ECE 653 - A2

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1 Question 1

1.1 (a)

Four paths.
path 1: 1,2,3,4,9,11,12,13,17
path 2: 1,2,5,6,7,9,11,12,13,17
path 3: 1,2,3,4,9,11,14,15,16,17
path 4: 1,2,5,6,7,9,11,14,15,16,17

1.2 (b)

Path 1:

Edge	Symbolic State (PV)	Path Condition (PC)
$1 \to 2$	$x \mapsto X_0, y \mapsto Y_0$	true
$2 \to 3$	$x \mapsto X_0 + 7, y \mapsto Y_0$	$X_0 + Y_0 > 15$
$3 \to 4$	$x \mapsto X_0 + 7, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$4 \to 9$	$x \mapsto X_0 + 9, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$9 \to 11$	$x \mapsto X_0 + 9, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$11 \to 12$	$x \mapsto 3(X_0 + 9), y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15 \land 2(X_0 + 9 + Y_0 - 12) > 21$
$12 \to 13$	$x \mapsto 3(X_0 + 9), y \mapsto 2(Y_0 - 12)$	$X_0 + Y_0 > 15 \land 2(X_0 + 9 + Y_0 - 12) > 21$
$13 \to 17$	$x \mapsto 3(X_0 + 9), y \mapsto 2(Y_0 - 12)$	$X_0 + Y_0 > 15 \land 2(X_0 + 9 + Y_0 - 12) > 21$

Path 2:

Edge	Symbolic State (PV)	Path Condition (PC)
$1 \to 2$	$x \mapsto X_0, y \mapsto Y_0$	true
$2 \to 5$	$x \mapsto X_0, y \mapsto Y_0$	$X_0 + Y_0 \leq 15$
$5 \to 6$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$6 \to 7$	$x \mapsto X_0 - 2, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$7 \to 9$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$9 \to 11$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$11 \to 12$	$x \mapsto 3 \cdot X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) > 21$
$12 \to 13$	$x \mapsto 3 \cdot X_0, y \mapsto 2 \cdot (Y_0 + 10)$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) > 21$
$13 \to 17$	$x \mapsto 3 \cdot X_0, y \mapsto 2 \cdot (Y_0 + 10)$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) > 21$

Path 3:

Edge	Symbolic State (PV)	Path Condition (PC)
$1 \to 2$	$x \mapsto X_0, y \mapsto Y_0$	true
$2 \to 3$	$x \mapsto X_0 + 7, y \mapsto Y_0$	$X_0 + Y_0 > 15$
$3 \to 4$	$x \mapsto X_0 + 7, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$4 \to 9$	$x \mapsto X_0 + 9, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$9 \to 11$	$x \mapsto X_0 + 9, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15$
$11 \to 14$	$x \mapsto X_0 + 9, y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15 \wedge 2(X_0 + 9 + Y_0 - 12) \leq 21$
$14 \to 15$	$x \mapsto 4 \cdot (X_0 + 9), y \mapsto Y_0 - 12$	$X_0 + Y_0 > 15 \wedge 2(X_0 + 9 + Y_0 - 12) \leq 21$
$15 \to 16$	$x \mapsto 4 \cdot (X_0 + 9), y \mapsto 3 \cdot (Y_0 - 12) + 4 \cdot (X_0 + 9)$	$X_0 + Y_0 > 15 \wedge 2(X_0 + 9 + Y_0 - 12) \leq 21$
$16 \to 17$	$x \mapsto 4 \cdot (X_0 + 9), y \mapsto 3 \cdot (Y_0 - 12) + 4 \cdot (X_0 + 9)$	$X_0 + Y_0 > 15 \wedge 2(X_0 + 9 + Y_0 - 12) \leq 21$

Path 4:

Edge	Symbolic State (PV)	Path Condition (PC)
$1 \to 2$	$x \mapsto X_0, y \mapsto Y_0$	true
$2 \to 5$	$x \mapsto X_0, y \mapsto Y_0$	$X_0 + Y_0 \leq 15$
$5 \to 6$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$6 \to 7$	$x \mapsto X_0 - 2, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$7 \to 9$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$9 \to 11$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15$
$11 \to 14$	$x \mapsto X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) \leq 21$
$14 \to 15$	$x \mapsto 4 \cdot X_0, y \mapsto Y_0 + 10$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) \leq 21$
$15 \to 16$	$x \mapsto 4 \cdot X_0, y \mapsto 3 \cdot (Y_0 + 10) + 4 \cdot X_0$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) \leq 21$
$16 \to 17$	$x \mapsto 4 \cdot X_0, y \mapsto 3 \cdot (Y_0 + 10) + 4 \cdot X_0$	$X_0 + Y_0 \leq 15 \wedge 2(X_0 + Y_0 + 10) \leq 21$

1.3 (c)

Path 1: Feasible : $X_0 = 10, Y_0 = 15$
Path 2: Feasible : $X_0 = 3, Y_0 = 6$
Path 3: Not Feasible. It's not possible to find X_0 and Y_0 that satisfy the path condition.
Path 4: Feasible : $X_0 = -1, Y_0 = 0$

2 Question 2

2.1 (a)

$\neg(a_1 \wedge a_2) \wedge \neg(a_1 \wedge a_3) \wedge \neg(a_1 \wedge a_4) \wedge \neg(a_2 \wedge a_3) \wedge \neg(a_2 \wedge a_4) \wedge \neg(a_3 \wedge a_4)$

2.2 (b)

The sentence is valid.

Left to Right: if the left hand side of the sentence is true, then $\forall x \cdot \exists y$ such that either $P(x)$ is true or $Q(y)$ is true.

Case 1: If in any case that $Q(y)$ is true, that is, $\exists y Q(y)$ is true, the right hand side is true.

Case 2: If $Q(y)$ is never true, then $\forall x P(x)$ is always true ($P(x)$ is not dependent on y so we don't have to consider y here.) Therefore, the right hand side is true.

Right to Left: Suppoese the right hand side is true, then either $\forall x P(x)$ is true or $\exists y Q(y)$ is true.

Case 1: If $\exists y Q(y)$ is true, then $\forall x \cdot \exists y$ such that $Q(y)$ is true, therefore $\forall x \cdot \exists y(P(x) \vee Q(y))$ is true.

Case 2: If $\forall x P(x)$ is true, then $\forall x(P(x) \vee Q(y))$ is always true no matter what value y takes. Therefore, $\forall x \cdot \exists y(P(x) \vee Q(y))$ is true.

2.3 (c)

The sentence is not valid.

Suppose there exists a model $M = \{1, 2, P, Q\}$, $M(P) = \{(1, 1), (1, 2)\}$, $M(Q) = \{(2, 1), (2, 2)\}$. According to the model, the values of x and y can either be 1 or 2.

When $x = 1$, $y = 1$ or $y = 2$ makes $P(x, y)$ true, and therefore makes $P(x, y) \vee Q(x, y)$ true.

When $x = 2$, $y = 1$ or $y = 2$ makes $Q(x, y)$ true, and therefore makes $P(x, y) \vee Q(x, y)$ true.

Therefore, Model M makes the left hand side of the sentence true.

However, When $x = 1$, there doesn't exists a y value such that $Q(x, y)$ is true, so $(\forall x \cdot \exists y \cdot Q(x, y))$ is false. When $x = 2$, there doesn't exists a y value such that $P(x, y)$ is true, so $(\forall x \cdot \exists y \cdot P(x, y))$ is false.

Therefore, the right hand side is false under model M. The sentence is not valid.

2.4 (d)

(a) Yes.

According to M_1, $P(x, y)$ is true if $x < y$.

If $P(x, y)$ is true, then $x < y$.

If $P(z, y)$ is true, then $z < y$.

If $P(x, z)$ is true, then $x < z$.

If $\neg P(z, x)$ is true, then $P(z, x)$ is false, $x \leq z$.

$x < z < y$ satisfies the 4 constraints at the same time, and thus $M_1 \models \Phi$.

(b) No.

According to M_2, $P(x, y)$ is true if $y = x + 1$.

If $P(x, y)$ is true, then $y = x + 1$.

If $P(z, y)$ is true, then $y = z + 1$.

If $P(x, z)$ is true, then $z = x + 1$.

If $\neg P(z, x)$ is true, then $P(z, x)$ is false, $x \neq z + 1$.

It's not possible to satisfy the 4 constraints at the same time, and thus M_2 violates the formula Φ.

(c) Yes.

According to M_2, $P(x, y)$ is true if $x \subseteq y$.

If $P(x, y)$ is true, then $x \subseteq y$.

If $P(z, y)$ is true, then $z \subseteq y$.

If $P(x, z)$ is true, then $x \subseteq z$.

If $\neg P(z, x)$ is true, then $P(z, x)$ is false, $z \nsubseteq x$.

$x \subset z, z \subseteq y$ satisfies the 4 constraints at the same time, and thus $M_3 \models \Phi$.

3 Question 3

3.1 (a)

Let function f_x denotes the xth integer in the magic square from the upper left corner to the lower right corner.

Let function $m_{i,j}$ denotes the integer in the magic square located at row i, column j.

Let function $S(n_1, n_2, ..n_k)$ denotes the sum of k integers.

Let a boolean function D(n_1,n_2) denotes that two integers n_1 and n_2 are not equal.

3.1.1 Condition 1: All integers in the magic square are in the range from 1 to n^2.

$$\bigwedge_{0 \leq i < n} 1 \leq f_i \leq n^2$$

3.1.2 Condition 2: All integers in the magic square are distinct.

$$\bigwedge_{0 \leq i < j < n} D(f_i, f_j)$$

3.1.3 Condition 3: The sums of integers in each row are the same.

$$\bigwedge_{0 \leq i < n-1} S(m_{i,0}, m_{i,1}, ..., m_{i,n-1}) = S(m_{i+1,0}, m_{i+1,1}, ..., m_{i+1,n-1})$$

3.1.4 Condition 4: The sums of integers in each column are the same.

$$\bigwedge_{0 \leq i < n-1} S(m_{0,i}, m_{1,i}, ..., m_{n-1,i}) = S(m_{0,i+1}, m_{1,i+1}, ..., m_{n-1,i+1})$$

3.1.5 Condition 5: The sums of two diagonals are equal.

$S(m_{0,0}, m_{1,1}, ...m_{n-1,n-1}) = S(m_{0,n-1}, m_{1,n-2}, ...m_{n-1,0})$

3.1.6 Condition 6: The sums of each row, column and diagonal are the same.

$S(m_{0,0}, m_{0,1}, ...m_{0,n-1}) = S(m_{0,0}, m_{1,0}, ..., m_{n-1,0}) \wedge$
$S(m_{0,0}, m_{1,0}, ..., m_{n-1,0}) = S(m_{0,0}, m_{1,1}, ...m_{n-1,n-1})$