

Fall Real Analysis

Azure

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1 The Real Number System

1.1 Number Systems

Natural : $\mathbb{N} = \{1, 2, 3, \dots\}$

Integer : $\mathbb{Z} = \{-2, -1, 0, 1, 2, \dots\}$

Rational : $\mathbb{Q} = \frac{p}{q}$ where $p, q \in \mathbb{N}$

*** \mathbb{Q} is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.

Let set $A = \{x \in \mathbb{Q} \text{ for } x^2 < 2\}$ where $A \neq \emptyset$ and 2 is an upper bound for A .

But, A has no least upper bound in \mathbb{Q} , but A has a least upper bound in \mathbb{R} .

1.2 Real Number System

\mathbb{R} is the unique ordered field with the least upper bound property.

Also, \mathbb{R} exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- **Trichotomy**: For all $x, y \in S$, only one holds true:
 - $x < y$
 - $x = y$
 - $x > y$
- **Transitivity**: If $x < y$ and $y < z$, then $x < z$.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and $E \subset S$.

An upper bound of E is a $\beta \in S$ if $x \leq \beta$ for all $x \in E$.

If such a β exists, then E is bounded from above.

A lower bound of E is a $\alpha \in S$ if $x \geq \alpha$ for all $x \in E$.

If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.

Let $E \subset S$ be bounded from above. Least upper bound $\beta \in S$ exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E .
Then $\beta = \sup(E)$.

Let $E \subset S$ be bounded from below. Greatest lower bound $\alpha \in S$ exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E .
Then $\alpha = \inf(E)$.

Example 1.2.5

Let $S = (1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from \mathbb{R} . For subsets E of S :

- $E = (1, 2)$ is bounded above and $\sup(E) = 2$
- $E = (5, 6)$ is not bounded above so $\sup(E) = \text{DNE}$
- $E = [3, 4)$ is bounded below $\inf(E) = 3$ and $\sup(E) = \text{DNE}$

Observations on the Least Upper Bound

If $\sup(E)$ exists, it may or may not exist at S .

If $\sup(E)$ exists, then $\sup(E)$ is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property**Theorem 1.3.1: Least Upper Bound Property**

An ordered set S has a least upper bound property if:

For every nonempty subset $E \subset S$ that is bounded from above:
 $\sup(E)$ exists in S .

Example 1.3.2

\mathbb{Q} doesn't have a least upper bound property. For example, $z = \sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If $y^2 - 2 < 0$, then $z > y$ where $z \in A$. So, y is not an upper bound.
Since for any y , there is $z > y$ where $z \in A$, then $\sup(A)$ doesn't exist in \mathbb{Q} .
- If $y^2 - 2 > 0$, then $z < y$ where $z \in B$. So, y is an upper bound, but not $\sup(E)$.
Since for any y , there is $z < y$ where $z \in B$, then $\inf(B)$ doesn't exist in \mathbb{Q} .

Thus, \mathbb{Q} doesn't have the least upper bound or greatest lower bound property.

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be an ordered set with the least upper bound property.

Let non-empty $B \subset S$ be bounded below.

Let L be the set of all lower bounds of B .

Then $\alpha = \sup(L)$ exists in S .

Proof

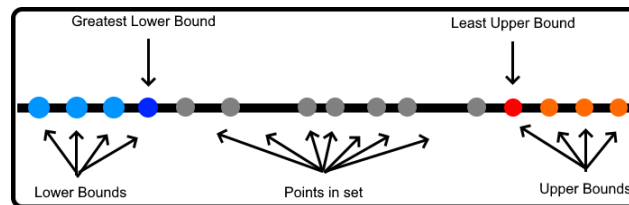
L is non-empty since B is bounded from below.

Thus, by the least upper bound property of S , $\alpha = \sup(L)$ exists in S .

We claim that $\alpha = \inf(B)$.

If $\gamma < \alpha$, then γ is not an upper bound for L so $\gamma \notin B$ since all upper bounds for L are in B . Thus, for every $x \in B$, $\alpha \leq x$.

If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in B$. Thus, $\inf(B) = \alpha$.



2.2 Fields

Addition Axioms

- If $x, y \in F$, then $x+y \in F$
- $x+y = y+x$ for all $x, y \in F$
- $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$
- There exists $0 \in F$ such that $0+x = x$ for all $x \in F$
- For every $x \in F$, there is $-x \in F$ where $x+(-x) = 0$

Multiplicative Axioms

- If $x, y \in F$, then $xy \in F$
- $yx = xy$ for all $x, y \in F$
- $(xy)z = x(yz)$ for all $x, y, z \in F$
- There exists $1 \neq 0 \in F$ such that $1x = x$ for all $x \in F$
- If $x \neq 0 \in F$, there is $\frac{1}{x} \in F$ where $x(\frac{1}{x}) = 1$

Distributive Law

$$x(y+z) = xy + xz \text{ hold for all } x, y, z \in F.$$

Propositions 2.2.1

- (a) If $x+y = x+z$, then $y = z$

Proof

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If
- $x+y = x$
- , then
- $y = 0$

ProofFrom (a), let $z = 0$.

- (c) If
- $x+y = 0$
- , then
- $y = -x$

ProofFrom (a), let $z = -x$.

- (d)
- $-(-x) = x$

ProofFrom (c), let $x = -x$ and $y = x$.

- (e) If
- $x \neq 0$
- and
- $xy = xz$
- , then
- $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If
- $x \neq 0$
- and
- $xy = x$
- , then
- $y = 1$

ProofFrom (e), let $z = 1$.

- (g) If
- $x \neq 0$
- and
- $xy = 1$
- , then
- $y = \frac{1}{x}$

ProofFrom (e), let $z = \frac{1}{x}$.

- (h) If
- $x \neq 0$
- , then
- $\frac{1}{1/x} = x$

ProofFrom (g), let $x = \frac{1}{x}$ and $y = x$.

- (i)
- $0x = 0$

ProofSince $0x + 0x = (0+0)x = 0x = 0x + 0$, then $0x = 0$.

- (j) If
- $x, y \neq 0$
- , then
- $xy \neq 0$

Proof

Suppose $xy = 0$, then $1 = \frac{1}{y} \frac{1}{x} xy = \frac{1}{y} \frac{1}{x} 0 = 0$.
 $0 = 1$ is a contradiction.

- (k)
- $(-x)y = -(xy) = x(-y)$

Proof

$xy + (-x)y = (x+(-x))y = 0y = 0$. Then by part (c), $(-x)y = -(xy)$.
 $xy + x(-y) = x(y+(-y)) = x0 = 0$. Then by part (c), $x(-y) = -(xy)$.

- (l)
- $(-x)(-y) = xy$

Proof

By part (k), then $(-x)(-y) = -[x(-y)] = -[-(xy)]$.
 By part (d), $-[-(xy)] = xy$.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all $x, y, z \in F$.

- If $y < z$, then $y+x < z+x$
- If $x, y > 0$, then $xy > 0$

Definition 2.3.1: \mathbb{Q} and \mathbb{R} are ordered fields

\mathbb{Q} , \mathbb{R} are ordered fields, but \mathbb{C} is not an ordered field since $i^2 = -1 \not> 1$.

Propositions 2.3.2

Let F be an ordered field. For all $x, y, z \in F$.

- (a) If $x > 0$, then $-x < 0$ and vice versa

Proof

$$-x = -x + 0 < -x + x = 0$$

- (b) If $x > 0$ and $y < z$, then $xy < xz$

Proof

$$\text{Since } z - y > 0, \text{ then } 0 < x(z - y) = xz - xy$$

- (c) If $x < 0$ and $y < z$, then $xy > xz$

Proof

$$\text{Since } -x > 0 \text{ and } z - y > 0, \text{ then } 0 < -x(z - y) = xy - xz$$

- (d) If $x \neq 0$, $x^2 > 0$

Proof

$$\begin{aligned} \text{If } x > 0, \text{ then } x^2 &= x \cdot x > 0 \\ \text{If } x < 0, \text{ then } (-x)^2 &= (-x) \cdot (-x) = x \cdot x = x^2 > 0 \end{aligned}$$

- (e) If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

$$\begin{aligned} \text{Since } (\frac{1}{y})y &= 1 > 0, \text{ then } (\frac{1}{y}) > 0 \\ \text{Since } x < y, \text{ then } \frac{1}{y} &= (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x} \end{aligned}$$

Theorem 2.3.3: \mathbb{R} is an ordered field with $<$

There exists a unique ordered field \mathbb{R} with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$ so \mathbb{Q} is also an ordered field.

Theorem 2.3.4

For all $x, y \in \mathbb{R}$:

- **Archimedean Property:** If $x > 0$, there is $n \in \mathbb{Z}$ such that $nx > y$.

Proof

Fix $x > 0$. Suppose there is a y such that the property fails.
 Let $A = \{ nx : n = 1, 2, 3, \dots \}$.
 Then, A is nonempty and bounded from above by y .
 Then by the least upper bound property of \mathbb{R} , $\alpha = \sup(A)$ exists in \mathbb{R} .
 Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$.
 So $\alpha - x$ is not an upper bound of A .
 So there is a $mx \in A$ such that $mx > \alpha - x$.
 Then $\alpha < (m+1)x$, but $(m+1)x \in A$ contradicting α is an upper bound for A .

- **\mathbb{Q} is dense in \mathbb{R} :** If $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof

Since $x < y$, then $y - x > 0$. Then by the Archimedean Property, there exists a $n \in \mathbb{Z}$ such that $n(y - x) > 1$. Thus, $ny > nx + 1 > nx$.
 By the well-ordering principle, there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$.
 Then, $m > nx \geq m - 1$ so $nx + 1 \geq m > nx$.
 Since $ny > nx + 1 \geq m > nx$, then $y > m/n > x$.

3 Roots & Complex Field

3.1 nth Root

- (a) If $0 < t \leq 1$, then $t^n \leq t$.

Proof

Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
 Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
 Applying the process n times, then $t^n \leq t$.

- (b) If $t \geq 1$, $t^n \geq t$.

Proof

Since $0 < 1 \leq t$, then $t \leq t^2$.
 Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
 Applying the process n times, $t \leq t^n$.

- (c) If $0 < s < t$, then $s^n < t^n$.

Proof

$$\underbrace{s \cdot s \cdot \dots \cdot s}_n < t \cdot s \cdot \dots \cdot s < t \cdot t \cdot \dots \cdot s < \dots < \underbrace{t \cdot \dots \cdot t}_n$$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = y_1^n < y_2^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$.

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < t_1^n < t_1 < x$ so $t_1 \in A$.

A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

So $x+1$ is an upper bound of A .

Thus by the least upper bound property, $y = \sup(A)$ exists.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

*** (Not an upper bound of A if $<$ and not a least upper bound of A if $>$) ***

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\beta^{n-1} + \beta^{n-2}\alpha^1 + \dots + \alpha^{n-1})}_{\substack{\beta^{n-1} < \beta^{n-1} < \beta^{n-1}}} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$.

From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.

Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus, $t^n > x$ so $t \notin A$.

Thus, $y-k$ is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 3.1.2: n-th root of product = product of n-th root

If $a, b > 0$ and $n \in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Proof

Let $A = a^{\frac{1}{n}}$ and $B = b^{\frac{1}{n}}$.

Then by **theorem 3.1.1**, since A is a solution to $y_1^n = a$, then $A^n = a$.

Similarly, B is a solution of $y_2^n = b$ so $B^n = b$. Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 \dots A_n B_1 B_2 \dots B_n \\ &= A_1 A_2 \dots B_1 A_n B_2 \dots B_n = \dots = A_1 B_1 A_2 \dots A_{n-1} A_n B_3 \dots B_n \\ &= \dots = A_1 B_1 A_2 B_2 \dots A_n B_n = (AB)^n \end{aligned}$$

Then again by **theorem 3.1.1**, there is a unique $(ab)^{\frac{1}{n}} = AB = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.

Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x$

Let E be the set of d_k for $k = 0, 1, \dots, \infty$. Then, $x = \sup(E)$.

3.3 Extended Reals

The extended real number system consist of \mathbb{R} and $\pm\infty$ such that:

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}$$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If $x > 0$, then $x(\pm\infty) = \pm\infty$
- If $x < 0$, then $x(\pm\infty) = \mp\infty$

3.4 Complex Numbers**Definition 3.4.1: Complex**

A complex number is an ordered pair (a, b) where $a, b \in \mathbb{R}$. For $x, y \in \mathbb{C}$

- $x + y = (a, b) + (c, d) = (a + c, b + d)$
- $xy = (a, b)(c, d) = (ac - bd, ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)^{-1}(a, -b)$

Thus, the axioms form a field where $(0, 0) = 0$ and $(1, 0) = 1$ and $(0, 1) = i$.

Definition 3.4.2: Imaginary i

Let $i = (0, 1)$. Then, $i^2 = -1$.

Proof

$$i^2 = (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1$$

Definition 3.4.3: Form $a + bi$

$$(a, b) = a + bi$$

Proof

$$(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi$$

Definition 3.4.4: Conjugate

Let conjugate: $\bar{z} = a - bi$ where $\text{Re}(z) = a$, $\text{Im}(z) = b$.

Let $z = (a,b)$ and $w = (c,d)$:

(a) $\overline{z+w} = \bar{z} + \bar{w}$

Proof

$$\overline{z+w} = \overline{(a+c, b+d)} = (a+c, -b-d) = (a, -b) + (c, -d) = \bar{z} + \bar{w}$$

(b) $\overline{zw} = \bar{z} \bar{w}$

Proof

$$\overline{zw} = \overline{(ac-bd, ad+bc)} = (ac-bd, -ad-bc) = (a, -b)(c, -d) = \bar{z} \bar{w}$$

(c) $z + \bar{z} = 2 \text{Re}(z)$ $z - \bar{z} = 2i \text{Im}(z)$

Proof

$$\begin{aligned} z + \bar{z} &= (a, b) + (a, -b) = (2a, 0) = 2 \text{Re}(z) \\ z - \bar{z} &= (a, b) - (a, -b) = (0, 2b) = (0, 2) b = 2i \text{Im}(z) \end{aligned}$$

(d) $z\bar{z} \geq 0$

Proof

$$z\bar{z} = (a, b)(a, -b) = (a^2 + b^2, -ab+ab) = a^2 + b^2 \geq 0$$

Definition 3.4.5: Absolute Value

Let absolute value: $|z| = \sqrt{z\bar{z}}$

Let $z = (a,b)$ and $w = (c,d)$:

(a) If $z \neq 0$, then $|z| > 0$.

Proof

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2} \geq 0 \text{ where } |z| = 0 \text{ only if } a, b = 0 \text{ so only if } z = (0, 0).$$

(b) $|\bar{z}| = |z|$

Proof

$$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

(c) $|zw| = |z| |w|$

Proof

$$\begin{aligned} |zw| &= |(ac-bd, ad+bc)| = \sqrt{(ac-bd)^2 + (ad+bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z| |w| \end{aligned}$$

(d) $|\text{Re}(z)| \leq |z|$

Proof

$$|\text{Re}(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

(e) $|z+w| \leq |z| + |w|$

Proof

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + |w|^2 + 2 \text{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2 \end{aligned}$$

4 Euclidean Spaces & Cauchy-Schwarz

4.1 Euclidean Spaces

For each positive integer k , let \mathbb{R}^k be the set of all ordered k -tuples:

$$x = (x_1, \dots, x_k) \quad \text{for each } x_i \in \mathbb{R}$$

with the properties:

- $x+y = (x_1 + y_1, \dots, x_k + y_k) \in \mathbb{R}^k$
- $cx = (cx_1, \dots, cx_k) \in \mathbb{R}^k$

So, \mathbb{R}^n has a vector space structure. Similarly, for \mathbb{C}^n .

Definition 4.1.1: Inner Product for \mathbb{R}^k

$$x \cdot y = x_1 y_1 + \dots + x_k y_k \in \mathbb{R}$$

Definition 4.1.2: Norm

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$$

Definition 4.1.3: Extension to \mathbb{C}^k

For $z, w \in \mathbb{C}^n$

- $z \cdot w = z_1 \overline{w_1} + \dots + z_k \overline{w_k}$
- $z \cdot z = z_1 \overline{z_1} + \dots + z_k \overline{z_k} = |z_1|^2 + \dots + |z_n|^2 = |z|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $b_1, \dots, b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^n \alpha_j (b_j)|^2 \leq \sum_{j=1}^n |\alpha_j|^2 \sum_{j=1}^n |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j \overline{b_j}$.

If $B = 0$, then $b_1 = \dots = b_n = 0$. Thus, $0 \leq A(0)$ holds true.

Suppose $B > 0$. Then:

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) \\ &= \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B \overline{C} a_j \overline{b_j} - B C \overline{a_j} b_j + C \overline{C} b_j \overline{b_j} \\ &= B^2 \sum |a_j|^2 - B \overline{C} \sum a_j \overline{b_j} - B C \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B \overline{C} C - B C \overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.

Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i \overline{z_i}$, then $\sum z_i \overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$$|z \cdot w|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2 |w|^2$$

Thus, $|z \cdot w| \leq |z| |w|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

- (a) $|x| \geq 0$ where $|x| = 0$ only if $x = 0$

Proof

$$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = \dots = x_k = 0$$

- (b) $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2} \sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$$

- (c) $|x + y| \leq |x| + |y|$

Proof

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

- (d) $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

5 Construction of \mathbb{R} : **Theorem 2.3.3**

There exists an ordered field \mathbb{R} which has the least upper bound property.
Also, \mathbb{R} contains \mathbb{Q} as a subfield.

Definition 5.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$
- If $p \in \alpha$ and $q \in \mathbb{Q} < p$, then $q \in \alpha$
- If $p \in \alpha$, then $p < r \in \mathbb{Q}$ for some $r \in \alpha$

Proposition 5.2: Order of $\mathbb{R} \rightarrow$ ordered set \mathbb{R}

Define $\alpha < \beta$ if α is a proper subset of β .

- If $\alpha \not\subseteq \beta$, then β is not a subset of α .
Then there is a $p \in \beta$ such that $p \notin \alpha$.
Then for any $q \in \alpha$, $q < p$ and thus, $q \in \beta$.
Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.
- If $\alpha \not\subseteq \beta$ and $\alpha \not\supset \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
If $\alpha \neq \beta$, there are p, q such that $p \in \alpha$, but $p \notin \beta$ and $q \in \beta$, but $q \notin \alpha$.
But if $p \notin \beta$, then for any $b \in \beta$, $b < p$. Thus, $q < p$.
Similarly, if $q \notin \alpha$, then for any $a \in \alpha$, $a < q$. Thus, $p < q$.
Thus, there is a contradiction since $p > q$ and $p < q$ so $\alpha = \beta$.
- If $\alpha \not\subseteq \beta$, then α is not a subset of β .
Then there is a $p \in \alpha$ such that $p \notin \beta$.
Then for any $q \in \beta$, $q < p$ and thus, $q \in \alpha$.
Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.
- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ , then α is a proper subset of γ . Thus, $\alpha < \gamma$.

Thus, \mathbb{R} is an ordered set with such an order $<$.

Proposition 5.3: Least Upper Bound of $\mathbb{R} \rightarrow$ Least Upper Bound Property

Let $A \subset \mathbb{R}$ and β be an upper bound for A . Let γ be the union of all $\alpha \in A$.
Thus, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$.

γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in A$ where α_0 is nonempty.
Since α_0 is nonempty, then γ is nonempty.
Since every $\alpha \in A$ is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.
- If $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$ so $q \in A$.
- If $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$. Thus, there is a $r \in \alpha_1$ such that $r > p$ so $r \in \gamma$. Thus, there is a $r \in \gamma$ where $r > p$.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in A \subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A .

Suppose $\delta < \gamma$. Then there is a $s \in \gamma$ such that $s \notin \delta$. Since $s \in \gamma$, then there is a $\alpha \in A$ such that $s \in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A .

Thus, $\gamma = \sup(A)$.

Proposition 5.4: \mathbb{R} is a field

If $\alpha, \beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Also, let 0^* be the set of all negative rational numbers which is a cut since:

- 0^* is nonempty and $0^* \neq \mathbb{Q}$
- If $p \in 0^*$, then any $q \in \mathbb{Q} < p$ is a negative rational and thus, $q \in 0^*$.
- Since \mathbb{Q} is dense in \mathbb{R} , then for any $p \in 0^*$, there is a $r \in \mathbb{Q}$ where $p < r < 0$ so r is a negative rational so $r \in 0^*$.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take $r' \notin \alpha, s' \notin \beta$, then $r' + s' > r + s$ for $r \in \alpha, s \in \beta$. Thus, $r' + s' \notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If $p \in \alpha + \beta$, then $p = r + s$ where $r \in \alpha$ and $s \in \beta$.
If $q < p$, then $q - s < p - s = (r + s) - s = r$ so $q - s \in \alpha$.
Since $q - s \in \alpha$ and $s \in \beta$, then $(q - s) + s = q \in \alpha + \beta$.
- If $r \in \alpha$, then there is a $t \in \alpha$ such that $t > r$. Let $s \in \beta$.
Thus, for any $p = r + s \in \alpha + \beta$, there is a $q = t + s \in \alpha + \beta$ such that $p = r + s < t + s = q$.

$\alpha + \beta = \beta + \alpha$

If $p = r + s \in \alpha + \beta$ where $r \in \alpha, s \in \beta$, then $s + r = r + s = p \in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If $r \in \alpha, s \in \beta, t \in \gamma$, then $r + s + t = (r + s) + t \in (\alpha + \beta) + \gamma$ and $r + s + t = r + (s + t) \in \alpha + (\beta + \gamma)$.

$\alpha + 0^* = \alpha$

If $r \in \alpha, s \in 0^*$, then $r + s < r$. Thus, $r + s \in \alpha$. Thus, $\alpha + 0^* \subset \alpha$.

If $p \in \alpha$, there is a $r \in \alpha$ where $r > p$. Thus, $p - r \in 0^*$.

Since $p = r + (p - r) \in \alpha + 0^*$, then $\alpha \subset \alpha + 0^*$. Thus, $\alpha + 0^* = \alpha$.

There is a $-\alpha$ such that $\alpha + (-\alpha) = 0^*$

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is $r > 0$ such that $-p - r \notin \alpha$.

$\beta \in \mathbb{R}$ since β is a cut:

- If $s \notin \alpha$ and $p = -s - 1$, then $-p - 1 \notin \alpha$. Thus, $p \in \beta$ so β is nonempty. If $q \in \alpha$, then $-q \notin \beta$ so $\beta \neq \mathbb{R}$.
- If $p \in \beta$, let $r > 0$ so $-p - r \notin \alpha$. If $q < p$, then $-q - r > -p - r$ and thus, $-q - r \notin \alpha$ so $q \in \beta$.
- If $p \in \beta$, let $t = p + (r/2)$. Then $-t - (r/2) = -p - r \notin \alpha$ and thus, $t \in \beta$ where $p < t$.

If $r \in \alpha, s \in \beta$, then $s \notin \alpha$. Thus, $r < -s$ so $r + s < 0$. Thus, $\alpha + \beta \subset 0^*$.

Let $v \in 0^*$ and let $w = -v/2$ so $w > 0$.

Thus, by the Archimedean property, there is an integer n such that $nw \in \alpha$, but $(n+1)w \notin \alpha$. Let $p = -(n+2)w$ so $-p - w = (n+1)w \notin \alpha$ so $p \in \beta$.

Then, $v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p \in \alpha + \beta$.

Since $v \in 0^*$, then $0^* \subset \alpha + \beta$. Thus, $\alpha + \beta = 0^*$. Then, let $-\alpha = \beta$.

Thus, if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Thus, if $\alpha > 0^*$, then $-\alpha = -\alpha + 0^* < -\alpha + \alpha = 0^*$ so $-\alpha < 0^*$.

If $\alpha, \beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that $p \leq rs$ for $r \in \alpha, s \in \beta$.

Define 1^* as the set of all $q < 1$. Then all multiplication axioms holds with similar proofs as addition. Also, note since $\alpha, \beta > 0^*$, then $\alpha\beta > 0^*$.

Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were $\alpha, \beta, \gamma >, < 0^*$.

6 Cardinality

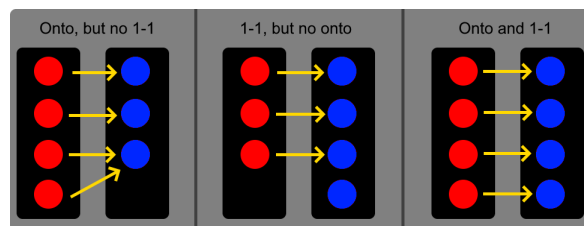
6.1 Cardinality

Definition 6.1.1: Onto and 1-1 Mapping

Suppose for every $x \in A$, there is an associated $f(x) \in B$.

Then f maps A into $B = f: A \rightarrow B$.

- If $f(A) = B$, then f maps A onto B .
- If for each $y \in B$, $f^{-1}(y)$ consist of at most one $x \in A$ where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B .



Definition 6.1.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function $f: A \rightarrow B$. (1-1 correspondence between A and B) Then:

$$A \sim B$$

If $f: A \rightarrow B$ is 1-1 and onto, then there is a $f^{-1}: B \rightarrow A$ that is 1-1 and onto.

Definition 6.1.3: Countability

- A is finite if $A \sim J_n = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$
- A is infinite if A is not finite
- A is countably infinite if $A \sim J = \mathbb{Z}_+$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite

Example 6.1.4

\mathbb{Z} is countably infinite

Proof

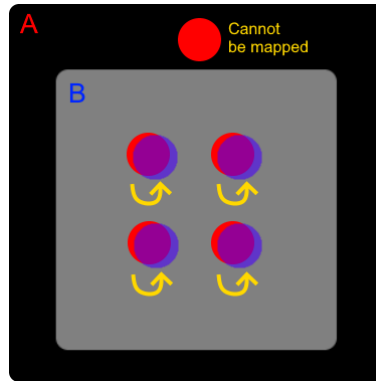
Let $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n-1}{2} & n \text{ is odd} \end{cases}$$

So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto -1$, $4 \mapsto 2$, $5 \mapsto -2$, etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 6.1.5: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A .

**Theorem 6.1.6: Infinite subsets of countable sets are countable**

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let $E \subset A$ be an infinite subset. For every distinct $x_i \in A$, let $x = \{x_1, x_2, \dots\}$.
 Let n_1 be smallest integer such that $x_{n_1} \in E$.
 Then let n_2 be the smallest integer where $n_2 > n_1$ such that $x_{n_2} \in E$.
 Repeat the process to create sequence $f(k) = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$.
 Thus, there is a 1-1 correspondence between E and \mathbb{Z}_+ so E is countably infinite.

**6.2 Set of Sets****Definition 6.2.1: Union and Intersection**

Let sets Ω, B be such that for each $x \in \Omega$, there is an associated $E_x \subset B$.

- $E = \bigcup_{x=1}^n E_x$ only if for every $x \in E$, $x \in E_x$ for at least one $x \in \Omega$.
- $P = \bigcap_{x=1}^n E_x$ only if for every $x \in P$, $x \in E_x$ for all $x \in \Omega$.

with properties:

- | | |
|---|---|
| (a) $A \cup B = B \cup A$ | (a) $A \cap B = B \cap A$ |
| (b) $(A \cup B) \cup C = A \cup (B \cup C)$ | (b) $(A \cap B) \cap C = A \cap (B \cap C)$ |
| (c) $A \subset A \cup B$ | (c) $(A \cap B) \subset A$ |
| (d) If $A \subset B$, then $A \cup B = B$ and $A \cap B = A$ | |

Proof

If $x \in A \cup B$, then $x \in A$ or/and $x \in B$.

- If $x \in A$, since $A \subset B$, then $x \in B$. Then, $(A \cup B) \subset B$.
- If $x \in B$, then immediately $(A \cup B) \subset B$.

If $x \in B$, then $x \in A \cup B$ so $B \subset (A \cup B)$. Thus, $A \cup B = B$.

If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus, $(A \cap B) \subset A$.

If $x \in A$, since $A \subset B$, then $x \in B$ so $x \in A \cap B$. Thus, $A \subset (A \cap B)$.
 Thus, $A \cap B = A$.

$$(e) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof

If $x \in A \cap (B \cup C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

- If $x \in B$, then $x \in (A \cap B)$ so $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in C$, then $x \in (A \cap C)$ so $x \in (A \cap B) \cup (A \cap C)$.

Thus, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

If $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ and ($x \in B$ or/and $x \in C$).

Thus, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$$(f) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

If $x \in A \cup (B \cap C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).

- If $x \in A$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
- If $x \in B, C$, then $x \in (A \cup B)$ and $x \in (A \cup C)$ so $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

If $x \in (A \cup B) \cap (A \cup C)$, then $x \in A$ or/and ($x \in B$ and $x \in C$).

Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 6.2.2: Union of countably infinite sets is countably infinite

If E_1, E_2, \dots are countably infinite sets, then $S = \bigcup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

For each E_n , there is a sequence $\{x_{n1}, x_{n2}, \dots\}$. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence $S^* = \{x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; \dots\}$.

Since $S^* \sim S$ so S is at most countable and S is infinite since E_1, E_2, \dots are infinite, then S cannot be finite and thus, countably infinite.

Alternative Proof

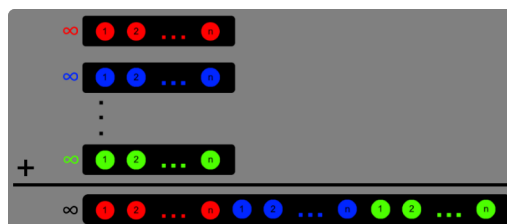
For each E_n , let set $\widetilde{E}_n = E_n - \bigcup_{m=1}^{\infty} E_m$ where $m \neq n$. Thus, $S = \bigcup_{n=1}^{\infty} \widetilde{E}_n$.

Since each E_n is countably infinite, there exists a 1-1 mapping $\delta_n: E_n \rightarrow \mathbb{Z}_+$.

Thus, for each \widetilde{E}_n , there is a 1-1 mapping $\delta_n: \widetilde{E}_n \rightarrow A \subset \mathbb{Z}_+$.

Let p_1, p_2, \dots be distinct primes. Since for $s \in S$, there exists a unique \widetilde{E}_i such that $s \in \widetilde{E}_i$, then let $f(s) = p_1^{\delta_1(s)} p_2^{\delta_2(s)} \dots$ where $p_k^{\delta_k(s)} = 1$ if $k \neq i$.

Then, by the Fundamental theorem of arithmetic, f maps s to a unique $z \in \mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable. Since any $E_n \subset S$ is countably infinite, then S cannot be finite and thus, S is countably infinite.



Theorem 6.2.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B_n be the set of all n -tuples (a_1, \dots, a_n) where $a_k \in A$. Then B_n is countably infinite.

Proof

The base case B_1 is countably infinite since $B_1 = A$.

Suppose B_{n-1} is countably infinite. Then for every $x \in B$:

$$x = (b, a) \quad b \in B_{n-1} \text{ and } a \in A$$

Since for every fixed b , $(b, a) \sim A$ and thus, countably infinite.

Since B is a set of countably infinite sets, then B_n is countably infinite.

Definition 6.2.4: \mathbb{Q} is countable

The set of rational numbers, \mathbb{Q} , is countably infinite.

Proof

Since elements of \mathbb{Q} are of form $\frac{a}{b}$ which is a 2-tuple, then by the **theorem 6.2.3**, \mathbb{Q} is countably infinite.

Alternative Proof

For every $x \in \mathbb{Q}$, let $x = (-1)^i \frac{p}{q}$ where $p, q \in \mathbb{Z}_+$.

Let $f(x) = 2^i 3^p 5^q$. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to $E \subset \mathbb{Z}_+$.

Thus, \mathbb{Q} is at most countable, but since $p, q \in \mathbb{Z}_+$, then \mathbb{Q} cannot be finite and thus, is countably infinite.

Example 6.2.5: Sequences of 0 and 1 are uncountable

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences s_1, s_2, \dots . Then construct a sequence s as follows:

If the n -th digit in s_n is 1, then let the n -th digit of s be 0 and vice versa.

Thus, s differs from every $s_n \in E$ so $s \notin E$.

But, $s \in A$ so E is a proper subset of A .

Thus, every countably infinite subset of A is a proper subset of A .

If A is countably infinite, then A is a proper subset of A which is a contradiction.

7 Metric Spaces & Closed/Open

7.1 Metric Spaces

Definition 7.1.1: Metric Spaces

A set X is a metric space if for any $p, q \in X$, there is an associated $d(p, q) \in \mathbb{R}$ such that:

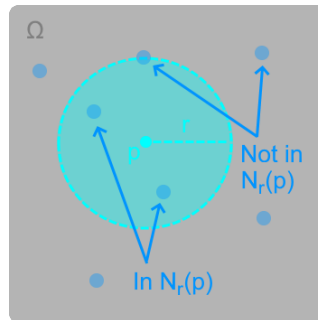
- $d(p, q) > 0$ if $p \neq q$
- $d(p, q) = 0$ if and only if $p = q$
- **Symmetry**: $d(p, q) = d(q, p)$
- **Triangle Inequality**: $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

For euclidean spaces \mathbb{R}^k , $d(x, y) = |x - y|$ where $x, y \in \mathbb{R}^k$.

Definition 7.1.2: Types of Points and Sets

(a) Neighborhood

For $p \in \mathbb{R}^k$ and $r > 0$, $N_r(p)$ is the set of all $q \in X$ where $d(q, p) < r$



(b) Limit Points and Closed Sets

Closed set E contain all $p \in X$ where every $N_r(p)$ contain a $q \neq p \in E$

• Limit Points

For point $p \in X$, every $N_r(p)$ contains a $q \neq p \in E$

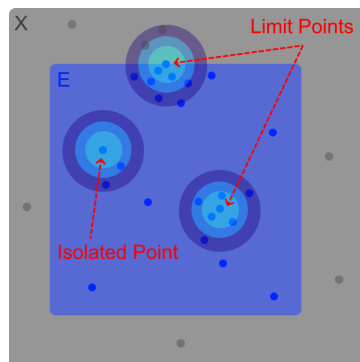
The set of all limit points of $E = E'$

• Isolated Points

If $p \in E$ is not a limit point of E

• Closed

If every limit point p of E is a $p \in E$



(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset E$

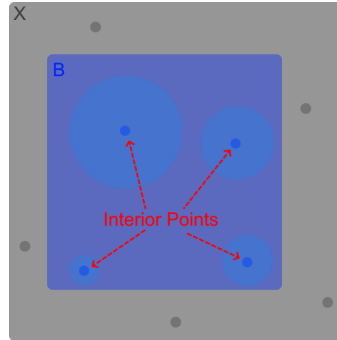
- Interior Point

For $p \in X$, there is a $N_r(p) \subset E$

The set of all interior points = E°

- Open

If every $p \in E$ is an interior point of E



(d) More about Sets

- Bounded

If there is $M \in \mathbb{R}$, $q \in X$ such that $d(p, q) < M$ for all $p \in E$

- Complement

From E , E^c is the set of all $p \in X$ such that $p \notin E$

- Perfect

If E is closed and if every $p \in E$ is a limit point of E

- Dense

If every $p \in X$ is a limit point of E or/and $p \in E$

- Boundary Point

For $p \in X$, if every $N_r(p)$ contains a $x \in E$ and $y \in E^c$

The set of all boundary points = ∂E

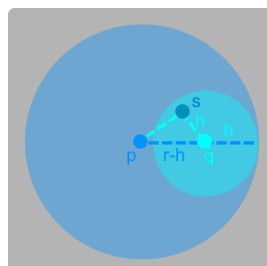
For a metric space X , $\{X, \emptyset\}$ are both open and closed.

Theorem 7.1.3: $N_r(p)$ is open

Every neighborhood is an open set.

Proof

Let $q \in N_r(p)$. Then there is a $h > 0 \in \mathbb{R}$ such that $d(q, p) = r - h$.
 Then for any $s \in N_h(q)$, $d(s, p) \leq d(s, q) + d(q, p) = h + (r - h) = r$.
 Thus, for any $q \in N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

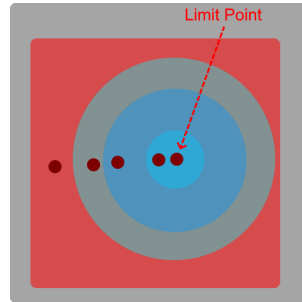


Theorem 7.1.4: If a set has a limit point, there are infinite $q \in E$ in $N_r(p)$

If p is a limit point of set E , then every $N_r(p)$ contains infinitely many $q \in E$.

Proof

Suppose there is $N_{r_1}(p)$ which contains finitely many $q = \{q_1, \dots, q_n\}$.
 Let $r = \min_{m \in [1, n]} d(p, q_m)$. Then $N_r(p)$ contains no $q \in E$ such that $q \neq p$.
 So, p is not a limit point of E which is a contradiction since p is a limit point of E .



Corollary 7.1.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since $\emptyset \in A$, all finite set must be closed.

Proof

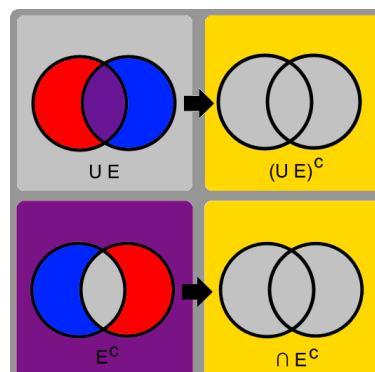
Let p be a limit point of finite set E . By **theorem 7.1.4**, then any $N_r(p)$ contain infinite $q \in E$ so E is an infinite set which is a contradiction since E is finite.
 So p cannot be limit point of E and thus, E has no limit points.

Theorem 7.1.6: De Morgan's Laws

Let E_1, E_2, \dots be a collection of sets. Then, $(\cup E_x)^c = \cap (E_x^c)$.

Proof

If $p \in (\cup E_x)^c$, then $p \notin (\cup E_x)$.
 Thus, $p \notin E_x$ for any x so $p \in E_x^c$ for all x . Thus, $p \in \cap (E_x^c)$ so $(\cup E_x)^c \subset \cap (E_x^c)$.
 If $p \in \cap (E_x^c)$, then $p \in E_x^c$ for all x .
 Thus, $p \notin E_x$ for any x so $p \notin \cup E_x$. Thus, $p \in (\cup E_x)^c$ so $\cap (E_x^c) \subset (\cup E_x)^c$.
 Thus, $(\cup E_x)^c = \cap (E_x^c)$.



Theorem 7.1.7: Open set \rightarrow Closed complement

A set E is open if and only if E^c is closed.

Proof

Suppose E is open. Let x be a limit point of E^c .

Then for every $r > 0$, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.

Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.

Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.

Since $x \notin E^c$, x is not a limit point of E^c . Then there exists a $r > 0$ such that any $p \in N_r(x)$ is not in E^c . Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E . Since any $x \in E$ is an interior point of E , then E is open.

Corollary 7.1.8: Closed set \rightarrow Open complement

A set F is closed if and only if F^c is open.

Proof

From **theorem 7.1.7**, let $E = F^c$.

Theorem 7.1.9: Union open \rightarrow open and Intersection closed \rightarrow closed

- (a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup G_x$ is open.

Proof

If $p \in \cup G_x$, then $p \in G_x$ for at least one x . Let \bar{x} be such an x .

Since $G_{\bar{x}}$ is open, then p is an interior point of $G_{\bar{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\bar{x}} \subset \cup G_x$. So p is an interior point of $\cup G_x$. Since any $p \in \cup G_x$ is an interior point, then $\cup G_x$ is open.

- (b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap F_x$ is closed.

Proof

By **theorem 7.1.7**, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup F_x^c$ is open.

Thus, again by **theorem 7.1.7**, $(\cup F_x^c)^c$ is closed.

By **theorem 7.1.6**, $(\cup F_x^c)^c = \cap (F_x^c)^c = \cap F_x$.

- (c) If G_1, \dots, G_n is a finite collection of open sets, then $\cap_{x=1}^n G_x$ is open.

Proof

If $p \in \cap_{x=1}^n G_x$, then $p \in G_x$ for all G_x for $x = \{1, 2, \dots, n\}$.

Since each G_x is open, then for any G_x , there is a $N_{r_x}(p) \subset G_x$.

Let $r = \min(r_1, r_2, \dots, r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x .

So, $N_r(p) \subset \cap_{x=1}^n G_x$ and thus, p is an interior point of $\cap_{x=1}^n G_x$ so $\cap_{x=1}^n G_x$ is open.

Infinite + Closed: $G_i = (-1/i, 1/i)$

Infinite + Open: $G_i = (-i, i)$

- (d) If F_1, \dots, F_n is a finite collection of closed sets, then $\cup_{x=1}^n F_x$ is closed.

Proof

By **theorem 7.1.7**, any F_x^c is open. Since F_1^c, \dots, F_n^c is a finite collection of open set, then by part (c), $\cap_{x=1}^n F_x^c$ is open.

Thus, again by **theorem 7.1.7**, $(\cap_{x=1}^n F_x^c)^c$ is closed.

By **theorem 7.1.6**, $(\cap_{x=1}^n F_x^c)^c = \cup_{x=1}^n (F_x^c)^c = \cup_{x=1}^n F_x$.

Infinite + Closed: $F_i = [-1/i, 1/i]$

Infinite + Open: $F_i = [1/i, \infty)$

Theorem 7.1.10: E' is closed

Let $E \subset X$. Then, $(E')' \subset E'$. Thus, E' is closed.

Proof

If $x \in (E')'$, then for every $N_{r_1}(x)$, there is a $y \neq x$ where $y \in E'$.
 Since $y \in E'$, then for every $N_{r_2}(y)$, there is a $z \neq y$ where $z \in E$.
 Let $r = r_1 + r_2$.
 Then for every $N_r(x)$, there exists a $z \neq x$ where $z \in E$. Thus, $x \in E'$ so $(E')' \subset E'$.

Theorem 7.1.11: E° is open

Let $E \subset X$. Then, E° is open.

Proof

If $p \in E^\circ$, there is a $r > 0$ such that $N_r(p) \subset E$.
 Then for $0 < s < r$, $N_s(p) \subset N_r(p)$ so any $q \in N_s(p)$ is $q \in E^\circ$.
 Since any $p \in E^\circ$ have a $N_s(p) \subset E^\circ$, then E° is open.

7.2 Intervals and Balls

Definition 7.2.1: Segments and Intervals

In \mathbb{R} , a **segment** is an open interval $(a,b) = \{x \in \mathbb{R} : a < x < b\}$

In \mathbb{R} , a **interval** is a closed interval $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

Definition 7.2.2: Open Balls

In \mathbb{R}^k , an **open ball** of radius $r > 0$ centered at p is:

$$N_r(p) = \{x \in \mathbb{R}^k : |x - p| < r\} = \{x \in \mathbb{R}^k : d(x,p) < r\}$$

A **closed ball** has $d(x,p) \leq r$.

Definition 7.2.3: Convex

$E \subset \mathbb{R}^k$ is **convex** if for all $x, y \in E$ and $t \in [0,1]$, $tx + (1-t)y \in E$.

Example 7.2.4: Balls are convex

Balls in \mathbb{R}^k are convex.

Proof

Let $x, y \in$ open ball $N_r(p)$. Let $z = tx + (1-t)y$ for $t \in [0,1]$.
 Since $|x - p| < r$ and $|y - p| < r$:

$$\begin{aligned} |z - p| &= |tx + (1-t)y - p| = |tx + (1-t)y - tp + (t-1)p| \\ &= |t(x-p) + (1-t)(y-p)| \leq t|x-p| + (1-t)|y-p| \\ &< tr + (1-t)r = r \end{aligned}$$

 Thus, $z \in N_r(p)$ so balls are convex. Same proof applies to closed balls.

Definition 7.2.5: Dense

$E \subset X$ is dense if every $x \in X$ is either in E or a limit point of E .

Example 7.2.6: \mathbb{Q} is dense in \mathbb{R}

Let $X = \mathbb{R}$. Then, $E = \mathbb{Q}$ is dense in \mathbb{R} .

Proof

Fix $x \in \mathbb{R}$ and $r > 0$. There is a $q \in \mathbb{Q}$ such that $x-r < q < x$. So for any $r > 0$ and $q \in \mathbb{Q}$, $q \neq x$ and $q \in N_r(x)$. Thus, every $x \in \mathbb{R}$ is a limit point of \mathbb{Q} .

8 Closure, Open Relative, & Compact

8.1 Closure

Definition 8.1.1: Closure

Let $E \subset$ metric space X and E' be the set of all limit points of E in X .

Then the closure of E : $\overline{E} = E \cup E'$

with the properties:

- (a) \overline{E} is closed
- (b) $E = \overline{E}$ if and only if E is closed
- (c) $\overline{E} \subset F$ for every closed $F \subset X$ such that $E \subset F$

Proof

Suppose $x \in X$, but $x \notin \overline{E}$. Thus, $x \in \overline{E}^c$.

Thus, there is a $N_r(x) \subset \overline{E}^c$ since else there is always a $p \in N_r(x)$ where $p \in \overline{E}$ so x is a limit point of \overline{E} so $x \in \overline{E}$. Thus, \overline{E}^c is open so \overline{E} is closed by [theorem 7.1.7](#).

If $E = \overline{E}$, then by part (a), E is closed.

If E is closed, then $E' \subset E$ so $E = E \cup E' = \overline{E}$.

If closed set F , then $F' \subset F$ and since $E \subset F$, then $E' \subset F' \subset F$. Thus, $\overline{E} \subset F$.

Theorem 8.1.2: $\sup(E) \in \overline{E}$

Let non-empty set of real numbers, E , be bounded above. Let $y = \sup(E)$.

Then, $y \in \overline{E}$. Thus, $y \in E$ if E is closed and $y \notin E$ if E is open in \mathbb{R} .

Proof

If $y \in E$, then $y \in \overline{E}$. Suppose $y \notin E$.

For every $h > 0$, there exists a $x \in E$ such that $y-h < x < y$ otherwise $y-h$ is an upper bound for E which is a contradiction since $y = \sup(E)$.

Thus, y is a limit point of E so $y \in E'$.

If E is closed, then $y \in E$ since $y \in E'$. Also, $y \in \overline{E}$.

If E is open, then any $N_r(y) \not\subset E$ since $N_r(y)$ in \mathbb{R} must contain a $\gamma > y$ so $y \notin E^\circ$.

8.2 Open Relative

Definition 8.2.1: Open Relative

Suppose $E \subset Y \subset$ metric space X .

Then E is open relative to Y if for each $p \in E$, there is an $r > 0$ such that for any $q \in Y$, then $q \in E$ if $d(q,p) < r$.

Theorem 8.2.2: E is open relative to $Y \subset X$ if $E = Y \cap G$ and G is open in X

Suppose $E \subset Y \subset X$.

E is open relative to Y if and only if $E = Y \cap G$ for some open $G \subset X$.

Proof:

Suppose E is open relative to Y .

Then for each $p \in E$, there is a $r_p > 0$ such that for any $q \in Y$ where $d(p, q) < r_p$, then $q \in E$.

Since $Y \subset X$, let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$ and define $G = \bigcup_{p \in E} V_p$. Since V_p is open by **theorem 7.1.3**, then by **theorem 7.1.9a**, open $G \subset X$. Since $p \in V_p$ for all $p \in E$, then $E \subset G \cap Y$. Also, by construction, then $V_p \cap Y \subset E$ so $G \cap Y \subset E$. Thus, $E = Y \cap G$.

If G is open in X and $E = G \cap Y$, then every $p \in E$ has a $V_p \subset G$.

Then, $V_p \cap Y \subset G \cap Y = E$ so E is open relative to Y .

8.3 Compact Sets

Definition 8.3.1: Open Cover

An open cover of set $E \subset X$ is a collection of open $G_1, G_2, \dots \subset X$ such that $E \subset \bigcup G_i$.

Definition 8.3.2: Compact

$K \subset X$ is compact if every open cover of K contains a finite subcover.

If G_1, G_2, \dots is an open cover of K , then $K \subset \bigcup_{i=1}^n G_i$ for some n .

Theorem 8.3.3: A compact set is compact in every metric space

Suppose $K \subset Y \subset X$.

Then K is compact relative to X if and only if K is compact relative to Y .

Proof

Suppose K is compact relative to X .

Let V_1, V_2, \dots be sets open relative to Y such that $K \subset \bigcup V_x$. Then by **theorem 8.2.2** for each V_x , there is a G_x open relative to X where $V_x = Y \cap G_x$.

Since K is compact relative to X , then there is a n such that $K \subset G_{x_1} \cup \dots \cup G_{x_n}$.

Thus, $K = K \cap Y \subset (\bigcup_{i=1}^n G_{x_i}) \cap Y = (\bigcup_{i=1}^n (G_{x_i} \cap Y)) = \bigcup_{i=1}^n V_{x_i}$.

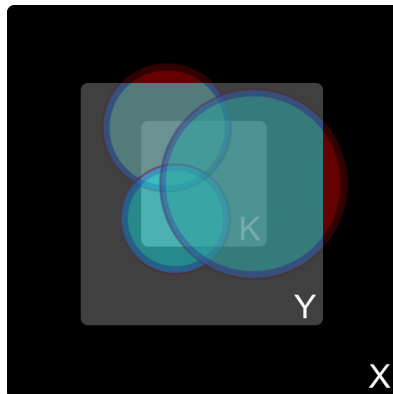
Since there are open V_{x_1}, \dots, V_{x_n} where $K \subset \bigcup_{i=1}^n V_{x_i}$ so K is compact relative to Y .

Suppose K is compact relative to Y .

Let open $G_1, G_2, \dots \subset X$ such that $X \subset \bigcup G_x$. For each G_x , let $V_x = Y \cap G_x \subset Y$.

Since K is compact relative to Y , there is a n such that $K \subset \bigcup_{i=1}^n V_{x_i}$.

Thus, $K \subset \bigcup_{i=1}^n V_{x_i} = \bigcup_{i=1}^n (Y \cap G_{x_i}) \subset \bigcup_{i=1}^n G_{x_i}$ so K is compact relative to X .

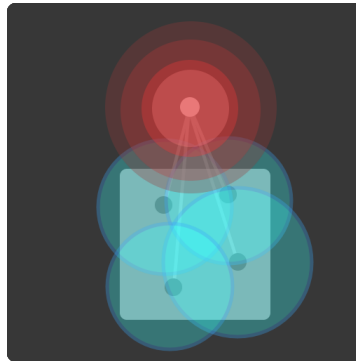


Theorem 8.3.4: A compact set is closed

Compact subsets of metric spaces are closed.

Proof

Let compact $K \subset X$. Suppose $p \in X$, but $p \notin K$ so $p \in K^c$.
 If $q \in K$, let W_q be a neighborhood of q with $r < \frac{1}{2}d(p,q)$. Let $V_{p,q}$ be a neighborhood of p with $r < \frac{1}{2}d(p,q)$. Since K is compact, then there are finite points q_1, \dots, q_n such that $K \subset W$ where $W = W_{q_1} \cup \dots \cup W_{q_n}$.
 Let $V = V_{p,q_1} \cap \dots \cap V_{p,q_n}$, then $K \cap V \subset W \cap V = \emptyset$ so $V \subset K^c$.
 Since there is a neighborhood V for $p \in K^c$ where $V \subset K^c$, then every $p \in K^c$ is an interior point so K^c is open. Then by **theorem 7.1.7**, K is closed.

**Theorem 8.3.5: If closed $E \subset$ compact set K , E is compact**

Closed subsets of compact sets are compact.

Proof

Suppose $F \subset K \subset X$ where F is closed relative to X and K is compact.
 Let V_1, V_2, \dots be an open cover for F . Let open set F^c be all $k \in K$ where $k \notin F$.
 $K = F \cup F^c \subset V_1 \cup V_2 \cup \dots \cup F^c$
 Thus, $V_1 \cup V_2 \cup \dots \cup F^c$ is an open cover for K .
 Since K is compact, there is a finite subcover Ω that covers K and thus, finite subcover Ω covers $F \cup F^c$.
 Remove F^c from Ω . Since finite subcover $\Omega - F^c$ covers F , then F is compact.

Corollary 8.3.6: Closed $F \cap$ compact $K =$ compact

If F is closed and K is compact, then $F \cap K$ is compact.

Proof

Since K is compact, then K is closed by **theorem 8.3.4**.
 Then, by **7.1.9b**, $F \cap K$ is closed.
 Since $F \cap K \subset K$, then by **theorem 8.3.5**, $F \cap K$ is compact.

Theorem 8.3.7: Nonempty $\cap_{i=1}^n K_i \rightarrow$ nonempty $\cap K_i$

For compact sets $K_1, K_2, \dots \subset X$ where any intersection of finite K_i is nonempty, then $\cap K_i$ is nonempty.

Proof

Fix K_1 . If there is a $k \in K_1$ where $k \in K_i$ for all i , then $k \in \cap K_i$ so $\cap K_i \neq \emptyset$.
 Suppose for every $k \in K_1$, $k \notin K_i$ for some i .
 Then for every $k \in K_1$, there is a K_i such that $k \notin K_i$ so $k \in K_i^c$.
 Thus, K_2^c, K_3^c, \dots form an open cover for K_1 .
 Since K_1 is compact, there is a n where $K_1 \subset K_{i_1}^c \cup \dots \cup K_{i_n}^c$.
 But then, $K_1 \cap K_{i_1} \cap \dots \cap K_{i_n} = \emptyset$ which is a contradiction.

Corollary 8.3.8: Nonempty K_i where $K_{i+1} \subset K_i \rightarrow$ nonempty $\cap K_i$

If K_1, K_2, \dots is a sequence of nonempty compact sets such that $K_{i+1} \subset K_i$, then $\cap K_i$ is nonempty.

Proof

Since each K_i is nonempty and if $i_1 < \dots < i_n$, then $K_{i_1} \cap \dots \cap K_{i_n} = K_{i_n}$ is nonempty, then by **theorem 8.3.7**, $\cap K_i$ is nonempty.

Theorem 8.3.9: Nonempty intervals I_n where $I_{n+1} \subset I_n \rightarrow$ nonempty $\cap I_n$

If I_1, I_2, \dots is a sequence of intervals in \mathbb{R}^1 such that $I_{n+1} \subset I_n$, then $\cap I_n$ is nonempty.

Proof

Let $I_n = [a_n, b_n]$ and thus, each I_n is nonempty. If $n_1 < \dots < n_m$, then $I_{n_1} \cap \dots \cap I_{n_m} = [a_{n_m}, b_{n_m}]$ is nonempty. Thus, by **theorem 8.3.7**, $\cap I_n$ is nonempty.

Theorem 8.3.10: $p \in E'$ exists if infinite $E \subset$ compact K

If E is an infinite subset of compact set K , then E has a limit point in K .

Proof

If no $p \in K$ is a $p \in E'$, then each p would have a neighborhood V_p contains at most $p \in E$ if $p \in E$. Thus, there is no finite subcover that covers E and thus, there is no finite subcover that covers K since $E \subset K$ which contradicts K is compact.

Definition 8.3.11: K-cells

The set of all $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.

Theorem 8.3.12: K-cells are compact

Every k -cell is compact.

Proof

Let k -cell I consists of all $x = (x_1, \dots, x_k)$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.
 Let $\delta = \sqrt{\sum_{i=1}^k (b_i - a_i)^2}$. Thus, $|x - y| \leq \delta$ for $x, y \in I$.
 Suppose there exists an open cover G_1, G_2, \dots of I which contain no finite subcover.
 Let $c_i = \frac{a_i + b_i}{2}$. Then each interval splits into $[a_i, c_i]$ and $[c_i, b_i]$ for $i \in [1, k]$ so there now exists 2^k k -cells Q_i whose union is I .
 At least one Q_i cannot be covered else I would be covered. Then subdivide Q_i as before and repeating the process so $Q_{i+1} \subset Q_i$ and each are not covered.
 However, there is a point $x^* \in Q_{i_j}$ for all j such that $N_r(x^*) \subset G$ so Q_{i_1} is covered which is a contradiction.

Theorem 8.3.13: Heine-Borel Theorem

If a set $E \subset \mathbb{R}^k$ has one of the three properties, then it has the other two:

- (a) E is closed and bounded
- (b) E is compact
- (c) Every infinite subset of E has a limit point in E

Proof

Suppose E is closed and bounded.

Then there exists a $M \in \mathbb{R}$ and $q \in \mathbb{R}^k$ such that $d(p, q) < M$ for all $p \in E$.

Thus, there is a k -cell $K = [-M + q_1, q_1 + M] \times \dots \times [-M + q_k, q_k + M]$ such that $E \subset K$. Then by [theorem 8.3.12](#), K is compact and thus by [theorem 8.3.5](#), E is compact so (a) \rightarrow (b).

Then by [theorem 8.3.10](#), any infinite subset of E has a limit point in E so (b) \rightarrow (c). Suppose E is not bounded.

Then there exists $p \in E$ such that $d(p, q) > M$ for any $M \in \mathbb{R}$ and $q \in \mathbb{R}^k$.

Let $S \subset E$ be such points p .

Then S is infinite else there is a maximal p and thus, p is bounded. Thus, S is infinite and contains no limit points in E since any $d(p_1, p_2) > M$ which contradicts that every infinite subset of E has a limit point in E . Thus, E is bounded.

Suppose E is not closed.

Then there exists a $p \in E'$, but $p \notin E$. Since p is a limit point, then there is a $q \in E$ such that $\frac{1}{n+1} < d(q, p) < \frac{1}{n}$ for $n = \{1, 2, \dots\}$.

Let $S \subset E$ be such points q .

Thus, p is the only limit point of S since for $r < \frac{1}{n}$, any $N_r(q_i)$ contains no points of S other than q_i since $d(q_i, q_j) > \frac{1}{n}$ for any $q_1, q_2 \in S$.

Thus, S is infinite, but the only $p \in S'$ is $p \notin E$ which contradicts that every infinite subset of E has a limit point in E . Thus, E is closed. So, (c) \rightarrow (a).

Theorem 8.3.14: Weierstrass Theorem

Every bounded infinite set $E \subset \mathbb{R}^k$ has a limit point in \mathbb{R}^k .

Proof

Since E is bounded, then there exists a k -cell K such that $E \subset K$. Since K is compact, then by [theorem 8.3.10](#), E has a limit point in K and thus, in \mathbb{R}^k .

9 Perfect and Connected Sets

9.1 Perfect Sets

Definition 9.1.1: Perfect Set

$E \subset X$ is perfect if E is closed and if every $p \in E$ is $p \in E'$.

Theorem 9.1.2: Perfect sets are uncountable

Let P be a nonempty perfect set in \mathbb{R}^k . Then, P is uncountable.

Proof

Since P has limit points, then by [theorem 7.1.4](#), P is infinite.
 Suppose P is countable. Then let $x_1, x_2, \dots \in P$.
 Let V_i be a neighborhood of x_i where $y \in V_i$ for any $y \in \mathbb{R}^k$ such that $|y - x_i| < r$.
 Thus, the \bar{V}_i is the set of all $y \in \mathbb{R}^k$ such that $|y - x_i| \leq r$.
 Since every x_i are limit points, then any $V_i \cap P$ is not empty where there is a V_{i+1}
 (a) $\bar{V}_{i+1} \subset V_i$
 (b) $x_i \notin \bar{V}_{i+1}$
 (c) $V_{i+1} \cap P$ is nonempty
 Let $K_i = \bar{V}_i \cap P$. Since \bar{V}_i is closed and bounded, then by [theorem 8.3.11](#), \bar{V}_i is compact. Since $x_i \notin K_{i+1}$, then no $x_i \in P$ is $x_i \in \cap K_i$. Since $K_n \subset P$, then $\cap K_i$ is nonempty which contradicts [corollary 8.3.8](#) since each K_i is empty and $K_{i+1} \subset K_i$.

Corollary 9.1.3: \mathbb{R} is not countable

Every interval $[a, b]$ is uncountable. Thus, \mathbb{R} is uncountable.

Proof

Since $[a, b]$ is closed and every $p \in [a, b]$ is a limit point, then nonempty set $[a, b]$ is perfect. Thus, by [theorem 9.1.2](#), $[a, b]$ is uncountable.

Definition 9.1.4: Cantor Sets

There exists perfect segments in \mathbb{R}^1 which contain no segment.

Let $E_0 = [0, 1]$.

For E_1 , remove $(\frac{1}{3}, \frac{2}{3})$. Thus, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

For E_2 , remove $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. Thus, $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

Continuing such a sequence, the set of compact sets E_n are such that:

(a) $E_{n+1} \subset E_n$

(b) E_n is the union of 2^n intervals each of length 3^{-n} .

$P = \cap E_n$ is called the Cantor set. P is compact and nonempty.

Thus, any segment of form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ where $k, m \in \mathbb{Z}_+$ has no points in common with P . Since any segment (a, b) contain a segment of such a form since $3^{-m} < \frac{b-a}{6}$, then P contains no segment.

Let $x \in P$ and segment S contain x . Let I_n be an interval of E_n containing x . Then choose a large enough n so $I_n \subset S$.

Let x_n be an endpoint of I_n where $x_n \neq x$ and thus, x is a limit point. Since P is closed and every $p \in P$ is $p \in P'$, then P is perfect.

9.2 Connected Sets

Definition 9.2.1: Connected Set

$A, B \subset X$ are separated if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.

$E \subset X$ is connected if E is not the union of two nonempty separated sets.

Separated sets are disjoint, but disjoint sets need not be separated.

Theorem 9.2.2: All points between points in connected sets exists

$E \subset \mathbb{R}^1$ is connected if and only if:

If $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof

If there exists $x, y \in E$ and $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where $A_z = E \cap (-\infty, z)$ and $B_z = E \cap (z, \infty)$.

Since $x \in A_z$ and $y \in B_z$, then A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z = (z, \infty)$, then A_z and B_z are separated. Thus, E is not connected.

Suppose E is not connected. Then, there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$ where $x < y$. Let $z = \sup(A \cap [x, y])$.

Since, $z \in \overline{A}$ so $z \notin B$, then $x \leq z < y$. If $z \notin A$, then $x < z < y$ so $z \notin E$.

If $z \in A$, then $z \notin \overline{B}$ and thus, there exists a z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then, $x < z_1 < y$ so $z_1 \notin E$.

10 Convergent and Cauchy Sequences

10.1 Convergent Sequences

Definition 10.1.1: Convergent Sequence

A sequence $\{x_n\}$ in metric space X converge if there is a $x \in X$ such that:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for all $n \geq N$, $d(x_n, x) < \epsilon$

Then, $\{x_n\}$ converges to x : $\lim_{n \rightarrow \infty} x_n = x$

If $\{x_n\}$ does not converge, then it diverges.

Example 10.1.2

- (a) Let $x_n = \frac{1}{n}$ in \mathbb{R}^2 . Then, $\lim_{n \rightarrow \infty} x_n = 0$

Proof

For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$. Then:
 $d(x_n, 0) = |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$

- (b) Let $x_n = (-1)^n + \frac{1}{n}$ in \mathbb{R}^2 . Then, $\{x_n\}$ diverges.

Proof

$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (-1)^n + \lim_{n \rightarrow \infty} \frac{1}{n} = \lim_{n \rightarrow \infty} (-1)^n$
 Since $(-1)^n$ alternates between -1 and 1, then $\{x_n\}$ diverges.

Theorem 10.1.3: A convergent sequence is unique

- (a) $\{p_n\}$ converges to $p \in X$ if and only if every $N_r(p)$ contains p_n for all, but finitely many n .

Proof

Suppose $p_n \rightarrow p$. Then for $N_\epsilon(p)$, any $q \in X$ such that $d(q, p) < \epsilon$ is $q \in N_\epsilon(p)$. Since $p_n \rightarrow p$, there is a N such that for $n \geq N$, $d(p_n, p) < \epsilon$. Thus, for $n \geq N$, $p_n \in N_\epsilon(p)$.
 Suppose every $N_r(p)$ contains p_n for all, but finitely many n .
 For $\epsilon > 0$, let $N_\epsilon(p)$ be the set of all $q \in X$ such that $d(p, q) < \epsilon$. Thus, there exists a N such that $p_n \in N_\epsilon(p)$ if $n \geq N$.
 Thus, $d(p_n, p) < \epsilon$ so $p_n \rightarrow p$.

- (b) If $p, p' \in X$ and $\{p_n\}$ converges to p and p' , then $p = p'$.

Proof

For $\epsilon > 0$, there exists N, N' such that:
 $d(p_n, p) < \frac{\epsilon}{2}$ for $n \geq N$ $d(p_n, p') < \frac{\epsilon}{2}$ for $n \geq N'$
 Then for $n \geq \max(N, N')$, $d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon$.
 Thus, $p = p'$.

- (c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

Proof

If $\{p_n\} \rightarrow p$, there is a N such that for $n > N$, $d(p_n, p) < 1$.
 Let $r = \max(1, d(p_1, p), \dots, d(p_N, p))$. Thus for all n , $d(p_n, p) \leq r$.

- (d) If $E \subset X$ and $p \in E'$, there is a $\{p_n\}$ in E such that $p = \lim_{n \rightarrow \infty} p_n$.

Proof

Since $p \in E'$, then for each $n \in \mathbb{Z}_+$, there is a $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$ so for $n \geq N$, $d(p_n, p) < \frac{1}{n} < \frac{1}{N} < \epsilon$.
 Thus, $p = \lim_{n \rightarrow \infty} p_n$.

Theorem 10.1.4: Arithmetic Operations for Sequences

Suppose $\{s_n\}, \{t_n\} \in \mathbb{C}$ where $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} t_n = t$.

- (a) $\lim_{n \rightarrow \infty} s_n + t_n = s + t$

Proof

For $\epsilon > 0$, there exists N_1, N_2 such that
 $|s_n - s| < \frac{\epsilon}{2}$ for $n \geq N_1$ $|t_n - t| < \frac{\epsilon}{2}$ for $n \geq N_2$
 If $N = \max(N_1, N_2)$, then for $n \geq N$:
 $|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$

- (b) $\lim_{n \rightarrow \infty} cs_n = cs$ and $\lim_{n \rightarrow \infty} c + s_n = c + s$

Proof

For $\epsilon > 0$, there exists a N such that
 $|s_n - s| < \frac{\epsilon}{c}$ for $n \geq N$
 $|cs_n - cs| \leq c \cdot |s_n - s| < \epsilon$

- (c) $\lim_{n \rightarrow \infty} s_n t_n = st$

Proof

Note $s_n t_n - st = (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$.
 For $\epsilon > 0$, there exists N_1, N_2 such that
 $|s_n - s| < \sqrt{\epsilon}$ for $n \geq N_1$ $|t_n - t| < \sqrt{\epsilon}$ for $n \geq N_2$
 If $N = \max(N_1, N_2)$, then for $n \geq N$, $|(s_n - s)(t_n - t)| < \epsilon$.
 Thus, $\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0$.
 $\lim_{n \rightarrow \infty} (s_n t_n - st) = \lim_{n \rightarrow \infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$
 $= 0 + t \cdot 0 + s \cdot 0 = 0$

- (d) $\lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}$ where $s_n, s \neq 0$

Proof

Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$ so $|s_n| > \frac{1}{2}|s|$ for $n \geq m$.
 For $\epsilon > 0$, there is a $N > m$ such that for $n \geq N$, $|s_n - s| < \frac{1}{2}|s|^2 \epsilon$.
 Thus, for $n \geq N$, $|\frac{1}{s_n} - \frac{1}{s}| = |\frac{s_n - s}{s_n s}| < \frac{2}{|s|^2} |s_n - s| < \epsilon$.

Theorem 10.1.5: Extension to \mathbb{R}^k

- (a) Suppose $x_n \in \mathbb{R}^k$ and $x_n = (\alpha_{n_1}, \dots, \alpha_{n_k})$. Then $\{x_n\}$ converges to $x = (\alpha_1, \dots, \alpha_k)$ if and only if $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ for $i \in [1, k]$.

Proof

Suppose $\{x_n\}$ converges to $x = (\alpha_1, \dots, \alpha_k)$.
 Since for any $i \in [1, k]$, $|\alpha_{n_i} - \alpha_i| \leq |x_n - x| < \epsilon$. Then, $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$.
 Suppose $\lim_{n \rightarrow \infty} \alpha_{n_i} = \alpha_i$ for $i \in [1, k]$.
 Then for $\epsilon > 0$, there is an N such that for $n \geq N$:
 $|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}}$ for $i \in [1, k]$
 $|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot (\frac{\epsilon}{\sqrt{k}})^2} = \epsilon$

- (b) Suppose $\{x_n\}, \{y_n\} \in \mathbb{R}^k$ and $\{\beta_n\} \in \mathbb{R}$ and $x_n \rightarrow x, y_n \rightarrow y, \beta_n \rightarrow \beta$.
 $\lim_{n \rightarrow \infty} x_n + y_n = x + y$ $\lim_{n \rightarrow \infty} x_n \cdot y_n = x \cdot y$ $\lim_{n \rightarrow \infty} \beta_n x_n = \beta x$

Proof

By part a, then $\lim_{n \rightarrow \infty} x_{n_i} + y_{n_i} = x_i + y_i$ so $\{x_n + y_n\} \rightarrow x + y$.
 Also, $\lim_{n \rightarrow \infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$ so $\{x_n \cdot y_n\} \rightarrow x \cdot y$.
 Also, $\lim_{n \rightarrow \infty} \beta_i x_{n_i} = \beta_i x_i$ so $\{\beta_n x_n\} \rightarrow \beta x$.

10.2 Subsequences

Definition 10.2.1: Subsequence

For sequence $\{p_n\}$, let $\{n_k\} \in \mathbb{Z}_+$ where $n_k < n_{k+1}$.

Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.

If $\{p_{n_k}\}$ converges, then its limit is called a subsequential limit.

Theorem 10.2.2: $\{p_n\} \rightarrow p \iff \{p_{n_k}\} \rightarrow p$

$\{p_n\}$ converges to p if and only if every subsequence converges to p .

Proof

Suppose $\{p_n\}$ converges to p .

Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $|p_n - p| < \epsilon$.

Let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$.

Then for $n_k \geq N$, $|p_{n_k} - p| < \epsilon$. Thus, every $\{p_{n_k}\} \rightarrow p$.

Suppose every subsequence converges to p .

Since $\{p_n\}$ is a subsequence of itself, then $\{p_n\}$ converges to p .

Theorem 10.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \rightarrow p$

- (a) If $\{p_n\}$ is a sequence in a compact metric space X , then some subsequence converges to $p \in X$.

Proof

Let E be the range of $\{p_n\}$.

If E is finite, there is a $p \in E$ and sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $p_{n_1} = p_{n_2} = \dots = p$. Thus, $\{p_{n_k}\} \rightarrow p$.

If E is infinite, then by **theorem 8.3.10**, then there exists a $p \in E'$.

Then there are n_k such that $d(p_{n_k}, p) < \frac{1}{k}$. Thus, $\{p_{n_k}\} \rightarrow p$.

- (b) Every bounded sequence in \mathbb{R}^k contains a convergent subsequence.

Proof

Let E be a bounded sequence in \mathbb{R}^k . Since $E \cup E'$ is bounded and closed, then by **theorem 8.3.13**, $E \cup E'$ is compact.

Thus by part a, E contains a convergent subsequence.

Theorem 10.2.4: The set of subsequential limits is closed

The subsequential limits of $\{p_n\}$ in metric space X form a closed subset of X .

Proof

Let E be the range of the set of all subsequential limits of $\{p_n\}$.

If E is empty, then E is closed. If E is finite, then E' is empty so E is closed.

Suppose E is infinite. Then, let $q \in E'$.

Choose n_1 so $p_{n_1} \neq q$. Let $\frac{\epsilon}{2} = d(p_{n_1}, q)$.

Since $q \in E'$, there is a $x \in E$ where $d(x, q) < \frac{\epsilon}{2}$.

Since $x \in E$, there is a $\{p_{n_k}\} \rightarrow x$ so there is a N such that for $n \geq N$, $d(p_{n_k}, x) < \frac{\epsilon}{2}$.

Thus, $d(p_{n_k}, q) \leq d(p_{n_k}, x) + d(x, q) < \epsilon$ so q is a subsequential limit of $\{p_n\}$.

Thus, $q \in E$ so E is closed.

10.3 Cauchy Sequences

Definition 10.3.1: Metric Spaces

Sequence $\{p_n\} \in X$ is a Cauchy sequence if:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$

Let nonempty $E \subset X$ and $S \subset \mathbb{R}$ of $d(p, q)$ where $p, q \in E$.

Let $\sup(S) = \text{diam}(E)$. If $\{p_n\} \in X$, and $p_N, p_{N+1}, \dots \in E_N$, then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N \rightarrow \infty} \text{diam}(E_N) = 0$.

Theorem 10.3.2: Cauchy sequences and its closure have the same diam

- (a) If $\overline{E} \subset X$, then $\text{diam}(\overline{E}) = \text{diam}(E)$.

Proof

Since $E \subset \overline{E}$, then $\text{diam}(E) \leq \text{diam}(\overline{E})$.

For $\epsilon > 0$, let $p, q \in E'$.

Thus, there are $p', q' \in E$ such that $d(p', p) < \epsilon$ and $d(q', q) < \epsilon$. Thus:
 $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < 2\epsilon + d(p', q') \leq 2\epsilon + \text{diam}(E)$.
 Thus, $\text{diam}(\overline{E}) \leq 2\epsilon + \text{diam}(E)$ so $\text{diam}(\overline{E}) = \text{diam}(E)$.

- (b) If K_n is a sequence of compact sets of X such that $K_{n+1} \subset K_n$ and $\lim_{n \rightarrow \infty} \text{diam}(K_n) = 0$, then $\cap K_n$ consist of only one point.

Proof

Let $K = \cap K_n$. Since K_n is a sequence of compact sets, then by [Corollary 8.3.8](#), K is nonempty.

If K contains more than one point, then $\text{diam}(K) > 0$.

But since $K \subset K_n$, then $\text{diam}(K) \leq \text{diam}(K_n)$ which contradicts that $\text{diam}(K_n) \rightarrow 0$.

Theorem 10.3.3: Cauchy sequences are convergent

- (a) In every metric space, every convergent sequence is a a Cauchy sequence.

Proof

If $p_n \rightarrow p$ and $\epsilon > 0$, there is a N such that for all $n \geq N$, $d(p, p_n) < \frac{\epsilon}{2}$.

Thus, for $m, n \geq N$:

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon.$$

Thus, $\{p_n\}$ is a Cauchy sequence.

- (b) If $\{p_n\}$ is a Cauchy sequence in compact metric space X , then $\{p_n\}$ converges to some $p \in X$.

Proof

Let $\{p_n\}$ be a Cauchy sequence in compact space X .

Let $p_N, p_{N+1}, \dots \in E_N$.

Since $\{p_n\}$ is a Cauchy sequence, then $\lim_{N \rightarrow \infty} \text{diam}(\overline{E_N}) = 0$. Since $\overline{E_N}$ is closed in compact X , then by [theorem 8.3.5](#), $\overline{E_N}$ is compact.

Since $E_{N+1} \subset E_N$, then $\overline{E_{N+1}} \subset \overline{E_N}$ and thus, by [theorem 10.3.2b](#), then there is a unique $p \in \overline{E_N}$ for every N .

Since $p \in \overline{E_N}$, then $d(p, q) < \epsilon$ for every $q \in \overline{E_N}$ so every $q \in E_N$.

Then for $\epsilon > 0$, there is a N_0 such that for $N \geq N_0$, $\text{diam}(\overline{E_N}) < \epsilon$.

Thus, $d(p_n, p) < \epsilon$ for $n \geq N_0$ so $\{p_n\} \rightarrow p$.

(c) In \mathbb{R}^k , every Cauchy sequence converges.

Proof

Let $\{x_n\}$ be a Cauchy sequence in \mathbb{R}^k . Let $x_N, x_{N+1}, \dots \in E_N$. Then for some N , $\text{diam}(E_N) < 1$. Thus, the range of $\{x_n\} = E_N \cup \{x_1, \dots, x_{N-1}\}$. Thus, $\{x_n\}$ is bounded. Thus, the $\overline{\{x_n\}}$ is closed and bounded so by **theorem 8.3.13**, $\overline{\{x_n\}}$ is compact. Thus, by part b, $\{x_n\}$ converges to some $p \in \mathbb{R}^k$.

Definition 10.3.4: Complete

A metric space where every Cauchy sequence converges is complete.

Thus, by **theorem 10.3.3**, all compact and Euclidean metric spaces are complete.

Definition 10.3.5: Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is:

- (a) monotonically increasing if $s_n \leq s_{n+1}$
- (b) monotonically decreasing if $s_n \geq s_{n+1}$

Theorem 10.3.6: Monotonic sequences converge if bounded

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof

Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$.
 Suppose $\{s_n\}$ is bounded.
 Let $s = \sup(E)$ so $s_n \leq s$. For every $\epsilon > 0$, there is a N such that $s - \epsilon < s_N \leq s$ else $s - \epsilon$ would be an upper bound of E which contradicts $s = \sup(E)$.
 Since $\{s_n\}$ increases, then for $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s$ so $\{s_n\} \rightarrow s$.
 Suppose $\{s_n\}$ converges to s .
 Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $s - \epsilon < s_N \leq s_n \leq s$.
 Thus, $\{s_n\}$ is bounded from above.
 Suppose $s_n \geq s_{n+1}$. Let E be the range of $\{s_n\}$.
 Suppose $\{s_n\}$ is bounded.
 Let $s = \inf(E)$ so $s_n \geq s$. For every $\epsilon > 0$, there is a N such that $s \leq s_N < s + \epsilon$ else $s + \epsilon$ would be a lower bound of E which contradicts $s = \inf(E)$.
 Since $\{s_n\}$ decreases, then for $n \geq N$, $s \leq s_n \leq s_N < s + \epsilon$ so $\{s_n\} \rightarrow s$.
 Suppose $\{s_n\}$ converges to s .
 Then for $\epsilon > 0$, there is a N such that for $n \geq N$, $s \leq s_n < s + \epsilon$.
 Thus, $\{s_n\}$ is bounded from below.

11 Limits and Special Sequences

11.1 Upper and Lower Limits

Definition 11.1.1: No limit

Let $\{s_n\}$ be a sequence of real numbers such that:

For every real M , there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \geq M$.

Then, $s_n \rightarrow +\infty$.

For every real M , there is a $N \in \mathbb{Z}$ such that for $n \geq N$, $s_n \leq M$.

Then, $s_n \rightarrow -\infty$.

Definition 11.1.2: Upper and Lower Limits

Let $\{s_n\}$ be a sequence of real numbers.

Let E contain all subsequential limits of $\{s_n\}$ plus possibly $\pm\infty$.

Then, the upper limit of $\{s_n\}$:

$$s^* = \sup(E) \quad \lim_{n \rightarrow \infty} \sup(s_n) = s^*$$

Then, the lower limit of $\{s_n\}$:

$$s_* = \inf(E) \quad \lim_{n \rightarrow \infty} \inf(s_n) = s_*$$

Theorem 11.1.3: Upper and Lower limits are unique

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and s^* be the upper limit of $\{s_n\}$. Then:

(a) $s^* \in E$

Proof

If $s^* = +\infty$, then there is a $\{s_{n_k}\} \rightarrow +\infty$ so E is not bounded above.

If $s^* \in \mathbb{R}$, then E is bounded above so $s^* \in E$.

Then by [theorem 10.2.4](#), $s^* \in E$.

If $s^* = -\infty$, then there are no subsequential limits in E . Thus, for every M , there is a N such that for $n \geq N$, $s_n \leq M$ so $-\infty \in E$.

(b) If $x > s^*$, there is a N such that for $n \geq N$, $s_n < x$

Proof

Suppose there is a $x > s^*$ such that $s_n \geq x$ for infinitely many n .

Then, there is a $y \in E$ where $y \geq x > s^*$ which contradicts $s^* = \sup(E)$.

(c) s^* is the only number that satisfies (a) and (b)

Proof

Suppose p, q satisfy part a and b where $p < q$.

Choose x where $p < x < q$. Since p satisfies b, then $s_n < x$ for $n \geq N$.

Thus, q cannot satisfy a.

The same properties are analogous for s_* .

Theorem 11.1.4: Inf & Sup of $s_n \leq t_n$

If $s_n \leq t_n$ for $n \geq$ fixed N , then

$$\lim_{n \rightarrow \infty} \inf(s_n) \leq \lim_{n \rightarrow \infty} \inf(t_n)$$

$$\lim_{n \rightarrow \infty} \sup(s_n) \leq \lim_{n \rightarrow \infty} \sup(t_n)$$

Proof

Let E_1 be the set of extended reals x such that $\{s_{n_k}\} \rightarrow x$ for some $\{s_{n_k}\}$.

Let E_2 be the set of extended reals y such that $\{t_{n_k}\} \rightarrow y$ for some $\{s_{n_k}\}$.

Let $s^* = \sup(E_1)$, $s_* = \inf(E_1)$, $t^* = \sup(E_2)$, and $t_* = \inf(E_2)$.

Since there is a N such that $s_n \leq t_n$ for $n \geq N$, then:

$$x \leftarrow \{s_N, s_{N+1}, \dots\} \leq \{t_N, t_{N+1}, \dots\} \rightarrow y$$

Thus, for $n \geq N$, $\inf(s_n) \leq \inf(t_n)$ and $\sup(s_n) \leq \sup(t_n)$.

11.2 Special Sequences**Theorem 11.2.1: Special Sequences**

- (a) If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$

Proof

For $\epsilon > 0$, let $N > \sqrt[p]{\frac{1}{\epsilon}}$.

Then for $n \geq N$, $\lim_{n \rightarrow \infty} \frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{\sqrt[p]{\frac{1}{\epsilon}}} = \epsilon$

- (b) If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$

Proof

If $p > 1$, then let $x_n = \sqrt[p]{p} - 1 > 0$.

$$p = (x_n + 1)^n = x_n^n + nx_n^{n-1} + \dots + nx_n + 1 \geq nx_n + 1$$

Thus, $0 < x_n \leq \frac{p-1}{n}$ so $\{x_n\} \rightarrow 0$ and thus, $\{\sqrt[p]{p}\} \rightarrow 1$.

If $p = 1$, then $\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} 1 = 1$.

If $0 < p < 1$, then $\frac{1}{p} > 1$. From the proof above for $p > 1$, $\{\sqrt[p]{\frac{1}{p}}\} \rightarrow 1$.

Thus, $\{\frac{1}{\sqrt[p]{p}}\} \rightarrow 1$ so $\{\sqrt[p]{p}\} \rightarrow 1$.

- (c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$

Proof

Let $x_n = \sqrt[n]{n} - 1 \geq 0$.

$$n = (x_n + 1)^n \geq \frac{n(n-1)}{2} x_n^2$$

Thus, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ so $\{x_n\} \rightarrow 0$ and thus, $\{\sqrt[n]{n}\} \rightarrow 1$.

- (d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$

Proof

Let $k \in \mathbb{Z}$ such that $k > \alpha$ and $k > 0$. For $n > 2k$:

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

Thus, $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k}$.

Since $\alpha - k < 0$, then $\{n^{\alpha-k}\} \rightarrow 0$ so $\{\frac{n^\alpha}{(1+p)^n}\} \rightarrow 0$.

- (e) If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$

Proof

From part d, let $\alpha = 0$.

Thus, $\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = 0$ and since $p > 0$, then $\frac{1}{(1+p)^n} = (\frac{1}{1+p})^n < 1$.

Also, $-\lim_{n \rightarrow \infty} \frac{1}{(1+p)^n} = \lim_{n \rightarrow \infty} \frac{-1}{(1+p)^n} = 0$ so $\frac{-1}{(1+p)^n} = (\frac{-1}{1+p})^n > -1$.

12 Series and Comparison Test

12.1 Series

Definition 12.1.1: Series

For sequence $\{a_n\}$, define $\sum_{n=p}^q a_n = a_p + a_{p+1} + \dots + a_q$.

Then associate $\{a_n\}$ with a sequence $\{s_n\}$ such that $s_n = \sum_{k=1}^n a_k$.

Then $\{s_n\}$ is a series with partial sums s_n .

If $\{s_n\} \rightarrow s$, then $\sum_{n=1}^{\infty} a_n = s$ is the sum of the convergent series.

Note $a_1 = s_1$ and $a_n = s_n - s_{n-1}$.

Theorem 12.1.2: Cauchy Criterion Redefined

$\sum a_n$ converges if and only if:

For every $\epsilon > 0$, there is a $N \in \mathbb{Z}$ such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$

Proof

Suppose $\sum_{k=1}^n a_k$ converges.

Then by [theorem 10.3.3a](#), $\sum_{k=1}^n a_k$ is a Cauchy sequence.

Then for $\epsilon > 0$, there is a N such that for $m \geq n \geq N$:

$$d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = |\sum_{k=n}^m a_k| \leq \epsilon$$

Suppose for every $\epsilon > 0$, there is a N such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$.

$$|\sum_{k=n}^m a_k| = |\sum_{k=1}^m a_k - \sum_{k=1}^n a_k| = d(\sum_{k=1}^n a_k, \sum_{k=1}^m a_k) \leq \epsilon$$

Thus, $\sum_{k=1}^n a_k$ is a Cauchy sequence and thus, convergent.

Theorem 12.1.3: Convergent $\sum a_n \Rightarrow \{a_n\} \rightarrow 0$

If $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof

Since $\sum a_n$ converges, then by [theorem 12.1.2](#), for $\epsilon > 0$, there is a N such that for $m \geq n \geq N$, $|\sum_{k=n}^m a_k| \leq \epsilon$. Then if $m = n \geq N$, $|\sum_{k=n}^m a_k| = |a_n| \leq \epsilon$ so $\{a_n\} \rightarrow 0$.

Example 12.1.4: $\{a_n\} \rightarrow 0 \not\Rightarrow$ Convergent $\sum a_n$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \dots + \frac{1}{8}) + (\frac{1}{9} + \dots) \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

Thus, $s_{2^k} = \sum_{n=1}^{2^k} a_n \geq 1 + k \cdot \frac{1}{2}$ which is unbounded and thus, not convergent.

Theorem 12.1.5: Convergent series \Leftrightarrow Bounded sequence

A series of nonnegative terms converge if and only if its partial sums form a bounded sequence.

Proof

Suppose $\sum a_n$ converges where $a_n \geq 0$.

Since $a_n \geq 0$, then $\{s_n\}$ is monotonic so by [theorem 10.3.6](#), $\{s_n\}$ is bounded above.

Suppose $\{s_n\}$ is bounded where $a_n \geq 0$.

Since $\{s_n\}$ is monotonic and bounded, then by [theorem 10.3.6](#), $\{s_n\}$ converges.

Theorem 12.1.6: Comparison Test

- (a) If $|a_n| \leq c_n$ for $n \geq N_0$ and $\sum c_n$ converges, then $\sum a_n$ converges.

Proof

For $\epsilon > 0$, there exists a $N \geq N_0$ such that for $m \geq n \geq N$, $\sum_{k=n}^m c_k \leq \epsilon$.
 $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \epsilon$
 Thus, $\sum a_n$ converges.

- (b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$ and $\sum d_n$ diverges, then $\sum a_n$ diverges.

Proof

Suppose $\sum a_n$ converges.
 Then from part a, $\sum d_n$ converges which contradicts that $\sum a_n$ diverges.
 Thus, $\sum a_n$ diverges.

12.2 Series of Nonnegative Terms**Theorem 12.2.1: Infinite Geometric Series**

If $x \in [0,1)$, then:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geq 1$, the series diverges.

Proof

If $x \neq 1$, then using the geometric series:

$$s_n = \sum_{k=0}^n x^k = \frac{1-x^{n+1}}{1-x}$$

Let $n \rightarrow \infty$.

If $x \in [0,1)$, then by **theorem 11.2.1e**, $s_n = \frac{1}{1-x} (1 - x^{n+1}) = \frac{1}{1-x} (1 - 0) = \frac{1}{1-x}$.

Also, by **theorem 11.2.1e**, if $x \geq 1$, then the series diverges.

Theorem 12.2.2: Cauchy's Convergence Criterion

Suppose $0 \leq a_{i+1} \leq a_i$.

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots \text{ converges.}$$

Proof

Let $s_n = a_1 + a_2 + \dots + a_n$ and $t_k = a_1 + 2a_2 + \dots + 2^k a_{2^k}$.

For $n < 2^k$:

$$\begin{aligned} s_n &\leq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \dots + a_{2^k} \\ &\leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k} + \dots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \dots + 2^k a_{2^k} = t_k \end{aligned}$$

Thus, by the **comparison test**, if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

For $n > 2^k$:

$$\begin{aligned} s_n &\geq a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots + a_{2^k} \\ &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \dots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k \end{aligned}$$

Thus, by the **comparison test**, if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem 12.2.3: P-series

$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof

If $p \leq 0$, then by **theorem 12.1.3**, $\sum \frac{1}{n^p}$ diverges.
 If $p > 0$, then by **theorem 12.2.2**, $\sum \frac{1}{n^p}$ converges only if $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges.
 Since $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^{k(1-p)}$, then by **theorem 12.2.1**, $\sum_{k=0}^{\infty} 2^{k(1-p)}$ converges if $2^{1-p} < 1$ so if $1-p < 0$ so $p > 1$.

Theorem 12.2.4: Log P-series

$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof

Since $\frac{1}{n(\log(n))^p}$ decreases, then by **theorem 12.2.2**,
 $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p}$ converges.
 $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k \log(2)^p} = \frac{1}{\log(2)^p} \sum_{k=1}^{\infty} \frac{1}{k}$
 Then by **theorem 12.2.3**, $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.
 Thus, $\sum_{n=0}^{\infty} \frac{1}{n(\log(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Corollary 12.2.5: Log P-series extended

$\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof

From **theorem 12.2.4**, replace $n = \log(n)$ and multiplying by $\frac{1}{n} \rightarrow \frac{1}{n \log(n)(\log(\log(n)))^p}$.
 Since $\frac{1}{n \log(n)(\log(\log(n)))^p}$ decreases, by **theorem 12.2.2** $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k \log(2^k)(\log(\log(2^k)))^p}$:
 $\sum_{k=1}^{\infty} \frac{1}{\log(2^k)(\log(\log(2^k)))^p} = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{1}{k(\log(k \log(2)))^p} < \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$
 Since $\sum_{k=2}^{\infty} \frac{1}{k(\log(k))^p}$ converges by **theorem 12.2.4**, $\sum_{n=3}^{\infty} \frac{1}{n \log(n)(\log(\log(n)))^p}$ converges.

12.3 The Number e**Definition 12.3.1: Summation equivalence to e**

$$s_m = \sum_{n=0}^m \frac{1}{n!} = 1 + \sum_{n=1}^m \frac{1}{n!} < 1 + \sum_{n=1}^m \frac{1}{2^{n-1}} < 3$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Theorem 12.3.2: Limit equivalence to e

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Proof

Let $s_n = \sum_{k=0}^n \frac{1}{k!}$ and $t_n = \left(1 + \frac{1}{n}\right)^n$. Using the binomial theorem:
 $t_n = \sum_{k=0}^n \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^n \frac{n(n-1)\dots(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$
 Thus, $t_n \leq s_n$ so $\lim_{n \rightarrow \infty} \sup(t_n) \leq e$.
 If $n \geq m$, then $t_n \geq \sum_{k=0}^m \frac{1}{k!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right)$.
 As $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} \inf(t_n) \geq \sum_{k=0}^m \frac{1}{k!} = s_m$. As $m \rightarrow \infty$, $\lim_{n \rightarrow \infty} \inf(t_n) \geq e$.

Definition 12.3.3: Rapidity of convergence of e

$$0 < e - s_n < \frac{1}{n!n}$$

Proof

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) = \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!n}$$

Theorem 12.3.4: e is irrational

e is irrational

Proof

Suppose r is rational. Then let $e = \frac{p}{q}$ for $p, q \in \mathbb{Z}_+$.
 Thus, by [definition 12.3.3](#), $0 < e - s_q < \frac{1}{q!q}$ so $0 < q!(e - s_q) < \frac{1}{q}$.
 Since $e = \frac{p}{q}$, then $q!e$ is an integer and $q!s_q = q!(1 + \frac{1}{2!} + \dots + \frac{1}{q!})$ is an integer.
 Thus, $q!(e - s_q)$ is an integer which is between 0 and $\frac{1}{q}$ and thus, a contradiction.

12.4 Root and Ratio Tests**Theorem 12.4.1: Root Test**

For $\sum a_n$, let $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|})$.

- (a) If $\alpha < 1$, $\sum a_n$ converges
- (b) If $\alpha > 1$, $\sum a_n$ diverges
- (c) If $\alpha = 1$, unclear

Proof

If $\alpha < 1$, choose β such that $\beta \in (\alpha, 1)$ and $N \in \mathbb{Z}$ such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$.
 Since $\beta \in (0, 1)$, then by [theorem 12.2.1](#), $\sum \beta^n$ converges. Then by the [comparison test](#), $\sum a_n$ converges.
 If $\alpha > 1$, then there is a a_{n_k} such that $\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha$.
 Thus, $|a_n| > 1$ for infinitely many n so by [theorem 12.1.3](#), $\sum a_n$ doesn't converge.
 For $\alpha = 1$, both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ have $\alpha = 1$, but $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges by [theorem 12.2.3](#).

Theorem 12.4.2: Ratio Test

- (a) $\sum a_n$ converges if $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$
- (b) $\sum a_n$ diverges if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq n_0$ for $n_0 \in \mathbb{Z}$

Proof

If $\lim_{n \rightarrow \infty} \sup(|\frac{a_{n+1}}{a_n}|) < 1$, there is a $\beta < 1$ and N such that for $n \geq N$, $|\frac{a_{n+1}}{a_n}| < \beta$.
 Then $|a_{N+1}| < \beta|a_N|$ so $|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N|$.
 Thus, $|a_{N+p}| < \beta^p|a_N|$ so $|a_n| < |a_N|\beta^{-N}\beta^n$.
 Thus, by the [comparison test](#), $\sum a_n$ converges.
 If $|a_{n+1}| \geq |a_n| > 0$ for $n \geq n_0$, then by [theorem 12.1.3](#), $\sum a_n$ diverges.

Theorem 12.4.3: Ratio convergence \rightarrow Root convergence

$$\lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n}) \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$$

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$$

Proof

Let $\alpha = \lim_{n \rightarrow \infty} \inf(\frac{c_{n+1}}{c_n})$. If $\alpha = -\infty$, then $-\infty \leq \lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n})$ holds true.
 If α is finite, there is a $\beta \leq \alpha$ and N such that for $n \geq N$, $\frac{c_{n+1}}{c_n} \geq \beta$ so $c_{N+p} \geq \beta^p c_N$.
 Then, $c_n \geq c_N \beta^{-N} \beta^n$ so $\sqrt[n]{c_n} \geq \sqrt[n]{c_N \beta^{-N} \beta^n}$. Thus, $\lim_{n \rightarrow \infty} \inf(\sqrt[n]{c_n}) \geq \beta = \alpha$.
 Let $\alpha = \lim_{n \rightarrow \infty} \sup(\frac{c_{n+1}}{c_n})$. If $\alpha = \infty$, then $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \infty$ holds true.
 If α is finite, there is a $\beta \geq \alpha$ and N such that for $n \geq N$, $\frac{c_{n+1}}{c_n} \leq \beta$ so $c_{N+p} \leq \beta^p c_N$.
 Then, $c_n \leq c_N \beta^{-N} \beta^n$ so $\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N} \beta^n}$. Thus, $\lim_{n \rightarrow \infty} \sup(\sqrt[n]{c_n}) \leq \beta = \alpha$.

13 Power Series

13.1 Power Series

Definition 13.1.1: Power Series

For a sequence $\{c_n\} \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is a power series.
 c_n are the coefficients and $z \in \mathbb{C}$.

Theorem 13.1.2: Radius of Convergence

For power series $\sum c_n z^n$, let $\alpha = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|})$ and $R = \frac{1}{\alpha}$.
 Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Proof

Let $a_n = c_n z^n$. Using the **root test**,

$$\lim_{n \rightarrow \infty} \sup(\sqrt[n]{|a_n|}) = \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n z^n|})$$

$$= |z| \lim_{n \rightarrow \infty} \sup(\sqrt[n]{|c_n|}) = \frac{|z|}{R}$$
 Thus, $\sum c_n z^n$ converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$

13.2 Summation By Parts

Theorem 13.2.1: Summation By Parts

For sequences $\{a_n\}$, $\{b_n\}$, let $A_n = \sum_{k=0}^n a_k$. Then for $0 \leq p \leq q$:

$$\sum_{n=p}^q a_n b_n = (\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p$$

Proof

$$\begin{aligned} \sum_{n=p}^q a_n b_n &= \sum_{n=p}^q (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^q A_n b_n - \sum_{n=p}^q A_{n-1} b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\ &= \sum_{n=p}^{q-1} A_n b_n - \sum_{n=p}^{q-1} A_n b_{n+1} + A_q b_q - A_{p-1} b_p \\ &= (\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p \end{aligned}$$

Theorem 13.2.2: Conditions for convergent $\sum a_n b_n$

Suppose for $\{a_n\}$, $\{b_n\}$:

- partial sums A_n of $\sum a_n$ form a bounded sequence
- $b_i \geq b_{i+1}$
- $\lim_{n \rightarrow \infty} b_n = 0$

Then $\sum a_n b_n$ converges.

Proof

Since $\{A_n\}$ is bounded, $|A_n| \leq M$ for all n .
 Since $\{b_n\}$ is monotonically decreasing and $\lim_{n \rightarrow \infty} b_n = 0$, then for $\epsilon > 0$, there is a N such that $b_N \leq \frac{\epsilon}{2M}$. Then for $N \leq p \leq q$:

$$\begin{aligned} |\sum_{n=p}^q a_n b_n| &= |(\sum_{n=p}^{q-1} A_n (b_n - b_{n+1})) + A_q b_q - A_{p-1} b_p| \\ &\leq M |\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p| = 2M b_p \leq 2M b_N \leq \epsilon \end{aligned}$$

Corollary 13.2.3: Convergent Series of Alternating Sequences

Suppose for $\{c_n\}$:

- $|c_i| \geq |c_{i+1}|$
- $c_{2i-1} \geq 0$ and $c_{2i} \leq 0$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then $\sum c_n$ converges.

Proof

From **theorem 13.2.2**, let $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

Corollary 13.2.4: Convergent Power Series

Suppose for $\{c_n\}$:

- Radius of convergence of $\sum c_n z^n$ is 1
- $c_i \geq c_{i+1}$
- $\lim_{n \rightarrow \infty} c_n = 0$

Then $\sum c_n z^n$ converges at every point where $|z| = 1$ except possibly $z = 1$.

Proof

From **theorem 13.2.2**, let $a_n = z^n$ and $b_n = c_n$.
 A_n of $\sum a_n$ form a bounded sequence since $|A_n| = |\sum_0^n z^n| = \left| \frac{1-z^{n+1}}{1-z} \right| \leq \frac{2}{|1-z|}$.

13.3 Absolute Convergence**Definition 13.3.1: Absolute Convergence**

$\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ converges non-absolutely.

Theorem 13.3.2: Absolute convergence \rightarrow convergence

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof

Since $\sum a_n$ converges absolutely, then for every $\epsilon > 0$, there is an integer N such that for $m \geq n \geq N$, $|\sum_{k=n}^m |a_k|| = \sum_{k=n}^m |a_k| \leq \epsilon$.
 Thus, $|\sum_{k=n}^m a_k| \leq \sum_{k=n}^m |a_k| \leq \epsilon$ so $\sum a_n$ converges.

13.4 Addition & Multiplication of Series**Theorem 13.4.1: Addition and Scalar Multiplication**

If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$ and $\sum ca_n = cA$.

Proof

Let $A_n = \sum_{k=0}^n a_k$ and $B_n = \sum_{k=0}^n b_k$.
 Then $A_n + B_n = \sum_{k=0}^n a_k + b_k$ so $\lim_{n \rightarrow \infty} A_n + B_n = A + B$.
 Then $\lim_{n \rightarrow \infty} cA_n = \underbrace{A + \dots + A}_c = cA$

Definition 13.4.2: Cauchy Product

For $\sum a_n$ and $\sum b_n$, let $c_n = \sum_{k=0}^n a_k b_{n-k}$ and the product as $\sum c_n$.

$$\begin{aligned} \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n) (b_0 + b_1 z + b_2 z^2 + \dots + b_n z^n) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \end{aligned}$$

Theorem 13.4.3: Conditions $\sum c_n = AB$

Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(b) $\sum_{n=0}^{\infty} a_n = A$

(c) $\sum_{n=0}^{\infty} b_n = B$

(d) $c_n = \sum_{k=0}^{\infty} a_k b_{n-k}$

Then $\sum_{n=0}^{\infty} c_n = AB$.**Proof**Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$, and $\beta_n = B_n - B$.

$$\begin{aligned}
C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + \dots + a_n b_0) \\
&= a_0 B_n + a_1 B_{n-1} + \dots + a_n B_0 \\
&= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \dots + a_n (B + \beta_0) \\
&= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0
\end{aligned}$$

Let $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \dots + a_n \beta_0$ so $C_n = A_n B + \gamma_n$.Since a_n converges absolutely, then $\sum_{n=0}^{\infty} |a_n| = \alpha$.Since $\sum_{n=0}^{\infty} b_n = B$, then $\beta_n \rightarrow 0$.Then for $\epsilon > 0$, there is a N such that $|\beta_n| \leq \frac{\epsilon}{\alpha}$ for $n \geq N$.

$$\begin{aligned}
|\gamma_n| &\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \dots + \beta_n a_0| \\
&\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + |a_{n-N-1} + \dots + a_0| \frac{\epsilon}{\alpha} \\
&\leq |\beta_0 a_n + \dots + \beta_N a_{n-N}| + \alpha \frac{\epsilon}{\alpha}
\end{aligned}$$

Thus, with a fixed N , since $a_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} |\gamma_n| \leq \epsilon$ so $\lim_{n \rightarrow \infty} \gamma_n = 0$.Thus, $\lim_{n \rightarrow \infty} C_n = \lim_{n \rightarrow \infty} A_n B + \gamma_n = AB$.**Theorem 13.4.4: By Cauchy Product, $\sum c_n = C$ implies $C = AB$** If $\sum a_n = A$, $\sum b_n = B$, $\sum c_n = C$ where $c_n = a_0 b_n + \dots + a_n b_0$, then $C = AB$.**13.5 Rearrangements****Definition 13.5.1: Rearrangements**Let $a'_n = a_{k_n}$. Then $\sum a'_n$ is a rearrangement of $\sum a_n$.**Theorem 13.5.2: Rearrangements can converge or diverge**Let $\sum a_n \in \mathbb{R}$ converge non-absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$.Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that:

$$\lim_{n \rightarrow \infty} \inf(s'_n) = \alpha \quad \lim_{n \rightarrow \infty} \sup(s'_n) = \beta$$

Proof

Let $p_n = \frac{|a_n| + a_n}{2}$ and $q_n = \frac{|a_n| - a_n}{2}$. Since $\sum |a_n|$ diverge, then $\sum p_n$ and $\sum q_n$ diverges. Let P_1, P_2, P_3, \dots be the nonnegative terms of $\sum a_n$ in order and Q_1, Q_2, Q_3, \dots be the absolute values of the negative terms of $\sum a_n$ in order. Thus, $\sum P_n$ and $\sum Q_n$ differ from $\sum p_n$ and $\sum q_n$ only by the zero terms and thus, are divergent.

Choose real-valued sequences $\{\alpha_n\} \rightarrow \alpha$, $\{\beta_n\} \rightarrow \beta$ such that $\alpha_n < \beta_n$ and $\beta_1 > 0$.Let m_1, k_1 be the smallest integers such that:

$$P_1 + \dots + P_{m_1} > \beta_1 \quad P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} < \alpha_1$$

Let m_2, k_2 be the smallest integers such that:

$$\begin{aligned}
P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} &< \beta_2 \\
P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} &< \alpha_2
\end{aligned}$$

Continuing such a process, then $\lim_{n \rightarrow \infty} \inf(s'_n) = \alpha$ and $\lim_{n \rightarrow \infty} \sup(s'_n) = \beta$.

Theorem 13.5.3: Absolute rearrangements converges uniquely

If $\sum a_n \in \mathbb{C}$ converges absolutely, then every rearrangement of $\sum a_n$ converges to the same sum.

Proof

Let $\sum a'_n$ be a rearrangement with partial sums s'_n .
For $\epsilon > 0$, there is a N such that for $m \geq n \geq N$, $\sum_{i=n}^m |a_i| \leq \epsilon$.
Let p be the maximum index of $\{a_1, a_2, \dots, a_N\}$ in a'_n and a_n .
Since if $n > p$, then a_1, a_2, \dots, a_N will cancel in $s_n - s'_n$ and thus, $|s_n - s'_n| \leq \epsilon$.
Thus, every $\{s'_n\}$ converges to $\{s_n\}$.

14 Continuity

14.1 Limits of Functions

Definition 14.1.1: Limits of Functions

For metric spaces X, Y , let $E \subset X$, f maps E into Y , and $p \in E'$.

Then $\lim_{x \rightarrow p} f(x) = q$ if there is a $q \in Y$ such that:

For every $\epsilon > 0$, there is a $\delta > 0$ where $d_Y(f(x), q) < \epsilon$
for all $x \in E$ where $d_X(x, p) < \delta$

Theorem 14.1.2: Sequence definition of $\lim_{x \rightarrow p} f(x) = q$

$\lim_{x \rightarrow p} f(x) = q$ if and only if $\lim_{n \rightarrow \infty} f(p_n) = q$ for every sequence $\{p_n\} \in E$
where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Proof

Suppose $\lim_{x \rightarrow p} f(x) = q$. For $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ if $x \in E$ and $d_X(x, p) < \delta$.

Choose $\{p_n\} \in E$ such that $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Thus, for $n > N$, $d_Y(f(p_n), q) < \epsilon$.

Suppose $\lim_{x \rightarrow p} f(x) \neq q$. Then there is a $\epsilon > 0$ such that for every $\delta > 0$, there is a $x \in E$ where $d_Y(f(x), q) \geq \epsilon$, but $d_X(x, p) < \delta$. Let $\delta_n = \frac{1}{n}$ and thus, there is a $\{p_n\}$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$, but $\lim_{n \rightarrow \infty} f(p_n) \neq q$.

Corollary 14.1.3: A limit of a function is unique

If f has a limit at p , this limit is unique.

Proof

If $\lim_{x \rightarrow p} f(x) = q$, then by **theorem 14.1.2**, $\lim_{n \rightarrow \infty} f(p_n) = q$ for every $\{p_n\} \in E$
where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$.

Thus, if there exists $\lim_{x \rightarrow p} f(x) = q'$, then there is a $\{p_n\} \in E$ where $p_n \neq p$ and $\lim_{n \rightarrow \infty} p_n = p$, but $\lim_{n \rightarrow \infty} f(p_n) = q'$ which is a contradiction.

Theorem 14.1.4: Arithmetic operations on functions of limits

Let $E \subset X$, $p \in E'$, and $f(x), g(x) \in \mathbb{C}$ so $\lim_{x \rightarrow p} f(x) = A$, $\lim_{x \rightarrow p} g(x) = B$.

$$(a) \lim_{x \rightarrow p} (f + g)(x) = A + B$$

$$(b) \lim_{x \rightarrow p} (fg)(x) = AB$$

$$(c) \lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$$

14.2 Continuous Functions

Definition 14.2.1: Continuous functions on a set

Suppose X, Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y .

f is continuous at p if:

For every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$
for all $x \in E$ where $d_X(x, p) < \delta$.

$f(p)$ have to be defined to be continuous.

If f is continuous at every $p \in E$, then f is continuous on E .

f is continuous at isolated points since regardless of ϵ , there is a $\delta > 0$ such
that $d_X(x, p) < \delta$ is $x = p$ so $d_Y(f(x), f(p)) = 0 < \epsilon$.

Theorem 14.2.2: Continuity at limit points

Suppose $E \subset X$, $p \in E$, and f maps E into Y . Let $p \in E'$.

Then f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.

Proof

If f is continuous at p , then for every $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all $x \in E$ where $d_X(x, p) < \delta$. Thus, $\lim_{x \rightarrow p} f(x) = f(p)$.

If $\lim_{x \rightarrow p} f(x) = f(p)$, then for every $\epsilon > 0$, there is a $\delta > 0$ where $d_Y(f(x), f(p)) < \epsilon$ for all $x \in E$ where $d_X(x, p) < \delta$. Thus, f is continuous at p .

Theorem 14.2.3: Continuity with Chain Rule

Suppose $E \subset X$, $f: E \rightarrow Y$, $g: f(E) \rightarrow Z$, and $h: E \rightarrow Z$ where $h(x) = g(f(x))$.

If f is continuous at p and g is continuous at $f(p)$, then h is continuous at p .

Proof

Since g is continuous at $f(p)$, then for $\epsilon > 0$, there is a δ_1 such that:

$$d_Z(g(y), g(f(p))) < \epsilon \text{ for } d_Y(y, f(p)) < \delta_1 \text{ where } y \in f(E)$$

Since f is continuous at p , there is a $\delta_2 > 0$ such that:

$$d_Y(f(x), f(p)) < \delta_1 \text{ for } d_X(x, p) < \delta_2 \text{ where } x \in E$$

Thus:

$$d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \epsilon \text{ for } d_X(x, p) < \delta_2 \text{ where } x \in E$$

Thus, h is continuous at p .

Theorem 14.2.4: Continuous functions map open sets to open sets

$f: X \rightarrow Y$ is continuous on X if and only if:

$$f^{-1}(V) \text{ is open in } X \text{ for every open set } V \text{ in } Y.$$

Proof

Suppose f is continuous on X and V is an open set in Y .

Suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\epsilon > 0$ such that $y \in V$ if $d_Y(y, f(p)) < \epsilon$. Since f is continuous at p , there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for $d_X(x, p) < \delta$. Thus, $x \in f^{-1}(V)$ for $d_X(x, p) < \delta$.

Suppose $f^{-1}(V)$ is open in X for every open V in Y .

Fix $p \in X$ and $\epsilon > 0$. Let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \epsilon$ so V is open and thus, $f^{-1}(V)$ is open. Thus, there exists $\delta > 0$ such that $x \in f^{-1}(V)$ for $d_X(x, p) < \delta$. Since $x \in f^{-1}(V)$, then $f(x) \in V$ so $d_Y(f(x), f(p)) < \epsilon$.

Corollary 14.2.5: Continuous functions map closed sets to closed sets

$f: X \rightarrow Y$ is continuous on X if and only if:

$$f^{-1}(C) \text{ is closed in } X \text{ for every closed set } C \text{ in } Y.$$

Proof

By **theorem 14.2.4**, f is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y . Let $C = V^c$. Since V is open, then C is closed.

Since $f^{-1}(C) = f^{-1}(V^c) = (f^{-1}(V))^c$, then $f^{-1}(C)$ is closed since $f^{-1}(V)$ is open.

Theorem 14.2.6: Continuous functions

Let f, g be complex continuous functions on X .

Then $f+g$, fg , and $\frac{f}{g}$ where $g \neq 0$ for all $x \in X$ are continuous on X .

Proof

If x is an isolated point, $f+g$, fg , and $\frac{f}{g}$ are continuous by definition. If x is a limit point, then by **theorems 14.1.4 and 14.2.2**, $f+g$, fg , and $\frac{f}{g}$ are continuous since

- $\lim_{x \rightarrow p} (f + g)(x) = \lim_{x \rightarrow p} f(x) + \lim_{x \rightarrow p} g(x) = f(p) + g(p)$
- $\lim_{x \rightarrow p} (fg)(x) = \lim_{x \rightarrow p} f(x) \lim_{x \rightarrow p} g(x) = f(p)g(p)$
- $\lim_{x \rightarrow p} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \rightarrow p} f(x)}{\lim_{x \rightarrow p} g(x)} = \frac{f(p)}{g(p)}$

Theorem 14.2.7: Continuous Functions on \mathbb{R}^k

- (a) Let f_1, \dots, f_k be real functions on X and $f: X \rightarrow \mathbb{R}^k$ where $f(x) = (f_1(x), \dots, f_k(x))$ for $x \in X$.

Then f is continuous if and only if f_1, \dots, f_k are continuous.

- (b) If f and g are continuous mappings of X into \mathbb{R}^k , then $f+g$ and $f \cdot g$ are continuous on X .

Proof

Since $|f_i(x) - f_i(y)| \leq |f(x) - f(y)|$, then if f is continuous, then each f_i is continuous and vice versa.

By **theorem 14.2.6**, $f+g$ and $f \cdot g$ are continuous since each f_i, g_i are continuous by part a.

Thus, every polynomial, rational, and absolute value function is continuous since polynomials are $x_1 \cdot \dots \cdot x_k$ where each x_i is continuous, rationals are polynomials divided by polynomials, and $||x| - |y|| \leq |x - y|$ implies $|x|$ is continuous.

15 Properties of Continuity

15.1 Continuity and Compactness

Definition 15.1.1: Bounded Functions

$f: E \rightarrow \mathbb{R}^k$ is bounded if there is a real number M such that $f(x) \leq M$ for all $x \in E$.

Theorem 15.1.2: Continuous functions from compact spaces are compact

Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Proof

Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, then by [theorem 14.2.4](#), each $f^{-1}(V_\alpha)$ is open. Since X is compact, there is a n such that $X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n})$. Thus, $f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ so $f(X)$ is compact.

Theorem 15.1.3: Continuous functions from compact to \mathbb{R}^k are bounded

If f is a continuous mapping of a compact metric space X into \mathbb{R}^k , then $f(X)$ is closed and bounded.

Proof

By [theorem 15.1.2](#), $f(X)$ is compact. Then by [theorem 8.3.13](#), $f(X)$ is closed and bounded.

Theorem 15.1.4: Generalized extreme value theorem

Suppose f is a continuous real function of a compact metric space X such that $M = \sup_{p \in X} f(p)$ and $m = \inf_{p \in X} f(p)$. Then there exists $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proof

By [theorem 15.1.3](#), $f(X)$ is closed and bounded. Thus, $f(X)$ contains $M = \sup f(X)$ and $m = \inf f(X)$ for some $p, q \in X$.

Theorem 15.1.5: If f is continuous 1-1, then f^{-1} is continuous

Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y . Then f^{-1} is a continuous mapping of Y onto X .

Proof

Let V be an open set in X . Since V^c is closed and $V^c \subset$ compact set X , then by [theorem 8.3.5](#), V^c is compact. Thus by [theorem 15.1.2](#), $f(V^c)$ is a compact subset of Y so $f(V^c)$ is closed. Since f is 1-1 and onto, $f(V^c) = f(V)^c$ so $f(V)$ is open. Since from any open set V in X , $f(V)$ is open in Y , then by [theorem 14.2.4](#), f^{-1} is continuous.

Definition 15.1.6: Uniformly Continuous

Let $f: X \rightarrow Y$. Then f is uniformly continuous on X if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all $p, q \in X$ where $d_X(p, q) < \delta$.

Theorem 15.1.6: Continuous functions from compact are uniformly continuous

Let f be a continuous mapping of a compact metric space X into metric space Y . Then f is uniformly continuous on X .

Proof

For $\epsilon > 0$, since f is continuous, then for each $p \in X$, there is a $\phi(p)$ such that for all $q \in X$ where $d_X(q, p) < \phi(p)$, $d_Y(f(q), f(p)) < \frac{\epsilon}{2}$.

Let $J(p)$ be the set of all $q \in X$ where $d_X(q, p) < \frac{1}{2}\phi(p)$.

Since the set of all $J(p)$ is an open cover of X and since X is compact, then there is a n such that $X \subset J(p_1) \cup \dots \cup J(p_n)$. Let $\delta = \frac{1}{2} \min(\phi(p_1), \dots, \phi(p_n)) > 0$.

Then for $p, q \in X$ where $d_X(p, q) < \delta$, there is a m where $1 \leq m \leq n$ such that $p \in J(p_m)$ so $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$. Thus:

$$\begin{aligned} d_X(q, p_m) &\leq d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \leq \phi(p_m) \\ d_Y(f(p), f(q)) &\leq d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Theorem 15.1.7: Continuous functions from noncompact \nrightarrow uniformly continuous

Let E be a noncompact set in \mathbb{R}^1 .

- (a) There exists a continuous function which is not bounded.
- (b) There exists a continuous, bounded function which has no maximum.
- (c) If E is bounded, there exists a continuous function which is not uniformly continuous.

Proof

Suppose E is bounded so there is a $x_0 \in E'$, but $x_0 \notin E$.

Consider $f(x) = \frac{1}{x-x_0}$ which is continuous on E , but unbounded.

For $\epsilon > 0$ and $\delta > 0$, there is a $x \in E$ such that $|x - x_0| < \delta$. Take t close enough to x_0 so $|f(t) - f(x_0)| > \epsilon$, but $|t - x| < \delta$. Thus, f is not uniformly continuous.

Consider $g(x) = \frac{1}{1+(x-x_0)^2}$ which is continuous on E and bounded since $g(x) \in (0,1)$. Since $\sup_{x \in E} g(x) = 1$, but $g(x) < 1$ for all $x \in E$, then g has no maximum on E .

15.2 Continuity and Connectedness**Theorem 15.2.1: Continuous functions map connected to connected**

If f is a continuous mapping of X into Y and E is a connected subset of X , then $f(E)$ is connected.

Proof

Suppose $f(E) = A \cup B$ where A and B are nonempty separated subsets of Y .

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. Then $E = G \cup H$.

Since $A \subset \overline{A}$, $G \subset f^{-1}(\overline{A})$. Since f is continuous, then $f^{-1}(\overline{A})$ is closed so $\overline{G} \subset f^{-1}(\overline{A})$. Thus, $f(\overline{G}) \subset \overline{A}$.

Since $f(H) = B$ and $\overline{A} \cap B$ is empty, $\overline{G} \cap H$ is empty. Similarly, $G \cap \overline{H}$ is empty so G and H are separated which contradicts that $E = G \cup H$ is connected.

Theorem 15.2.2: Generalized intermediate Value Theorem

Let f be a continuous real function on $[a,b]$. If $f(a) < c < f(b)$, then there exists $x \in (a,b)$ such that $f(x) = c$.

Proof

Since $[a,b]$ is connected, then by **theorem 15.2.1**, $f([a,b])$ is a connected subset of \mathbb{R}^1 . Thus, by **theorem 9.2.2**, any c where $f(a) < c < f(b)$ is $c \in f(x)$ for some $x \in [a,b]$.

16 Discontinuities

References