# CENG 384 - Signals and Systems for Computer Engineers
# 20232
# Written Assignment 3 Solutions

May 22, 2024

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1.

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

For one full period:

$$x(t) = a_0 + a_1 e^{j\frac{\pi}{2}t} + a_2 e^{j\pi t} + a_3 e^{j\frac{3\pi}{2}t}$$
$$x(t) = -1 + e^{j\frac{\pi}{2}t} - e^{j\pi t} + e^{j\frac{3\pi}{2}t}$$

2. (a) The Fourier series coefficients $a_k$ are given by:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt$$

where $\omega_0 = \frac{2\pi}{T}$.
Here, $T = 4$, so $\omega_0 = \frac{\pi}{2}$.
The integral can be broken into two parts:

$$a_k = \frac{1}{4}\left(\int_0^2 2te^{-jk\frac{\pi}{2}t}\,dt + \int_2^4 (4-t)e^{-jk\frac{\pi}{2}t}\,dt\right)$$

Part 1: $0 \le t < 2$

$$\int_0^2 2te^{-jk\frac{\pi}{2}t}\,dt$$

Using integration by parts: Let $u = 2t$ and $dv = e^{-jk\frac{\pi}{2}t}\,dt$.
Then, $du = 2\,dt$ and $v = \frac{-2j}{k\pi}e^{-jk\frac{\pi}{2}t}$.

$$\int udv = uv - \int vdu$$

$$\begin{aligned}
\int_0^2 2te^{-jk\frac{\pi}{2}t}\,dt &= \frac{-4jt}{k\pi}e^{-jk\frac{\pi}{2}t}\Big|_0^2 + \frac{4j}{k\pi}\int_0^2 e^{-jk\frac{\pi}{2}t}\,dt \\
&= \left[\frac{-4jt}{k\pi}e^{-jk\frac{\pi}{2}t}\right]_0^2 + \frac{4j}{k\pi}\left[\frac{-2j}{k\pi}e^{-jk\frac{\pi}{2}t}\right]_0^2 \\
&= \left(\frac{-8j}{k\pi}e^{-jk\pi} - \frac{0}{k\pi}\right) + \frac{4j}{k\pi}\left(\frac{-2j}{k\pi}(e^{-jk\pi} - 1)\right) \\
&= \frac{-8j}{k\pi}(-1)^k + \frac{4j}{k\pi}\cdot\frac{-2j}{k\pi}((-1)^k - 1) \\
&= \frac{-8j(-1)^k}{k\pi} + \frac{8((-1)^k - 1)}{k^2\pi^2}
\end{aligned}$$

Part 2: $2 \le t < 4$

$$\int_2^4 (4-t)e^{-jk\frac{\pi}{2}t}\,dt$$

Using integration by parts: Let $u = 4 - t$ and $dv = e^{-jk\frac{\pi}{2}t}\,dt$.

Then, $du = -dt$ and $v = \frac{-2j}{k\pi}e^{-jk\frac{\pi}{2}t}$.

$$\int udv = uv - \int vdu$$

$$\int_2^4 (4-t)e^{-jk\frac{\pi}{2}t}\,dt = \frac{-8j+2jt}{k\pi}e^{-jk\frac{\pi}{2}t}\Big|_2^4 - \frac{2j}{k\pi}\left[\frac{-2j}{k\pi}e^{-jk\frac{\pi}{2}t}\right]_2^4$$

$$= \left(0 - \frac{-8j}{k\pi}e^{-jk\pi} + \frac{-4j}{k\pi}e^{-jk\pi}\right) - \frac{2j}{k\pi}\left(\frac{-2j}{k\pi}(e^{-jk\pi} - e^{-jk2\pi})\right)$$

$$= \frac{8j(-1)^k}{k\pi} - \frac{4j(-1)^k}{k\pi} - \frac{4j(-1)^k}{k^2\pi^2} + \frac{4}{k^2\pi^2}(-1)^{2k}$$

$$= \frac{4j(-1)^k}{k\pi} + \frac{4}{k^2\pi^2} + \frac{4j(-1)^k}{k^2\pi^2}$$

Combining Both Parts

$$a_k = \frac{1}{4}\left(\frac{-8j(-1)^k}{k\pi} + \frac{8((-1)^k-1)}{k^2\pi^2} + \frac{4j(-1)^k}{k\pi} + \frac{4}{k^2\pi^2} + \frac{4j(-1)^k}{k^2\pi^2}\right)$$

$$= \frac{1}{4}\left(\frac{-4j(-1)^k}{k\pi} + \frac{-4}{k^2\pi^2} + \frac{(8+4j)(-1)^k}{k^2\pi^2}\right)$$

$$= \frac{-j(-1)^k}{k\pi} + \frac{-1}{k^2\pi^2} + \frac{(2+j)(-1)^k}{k^2\pi^2}$$

So the Fourier series coefficients are:

$$a_k = \frac{-j(-1)^k}{k\pi} + \frac{-1}{k^2\pi^2} + \frac{(2+j)(-1)^k}{k^2\pi^2}$$

(b) The differentiation property of Fourier series states that if $x(t)$ has Fourier coefficients $a_k$, then $\frac{dx}{dt}$ has Fourier coefficients $b_k = jk\omega_0 a_k$, where $\omega_0$ is the fundamental angular frequency.
From part (a), we have the Fourier coefficients $a_k$:

$$a_k = \frac{-j(-1)^k}{k\pi} + \frac{-1}{k^2\pi^2} + \frac{(2+j)(-1)^k}{k^2\pi^2}$$

The fundamental angular frequency is:

$$\omega_0 = \frac{\pi}{2}$$

Using the differentiation property, the Fourier coefficients $b_k$ of $\frac{dx}{dt}$ are given by:

$$b_k = jk\omega_0 a_k = jk\frac{\pi}{2}\left(\frac{-j(-1)^k}{k\pi} + \frac{-1}{k^2\pi^2} + \frac{(2+j)(-1)^k}{k^2\pi^2}\right)$$

Simplifying:

$$b_k = jk\frac{\pi}{2}\left(\frac{-j(-1)^k}{k\pi} + \frac{-1}{k^2\pi^2} + \frac{(2+j)(-1)^k}{k^2\pi^2}\right)$$

$$b_k = jk\frac{\pi}{2}\cdot\frac{3j}{k\pi} + jk\frac{\pi}{2}\cdot\frac{4}{k^2\pi^2}$$

$$b_k = \frac{(-1)^k}{2} + \frac{-j}{2k\pi} + \frac{(2j-1)(-1)^k}{2k\pi}$$

3. Let's first name the coefficients for each signal.

$$x_1[n] \overset{\text{FS}}{\longleftrightarrow} a_k$$

$$x_2[n] \overset{\text{FS}}{\longleftrightarrow} b_k$$

$$x_1[n]x_2[n] = x_3[n] \overset{\text{FS}}{\longleftrightarrow} d_k$$

(a)

$$x_1[n] = \frac{1}{2}\left(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\right), \quad \omega_0 = \frac{\pi}{2}.$$

$$a_1 = a_{-1} = \frac{1}{2}.$$

$$x_2[n] = \frac{1}{2j}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right), \quad \omega_0 = \frac{\pi}{2}.$$

$$b_1 = \frac{1}{2j} = -\frac{j}{2}, \quad b_{-1} = -\frac{1}{2j} = \frac{j}{2}.$$

$$\begin{aligned} x_3[n] = x_1[n]x_2[n] &= \left(\sin\frac{\pi}{2}n\right)\left(\cos\frac{\pi}{2}n\right) \\ &= \frac{1}{2}\cancel{\underbrace{\sin \pi n}_{\text{always } 0}} \end{aligned}$$

Therefore

$$d_k = 0.$$

(b) Multiplication property:

$$x[n]y[n] \overset{\text{FS}}{\longleftrightarrow} \sum_{l=<N>} a_l b_{k-l}$$

$$\begin{aligned} d_k &= \sum_{l=0}^{3} a_l b_{k-l}, \quad \text{since} \quad N = 4 \\ &= \cancel{\underbrace{a_0 b_k}_{a_0=0}} + a_1 b_{k-1} + \cancel{\underbrace{a_2 b_{k-2}}_{a_2=0}} + a_3 b_{k-3} \\ &= a_1 b_{k-1} + \underbrace{a_3 b_{k-3}}_{a_3=a_{-1}} \\ &= a_1 b_{k-1} + a_{-1} b_{k-3} \end{aligned}$$

$$\begin{aligned} d_0 &= a_1 b_{-1} + a_{-1} b_{-3} \\ &= a_1 b_{-1} + a_{-1} b_1 \\ &= -\frac{1}{4j} + \frac{1}{4j} \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_2 = d_{-2} &= a_1 b_1 + a_{-1} b_{-1} \\ &= \frac{1}{4j} - \frac{1}{4j} \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_1 &= \cancel{\underbrace{a_1 b_0}_{b_0=0}} + \cancel{\underbrace{a_{-1} b_{-2}}_{b_{-2}=0}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_{-1} &= \cancel{\underbrace{a_1 b_{-2}}_{b_{-2}=0}} + \cancel{\underbrace{a_{-1} b_{-4}}_{b_{-4}=b_0=0}} \\ &= 0 \end{aligned}$$

As you can see the results are the same as the ones found in part a.

4. By Euler's Equation we have $a_k$ as

$$a_k = \frac{1}{2}\left(e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}}\right) + \frac{1}{2}\left(e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}\right).$$

And we know from the analysis equation

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}.$$

Here, by inspection, we see that $N = 24$ and $\omega_0 = \frac{\pi}{12}$. Therefore we got

$$a_k = \frac{1}{24}\sum x[n]e^{-jk\frac{\pi}{12}n}.$$

Using these equations we will now analyze $a_1$ to specify $x[n]$:

$$a_1 = \frac{1}{2}\left(e^{j\frac{4\pi}{12}} + e^{-j\frac{4\pi}{12}}\right) + \frac{1}{2}\left(e^{j\frac{3\pi}{12}} + e^{-j\frac{3\pi}{12}}\right) = \frac{1}{24}x[4]e^{-j\frac{4\pi}{12}} + \frac{1}{24}x[-4]e^{j\frac{4\pi}{12}} + \frac{1}{24}x[3]e^{-j\frac{3\pi}{12}} + \frac{1}{24}x[-3]e^{j\frac{3\pi}{12}}$$

$$x[4] = 12, \quad x[-4] = x[20] = 12, \quad x[3] = 12, \quad x[-3] = x[21] = 12.$$

So for $0 \le n \le 23$, we have $x[n]$ as

$$x[n] = 12\delta[n-3] + 12\delta[n-4] + 12\delta[n-20] + 12\delta[n-21].$$

5. (a) To find the fundamental period, we need to determine the smallest positive integer $N$ such that:

$$\frac{6\pi}{13}N = 2\pi k$$

for some integer $k$.

$$\frac{6N}{13} = 2k$$
$$6N = 26k$$
$$3N = 13k$$

Therefore, the fundamental period $N$ of the signal is 13.

(b) Given $N = 13$, we have:

$$x[n] = \sin\left(\frac{6\pi}{13}n + \frac{\pi}{2}\right) = \cos\left(\frac{6\pi}{13}\right) = \frac{1}{2}e^{j\frac{6\pi}{13}n} + \frac{1}{2}e^{-j\frac{6\pi}{13}n}$$

For a period $[-6, 6]$

$$a_3 = a_{-3} = \frac{1}{2}.$$

The magnitude of spectral coefficients:

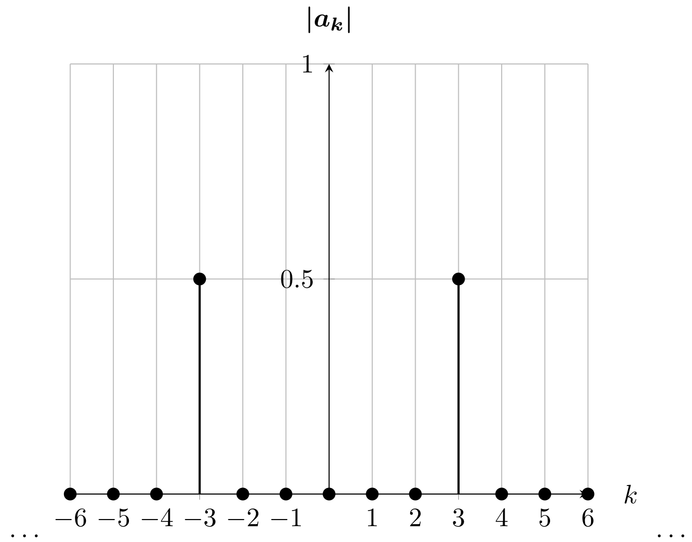



Figure 1: $k$ vs. $|a_k|$.

Phase of the spectral coefficients:

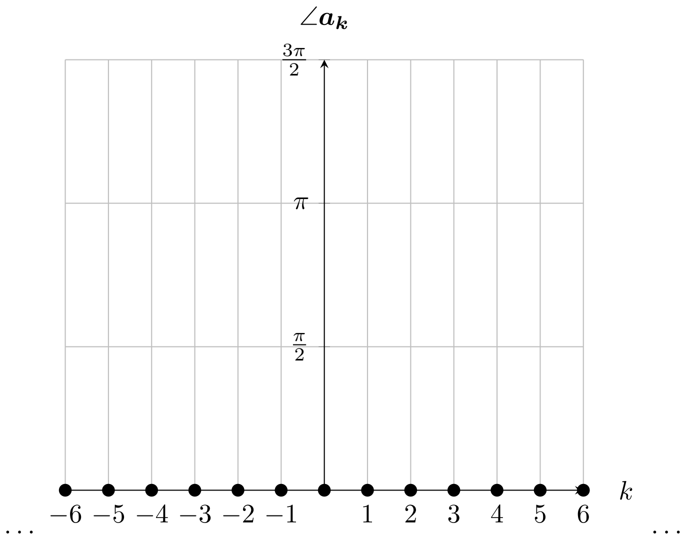



Figure 2: $k$ vs. $\angle a_k$.

6. (a)

$$H(j\omega) = \frac{1}{3+4j\omega} = \frac{1/4}{3/4+j\omega}.$$

Take IFT:

$$h(t) = \frac{1}{4}e^{-\frac{3}{4}t}u(t).$$

(b)

$$Y(j\omega) = \frac{1}{5+j\omega} - \frac{1}{10+j\omega},$$

$$X(j\omega) = \frac{Y(j\omega)}{H(j\omega)} = \frac{15+20j\omega}{(5+j\omega)(10+j\omega)},$$

Using partial fractions, we get:

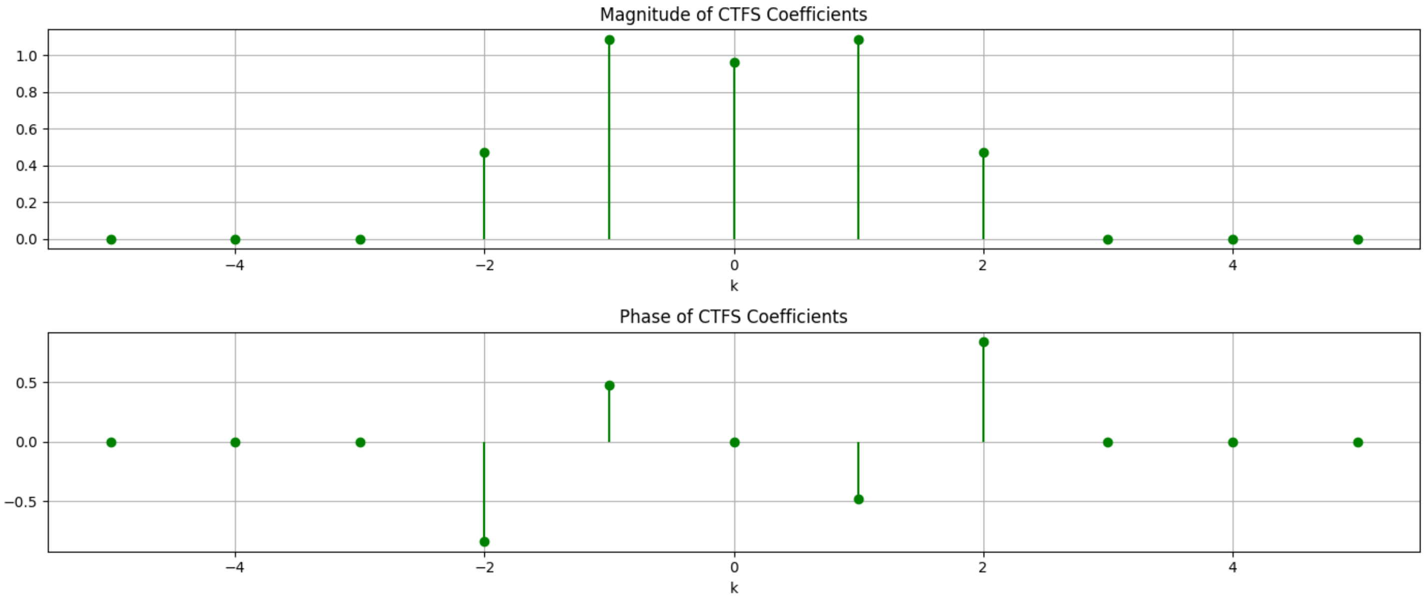
$$X(j\omega) = \frac{37}{10+j\omega} - \frac{17}{5+j\omega},$$

Take IFT:

$$x(t) = (37e^{-10t} - 17e^{-5t})u(t).$$

7.

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the signal parameters
T = 1
omega_0 = 2 * np.pi / T
t = np.arange(0, 1, 0.01)

# Define the signal
xt = 1 + np.sin(omega_0 * t) + 2 * np.cos(omega_0 * t) + np.cos(2 * omega_0 * t + np.pi / 4)

# Initialize the arrays for coefficients
a = []

# Calculate the coefficients
for k in range(0, 6):
    C = np.exp(-1j * omega_0 * t * k)
    a_k = np.trapz(xt * C, t) / T
    if np.abs(a_k) <= 0.1:
        a_k = 0
    a.append(a_k)

# Prepare the coefficients for plotting
a = np.array(a)
a_conj = np.conj(a)
ak = np.concatenate((a_conj[-1:0:-1], a))

# Calculate the magnitude and phase
Mag_ak = np.abs(ak)
Phase_ak = np.angle(ak)

# Plot the magnitude and phase of the coefficients
plt.figure(figsize=(14, 6))

```




```python
plt.subplot(2, 1, 1)
plt.stem(range(-5, 6), Mag_ak, 'g', markerfmt='go', basefmt=" ", use_line_collection=True)
plt.title('Magnitude of CTFS Coefficients')
plt.xlabel('k')
plt.grid(True)

plt.subplot(2, 1, 2)
plt.stem(range(-5, 6), Phase_ak, 'g', markerfmt='go', basefmt=" ", use_line_collection=True)
plt.title('Phase of CTFS Coefficients')
plt.xlabel('k')
plt.grid(True)

plt.tight_layout()
plt.show()
```