(x_1) \neq f(x_2)$ .

$\mathcal{A}$  vanishes at no point of  $E$ :

If for each  $x \in E$ , there is a  $g \in \mathcal{A}$  such that  $g(x) \neq 0$

**Theorem 14.7.6: Non-vashing, separate algebra contain all values**

Suppose algebra  $\mathcal{A}$  separates points and vanishes at no points on  $E$ . If  $x_1, x_2$  are distinct points, then for constants  $c_1, c_2$ , there is a  $f \in \mathcal{A}$  where:

$$f(x_1) = c_1 \text{ and } f(x_2) = c_2.$$

**Proof**

Since  $\mathcal{A}$  separates points and vanishes at no points on  $E$ , then there are  $g, h, k \in \mathcal{A}$ :

$$g(x_1) \neq g(x_2) \quad h(x_1) \neq 0 \quad k(x_2) \neq 0$$

Let  $u = k(g - g(x_1))$  and  $v = h(g - g(x_2))$  so  $u, v \in \mathcal{A}$  where  $u(x_1) = v(x_2) = 0$  and  $u(x_2), v(x_1) \neq 0$ . Then,  $f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$  have  $f(x_1) = c_1$  and  $f(x_2) = c_2$ .

**Theorem 14.7.7: Stone-Weierstrass Theorem**

If algebra of real continuous functions on compact  $K$ ,  $\mathcal{A}$ , separates points and vanishes at no points on  $K$ , then  $\mathcal{B}$  consist of all real continuous functions.

**Proof**

Claim: If  $f \in \mathcal{B}$ , then  $|f| \in \mathcal{B}$ .

Let  $a = \sup(|f(x)|)$ . By **Corollary 14.7.2**, for  $\epsilon > 0$ , there are  $c_1, \dots, c_n$  such that:

$$|\sum_{i=1}^n c_i y^i - |y|| < \epsilon \quad \text{for } y \in [-a, a]$$

Since  $\mathcal{B}$  is an algebra, then  $g = \sum_{i=1}^n c_i f^i \in \mathcal{B}$ . Thus:

$$|g(x) - |f(x)|| < \epsilon \quad \text{for } x \in K$$

Since  $\mathcal{B}$  is uniformly closed, then  $|f(x)| \in \mathcal{B}$ .

Claim: If  $f, g \in \mathcal{B}$ , then  $\min(f, g), \max(f, g) \in \mathcal{B}$ .

Since:

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2} \quad \min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$$

then  $\max(f, g), \min(f, g) \in \mathcal{B}$ .

Claim: For real, continuous  $f$  on  $K$  and  $\epsilon > 0$ , there exist  $g_x \in \mathcal{B}$  where  $g_x(x) = f(x)$  and  $g_x(t) > f(t) - \epsilon$  for  $t \in K$ .

Since  $\mathcal{A} \subset \mathcal{B}$  where  $\mathcal{A}$  separates points and vanishes at no points on  $E$ , then  $\mathcal{B}$  is the same. Then by **theorem 14.7.6**, for  $y \in K$ , there is a  $h_y \in \mathcal{B}$  where:

$$h_y(x) = f(x) \quad h_y(y) = f(y)$$

Since  $h_y$  is continuous, there is an open set  $J_y$  such that  $h_y(t) > f(t) - \epsilon$  for  $t \in J_y$ .

Since  $K$  is compact, there are finite  $y_1, \dots, y_n$  such that  $K \subset J_{y_1} \cup \dots \cup J_{y_n}$ .

Let  $g_x = \max(h_{y_1}, \dots, h_{y_n})$  so  $g_x \in \mathcal{B}$  where  $g_x(t) > f(t) - \epsilon$  for  $t \in K$ .

Claim: For real, continuous  $f$  on  $K$  and  $\epsilon > 0$ , there is a  $h \in \mathcal{B}$  where  $|h(x) - f(x)| < \epsilon$ .

Since  $g_x$  is continuous, there is an open set  $V_x$  where  $g_x(t) < f(t) + \epsilon$  for  $t \in V_x$ .

Since  $K$  is compact, there are finite  $x_1, \dots, x_m$  such that  $K \subset V_{x_1} \cup \dots \cup V_{x_m}$ .

Let  $h = \min(g_{x_1}, \dots, g_{x_m})$  so  $h \in \mathcal{B}$  where  $h(t) > f(t) - \epsilon$ . But,  $h(t) < f(t) + \epsilon$  so  $|h(x) - f(x)| < \epsilon$ . Since  $\mathcal{B}$  is uniformly closed, then the theorem holds true.

**Definition 14.7.8: Self-Adjoint**

$\mathcal{A}$  is self-adjoint if for every  $f \in \mathcal{A}$ , then  $\overline{f} \in \mathcal{A}$

**Theorem 14.7.9: Stone-Weierstrass for complex functions**

If self-adjoint algebra of complex continuous functions on compact  $K$ ,  $\mathcal{A}$ , separates points and vanishes at no points on  $K$ , then  $\mathcal{B}$  consist of all complex continuous functions on  $K$ . In other words,  $\mathcal{A}$  is dense in  $\mathcal{C}(K)$ .

**Proof**

Let  $\mathcal{A}_R$  be the set of all real functions on  $K$  in  $\mathcal{A}$ .

If  $f \in \mathcal{A}$  and  $f = u + iv$  for real  $u, v$  then  $2u = f + \overline{f} \in \mathcal{A}_R$ .

If  $x_1 \neq x_2$ , there exists  $f \in \mathcal{A}$  such that  $f(x_1) = 1$  and  $f(x_2) = 0$  so  $u(x_1) \neq u(x_2)$  so  $\mathcal{A}_R$  separates points.

If  $x \in K$ , then  $g(x) \neq 0$  for some  $g \in \mathcal{A}$  and there is a complex  $\lambda$  such that  $\lambda g(x) > 0$ .

If  $f = \lambda g$ , then  $u(x) > 0$  so  $\mathcal{A}_R$  vanishes at no point of  $K$ .

Then by [theorem 14.7.7](#), every real continuous function on  $K$  lies in  $\mathcal{B}_{\mathcal{A}_R}$  and since  $\mathcal{B}_{\mathcal{A}_R} \subset \mathcal{B}$ , then every real continuous function lies in  $\mathcal{B}$ . If  $f$  is complex continuous where  $f = u + iv$ , then  $f \in \mathcal{B}$  since  $u, v \in \mathcal{B}$ .

## 15 Special Functions

### 15.1 Power Series

#### Definition 15.1.1: Analytic Functions

Power series,  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

If  $f(x)$  converges for  $|x-a| < R$  for some  $R$ , then  $f$  is expanded in a power series about  $x = a$ .

#### Theorem 15.1.2: Convergent Power Series are differentiable

If  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  converges for  $|x| < R$ , then  $f(x)$  converges uniformly on  $[-R+\epsilon, R-\epsilon]$  for any  $\epsilon > 0$ .

Then,  $f$  is continuous and differentiable in  $(-R, R)$  where:

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

#### Proof

For  $\epsilon > 0$  and  $|x| \leq R - \epsilon$ :

$$|c_n x^n| \leq |c_n (R - \epsilon)^n|$$

Since  $\sum c_n (R - \epsilon)^n$  converges absolutely in  $[-R + \epsilon, R - \epsilon]$ , then  $f(x)$  uniformly converges on  $[-R + \epsilon, R - \epsilon]$ .

Since  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ , then:

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{n} |c_n|) = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$$

so  $f(x)$  and  $f'(x)$  have the same interval of convergence so  $f'(x)$  uniformly converges on  $[-R + \epsilon, R - \epsilon]$ . Since  $f'(x)$  exists, then  $f$  is differentiable and thus, continuous.

#### Corollary 15.1.3: Power Series have infinite derivatives

On  $(-R, R)$ ,  $f$  has derivatives of all orders:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)c_n x^{n-k}$$

$$f^{(k)}(0) = k! c_k$$

#### Proof

By theorem 15.1.2, apply derivative  $k$  times.

#### Theorem 15.1.4: Continuity of Power Series at endpoints

Suppose  $\sum c_n$  converges where  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  for  $x \in (-1, 1)$ .

Then  $\lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} c_n$ .

#### Proof

Let  $s_n = c_0 + \dots + c_n$ .

$$\begin{aligned} \sum_{n=0}^m c_n x^n &= \sum_{n=0}^m (s_n - s_{n-1}) x^n = \sum_{n=0}^m s_n x^n - \sum_{n=0}^m s_{n-1} x^n \\ &= \sum_{n=0}^m s_n x^n - \sum_{n=0}^{m-1} s_n x^{n+1} = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m \end{aligned}$$

Since  $|x| < 1$ , then as  $m \rightarrow \infty$ , then  $s_m x^m \rightarrow 0$ . Let  $s = \lim_{n \rightarrow \infty} s_n$ .

Thus, for  $\epsilon > 0$ , there is a  $N$  such that for  $n > N$ , then  $|s - s_n| < \frac{\epsilon}{2}$ .

Since  $(1-x) \sum_{n=0}^{\infty} x^n = (1-x) \frac{1}{1-x} = 1$ , then:

$$|f(x) - s| = |(1-x) \sum_{n=0}^{\infty} (s_n - s) x^n| \leq (1-x) \sum_{n=0}^N |s_n - s| |x|^n + \frac{\epsilon}{2}$$

Then choose  $\delta > 0$  such that  $(1-x) \sum_{n=0}^N |s_n - s| < \frac{\epsilon}{2}$  for  $x > 1 - \delta$ . Thus:

$$|\lim_{x \rightarrow 1} f(x) - s| < \epsilon$$

**Corollary 15.1.5: Cauchy Product**

If  $\sum a_n \rightarrow A$ ,  $\sum b_n \rightarrow B$ , and  $\sum c_n \rightarrow C$  where  $c_n = \sum_{k=0}^n a_k b_{n-k}$ , then:  
 $C = AB$

**Proof**

For  $x \in (0,1)$ , let:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad g(x) = \sum_{n=0}^{\infty} b_n x^n \quad h(x) = \sum_{n=0}^{\infty} c_n x^n$$

Then  $f, g, h$  absolutely converges. Note  $fg = h$ .

By **theorem 15.1.4**, then  $AB = \lim_{x \rightarrow 1} f(x)g(x) = \lim_{x \rightarrow 1} h(x) = C$ .

**Theorem 15.1.6:  $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$** 

Suppose  $\sum_{j=1}^{\infty} |a_{i,j}| = b_i$  where  $\sum b_i$  converges, then:

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j}$$

**Proof**

Let countable set  $E$  contain points  $x_n$  where  $x_n \rightarrow x_0$ . Let:

$$f_i(x_n) = \sum_{j=1}^n a_{i,j} \quad f_i(x_0) = \sum_{j=1}^{\infty} a_{i,j} \quad g(x) = \sum_{i=1}^{\infty} f_i(x)$$

Thus, each  $f_i$  is continuous at  $x_0$ . Since  $|f_i(x)| \leq b_i$ , then  $g(x)$  converges uniformly so  $g$  is continuous at  $x_0$ . Thus:

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} &= \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} f_i(x_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^{\infty} a_{i,j} = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} \sum_{i=1}^n a_{i,j} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{i,j} \end{aligned}$$

**Theorem 15.1.7: Extension to Taylor's Theorem**

If  $f(x) = \sum_{n=0}^{\infty} c_n x^n$  converges for  $|x| < R$  where  $a \in (-R, R)$ , then  $f$  is expanded in a power series about  $x = a$  which converges in  $|x - a| < R - |a|$  where:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

**Proof**

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n [(x - a) + a]^n = \sum_{n=0}^{\infty} c_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x - a)^m \\ &= \sum_{m=0}^{\infty} \left[ \sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} \right] (x - a)^m \end{aligned}$$

Then by **corollary 15.1.3**,  $\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} = \frac{f^{(m)}(a)}{m!}$  so  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$ .

**Theorem 15.1.8: Equivalent Power Series have the same coefficients**

If  $\sum a_n x^n, \sum b_n x^n$  converge in  $S = (-R, R)$ , let  $E$  be the set of all  $x \in S$  where  $\sum a_n x^n = \sum b_n x^n$ . If  $E$  has a limit point in  $S$ , then  $a_n = b_n$  for all  $n$ .

**Proof**

Let  $c_n = a_n - b_n$  and  $f(x) = \sum_{n=0}^{\infty} c_n x^n$ . Then  $f(x) = 0$  on  $E$ .

Let  $A = E'$  and  $B = S \setminus E'$ . Thus,  $B$  is open. If  $x_0 \in A$ , then:

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n \quad |x - x_0| < R - |x_0|$$

Suppose  $d_n \neq 0$  for some  $n$ . Let  $k$  be the smallest integer where  $d_k \neq 0$ . Then:

$$f(x) = (x - x_0)^k g(x) \quad |x - x_0| < R - |x_0| \text{ and } g(x) = \sum_{m=0}^{\infty} d_{k+m} (x - x_0)^m$$

Since  $g$  is continuous at  $x_0$  and  $g(x_0) = d_k \neq 0$ , there is a  $\delta > 0$  such that  $g(x) \neq 0$  for  $|x - x_0| < \delta$ .

Thus,  $f(x) \neq 0$  if  $|x - x_0| < \delta$  which contradicts that  $x_0$  is a limit point of  $E$ . Thus,  $d_n = 0$  for all  $n$  so  $f(x) = 0$  for all  $x$  so  $A$  is open. Thus,  $A$  and  $B$  are disjoint and thus, are separated. Since  $S = A \cup B$  and  $S$  is connected, then either  $A$  or  $B$  is empty. Since  $A$  cannot be empty, then  $B$  is empty so  $A = S$ . Since  $f$  is continuous in  $S$ , then  $A \subset S$  so  $E = S$  so  $c_n = 0$  for all  $n$ .

## 15.2 Exponential and Logarithmic Functions

### Definition 15.2.1: Exponential Function

Define  $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  for  $x \in \mathbb{C}$ .

By the ratio test:

$$\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) = \lim_{n \rightarrow \infty} \sup(|\frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}}|) = \lim_{n \rightarrow \infty} \sup(|\frac{z}{n+1}|) = 0 < 1$$

Thus,  $E(x)$  converges. Then by [corollary 15.1.5](#):

$$\begin{aligned} E(x)E(y) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{x^k y^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = E(x+y) \end{aligned}$$

As a result,  $E(x)E(-x) = E(0) = 1$ . As a consequence:

- (a)  $E(x) \neq 0$  for all  $x$
- (b) If  $x > 0$ , then  $E(x) > 0$  and thus,  $E(x) > 0$  for all  $x \in \mathbb{R}$
- (c)  $\lim_{x \rightarrow \infty} E(x) \rightarrow \infty$  so  $\lim_{x \rightarrow -\infty} E(x) \rightarrow 0$  for  $x \in \mathbb{R}$
- (d) For  $0 < x < y$ ,  $E(x) < E(y)$  so  $E(-y) = \frac{1}{E(y)} < \frac{1}{E(x)} = E(-x)$  so  $E(x)$  is strictly increasing on  $\mathbb{R}$
- (e)  $E'(x) = \lim_{h \rightarrow 0} \frac{E(x+h) - E(x)}{h} = \lim_{h \rightarrow 0} \frac{E(x)E(h) - E(x)}{h}$   
 $= E(x) \lim_{h \rightarrow 0} \frac{E(h) - 1}{h} = E(x) (\lim_{h \rightarrow 0} \frac{E(h)}{h} - \lim_{h \rightarrow 0} \frac{1}{h})$   
 $= E(x) (\lim_{h \rightarrow 0} \frac{1}{h} + 1 - \lim_{h \rightarrow 0} \frac{1}{h}) = E(x)$
- (f) For  $n > 0 \in \mathbb{Z}$ :  

$$E(n) = \underbrace{E(1) \dots E(1)}_n = e^n$$

For  $p = \frac{n}{m} > 0 \in \mathbb{Q}$ :

$$[E(p)]^m = E(mp) = E(n) = e^n \text{ so } E(p) = e^{n/m} = e^p$$

Since  $E(-p) = \frac{1}{E(p)} = e^{-p}$ , then  $E(p) = e^p$  hold for all  $p \in \mathbb{Q}$ .

For  $x \in \mathbb{R}$ , let  $e^x = \sup_{x > p} (e^p)$  for  $p \in \mathbb{Q}$ . Since  $E(x)$  is continuous and

monotonically increasing, for every  $\epsilon > 0$ , there is a  $\delta > 0$  where  $|x - p| < \delta$ , then  $|\sup_{x > p} (e^p) - e^p| < \epsilon$ . Thus:

$$e^x = \sup_{x > p} (e^p) = \lim_{p \rightarrow x} E(p) = E(x).$$

### Theorem 15.2.2: Properties of $e^x$

- (a)  $e^x$  is continuous and differentiable for all  $x \in \mathbb{R}$
- (b)  $(e^x)' = e^x$
- (c)  $e^x$  is strictly increasing where  $e^x > 0$
- (d)  $e^{x+y} = e^x e^y$
- (e)  $\lim_{x \rightarrow \infty} e^x = \infty$  and  $\lim_{x \rightarrow -\infty} e^x = 0$
- (f)  $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$  for every  $n > 0$

### Proof

Part (a) is proved by convergent power series while parts (c) to (e) is proved by properties of  $E(x)$  above. Since  $e^x > \frac{x^{n+1}}{(n+1)!}$  for  $x > 0$  and every  $n \in \mathbb{Z}_+$ , then:

$$0 \leq \lim_{x \rightarrow \infty} x^n e^{-x} < \lim_{x \rightarrow \infty} \frac{(n+1)!}{x} = 0$$

Thus,  $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$  for every  $n \in \mathbb{Z}_+$ . Since  $x^n e^{-x}$  is continuous and  $n \in \mathbb{Z}_+$  is dense in  $\mathbb{R}_+$ , then  $\lim_{x \rightarrow \infty} x^n e^{-x} = 0$  for every  $n > 0$ .

**Definition 15.2.3: Logarithmic Function**

Since  $y = E(x)$  is strictly increasing on  $\mathbb{R}$ , then  $E(x)$  is injective and thus, there exist an inverse function  $L(y)$  which is also strictly increasing. Since  $E(x)$  is differentiable, then  $L(y)$  is also differentiable. Then:

$$\begin{aligned} E(L(y)) &= y & \text{where } y > 0 \\ L(E(x)) &= x & \text{where } x \in \mathbb{R} \end{aligned}$$

Then:

$$L'(E(x))E'(x) = L'(y)E'(x) = L'(y)y = 1 \quad \Rightarrow \quad L'(y) = \frac{1}{y}$$

Since for  $x = 0$  have  $y = E(0) = 1$ , then  $L(1) = 0$ . Thus:

$$L(y) = \int_1^y L'(t) dt = \int_1^y \frac{1}{t} dt$$

As a consequence:

(a) Let  $y_1 = E(x_1)$  and  $y_2 = E(x_2)$ , then:

$$L(y_1 y_2) = L(E(x_1)E(x_2)) = L(E(x_1 + x_2)) = x_1 + x_2 = L(y_1) + L(y_2)$$

(b) Let  $\log(y) = L(y)$ . Then:

$$\text{Since } \lim_{x \rightarrow \infty} E(x) = \infty, \text{ then } \lim_{y \rightarrow \infty} L(y) = \infty.$$

$$\text{Since } \lim_{x \rightarrow -\infty} E(x) = 0, \text{ then } \lim_{y \rightarrow 0} L(y) = -\infty.$$

(c) For  $n \in \mathbb{Z}$ :

$$\text{If } n \geq 0, E(nL(y)) = E(\underbrace{L(y) + \dots + L(y)}_n) = E(L(y^n)) = y^n$$

$$\text{If } n < 0, E(nL(y)) = E(\underbrace{-(L(y) + \dots + L(y))}_{-n}) = [E(L(y^{-n}))]^{-1} = y^n$$

For  $p = \frac{a}{b} \in \mathbb{Q}$  where  $b > 0$ , let  $t^b = y$ :

$$\begin{aligned} E(pL(y)) &= \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(y))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}L(t^b))^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{b}bL(t))^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(aL(t))^n}{n!} = \sum_{n=0}^{\infty} \frac{(L(t^a))^n}{n!} = t^a = y^{\frac{a}{b}} = y^p \end{aligned}$$

For  $c \in \mathbb{R}$ , let  $y^c = \sup_{c > p} E(pL(y))$ . Since  $E(x), L(y)$  are continuous and

monotonically increasing, then for every  $\epsilon > 0$ , there is a  $\delta > 0$  where  $|c - p| < \delta$ , then  $|\sup_{c > p} (E(pL(y)) - E(pL(y)))| < \epsilon$ . Thus:

$$y^c = \sup_{c > p} E(pL(y)) = \lim_{p \rightarrow c} E(pL(y)) = E(cL(y))$$

(d) For  $y \in \mathbb{C}$  and  $c \neq 0 \in \mathbb{R}$ :

$$(y^c)' = E'(cL(y))cL'(y) = E(cL(y))c\frac{1}{y} = y^c c\frac{1}{y} = cy^{c-1}$$

Thus:

$$\text{If } c \neq -1, \text{ then } \int y^c dy = \int \frac{1}{c+1}(y^{c+1})' dy = \frac{1}{c+1}y^{c+1}$$

$$\text{If } c = -1, \text{ then } \int y^{-1} dy = \int L'(y) dy = L(y) = \log(y)$$

(e)  $\lim_{y \rightarrow \infty} y^{-c} \log(y) = 0$  for every  $c > 0$

For  $\epsilon \in (0, c)$  and  $y > 1$ :

$$y^{-c} \log(y) = y^{-c} \int_1^y t^{-1} dt < y^{-c} \int_1^y t^{\epsilon-1} dt = y^{-c} \frac{y^{\epsilon}-1}{\epsilon} < \frac{1}{y^{c-\epsilon}}$$

$$0 \leq \lim_{y \rightarrow \infty} y^{-c} \log(y) < \lim_{y \rightarrow \infty} \frac{1}{y^{c-\epsilon}} = 0$$

## 15.3 Trigonometric Function

### Definition 15.3.1: Trigonometric Function

Define for  $x \in \mathbb{C}$ :

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \quad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

Since  $E(\bar{x}) = \sum_{n=0}^{\infty} \frac{\bar{x}^n}{n!} = \sum_{n=0}^{\infty} \frac{\overline{x^n}}{n!} = \overline{\sum_{n=0}^{\infty} \frac{x^n}{n!}} = \overline{E(x)}$ , then for  $x \in \mathbb{R}$ :

$$C(x), S(x) \in \mathbb{R}$$

Also,  $E(ix) = C(x) + iS(x)$ . Then:

$$(a) |E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(0) = 1 \text{ so } |E(ix)| = 1$$

$$(b) C(0) = \frac{1}{2}[E(0) + E(0)] = 1$$

$$S(0) = \frac{1}{2i}[E(0) - E(0)] = 0$$

$$(c) C'(x) = \frac{1}{2}[E'(ix)i + E'(-ix)(-i)] = \frac{1}{2}[E(ix)i - E(-ix)i] = -S(x)$$

$$S'(x) = \frac{1}{2i}[E'(ix)i - E'(-ix)(-i)] = \frac{1}{2i}[E(ix)i + E(-ix)i] = C(x)$$

$$(d) \text{ There exists positive numbers such that } C(x) = 0.$$

If the claim is false, since  $C$  is continuous and  $C(0) = 1$ , then  $S'(x) = C(x) > 0$ . Then  $S(x)$  is strictly increasing and since  $S(0) = 0$ , then  $S(x) > 0$  for  $x > 0$ . Then for  $0 < x < y$ :

$$\begin{aligned} S(x)(y-x) &< \int_x^y S(t) dt = \int_x^y -C'(t) dt = C(x) - C(y) \\ &\leq |C(x) - C(y)| \leq |C(x)| + |C(y)| = 2 \end{aligned}$$

But if  $S(x) > 0$ , then  $S(x)(y-x) \not\leq 2$  for a large enough  $y$  for any  $S(x)$ . Thus by contradiction, there are positive numbers where  $C(x) = 0$ .

Since the set of zeros of a continuous function is closed, there exists a smallest positive number  $x_0$  such that  $C(x_0) = 0$ . Let  $\pi = 2x_0$ .

Then,  $C(\frac{\pi}{2}) = C(x_0) = 0$  and since  $|E(ix)| = |C(x) + iS(x)| = 1$ , then  $S(\frac{\pi}{2}) = \pm 1$ . Since  $C(x)$  is continuous where  $C(0) = 1$  and  $C(\frac{\pi}{2}) = 0$ , then  $S'(x) = C(x) > 0$  for  $x \in (0, \frac{\pi}{2})$  where  $S(0) = 0$  so  $S(\frac{\pi}{2}) = 1$ .

Thus,  $E(\frac{\pi}{2}i) = C(\frac{\pi}{2}) + iS(\frac{\pi}{2}) = 0 + i1 = i$ . Then:

$$-1 = i^2 = E(\frac{\pi}{2}i)E(\frac{\pi}{2}i) = E(\frac{\pi}{2}i + \frac{\pi}{2}i) = E(\pi i)$$

$$1 = (-1)^2 = E(\pi i)E(\pi i) = E(\pi i + \pi i) = E(2\pi i)$$

$$E(z) = E(z)1 = E(z)E(2\pi i) = E(z + 2\pi i)$$

### Theorem 15.3.2: Properties of $C(x)$ and $S(x)$

$$(a) E \text{ is periodic with period } 2\pi i$$

**Proof**

$$\text{Since } E(z) = E(z + 2\pi i), E \text{ has period } 2\pi i.$$

$$(b) C(x) \text{ and } S(x) \text{ are periodic with period } 2\pi$$

**Proof**

$$\text{Since } C(x) = \frac{1}{2}[E(ix) + E(-ix)] \text{ and } S(x) = \frac{1}{2i}[E(ix) - E(-ix)] \text{ where } E(x) \text{ have period } 2\pi i \text{ so } C(x) \text{ and } S(x) \text{ have period } 2\pi.$$

$$(c) \text{ If } t \in (0, 2\pi), \text{ then } E(it) \neq 1$$

**Proof**

$$\text{If } t \in (0, \frac{\pi}{2}) \text{ where } E(it) = x + iy, \text{ then } x, y \in (0, 1).$$

$$\text{Note } E(4it) = [E(it)]^4 = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

$$\text{If } E(4it) \text{ is real, then } x^2 - y^2 = 0. \text{ Thus, since } |E(ix)| = 1, \text{ then } x^2 + y^2 = 1 \text{ so } x^2 = y^2 = \frac{1}{2} \text{ and thus, } E(4it) = -1 \neq 1.$$

- (d) For  $z \in \mathbb{C}$  where  $|z| = 1$ , there is a unique  $t \in [0, 2\pi)$  such that  $E(it) = z$

**Proof**

By part (c), for  $0 \leq t_1 < t_2 < 2\pi$ :

$$E(it_2)[E(it_1)]^{-1} = E(it_2)[E(-it_1)] = E(it_2 - it_1) \neq 1$$

Thus,  $t \in [0, 2\pi)$  must be unique. Let fixed  $z = x + iy$  where  $|z| = 1$ .

For  $x, y \geq 0$ , since  $C(x)$  decreases from 1 to 0 on  $[0, \frac{\pi}{2}]$ , then  $C(t) = x$  for some  $t \in [0, \frac{\pi}{2}]$ . Since  $|E(it)| = C(t)^2 + S(t)^2 = 1$  and  $x^2 + y^2 = 1$ , then  $S(t) = y$  so  $E(it) = x + yi = z$ .

If  $x < 0$ ,  $y \geq 0$ , fix  $-iz$  instead of  $z$  and thus,  $E(it) = -iz$  for some  $t \in [0, \frac{\pi}{2}]$ . Since  $E(\frac{\pi}{2}i) = i$ , then  $z = -iz(i) = E(it)E(\frac{\pi}{2}i) = E(i(t + \frac{\pi}{2}))$ .

If  $x, y < 0$ , fix  $-z$  instead of  $z$  and thus,  $E(it) = -z$  for some  $t \in [0, \frac{\pi}{2}]$ . Since  $E(\pi i) = -1$ , then  $z = -z(-1) = E(it)E(\pi i) = E(i(t + \pi))$ .

If  $x \geq 0$ ,  $y < 0$ , fix  $iz$  instead of  $z$  and thus,  $E(it) = iz$  for some  $t \in [0, \frac{\pi}{2}]$ . Then  $z = iz(-1)(i) = E(it)E(\pi i)E(\frac{\pi}{2}i) = E(i(t + \frac{3\pi}{2}))$ .

### Definition 15.3.3: Unit Curve

Let  $\gamma(t) = E(it)$  for  $t \in [0, 2\pi]$ .

By **theorem 15.3.2(d)** and  $E(z) = E(z + 2\pi i)$ , then  $\gamma(t)$  is a simple closed curve whose range is the unit circle. Since  $\gamma'(t) = iE'(it) = iE(it)$ , the length of  $\gamma$ :

$$\Lambda(\gamma) = \int_0^{2\pi} |\gamma'(t)| dt = 2\pi$$

Thus,  $\pi = 2x_0$  defined earlier have the same geometric significance as  $\pi$ . Then consider the triangle with vertices at:

$$z_1 = 0 \quad z_2 = C(t_0) \quad z_3 = \gamma(t_0) = (C(t_0), S(t_0))$$

Thus,  $C(t) = \cos(t)$  and  $S(t) = \sin(t)$ .

## 15.4 Algebraic Completeness of the Complex Field

### Theorem 15.4.1: Every complex polynomial has a complex root

For  $a_0, \dots, a_n \neq 0 \in \mathbb{C}$  where  $n \geq 1$ , let  $P(z) = \sum_{k=0}^n a_k z^k$ .

Then  $P(z) = 0$  for some  $z \in \mathbb{C}$ .

**Proof**

Assume  $a_n = 1$ . Let  $\mu = \inf(|P(z)|)$ . If  $|z| = R$ , then:

$$|P(z)| \geq R^n(1 - |a_{n-1}|R^{-1} - \dots - |a_0|R^{-n})$$

Thus,  $\lim_{R \rightarrow \infty} |P(z)| = \infty$  so there is a  $R_0$  such that  $|R(z)| > \mu$  if  $|z| > R_0$ .

Since  $|P|$  is continuous, then for a closed  $N_{R_0}(0)$ , by the Extreme Value Theorem:

$$|P(z_0)| = \mu \quad \text{for some } z_0$$

Suppose  $\mu \neq 0$ . Let polynomial  $Q(z) = \frac{P(z+z_0)}{P(z_0)}$  where  $Q(0) = 1$ ,  $Q(z) \geq 1$  for all  $z$ .

Then there is a smallest integer  $k \leq n$  so  $b_k \neq 0$  so  $Q(z) = 1 + b_k z^k + \dots + b_n z^n$ .

By **theorem 15.3.2(d)**, there is a  $\theta \in \mathbb{R}$  such that  $e^{ik\theta} b_k = -|b_k|$ .

If  $r > 0$  and  $r^k |b_k| < 1$ , then  $|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|$ . Thus:

$$\begin{aligned} |Q(re^{i\theta})| &= |1 + b_k r^k e^{ik\theta} + b_{k+1} r^{k+1} e^{i\theta(k+1)} + \dots + b_n r^n e^{i\theta n}| \\ &\leq |1 + b_k r^k e^{ik\theta}| + |b_{k+1} r^{k+1} e^{i\theta(k+1)}| + \dots + |b_n r^n e^{i\theta n}| \\ &= 1 - r^k |b_k| + r^{k+1} |b_{k+1}| + \dots + r^n |b_n| = 1 - r^k (|b_k| - r |b_{k+1}| - \dots - r^{n-k} |b_n|) \end{aligned}$$

Thus, for a sufficiently small  $r$ ,  $|Q(re^{i\theta})| < 1$  which contradicts  $Q(z) \geq 1$  for all  $z$ .

Thus,  $\mu = 0$  so there is a  $z_0$  such that  $|P(z_0)| = \mu = 0$  so  $P(z_0) = 0$ .



## 15.5 Fourier Series

### Definition 15.5.1: Trigonometric Polynomial

A trigonometric polynomial is a finite sum where for  $x \in \mathbb{R}$ :

$$f(x) = a_0 + \sum_{n=1}^N [a_n \cos(nx) + b_n \sin(nx)] = \sum_{n=-N}^N c_n e^{inx}$$

A trigonometric series is then:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Thus:

(a)  $f(x)$  has period of  $2\pi$

(b) Since  $(\frac{1}{in} e^{inx})' = e^{inx}$  where  $\frac{1}{in} e^{inx}$  have period of  $2\pi$ , then for  $n \in \mathbb{Z}$ :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\frac{1}{in} e^{inx})' dx = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \dots \end{cases}$$

(c) For  $m \in \{-N, -N+1, \dots, N\}$ , then:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\sum_{n=-N}^N c_n e^{inx} e^{-imx}] dx \\ &= \sum_{n=-N}^N [\frac{1}{2\pi} \int_{-\pi}^{\pi} c_n e^{inx} e^{-imx} dx] = c_m \end{aligned}$$

(d) If  $f(x)$  is real, then:

$$\overline{c_m} = \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} dx = c_{-m}$$

Thus,  $f(x)$  is real if and only if  $c_{-n} = \overline{c_n}$  for  $n = \{0, 1, \dots, N\}$ .

If  $f(x)$  is integrable on  $[-\pi, \pi]$ , then  $c_m$  are called the Fourier coefficients and  $f(x)$  is a Fourier series of  $f$ .

### Definition 15.5.2: Orthogonal System of Functions

Let  $\{\phi_n\}$  be a sequence of complex functions on  $[a, b]$  such that:

$$\int_a^b \phi_n(x) \overline{\phi_m(x)} dx = 0 \quad \text{for } m \neq n$$

Then,  $\{\phi_n\}$  is an orthogonal system of functions on  $[a, b]$ . Additionally, if:

$$\int_a^b \phi_n(x) \overline{\phi_n(x)} dx = \int_a^b |\phi_n(x)|^2 dx = 1$$

for all  $n$ , then  $\{\phi_n\}$  is orthonormal.

If  $\{\phi_n\}$  is orthonormal on  $[a, b]$  and  $c_n = \int_a^b f(t) \overline{\phi_n(t)} dt$  for  $n = \{1, 2, \dots\}$ , then  $c_n$  is the  $n$ -th Fourier coefficient of  $f$  relative to  $\{\phi_n\}$ . Then:

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

**Theorem 15.5.3: Fourier Series of f is the best approximation to f**

For orthonormal  $\{\phi_n\}$  on  $[a,b]$ , let n-th partial sum of the Fourier series of f,  $\sum_{m=1}^n c_m \phi_m(x) = s_n(x)$ . Suppose  $f \in \mathcal{R}$  and  $t_n(x) = \sum_{m=1}^n \gamma_m \phi_m(x)$ . Then:

$$\int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx$$

where

$$\int_a^b |f - s_n|^2 dx = \int_a^b |f - t_n|^2 dx$$

if and only if  $\gamma_m = c_m$  for every  $m = \{1, \dots, n\}$ .

Also,  $\int |s_n(x)|^2 dx \leq \int |f(x)|^2 dx$ .

**Proof**

$$\int f(x) \overline{t_n(x)} dx = \int f(x) \sum [\overline{\gamma_m \phi_m(x)}] dx = \sum [\int f(x) \overline{\gamma_m \phi_m(x)} dx] = \sum c_m \overline{\gamma_m}$$

Since  $\{\phi_n\}$  is orthonormal, then:

$$\begin{aligned} \int |t_n(x)|^2 dx &= \int t_n(x) \overline{t_n(x)} dx = \int [\sum_m \gamma_m \phi_m(x)] [\sum_k \overline{\gamma_k \phi_k(x)}] dx \\ &= \sum_m \sum_k [\int \gamma_m \phi_m(x) \overline{\gamma_k \phi_k(x)} dx] = \sum |\gamma_m|^2 \end{aligned}$$

Thus:

$$\begin{aligned} \int |f(x) - t_n(x)|^2 dx &= \int |f(x)|^2 dx - \int f(x) \overline{t_n(x)} dx - \int \overline{f(x)} t_n(x) dx + \int |t_n(x)|^2 dx \\ &= \int |f(x)|^2 dx - \sum c_m \overline{\gamma_m} - \sum \overline{c_m} \gamma_m + \sum |\gamma_m|^2 \\ &= \int |f(x)|^2 dx - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2 \end{aligned}$$

Thus,  $\int |f(x) - t_n(x)|^2 dx$  is minimized if and only if  $\gamma_m = c_m$  for every  $m = \{1, \dots, n\}$ .

Let  $\gamma_m = c_m$  and since  $\int |f(x) - s_n(x)|^2 dx \geq 0$ , then:

$$\begin{aligned} \int |f(x) - s_n(x)|^2 dx &= \int |f(x)|^2 dx - \sum |c_m|^2 \\ \int |s_n(x)|^2 dx &= \sum |c_m|^2 \leq \int |f(x)|^2 dx \end{aligned}$$

**Theorem 15.5.4: Bessel Inequality**

For  $\{\phi_n\}$  is orthonormal on  $[a,b]$  and  $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$ , if  $f \in \mathcal{R}$ , then:

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n = 0$$

**Proof**

Since  $\{\phi_n\}$  is orthonormal on  $[a,b]$  and  $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$ , then by **theorem 15.5.3**, for any integer  $n > 1$ :

$$\sum_{m=1}^n |c_m|^2 \leq \int_a^b |f(x)|^2 dx$$

Thus, as  $n \rightarrow \infty$ , then  $\sum_{m=1}^{\infty} |c_m|^2 \leq \int_a^b |f(x)|^2 dx$ .

Since  $\sum_{m=1}^{\infty} |c_m|^2$  is monotonically increasing and bounded above, then  $\sum_{m=1}^{\infty} |c_m|^2$  converges and thus,  $\lim_{n \rightarrow \infty} c_n = 0$ .

**Definition 15.5.5: Trigonometric Series**

Consider functions  $f \in \mathcal{R}$  on  $[-\pi, \pi]$  with period  $2\pi$ . Let  $\phi_n(x) = e^{inx}$  which is orthogonal and orthonormal when  $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ .

Thus, the  $N$ -th partial sum of the Fourier series of  $f$  is:

$s_N(f; x) = \sum_{n=-N}^N c_n e^{inx}$   
where  $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$ . Then by **theorem 15.5.3**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(f; x)|^2 dx = \sum_{n=-N}^N |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

From the Dirichlet kernel,  $D_N(x) = \sum_{n=-N}^N e^{inx}$ .

$$(e^{ix} - 1)D_N(x) = \sum_{n=-N}^N [e^{i(n+1)x} - e^{inx}] = e^{i(N+1)x} - e^{-iNx}$$

$$\begin{aligned} D_N(x) &= \frac{e^{-\frac{1}{2}ix}(e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x})}{e^{-\frac{1}{2}ix}(e^{ix} - 1)} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} \\ &= \frac{2i \sin((N+\frac{1}{2})x)}{2i \sin(\frac{1}{2}x)} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)} \end{aligned}$$

Since  $e^{inx}$  is periodic for  $2\pi$  for each  $n \in [-N, N]$ , then  $D_N(x)$  is periodic for  $2\pi$ .

Thus, since  $f$  is also periodic for  $2\pi$ , then:

$$\begin{aligned} s_N(f; x) &= \sum_{n=-N}^N c_n e^{inx} = \sum_{n=-N}^N \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right] e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[ \sum_{n=-N}^N e^{in(x-t)} \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt \end{aligned}$$

**Theorem 15.5.6:** If  $f$  is continuous at some  $x$ , then Fourier Series of  $f$  converges to  $f$

If for some  $x$ , there are  $\delta > 0$  and  $M$  such that  $|f(x+t) - f(x)| \leq M|t|$  for all  $t \in (-\delta, \delta)$ , then:

$$\lim_{N \rightarrow \infty} s_N(f; x) = f(x)$$

**Proof**

Let  $g(t) = \frac{f(x-t) - f(x)}{\sin(\frac{1}{2}t)}$  for  $t \in [-\pi, \pi]$  where  $g(0) = 0$ . Then by **definition 15.5.1(b)**:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \sum_{n=-N}^N e^{inx} \right] dx = 1$$

Thus:

$$\begin{aligned} s_N(f; x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt - f(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt - f(x) \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)] D_N(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin((N+\frac{1}{2})t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) [\sin(Nt) \cos(\frac{1}{2}t) + \sin(\frac{1}{2}t) \cos(Nt)] dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \cos(\frac{1}{2}t)] \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \sin(\frac{1}{2}t)] \cos(Nt) dt \end{aligned}$$

Since  $g(t)$  and  $\cos(\frac{1}{2}t), \sin(\frac{1}{2}t)$  are bounded on  $[-\pi, \pi]$ , then  $g(t) \cos(\frac{1}{2}t)$  and  $g(t) \sin(\frac{1}{2}t)$  are bounded on  $[-\pi, \pi]$ . As  $N \rightarrow \infty$ , then  $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \cos(\frac{1}{2}t)] \sin(Nt) dt = 0$  and  $\frac{1}{2\pi} \int_{-\pi}^{\pi} [g(t) \sin(\frac{1}{2}t)] \cos(Nt) dt = 0$  so  $\lim_{N \rightarrow \infty} s_N(f; x) = f(x)$ .

**Corollary 15.5.7: Localization Theorem**

If  $f(x) = 0$  for all  $x$  in some segment  $J$ , then for every  $x \in J$ :

$$\lim_{N \rightarrow \infty} s_N(f; x) = 0$$

**Proof**

Let  $J = (a, b)$ . Then for  $x \in J$ , choose  $\delta$  such that  $(x - \delta, x + \delta) \subset J$ .

Thus for any  $t \in (-\delta, \delta)$ , then  $|f(x+t) - f(x)| = |0 - 0| = 0$ .

Then by **theorem 15.5.6**, for every  $x \in J$ ,  $\lim_{N \rightarrow \infty} s_N(f; x) = f(x) = 0$ .

**Corollary 15.5.8:** Equivalent functions on (a,b) have similar Fourier Series on (a,b)

If  $f(t) = g(t)$  for all  $t$  in some neighborhood of  $x$ , then:

$$\lim_{N \rightarrow \infty} [s_N(f; x) - s_N(g; x)] = 0$$

**Proof**

Since  $f(t) - g(t) = 0$  for all  $t \in (x - \delta, x + \delta)$ , then by **corollary 15.5.7**, then:

$$\lim_{N \rightarrow \infty} s_N(f - g; x) = 0$$

The Fourier series for  $f - g$ :

$$s_N(f - g; x) = \sum_{n=-N}^N c_n e^{inx} \quad \text{where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f - g)(t) e^{-int} dt$$

The Fourier series for  $f$  and  $g$ :

$$s_N(f; x) = \sum_{n=-N}^N a_n e^{inx} \quad \text{where } a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

$$s_N(g; x) = \sum_{n=-N}^N b_n e^{inx} \quad \text{where } b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-int} dt$$

Then  $s_N(f - g; x) = s_N(f; x) - s_N(g; x)$  and thus:

$$\lim_{N \rightarrow \infty} [s_N(f; x) - s_N(g; x)] = \lim_{N \rightarrow \infty} s_N(f - g; x) = 0$$

**Theorem 15.5.9:** There are Fourier Series that converge uniformly to continuous  $f$

If  $f$  is continuous with period  $2\pi$ , then for  $\epsilon > 0$ , there is a trigonometric polynomial  $P$  such that for all  $x \in \mathbb{R}$ :

$$|P(x) - f(x)| < \epsilon$$

**Proof**

Since  $f(x)$  has a period of  $2\pi$ , then for a fixed  $x \in \mathbb{R}$ ,  $f(x)$  on  $\mathbb{R}$  can be defined on compact  $[x, x+2\pi]$  which is the complex unit circle  $T$  by a mapping of  $x \rightarrow e^{ix}$ .

The set of trigonometric polynomials,  $P(x) = \sum_{n=-N}^N c_n e^{inx}$  for constants  $c_n \in \mathbb{C}$  and integer  $N \geq 0$ , is an algebra  $\mathcal{A}$  since for  $P_1(x) = \sum_{n=-N_1}^{N_1} a_n e^{inx}$  and  $P_2(x) = \sum_{n=-N_2}^{N_2} b_n e^{inx}$ , let  $N = \max(N_1, N_2)$  and  $a_n, b_n = 0$  if  $n \geq N_1, N_2$  respectively:

$$P_1(x) + P_2(x) = \sum_{n=-N}^N (a_n + b_n) e^{inx} \text{ so } P_1(x) + P_2(x) \in \mathcal{A}$$

$$P_1(x)P_2(x) = \sum_{n=-2N}^{2N} d_n e^{inx} \text{ where } d_n = \sum_{k=-N}^N a_k b_{n-k} \text{ so } P_1(x)P_2(x) \in \mathcal{A}$$

$$cP_1(x) = \sum_{n=-N_1}^{N_1} (ca_n) e^{inx} \text{ where } ca_n \in \mathbb{C} \text{ so } cP_1(x) \in \mathcal{A}$$

Also,  $\mathcal{A}$  is self-adjoint since:

$$\overline{P_1(x)} = \sum_{n=-N_1}^{N_1} \overline{a_n} e^{-inx} = \sum_{n=-N_1}^{N_1} \overline{a_{-n}} e^{inx} \text{ where } \overline{a_{-n}} \in \mathbb{C} \text{ so } \overline{P_1(x)} \in \mathcal{A}$$

Also,  $\mathcal{A}$  separates points on  $T$  since any two points on  $T$  are distinct and  $\mathcal{A}$  vanishes at no point of  $T$  since  $(0,0)$  does not exist on the complex unit circle. For  $\pi > \epsilon > 0$ , since the mapping  $x \rightarrow e^{ix}$  is 1-1 from  $[x+\epsilon, x+2\pi-\epsilon]$ , then  $\mathcal{A}$  separates points and vanishes at no point on  $[x+\epsilon, x+2\pi-\epsilon]$ .

Thus, by **theorem 14.7.9**, then  $\mathcal{B}$ , the set of all uniformly convergent  $P(x)$  from  $\mathcal{A}$ , consist of all complex continuous  $f$  on  $[x+\epsilon, x+2\pi-\epsilon]$ .

So there is a  $P(x)$  such that  $P(x)$  converges uniformly to  $f$  so for all  $t \in [x, x+2\pi]$ , then  $|P(t) - f(t)| < \epsilon$ . Since  $f$  has a period of  $2\pi$ , then for all  $x \in \mathbb{R}$ , then  $|P(t) - f(t)| < \epsilon$ .

**Definition 15.5.10:  $L^p$  Space**

For  $p \geq 1$ , let  $L^p = \{ f: [a,b] \rightarrow \mathbb{C} \mid \|f\|_p = [\int_a^b |f(x)|^p dx]^{\frac{1}{p}} < \infty \}$ .

For complex  $f, g \in \mathcal{R}$ :

- (a) **Holder's Inequality**: If  $\frac{1}{p} + \frac{1}{q} = 1$  where  $p, q \geq 1$ , then  $\|fg\|_1 \leq \|f\|_p \|g\|_q$

**Proof**

**Claim**: If  $a, b \geq 0$ , then  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$  and equality only if  $a^p = b^q$ .  
 Take  $y = f(x) = x^{p-1}$  for  $x \in [0, a]$  and  $x = f^{-1}(y) = \sqrt[p-1]{y}$  for  $y \in [0, b]$ .  
 The total area is  $\int_0^a x^{p-1} dx + \int_0^b y^{\frac{1}{p-1}} dy = \frac{a^p}{p} + \frac{p-1}{p} b^{\frac{p}{p-1}} = \frac{a^p}{p} + \frac{b^q}{q}$ .  
 Graphing each function on their respective axes, it is shown that regardless if  $a^{p-1} > b$  or  $a^{p-1} < b$ , the total area is greater than  $ab$  and equality holds only if  $a^{p-1} = b$  so  $b^q = a^{(p-1)q} = a^{(p-1)\frac{p}{p-1}} = a^p$ .

$$\begin{aligned} \frac{1}{\|f\|_p \|g\|_q} \|fg\|_1 &= \frac{1}{\|f\|_p \|g\|_q} \int |fg| dx = \frac{1}{\|f\|_p \|g\|_q} \int |f| |g| dx \\ &= \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} dx \leq \int \frac{|f|^p}{\|f\|_p^p} + \frac{|g|^q}{\|g\|_q^q} dx \\ &= \frac{1}{\|f\|_p^p} \int |f|^p dx + \frac{1}{\|g\|_q^q} \int |g|^q dx \\ &= \frac{1}{\|f\|_p^p} \|f\|_p^p + \frac{1}{\|g\|_q^q} \|g\|_q^q = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Since  $a = \frac{|f|}{\|f\|_p}$  and  $b = \frac{|g|}{\|g\|_q}$ , then equality holds only if  $\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q}$ .

- (b) **Minkowski's Inequality**:  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

**Proof**

Since  $f, g \in \mathcal{R}$ , then  $|f + g|^p \in \mathcal{R}$ . By Holder's Inequality:  

$$\begin{aligned} \|f + g\|_p^p &= \int_a^b |f(x) + g(x)|^p dx = \int_a^b |f(x) + g(x)| |f(x) + g(x)|^{p-1} dx \\ &\leq \int_a^b (|f(x)| + |g(x)|) |f(x) + g(x)|^{p-1} dx \\ &\leq \int_a^b |f(x)| |f(x) + g(x)|^{p-1} dx + \int_a^b |g(x)| |f(x) + g(x)|^{p-1} dx \\ &\leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} + (\int_a^b |g(x)|^p dx)^{\frac{1}{p}} (\int_a^b |f(x) + g(x)|^{p-1(\frac{p}{p-1})} dx)^{1-\frac{1}{p}} \\ &= (\|f\|_p + \|g\|_p) \|f + g\|_p^{p-1} \end{aligned}$$

**Theorem 15.5.11: For integrable  $f$ , there are continuous  $g$  where  $f, g \in L^2$** 

Let  $f \in \mathcal{R}$  on  $[a, b]$ . Then for  $\epsilon > 0$ , there is a continuous  $g$  where:

$$g(a) = f(a) \quad g(b) = f(b) \quad \|f(x) - g(x)\|_2 < \epsilon$$

**Proof**

Since  $f \in \mathcal{R}$ , then  $|f(x)| < M$ . For  $\epsilon > 0$ , there is a partition  $P = \{x_0, \dots, x_n\}$  of  $[a, b]$ :

$$U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i) \Delta x_i < \frac{\epsilon^2}{2M}$$

Let  $g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$  for  $t \in [x_{i-1}, x_i]$  which is continuous on  $[a, b]$  since:

$$g(x_i+) = f(x_i) = g(x_i-) \Rightarrow g(x_i) = f(x_i) \text{ so } g(a) = f(a), g(b) = f(b)$$

Thus, for  $t \in [x_{i-1}, x_i]$ :

$$\begin{aligned} |f(t) - g(t)| &= |f(t) - \frac{x_i - t}{\Delta x_i} f(x_{i-1}) - \frac{t - x_{i-1}}{\Delta x_i} f(x_i)| \\ &= |\frac{x_i - t}{\Delta x_i} [f(t) - f(x_{i-1})] + \frac{t - x_{i-1}}{\Delta x_i} [f(t) - f(x_i)]| \\ &\leq |\frac{x_i - t}{\Delta x_i}| |f(t) - f(x_{i-1})| + |\frac{t - x_{i-1}}{\Delta x_i}| |f(t) - f(x_i)| = M_i - m_i \end{aligned}$$

Since  $g$  is continuous, then  $g \in \mathcal{R}$  and thus,  $|f(x) - g(x)|^2 \in \mathcal{R}$ . Thus:

$$\begin{aligned} \|f(x) - g(x)\|_2 &= [\int_a^b |f(x) - g(x)|^2 dx]^{\frac{1}{2}} = \lim_{n \rightarrow \infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t) - g(t)|^2 dt]^{\frac{1}{2}} \\ &\leq \lim_{n \rightarrow \infty} [\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (M_i - m_i)^2 dt]^{\frac{1}{2}} \leq \lim_{n \rightarrow \infty} [\sum_{i=1}^n 2M \int_{x_{i-1}}^{x_i} (M_i - m_i) dt]^{\frac{1}{2}} \\ &= \lim_{n \rightarrow \infty} [2M \sum_{i=1}^n (M_i - m_i) \Delta x_i]^{\frac{1}{2}} < \lim_{n \rightarrow \infty} [2M \frac{\epsilon^2}{2M}]^{\frac{1}{2}} = \epsilon \end{aligned}$$

**Theorem 15.5.12: Parseval's Theorem**

For  $f, g \in \mathcal{R}$  with period of  $2\pi$  where:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$$

then:

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 dx = 0$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

**Proof**

Since  $f \in \mathcal{R}$  on  $[x, x+2\pi]$  for a fixed  $x \in \mathbb{R}$ , where  $f(x) = f(x+2\pi)$ , then by **theorem 15.5.11**, for  $\epsilon > 0$ , there is a continuous  $h$  such that:

$$\|f(x) - h(x)\|_2 < \epsilon$$

Also,  $h(x) = f(x)$  and  $h(x+2\pi) = f(x+2\pi)$  for any  $x \in \mathcal{R}$ , and since  $f(x) = f(x+2\pi)$ , then  $h$  has a period of  $2\pi$ . Then by **theorem 15.5.9**, there is a trigonometric polynomial  $P(x)$  such that for all  $x \in \mathbb{R}$ :

$$|h(x) - P(x)| < \epsilon \quad \Rightarrow \quad \|h(x) - P(x)\|_2 = \left[ \int_x^{x+2\pi} |h(x) - P(x)|^2 dx \right]^{\frac{1}{2}} < \sqrt{2\pi} \epsilon$$

Then by **theorem 15.5.3**:

$$\|h(x) - s_N(h; x)\|_2 \leq \|h(x) - P(x)\|_2 < \sqrt{2\pi} \epsilon$$

$$\|s_N(h; x) - s_N(f; x)\|_2 = \|s_N(h - f; x)\|_2 \leq \|h(x) - f(x)\|_2 < \epsilon$$

Thus:

$$\begin{aligned} \|f(x) - s_N(f; x)\|_2 &\leq \|f(x) - h(x)\|_2 + \|h(x) - s_N(h; x)\|_2 + \|s_N(h; x) - s_N(f; x)\|_2 \\ &< (2 + \sqrt{2\pi}) \epsilon \end{aligned}$$

Note  $\frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx = \sum_{n=-N}^N [c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \overline{g(x)} dx] = \sum_{n=-N}^N c_n \overline{\gamma_n}$ .

By Holder's Inequality:

$$\begin{aligned} & \left| \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx - \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx \right| \\ & \leq \int_{-\pi}^{\pi} |f(x) - s_N(f; x)| |g(x)| dx \\ & \leq \left[ \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 dx \right]^{\frac{1}{2}} \left[ \int_{-\pi}^{\pi} |g(x)|^2 dx \right]^{\frac{1}{2}} \\ & = \|f(x) - s_N(f; x)\|_2 \|g(x)\|_2 \end{aligned}$$

Since  $g \in \mathcal{R}$ , then  $|g|^2 \in \mathcal{R}$  and thus,  $\|g(x)\|_2$  is bounded.

Since  $\lim_{N \rightarrow \infty} \|f(x) - s_N(f; x)\|_2 = 0$ , then:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f; x) \overline{g(x)} dx = \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n \overline{\gamma_n} = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{f(x)} dx = \sum_{n=-\infty}^{\infty} c_n \overline{c_n} = \sum_{n=-\infty}^{\infty} |c_n|^2$$

## 16 Multivariable Functions

### 16.1 Linear Transformations

#### Definition 16.1.1: Vector Spaces

(a) **Vector Space**

A nonempty set  $X \subset \mathbb{R}^n$  is a vector space if for all  $x, y \in X$  and scalar  $c$ :

$$x+y \in X \quad cx \in X$$

Null vector  $0$  is also defined as  $0 = (0, \dots, 0) \in \mathbb{R}^k$ .

(b) **Linear Combinations and Span**

For scalars  $c_1, \dots, c_k$ , a linear combination of  $x_1, \dots, x_k \in \mathbb{R}^n$ :

$$c_1x_1 + \dots + c_kx_k$$

The span of  $x_1, \dots, x_k$  is the set of all linear combinations of  $x_1, \dots, x_k$ .

(c) **Independence and Dimension**

If  $c_1x_1 + \dots + c_kx_k = 0$  only if  $c_1 = \dots = c_k = 0$ , then  $x_1, \dots, x_k$  are independent. Any independent set does not contain  $0$  since  $c0 + c_1x_1 + \dots + c_kx_k = 0$  holds true for  $c, 0, \dots, 0$  where  $c$  is any number, not just  $0, 0, \dots, 0$ .

If vector space  $X$  have  $r$  independent vectors, but not  $r+1$  independent vectors, then  $\dim(X) = r$ . The set  $\{0\}$  has dimension  $0$ .

(d) **Basis**

If  $x_1, \dots, x_k \in X$  are independent and spans  $X$ , then  $x_1, \dots, x_k$  is a basis of  $X$ . Thus, for every  $x \in X$ :

Since  $x_1, \dots, x_k$  spans  $X$ , there exists  $c_1, \dots, c_k$  such that  $x = c_1x_1 + \dots + c_kx_k$ .

Since  $x_1, \dots, x_k$  are independent, then such  $c_1, \dots, c_k$  are unique else there are  $a_1, \dots, a_k$  where at least one  $a_i \neq c_i$  such that:

$$x = a_1x_1 + \dots + a_kx_k \quad \Rightarrow \quad 0 = x - x = (a_1 - c_1)x_1 + \dots + (a_k - c_k)x_k$$

where at least one  $(a_i - c_i) \neq 0$  contradicting  $x_1, \dots, x_k$  are independent.

The  $c_1, \dots, c_k$  are called the coordinates of  $x$  with respect to basis  $x_1, \dots, x_k$ .

(e) **Standard Basis of  $\mathbb{R}^k$**

Let  $e_i = (0, \dots, \underset{1}{0}, \dots, \underset{i-1}{0}, \underset{i}{1}, \underset{i+1}{0}, \dots, \underset{k}{0}) \in \mathbb{R}^k$ .

Thus,  $e_1, \dots, e_k$  is a basis for  $\mathbb{R}^k$  where any  $x = (x_1, \dots, x_k) = x_1e_1 + \dots + x_ke_k$ .

#### Theorem 16.1.2: $\dim(X) \leq (\# \text{ vectors that span } X)$

If vector space  $X$  is spanned by  $r$  vectors, then  $\dim(X) \leq r$ .

#### Proof

If  $\dim(X) > r$ , then there are at minimum  $r+1$  independent vectors that spans  $X$  which contradicts that  $X$  is spanned by  $r$  vectors.

Let  $X$  be spanned by  $x_1, \dots, x_r \neq 0$ . If  $x_1, \dots, x_r$  are independent, then  $\dim(X) = r$ .

If  $x_1, \dots, x_r$  are not independent, then there is at least two  $c_k \neq 0$  where:

$$0 = c_1x_1 + \dots + c_rx_r$$

since if only one  $c_k \neq 0$ , then  $0 = c_1x_1 + \dots + c_rx_r = c_kx_k$  which implies  $x_k = 0$  since  $c_k \neq 0$  which is a contradiction. Thus, for  $c_k, c_{i_1}, \dots, c_{i_n} \neq 0$ :

$$0 = c_1x_1 + \dots + c_rx_r = c_kx_k + c_{i_1}x_{i_1} + \dots + c_{i_n}x_{i_n} \quad \Rightarrow \quad x_k = \frac{-c_{i_1}}{c_k}x_{i_1} + \dots + \frac{-c_{i_n}}{c_k}x_{i_n}$$

Remove  $x_k$  from  $x_1, \dots, x_r$  and repeat the process until all  $x_i$  are independent and thus,  $\dim(X) = r - (\# x_i \text{ removed}) < r$ .

**Corollary 16.1.3:**  $\dim(X) = (\# \text{ vectors in a basis})$

If  $x_1, \dots, x_n$  is a basis for  $X$ , then  $\dim(X) = n$ .

Thus,  $\dim(\mathbb{R}^n) = n$ .

**Proof**

Since  $x_1, \dots, x_n$  is a basis for  $X$ , then  $x_1, \dots, x_n$  spans  $X$  and are independent.  
 Since  $x_1, \dots, x_n$  span  $X$ , then by **theorem 16.1.2**, then  $\dim(X) \leq n$ . Since  $x_1, \dots, x_n$  are independent, then  $\dim(X) \geq n$  since there might be another  $x_i$  independent to  $x_1, \dots, x_n$  and another and so on. Thus,  $\dim(\mathbb{R}^n) = n$ .  
 Since  $e_1, \dots, e_n$  is a basis for  $\mathbb{R}^n$ , then  $\dim(\mathbb{R}^n) = n$ .

**Theorem 16.1.4: Properties of Basis**

For vector space  $X$  where  $\dim(X) = n$ :

- (a)  $n$  vectors span  $X$  if and only if the  $n$  vectors are independent
- (b)  $X$  has a basis where every basis have only  $n$  vectors
- (c) For independent  $x_1, \dots, x_r$  where  $r \in \{1, \dots, n\}$ ,  $X$  has a basis with  $x_1, \dots, x_r$

**Intuition**

$x_1, \dots, x_m$  can span  $X$ , but not independent since there might be a  $x_i$  that is dependent on the other  $x_i$  (aka  $x_i = a_i x_i + \dots + a_{i-1} x_{i-1} + a_{i+1} x_{i+1} + \dots + a_m x_m$ ).  
 $x_1, \dots, x_k$  can be independent, but not span  $X$  since there might be another  $x$  that is independent to each  $x_i$  (aka  $x \neq b_1 x_1 + \dots + b_k x_k$  for any  $b_1, \dots, b_k$ ).  
 So to get a basis, either remove the dependent elements from  $x_1, \dots, x_m$  to get independent or add independent elements to  $x_1, \dots, x_k$  to get a span of  $X$ . Simply, a basis has a set amount of vectors, but  $x_1, \dots, x_m$  has too much while  $x_1, \dots, x_k$  has too few.

**Proof**

Let  $x_1, \dots, x_n$  span  $X$ . If  $x_1, \dots, x_n$  are not independent, then remove  $x_i$  until  $x_1, \dots, x_k$  are independent as performed in **theorem 16.1.2**. Thus,  $\dim(X) = k < n$  which is a contradiction and thus,  $x_1, \dots, x_n$  are independent.

For independent  $x_1, \dots, x_n$ , add  $y_1, \dots, y_k \in X$  so  $x_1, \dots, x_n, y_1, \dots, y_k$  span  $X$ . Since  $\dim(X) = n$ , then  $x_1, \dots, x_n, y_1, \dots, y_k$  are not independent. Since any non-independent set can remove elements in its span until it is independent and thus, preserves its span as performed in **theorem 16.1.2**, then each  $y_i$  can be removed to reach independent  $x_1, \dots, x_n$  which still spans  $X$ .

By part (a), any  $n$  independent vectors spans  $X$  so thus, forms a basis for  $X$ . For  $x_1, \dots, x_k$  where  $k > n$ , since  $\dim(X) = n$ , then  $x_1, \dots, x_k$  is non-independent and is thus, not a basis. For  $x_1, \dots, x_k$  where  $k < n$ , since  $\dim(X) = n$ , there is a  $x \in X$  such that  $x_1, \dots, x_k, x$  are independent. Then  $x \neq c_1 x_1 + \dots + c_k x_k$  for any  $c_1, \dots, c_k$  else

$$x = c_1 x_1 + \dots + c_k x_k \quad \Rightarrow \quad 0 = c_1 x_1 + \dots + c_k x_k + -x$$

so  $x_1, \dots, x_k, x$  are not independent. Thus, there is a  $x \in X$  that is not in the span of  $x_1, \dots, x_k$  so  $x_1, \dots, x_k$  does not span  $X$ .

For independent  $x_1, \dots, x_r$ , since  $\dim(X) = n$ , there are  $x_{r+1}, \dots, x_n$  such that  $x_1, \dots, x_n$  are independent. By part (a),  $x_1, \dots, x_n$  spans  $X$  so  $x_1, \dots, x_n$  forms a basis which contain  $x_1, \dots, x_r$ .



**Definition 16.1.5: Linear Transformation**

A mapping  $A$  of vector space  $X$  into vector space  $Y$  is a linear transformation if for all  $x_1, x_2 \in X$  and scalar  $c$ :

$$A(x_1 + x_2) = Ax_1 + Ax_2 \quad A(cx_1) = cAx_1$$

Since  $A0 + A0 = A(0+0) = A0$ , then  $A0 = 0$ .

If  $x_1, \dots, x_n$  is a basis for  $X$ , then for any  $x \in X$ , there is a unique set of  $c_1, \dots, c_n$  where  $x = c_1x_1 + \dots + c_nx_n$  such that:

$$Ax = A(c_1x_1 + \dots + c_nx_n) = c_1Ax_1 + \dots + c_nAx_n$$

Linear transformation that maps  $X$  into  $X$  are **linear operators**.

Additionally, if  $A$  is 1-1 and maps  $X$  onto  $X$ , then  $A$  is **invertible**.

Thus, there is a  $A^{-1}$  such that:

$$A^{-1}(Ax) = x \quad \text{for all } x \in X$$

Since  $A$  maps  $X$  onto  $X$ , for any  $x \in X$ , then  $Ax = y \in X$ .

Thus, for all  $y \in X$ , then  $x = A^{-1}(Ax) = A^{-1}y$ . Thus:

$$A(A^{-1}y) = Ax = y$$

Also, for any  $x_1, x_2 \in X$  and scalars  $c_1, c_2$  where  $Ax_1 = y_1$  and  $Ax_2 = y_2$ :

$$\begin{aligned} A^{-1}(c_1y_1 + c_2y_2) &= A^{-1}(c_1Ax_1 + c_2Ax_2) = A^{-1}(A(c_1x_1 + c_2x_2)) \\ &= c_1x_1 + c_2x_2 = c_1A^{-1}(y_1) + c_2A^{-1}(y_2) \end{aligned}$$

So,  $A^{-1}$  is a linear transformation.

**Theorem 16.1.6: Linear Operators imply 1-1  $\Leftrightarrow$  onto**

Linear operator  $A$  preserves independence if and only if  $A$  is 1-1.

Thus, linear operator  $A$  is 1-1 if and only if  $A(X) = X$ .

**Proof**

Let  $x_1, \dots, x_n$  be a basis for  $X$  where each  $Ax_i = y_i \in X$ . So for any  $y \in A(X)$ , there is  $x \in X$  where  $x = c_1x_1 + \dots + c_nx_n$  for a unique set of  $c_1, \dots, c_n$  such that:

$$y = Ax = A(c_1x_1 + \dots + c_nx_n) = c_1Ax_1 + \dots + c_nAx_n = c_1y_1 + \dots + c_ny_n$$

If  $A$  is 1-1, then there is only one such  $x$  so in respect to  $y_1, \dots, y_n$ , then any

$y = k_1y_1 + \dots + k_ny_n$  must have  $k_1 = c_1, \dots, k_n = c_n$ . Thus, for  $y = 0$ , since  $0 = A0$  and  $x_1, \dots, x_n$  are independent, then  $c_1 = \dots = c_n = 0$  so  $y_1, \dots, y_n$  are independent.

If  $A$  is not 1-1, then there is  $y$  where there are at least two distinct such  $x$  so in respect to  $y_1, \dots, y_n$ , then  $y = k_1y_1 + \dots + k_ny_n$  holds true for at least 2 distinct  $k_1, \dots, k_n$  so  $y_1, \dots, y_n$  are not independent. Thus,  $A$  is 1-1 if and only if  $y_1, \dots, y_n$  is independent. By **theorem 16.1.4a**,  $y_1, \dots, y_n$  span  $X$  so  $A(X) = X$  if and only if  $y_1, \dots, y_n$  are independent.

**Definition 16.1.7: Operations of Linear Transformations**

Let  $L(X, Y)$  be the set of all linear transformation of  $X$  into  $Y$ .

Let  $\Omega$  be the set of all invertible linear operators on  $\mathbb{R}^n$ .

- (a) If  $A_1, A_2 \in L(X, Y)$  and  $c_1, c_2$  are scalars, then for any  $x \in X$ , define:  
 $(c_1 A_1 + c_2 A_2)x = c_1 A_1 x + c_2 A_2 x$
- (b) For vector space  $Z$ , if  $A \in L(X, Y)$  and  $B \in L(Y, Z)$ , then for any  $x \in X$ , define:  
 $(BA)x = B(Ax) \in L(X, Z)$
- (c) For  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ , define the norm:  
 $\|A\| = \sup(|Ax| \mid x \in \mathbb{R}^n \text{ where } |x| \leq 1)$
- (d)  $|Ax| = |A(|x| \frac{x}{|x|})| = |A(\frac{x}{|x|})| |x| \leq \sup(|A(\frac{x}{|x|})|) |x| = \|A\| |x|$   
 If there is a  $\lambda$  such that  $|Ax| \leq \lambda |x|$  for all  $x \in \mathbb{R}^n$ , then  $\|A\| \leq \lambda |1| = \lambda$ .
- (e) For  $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ , the distance between  $A$  and  $B$  is defined  $\|A - B\|$

**Theorem 16.1.8: Operations of Norms of Linear Transformations**

- (a) If  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ , then  $\|A\| < \infty$ . Thus,  $A$  is uniformly continuous.

**Proof**

For  $|x| \leq 1$ :

$$|Ax| = |A(x_1 e_1 + \dots + x_n e_n)| \leq |x_1| |Ae_1| + \dots + |x_n| |Ae_n|$$

$$\leq |Ae_1| + \dots + |Ae_n| = M$$

Thus,  $\|Ax\| \leq |Ae_1| + \dots + |Ae_n| = M < \infty$ .

Let  $|x - y| < \epsilon$  and thus,  $|Ax - Ay| = |A(x - y)| \leq \|A\| |x - y| < M\epsilon$  so  $A$  is uniformly continuous.

- (b) If  $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$  and  $c$  is a scalar, then:

$$\|A + B\| \leq \|A\| + \|B\| \quad \|cA\| = |c| \|A\|$$

**Proof**

For  $|x| \leq 1$ ,  $|(A + B)x| \leq |Ax + Bx| \leq |Ax| + |Bx| \leq \|A\| + \|B\|$ .

Thus,  $\|A + B\| \leq \|A\| + \|B\|$ . Since  $|cAx| = |c| |Ax|$ , then  $\|cA\| = |c| \|A\|$ .

Also, for the distance between  $A$  and  $B$ , by part a:

$$\|A - B\| \leq \|A + B\| \leq \|A\| + \|B\| \leq M_1 + M_2$$

- (c) If  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$  and  $B \in L(\mathbb{R}^m, \mathbb{R}^k)$ , then:

$$\|BA\| \leq \|B\| \|A\|$$

**Proof**

For  $|x| \leq 1$ ,  $|BAx| = |B(Ax)| \leq \|B\| |Ax| \leq \|B\| \|A\| |x| \leq \|B\| \|A\|$ .

Thus,  $\|BA\| \leq \|B\| \|A\|$ .

**Theorem 16.1.9: Operations of Norms of Invertible Linear Operators**

- (a) If  $A \in \Omega$  and  $B \in L(\mathbb{R}^n, \mathbb{R}^n)$  where  $\|B - A\| \|A^{-1}\| < 1$ , then  $B \in \Omega$

**Proof**

$$\begin{aligned} \frac{1}{\|A^{-1}\|} |x| &= \frac{1}{\|A^{-1}\|} |A^{-1}Ax| \leq \frac{1}{\|A^{-1}\|} \|A^{-1}\| |Ax| \\ &= |Ax| \leq |(A - B)x| + |Bx| \leq \|A - B\| |x| + |Bx| \\ \text{Thus, } |Bx| &\geq \left( \frac{1}{\|A^{-1}\|} - \|A - B\| \right) |x| \geq \frac{2}{\|A^{-1}\|} |x| \geq 0 \text{ so } Bx \neq 0 \text{ if } x \neq 0 \text{ so } B \\ &\text{is 1-1. Then by theorem 16.1.4a, } B \text{ spans } \mathbb{R}^n \text{ so } B \text{ is invertible so } B \in \Omega. \end{aligned}$$

- (b)  $\Omega \subset L(\mathbb{R}^n, \mathbb{R}^n)$  is open and the mapping  $T: A \rightarrow A^{-1}$  is continuous on  $\Omega$

**Proof**

Since  $\|B - A\| < \frac{1}{\|A^{-1}\|}$  for any  $B \in \Omega$ , then for every  $B \in \Omega$ , there exist an open subset of  $L(\mathbb{R}^n, \mathbb{R}^n)$  that contains  $B$  so  $\Omega$  is open.

Since

$$\begin{aligned} |y| &= |BB^{-1}y| \geq \left( \frac{1}{\|A^{-1}\|} - \|A - B\| \right) |B^{-1}y| \\ &\geq \left( \frac{1}{\|A^{-1}\|} - \|A - B\| \right) \|B^{-1}\| |y| \end{aligned}$$

then  $\frac{1}{\|A^{-1}\| - \|A - B\|} \geq \|B^{-1}\|$ . Thus, by theorem 16.1.8:

$$\begin{aligned} \|B^{-1} - A^{-1}\| &= \|B^{-1}(A - B)A^{-1}\| \\ &\leq \|B^{-1}\| \|A - B\| \|A^{-1}\| \leq \frac{\|A - B\| \|A^{-1}\|}{\frac{1}{\|A^{-1}\|} - \|A - B\|} \end{aligned}$$

Since  $\lim_{B \rightarrow A} \|A - B\| \rightarrow 0$  so  $\|B^{-1} - A^{-1}\| \rightarrow 0$ , then  $T$  is continuous on  $\Omega$ .

**Definition 16.1.10: Matrices**

Let  $x_1, \dots, x_n$  be a basis for  $X$  and  $y_1, \dots, y_m$  be a basis for  $Y$ .

Then every  $A \in L(X, Y)$  determines a set of numbers  $a_{ij}$  such that:

$$Ax_j = \sum_{i=1}^m a_{ij} y_i \quad \text{for } j = \{1, \dots, n\}$$

Thus,  $A$  can be represented by an  $m$  by  $n$  matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Since the  $a_{ij}$  of  $Ax_j$  are from the  $j$ -th column  $[A]$ , then  $Ax_j$  is called the column vector of  $[A]$ . Thus, the  $\text{span}(A)$  is the span of the column vectors of  $[A]$ .

For any  $x \in X$ , there is a unique set of  $c_1, \dots, c_n$  such that  $x = c_1 x_1 + \dots + c_n x_n$ :

$$[Ax] = \begin{bmatrix} (y_1) \{ \overbrace{a_{11}}^{c_1} \overbrace{a_{12}}^{c_2} \dots \overbrace{a_{1n}}^{c_n} \\ (y_2) \{ a_{21} \quad a_{22} \quad \dots \quad a_{2n} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ (y_m) \{ a_{m1} \quad a_{m2} \quad \dots \quad a_{mn} \end{bmatrix}$$

$$\begin{aligned} Ax &= A(\sum_{j=1}^n c_j x_j) \\ &= \sum_{j=1}^n c_j Ax_j \\ &= \sum_{j=1}^n c_j \sum_{i=1}^m a_{ij} y_i \\ &= \sum_{j=1}^n \sum_{i=1}^m a_{ij} c_j y_i \\ &= \sum_{i=1}^m [\sum_{j=1}^n a_{ij} c_j] y_i \end{aligned}$$

So  $[\sum_{j=1}^n a_{1j} c_j], \dots, [\sum_{j=1}^n a_{mj} c_j]$  are  $Ax$ 's coordinates in respect to  $y_1, \dots, y_m$ .

Let  $A \in L(X, Y)$  and  $B \in L(Y, Z)$ . Then,  $BA \in L(X, Z)$ .

Let  $z_1, \dots, z_p$  be a basis for  $Z$  where:

$$By_i = \sum_{k=1}^p b_{ki} z_k \quad (BA)x_j = \sum_{k=1}^p c_{kj} z_k$$

Thus,  $B$  as a  $p$  by  $m$  matrix and  $BA$  as a  $p$  by  $n$  matrix can be represented:

$$[B] = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \quad [BA] = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix}$$

$$\begin{aligned} (BA)x_j &= B(Ax_j) = B(\sum_{i=1}^m a_{ij} y_i) \\ &= \sum_{i=1}^m a_{ij} By_i \\ &= \sum_{i=1}^m a_{ij} \sum_{k=1}^p b_{ki} z_k \\ &= \sum_{i=1}^m \sum_{k=1}^p b_{ki} a_{ij} z_k \\ &= \sum_{k=1}^p [\sum_{i=1}^m b_{ki} a_{ij}] z_k \end{aligned}$$

Thus,  $c_{kj} = \sum_{i=1}^m b_{ki} a_{ij}$  for  $j = \{1, \dots, n\}$  and  $k = \{1, \dots, p\}$ .

So to get matrix  $[BA]$  from  $[B]$  and  $[A]$ :

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m b_{1i} a_{i1} & \sum_{i=1}^m b_{1i} a_{i2} & \dots & \sum_{i=1}^m b_{1i} a_{in} \\ \sum_{i=1}^m b_{2i} a_{i1} & \sum_{i=1}^m b_{2i} a_{i2} & \dots & \sum_{i=1}^m b_{2i} a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m b_{pi} a_{i1} & \sum_{i=1}^m b_{pi} a_{i2} & \dots & \sum_{i=1}^m b_{pi} a_{in} \end{bmatrix}$$

For  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ , since  $Ax = \sum_{i=1}^m [\sum_{j=1}^n a_{ij}c_j] e_i$  where  $x = \sum_{j=1}^n c_j e_j$ , then by the Cauchy-Schwarz Inequality:

$$\begin{aligned} |Ax|^2 &= \sum_{i=1}^m [\sum_{j=1}^n a_{ij}c_j]^2 \\ &\leq \sum_{i=1}^m [(\sum_{j=1}^n a_{ij}^2) (\sum_{j=1}^n c_j^2)] \\ &= [\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2] (\sum_{j=1}^n c_j^2) \\ &= [\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2] |x|^2 \end{aligned}$$

Thus, for  $|x| \leq 1$ , then  $\|A\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$ .

### Theorem 16.1.11: A linear transformation of continuous functions is continuous

If each  $a_{ij}$  is a continuous function on  $S$  and for each  $p \in S$ , then  $A_p \in L(\mathbb{R}^n, \mathbb{R}^m)$  with entries  $a_{ij}(p)$ , then the mapping  $T: S \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$  is continuous.

#### Proof

Since each  $a_{i,j}$  is continuous, then for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for  $t, p \in S$  where  $|t - p| < \delta$ , then  $|a_{i,j}(t) - a_{i,j}(p)| < \frac{\epsilon}{\sqrt{mn}}$ . Thus, for  $|t - p| < \delta$ :

$$\|A_p - A_t\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n (a_{ij}(p) - a_{ij}(t))^2} < \sqrt{\sum_{i=1}^m \sum_{j=1}^n (\frac{\epsilon}{\sqrt{mn}})^2} = \epsilon$$

## 16.2 Differentiation

### Definition 16.2.1: Derivative Extended to Higher Dimensions

First, let's redefine the derivative such that it can be extended to higher dimensions. For  $f: (a,b) \subset \mathbb{R} \rightarrow \mathbb{R}^m$ , let  $f'(x) = y \in \mathbb{R}^m$  such that:

$$f(x+h) - f(x) = yh + r(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{r(h)}{h} = 0$$

Since  $y: h \rightarrow yh$  is a linear transformation from  $\mathbb{R}$  to  $\mathbb{R}^m$ , then  $f'(x) \in L(\mathbb{R}, \mathbb{R}^m)$ .

Now for derivatives in higher dimensions.

Let  $f: x \in \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

If there is an  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$  such that for any  $h \in E$ :

$$f(x+h) - f(x) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

then  $f$  is differentiable at  $x$ . Then differential of  $f$  at  $x$ ,  $f'(x) = A$ .

If  $f$  is differentiable at every  $x \in E$ , then  $f$  is differentiable on  $E$ .

### Theorem 16.2.2: The derivative of a function is unique

Let  $f: x \in \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Let  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$  such that for any  $h \in E$ :

$$f(x+h) - f(x) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

Suppose  $A = A_1$  and  $A = A_2$  satisfies such conditions. Then  $A_1 = A_2$ .

#### Proof

For any  $h \in \mathbb{R}^n$ :

$$\begin{aligned} |(A_2 - A_1)h| &= |[f(x+h) - f(x) - r_{A_1}(h)] - [f(x+h) - f(x) - r_{A_2}(h)]| \\ &= |r_{A_2}(h) - r_{A_1}(h)| \\ &\leq |r_{A_2}(h)| + |r_{A_1}(h)| \end{aligned}$$

Since  $A_1, A_2 \in L(\mathbb{R}^n, \mathbb{R}^m)$ , for any  $t$  where  $h$  is fixed, then:

$$\begin{aligned} |(A_2 - A_1)(th)| &\leq |r_{A_2}(th)| + |r_{A_1}(th)| \\ |t| |(A_2 - A_1)h| &\leq |r_{A_2}(th)| + |r_{A_1}(th)| \\ |(A_2 - A_1)h| &\leq \frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|} \end{aligned}$$

So as  $t \rightarrow 0$ , then  $\frac{|r_{A_2}(th)|}{|t|} + \frac{|r_{A_1}(th)|}{|t|} \rightarrow 0 + 0 = 0$ . Thus,  $A_1 = A_2$ .

**Theorem 16.2.3: Derivative of a Linear Transformation**

If  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$  and  $x \in \mathbb{R}^n$ , then:

$$A'(x) = A$$

**Proof**

Since  $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ , then let  $f(x) = Ax$ .

$$f(x+h) - f(x) = A(x+h) - Ax = Ax + Ah - Ax = Ah$$

Thus,  $r_A(h) = 0$  so  $\lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = \lim_{h \rightarrow 0} 0 = 0$ . Thus,  $A'(x) = f'(x) = A$ .

**Theorem 16.2.4: Chain Rule in Higher Dimensions**

Let  $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $x_0 \in E$  and  $g: f(E) \subset \text{open } H \subset \mathbb{R}^m \rightarrow \mathbb{R}^k$  be differentiable at  $f(x_0)$ .

Then  $F: E \rightarrow \mathbb{R}^k$  where  $F(x) = g(f(x))$  is differentiable at  $x_0$  such that:

$$F'(x_0) = g'(f(x_0)) f'(x_0)$$

**Proof**

Since  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ , then there is a  $A = f'(x_0)$  and  $B = g'(f(x_0))$  such that:

$$f(x_0+h) - f(x_0) = Ah + r_A(h) \quad \text{where } \lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$$

$$g(f(x_0)+k) - g(f(x_0)) = Bk + r_B(k) \quad \text{where } \lim_{k \rightarrow 0} \frac{|r_B(k)|}{|k|} = 0$$

Let  $k = f(x_0+h) - f(x_0)$ . Thus:

$$\begin{aligned} F(x_0+h) - F(x_0) - BAh &= g(f(x_0+h)) - g(f(x_0)) - BAh \\ &= g(f(x_0)+k) - g(f(x_0)) - BAh = Bk + r_B(k) - BAh \\ &= B(k - Ah) + r_B(k) = B(f(x_0+h) - f(x_0) - Ah) + r_B(k) \\ &= Br_A(h) + r_B(k) \end{aligned}$$

$$\frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} = \frac{|Br_A(h) + r_B(k)|}{|h|} \leq \frac{|Br_A(h)| + |r_B(k)|}{|h|} \leq \frac{\|B\| |r_A(h)| + |r_B(k)|}{|h|}$$

Since  $f$  is differentiable at  $x_0$ , then  $f$  is continuous at  $x_0$  and thus,  $\lim_{h \rightarrow 0} k = 0$ .

Since  $\lim_{h \rightarrow 0} \frac{|r_A(h)|}{|h|} = 0$  and  $\lim_{k \rightarrow 0} \frac{|r_B(k)|}{|k|} = 0$ , then:

$$\lim_{h \rightarrow 0} \frac{|F(x_0+h) - F(x_0) - BAh|}{|h|} \leq \lim_{h \rightarrow 0} \|B\| \frac{|r_A(h)|}{|h|} + \lim_{h \rightarrow 0} \frac{|r_B(k)|}{|h|} = 0 + 0 = 0$$

Thus,  $F'(x_0) = BA = g'(f(x_0)) f'(x_0)$ .

**Definition 16.2.5: Partial Derivatives: Derivatives along the standard basis**

Let  $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . The components of  $f$  are the  $f_1, \dots, f_m \in \mathbb{R}$  such that for  $x \in E$ , then  $f(x) = \sum_{i=1}^m f_i(x) e_i$ .

Since  $e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ , then  $f(x) \cdot e_i = [\sum_{i=1}^m f_i(x) e_i] \cdot e_i = f_i(x)$ .

Then for  $x \in E$  and  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$ , let the **partial derivative**  $\frac{\partial f_i}{\partial x_j} = D_j f_i$  be the derivative of  $f_i$  with respect to  $x_j$ . Then for  $t \in \mathbb{R}$ :

$$f_i(x + te_j) - f_i(x) = D_j f_i(te_j) + r_{D_j f_i}(te_j) \quad \text{where } \lim_{t \rightarrow 0} \frac{|r_{D_j f_i}(te_j)|}{|t|} = 0$$

**Theorem 16.2.6: Derivative of f is the sum of all partial derivatives**

Let  $f: \text{open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $x \in E$ . Then the partial derivatives  $(D_j f_i)(x)$  exists such that for  $j \in \{1, \dots, n\}$ :

$$f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i$$

**Proof**

For a fixed  $j$ , since  $f$  is differentiable at  $x$ , then:

$$f(x+te_j) - f(x) = f'(x)(te_j) + r(te_j) \quad \text{where } \lim_{t \rightarrow 0} \frac{|r(te_j)|}{|t|} = 0$$

Then  $f'(x)$  exist where:

$$\lim_{t \rightarrow 0} \frac{f(x+te_j) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{f'(x)(te_j)}{t} + \frac{r(te_j)}{t} = \lim_{t \rightarrow 0} t \frac{f'(x)e_j}{t} = f'(x)e_j$$

Since  $f(x) = \sum_{i=1}^m f_i(x)e_i$ , then:

$$\lim_{t \rightarrow 0} \frac{f(x+te_j) - f(x)}{t} = \lim_{t \rightarrow 0} \sum_{i=1}^m \frac{f_i(x+te_j) - f_i(x)}{t} e_i = f'(x)e_j$$

Since  $f'(x)$  exist and  $\lim_{t \rightarrow 0} \frac{f_i(x+te_j) - f_i(x)}{t} = D_j f_i(x)$ , then each  $D_j f_i(x)$  exists where:

$$f'(x)e_j = \sum_{i=1}^m \lim_{t \rightarrow 0} \frac{f_i(x+te_j) - f_i(x)}{t} e_i = \sum_{i=1}^m (D_j f_i)(x)e_i$$

**Definition 16.2.7: Matrix of the Differential of f**

By **theorem 16.2.6**,  $f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i$  where  $(D_j f_i)(x)$  is the derivative of the component  $f_i$  in respect to  $x_j$  for  $j = \{1, \dots, n\}$ .

Since  $f'(x)e_j$  is the  $j$ -th column of  $[f'(x)]$ , then:

$$[f'(x)] = \begin{bmatrix} \sum_{i=1}^m (D_1 f_i)(x)e_i & \sum_{i=1}^m (D_2 f_i)(x)e_i & \dots & \sum_{i=1}^m (D_n f_i)(x)e_i \end{bmatrix}$$

where each  $\sum_{i=1}^m (D_j f_i)(x)e_i$  is a column vector at the  $j$ -th column.

Since each  $\sum_{i=1}^m (D_j f_i)(x)e_i$  has a coordinate of  $(D_j f_i)(x)$  for  $e_i$  where each  $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^m$ , then:

$$[f'(x)] = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & \dots & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & \dots & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & \dots & (D_n f_m)(x) \end{bmatrix}$$

Thus, for  $x \in \mathbb{R}^n$  where  $x = x_1 e_1 + \dots + x_n e_n$ , then:

$$\begin{aligned} f'(x)x &= f'(x) \left[ \sum_{j=1}^n x_j e_j \right] \\ &= \sum_{j=1}^n x_j f'(x)e_j \\ &= \sum_{j=1}^n x_j \sum_{i=1}^m (D_j f_i)(x)e_i \\ &= \sum_{i=1}^m \left[ \sum_{j=1}^n x_j (D_j f_i)(x) \right] e_i \end{aligned}$$

**Definition 16.2.8: Gradient and Directional Derivative**

Let  $\gamma: (a,b) \subset \mathbb{R} \rightarrow \text{open } E \subset \mathbb{R}^n$  and  $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}$  both be differentiable. Then by **theorem 16.2.4**,  $g: \mathbb{R} \rightarrow \mathbb{R}$  defined as  $g(t) = f(\gamma(t))$  is differentiable for any  $t \in (a,b)$  such that:

$$g'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

Since  $f(\gamma(t)): E \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , by **theorem 16.2.6**, then:

$$f'(\gamma(t))e_j = (D_j f)(\gamma(t)) \text{ for } j = \{1, \dots, n\}$$

Since  $\gamma: (a,b) \subset \mathbb{R} \rightarrow \text{open } E \subset \mathbb{R}^n$ , then:

$$\gamma'(t) = \sum_{i=1}^n (D_i \gamma_i)(t)e_i = \sum_{i=1}^n \gamma'_i(t)e_i$$

Thus,  $g'(t) = \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t)$ .

For each  $x \in E$ , let the **gradient** of  $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}$  at  $x$ ,  $(\nabla f)(x)$ :

$$(\nabla f)(x) = \sum_{i=1}^n (D_i f)(x)e_i$$

Since  $e_i e_j = 1$  if  $i = j$ , but  $e_i e_j = 0$  if  $i \neq j$ , then:

$$\begin{aligned} [f(\gamma(t))]' &= g'(t) \\ &= \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t) \\ &= \sum_{i=1}^n [(D_i f)(\gamma(t))e_i \cdot \gamma'_i(t)e_i] \\ &= [\sum_{i=1}^n (D_i f)(\gamma(t))e_i] \cdot [\sum_{i=1}^n \gamma'_i(t)e_i] = (\nabla f)(\gamma(t)) \cdot \gamma'(t) \end{aligned}$$

For  $t \in (-\infty, \infty)$ , let  $\gamma(t) = x + tu$  where  $x \in E$  and unit vector  $u \in \mathbb{R}^n$ . Then:

$$\begin{aligned} (D_u f)(x) &= \lim_{t \rightarrow 0} \frac{f(x+tu) - f(x)}{t} = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} = g'(x) \\ &= (\nabla f)(\gamma(x)) \cdot \gamma'(x) = (\nabla f)(x) \cdot u \end{aligned}$$

Let  $(D_u f)(x)$  be the **directional derivative** of  $f$  at  $x$  in direction of  $u$ .

For  $u = u_1 e_1 + \dots + u_n e_n$ :

$$(D_u f)(x) = (\nabla f)(x) \cdot u = \sum_{i=1}^n (D_i f)(x)e_i \cdot \sum_{i=1}^n u_i e_i = \sum_{i=1}^n (D_i f)(x)u_i$$

Also, for a fixed  $f$  and  $x$ ,  $(D_u f)(x)$  is maximized when  $u = \lambda(\nabla f)(x)$  for  $\lambda > 1$  since  $x \cdot y = |x||y| \cos(\theta)$  where  $\theta$  is the angle between  $x$  and  $y$ .

**Theorem 16.2.9: A bounded derivative over a convex space have bounded range**

For differentiable  $f: \text{convex open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , there is a  $M \in \mathbb{R}$  such that  $\|f'(x)\| \leq M$  for every  $x \in E$ . Then for all  $a, b \in E$ :

$$|f(b) - f(a)| \leq M|b - a|$$

**Proof**

For fixed  $a, b \in E$ , let  $\gamma(t) = (1-t)a + tb$ . Since  $E$  is convex, for  $t \in [0,1]$ , then  $\gamma(t) \in E$ . Let  $g(t) = f(\gamma(t))$ . Then  $g'(t) = f'(\gamma(t))\gamma'(t) = f'(\gamma(t))(b-a)$ . Thus, for  $t \in [0,1]$ :

$$|g'(t)| = |f'(\gamma(t))(b-a)| \leq \|f'(\gamma(t))\| |b-a| \leq M|b-a|$$

Since  $g(0) = f(\gamma(0)) = f(a)$  and  $g(1) = f(\gamma(1)) = f(b)$ , then by the Mean Value Theorem, for  $x \in (0,1)$

$$|f(b) - f(a)| = |g(1) - g(0)| \leq (1-0)|g'(x)| \leq M|b-a|$$
**Corollary 16.2.10: If the derivative is 0, the function is constant**

For differentiable  $f: \text{convex open } E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $f'(x) = 0$  for all  $x \in E$ .

Then,  $f$  is constant.

**Proof**

Since  $\|f'(x)\| = 0$  for all  $x \in E$ , then by **theorem 7.2.9**, for all  $a, b \in E$ :

$$0 \leq |f(b) - f(a)| \leq 0(b-a) = 0$$

Thus,  $f(b) = f(a)$  for all  $a, b \in E$  so  $f$  is constant.



**Definition 16.2.11: Continuously Differentiable**

A differentiable  $f$ : open  $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuously differentiable in  $E$  if:

$f'$ :  $E \rightarrow L(\mathbb{R}^n, \mathbb{R}^m)$  is continuous

For  $\epsilon > 0$ , there is a  $\delta > 0$  such that for every  $x, y \in E$  where  $|x - y| < \delta$ , then:

$$\|f'(y) - f'(x)\| < \epsilon$$

If  $f$  is continuously differentiable, then  $f \in \mathcal{C}'(E)$ .

**Theorem 16.2.12: Continuously differentiable imply continuous partial derivatives**

Let  $f$ : open  $E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Then  $f \in \mathcal{C}'(E)$  if and only if each partial derivative

$D_j f_i$  exist and are continuous on  $E$ .

**Proof**

If  $f \in \mathcal{C}'(E)$ , then  $f$  is differentiable. Thus, by **theorem 16.2.6**, partial derivative  $D_j f_i$  where  $j = \{1, \dots, n\}$  exists for any  $x \in E$  such that:

$$f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)e_i \quad \Rightarrow \quad (D_j f_i)(x) = f'(x)e_j \cdot e_i$$

Thus, since  $f \in \mathcal{C}'(E)$ , then for  $|x - y| < \delta$ :

$$\begin{aligned} |(D_j f_i)(y) - (D_j f_i)(x)| &= |f'(y)e_j \cdot e_i - f'(x)e_j \cdot e_i| = |[f'(y) - f'(x)]e_j \cdot e_i| \\ &\leq |[f'(y) - f'(x)]e_j| |e_i| \leq \|f'(y) - f'(x)\| |e_j| |e_i| \\ &= \|f'(y) - f'(x)\| < \epsilon \end{aligned}$$

Thus, each  $D_j f_i$  is continuous.

Since each  $D_j f_i$  is continuous, then for  $\epsilon > 0$ , there is a  $\delta > 0$  such that for  $|y - x| < \delta$ , then for all  $j \in \{1, \dots, n\}$  and  $i \in \{1, \dots, m\}$ , then  $|D_j f_i(y) - D_j f_i(x)| < \epsilon$ .

Then for  $h = h_1 e_1 + \dots + h_n e_n$  where  $|x - h| < \delta$ :

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x) h_j] e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + \dots + h_n e_n) - f_i(x)] e_i - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x) h_j] e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [f_i(x+h_1 e_1 + \dots + h_n e_n) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x) h_j] e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{\left| \begin{array}{c} f_i(x + \sum_{k=1}^n h_k e_k) - f_i(x + \sum_{k=1}^{n-1} h_k e_k) \\ + f_i(x + \sum_{k=1}^{n-1} h_k e_k) - f_i(x + \sum_{k=1}^{n-2} h_k e_k) \\ + \dots + f_i(x + h_1) - f_i(x) - \sum_{j=1}^n (D_j f_i)(x) h_j \end{array} \right|}{|h|} e_i| \end{aligned}$$

Since each  $D_j f_i$  exist, then by the Mean Value Theorem, for each  $j = \{1, \dots, n\}$ , there is a  $t_j \in (0, h_j)$  such that:

$$f_i(x + \sum_{k=1}^j h_k e_k) - f_i(x + \sum_{k=1}^{j-1} h_k e_k) = D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j) h_j$$

Thus:

$$\begin{aligned} &\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \sum_{i=1}^m [\sum_{j=1}^n (D_j f_i)(x) h_j] e_i|}{|h|} \\ &= \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n D_n f_i(x + \sum_{k=1}^{j-1} h_k e_k + t_j e_j) h_j - \sum_{j=1}^n (D_j f_i)(x) h_j] e_i|}{|h|} \\ &< \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [\sum_{j=1}^n \epsilon h_j] e_i|}{|h|} \leq \lim_{h \rightarrow 0} \frac{|\sum_{i=1}^m [n \epsilon |h|] e_i|}{|h|} = \lim_{h \rightarrow 0} \frac{\sqrt{mn} \epsilon |h|}{|h|} = \sqrt{mn} \epsilon \end{aligned}$$

Thus,  $f(x)$  is differentiable where:

$$f'(x) = \begin{bmatrix} (D_1 f_1)(x) & (D_2 f_1)(x) & \dots & (D_n f_1)(x) \\ (D_1 f_2)(x) & (D_2 f_2)(x) & \dots & (D_n f_2)(x) \\ \vdots & \vdots & \ddots & \vdots \\ (D_1 f_m)(x) & (D_2 f_m)(x) & \dots & (D_n f_m)(x) \end{bmatrix}$$

Thus, for  $|y - x| < \delta$ :

$$\|f'(y) - f'(x)\| \leq \sqrt{\sum_{i=1}^m \sum_{j=1}^n [(D_j f_i)(y) - (D_j f_i)(x)]^2} < \sqrt{\sum_{i=1}^m \sum_{j=1}^n \epsilon^2} = \sqrt{mn} \epsilon$$

Thus,  $f \in \mathcal{C}'(E)$ .

## 16.3 The Contraction Principle

### Definition 16.3.1: Contraction

For metric space  $X$  with metric  $d$ , then  $\phi: X \rightarrow X$  is a **contraction** if there is  $c \in (0,1)$  such that for all  $x,y \in X$ :

$$d(\phi(x), \phi(y)) \leq c d(x,y)$$

### Theorem 16.3.2: Banach's Fixed Point Theorem

If  $X$  is a complete metric space and  $\phi$  is a contraction of  $X$  into  $X$ , then there is a unique  $x \in X$  such that  $\phi(x) = x$

#### Proof

Let  $\phi(x) = x$  and  $\phi(y) = y$ . Since  $\phi$  is a contraction, then  $d(x,y) = d(\phi(x), \phi(y)) \leq c d(x,y)$  would hold true only if  $d(x,y) = 0$  so  $x = y$ . Thus, such a  $\phi(x) = x$  is unique. For a fixed  $x_0 \in X$ , let  $\{x_n\}$  have  $x_{n+1} = \phi(x_n)$ . Thus, for some  $c \in (0,1)$ :

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\phi(x_n), \phi(x_{n-1})) \leq c d(x_n, x_{n-1}) \\ &= c d(\phi(x_{n-1}), \phi(x_{n-2})) = \dots = c^n d(x_1, x_0) \end{aligned}$$

Thus, for  $\epsilon > 0$ , choose  $N$  such that  $d(x_1, x_0) \frac{c^N}{(1-c)} < \epsilon$ . Then for  $m > n \geq N$ :

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} c^i d(x_1, x_0) \\ &\leq d(x_1, x_0) \frac{c^n}{1-c} \leq d(x_1, x_0) \frac{c^N}{1-c} < \epsilon \end{aligned}$$

Thus,  $\{x_n\}$  is a Cauchy Sequence and since  $X$  is complete, then  $\{x_n\}$  converges to a  $x \in X$ . Note a contraction is uniformly continuous so:

$$\phi(x) = \lim_{n \rightarrow \infty} \phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

#### Example

For  $y' = y$  where  $y(0) = 1$ , show  $y(x) = e^x$  for  $x$  near 0.

Take the metric space of continuous functions,  $C[a,b]$ , with the sup metric as defined in [definition 14.3.4](#) where  $0 \in [a,b]$ . By [theorem 14.3.5](#),  $C[a,b]$  is complete.

Then for each  $f \in C[a,b]$ , let  $Tf(x) = 1 + \int_0^x f(t) dt$  for  $x \in [a,b]$ .

$$\begin{aligned} |Tf(x) - Tg(x)| &= \left| \int_0^x f(t) - g(t) dt \right| \leq \int_{\min(0,x)}^{\max(0,x)} |f(t) - g(t)| dt \\ &\leq |x - 0| d(f,g) \leq (b-a) d(f,g) \end{aligned}$$

Thus,  $d(Tf(x), Tg(x)) \leq (b-a) d(f,g)$  so for  $b-a < 1$ , then  $T$  is a contraction. By [theorem 16.3.2](#), there is a unique  $y$  where  $y(x) = 1 + \int_0^x y(t) dt$ . To determine  $y$ , use the process defined in [theorem 16.3.2](#)'s proof referred as the Picard iteration. Using any continuous  $f(x)$ , let's take  $f(x) = 1$ . Then:

$$T(1) = 1 + \int_0^x 1 dt = 1 + x$$

$$T(T(1)) = 1 + \int_0^x 1+t dt = 1 + x + \frac{1}{2}x^2$$

$$T(T(T(1))) = 1 + \int_0^x 1+t+\frac{1}{2}t^2 dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Thus, by [definition 15.2.1](#),  $y(x) = \lim_{n \rightarrow \infty} T^n(1) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ .

## 16.4 Inverse Function Theorem

### Theorem 16.4.1: Inverse Function Theorem

Suppose  $f: E \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\mathcal{C}'(E)$  where  $f(a) = b$  and  $Df(a)$  is invertible for some  $a \in E$ .

Then there exists open  $U, V \subset \mathbb{R}^n$  such that  $f: a \in U \rightarrow b \in V$  is invertible.

#### Proof

Since  $Df(a)$  is invertible, then choose  $\lambda$  such that  $2\lambda\| [Df(a)]^{-1} \| = 1$ . Since  $f \in \mathcal{C}'(E)$ , there is a  $N_r(a) \subset E$  such that for all  $x \in N_r(a)$ , then  $\|Df(x) - Df(a)\| < \lambda$ .

Let  $\phi(x) = x + [Df(a)]^{-1}(y - f(x))$  for  $x \in E$  and  $y \in \mathbb{R}^n$ . Thus,  $\phi(x) = x$  if and only if  $y = f(x)$ . Since  $f$  is differentiable, then  $\phi$  is differentiable where:

$$\phi'(x) = I_{n \times n} - [Df(a)]^{-1}Df(x) = [Df(a)]^{-1}(Df(a) - Df(x))$$

$$\|\phi'(x)\| = \|[Df(a)]^{-1}(Df(a) - Df(x))\| \leq \|[Df(a)]^{-1}\| \|Df(a) - Df(x)\| < \frac{1}{2\lambda}\lambda = \frac{1}{2}$$

Thus, by [theorem 16.2.9](#), then for all  $x_1, x_2 \in N_r(a)$ :

$$|\phi(x_2) - \phi(x_1)| \leq \frac{1}{2}|x_2 - x_1|$$

so  $\phi$  is a contraction and since  $N_r(a) \subset \mathbb{R}^n$  is complete, then by [theorem 16.3.2](#), there is a unique  $x \in N_r(a)$  where  $\phi(x) = x$  so  $y = f(x)$  for some unique  $x$ .

Let  $U = N_r(a)$  and  $V = f(N_r(a))$  so  $f$  is 1-1 in  $U$  and onto  $V$  so  $f$  is invertible on  $U$ .

To show  $V$  is open, let  $y_0 \in V$  where  $y_0 = f(x_0)$  for some  $x_0 \in U$ .

Since  $f$  is invertible on  $U$ , then for  $y$  where  $|y - y_0| < \lambda r$ :

$$|\phi(x_0) - x_0| = |[Df(x_0)]^{-1}(y - f(x_0))| \leq \|[Df(x_0)]^{-1}\| |y - y_0| < \frac{1}{2\lambda}\lambda r = \frac{r}{2}$$

Thus, for  $x \in N_r(x_0)$ :

$$|\phi(x) - x_0| \leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| < \frac{1}{2}|x - x_0| + \frac{r}{2} < \frac{1}{2}r + \frac{r}{2} = r$$

so  $\phi(x) \in N_r(x_0)$ . Thus, as shown above, there is a unique  $x \in N_r(x_0)$  where  $\phi(x) = x$  so  $y = f(x) \in V$  where also  $y \in N_{\lambda r}(y_0)$ .

## References

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