CENG 384 - Signals and Systems for Computer Engineers
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Homework 3

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1. We know the formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0t}.$$

And, we know that

$$T = 4 \ \rightarrow w_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Thus, $x(t)$ becomes

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{2}t}$$

$$= \begin{cases} \sum_{k=-\infty}^{\infty} -e^{jk\frac{\pi}{2}t} & \text{k is even} \\ \sum_{k=-\infty}^{\infty} e^{jk\frac{\pi}{2}t} & \text{k is odd} \end{cases}$$

$$= \sum_{k=-\infty}^{\infty} a_{2k} e^{jk\pi t} + \sum_{k=-\infty}^{\infty} a_{2k+1} e^{j(2k+1)\pi t}$$

$$= -\sum_{k=-\infty}^{\infty} e^{jk\pi t} + e^{j\frac{\pi}{2}t} \sum_{k=-\infty}^{\infty} e^{jk\pi t}$$

$$= (e^{j\frac{\pi}{2}t} - 1) \sum_{k=-\infty}^{\infty} e^{jk\pi t}$$

$$= \ldots\, e^{-j3\frac{\pi}{2}t} - e^{-j\pi t} + e^{-j\frac{\pi}{2}t} - 1 + e^{j\frac{\pi}{2}t} - e^{j\pi t} + e^{j3\frac{\pi}{2}t} - \ \ldots$$

2. (a) We know the formula

$$a_k = \frac{1}{T}\int_T x(t) e^{-jkw_0t} dt.$$

And, we know that

$$T = 4 \ \rightarrow \ w_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Hence, we have

$$a_k = \frac{1}{4}\int_0^2 2te^{-jk\frac{\pi}{2}t}dt + \frac{1}{4}\int_2^4 (4-t)e^{-jk\frac{\pi}{2}t}dt$$

$$= \frac{1}{2}\int_0^2 te^{-jk\frac{\pi}{2}t}dt + \int_2^4 e^{-jk\frac{\pi}{2}t}dt - \frac{1}{4}\int_2^4 te^{-jk\frac{\pi}{2}t}dt$$

$$= \frac{1}{2}I_1 + I_2 - \frac{1}{4}I_3.$$

For I_1, let $u = t,\ dv = e^{-jk\frac{\pi}{2}t}dt$. Then, $du = dt,\ v = \frac{-2}{jk\pi}e^{-jk\frac{\pi}{2}t}$. By Integration By Parts, we have

$$I_1 = uv - \int_0^2 vdu = \frac{-2t}{jk\pi} - \int_0^2 \frac{-2}{jk\pi} e^{-jk\frac{\pi}{2}t}dt$$

$$= \frac{-2t}{jk\pi} + \frac{-4}{j^2k^2\pi^2} e^{-jk\frac{\pi}{2}t}\Big|_0^2 = \frac{-2t}{jk\pi} + \frac{4}{k^2\pi^2}(e^{-jk\pi} - 1).$$

Calculating I_2,

$$I_2 = \int_2^4 e^{-jk\frac{\pi}{2}t}dt = \frac{-2}{jk\pi}e^{-jk\frac{\pi}{2}t}\Big|_2^4 = \frac{-2}{jk\pi}(e^{-jk2\pi} - e^{-jk\pi}).$$

Since I_3 is similar to I_1, we can directly write

$$I_3 = \frac{-2t}{jk\pi} + \frac{4}{k^2\pi^2}(e^{-jk2\pi} - e^{-jk\pi}).$$

Thus,

$$a_k = \frac{1}{2}I_1 + I_2 - \frac{1}{4}I_3$$

$$= \frac{-t}{jk\pi} + \frac{2}{k^2\pi^2}e^{-jk\pi} - \frac{2}{k^2\pi^2} - \frac{2}{jk\pi}e^{-jk2\pi} + \frac{2}{jk\pi}e^{-jk\pi} + \frac{t}{2jk\pi} - \frac{1}{k^2\pi^2}e^{-jk2\pi} + \frac{1}{k^2\pi^2}e^{-jk\pi}$$

$$= \frac{-t}{2jk\pi} - \frac{2}{k^2\pi^2} + \frac{2}{jk\pi}(e^{-jk\pi} - e^{-jk2\pi}) + \frac{1}{k^2\pi^2}(3e^{-jk\pi} - e^{-jk2\pi})$$

$$= \frac{2}{jk\pi}(e^{-jk\pi} - e^{-jk2\pi} - \frac{t}{4}) + \frac{1}{k^2\pi^2}(3e^{-jk\pi} - e^{-jk2\pi} - 2).$$

(b) The Differentiation Property states that, if

$$x(t) \leftrightarrow a_k$$

then,

$$\frac{dx(t)}{dt} \leftrightarrow jkw_0a_k.$$

Applying this property, we get

$$a_k = \frac{jk\pi}{2}\frac{2}{jk\pi}(e^{-jk\pi} - e^{-jk2\pi} - \frac{t}{4}) + \frac{jk\pi}{2}\frac{1}{k^2\pi^2}(3e^{-jk\pi} - e^{-jk2\pi} - 2)$$

$$= e^{-jk\pi} - e^{-jk2\pi} - \frac{t}{4} + \frac{j}{2k\pi}(3e^{-jk\pi} - e^{-jk2\pi} - 2).$$

3. (a) Let's start with the first DT signal $x_1[n]$. Its period can be found as follows:

$$cos(\frac{\pi}{2}n) = cos(\frac{\pi}{2}n + \frac{\pi}{2}N)$$
$$\frac{\pi}{2}n + 2\pi m = \frac{\pi}{2}n + \frac{\pi}{2}N$$

Then, the equations above drives us to find $m = 1$ and $N = 4$. According to Table 7.2 in the book, the spectral coefficients of $x_1[n]$, which we define as a_k will be as follows:

$$a_k = \begin{cases} \frac{1}{2} & k = \pm1, \pm1 \pm 4, \pm1 \pm 8 \\ 0 & \text{otherwise} \end{cases}$$

When applying the same process for $x_2[n]$, the following is obtained:

$$sin(\frac{\pi}{2}n) = sin(\frac{\pi}{2}n + \frac{\pi}{2}N)$$
$$\frac{\pi}{2}n + 2\pi m = \frac{\pi}{2}n + \frac{\pi}{2}N$$

Then, m and N values are 1 and 4 respectively. According to Table 7.2 in the book, the spectral coefficients of $x_2[n]$, which we define as b_k will be as follows:

$$b_k = \begin{cases} \frac{1}{2j} & k = \pm1, \pm1 \pm 4, \pm1 \pm 8, ... \\ -\frac{1}{2j} & k = -1, -1 \pm 4, -1 \pm 8, ... \end{cases}$$

As in the question, $x_3[n]$ is the product of the first two signals. After the following algebraic manipulation, we get a sine-based signal.

$$cos(\frac{\pi}{2}n)sin(\frac{\pi}{2}n) = \frac{1}{2}sin(\pi n)$$

Utilizing the method of linearity and Table 7.2 in the book, c_k is found below.

$$c_k = \begin{cases} \frac{1}{4j} & k = \pm 1, \pm 1 \pm 2, \pm 1 \pm 4, ... \\ -\frac{1}{4j} & k = -1, -1 \pm 2, -1 \pm 4, ... \end{cases}$$

(b) The multiplication property states that, multiplication operation in time domain corresponds to convolution operation in frequency domain. Using this fact, we can define c_k as follows:

$$x_3[n] \leftrightarrow c_k$$
$$x_1[n]x_2[n] \leftrightarrow a_k * b_k$$

So, we have

$$c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

Using the spectral coefficients we found in previous part, this sum is equal to

$$c_k = \frac{1}{2} b_{k-1} + \frac{1}{2} b_{k-3}$$

Notice that the magnitude of c_k is half compared to b_k, and its frequency is twice b_k's since the two terms in the expression. Then, c_k is below.

$$c_k = \begin{cases} \frac{1}{4j} & k = \pm 1, \pm 1 \pm 2, \pm 1 \pm 4, ... \\ -\frac{1}{4j} & k = -1, -1 \pm 2, -1 \pm 4, ... \end{cases}$$

To wrap up, when comparing the c_k signals found in this part and the previous part, they are identical.

4. Using the linearity property, we can split a_k into two terms, as $a_k = a_k^{(1)} + a_k^{(2)}$ where $a_k^{(1)} = cos(k\frac{\pi}{3})$ and $a_k^{(2)} = cos(k\frac{\pi}{4})$. Each of them corresponds a signal in time domain. Those are defined below.

$$x_1[n] \leftrightarrow a_k^{(1)}$$
$$x_2[n] \leftrightarrow a_k^{(2)}$$

The linearity property implies that,

$$x[n] = x_1[n] + x_2[n] \leftrightarrow a_k = a_k^{(1)} + a_k^{(2)}$$

The spectral coefficients $a_k^{(1)}$ and $a_k^{(2)}$ can be written in terms of complex exponential functions.

$$a_k^{(1)} = cos(k\frac{\pi}{3}) = \frac{e^{jk\frac{\pi}{3}} + e^{-jk\frac{\pi}{3}}}{2}$$
$$a_k^{(2)} = cos(k\frac{\pi}{4}) = \frac{e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}}{2}$$

The angular frequency of $a_k^{(1)}$ and $a_k^{(2)}$ are $\frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively. The analysis equations for both can be written as follows:

$$a_k^{(1)} = \frac{1}{6} \sum_{n=-2}^{3} x_1[n] e^{-jk\frac{\pi}{3}n} = \frac{1}{2} e^{jk\frac{\pi}{3}} + \frac{1}{2} e^{-jk\frac{\pi}{3}}$$

$$a_k^{(2)} = \frac{1}{8} \sum_{n=-3}^{4} x_2[n] e^{-jk\frac{\pi}{4}n} = \frac{1}{2} e^{jk\frac{\pi}{4}} + \frac{1}{2} e^{-jk\frac{\pi}{4}}$$

Comparing the left hand side and the right hand side of the equations above,

$$x_1[1] = x_1[-1] = 3 \to N = 6$$
$$x_1[1] = x_1[-1] = 4 \to N = 8$$

Note that $x[n]$ is periodic, and its period is the LCM of $x_1[n]$'s and $x_2[n]$'s, which is 24. In the interval $-1 \leq n \leq 23$, $x_1[n]$ and $x_2[n]$ are written as,

$$x_1[-1] = x_1[1] = x_1[5] = x_1[7] = x_1[11] = x_1[13] = x_1[17] = x_1[19] = x_1[23] = 3$$
$$x_2[-1] = x_2[1] = x_2[7] = x_2[9] = x_2[15] = x_2[17] = x_2[23] = 4$$

Hence, $x[n]$ can be written as follows:

$$x[n] = x_1[n] + x_2[n]$$
$$x[n] = 7\delta[n+1] + 7\delta[n-1] + 3\delta[n-5] + 7\delta[n-7] + 4\delta[n-9] + 3\delta[n-11] + 3\delta[n-13] + 4\delta[n-15] + 7\delta[n-17] + 3\delta[n-19] + 7\delta[n-23]$$

Notice that $x[n]$ is a periodic function, where $x[n] = x[n \pm 24]$ for all $n \in \mathbb{Z}$.

5. (a) Notice that the signal is a discrete time signal. Let's apply the steps to find its period.

$$x[n] = sin(\frac{6\pi}{13}n + \frac{\pi}{2})$$
$$sin(\frac{6\pi}{13}n + \frac{\pi}{2}) = sin(\frac{6\pi}{13}n + \frac{6\pi}{13}N + \frac{\pi}{2})$$
$$\frac{6\pi}{13}n + \frac{\pi}{2} + 2\pi m = \frac{6\pi}{13}n + \frac{6\pi}{13}N + \frac{\pi}{2}$$

Then, we found that $N = 13$ for $m = 3$. That drives us $\omega_0 = \frac{2\pi}{N}m = \frac{6\pi}{13}$.

(b) From Table 7.2, we see that spectral coefficients of $x'[n] = sin(\omega_0 n)$ for $\omega_0 = \frac{6\pi}{13}$ is,

$$a'_k = \begin{cases} \frac{1}{2j} & k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, ... \\ -\frac{1}{2j} & k = -3, -3 \pm 13, -3 \pm 26, ... \end{cases}$$

Time-shifting property of DT Fourier Series states that, the time shift operation comes along with a multiplicative exponential factor in the frequency domain.

$$x[n] = x'[n - n_0] \leftrightarrow a'_k e^{-jk\omega_0 n_0}$$
$$x[n] = sin(\frac{6\pi}{13}n + \frac{\pi}{2}) = sin(\frac{6\pi}{13}(n + \frac{13}{12}))$$

Hence, $n_0 = -\frac{13}{12}$. Replacing the values, we obtain

$$a_k = \begin{cases} \frac{1}{2j}e^{jk\frac{\pi}{2}} & k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, ... \\ -\frac{1}{2j}e^{jk\frac{\pi}{2}} & k = -3, -3 \pm 13, -3 \pm 26, ... \end{cases}$$

Since $e^{\frac{j\pi}{2}} = j$ and $e^{\frac{-j\pi}{2}} = -j$, we can rewrite a_k as

$$a_k = \begin{cases} \frac{1}{2j}j^k & k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, ... \\ \frac{-1}{2j}(-j)^k & k = -3, -3 \pm 13, -3 \pm 26, ... \end{cases}$$

As we can see, a_k always takes the values $\pm\frac{1}{2j}$ and $\pm\frac{1}{2}$, of which magnitude are $\frac{1}{2}$. Hence, the magnitude spectrum is,

$$|a_k| = \frac{1}{2} \text{ for } k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, ...$$

and the phase spectrum is,

$$\angle a_k = \begin{cases} (k-1)\frac{\pi}{2} \text{ for } k = \pm 3, \pm 3 \pm 13, \pm 3 \pm 26, ... \\ -(k-1)\frac{\pi}{2} \text{ for } k = -3, -3 \pm 13, -3 \pm 26, ... \end{cases}$$

Here are the plots:

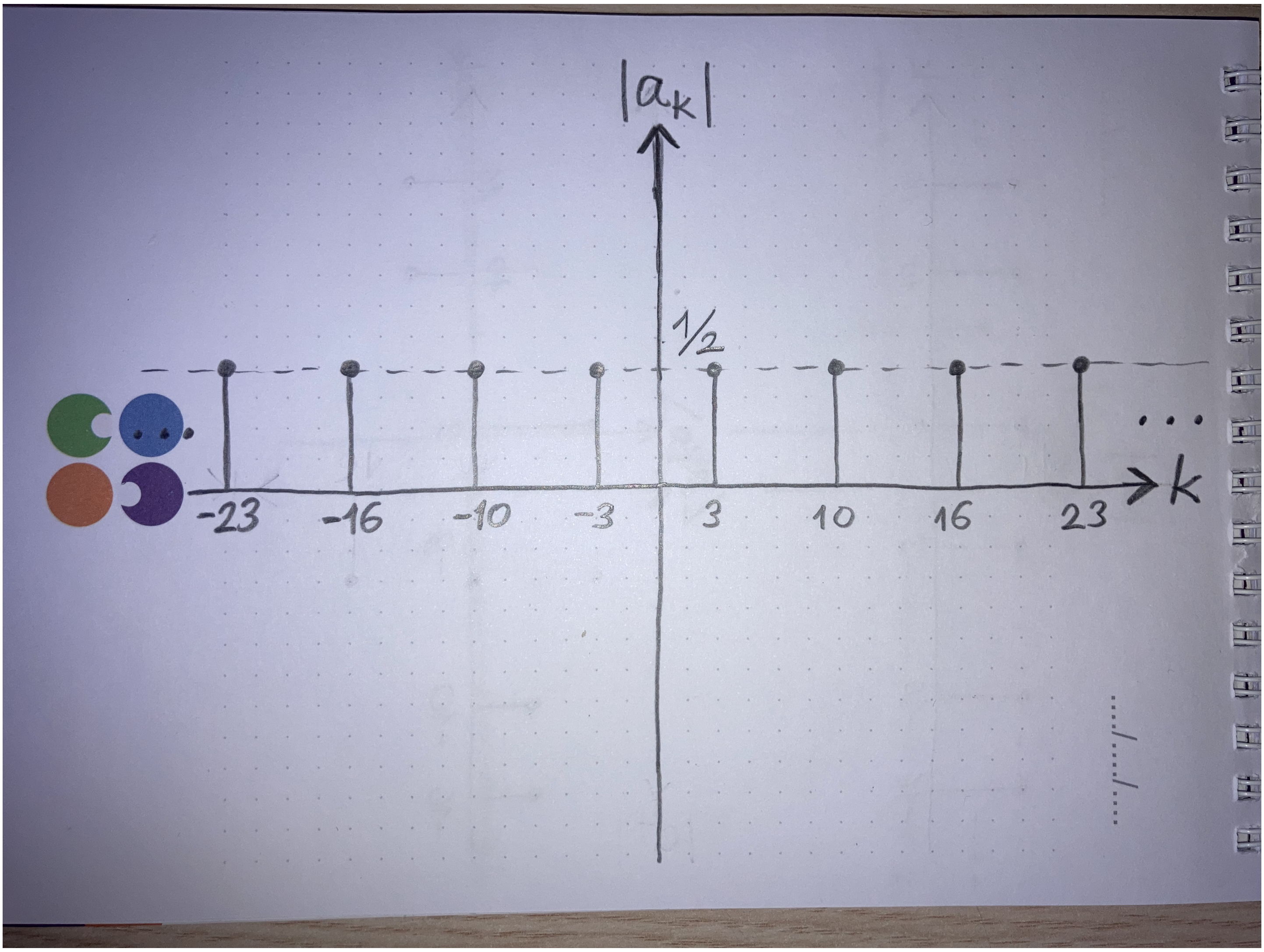

Figure 1: Magnitude plot

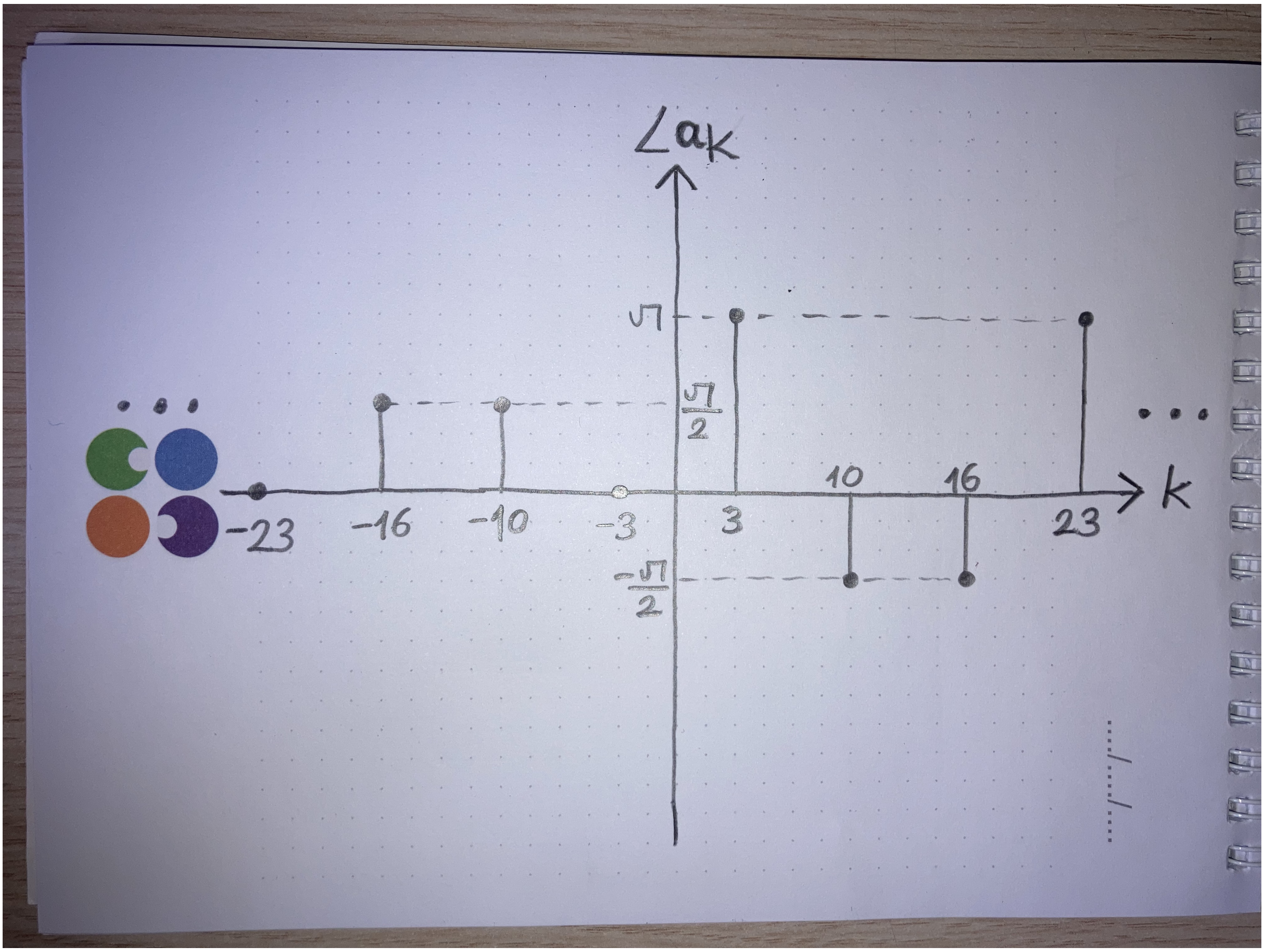

Figure 2: Phase plot

6. (a) We have

$$H(jw) = \frac{1}{3+4jw} = \frac{1}{4}\frac{1}{\frac{3}{4}+jw}.$$

We know that

$$\frac{1}{a+jw} \leftrightarrow e^{-at}u(t).$$

Therefore,

$$\frac{1}{\frac{3}{4}+jw} \leftrightarrow e^{\frac{-3t}{4}}u(t).$$

By Linearity Property of Fourier Transform, we get

$$H(jw) = \frac{1}{4}\frac{1}{\frac{3}{4}+jw} \leftrightarrow \frac{1}{4}e^{\frac{-3t}{4}}u(t).$$

(b) By Convolution Property of Fourier Transform,

$$y(t) = x(t) * h(t) \;\leftrightarrow\; Y(jw) = X(jw)H(jw).$$

We know the following property:

$$e^{-at}u(t) \;\leftrightarrow\; \frac{1}{a+jw}.$$

Thus, we have

$$Y(jw) = \frac{1}{5+jw} - \frac{1}{10+jw} = \frac{5}{(10+jw)(5+jw)} = X(jw)H(jw).$$

Hence,

$$X(jw) = \frac{Y(jw)}{H(jw)} = \frac{\frac{5}{(10+jw)(5+jw)}}{\frac{1}{3+4jw}} = \frac{15+20jw}{(10+jw)(5+jw)}.$$

To find the $x(t)$, the partial fraction method can be applied.

$$\frac{15+20jw}{(10+jw)(5+jw)} = \frac{A}{10+jw} + \frac{B}{5+jw}.$$

After some algebraic operations, we obtain the following equations:

$$A + B = 20$$
$$5A + 10B = 15$$
$$A = 37 \text{ and } B = -17$$

Then, the expression below is obtained,

$$X(jw) = \frac{37}{10+jw} + \frac{-17}{5+jw}.$$

Since we know the conversion formula, $x(t)$ will be as follows:

$$X(jw) \leftrightarrow x(t) = (37e^{-10t} - 17e^{-5t})u(t).$$

7. In order to find T,

Fundamental period of $cos(\frac{\pi t}{3}) = \frac{2\pi}{\frac{\pi}{3}} = 6$.

Fundamental period of $2cos(\pi t + \frac{\pi}{2}) = \frac{2\pi}{\pi} = 2$.

Since they are added, their LCM gives the fundamental period of $x(t)$, which is $T = 6$.

Therefore, $w_0 = \frac{2\pi}{T} = \frac{\pi}{3}$.

Rewrite $x(t)$ by using Euler's Formula:

$$x(t) = \frac{1}{2}e^{\frac{j\pi t}{3}} + \frac{1}{2}e^{\frac{-j\pi t}{3}} + e^{j(\pi t + \frac{\pi}{2})} + e^{-j(\pi t + \frac{\pi}{2})}.$$

Since $e^{\frac{j\pi}{2}} = cos(\frac{j\pi}{2}) + jsin(\frac{j\pi}{2}) = 0 + j = j$ and $e^{\frac{-j\pi}{2}} = cos(\frac{-j\pi}{2}) + jsin(\frac{-j\pi}{2}) = 0 - j = -j$, we have

$$x(t) = \frac{1}{2}e^{\frac{j\pi t}{3}} + \frac{1}{2}e^{\frac{-j\pi t}{3}} + je^{j\pi t} - je^{-j\pi t}.$$

We know the formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jkw_0 t}.$$

Putting in the natural frequency, we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{\pi}{3}t}.$$

Therefore, Fourier series coefficients are $a_{-1} = \frac{1}{2}$, $a_1 = \frac{1}{2}$, $a_{-3} = j$, $a_3 = -j$, and $a_k = 0$ for all other k.

The code for plotting magnitude and phase of a_k is below:

```python
import numpy as np
import matplotlib.pyplot as plt

a_negative1 = 0.5
a_1 = 0.5
a_negative3 = 1j
a_3 = -1j

X = np.arange(-5,6)
Y = np.array([0, 0, a_negative3, 0, a_negative1, 0, a_1, 0, a_3, 0, 0])
Y_magnitude = np.abs(Y)
Y_phase = np.angle(Y, deg=True)

figs, axs = plt.subplots(1, 2, figsize=(20,10))
axs[0].scatter(X, Y_magnitude)
axs[0].set_xticks(np.arange(-5,6))
axs[0].set_yticks(np.arange(0.0, 1.1, 0.1))
axs[0].set_xlabel("k")
axs[0].set_ylabel("Magnitude of a_k")
axs[0].set_title("Magnitude plot")
axs[1].scatter(X, Y_phase)
axs[1].set_xticks(np.arange(-5,6))
axs[1].set_yticks(np.arange(-180, 181, 30))
axs[1].set_xlabel("k")
axs[1].set_ylabel("Phase of a_k (in degrees)")
axs[1].set_title("Phase plot")
plt.show()

print("The fundamental period is: 6")
print("The simplified Fourier series representation is:")
print(f"\t a_(-1) = {a_negative1},")
print(f"\t a_(1) = {a_1},")
print(f"\t a_(-3) = {a_negative3},")
print(f"\t a_(3) = {a_3},")
print("\t and a_k = 0 for all other k.")
```