Fall Real Analysis

Willie Xie

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Contents

1 The Real Number System

1.1 Number Systems

Integer: Z = {-2, -1, 0, 1, 2, ... }

Rational: Q = $\text{p}_{\overline{q \text{ where p,q} \in \mathbb{N}}}$

*** Q is countable, but fails to have the least upper bound property ***

Example 1.1.1

Let $\alpha \in \mathbb{R}$ where $\alpha^2 = 2$. Then α cannot be rational.

Proof

Let $\alpha = \frac{p}{q}$ where p and q cannot both be even.
Let set A = $\{x \in \mathbb{Q}$ for $x^2 < 2\}$ where A $\neq \emptyset$ and 2 is an upper bound for A.
A has no least upper bound in Q, but A has a least upper bound in R.

1.2 Real Number System

$\mathbb{R}$ is the unique ordered field with the least upper bound property.
$\mathbb{R}$ exists and unique.

Definition 1.2.1

Let S be a set. An order on S is a relation < satisfying two axioms:

- Trichotomy: For all x,y $\in$ S, only one holds true:
  - x < y
  - x = y
  - x > y
- Transitivity: If x < y and y < z, then x < z.

Definition 1.2.2

An ordered set is a set with an order.

Definition 1.2.3

Let S be an ordered set. Let E $\subset$ S.
An upper bound of E is a $\beta \in$ S if x $\leq \beta$ for all x $\in$ E.
If such a β exists, then E is bounded from above.

Definition 1.2.4

Let S be an ordered set. Let E $\subset$ S be bounded from above.
Then, there exists a least upper bound α if:

- α is an upper bound for E
- If $\gamma < \alpha$, then γ is not an upper bound for E.

Then α = sup(E).
*** Greatest Lower Bound: inf(E) ***

Example 1.2.5

Let S = $(1, 2) \cup [3, 4) \cup (5, 6)$ with the order $<$ from $\mathbb{R}$. For subsets E of S:

- E = (1,2) is bounded above and sup(E) = 3
- E = (5,6) is not bounded above so sup(E) = DNE
- E = [3,4) is bounded below inf(E) = 3 and sup(E) = DNE

Observations on the Least Upper Bound

If sup E exists, it may or may not exists at E.
If α exists, then α is unique. If $\gamma \neq \alpha$, then $\gamma < \alpha$ or $\gamma > \alpha$.

1.3 Least Upper Bound Property

Theorem 1.3.1

An ordered set of S has a least upper bound property if:
For every nonempty subset E $\subset$ S that is bounded from above:
sup(E) exists in S.

Example 1.3.2

$\mathbb{Q}$ doesn't have a least upper bound property. For example, z = $\sqrt{2}$.

Proof

Let $z = y - \frac{y^2-2}{y+2} = \frac{2y+2}{y+2}$, then take $z^2 - 2 = \frac{2(y^2-2)}{(y+2)^2}$.
Let set A = $\{y > 0 \in \mathbb{Q}$ where $y^2 < 2\}$ and set B = $\{y > 0 \in \mathbb{Q}$ where $y^2 > 2\}$

- If $y^2 - 2 < 0$, then y is not an upper bound for E.
- If $y^2 - 2 > 0$, y is an upper bound for E, but not the sup(E).

Thus, E has no least upper bound in $\mathbb{Q}$.
However in $\mathbb{R}$, $\sqrt{2}$ is in E.

2 Day 2: Fields

2.1 Greatest Upper Bound Property

Theorem 2.1.1: Least Upper Bound implies Greatest Upper Bound
Let S be a ordered set with the least upper bound property.
Let non-empty B $\subset$ S be bounded below.
Let L be the set of all lower bounds of B.
Then α = sup(L) exists in S and $\alpha \in$ B.

Proof
L is non-empty since B is bounded from below.
Thus, by the least upper bound property of S, α = sup(L) exists in S.
We claim that α = inf(B).
If $\gamma < \alpha$, then γ is not an upper bound for L so y $\notin$ B.
Thus, for every x $\in$ B, $\alpha \leq$ x.
If $\gamma \geq \alpha$, then γ is an upper bound of L so $\gamma \in$ B. Thus, inf(B) = α.

2.2 Fields

Addition Axioms

- If x,y $\in$ F, then x+y $\in$ F
- x+y = y+x for all x,y $\in$ F
- (x+y)+z = x+(y+z) for all x,y,z $\in$ F
- There exists 0 $\in$ F such that 0+x = x for all x $\in$ F
- For every x $\in$ F, there is -x $\in$ F where x+(-x) = 0

Multiplicative xioms

- If x,y $\in$ F, then xy $\in$ F
- yx = xy for all x,y $\in$ F
- (xy)z = x(yx) for all x,y,z $\in$ F
- There exists 1 $\neq$ 0 $\in$ F such that 1x = x for all x $\in$ F
- If x $\neq$ 0 $\in$ F, there is $\frac{1}{x} \in$ F where x($\frac{1}{x}$) = 1

Distributive Law
x(y+z) = xy + xz hold for all x,y,z $\in$ F.

Definition 2.2.1

(a) If x+y = x+z, then y = z

Proof

y = 0+y = (-x)+x+y = (-x)+x+z = 0+z = z

(b) If x+y = x, then y = 0

Proof

From (a), let z = 0.

(c) If x+y = 0, then y = -x

Proof

From(a), let z = -x.

(d) -(-x) = x

Proof

From (a), let x = -x and y = x.

(e) If x $\neq$ 0 and xy = xz, then y = z

Proof

y = 1y = $\frac{1}{x}$xy = $\frac{1}{x}$xz = 1z = z

(f) If x $\neq$ 0 and xy = x, then y = 1

Proof

From (e), let z = 1.

(g) If x $\neq$ 0 and xy = 1, then y = $\frac{1}{x}$

Proof

From (e), let z = $\frac{1}{x}$.

(h) If x $\neq$ 0, then $\frac{1}{1/x}$ = x

Proof

From (e), let x = $\frac{1}{x}$ and y = x.

(i) 0x = 0

Proof

Since 0x + 0x = (0+0)x = 0x, then 0x = 0.

(j) If x,y $\neq$ 0, then xy $\neq$ 0

Proof

Suppose xy = 0, then $\frac{1}{y}\frac{1}{x}$xy = $\frac{1}{y}$1y = $\frac{1}{y}$y = 1.

xy = 0 = 1 is a contradiction.

(k) (-x)y = -(xy) = x(-y)

Proof

xy + (-x)y = (x+(-x))y = 0y = 0.

Then by part (c), (-x)y = -(xy).

Similarly, xy + x(-y) = x(y+(-y)) = x0 = 0.

Then by part (c), x(-y) = -(xy).

(l) (-x)(-y) = xy

Proof

By part (k), then (-x)(-y) = -[x(-y)] = -[-(xy)].

By part (d), -[-(xy)] = xy.

2.3 Ordered Fields

An ordered field F is a field F which is also an ordered set for all x,y,z $\in$ F.

- If y < z, then y+x < z+x
- If x,y > 0, then xy > 0

Definition 2.3.1: $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

$\mathbb{Q}$, $\mathbb{R}$ are ordered fields, but $\mathbb{C}$ is not an ordered field.

Definition 2.3.2

Let F be an ordered field. For all x,y,z $\in$ F.

- If x > 0, -x < 0 and vice versa
- If x > 0 and y < z, then xy < xz
- If x < 0 and y < z, then xy > xz
- If x $\neq$ 0, $x^2 > 0$
- If $0 < x < y$, then $0 < 1/y < 1/x$

Theorem 2.3.3: R is a ordered field with <

There exists a unique ordered field $\mathbb{R}$ with the least upper bound property.
Also, $\mathbb{Q} \subset \mathbb{R}$.

Theorem 2.3.4
For all x,y $\in \mathbb{R}$:

- Archimedean Property: If x > 0, there is n $\in \mathbb{Z}$ such that nx > y.

  Proof

  Fix x > 0. Suppose there is a y such that the property fails.

  Let A = { nx: n = 1, 2, 3, ... }.

  Then, A is nonempty and bounded from above by y.

  Then by the least upper bound property by $\mathbb{R}$, α = sup(A) exists in $\mathbb{R}$.

  Since x > 0, then -x < 0 so $\alpha - x < \alpha - 0 = \alpha$.

  So $\alpha - x$ is not an upper bound of A.

  So there is a mx $\in$ A such that mx > $\alpha - x$

  But then α < (m+1)x where (m+1)x $\in$ A which contradicts α is an upper bound for A.

- $\mathbb{Q}$ is dense in $\mathbb{R}$: If x < y, there is a p $\in \mathbb{Q}$ such that x < p < y.

  Proof

  Since x < y, then y-x > 0. Then by the Archimedean Property, there exists a n $\in$ Z such that n(y-x) > 1. Thus, ny > nx+1 > nx

  By the well-ordering principle, there is a smallest m $\in \mathbb{Z}_+$ such that m > nx.

  Then, m > nx $\geq$ m-1 so nx+1 $\geq$ m > nx.

  Since ny > nx+1 $\geq$ m > ny, then y > m/n > x.

References