

Fall Real Analysis

Willie Xie

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1 The Real Number System

1.1 Number Systems

Natural : $\mathbb{N} = \{1, 2, 3, \dots\}$

Integer : $\mathbb{Z} = \{-2, -1, 0, 1, 2, \dots\}$

Rational : $\mathbb{Q} = \frac{p}{q}$ where $p, q \in \mathbb{N}$

*** \mathbb{Q} is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.

Let set $A = \{x \in \mathbb{Q} \text{ for } x^2 < 2\}$ where $A \neq \emptyset$ and 2 is an upper bound for A . But, A has no least upper bound in \mathbb{Q} , but A has a least upper bound in \mathbb{R} .

1.2 Real Number System

\mathbb{R} is the unique ordered field with the least upper bound property.
Also, \mathbb{R} exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- **Trichotomy**: For all $x, y \in S$, only one holds true:
 - $x < y$
 - $x = y$
 - $x > y$
- **Transitivity**: If $x < y$ and $y < z$, then $x < z$.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and $E \subset S$.

An upper bound of E is a $\beta \in S$ if $x \leq \beta$ for all $x \in E$.

If such a β exists, then E is bounded from above.

A lower bound of E is a $\alpha \in S$ if $x \geq \alpha$ for all $x \in E$.

If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.

Let $E \subset S$ be bounded from above. Least upper bound $\beta \in S$ exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E .

Then $\beta = \sup(E)$.

Let $E \subset S$ be bounded from below. Greatest lower bound $\alpha \in S$ exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E .

Then $\alpha = \inf(E)$.

Example 1.2.5

Let $S = (1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from \mathbb{R} . For subsets E of S :

- $E = (1, 2)$ is bounded above and $\sup(E) = 2$
- $E = (5, 6)$ is not bounded above so $\sup(E) = \text{DNE}$
- $E = [3, 4)$ is bounded below $\inf(E) = 3$ and $\sup(E) = \text{DNE}$

Observations on the Least Upper Bound

If $\sup(E)$ exists, it may or may not exist in S .

If $\sup(E)$ exists, then $\sup(E)$ is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property**Theorem 1.3.1: Least Upper Bound Property**

An ordered set S has a least upper bound property if:

For every nonempty subset $E \subset S$ that is bounded from above:
 $\sup(E)$ exists in S .

Example 1.3.2

\mathbb{Q} doesn't have a least upper bound property. For example, $z = \sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set $A = \{y > 0 \in \mathbb{Q} \text{ where } y^2 < 2\}$ and set $B = \{y > 0 \in \mathbb{Q} \text{ where } y^2 > 2\}$

- If $y^2 - 2 < 0$, then $z > y$ where $z \in A$. So, y is not an upper bound.
 Since for any y , there is $z > y$ where $z \in A$, then $\sup(A)$ doesn't exist in \mathbb{Q} .
- If $y^2 - 2 > 0$, then $z < y$ where $z \in B$. So, y is an upper bound, but not $\sup(E)$.
 Since for any y , there is $z < y$ where $z \in B$, then $\inf(B)$ doesn't exist in \mathbb{Q} .

Thus, \mathbb{Q} doesn't have the least upper bound.

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be an ordered set with the least upper bound property.

Let non-empty $B \subset S$ be bounded below.

Let L be the set of all lower bounds of B .

Then $\alpha = \sup(L)$ exists in S .

Proof

L is non-empty since B is bounded from below.

Thus, by the least upper bound property of S , $\alpha = \sup(L)$ exists in S .

We claim that $\alpha = \inf(B)$.

If $\gamma < \alpha$, then γ is not an upper bound for L so $\gamma \notin B$.

Thus, for every $x \in B$, $\alpha \leq x$.

If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in B$. Thus, $\inf(B) = \alpha$.

2.2 Fields

Addition Axioms

- If $x, y \in F$, then $x+y \in F$
- $x+y = y+x$ for all $x, y \in F$
- $(x+y)+z = x+(y+z)$ for all $x, y, z \in F$
- There exists $0 \in F$ such that $0+x = x$ for all $x \in F$
- For every $x \in F$, there is $-x \in F$ where $x+(-x) = 0$

Multiplicative axioms

- If $x, y \in F$, then $xy \in F$
- $yx = xy$ for all $x, y \in F$
- $(xy)z = x(yz)$ for all $x, y, z \in F$
- There exists $1 \neq 0 \in F$ such that $1x = x$ for all $x \in F$
- If $x \neq 0 \in F$, there is $\frac{1}{x} \in F$ where $x(\frac{1}{x}) = 1$

Distributive Law

$x(y+z) = xy + xz$ hold for all $x, y, z \in F$.

Propositions 2.2.1

- (a) If $x+y = x+z$, then $y = z$

Proof

$$y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z$$

- (b) If $x+y = x$, then $y = 0$

Proof

From (a), let $z = 0$.

- (c) If $x+y = 0$, then $y = -x$

Proof

From (a), let $z = -x$.

- (d) $-(-x) = x$

Proof

From (c), let $x = -x$ and $y = x$.

- (e) If $x \neq 0$ and $xy = xz$, then $y = z$

Proof

$$y = 1y = \frac{1}{x}xy = \frac{1}{x}xz = 1z = z$$

- (f) If $x \neq 0$ and $xy = x$, then $y = 1$

Proof

From (e), let $z = 1$.

- (g) If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$

Proof

From (e), let $z = \frac{1}{x}$.

- (h) If $x \neq 0$, then $\frac{1}{1/x} = x$

Proof

From (g), let $x = \frac{1}{x}$ and $y = x$.

- (i) $0x = 0$

Proof

Since $0x + 0x = (0+0)x = 0x$, then $0x = 0$.

- (j) If $x, y \neq 0$, then $xy \neq 0$

Proof

Suppose $xy = 0$, then $\frac{1}{y}\frac{1}{x}xy = \frac{1}{y}1y = \frac{1}{y}y = 1$.

$xy = 0 = 1$ is a contradiction.

- (k) $(-x)y = -(xy) = x(-y)$

Proof

$$xy + (-x)y = (x+(-x))y = 0y = 0.$$

Then by part (c), $(-x)y = -(xy)$.

$$\text{Similarly, } xy + x(-y) = x(y+(-y)) = x0 = 0.$$

Then by part (c), $x(-y) = -(xy)$.

$$(l) \quad (-x)(-y) = xy$$

Proof

By part (k), then $(-x)(-y) = -[x(-y)] = -[-(xy)]$.

By part (d), $-[-(xy)] = xy$.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all $x, y, z \in F$.

- If $y < z$, then $y+x < z+x$
- If $x, y > 0$, then $xy > 0$

Definition 2.3.1: \mathbb{Q} and \mathbb{R} are ordered fields

\mathbb{Q} , \mathbb{R} are ordered fields, but \mathbb{C} is not an ordered field since $i^2 = -1 \not> 1$.

Propositions 2.3.2

Let F be an ordered field. For all $x, y, z \in F$.

- If $x > 0$, $-x < 0$ and vice versa

Proof

$$-x = (-x) + 0 < (-x) + x = 0$$

- If $x > 0$ and $y < z$, then $xy < xz$

Proof

$$\text{Since } z-y > 0, \text{ then } 0 < x(z-y) = xz - xy$$

- If $x < 0$ and $y < z$, then $xy > xz$

Proof

$$\text{Since } -x > 0 \text{ and } z-y > 0, \text{ then } 0 < -x(z-y) = xy - xz$$

- If $x \neq 0$, $x^2 > 0$

Proof

$$\text{If } x > 0, \text{ then } x^2 = x \cdot x > 0$$

$$\text{If } x < 0, \text{ then } x^2 = (-x) \cdot (-x) > 0$$

- If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

$$\text{Since } (\frac{1}{y})y = 1 > 0, \text{ then } (\frac{1}{y}) > 0$$

$$\text{Since } x < y, \text{ then } \frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$$

Theorem 2.3.3: \mathbb{R} is a ordered field with $<$

There exists a unique ordered field \mathbb{R} with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$ so \mathbb{Q} is also an ordered field.

Theorem 2.3.4

For all $x, y \in \mathbb{R}$:

- **Archimedean Property:** If $x > 0$, there is $n \in \mathbb{Z}$ such that $nx > y$.

Proof

Fix $x > 0$. Suppose there is a y such that the property fails.

Let $A = \{ nx : n = 1, 2, 3, \dots \}$.

Then, A is nonempty and bounded from above by y .

Then by the least upper bound property by \mathbb{R} , $\alpha = \sup(A)$ exists in \mathbb{R} .

Since $x > 0$, then $-x < 0$ so $\alpha - x < \alpha - 0 = \alpha$.

So $\alpha - x$ is not an upper bound of A .

So there is a $mx \in A$ such that $mx > \alpha - x$

But then $\alpha < (m+1)x$ where $(m+1)x \in A$ which contradicts α is an upper bound for A .

- **\mathbb{Q} is dense in \mathbb{R} :** If $x < y$, there is a $p \in \mathbb{Q}$ such that $x < p < y$.

Proof

Since $x < y$, then $y - x > 0$. Then by the Archimedean Property, there exists a $n \in \mathbb{Z}$ such that $n(y - x) > 1$. Thus, $ny > nx + 1 > nx$

By the well-ordering principle, there is a smallest $m \in \mathbb{Z}_+$ such that $m > nx$.

Then, $m > nx \geq m - 1$ so $nx + 1 \geq m > nx$.

Since $ny > nx + 1 \geq m > nx$, then $y > m/n > x$.

3 Roots & Complex Field

3.1 nth Root

- If $0 < t \leq 1$, then $t^n \leq t$.

Proof

Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.

Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.

Applying the process n times, then $t^n \leq t$.

- If $t \geq 1$, $t^n \geq t$.

Proof

Since $0 < 1 \leq t$, then $t \leq t^2$.

Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.

Applying the process n times, $t \leq t^n$.

- If $0 < s < t$, $s^n < t^n$.

Proof

$$\underbrace{s \cdot s \cdot \dots \cdot s}_n < t \cdot s \cdot \dots \cdot s < t \cdot t \cdot \dots \cdot s < \dots < \underbrace{t \cdot \dots \cdot t}_n$$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix n . For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = y_1^n < y_2^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$

$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ and $< x$ and thus, $0 < t_1^n < t_1 < x$ so $t_1 \in A$.

A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.

So $x+1$ is an upper bound of A .

Thus by the least upper bound property, $y = \sup(A)$ exists.

For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.

*** (Not an upper bound of A if $<$ and not a least upper bound of A if $>$)***

For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha) \underbrace{(\beta^{n-1} + \beta^{n-2}\alpha + \dots + \alpha^{n-1})}_{\substack{\beta^{n-1} < \beta^{n-1} < \beta^{n-1}}} < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x - y^n}{n(y+1)^{n-1}}$.

From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.

Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$

Thus, $t^n > x$ so $t \notin A$.

Thus, $y-k$ is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for $x > 0 \in \mathbb{R}$.

Then let n_k be the largest integer such that:

$$d_k = n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x$$

Let E be the set of d_k for $k = 0, 1, \dots, \infty$. Then, $x = \sup(E)$.

3.3 Extended Reals

The extended real number system consist of \mathbb{R} and $\pm\infty$ such that:

$$-\infty < x < \infty \quad \text{for every } x \in \mathbb{R}$$

with the properties:

- $x \pm \infty = \pm\infty$
- $x / \pm\infty = 0$
- If $x > 0$, then $x(\pm\infty) = \pm\infty$
- If $x < 0$, then $x(\pm\infty) = \mp\infty$

3.4 Complex Numbers

Definition 3.3.1: Complex

A complex number is an ordered pair (a,b) where $a,b \in \mathbb{R}$. For $x,y \in \mathbb{C}$

- $x + y = (a,b) + (c,d) = (a + c, b + d)$
- $xy = (a,b)(c,d) = (ac - bd, ad + bc)$
- $\frac{1}{x} = (a^2 + b^2)^{-1}(a,-b)$

Thus, the axioms form a field where $(0,0) = 0$ and $(1,0) = 1$ and $(0,1) = i$.

Definition 3.3.2: Imaginary i

Let $i = (0,1)$. Then, $i^2 = -1$.

Proof

$$i^2 = (0,1)(0,1) = (0-1, 0+0) = (-1,0) = -1$$

Definition 3.3.3: Form $a + bi$

$$(a,b) = a + bi$$

Proof

$$(a,b) = (a,0) + (0,b) = (a,0) + (b,0)(0,1) = a + bi$$

Definition 3.3.4: Conjugate

Let conjugate: $\bar{z} = a - bi$ where $\text{Re}(z) = a$, $\text{Im}(z) = b$

Let $z = (a,b)$ and $w = (c,d)$:

- $\overline{z + w} = \bar{z} + \bar{w}$

Proof

$$\overline{z + w} = \overline{(a + c, b + d)} = (a + c, -b - d) = (a + c, -b - d) = (a, -b) + (c, -d) = \bar{z} + \bar{w}$$

- $\overline{z\overline{w}} = \overline{z} \overline{\overline{w}}$

Proof

$$\overline{z\overline{w}} = \overline{(ac - bd, ad + bc)} = (ac - bd, -ad - bc) = (a, -b) (c, -d) = \overline{z} \overline{w}$$

- $z + \overline{z} = 2 \operatorname{Re}(z) \quad z - \overline{z} = 2i \operatorname{Im}(z)$

Proof

$$z + \overline{z} = (a, b) + (a, -b) = (2a, 0) = 2 \operatorname{Re}(z)$$

$$z - \overline{z} = (a, b) - (a, -b) = (0, 2b) = (0, 2) b = 2i \operatorname{Im}(z)$$

- $z\overline{z} \geq 0$

Proof

$$z\overline{z} = (a, b)(a, -b) = (a^2 + b^2, -ab + ab) = a^2 + b^2 \geq 0$$

Definition 3.3.5: Absolute Value

Let absolute value: $|z| = \sqrt{z\overline{z}}$

Let $z = (a, b)$ and $w = (c, d)$:

- If $z \neq 0$, then $|z| > 0$.

Proof

$$\sqrt{z\overline{z}} = \sqrt{a^2 + b^2} \geq 0 \text{ where } |z| = 0 \text{ only if } a, b = 0 \text{ so only if } z = (0, 0).$$

- $|\overline{z}| = |z|$

Proof

$$|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$$

- $|zw| = |z| |w|$

Proof

$$\begin{aligned} |zw| &= |(ac - bd, ad + bc)| = \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = |z| |w| \end{aligned}$$

- $|\operatorname{Re}(z)| \leq |z|$

Proof

$$|\operatorname{Re}(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

- $|z + w| \leq |z| + |w|$

Proof

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + |w|^2 + 2 \operatorname{Re}(z\overline{w}) \leq |z|^2 + |w|^2 + 2|z\overline{w}| = |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2 \end{aligned}$$

4 Euclidean Spaces

4.1 Euclidean Spaces

For each positive integer k , let \mathbb{R}^k be the set of all ordered k -tuples:

$$\mathbf{x} = (x_1, \dots, x_k) \quad \text{for each } x_i \in \mathbb{R}$$

with the properties:

- $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_k + y_k) \in \mathbb{R}^k$
- $c\mathbf{x} = (cx_1, \dots, cx_k) \in \mathbb{R}^k$

So, \mathbb{R}^n has a vector space structure. Similarly, for \mathbb{C}^n .

Definition 4.1.1: Inner Product

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_k y_k \in \mathbb{R}$$

Definition 4.1.2: Norm

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$$

Definition 4.1.3: Extension to \mathbb{C}^k

For $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$

- $\mathbf{z} \cdot \mathbf{w} = z_1 \overline{w_1} + \dots + z_k \overline{w_k}$
- $\mathbf{z} \cdot \mathbf{z} = z_1 \overline{z_1} + \dots + z_k \overline{z_k} = |z_1|^2 + \dots + |z_n|^2 = |\mathbf{z}|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $b_1, \dots, b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^n \alpha_j (\overline{b_j})|^2 \leq \sum_{j=1}^n |\alpha_j|^2 \sum_{j=1}^n |b_j|^2$$

Proof

Let $A = \sum |\alpha_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum \alpha_j (\overline{b_j})$.

If $B = 0$, then $b_1 = \dots = b_n = 0$. Thus, $0 \leq A(0)$ holds true.

Suppose $B > 0$. Then:

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j) \overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) \\ &= \sum (Ba_j - Cb_j) (\overline{B} \overline{a_j} - \overline{C} \overline{b_j}) = \sum B^2 a_j \overline{a_j} - B \overline{C} a_j \overline{b_j} - B C \overline{a_j} b_j + C \overline{C} b_j \overline{b_j} \\ &= B^2 \sum |a_j|^2 - B \overline{C} \sum a_j \overline{b_j} - B C \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B \overline{C} C - B C \overline{C} + |C|^2 B = B^2 A - 2|C|^2 B + |C|^2 B = B^2 A - |C|^2 B \\ &= B(AB - |C|^2) \end{aligned}$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.

Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i \overline{z_i}$, then $\sum z_i \overline{z_i} = \sum |z_i|^2 = |\mathbf{z}|^2$. Thus:

$$|\mathbf{z} \cdot \mathbf{w}|^2 = |\sum z_i \overline{w_i}|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |\mathbf{z}|^2 |\mathbf{w}|^2$$

Thus, $|\mathbf{z} \cdot \mathbf{w}| \leq |\mathbf{z}| |\mathbf{w}|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

- $|x| \geq 0$ where $|x| = 0$ only if $x = 0$

Proof

$$|x| = \sqrt{\sum_{i=1}^k x_i^2} \geq 0 \text{ where } |x| = 0 \text{ only if } x_1 = \dots = x_k = 0$$

- $|\alpha x| = |\alpha||x|$

Proof

$$|\alpha x| = \sqrt{\sum_{i=1}^k (\alpha x_i)^2} = \sqrt{\alpha^2} \sqrt{\sum_{i=1}^k x_i^2} = |\alpha||x|$$

- $|x + y| \leq |x| + |y|$

Proof

$$\begin{aligned} |x + y|^2 &= (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 \end{aligned}$$

- $|x - y| \leq |x - z| + |y - z|$

Proof

$$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$$

4.3 Cardinality**Definition 4.3.1: Onto and 1-1 Mapping**

Suppose for every $x \in A$, there is an associated $f(x) \in B$.

Then f maps A into $B = f: A \rightarrow B$.

- If $f(A) = B$, then f maps A onto B .
- If for each $y \in B$, $f^{-1}(y)$ consist of at most one $x \in A$ where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B .

Definition 4.3.2: 1-1 Correspondence

Sets A and B are equivalent (**have the same cardinality**) if there is a 1-1 onto function $f: A \rightarrow B$. (**1-1 correspondence between A and B**) Then:

$$A \sim B$$

If $f: A \rightarrow B$ is 1-1 and onto, then there is a $f^{-1}: B \rightarrow A$ that is 1-1 and onto.

Definition 4.3.3: Countability

- A is finite if $A \sim J_n = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$
- A is infinite if A is not finite
- A is countably infinite if $A \sim \mathbb{Z}_+ = \mathbb{N}$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite.

Example 4.3.4

\mathbb{Z} is countably infinite

Proof

Let $f: \mathbb{Z}_+ \rightarrow \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ is even} \\ -\frac{n-1}{2} & n \text{ is odd} \end{cases}$$

So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto -1$, $4 \mapsto 2$, $5 \mapsto -2$, etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 4.3.4: Pigeon Hole Principle

If A is finite, A is not equivalent to any proper set of A .

Theorem 4.3.6: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let $E \subset A$ be an infinite subset.

For every distinct $x_i \in A$, let $x = \{x_1, x_2, \dots\}$.

Let n_1 be smallest integer such that $x_{n_1} \in E$.

Then let n_2 be the smallest integer where $n_2 > n_1$ such that $x_{n_2} \in E$.

Repeat the process to create sequence $f(k) = \{x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots\}$.

Thus, there is a 1-1 correspondence between E and \mathbb{Z}_+ so E is countably infinite.

5 Temp

References