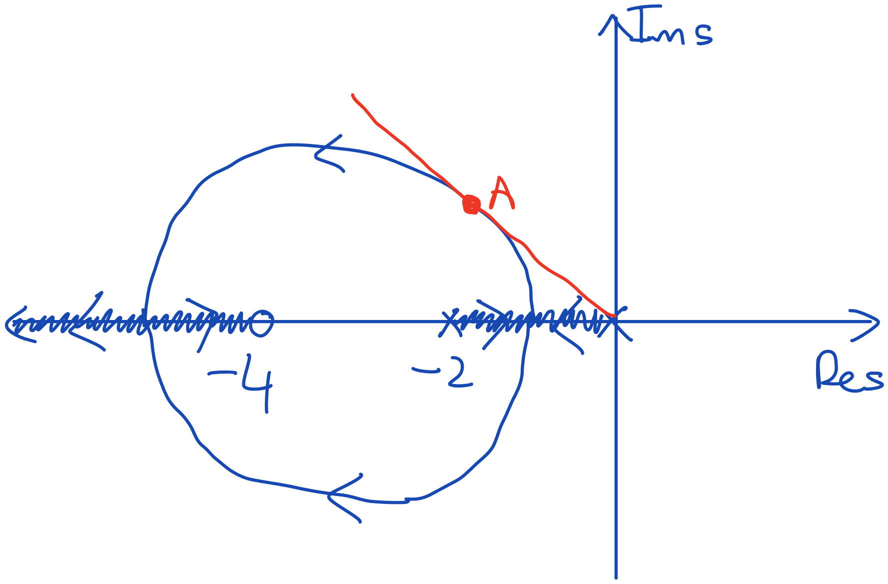
HW2 SOLUTIONS

Q1-) a-)

$$OLTF = K\frac{s+4}{s(s+2)}$$

OL poles: $p_1=0$ $p_2=-2$ $(n=2)$

OL zeros: $z_1=-4$ $(m=1)$



$$1+K\frac{s+4}{s(s+2)}=0$$

$$\Rightarrow s^2+2s+Ks+4K=0$$

$$\underbrace{s^2+(K+2)s+4K=0}_{\text{Characteristic Polynomial of the CL System}}$$

# of asymptotes $= n-m = 1$ Angle of the asymptote $=180^\circ$

Imaginary axis crossings: $s=j\omega$

$$-\omega^2+(K+2)j\omega+4K=0$$

$$\Rightarrow 4K-\omega^2+(K+2)j\omega=0$$

$4K-\omega^2=0$ $(K+2)\omega=0$

$\omega=0$ $\cancel{K=-2}$

$K=0$

$\Rightarrow$ The only imaginary axis crossing happens at $K=0$ $\omega=0$.

Break Away/In Points

$$K=\underbrace{-\frac{s(s+2)}{s+4}}=-\frac{D(s)}{N(s)}=-\frac{D'(s)}{N'(s)}$$

$$\frac{dK}{ds}=0 \Rightarrow (2s+2)(s+4)-(s^2+2s)=0$$

$$2(s+1)(s+4)-s^2-2s=0$$

$$2(s^2+5s+4)-s^2-2s=0$$

$$s^2+8s+8=0$$

$$(s+4)^2-8=0 \Rightarrow s=-4\mp2\sqrt{2}$$

Break Away point: $s=-4+2\sqrt{2}$

$$K=-\left.\frac{2s+2}{1}\right|_{s=-4+2\sqrt{2}} = 6-4\sqrt{2}$$

Break In point: $s=-4-2\sqrt{2}$

$$K=-(2s+2)\Big|_{s=-4-2\sqrt{2}} = 6+4\sqrt{2}$$

b-) See the point A on the root locus above.

c-) $s^2+(K+2)s+4K = s^2+2\zeta\omega_n s+\omega_n^2$

$$\Rightarrow \omega_n^2=4K \Rightarrow \omega_n = 2\sqrt{K}$$

$$2\zeta\omega_n = K+2 \Rightarrow \zeta = \frac{K+2}{4\sqrt{K}}$$

We can calculate $\frac{d\zeta}{dK}$, equate the result to zero and solve for K.

Since ζ is positive, an equivalent solution method is to calculate $\frac{d\zeta^2}{dK}$, equate it to zero and solve for K.

$$\zeta^2 = \frac{(K+2)^2}{16K} \qquad \frac{d\zeta^2}{dK} = \frac{1}{16}\left(2(K+2)K-(K+2)^2\right) = 0$$

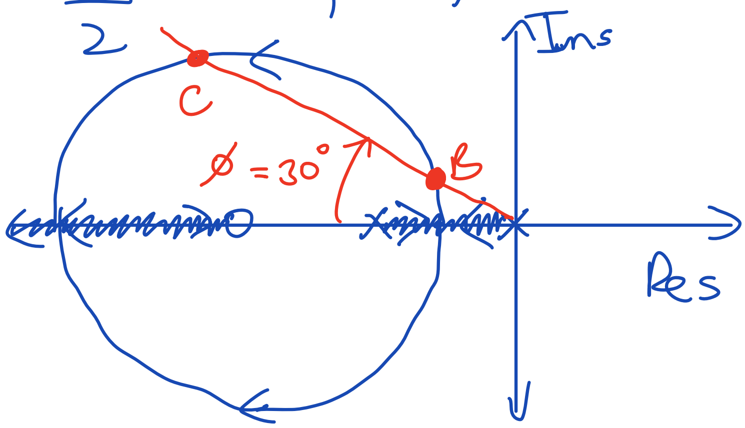
$$\Rightarrow 2\cancel{(K+2)}K = (K+2)^{\cancel{2}}$$

Solutions are $\cancel{K=-2}$ and $2K=K+2$

negative K

$$\boxed{K=2}$$

d-) $\zeta = \frac{\sqrt{3}}{2} \Rightarrow \zeta = \cos\phi \Rightarrow \phi = 30^\circ$



The CL poles would have the form

$$s = -\frac{\sqrt{3}}{2}\alpha \mp j\frac{1}{2}\alpha$$

for some $\alpha > 0$

$$\zeta^2 = \frac{(K+2)^2}{16K} \Rightarrow \frac{3}{\cancel{4}} = \frac{(K+2)^2}{\cancel{16}_4 K}$$

$$\Rightarrow 12K = K^2 + 4K + 4 \Rightarrow K^2 - 8K + 4 = 0$$

$$\Rightarrow (K-4)^2 - 12 = 0$$

$$\Rightarrow K = 4 \mp 2\sqrt{3}$$

Point B: $K = 4 - 2\sqrt{3}$

Point C: $K = 4 + 2\sqrt{3}$

$\Rightarrow$ At point C, the CL poles are further away from the imaginary axis than at point B. Hence the settling time at C is smaller than that at B $\Rightarrow$ We would choose point C.

$$\Rightarrow K = 4 + 2\sqrt{3}$$

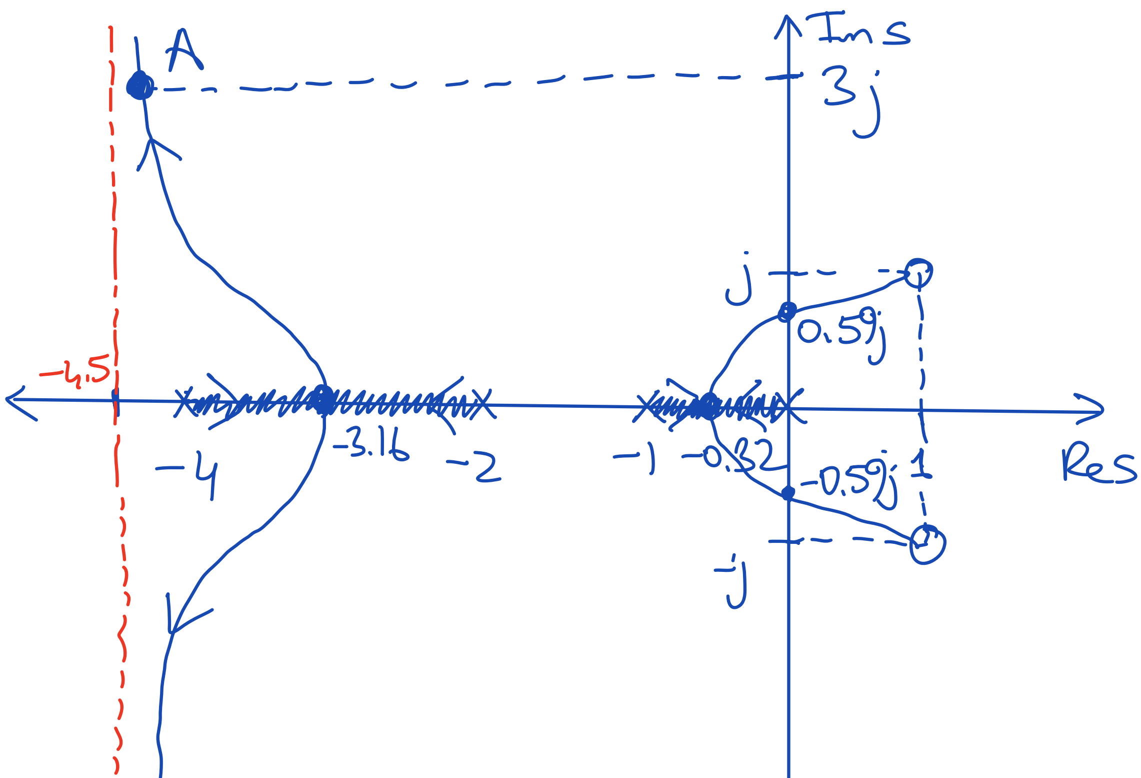
Q2-) $G(s) = \dfrac{K(s^2-2s+2)}{s(s+1)(s+2)(s+4)} = \dfrac{K((s-1)^2+1)}{s(s+1)(s+2)(s+4)}$

$$= K\,\frac{(s^2-2s+2)}{s^4+7s^3+14s^2+8s}$$

$1+K\dfrac{(s^2-2s+2)}{s^4+7s^3+14s^2+8s} = 0 \qquad s^4+7s^3+(K+14)s^2+(8-2K)s+2K=0$

a-) OL zeros at $z_1 = 1+j \quad z_2 = 1-j \quad (m=2)$

OL poles at $p_1=0 \quad p_2=-1 \quad p_3=-2 \quad p_4=-4 \quad (n=4)$



# of asymptotes $= n-m = 2$

Angle of the 1^{st} asymptote $= \dfrac{180}{2} = 90^\circ$

Angle between asymptotes $= \dfrac{360}{2} = 180^\circ$

Intersection point of the asymptotes $\sigma_0 = \dfrac{\sum p_i - \sum z_i}{n-m} = \dfrac{-7-2}{2} = -\dfrac{9}{2}$

Imaginary axis crossings: $s = j\omega$.

$$w^4 - 7jw^3 - (K+14)w^2 + (8-2K)jw + 2K = 0$$

$$w^4 - (K+14)w^2 + 2K = 0 \qquad jw(8-2K-7w^2) = 0$$

$$w = 0 \Downarrow K = 0 \qquad 7w^2 = 8-2K$$

$$w^2 = \frac{8-2K}{7}$$

$$\Rightarrow (w^2)^2 - (K+14)w^2 + 2K = 0$$

$$\Rightarrow \frac{(8-2K)^2}{49} - \frac{(K+14)(8-2K)}{7} + 2K = 0$$

$$\Rightarrow 4(K-4)^2 + 14(K+14)(K-4) + 98K = 0$$

$$4(K^2 - 8K + 16) + 14(K^2 + 10K - 56) + 98K = 0$$

$$4K^2 - 32K + 64 + 14K^2 + 140K - 784 + 98K = 0$$

$$18K^2 + 206K - 720 = 0$$

$$\Rightarrow 18K^2 + 206K - 720 = 0$$

$$9K^2 + 103K - 360 = 0$$

$$K \cong 2.8 \qquad \cancel{K \cong -14.25}$$

$$w^2 = \frac{8-5.6}{7} = \frac{2.4}{7} \cong 0.34 \Rightarrow w = \mp 0.59$$

Break-Away/In Points

$$K = -\frac{s^4 + 7s^3 + 14s^2 + 8s}{s^2 - 2s + 2} = \frac{D(s)}{N(s)} = \frac{-D'(s)}{N'(s)}$$

$$\Rightarrow \frac{dK}{ds} = -\Big((4s^3+21s^2+28s+8)(s^2-2s+2) - (2s-2)(s^4+7s^3+14s^2+8s) \Big) = 0$$

$$\begin{aligned}
&4s^5 - 8s^4 + 8s^3 \\
&\quad + 21s^4 - 42s^3 + 42s^2 \\
&\qquad + 28s^3 - 56s^2 + 56s \\
&\qquad\quad + 8s^2 - 16s + 16 \\
&-2(s^5 + 7s^4 + 14s^3 + 8s^2 \\
&\quad - s^4 - 7s^3 - 14s^2 - 8s) \\
+ &\overline{\qquad\qquad\qquad\qquad\qquad\qquad} \\
&2s^5 + s^4 - 20s^3 + 6s^2 + 56s + 16 = 0
\end{aligned}$$

$\boxed{s \cong -3.16}$ $\quad$ $\cancel{s \cong 2.2 \mp 0.91j}$ $\quad$ $\cancel{s \cong 1.45}$ $\quad$ $\boxed{s \cong -0.31}$

$$K = -\frac{4s^3+21s^2+28s+8}{2s-2}\Bigg|_{s=-3.16} \cong 0.36$$

$$K = -\frac{4s^3+21s^2+28s+8}{2s-2}\Bigg|_{s=-0.31} \cong 0.46$$

b-) At $K=0$ there is a CL pole at $s=0$. The CL poles at $K=0$ are at $s=0,-1,-2,-4$. (which are the OL poles)

At $K \cong 2.8$ there are CL poles at $s \cong \mp 0.59j$.

$\Rightarrow$ The polynomial $(s^2+0.59^2) = s^2+0.34$ should divide the characteristic polynomial without a remainder.

$s^4+7s^3+(K+14)s^2+(8-2K)s+2K \Big|_{K=2.8}$

$$
\begin{array}{l|l}
= s^4+7s^3+16.8s^2-2.4s+5.6 & s^2+0.34 \\
\hline
-\; s^4+0s^3+0.34s^2 & s^2+7s+16.46 \\
\quad 7s^3+16.46s^2-2.4s+5.6 & \\
-\; 7s^3+0s^2-2.4s & \\
\quad 16.46s^2+0s+5.6 &
\end{array}
$$

$\Rightarrow \quad s^2+7s+16.46 \Longrightarrow s_{3,4} \cong -3.5 \mp 2j$

$\Rightarrow$ CL poles are at $\quad s_{1,2} = \mp 0.59j \quad s_{3,4} = -3.5 \mp 2j$

c-) $0 < K < 2.8$

d-) For no oscillations all CL poles should be real which is the case for $\quad 0 \leq K < 0.36$

e-) The CL poles with real part -4 $(-4 \mp j\omega)$ are the roots of the polynomial $\quad (s+4)^2+\omega^2 = s^2+8s+\omega^2+16$

$\Rightarrow$ The characteristic polynomial should be divisible by $s^2+8s+\omega^2+16$ without a remainder

$$
\begin{array}{l|l}
s^4+7s^3+(K+14)s^2+(8-2K)s+2K & s^2+8s+w^2+16 \\ \hline
s^4+8s^3+(w^2+16)s^2 & s^2-s+(K-w^2+6)
\end{array}
$$

$$\overline{\quad -s^3+(K-w^2-2)s^2+(8-2K)s+2K \quad}$$

$$-s^3-8s^2 \qquad -(w^2+16)s$$

$$\overline{\quad (K-w^2+6)s^2+(w^2-2K+24)s+2K \quad}$$

$$-\;(K-w^2+6)s^2+8(K-w^2+6)s+(w^2+16)(K-w^2+6)$$

$$\overline{\big(w^2-2K+24-8(K-w^2+6)\big)s+\;2K-(w^2+16)(K-w^2+6)}$$

$$\Rightarrow \quad w^2-2K+24=8(K-w^2+6) \qquad 2K=(w^2+16)(K-w^2+6)$$

$$2K=K(w^2+16)+(w^2+16)(-w^2+6)$$

$$(w^2+16)(w^2-6)=K(w^2+14)$$

$$10K=w^2+24+8w^2-48=9w^2-24$$

$$10K(w^2+14)=(9w^2-24)(w^2+14)$$

$$10(w^2+16)(w^2-6)=(9w^2-24)(w^2+14)$$

$$a \triangleq w^2 \qquad\qquad 4)$$

$$\Rightarrow \quad 10(a+16)(a-6)-(9a-24)(a+14)$$

$$=(a^2-2a-624)=0$$

$$\Rightarrow \quad (a-1)^2 \approx 25^2 \quad \Rightarrow \quad a-1 \approx \mp 25$$

$\Rightarrow \quad \cancel{a = -24} \qquad a \cong 26$

$w^2 = 26 \Rightarrow \quad w = \mp\sqrt{26}$

$\Rightarrow$ CL poles are at $\quad s_{1,2} = -4 \mp j\sqrt{26}$

$10K = 9w^2 - 24 \cong 210 \quad \Longrightarrow \quad K \approx 21$

The other CL poles are the roots of

$$s^2 - s + (K - w^2 + 6) = s^2 - s + 1$$

$$\Rightarrow \quad s_{3,4} = \frac{1 \mp \sqrt{3}\,j}{2}$$

$\Rightarrow$ Frequency of oscillations observed are

$$w_1 \approx \sqrt{26} \text{ rad/sec}$$

$$w_2 = \frac{\sqrt{3}}{2} \text{ rad/sec}$$

f-) The related CL poles are now at $\quad s_{1,2} = -\alpha \mp j3$

$\Rightarrow$ We have to do polynomial division with the polynomial

$(s+\alpha)^2 + 9 = s^2 + 2\alpha s + \alpha^2 + 9$

and equate the remainder to zero (See point A in the root locus and then solve for the unknowns α & K.

Q3.

Let's assume that $c_1 = c_2 = c_3 = c_4 = 1$

A. First let's find the transfer function from the error signal to the output signal

$$G(s) = \frac{K_{out}}{s-1}\frac{\frac{1}{s-1}}{1+\frac{K_{in}}{s-1}} = \frac{K_{out}}{(s-1)(s+(K_{in}-1))} = \frac{K_{out}}{s^2+(K_{in}-2)s+(K_{in}-1)}$$

Now let's compute the closed loop transfer function

$$T(s) = \frac{K_{out}}{s^2+(K_{in}-2)s+(K_{in}+K_{out}-1)}$$

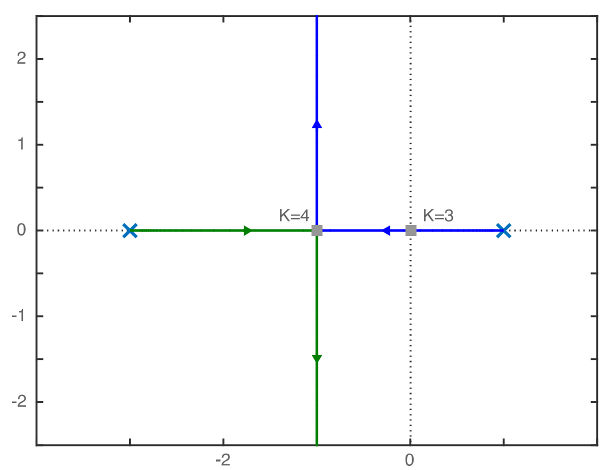
The denominator is a second order polynomial if we apply Routh Hurwitz, we can find the following conditions for guaranteeing the closed-loop (BIBO) stability.

$$\boldsymbol{K_{in} > 2 \quad \& \quad K_{out} > 1 - K_{in}}$$

B. Let $K_{in} = 4$, then open-loop transfer function takes the form

$$G_{OL}(s) = \frac{K_{out}}{(s-1)(s+3)}$$

Based in this result we can draw the root locus and find the important gains and associated pole locations as



C. This is a non-standard root-locus problem. We need to first re-arrange the denominator of the closed loop transfer function.

$$s^2+(K_{in}-2)s+(K_{in}+K_{out}-1) = s^2-2s+(K_{out}-1)+K_{in}(s+1)$$

Let $K_{out} = 1$ then

$$D_{CL}(s) = s^2 - 2s + K_{in}(s+1)$$

Note that the location of the roots corresponds to the points where

$$D_{CL}(s) = s^2 - 2s + K_{in}(s+1) = 0$$

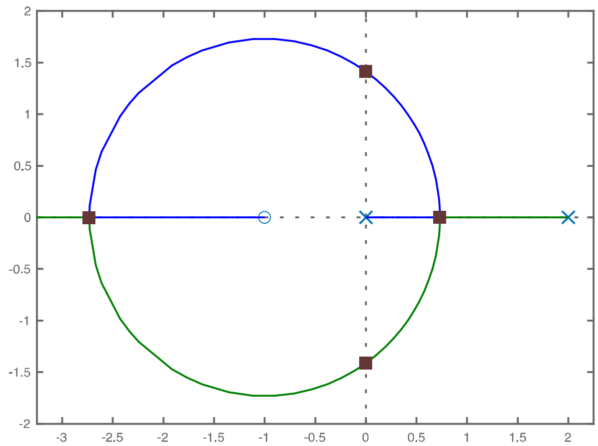
If we manipulate the equation we can find

$$1 + K_{in}\frac{(s+1)}{s^2 - 2s} = 0$$

Which puts the equation into standard root-locus form, i.e.

$$\tilde{G}_{OL}(s) = \frac{(s+1)}{s^2 - 2s}$$

And the root-locus will take the following form



Now we need to find special locations and their associated gains

We know from the stability analysis that closed-loop system is stable when $\boldsymbol{K_{in} > 2}$. Thus imaginary axis crossing occurs when $\boldsymbol{K_{in} = 2}$. We can find the locations of poles on the imganiay axis by simply using the closed-loop transfer functions denominator

$$D(s) = s^2 - 2s + K_{in}(s+1) = s^2 + 1 \rightarrow s_{1.2} = \pm j$$

Final step is finding the break-away and break-in points and associated gains

$$\frac{d}{ds}\left(\frac{(s+1)}{s^2 - 2s}\right) = \frac{(s^2 - 2s) - (s+1)(2s-2)}{(s^2 - 2s)^2} = \frac{-(s^2 + 2s - 2)}{(s^2 - 2s)^2}$$

$$(\sigma^2 + 2\sigma - 2) = 0 \rightarrow \sigma_{ba} = 0.7321 \; \& \; \sigma_{bi} = -2.7321$$

We can find the associated gains using magnitude condition

$$K_{ba} = 0.536 \; \& \; K_{bi} = 7.46$$