Fall Real Analysis

Azure

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Contents

1 The Real Number System

1.1 Number Systems

$$\text{Natural} : \mathbb{N} = \{1, 2, 3, ...\}$$
$$\text{Integer} : \mathbb{Z} = \{-2, -1, 0, 1, 2, ...\}$$
$$\text{Rational} : \mathbb{Q} = \frac{p}{q} \text{ where p,q} \in \mathbb{N}$$

*** $\mathbb{Q}$ is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A $= \{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
But, A has no least upper bound in $\mathbb{Q}$, but A has a least upper bound in $\mathbb{R}$.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
Also, $\mathbb{R}$ exists and unique.

Definition 1.2.1: Order

Let S be a set. An order on S is a relation $<$ satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x $<$ y
  - x $=$ y
  - x $>$ y
- Transitivity: If x $<$ y and y $<$ z, then x $<$ z.

Definition 1.2.2: Ordered Set

An ordered set is a set with an order.

Definition 1.2.3: Bounds

Let S be an ordered set and E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.
A lower bound of E is a $\alpha \in$ S if x $\geq \alpha$ for all x $\in$ E.
If such a α exists, then E is bounded from below.

Definition 1.2.4: Infimum & Supremum

Let S be an ordered set.

Let E ⊂ S be bounded from above. Least upper bound $\beta \in$ S exists if:

- β is an upper bound for E
- If $\gamma < \beta$, then γ is not an upper bound for E.

Then β = sup(E).

Let E ⊂ S be bounded from below. Greatest lower bound $\alpha \in$ S exists if:

- α is a lower bound for E
- If $\gamma > \alpha$, then γ is not a lower bound for E.

Then α = inf(E).

Example 1.2.5

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup(E) exists, it may or may not exists at S.

If sup(E) exists, then sup(E) is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1: Least Upper Bound Property

An ordered set S has a least upper bound property if:

For every nonempty subset E ⊂ S that is bounded from above:

sup(E) exists in S.

Example 1.3.2

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.

Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then z > y where z ∈ A. So, y is not an upper bound.
  Since for any y, there is z > y where z ∈ A, then sup(A) doesn't exists in $\mathbb{Q}$.
- If $y^2 - 2 > 0$, then z < y where z ∈ B. So, y is an upper bound, but not sup(E).
  Since for any y, there is z < y where z ∈ B, then inf(B) doesn't exists in $\mathbb{Q}$.

Thus, $\mathbb{Q}$ doesn't have the least upper bound or greatest lower bound property.

2 Day 2: Fields

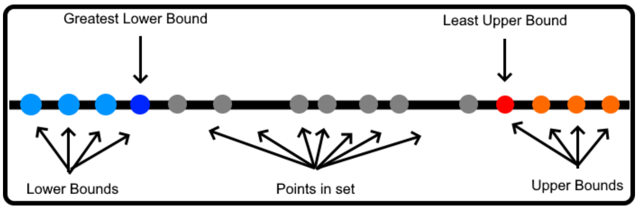
2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound + Lower Bound implies Greatest Upper Bound

Let S be a ordered set with the least upper bound property.
Let non-empty B $\subset$ S be bounded below.
Let L be the set of all lower bounds of B.
Then α = sup(L) exists in S.

Proof

L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, α = sup(L) exists in S.
We claim that α = inf(B).
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B since all upper bounds for L are in B. Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, inf(B) = α.



2.2 Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F
- x+y = y+x for all x,y $\in$ F
- (x+y)+z = x+(y+z) for all x,y,z $\in$ F
- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F
- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative Axioms

- If x,y $\in$ F, then xy $\in$ F
- yx = xy for all x,y $\in$ F
- (xy)z = x(yx) for all x,y,z $\in$ F
- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F
- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x}$ $\in$ F where x($\frac{1}{x}$) = 1

Distributive Law

x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Propositions 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0
Proof
From (a), let z = 0.

(c) If x+y = 0, then y = -x
Proof
From (a), let z = -x.

(d) -(-x) = x
Proof
From (c), let x = -x and y = x.

(e) If x $\neq$ 0 and xy = xz, then y = z
Proof
y = 1y = $\frac{1}{x}$xy = $\frac{1}{x}$xz = 1z = z

(f) If x $\neq$ 0 and xy = x, then y = 1
Proof
From (e), let z = 1.

(g) If x $\neq$ 0 and xy = 1, then y = $\frac{1}{x}$
Proof
From (e), let z = $\frac{1}{x}$.

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x
Proof
From (g), let x = $\frac{1}{x}$ and y = x.

(i) 0x = 0
Proof
Since 0x + 0x = (0+0)x = 0x = 0x + 0, then 0x = 0.

(j) If x,y $\neq$ 0, then xy $\neq$ 0
Proof
Suppose xy = 0, then 1 = $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}\frac{1}{x}$0 = 0.
0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)
Proof
xy + (-x)y = (x+(-x))y = 0y = 0.
Then by part (c), (-x)y = -(xy).
Similarly, xy + x(-y) = x(y+(-y)) = x0 = 0.
Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy
Proof
By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].
By part (d), -[-(xy)] = xy.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x
- If x,y > 0, then xy > 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field since i^2 = -1 $\not> 1$.

Propositions 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

(a) If x > 0, then -x < 0 and vice versa

Proof

-x = -x + 0 < -x + x = 0

(b) If x > 0 and y < z, then xy < xz

Proof

Since z-y > 0, then 0 < x(z-y) = xz - xy

(c) If x < 0 and y < z, then xy > xz

Proof

Since -x > 0 and z-y > 0, then 0 < -x(z-y) = xy - xz

(d) If x $\neq$ 0, $x^2 > 0$

Proof

If x > 0, then x^2 = x $\cdot$ x > 0

If x < 0, then $(-x)^2$ = (-x) $\cdot$ (-x) = x $\cdot$ x = x^2 > 0

(e) If $0 < x < y$, then $0 < 1/y < 1/x$

Proof

Since $(\frac{1}{y})y = 1 > 0$, then $(\frac{1}{y}) > 0$

Since x < y, then $\frac{1}{y} = (\frac{1}{y})(\frac{1}{x})x < (\frac{1}{y})(\frac{1}{x})y = \frac{1}{x}$

Theorem 2.3.3: $\mathbb{R}$ is an ordered field with <

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.

Also, $\mathbb{Q} \subset \mathbb{R}$ so $\mathbb{Q}$ is also an ordered field.

Theorem 2.3.4

For all x,y $\in \mathbb{R}$:

- Archimedean Property: If x > 0, there is n $\in \mathbb{Z}$ such that nx > y.

  Proof

  Fix x > 0. Suppose there is a y such that the property fails.

  Let A = { nx: n = 1, 2, 3, ... }.

  Then, A is nonempty and bounded from above by y.

  Then by the least upper bound property of $\mathbb{R}$, α = sup(A) exists in $\mathbb{R}$.

  Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$.

  So $\alpha - x$ is not an upper bound of A.

  So there is a mx $\in$ A such that mx > $\alpha - x$.

  Then α < (m+1)x, but (m+1)x $\in$ A contradicting α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If x < y, there is a p $\in \mathbb{Q}$ such that x < p < y.

  Proof

  Since x < y, then y-x > 0. Then by the Archimedean Property, there exists a n $\in$ Z such that n(y-x) > 1. Thus, ny > nx+1 > nx

  By the well-ordering principle, there is a smallest m $\in \mathbb{Z}_+$ such that m > nx.

  Then, m > nx $\geq$ m-1 so nx+1 $\geq$ m > nx.

  Since ny > nx+1 $\geq$ m > nx, then y > m/n > x.

3 Roots & Complex Field

3.1 nth Root

(a) If $0 < t \leq 1$, then $t^n \leq t$.
Proof
Since $t > 0$ and $t \leq 1$, then $t^2 \leq t$.
Since $t^2 \leq t$, then $t^3 \leq t^2$ so $t^3 \leq t^2 \leq t$.
Applying the process n times, then $t^n \leq t$.

(b) If $t \geq 1$, $t^n \geq t$.
Proof
Since $0 < 1 \leq t$, then $t \leq t^2$.
Since $t \leq t^2$, then $t^2 \leq t^3$ so $t \leq t^2 \leq t^3$.
Applying the process n times, $t \leq t^n$.

(c) If $0 < s < t$, then $s^n < t^n$.
Proof
$\underbrace{s \cdot s \cdot ... \cdot s}_{n} < t \cdot s \cdot ... \cdot s < t \cdot t \cdot ... \cdot s < ... < \underbrace{t \cdot ... \cdot t}_{n}$

Theorem 3.1.1: $y^n = x$ has a unique y

Fix $n \in \mathbb{Z}_+$. For every $x > 0$, there exists a unique $y \in \mathbb{R}$ such that $y^n = x$.
Also, such a y is written as $y = \sqrt[n]{x} = x^{\frac{1}{n}}$.

Proof

Uniqueness:

y is unique since if $y_1 < y_2$, then $x = {y_1}^n < {y_2}^n \neq x$.

Existence:

Let set $A = \{ t > 0 : t^n < x \}$.
$A \neq \emptyset$ since let $t_1 = \frac{x}{x+1} < 1$ so $t_1 < x$ and thus, $0 < {t_1}^n < t_1 < x$ so $t_1 \in A$.
A is bounded above since if $t \geq x+1$, then $t > 1$ so $t^n \geq t \geq x+1 > x$ so $t \notin A$.
So x+1 is an upper bound of A.
Thus by the least upper bound property, $y = \sup(A)$ exists.
For $y^n = x$, show $y^n < x$ and $y^n > x$ cannot hold true.
(Not an upper bound of A if < and not a least upper bound of A if >)
For $0 < \alpha < \beta$:

$$\beta^n - \alpha^n = (\beta - \alpha)\ (\underbrace{\beta^{n-1}}_{\beta^{n-1}} + \underbrace{\beta^{n-2}\alpha^1}_{<\beta^{n-1}} + ... + \underbrace{\alpha^{n-1}}_{<\beta^{n-1}}) < (\beta - \alpha)n\beta^{n-1}$$

Suppose $y^n < x$. Pick $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.
From inequality, let $\beta = y+h$ and $\alpha = y$

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n$$

Thus, $(y+h)^n < x$ so $y+h \in A$ and thus, not an upper bound of A which is a contradiction since $y = \sup(A)$.

Suppose $y^n > x$. Pick $0 < k = \frac{y^n - x}{ny^{n-1}} < \frac{y^n}{ny^{n-1}} = \frac{1}{n}y < y$.
Consider $t \geq y-k$, then: $y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x$
Thus, $t^n > x$ so $t \notin A$.
Thus, y-k is an upper bound of A which is a contradiction since $y = \sup(A)$.

Since $y^n < x$ and $y^n > x$, then $y^n = x$.

Corollary 3.1.2: n-th root of product = product of n-th root

If a,b > 0 and n $\in \mathbb{Z}_+$, then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.

Proof

Let A = $a^{\frac{1}{n}}$ and B = $b^{\frac{1}{n}}$.

Then by theorem 3.1.1, since A is a solution to y_1^n = a, then A^n = a.

Similarly, B is a solution of y_2^n = b so B^n = b. Thus:

$$\begin{aligned} ab &= A^n B^n = A_1 A_2 ... A_n B_1 B_2 ... B_n \\ &= A_1 A_2 ... B_1 A_n B_2 ... B_n = ... = A_1 B_1 A_2 ... A_{n-1} A_n B_3 ... B_n \\ &= ... = A_1 B_1 A_2 B_2 ... A_n B_n = (AB)^n \end{aligned}$$

Then again by theorem 3.1.1, there is a unique $(ab)^{\frac{1}{n}}$ = AB = $a^{\frac{1}{n}} b^{\frac{1}{n}}$.

3.2 Decimals

Let n_0 be the largest integer such that $n_0 \leq x$ for x > 0 $\in \mathbb{R}$.

Then let n_k be the largest integer such that $d_k = n_0 + \frac{n_1}{10} + ... + \frac{n_k}{10^k} \leq x$

Let E be the set of d_k for k = 0, 1, ... ∞. Then, x = sup(E).

3.3 Extended Reals

The extended real number system consist of $\mathbb{R}$ and $\pm\infty$ such that:

$-\infty < x < \infty$ for every x $\in \mathbb{R}$

with the properties:

- x $\pm\ \infty = \pm\infty$
- x / $\pm\infty$ = 0
- If x > 0, then x($\pm\infty$) = $\pm\infty$
- If x < 0, then x($\pm\infty$) = $\mp\infty$

3.4 Complex Numbers

Definition 3.4.1: Complex

A complex number is an ordered pair (a,b) where a,b $\in \mathbb{R}$. For x,y $\in \mathbb{C}$

- x + y = (a,b) + (c,d) = (a + c , b + d)
- xy = (a,b) (c,d) = (ac - bd , ad + bc)
- $\frac{1}{x}$ = $(a^2 + b^2)$(a,-b)

Thus, the axioms form a field where (0,0) = 0 and (1,0) = 1 and (0,1) = i.

Definition 3.4.2: Imaginary i

Let i = (0,1). Then, i^2 = -1.

Proof

i^2 = (0,1)(0,1) = (0-1,0+0) = (-1,0) = -1

Definition 3.4.3: Form a + bi

(a,b) = a + bi

Proof

(a,b) = (a,0) +(0,b) = (a,0) + (b,0)(0,1) = a + bi

Definition 3.4.4: Conjugate

Let conjugate: $\bar{z}$ = a - bi where Re(z) = a , Im(z) = b

Let z = (a,b) and w = (c,d):

(a) $\overline{z+w} = \overline{z} + \overline{w}$

Proof

$\overline{z+w} = \overline{(a+c, b+d)}$ = (a+c,-b-d) = (a,-b) + (c,-d) = $\overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\,\overline{w}$

Proof

$\overline{zw} = \overline{(ac-bd, ad+bc)}$ = (ac-bd,-ad-bc) = (a,-b) (c,-d) = $\overline{z}\,\overline{w}$

(c) z + $\overline{z}$ = 2 Re(z)   z - $\overline{z}$ = 2i Im(z)

Proof

z + $\overline{z}$ = (a,b) + (a,-b) = (2a,0) = 2 Re(z)

z - $\overline{z}$ = (a,b) - (a,-b) = (0,2b) = (0,2) b = 2i Im(z)

(d) z$\overline{z}$ $\geq$ 0

Proof

z$\overline{z}$ = (a,b)(a,-b) = (a^2 + b^2 , -ab+ab) = a^2 + b^2 $\geq$ 0

Definition 3.4.5: Absolute Value

Let absolute value: | z | = $\sqrt{z\overline{z}}$

Let z = (a,b) and w = (c,d):

(a) If z $\neq$ 0, then | z | > 0.

Proof

$\sqrt{z\overline{z}} = \sqrt{a^2+b^2} \geq 0$ where | z | = 0 only if a,b = 0 so only if z = (0,0).

(b) | $\overline{z}$ | = | z |

Proof

| $\overline{z}$ | = $\sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$ = | z |

(c) | zw | = | z | | w |

Proof

| zw | = | (ac-bd,ad+bc) | = $\sqrt{(ac-bd)^2+(ad+bc)^2}$

= $\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$

= $\sqrt{a^2+b^2}\,\sqrt{c^2+d^2}$ = | z | | w |

(d) | Re(z) | $\leq$ | z |

Proof

| Re(z) | = | a | = $\sqrt{a^2} \leq \sqrt{a^2+b^2}$ = | z |

(e) | z+w | $\leq$ | z | + | w |

Proof

$|z+w|^2$ = (z+w)$\overline{(z+w)}$ = (z+w)$(\overline{z}+\overline{w})$ = z$\overline{z}$ + z$\overline{w}$ + w$\overline{z}$ + w$\overline{w}$

= $|z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}) \leq |z|^2 + |w|^2 + 2\,|z\overline{w}|$

= $|z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$

4 Euclidean Spaces & Cauchy-Schwarz

4.1 Euclidean Spaces

For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples:

x = $(x_1, ..., x_k)$ for each $x_i \in \mathbb{R}$

with the properties:

- x+y = $(x_1 + y_1, ..., x_k + y_k) \in \mathbb{R}^k$
- cx = $(cx_1, ..., cx_k) \in \mathbb{R}^k$

So, $\mathbb{R}^n$ has a vector space structure. Similarly, for $\mathbb{C}^n$.

Definition 4.1.1: Inner Product for $\mathbb{R}^k$

$x \cdot y = x_1y_1 + ... + x_ky_k \in \mathbb{R}$

Definition 4.1.2: Norm

$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$

Definition 4.1.3: Extension to $\mathbb{C}^k$

For z,w $\in \mathbb{C}^n$

- $z \cdot w = z_1\overline{w_1} + ... + z_k\overline{w_k}$
- $z \cdot z = z_1\overline{z_1} + ... + z_k\overline{z_k} = |z_1|^2 + ... + |z_n|^2 = |z|^2$

4.2 Cauchy-Schwarz

Theorem 4.2.1: Cauchy-Schwarz

If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $b_1, ..., b_n \in \mathbb{C}$, then:

$$|\sum_{j=1}^{n} a_j(\overline{b_j})|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof

Let $A = \sum |a_j|^2$ and $B = \sum |b_j|^2$ and $C = \sum a_j(\overline{b_j})$.

If $B = 0$, then $b_1 = ... = b_n = 0$. Thus, $0 \leq A(0)$ holds true.

Suppose $B > 0$. Then:

$$\begin{aligned}
&\sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)} = \sum (Ba_j - Cb_j)(\overline{B}\,\overline{a_j} - \overline{C}\,\overline{b_j}) \\
&= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\,\overline{b_j}) = \sum B^2a_j\overline{a_j} - B\overline{C}a_j\overline{b_j} - BC\overline{a_j}b_j + C\overline{C}b_j\overline{b_j} \\
&= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j\overline{b_j} - BC \sum \overline{a_j}b_j + |C|^2 \sum |b_j|^2 \\
&= B^2A - B\overline{C}C - BC\overline{C} + |C|^2B = B^2A - 2|C|^2B + |C|^2B = B^2A - |C|^2B \\
&= B(AB - |C|^2)
\end{aligned}$$

Since $|Ba_j - Cb_j| \geq 0$, then $B(AB - |C|^2) \geq 0$.

Since $B > 0$, then $AB - |C|^2 \geq 0$ so $AB \geq |C|^2$.

Definition 4.2.2: Consequence of the Cauchy-Schwarz

Since $|z_i|^2 = z_i\overline{z_i}$, then $\sum z_i\overline{z_i} = \sum |z_i|^2 = |z|^2$. Thus:

$$|z \cdot w|^2 = |\sum z_i\overline{w_i}\,|^2 \leq \sum |z_i|^2 \sum |w_i|^2 = |z|^2\,|w|^2$$

Thus, $|z \cdot w| \leq |z||w|$.

Propositions 4.2.3

Let $x, y, z \in \mathbb{R}^k$ where $\alpha \in \mathbb{R}$:

(a) $|x| \geq 0$ where $|x| = 0$ only if x = 0

Proof

$|x| = \sqrt{\sum_{i=1}^{k} x_i^2} \geq 0$ where $|x| = 0$ only if $x_1 = ... = x_k = 0$

(b) $|\alpha x| = |\alpha||x|$

Proof

$|\alpha x| = \sqrt{\sum_{i=1}^{k} (\alpha x_i)^2} = \sqrt{\alpha^2}\sqrt{\sum_{i=1}^{k} x_i^2} = |\alpha||x|$

(c) $|x + y| \leq |x| + |y|$

Proof

$|x + y|^2 = (x + y) \cdot (x + y) = |x|^2 + 2(x \cdot y) + |y|^2$

$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$

(d) $|x - y| \leq |x - z| + |y - z|$

Proof

$|x - y| = |x - z + z - y| \leq |x - z| + |z - y| = |x - z| + |y - z|$

5 Construction of $\mathbb{R}$: Theorem 2.3.3

There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Also, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Definition 5.1: Cuts

Define a cut as any set $\alpha \subset \mathbb{Q}$ with the properties:

- α is not empty and $\alpha \neq \mathbb{Q}$
- If p $\in \alpha$ and q $\in \mathbb{Q} <$ p, then q $\in \alpha$
- If p $\in \alpha$, then p $<$ r $\in \mathbb{Q}$ for some r $\in \alpha$

Proposition 5.2: Order of $\mathbb{R} \to$ ordered set $\mathbb{R}$

Define $\alpha < \beta$ if α is a proper subset of β.

- If $\alpha \not\geq \beta$, then β is not a subset of α.
  Then there is a p $\in \beta$ such that p $\notin \alpha$.
  Then for any q $\in \alpha$, q $<$ p and thus, q $\in \beta$.
  Thus, $\alpha \subset \beta$ and since $\alpha \neq \beta$, then $\alpha < \beta$.
- If $\alpha \nless \beta$ and $\alpha \ngtr \beta$, then either $\alpha = \beta$ or $\alpha \neq \beta$.
  If $\alpha \neq \beta$, there are p,q such that p $\in \alpha$, but p $\notin \beta$ and q $\in \beta$, but q $\notin \alpha$.
  But if p $\notin \beta$, then for any b $\in \beta$, b $<$ p. Thus, q $<$ p.
  Similarly, if q $\notin \alpha$, then for any a $\in \alpha$, a $<$ q. Thus, p $<$ q.
  Thus, there is a contradiction since p $>$ q and p $<$ q so $\alpha = \beta$.
- If $\alpha \not\leq \beta$, then α is not a subset of β.
  Then there is a p $\in \alpha$ such that p $\notin \beta$.
  Then for any q $\in \beta$, q $<$ p and thus, q $\in \alpha$.
  Thus, $\beta \subset \alpha$ and since $\alpha \neq \beta$, then $\beta < \alpha$.
- If $\alpha < \beta$ and $\beta < \gamma$, then since α is a proper subset of β and β is a proper subset of γ, then α is a proper subset of γ. Thus, $\alpha < \gamma$.

Thus, $\mathbb{R}$ is an ordered set with such an order $<$.

Proposition 5.3: Least Upper Bound of $\mathbb{R} \to$ Least Upper Bound Property

Let A $\subset \mathbb{R}$ and β be an upper bound for A. Let γ be the union of all $\alpha \in$ A.
Thus, p $\in \gamma$ if and only if p $\in \alpha$ for some $\alpha \in$ A.
γ defines a cut since:

- Since A is nonempty, there exists a $\alpha_0 \in$ A where α_0 is nonempty.
  Since α_0 is nonempty, then γ is nonempty.
  Since every $\alpha \in$ A is $\alpha < \beta$, then $\gamma < \beta$ so $\gamma \subset \beta$ and thus, $\gamma \neq \mathbb{Q}$.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. If q $<$ p, then q $\in \alpha_1$ so q $\in$ A.
- If p $\in \gamma$, then p $\in \alpha_1$ for some $\alpha_1 \in$ A. Thus, there is a r $\in \alpha_1$ such that r $>$ p so r $\in \gamma$. Thus, there is a r $\in \gamma$ where r $>$ p.

Since γ defines a cut, then $\gamma \in \mathbb{R}$. Since every $\alpha \in$ A $\subset \gamma$, then $\alpha \leq \gamma$ so γ is an upper bound for A.
Suppose $\delta < \gamma$. Then there is a s $\in \gamma$ such that s $\notin \delta$. Since s $\in \gamma$, then there is a $\alpha \in$ A such that s $\in \alpha$. Since $\delta < \alpha$, then δ is not an upper bound of A.
Thus, $\gamma = \sup(\text{A})$.

Proposition 5.4: $\mathbb{R}$ is a field

If α, $\beta \in \mathbb{R}$, define $\alpha + \beta$ as the set of all sums r + s where r $\in \alpha$ and s $\in \beta$. Also, let 0* be the set of all negative rational numbers which is a cut since:

- 0* is nonempty and $0^* \neq \mathbb{Q}$
- If p $\in$ 0*, then any q $\in \mathbb{Q}$ < p is a negative rational and thus, q $\in$ 0*.
- Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then for any p $\in$ 0*, there is a r $\in \mathbb{Q}$ where p < r < 0 so r is a negative rational so r $\in$ 0*.

$\alpha + \beta \in \mathbb{R}$ since $\alpha + \beta$ is a cut:

- $\alpha + \beta$ is non-empty since α, β are non-empty. Take r' $\notin \alpha$, s' $\notin \beta$, then r' + s' > r + s for r $\in \alpha$, s $\in \beta$. Thus, r' + s' $\notin \alpha + \beta$ so $\alpha + \beta \neq \mathbb{Q}$.
- If p $\in \alpha + \beta$, then p = r + s where r $\in \alpha$ and s $\in \beta$.
  If q < p, then q - s < p - s = (r + s) - s = r so q - s $\in \alpha$.
  Since q - s $\in \alpha$ and s $\in \beta$, then (q - s) + s = q $\in \alpha + \beta$.
- If r $\in \alpha$, then there is a t $\in \alpha$ such that t > r. Let s $\in \beta$.
  Thus, for any p = r + s $\in \alpha + \beta$, there is a q = t + s $\in \alpha + \beta$ such that p = r + s < t + s = q.

$\alpha + \beta = \beta + \alpha$

If p = r + s $\in \alpha + \beta$ where r $\in \alpha$, s $\in \beta$, then s + r = r + s = p $\in \beta + \alpha$.

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

If r $\in \alpha$, s $\in \beta$, t $\in \gamma$, then r + s + t = (r + s) + t $\in (\alpha + \beta) + \gamma$ and r + s + t = r + (s + t) $\in \alpha + (\beta + \gamma)$.

$\alpha + 0^* = \alpha$

If r $\in \alpha$, s $\in$ 0*, then r + s < r. Thus, r + s $\in \alpha$. Thus, $\alpha + 0^* \subset \alpha$.
If p $\in \alpha$, there is a r $\in \alpha$ where r > p. Thus, p - r $\in$ 0*.
Since p = r + (p - r) $\in \alpha$ + 0*, then $\alpha \subset \alpha + 0^*$. Thus, $\alpha + 0^* = \alpha$.

There is a -α such that α + -α = 0*

Fix $\alpha \in \mathbb{R}$. Let set β be all p where there is r > 0 such that -p - r $\notin \alpha$.
$\beta \in \mathbb{R}$ since β is a cut:

- If s $\notin \alpha$ and p = -s - 1, then -p - 1 $\notin \alpha$. Thus, p $\in \beta$ so β is nonempty. If q $\in \alpha$, then -q $\notin \beta$ so $\beta \neq \mathbb{R}$.
- If p $\in \beta$, let r > 0 so -p - r $\notin \alpha$. If q < p, then -q - r > -p - r and thus, -q - r $\notin \alpha$ so q $\in \beta$.
- If p $\in \beta$, let t = p + (r/2). Then -t - (r/2) = -p - r $\notin \alpha$ and thus, t $\in \beta$ where p < t.

If r $\in \alpha$, s $\in \beta$, then s $\notin \alpha$. Thus, r < -s so r + s < 0. Thus, $\alpha + \beta \subset 0^*$.
Let v $\in$ 0* and let w = -v/2 so w > 0.
Thus, by the Achimedean property, there is an integer n such that nw $\in \alpha$, but (n+1)w $\notin \alpha$. Let p = -(n+2)w so -p - w = (n+1)w $\notin \alpha$ so p $\in \beta$.
Then, v = -2w = nw + -nw - 2w = nw + -(n+2)w = nw + p $\in \alpha + \beta$.
Since v $\in$ 0*, then $0^* \subset \alpha + \beta$. Thus, $\alpha + \beta = 0^*$. Then, let -α = β.

Thus, if α, β, $\gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
Thus, if α > 0*, then -α = -α + 0* < -α + α = 0* so -α < 0*.
If α, $\beta \in \mathbb{R}_+$, define $\alpha\beta$ as the set of all p such that p $\leq$ rs for r $\in \alpha$, s $\in \beta$.
Define 1* as the set of all q < 1. Then all multiplication axioms holds with similar proofs as addition. Also, note since α, β > 0*, then $\alpha\beta$ > 0*.
Also, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ holds through cases were α, β, γ >,< 0*.

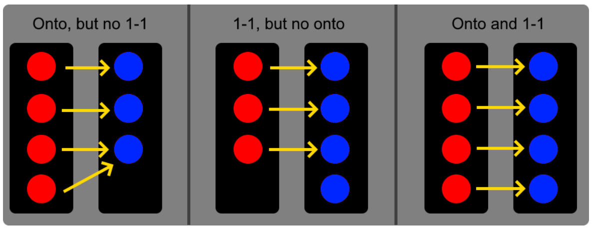
6 Cardinality

6.1 Cardinality

Definition 6.1.1: Onto and 1-1 Mapping

Suppose for every $x \in A$, there is an associated $f(x) \in B$.
Then f maps A into B = f: $A \to B$.

- If $f(A) = B$, then f maps A onto B.
- If for each $y \in B$, $f^{-1}(y)$ consist of at most one $x \in A$ where $f^{-1}(y_1) = x_1 \neq x_2 = f^{-1}(y_2)$ for $y_1 \neq y_2$, then f is a 1-1 mapping of A into B.



Definition 6.1.2: 1-1 Correspondence

Sets A and B are equivalent (have the same cardinality) if there is a 1-1 onto function f: $A \to B$. (1-1 correspondence between A and B) Then:

$A \sim B$

If f: $A \to B$ is 1-1 and onto, then there is a f^{-1}: $B \to A$ that is 1-1 and onto.

Definition 6.1.3: Countability

- A is finite if $A \sim J_n = \{0, 1, ..., n\}$ for some $n \in \mathbb{N}$
- A is infinite if A is not finite
- A is countably infinite if $A \sim J = \mathbb{Z}_+$
- A is uncountable if A is not finite or countably infinite
- A is at most countable if A is finite or countably infinite

Example 6.1.4

$\mathbb{Z}$ is countably infinite

Proof

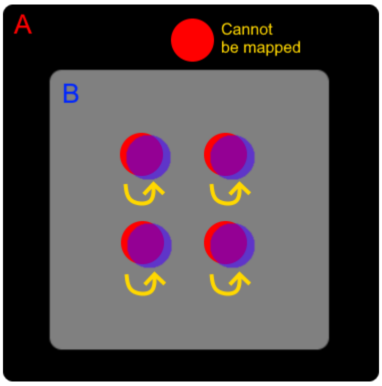
Let f: $\mathbb{Z}_+ \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{n is even} \\ -\frac{n-1}{2} & \text{n is odd} \end{cases}$$

So $1 \mapsto 0$, $2 \mapsto 1$, $3 \mapsto -1$, $4 \mapsto 2$, $5 \mapsto -2$, etc. Thus, $\mathbb{Z} \sim \mathbb{Z}_+$.

Definition 6.1.5: Pigeonhole Principle

If A is finite, A is not equivalent to any proper set of A.

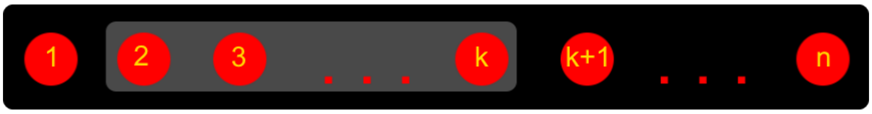


Theorem 6.1.6: Infinite subsets of countable sets are countable

An infinite subset E of a countably infinite set A is countably infinite.

Proof

Let E $\subset$ A be an infinite subset. For every distinct $x_i \in$ A, let x = { $x_1, x_2, \ldots$ }.
Let n_1 be smallest integer such that $x_{n_1} \in$ E.
Then let n_2 be the smallest integer where $n_2 > n_1$ such that $x_{n_2} \in$ E.
Repeat the process to create sequence f(k) = { $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ }.
Thus, there is a 1-1 correspondence between E and $\mathbb{Z}_+$ so E is countably infinite.



6.2 Set of Sets

Definition 6.2.1: Union and Intersection

Let sets Ω,B be such that for each x $\in$ Ω, there is an associated $E_x \subset$ B.

- E = $\cup_{x=1}^{n}$ E_x only if for every x $\in$ E, x $\in$ E_x for at least one x $\in$ Ω.
- P = $\cap_{x=1}^{n}$ E_x only if for every x $\in$ P, x $\in$ E_x for all x $\in$ Ω.

with properties:

(a) A $\cup$ B = B $\cup$ A A $\cap$ B = B $\cap$ A

(b) (A $\cup$ B) $\cup$ C = A $\cup$ (B $\cup$ C) (A $\cap$ B) $\cap$ C = A $\cap$ (B $\cap$ C)

(c) A $\subset$ A $\cup$ B (A $\cap$ B) $\subset$ A

(d) If A $\subset$ B, then A $\cup$ B = B and A $\cap$ B = A

Proof

If x $\in$ A $\cup$ B, then x $\in$ A or/and x $\in$ B.

- If x $\in$ A, since A $\subset$ B, then x $\in$ B. Then, (A $\cup$ B) $\subset$ B.
- If x $\in$ B, then immediately (A $\cup$ B) $\subset$ B.

If x $\in$ B, then x $\in$ A $\cup$ B so B $\subset$ (A $\cup$ B). Thus, A $\cup$ B = B.

If x $\in$ A $\cap$ B, then x $\in$ A and x $\in$ B. Thus, (A $\cap$ B) $\subset$ A.
If x $\in$ A, since A $\subset$ B, then x $\in$ B so x $\in$ A $\cap$ B. Thus, A $\subset$ (A $\cap$ B).
Thus, A $\cap$ B = A.

(e) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)
Proof
If x ∈ A ∩ (B ∪ C), then x ∈ A and (x ∈ B or/and x ∈ C).
- If x ∈ B, then x ∈ (A ∩ B) so x ∈ (A ∩ B) ∪ (A ∩ C).
- If x ∈ C, then x ∈ (A ∩ C) so x ∈ (A ∩ B) ∪ (A ∩ C).

Thus, A ∩ (B ∪ C) ⊂ (A ∩ B) ∪ (A ∩ C).
If x ∈ (A ∩ B) ∪ (A ∩ C), then x ∈ A and (x ∈ B or/and x ∈ C).
Thus, (A ∩ B) ∪ (A ∩ C) ⊂ A ∩ (B ∪ C).
Thus, A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).

(f) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)
Proof
If x ∈ A ∪ (B ∩ C), then x ∈ A or/and (x ∈ B and x ∈ C).
- If x ∈ A, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).
- If x ∈ B,C, then x ∈ (A ∪ B) and x ∈ (A ∪ C) so A ∪ (B ∩ C) ⊂ (A ∪ B) ∩ (A ∪ C).

If x ∈ (A ∪ B) ∩ (A ∪ C), then x ∈ A or/and (x ∈ B and x ∈ C).
Thus, (A ∪ B) ∩ (A ∪ C) ⊂ A ∪ (B ∩ C).
Thus, A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).

Theorem 6.2.2: Union of countably infinite sets is countably infinite

If $E_1, E_2, ...$ are countably infinite sets, then S = $\cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof

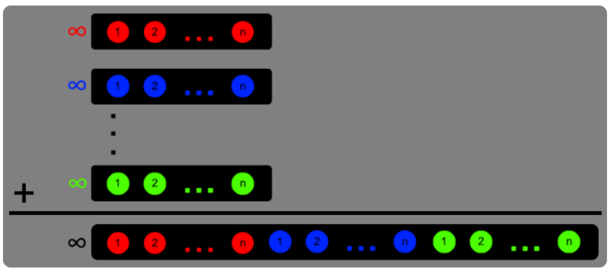
For each E_n, there is a sequence { x_{n1}, x_{n2}, ... }. Then construct an array as such:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Take elements diagonally, then sequence S* = { x_{11} ; x_{21}, x_{12} ; x_{31}, x_{32}, x_{33} ; ... }.
Since S* ∼ S so S is at most countable and S is infinite since $E_1, E_2, ...$ are infinite, then S cannot be finite and thus, countably infinite.

Alternative Proof

For each E_n, let set $\widetilde{E_n} = E_n$ - $\cup_{m=1}^{\infty} E_m$ where m ≠ n. Thus, S = $\cup_{n=1}^{\infty} \widetilde{E_n}$.
Since each E_n is countably infinite, there exists a 1-1 mapping δ_n: $E_n \to \mathbb{Z}_+$.
Thus, for each $\widetilde{E_n}$, there is a 1-1 mapping δ_n: $\widetilde{E_n} \to A \subset \mathbb{Z}_+$.
Let $p_1, p_2, ...$ be distinct primes. Since for s ∈ S, there exists a unique $\widetilde{E_i}$ such that s ∈ $\widetilde{E_i}$, then let f(s) = $p_1^{\delta_1(s)} p_2^{\delta_2(s)} ...$ where $p_k^{\delta_k(s)} = 1$ if k ≠ i.
Then, by the Fundamental theorem of arithmetic, f maps s to a unique z ∈ $\mathbb{Z}_+$ and thus, f is a 1-1 function so S is at most countable. Since any $E_n \subset$ S is countably infinite, then S cannot be finite and thus, S is countably infinite.



Theorem 6.2.3: The set of countable n-tuples are countable

Let A be a countably infinite set and B_n be the set of all n-tuples (a_1,...,a_n) where $a_k \in$ A. Then B_n is countably infinite.

Proof

The base case B_1 is countably infinite since B_1 = A.

Suppose B_{n-1} is countably infinite. Then for every x $\in$ B:

x = (b,a) b $\in B_{n-1}$ and a $\in$ A

Since for every fixed b, (b,a) $\sim$ A and thus, countably infinite.

Since B is a set of countably infinite sets, then B_n is countably infinite.

Definition 6.2.4: $\mathbb{Q}$ is countable

The set of rational numbers, $\mathbb{Q}$, is countably infinite.

Proof

Since elements of $\mathbb{Q}$ are of form $\frac{a}{b}$ which is a 2-tuple, then by the theorem 6.2.3, $\mathbb{Q}$ is countably infinite.

Alternative Proof

For every x $\in \mathbb{Q}$, let x = $(-1)^i\ \frac{p}{q}$ where p,q $\in \mathbb{Z}_+$.

Let f(x) = $2^i\ 3^p\ 5^q$. Then by the Fundamental theorem of arithmetic, f is a 1-1 mapping of x to E $\subset \mathbb{Z}_+$.

Thus, $\mathbb{Q}$ is at most countable, but since p,q $\in \mathbb{Z}_+$, then $\mathbb{Q}$ cannot be finite and thus, is countably infinite.

Example 6.2.5: Sequences of 0 and 1 are uncountable

Let A be the set of all sequences whose elements are digits 0 and 1. Then A is uncountable.

Proof: Cantor's Diagonalization Proof

Let set E be a countably infinite subset of A which consist of sequences $s_1, s_2, \ldots$.

Then construct a sequence s as follows:

If the n-th digit in s_n is 1, then let the n-th digit of s be 0 and vice versa.

Thus. s differs from every $s_n \in$ E so s $\notin$ E.

But, s $\in$ A so E is a proper subset of A.

Thus, every countably infinite subset of A is a proper subset of A.

If A is countably infinite, then A is a proper subset of A which is a contradiction.

7 Metric Spaces & Closed/Open

7.1 Metric Spaces

Definition 7.1.1: Metric Spaces

A set X is a metric space if for ant p,q $\in$ X, there is an associated d(p,q) $\in \mathbb{R}$ such that:

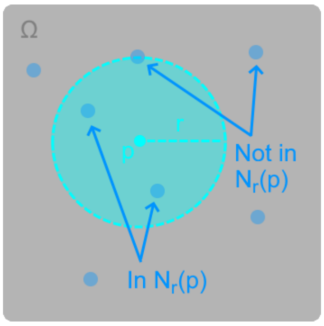
- d(p,q) > 0 if p $\neq$ q
- d(p,q) = 0 if and only if p = q
- Symmetry: d(p,q) = d(q,p)
- Triangle Inequality: d(p,q) $\leq$ d(p,r) + d(r,q) for any r $\in$ X.

For euclidean spaces $\mathbb{R}^k$, d(x,y) = $|x - y|$ where x,y $\in \mathbb{R}^k$.

Definition 7.1.2: Types of Points and Sets

(a) Neighborhood

For p $\in \mathbb{R}^k$ and r > 0, N$_r(p)$ is the set of all q $\in$ X where d(q,p) < r



(b) Limit Points and Closed Sets

Closed set E contain all p $\in$ X where every N$_r(p)$ contain a q $\neq$ p $\in$ E

- Limit Points

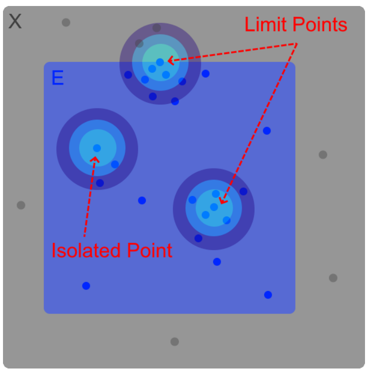
  For point p $\in$ X, every N$_r(p)$ contains a q $\neq$ p $\in$ E
  The set of all limit points of E = E'

- Isolated Points

  If p $\in$ E is not a limit point of E

- Closed

  If every limit point p of E is a p $\in$ E



(c) Interior Points and Open Sets

Open set E contains all its p which has a $N_r(p) \subset$ E

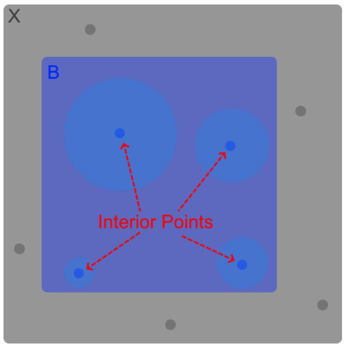
- Interior Point

  For p $\in$ X, there is a $N_r(p) \subset$ E

  The set of all interior points = E^o

- Open

  If every p $\in$ E is an interior point of E



(d) More about Sets

- Bounded

  If there is M $\in \mathbb{R}$, q $\in$ X such that d(p,q) < M for all p $\in$ E

- Complement

  From E, E^c is the set of all p $\in$ X such that p $\notin$ E

- Perfect

  If E is closed and if every p $\in$ E is a limit point of E

- Dense

  If every p $\in$ X is a limit point of E or/and p $\in$ E

- Boundary Point

  For p $\in$ X, if every $N_r(p)$ contains a x $\in$ E and y $\in$ E^c

  The set of all boundary points = ∂E

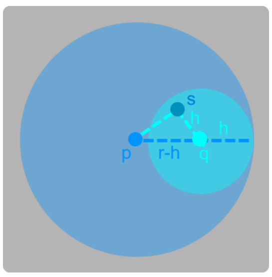
For a metric space X, {X,$\emptyset$} are both open and closed.

Theorem 7.1.3: $N_r(p)$ is open

Every neighborhood is an open set.

Proof

Let q $\in N_r(p)$. Then there is a h > 0 $\in \mathbb{R}$ such that d(q,p) = r - h.
Then for any s $\in N_h(q)$, d(s,p) $\leq$ d(s,q) + d(q,p) = h + (r - h) = r.
Thus, for any q $\in N_r(p)$, there exists a $N_h(q) \subset N_r(p)$.

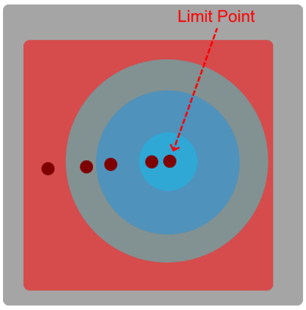


Theorem 7.1.4: If a set has a limit point, there are infinite q $\in$ E in N$_r(p)$

If p is a limit point of set E, then every N$_r(p)$ contains infinitely many q $\in$ E.

Proof

Suppose there is N$_{r_1}(p)$ which contains finitely many q = { q$_1$, ... , q$_n$ }.
Let r = min$_{m\in[1,n]}$ d(p,q$_m$). Then N$_r(p)$ contains no q $\in$ E such that q $\neq$ p.
So, p is not a limit point of E which is a contradiction since p is a limit point of E.



Corollary 7.1.5: Limit points do not exist in finite sets

A finite set E has no limit points. Since $\emptyset \in$ A, all finite set must be closed.

Proof

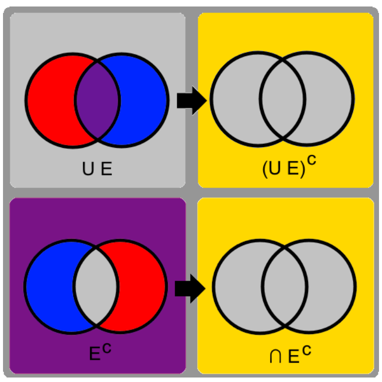
Let p be a limit point of finite set E. By theorem 7.1.4, then any N$_r(p)$ contain infinite q $\in$ E so E is an infinite set which is a contradiction since E is finite.
So p cannot be limit point of E and thus, E has no limit points.

Theorem 7.1.6: De Morgan's Laws

Let $E_1, E_2, \ldots$ be a collection of sets. Then, $(\cup\ E_x)^c = \cap\ (E_x{}^c)$.

Proof

If p $\in (\cup\ E_x)^c$, then p $\notin (\cup\ E_x)$.
Thus, p $\notin E_x$ for any x so p $\in E_x^c$ for all x. Thus, p $\in \cap\ (E_x^c)$ so $(\cup\ E_x)^c \subset \cap\ (E_x^c)$.
If p $\in \cap\ (E_x^c)$, then p $\in E_x^c$ for all x.
Thus, p $\notin E_x$ for any x so p $\notin \cup\ E_x$. Thus, p $\in (\cup\ E_x)^c$ so $\cap\ (E_x^c) \subset (\cup\ E_x)^c$.
Thus, $(\cup\ E_x)^c = \cap\ (E_x^c)$.



Theorem 7.1.7: Open set $\rightarrow$ Closed complement

A set E is open if and only if E^c is closed.

Proof

Suppose E is open. Let x be a limit point of E^c.
Then for every r > 0, $N_r(x)$ must contain a $p \in E^c$ such that $p \neq x$.
Then, $N_r(x) \not\subset E$ so x is not an interior point of E and thus, $x \notin E$ so $x \in E^c$.
Since any limit point x of E^c is a $x \in E^c$, then E^c is closed.

Suppose E^c is closed. Let $x \in E$.
Since $x \notin E$, x is not a limit point of E.
Then there exists a r > 0 such that any $p \in N_r(x)$ is not in E.
Thus, every $p \in N_r(x)$ is $p \in E$ so $N_r(x) \subset E$ and thus, x is an interior point of E.
Since any $x \in E$ is an interior point of E, then E is open.

Corollary 7.1.8: Closed set $\rightarrow$ Open complement

A set F is closed if only only if F^c is open.

Proof

From theorem 7.1.7, let $E = F^c$.

Theorem 7.1.9: Union open $\rightarrow$ open and Intersection closed $\rightarrow$ closed

(a) If $\{G_x\}$ is a finite or infinite collection of open sets, then $\cup\, G_x$ is open.
Proof
If $p \in \cup\, G_x$, then $p \in G_x$ for at least one x. Let $\overline{x}$ be such an x.
Since $G_{\overline{x}}$ is open, then p is an interior point of $G_{\overline{x}}$ and thus, there is a $N_r(p)$ such that $N_r(p) \subset G_{\overline{x}} \subset \cup\, G_x$. So p is an interior point of $\cup\, G_x$.
Since any $p \in \cup\, G_x$ is an interior point, then $\cup\, G_x$ is open.

(b) If $\{F_x\}$ is a finite or infinite collection of closed sets, then $\cap\, F_x$ is closed.
Proof
By theorem 7.1.7, any F_x^c is open. Since $\{F_x^c\}$ is a finite or infinite collection of open set, then by part (a), $\cup\, F_x^c$ is open.
Thus, again by theorem 7.1.7, $(\cup\, F_x^c)^c$ is closed.
By theorem 7.1.6, $(\cup\, F_x^c)^c = \cap\, (F_x^c)^c = \cap\, F_x$.

(c) If $G_1, ..., G_n$ is a finite collection of open sets, then $\cap_{x=1}^n G_x$ is open.
Proof
If $p \in \cap_{x=1}^n G_x$, then $p \in G_x$ for all G_x for x = {1, 2, ... , n}.
Since each G_x is open, then for any G_x, there is a $N_{r_x}(p) \subset G_x$.
Let $r = \min(r_1, r_2, ..., r_n)$. Thus, $p \in N_r(p) \subset N_{r_x}(p)$ for all x.
So, $N_r(p) \subset \cap_{x=1}^n G_x$ and thus, p is an interior point of $\cap_{x=1}^n G_x$ so $\cap_{x=1}^n G_x$ is open.
Infinite + Closed: $G_i = (-1/i, 1/i)$ Infinite + Open: $G_i = (-i, i)$

(d) If $F_1, ..., F_n$ is a finite collection of closed sets, then $\cup_{x=1}^n F_x$ is closed.
Proof
By theorem 7.1.7, any F_x^c is open. Since $F_1^c, ..., F_n^c$ is a finite collection of open set, then by part (c), $\cap_{x=1}^n F_x^c$ is open.
Thus, again by theorem 7.1.7, $(\cap_{x=1}^n F_x^c)^c$ is closed.
By theorem 7.1.6, $(\cap_{x=1}^n F_x^c)^c = \cup_{x=1}^n (F_x^c)^c = \cup_{x=1}^n F_x$.
Infinite + Closed: $F_i = [-1/i, 1/i]$ Infinite + Open: $F_i = [1/i, \infty)$

Theorem 7.1.10: E' is closed

Let E ⊂ X. Then, (E')' ⊂ E'. Thus, E' is closed.

Proof

If x ∈ (E')', then for every $N_{r_1}(x)$, there is a y ≠ x where y ∈ E'.
Since y ∈ E', then for every $N_{r_2}(y)$, there is a z ≠ y where z ∈ E.
Let r = $r_1 + r_2$.
Then for every $N_r(x)$, there exists a z ≠ x where z ∈ E. Thus, x ∈ E' so (E')' ⊂ E'.

Theorem 7.1.11: E^o is open

Let E ⊂ X. Then, E^o is open.

Proof

If p ∈ E^o, there is a r > 0 such that $N_r(p)$ ⊂ E.
Then for 0 < s < r, $N_s(p) \subset N_r(p)$ so any q ∈ $N_s(p)$ is q ∈ E^o.
Since any p ∈ E^o have a $N_s(p) \subset E^o$, then E^o is open.

7.2 Intervals and Balls

Definition 7.2.1: Segments and Intervals

In ℝ, a segement is an open interval (a,b) = { x ∈ ℝ : a < x < b }
In ℝ, a interval is a closed interval [a,b] = { x ∈ ℝ : a ≤ x ≤ b }

Definition 7.2.2: Open Balls

In $\mathbb{R}^k$, an open ball of radius r > 0 centered at p is:

$N_r(p)$ = { x ∈ $\mathbb{R}^k$: $|x-p|$ < r } = { x ∈ $\mathbb{R}^k$: d(x,p) < r }

A closed ball has d(x,p) ≤ r.

Definition 7.2.3: Convex

E ⊂ $\mathbb{R}^k$ is convex if for all x,y ∈ E and t ∈ [0,1], tx + (1-t)y ∈ E.

Example 7.2.4: Balls are convex

Balls in $\mathbb{R}^k$ are convex.

Proof

Let x,y ∈ open ball $N_r(p)$. Let z = tx + (1-t)y for t ∈ [0,1].
Since $|x-p| < r$ and $|y-p| < r$:

$$\begin{aligned} |z-p| &= |tx+(1-t)y-p| = |tx+(1-t)y-tp+(t-1)p| \\ &= |t(x-p)+(1-t)(y-p)| \leq t|(x-p)| + (1-t)|(y-p)| \\ &< \text{tr} + (1\text{-t})\text{r} = \text{r} \end{aligned}$$

Thus, z ∈ $N_r(p)$ so balls are convex. Same proof applies to closed balls.

Definition 7.2.5: Dense

E ⊂ X is dense if every x ∈ X is either in E or a limit point of E.

Example 7.2.6: ℚ is dense in ℝ

Let X = ℝ. Then, E = ℚ is dense in ℝ.

Proof

Fix x ∈ ℝ and r > 0. There is a q ∈ ℚ such that x-r < q < x. So for any r > 0 and q ∈ ℚ, q ≠ x and q ∈ $N_r(x)$. Thus, every x ∈ ℝ is a limit point of ℚ.

8 Closure, Open Relative, & Compact

8.1 Closure

Definition 8.1.1: Closure

Let E $\subset$ metric space X and E' be the set of all limit points of E in X.

Then the closure of E: $\overline{E}$ = E $\cup$ E'

with the properties:

(a) $\overline{E}$ is closed

(b) E = $\overline{E}$ if and only if E is closed

(c) $\overline{E}$ $\subset$ F for every closed F $\subset$ X such that E $\subset$ F

Proof

Suppose x $\in$ X, but x $\notin$ $\overline{E}$. Thus, x $\in$ $\overline{E}^c$.
Thus, there is a N$_r(x)$ $\subset$ $\overline{E}^c$ since else there is always a p $\in$ N$_r(x)$ where p $\in$ $\overline{E}$ so x is a limit point of $\overline{E}$ so x $\in$ $\overline{E}$. Thus, $\overline{E}^c$ is open so $\overline{E}$ is closed by theorem 7.1.7.

If E = $\overline{E}$, then by part (a), E is closed.
If E is closed, then E' $\subset$ E so E = E $\cup$ E' = $\overline{E}$.

If closed set F, then F' $\subset$ F and since E $\subset$ F, then E' $\subset$ F' $\subset$ F. Thus, $\overline{E}$ $\subset$ F.

Theorem 8.1.2: sup(E) $\in$ $\overline{E}$

Let non-empty set of real numbers, E, be bounded above. Let y = sup(E).
Then, y $\in$ $\overline{E}$. Thus, y $\in$ E if E is closed and y $\notin$ E if E is open in $\mathbb{R}$.

Proof

If y $\in$ E, then y $\in$ $\overline{E}$. Suppose y $\notin$ E.
For every h > 0, there exists a x $\in$ E such that y-h < x < y otherwise y-h is an upper bound for E which is a contradiction since y = sup(E).
Thus, y is a limit point of E so y $\in$ E'.
If E is closed, then y $\in$ E since y $\in$ E'. Also, y $\in$ $\overline{E}$.
If E is open, then any N$_r(y)$ $\not\subset$ E since N$_r(y)$ in $\mathbb{R}$ must contain a γ > y so y $\notin$ E^o.

8.2 Open Relative

Definition 8.2.1: Open Relative

Suppose E $\subset$ Y $\subset$ metric space X.
Then E is open relative to Y if for each p $\in$ E, there is an r > 0
such that for any q $\in$ Y, then q $\in$ E if d(q,p) < r.

Theorem 8.2.2: E is open relative to Y ⊂ X if E = Y ∩ G and G is open in X

Suppose E ⊂ Y ⊂ X.

E is open relative to Y if and only if E = Y ∩ G for some open G ⊂ X.

Proof:

Suppose E is open relative to Y.

Then for each p ∈ E, there is a $r_p > 0$ such that for any q ∈ Y where $d(p,q) < r_p$, then q ∈ E.

Since Y ⊂ X, let V_p be the set of all q ∈ X such that $d(p,q) < r_p$ and define $G = \cup_{p\in E} V_p$. Since V_p is open by theorem 7.1.3, then by theorem 7.1.9a, open G ⊂ X.

Since $p \in V_p$ for all p ∈ E, then E ⊂ G ∩ Y. Also, by construction, then $V_p \cap Y \subset E$ so G ∩ Y ⊂ E. Thus, E = Y ∩ G.

If G is open in X and E = G ∩ Y, then every p ∈ E has a $V_p \subset G$.

Then, $V_p \cap Y \subset G \cap Y = E$ so E is open relative to Y.

8.3 Compact Sets

Definition 8.3.1: Open Cover

An open cover of set E ⊂ X is a collection of open $G_1, G_2, ... \subset X$ such that $E \subset \cup\, G_i$.

Definition 8.3.2: Compact

K ⊂ X is compact if every open cover of K contains a finite subcover.

If $G_1, G_2, ...$ is an open cover of K, then $K \subset \cup_{i=1}^{n} G_i$ for some n.

Theorem 8.3.3: A compact set is compact in every metric space

Suppose K ⊂ Y ⊂ X.

Then K is compact relative to X if and only if K is compact relative to Y.

Proof

Suppose K is compact relative to X.

Let $V_1, V_2, ...$ be sets open relative to Y such that $K \subset \cup\, V_x$. Then by theorem 8.2.2 for each V_x, there is a G_x open relative to X where $V_x = Y \cap G_x$.

Since K is compact relative to X, then there is a n such that $K \subset G_{x_1} \cup ... \cup G_{x_n}$.

Thus, $K = K \cap Y \subset (\cup_{i=1}^{n} G_{x_i}) \cap Y = (\cup_{i=1}^{n} G_{x_i} \cap Y) = \cup_{i=1}^{n} V_{x_i}$.

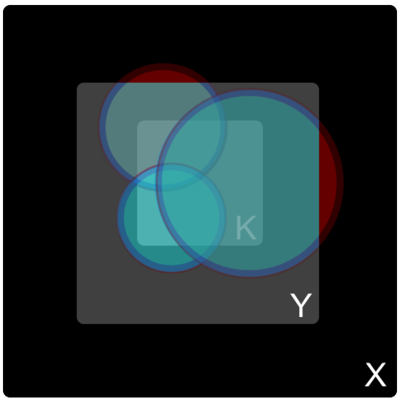
Since there are open $V_{x_1}, ..., V_{x_n}$ where $K \subset \cup_{i=1}^{n} V_{x_i}$ so K is compact relative to Y.

Suppose K is compact relative to Y.

Let open $G_1, G_2, ... \subset X$ such that $X \subset \cup\, G_x$. For each G_x, let $V_x = Y \cap G_x \subset Y$.

Since K is compact relative to Y, there is a n such that $K \subset \cup_{i=1}^{n} V_{x_i}$.

Thus, $K \subset \cup_{i=1}^{n} V_{x_i} = \cup_{i=1}^{n} (Y \cap G_{x_i}) \subset \cup_{i=1}^{n} G_{x_i}$ so K is compact relative to X.

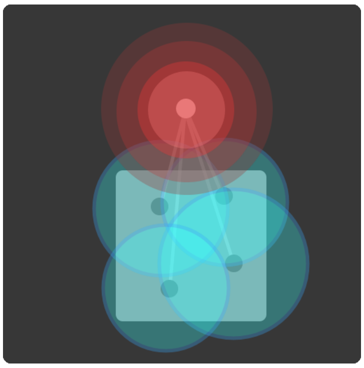


Theorem 8.3.4: A compact set is closed

Compact subsets of metric spaces are closed.

Proof

Let compact K ⊂ X. Suppose p ∈ X, but p ∉ K so p ∈ K^c.
If q ∈ K, let W_q be a neighborhood of q with r < $\frac{1}{2}$d(p,q). Let $V_{p,q}$ be a neighbohood of p with r < $\frac{1}{2}$d(p,q). Since K is compact, then there are finite points $q_1, ..., q_n$ such that K ⊂ W where W = $W_{q_1} \cup ... \cup W_{q_n}$.
Let V = $V_{p,q_1} \cap ... \cap V_{p,q_n}$, then K ∩ V ⊂ W ∩ V = ∅ so V ⊂ K^c.
Since there is a neighborhood V for p ∈ K^c where V ⊂ K^c, then every p ∈ K^c is an interior point so K^c is open. Then by theorem 7.1.7, K is closed.



Theorem 8.3.5: If closed E ⊂ compact set K, E is compact

Closed subsets of compact sets are compact.

Proof

Suppose F ⊂ K ⊂ X where F is closed relative to X and K is compact.
Let $V_1, V_2, ...$ be an open cover for F. Let open set F^c be all k ∈ K where k ∉ F.

K = F ∪ F^c ⊂ $V_1 \cup V_2 \cup ... \cup F^c$

Thus, $V_1 \cup V_2 \cup ... \cup F^c$ is an open cover for K.
Since K is compact, there is a finite subcover Ω that covers K and thus, finite subcover Ω covers F ∪ F^c.
Remove F^c from Ω. Since finite subcover Ω - F^c covers F, then F is compact.

Corollary 8.3.6: Closed F ∩ compact K = compact

If F is closed and K is compact, then F ∩ K is compact.

Proof

Since K is compact, then K is closed by theorem 8.3.4.
Then, by 7.1.9b, F ∩ K is closed.
Since F ∩ K ⊂ K, then by theorem 8.3.5, F ∩ K is compact.

Theorem 8.3.7: Nonempty $\cap_{i=1}^{n} K_i \to$ nonempty $\cap K_i$

For compact sets $K_1, K_2, ... \subset$ X where any intersection of finite K_i is nonempty, then $\cap K_i$ is nonempty.

Proof

Fix K_1. If there is a k $\in K_1$ where k $\in K_i$ for all i, then k $\in \cap K_i$ so $\cap K_i \neq \emptyset$.
Suppose for every k $\in K_1$, k $\notin K_i$ for some i.
Then for every k $\in K_1$, there is a K_i such that p $\notin K_i$ so p $\in K_i^c$.
Thus, $K_2^c, k_3^c, ...$ form an open cover for K_1.
Since K_1 is compact, there is a n where $K_1 \subset K_{i_1}^c \cup ... \cup K_{i_n}^c$.
But then, $K_1 \cap K_{i_1} \cap ... \cap K_{i_n} = \emptyset$ which is a contradiction.

Corollary 8.3.8: Nonempty K_i where $K_{i+1} \subset K_i \to$ nonempty $\cap K_i$

If $K_1, K_2, ...$ is a sequence of nonempty compact sets such that $K_{i+1} \subset K_i$, then $\cap K_i$ is nonempty.

Proof

Since each K_i is nonempty and if $i_1 < ... < i_n$, then $K_{i_1} \cap ... \cap K_{i_n} = K_{i_n}$ is nonempty, then by theorem 8.3.7, $\cap K_i$ is nonempty.

Theorem 8.3.9: Nonempty intervals I_n where $I_{n+1} \subset I_n \to$ nonempty $\cap I_n$

If $I_1, I_2, ...$ is a sequence of intervals in $\mathbb{R}^1$ such that $I_{n+1} \subset I_n$,
then $\cap I_n$ is nonempty.

Proof

Let $I_n = [a_n, b_n]$ and thus, each I_n is nonempty. If $n_1 < ... < n_m$, then
$I_{n_1} \cap ... \cap I_{n_m} = [a_{n_m}, b_{n_m}]$ is nonempty. Thus, by theorem 8.3.7, $\cap I_n$ is nonempty.

Theorem 8.3.10: p $\in$ E' exists if infinite E $\subset$ compact K

If E is an infinite subset of compact set K, then E has a limit point in K.

Proof

If no p $\in$ K is a p $\in$ E', then each p would have a neighbohood V_p contains at most p $\in$ E if p $\in$ E. Thus, there is no finite subcover that covers E and thus, there is no finite subcover that covers K since E $\subset$ K which contradicts K is compact.

Definition 8.3.11: K-cells

The set of all x = $(x_1, ..., x_k) \in \mathbb{R}^k$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.

Theorem 8.3.12: K-cells are compact

Every k-cell is compact.

Proof

Let k-cell I consists of all x = $(x_1, ..., x_k)$ where $x_i \in [a_i, b_i]$ for fixed $a_i, b_i \in \mathbb{R}$.
Let $\delta = \sqrt{\sum_{i=1}^{k} (b_i - a_i)^2}$. Thus, $|x - y| \leq \delta$ for $x, y \in$ I.
Suppose there exists an open cover $G_1, G_2, ...$ of I which contain no finite subcover.
Let $c_i = \frac{a_i + b_i}{2}$. Then each interval splits into $[a_i, c_i]$ and $[c_i, b_i]$ for i $\in$ [1,k] so there now exists 2^k k-cells Q_i whose union is I.
At least one Q_i cannot be covered else I would be covered. Then subdivide Q_i as before and repeating the process so $Q_{i+1} \subset Q_i$ and each are not covered.
However, there is a point x* $\in Q_{i_j}$ for all j such that $N_r(x^*) \subset G$ so Q_{i_1} is covered which is a contradiction.

Theorem 8.3.13: Heine-Borel Theorem

If a set E $\subset \mathbb{R}^k$ has one of the three properties, then it has the other two:

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

Proof

Suppose E is closed and bounded.
Then there exists a M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$ such that d(p,q) < M for all p $\in$ E.
Thus, there is a k-cell K = $[-M+q_1,q_1+M] \times ... \times [-M+q_k,q_k+M]$ such that E $\subset$ K. Then by theorem 8.3.12, K is compact and thus by theorem 8.3.5, E is compact so (a) $\rightarrow$ (b).
Then by thereom 8.3.10, any infinite subset of E has a limit point in E so (b) $\rightarrow$ (c).
Suppose E is not bounded.
Then there exists p $\in$ E such that d(p,q) > M for any M $\in \mathbb{R}$ and q $\in \mathbb{R}^k$.
Let S $\subset$ E be such points p.
Then S is infinite else there is a maximal p and thus, p is bounded. Thus, S is infinite and contains no limit points in E since any d(p_1,p_2) > M which contradicts that every infinite subset of E has a limit point in E. Thus, E is bounded.
Suppose E is not closed.
Then there exists a p $\in$ E', but p $\notin$ E. Since p is a limit point, then there is a q $\in$ E such that $\frac{1}{n+1}$ < d(q,p) < $\frac{1}{n}$ for n = {1, 2, ...}.
Let S $\subset$ E be such points q.
Thus, p is the only limit point of S since for r < $\frac{1}{n}$, any N$_r(q_i)$ contains no points of S other than q_i since d(q_i,q_j) > $\frac{1}{n}$ for any q_1,q_2 $\in$ S.
Thus, S is infinite, but the only p $\in$ S' is p $\notin$ E which contradicts that every infinite subset of E has a limit point in E. Thus, E is closed. So, (c) $\rightarrow$ (a).

Theorem 8.3.14: Weierstrass Theorem

Every bounded infinite set E $\subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof

Since E is bounded, then there exists a k-cell K such that E $\subset$ K. Since K is compact, then by theorem 8.3.10, E has a limit point in K and thus, in $\mathbb{R}^k$.

9 Perfect and Connected Sets

9.1 Perfect Sets

Definition 9.1.1: Perfect Set

E ⊂ X is perfect if E is closed and if every p ∈ E is p ∈ E'.

Theorem 9.1.2: Perfect sets are uncountable

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then, P is uncountable.

Proof

Since P has limit points, then by theorem 7.1.4, P is infinite.
Suppose P is countable. Then let $x_1, x_2, ... \in$ P.
Let V_i be a neighborhood of x_i where y ∈ V_i for any y ∈ $\mathbb{R}^k$ such that $|y - x_i| <$ r.
Thus, the $\overline{V_i}$ is the set of all y ∈ $\mathbb{R}^k$ such that $|y - x_i| \leq$ r.
Since every x_i are limit points, then any $V_i \cap$ P is not empty where there is a V_{i+1}

(a) $\overline{V_{i+1}} \subset V_i$

(b) $x_i \notin \overline{V_{i+1}}$

(c) $V_{i+1} \cap$ P is nonempty

Let $K_i = \overline{V_i} \cap$ P. Since $\overline{V_i}$ is closed and bounded, then by theorem 8.3.11, $\overline{V_i}$ is compact. Since $x_i \notin K_{i+1}$, then no $x_i \in$ P is $x_i \in \cap K_i$. Since $K_n \subset$ P, then $\cap K_i$ is nonempty which contradicts corollary 8.3.8 since each K_i is empty and $K_{i+1} \subset K_i$.

Corollary 9.1.3: $\mathbb{R}$ is not countable

Every interval [a,b] is uncountable. Thus, $\mathbb{R}$ is uncountable.

Proof

Since [a,b] is closed and every p ∈ [a,b] is a limit point, then nonempty set [a,b] is perfect. Thus, by theorem 9.1.2, [a,b] is uncountable.

Definition 9.1.4: Cantor Sets

There exists perfect segments in $\mathbb{R}^1$ which contain no segment.
Let E_0 = [0,1].
For E_1, remove $(\frac{1}{3},\frac{2}{3})$. Thus, $E_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$.
For E_2, remove $(\frac{1}{9},\frac{2}{9})$ and $(\frac{7}{9},\frac{8}{9})$. Thus, $E_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{3}{9}] \cup [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},1]$.
Continuing such a sequence, the set of compact sets E_n are such that:

(a) $E_{n+1} \subset E_n$

(b) E_n is the union of 2^n intervals each of length 3^{-n}.

P = $\cap E_n$ is called the Cantor set. P is compact and nonempty.
Thus, any segment of form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$ where k,m ∈ $\mathbb{Z}_+$ has no points in common with P. Since any segment (a,b) contain a segment of such a form since $3^{-m} < \frac{b-a}{6}$, then P contains no segment.
Let x ∈ P and segment S contain x. Let I_n be an interval of E_n containing x.
Then choose a large enough n so $I_n \subset$ S.
Let x_n be an endpoint of I_n where $x_n \neq$ x and thus, x is a limit point. Since P is closed and every p ∈ P is p ∈ P', then P is perfect.

9.2 Connected Sets

Definition 9.2.1: Connected Set

A, B $\subset$ X are separated if both A $\cap$ $\overline{B}$ and $\overline{A}$ $\cap$ B are empty.
E $\subset$ X is connected if E is not the union of two nonempty separated sets.
Separated sets are disjoint, but disjoint sets need not be separated.

Theorem 9.2.2: All points between points in connected sets exists

E $\subset$ $\mathbb{R}^1$ is connected if and only if:

If x,y $\in$ E and x $<$ z $<$ y, then z $\in$ E.

Proof

If there exists x,y $\in$ E and z $\in$ (x,y) such that z $\notin$ E, then E $=$ $A_z \cup B_z$ where A_z $=$ E $\cap$ $(-\infty, z)$ and B_z $=$ E $\cap$ (z, ∞).
Since x $\in$ A_z and y $\in$ B_z, then A and B are nonempty. Since $A_z \subset (-\infty, z)$ and B_z $=$ (z,∞), then A_z and B_z are separated. Thus, E is not connected.

Suppose E is not connected. Then, there are nonempty separated sets A and B such that A $\cup$ B $=$ E. Pick x $\in$ A, y $\in$ B where x $<$ y. Let z $=$ sup(A $\cap$ [x,y]).
Since, z $\in$ $\overline{A}$ so z $\notin$ B, then x $\leq$ z $<$ y. If z $\notin$ A, then x $<$ z $<$ y so z $\notin$ E.
If z $\in$ A, then z $\notin$ $\overline{B}$ and thus, there exists a z_1 such that z $<$ z_1 $<$ y and z_1 $\notin$ B.
Then, x $<$ z_1 $<$ y so z_1 $\notin$ E.

10 Convergent and Cauchy Sequences

10.1 Convergent Sequences

Definition 10.1.1: Convergent Sequence

A sequence $\{x_n\}$ in metric space X converge if there is a x $\in$ X such that:

For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for all n $\geq$ N, d(x_n,x) $< \epsilon$

Then, $\{x_n\}$ converges to x: $\lim_{n\to\infty} x_n$ = x

If $\{x_n\}$ does not converge, then it diverges.

Example 10.1.2

(a) Let $x_n = \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\lim_{n\to\infty} x_n = 0$

Proof

For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$. Then:

d(x_n,0) $= |x_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$

(b) Let $x_n = (-1)^n + \frac{1}{n}$ in $\mathbb{R}^2$. Then, $\{x_n\}$ diverges.

Proof

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (-1)^n + \lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} (-1)^n$

Since $(-1)^n$ alternates between -1 and 1, then $\{x_n\}$ diverges.

Theorem 10.1.3: A convergent sequence is unique

(a) $\{p_n\}$ converges to p $\in$ X if and only if every N$_r(p)$ contains p_n for all, but finitely many n.

Proof

Suppose $p_n \to$ p. Then for N$_\epsilon(p)$, any q $\in$ X such that d(q,p) $< \epsilon$ is q $\in$ N$_\epsilon(p)$. Since $p_n \to$ p, there is a N such that for n $\geq$ N, d(p_n,p) $< \epsilon$. Thus, for n $\geq$ N, $p_n \in$ N$_\epsilon(p)$.

Suppose every N$_r(p)$ contains p_n for all, but finitely many n.

For $\epsilon > 0$, let N$_\epsilon(p)$ be the set of all q $\in$ X such that d(p,q) $< \epsilon$. Thus, there exists an N such that $p_n \in$ N$_\epsilon(p)$ if n $\geq$ N.

Thus, d(p_n,p) $< \epsilon$ so $p_n \to$ p.

(b) If p,p' $\in$ X and $\{p_n\}$ converges to p and p', then p = p'.

Proof

For $\epsilon > 0$, there exists N,N' such that:

d(p_n,p) $< \frac{\epsilon}{2}$ for n $\geq$ N   d(p_n,p') $< \frac{\epsilon}{2}$ for n $\geq$ N'

Then for n $\geq$ max(N,N'):

d(p,p') $\leq$ d(p,p_n) + d(p_n,p') $< \epsilon$

Thus, p = p'.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.

Proof

If $\{p_n\} \to$ p, there is a N such that for n > N, d(p_n,p) < 1.

Let r = max(1, d(p_1,p), ... , d(p_N,p)). Thus for all n, d(p_n,p) $\leq$ r.

(d) If E $\subset$ X and p $\in$ E', there is a $\{p_n\}$ in E such that p = $\lim_{n\to\infty} p_n$.

Proof

Since p $\in$ E', then for each n $\in \mathbb{Z}_+$, there is a $p_n \in$ E such that d(p_n,p) $< \frac{1}{n}$. For $\epsilon > 0$, there is a $\frac{1}{N} < \epsilon$ so for n $\geq$ N, d(p_n,p) $< \frac{1}{n} < \frac{1}{N} < \epsilon$.

Thus, p = $\lim_{n\to\infty} p_n$.

Theorem 10.1.4: Arithmetic Operations for Sequences

Suppose $\{s_n\}$,$\{t_n\} \in \mathbb{C}$ where $\lim_{n\to\infty} s_n$ = s and $\lim_{n\to\infty} t_n$ = t.

(a) $\lim_{n\to\infty} s_n + t_n$ = s + t

Proof

For $\epsilon > 0$, there exists N_1, N_2 such that

$|s_n - s| < \frac{\epsilon}{2}$ for n $\geq N_1$ $\quad$ $|t_n - t| < \frac{\epsilon}{2}$ for n $\geq N_2$

If N = max(N_1,N_2), then for n $\geq$ N:

$|s_n + t_n - s + t| \leq |s_n - s| + |t_n - t| < \epsilon$

(b) $\lim_{n\to\infty} cs_n$ = cs and $\lim_{n\to\infty} c + s_n$ = c + s

Proof

For $\epsilon > 0$, there exists a N such that

$|s_n - s| < \frac{\epsilon}{c}$ for n $\geq$ N

$|cs_n - cs| \leq$ c $\cdot$ $|s_n - s| < \epsilon$

(c) $\lim_{n\to\infty} s_n t_n$ = st

Proof

Note $s_n t_n$ - st = $(s_n - s)(t_n - t)$ + t(s_n - s) + s(t_n - t).

For $\epsilon > 0$, there exists N_1,N_2 such that

$|s_n - s| < \sqrt{\epsilon}$ for n $\geq N_1$ $\quad$ $|t_n - t| < \sqrt{\epsilon}$ for n $\geq N_2$

If N = max(N_1,N_2), then for n $\geq$ N, $|(s_n - s)(t_n - t)| < \epsilon$.

Thus, $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$.

$$\lim_{n\to\infty} (s_n t_n - st) = \lim_{n\to\infty} (s_n - s)(t_n - t) + t(s_n - s) + s(t_n - t)$$
$$= 0 + \text{t} \cdot 0 + \text{s} \cdot 0 = 0$$

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ where $s_n, s \neq 0$

Proof

Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if n $\geq$ m so $|s_n| > \frac{1}{2}|s|$ for n $\geq$ m.

For $\epsilon > 0$, there is a N > m such that for n $\geq$ N, $|s_n - s| < \frac{1}{2}|s|^2\epsilon$.

Thus, for n $\geq$ N, $|\frac{1}{s_n} - \frac{1}{s}| = \frac{s_n - s}{s_n s} < \frac{2}{|s|^2}|s_n - s| < \epsilon$.

Theorem 10.1.5: Extension to $\mathbb{R}^k$

(a) Suppose $x_n \in \mathbb{R}^k$ and $x_n = (\alpha_{n_1}, \dots , \alpha_{n_k})$. Then $\{x_n\}$ converges to x = $(\alpha_1, \dots , \alpha_k)$ if and only if $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].

Proof

Suppose $\{x_n\}$ converges to x = $(\alpha_1, \dots , \alpha_k)$.

Since for any i $\in$ [1,k], $|\alpha_{n_i} - \alpha_i| \leq |x_n - x| < \epsilon$. Then, $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$.

Suppose $\lim_{n\to\infty} \alpha_{n_i} = \alpha_i$ for i $\in$ [1,k].

Then for $\epsilon > 0$, there is an N such that for n $\geq$ N:

$|\alpha_{n_i} - \alpha_i| < \frac{\epsilon}{\sqrt{k}}$ for i $\in$ [1,k]

$|x_n - x| = \sqrt{\sum_{i=1}^k |\alpha_{n_i} - \alpha_i|^2} < \sqrt{k \cdot (\frac{\epsilon}{\sqrt{k}})^2} = \epsilon$

(b) Suppose $\{x_n\}$,$\{y_n\} \in \mathbb{R}^k$ and $\{\beta_n\} \in \mathbb{R}$ and $x_n \to$ x, $y_n \to$ y, $\beta_n \to \beta$.

$\lim_{n\to\infty} x_n + y_n$ = x+y $\quad$ $\lim_{n\to\infty} x_n \cdot y_n$ = x$\cdot$y $\quad$ $\lim_{n\to\infty} \beta_n x_n = \beta$x

Proof

By part a, then $\lim_{n\to\infty} x_{n_i} + y_{n_i} = x_i + y_i$ so $\{x_n + y_n\} \to$ x+y.

Also, $\lim_{n\to\infty} \sum_{i=1}^k x_{n_i} y_{n_i} = \sum_{i=1}^k x_i y_i$ so $\{x_n \cdot y_n\} \to$ x$\cdot$y.

Also, $\lim_{n\to\infty} \beta_i x_{n_i} = \beta_i x_i$ so $\{\beta_n x_n\} \to \beta x$.

10.2 Subsequences

Definition 10.2.1: Subsequence

For sequence $\{p_n\}$, let $\{n_k\} \in \mathbb{Z}_+$ where $n_k < n_{k+1}$.
Then $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$.
If $\{p_{n_k}\}$ converges, then its limit is called a subsequential limit.

Theorem 10.2.2: $\{p_n\} \to$ p $\rightleftharpoons \{p_{n_k}\} \to$ p

$\{p_n\}$ converges to p if and only if every subsequence converges to p.

Proof

Suppose $\{p_n\}$ converges to p.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, $|p_n - p| < \epsilon$.
Let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$.
Then for $n_k \geq$ N, $|p_{n_k} - p| < \epsilon$. Thus, every $\{p_{n_k}\} \to$ p.
Suppose every subsequence converges to p.
Since $\{p_n\}$ is a subsequence of itself, then $\{p_n\}$ converges to p.

Theorem 10.2.3: $\{p_n\}$ in compact space have $\{p_{n_k}\} \to$ p

(a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence converges to p $\in$ X.

Proof

Let E be the range of $\{p_n\}$.
If E is finite, there is a p $\in$ E and sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $p_{n_1} = p_{n_2} = ... =$ p. Thus, $\{p_{n_k}\} \to$ p.
If E is infinite, then by theorem 8.3.10, then there exists a p $\in$ X.
Then choose n_k such that d(p_{n_k},p) $< \frac{1}{k}$. Thus, $\{p_{n_k}\} \to$ p.

(b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

Proof

Since every bounded set lies in a compact space in $\mathbb{R}^k$, then by part a, every bounded sequence contains a convergent subsequence.

Theorem 10.2.4: The set of subsequential limits is closed

The subsequential limits of $\{p_n\}$ in metric space X form a closed subset of X.

Proof

Let E be the range of the set of all subsequential limits of $\{p_n\}$.
If E is empty, then E is closed. If E is finite, then E' is empty so E is closed.
Suppose E is infinite. Then, let q $\in$ E'.
Choose n_1 so $p_{n_1} \neq$ q. Let $\frac{\epsilon}{2} =$ d(p_{n_1},q).
Since q $\in$ E', there is a x $\in$ E where d(x,q) $< \frac{\epsilon}{2}$.
Since x $\in$ E, then there is a $\{p_{n_k}\} \to$ x so d(p_{n_k},x) $< \frac{\epsilon}{2}$.
Thus, d(p_{n_k},q) $\leq$ d(p_{n_k},x) + d(x,q) $< \epsilon$ so q is a subsequential limit of $\{p_n\}$.
Thus, q $\in$ E so E is closed.

10.3 Cauchy Sequences

Definition 10.3.1: Metric Spaces

Sequence $\{p_n\} \in$ X is a Cauchy sequence if:

For every $\epsilon > 0$, there is a N $\in \mathbb{Z}$ such that for all n,m $\geq$ N, d(p_n,p_m) $< \epsilon$

Let nonempty E $\subset$ X and S $\subset \mathbb{R}$ of d(p,q) where p,q $\in$ E.

Let sup(S) = diam(E). If $\{p_n\} \in$ X, and $p_N, p_{N+1}, ... \in E_N$, then $\{p_n\}$ is a Cauchy sequence if and only if $\lim_{N\to\infty}$ diam(E_N) = 0.

Theorem 10.3.2: Cauchy sequences and its closure have the same diam

(a) If $\overline{E} \subset$ X, then diam($\overline{E}$) = diam(E).

Proof

Since E $\subset \overline{E}$, then diam(E) $\leq$ diam($\overline{E}$).

For $\epsilon > 0$, let p,q $\in$ E'.

Thus, there are p',q' $\in$ E such that d(p',p) $< \epsilon$ and d(q',q) $< \epsilon$. Thus:

d(p,q) $\leq$ d(p,p') + d(p',q') + d(q',q) $< 2\epsilon$ + d(p',q') $\leq 2\epsilon$ + diam(E).

Thus, diam($\overline{E}$) $\leq 2\epsilon$ + diam(E) so diam($\overline{E}$) = diam(E).

(b) If K_n is a sequence of compact sets of X such that $K_{n+1} \subset K_n$ and $\lim_{n\to\infty}$ diam(K_N) = 0, then $\cap\, K_n$ consist of only one point.

Proof

Let K = $\cap\, K_n$. Since K_n is a sequence of compact sets, then by Corollary 8.3.8 , K is nonempty.

If K contains more than one point, then diam(K) $>$ 0.

But since K $\subset K_n$, then diam(K) $\leq$ diam(K_n) which contradicts that diam(K_n) $\to$ 0.

Theorem 10.3.3: Cauchy sequences are convergent

(a) In every metric space, every convergent sequence is a a Cauchy sequence.

Proof

If $p_n \to$ p and $\epsilon > 0$, there is a N such that for all n $\geq$ N, d(p,p_n) $< \epsilon$.

Thus, for m,n $\geq$ N:

d(p_n,p_m) $\leq$ d(p_n,p) + d(p,p_m) $< 2\epsilon$.

Thus, $\{p_n\}$ is a Cauchy sequence.

(b) If $\{p_n\}$ is a Cauchy sequence in compact metric space X, then $\{p_n\}$ converges to some p $\in$ X.

Proof

Let $\{p_n\}$ be a Cauchy sequence in compact space X.

Let $p_N, p_{N+1}, ... \in E_N$.

Since $\{p_n\}$ is a Cauchy sequence, then $\lim_{N\to\infty}$ diam($\overline{E_N}$) = 0. Since $\overline{E_N}$ is closed in a compact set, then by theorem 8.3.5, $\overline{E_N}$ is compact.

Since $E_{N+1} \subset E_N$, then $\overline{E_{N+1}} \subset \overline{E_N}$ and thus, by theorem 10.3.2b, then there is a unique p $\in \overline{E_N}$ for every N.

Then for $\epsilon > 0$, there is a N_0 such that for N $\geq N_0$ diam($\overline{E_{N_0}}$) $< \epsilon$.

Since p $\in \overline{E_N}$, then d(p,q) $< \epsilon$ for every q $\in \overline{E_N}$ so every q $\in E_N$.

Thus, $\{p_n\} \to$ p.

(c) In $\mathbb{R}^k$, every Cauchy sequence converges.
Proof
Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Let $x_N, x_{N+1}, ... \in E_N$.
Then for some N, diam(E_N) < 1. Thus, the range of $\{x_n\} = E_N \cup \{x_1, ..., x_{N-1}\}$. Thus, $\{x_n\}$ is bounded.
Thus, the $\overline{\{x_n\}}$ is closed and bounded so by theorem 8.3.13, $\overline{\{x_n\}}$ is compact. Thus, by part b, $\{x_n\}$ converges to some p $\in \mathbb{R}^k$.

Definition 10.3.4: Complete

A metric space where every Cauchy sequence converges is complete.
Thus, by theorem 10.3.3, all compact and Euclidean metric spaces are complete.

Definition 10.3.5: Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is:
(a) monotonically increasing if $s_n \leq s_{n+1}$
(b) monotonically decreasing if $s_n \geq s_{n+1}$

Theorem 10.3.6: Monotonic sequences converge if bounded

Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof
Suppose $s_n \leq s_{n+1}$. Let E be the range of $\{s_n\}$.
Suppose $\{s_n\}$ is bounded.
Let s = sup(E) so $s_n \leq$ s. For every $\epsilon > 0$, there is a N such that s - $\epsilon < s_N \leq$ s else s-ϵ would be an upper bound of E which contradicts s = sup(E).
Since $\{s_n\}$ increases, then for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s so $\{s_n\} \to$ s.
Suppose $\{s_n\}$ converges to s.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s - $\epsilon < s_N \leq s_n \leq$ s.
Thus, $\{s_n\}$ is bounded from above.

Suppose $s_n \geq s_{n+1}$. Let E be the range of $\{s_n\}$.
Suppose $\{s_n\}$ is bounded.
Let s = inf(E) so $s_n \geq$ s. For every $\epsilon > 0$, there is a N such that s $\leq s_N <$ s + ϵ else s+ϵ would be a lower bound of E which contradicts s = inf(E).
Since $\{s_n\}$ decreases, then for n $\geq$ N, s $\leq s_n \leq s_N <$ s + ϵ so $\{s_n\} \to$ s.
Suppose $\{s_n\}$ converges to s.
Then for $\epsilon > 0$, there is a N such that for n $\geq$ N, s $\leq s_n < s_N <$ s + ϵ.
Thus, $\{s_n\}$ is bounded from below.

11 Limits and Special Sequences

11.1 Upper and Lower Limits

Definition 11.1.1: No limit

Let $\{s_n\}$ be a sequence of real numbers such that:

For every real M, there is a N $\in \mathbb{Z}$ such that for n $\geq$ N, $s_n \geq$ M.

Then, $s_n \to +\infty$.

For every real M, there is a N $\in \mathbb{Z}$ such that for n $\geq$ N, $s_n \leq$ M.

Then, $s_n \to -\infty$.

Definition 11.1.2: Upper and Lower Limits

Let $\{s_n\}$ be a sequence of real numbers.

Let E contain all subsequential limits of $\{s_n\}$ plus possibly $\pm\infty$.

Then, the upper limit of $\{s_n\}$:

$s^* = \sup(E)$ $\lim_{n\to\infty} \sup(s_n) = s^*$

Then, the lower limit of $\{s_n\}$:

$s_* = \inf(E)$ $\lim_{n\to\infty} \inf(s_n) = s_*$

Theorem 11.1.3: Upper and Lower limits are unique

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and s^* be the upper limit of $\{s_n\}$. Then:

(a) $s^* \in$ E

Proof

If $s^* = +\infty$, then there is a $\{s_{n_k}\} \to +\infty$ so E is not bounded above.
If $s^* \in \mathbb{R}$, then E is bounded above so $s^* \in$ E'.
Then by theorem 10.2.4, $s^* \in$ E.
If $s^* = -\infty$, then there are no subsequential limits in E since there are no upper bounds. Thus, for every M, there is a N such that for n $\geq$ N, $s_n \to -\infty$ so $-\infty \in$ E.

(b) If x $> s^*$, there is a N such that for n $\geq$ N, $s_n <$ x

Proof

Suppose there is a x $> s^*$ such that $s_n \geq$ x for infinitely many n.
Then, there is a y $\in$ E such that y $\geq$ x $> s^*$ which contradicts $s^* =$ sup(E).

(c) s^* is the only number that satisfies (a) and (b)

Proof

Suppose p,q statisfy part a and b where p $<$ q.
Choose x such that p $<$ x $<$ q.
Since p satisfies b, then $s_n <$ x for n $\geq$ N.
Thus, q cannot satisfy a.

References