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Homework 2 / 24.12.2023

Q1a OLTF $+1=0$

$$\Rightarrow \frac{K(s+4)}{s(s+2)}+1=0 \Rightarrow K(s+4)=-s(s+2)$$

$$\Rightarrow \underbrace{s(s+2)}_{\bar{D}(s)}+\underbrace{K(s+4)}_{\bar{N}(s)}=0$$

roots of $\bar{D}(s)$: $s_1=0,\ s_2=-2$ } $n=2$

roots of $\bar{N}(s)$: $s_1=-4$ } $m=1$.

angle 1^{st} asymptote: $\frac{180}{n-m}=180°$

angle between asymptotes: $\frac{360}{n-m}=360°$

# of asymptotes: $n-m=1$.

$$\sigma_0=\frac{\Sigma p_i-\Sigma z_i}{n-m}=\frac{-2-0-(-4)}{2-1}=2$$

Break away/in pts: $\bar{N}'\bar{D}-\bar{N}\bar{D}'=0$

$$\bar{N}'(s)=1,\quad \bar{D}'(s)=2s+2$$

$$\Rightarrow 1\cdot(s^2+2s)-(s+4)(2s+2)=0$$

$$\Rightarrow s^2+2s-(2s^2+10s+8)=0$$

$$\Rightarrow s^2+8s+8=0.$$

$$\Delta=64-32=32$$

$$s_1=\frac{-8-\sqrt{32}}{2}=-4-2\sqrt{2}=-6.83$$

$$s_2=\frac{-8+\sqrt{32}}{2}=-4+2\sqrt{2}=-1.17$$

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Q1a) S_2 is the break-away point because it is between the poles $s=0$ and $s=-2$. S_1 is the break-in point because it is the other root of $\bar{N}'\bar{D}-\bar{N}\bar{D}'=0$.

Imaginary axis crossings: $G(s)H(s)+1 \Big|_{s=j\omega} = 0.$

$$\Rightarrow s^2+2s+K(s+4)\Big|_{s=j\omega}=0.$$

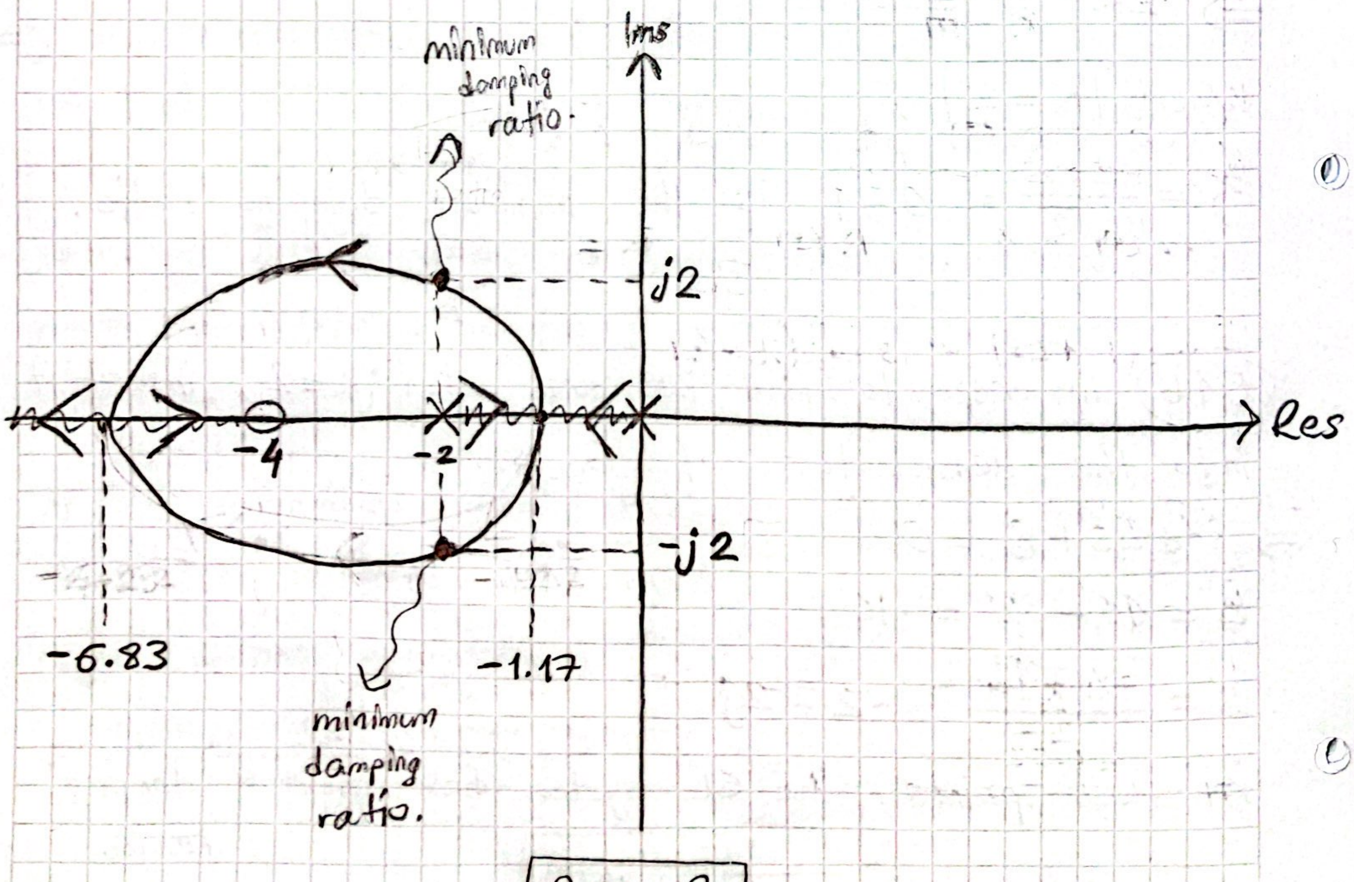
$$\Rightarrow -\omega^2+2j\omega+K(j\omega+4)=0$$

$$\Rightarrow -\omega^2+4K+j(2\omega+K\omega)=0$$

$$(K+2)\omega=0\ ,\quad 4K=\omega^2$$

$\cancel{K=-2}$

$\omega=0$, $K=0$ } When $K=0$, the pole $s=0$ intersects the imaginary axis. Except this, there is no imaginary axis crossing.



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Q1c $G_{CL}(s) = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{\frac{K(s+4)}{s(s+2)}}{1+\frac{K(s+4)}{s(s+2)}} = \dfrac{K(s+4)}{s^2+(K+2)s+4K}$

$2\zeta\omega_n s = (K+2)s \Rightarrow \zeta\omega_n = \dfrac{K+2}{2}$

$\omega_n^2 = 4K \Rightarrow \omega_n = \mp 2\sqrt{K}$

Hence,

$\zeta = \dfrac{K+2}{4\sqrt{K}}$

In order to minimize ζ, take its derivative with respect to K.

$\zeta' = \dfrac{4\sqrt{K} - \frac{(K+2)2}{\sqrt{K}}}{16K} = \dfrac{4K-2K-4}{16K\sqrt{K}} = 0$

$\Rightarrow 2K-4 = 0.$

When $\boxed{K=2}$,

$\zeta = \dfrac{2+2}{4\sqrt{2}} = 0.71$ is the minimum damping ratio.

Q1b In order to find the poles when K=2, insert it into the denominator of $G_{CL}(s)$.

$\Rightarrow s^2+4s+8 = 0.$

$\Delta = 16-32 = -16$

$s = \dfrac{-4\mp j4}{2} = -2\mp 2j$

At these points, the CL system has minimum damping ratio.

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Q1d In Q1c, we have found that $\zeta = \frac{K+2}{4\sqrt{K}}$.

$$\Rightarrow \frac{K+2}{4\sqrt{K}} = \frac{\sqrt{3}}{2} \Rightarrow 2K+4 = 4\sqrt{3K} \Rightarrow 4K^2+16K+16 = 48K$$

$$\Rightarrow 4K^2-32K+16=0 \Rightarrow K^2-8K+4=0.$$

$$\Delta = 64-16 = 48$$

$$K = \frac{8 \mp \sqrt{48}}{2} = 4 \mp 2\sqrt{3} \Rightarrow K_1 = 7.46,\ K_2 = 0.54$$

So, if the CL system has the damping ratio $\zeta = \frac{\sqrt{3}}{2}$, we have either $K=7.46$ or $K=0.54$. When $K=0.54$, $t_s(5\%)$ is bigger because poles are closer to the imaginary axis. So, $\boxed{K=7.46}$.

Q2a OLTF$+1=0$.

$$\Rightarrow \frac{K(s^2-2s+2)}{s(s+1)(s+2)(s+4)} + 1 = 0$$

$$\Rightarrow \underbrace{s(s+1)(s+2)(s+4)}_{\bar{D}(s)} + K\underbrace{(s^2-2s+2)}_{\bar{N}(s)} = 0$$

roots of $\bar{D}(s)$: $s_1=0,\ s_2=-1,\ s_3=-2,\ s_4=-4 \ \} \ n=4$

roots of $\bar{N}(s)$: $s^2-2s+2 = (s-1)^2+1=0$

$\Rightarrow s_1 = 1+j,\ s_2 = 1-j \ \} \ m=2.$

# of asymptotes: $|n-m| = 2$

Angle 1st asymptote: $\frac{180}{n-m} = 90^\circ$

Angle between asymptotes: $\frac{360}{n-m} = 180^\circ$

$$\sigma_0 = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-7-(2)}{2} = -4.5$$

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Q2a Break away/in points: $\bar{N}'\bar{D} - \bar{N}\bar{D}' = 0$.

$\bar{N}'(s) = 2s-2$

$\bar{D}(s) = s(s+1)(s+2)(s+4) = s^4+7s^3+14s^2+8s$

$\bar{D}'(s) = 4s^3+21s^2+28s+8$

$\Rightarrow (2s-2)(s^4+7s^3+14s^2+8s) - (s^2-2s+2)(4s^3+21s^2+28s+8) = 0$.

Roots of this polynomial are: $s_1 = -0.31$, $s_2 = -3.16$, $s_3 = -1.45$

Imaginary axis crossings: $G(s)H(s) + 1 \Big|_{s=jw} = 0$

$$\Rightarrow s^4+7s^3+14s^2+8s+K(s^2-2s+2)\Big|_{s=jw} = 0.$$

$$\Rightarrow w^4 - 7jw^3 - 14w^2 + 8jw - Kw^2 - 2Kjw + 2K = 0$$

$$\Rightarrow w^4 - (14+K)w^2 + 2K + j\left(-7w^3 + (8-2K)w\right) = 0.$$

Let $a = w^2$. Then

$a^2 - (14+K)a + 2K = 0$.

$\Delta = K^2+28K+196-8K = K^2+20K+196$

$$a = \frac{14+K-\sqrt{K^2+20K+196}}{2}$$

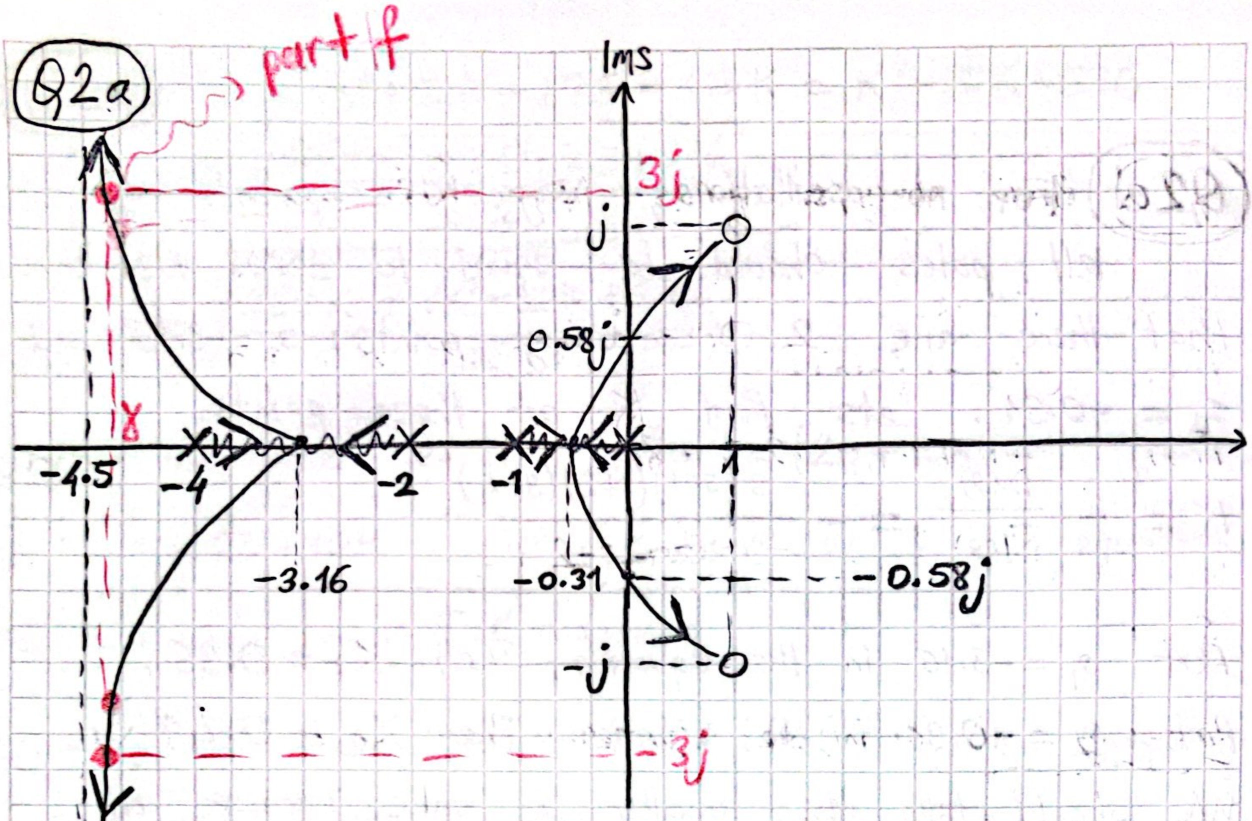
$$\Rightarrow w = \sqrt{\frac{14+K-\sqrt{K^2+20K+196}}{2}}$$

Put w in $-7w^3+(8-2K)w = 0$.

From that, we find $K_1 = 0$, $K_2 = 2.81$, $K_3 = -13.62$.

Hence, we see that there is an imaginary axis crossing when $\boxed{K = 2.81}$, and the intersection points are $\boxed{\mp jw = \mp j \cdot 0.58}$

Q2a



Q2b As we have found in part a, when $K=2.81$, there are 2 poles on the imaginary axis:

$\boxed{s_1 = 0.58j}$, $\boxed{s_2 = -0.58j}$.

Now we need to find the other poles.

$$G_{CL}(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{K(s^2-2s+2)}{s(s+1)(s+2)(s+4)+K(s^2-2s+2)}$$

Put $K=2.81$ in the denominator and equate it to zero.

$\boxed{s_3 = -3.50 - 2.05j}$ $\boxed{s_4 = -3.50 + 2.05j}$

Q2c When $K \geq 2.81$, 2 poles exit from open left half plane, which makes the CL system unstable.

So, $\boxed{0 \leq K < 2.81}$

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Q2d For no oscillations, imaginary components of all poles should be equal to zero. We know that there are 2 break-away points: $s_1 = -3.16$ and $s_2 = -0.31$. Lets find K at these points.

$$K = -\frac{\bar{D}(s)}{\bar{N}(s)} = -\frac{s(s+1)(s+2)(s+4)}{s^2-2s+2}$$

Put $s_1 = -3.16$ in the equation. Then $K_1 = 0.36$.
Put $s_2 = -0.31$ in the equation. Then $K_2 = 0.49$.
We should take the smaller K value because all poles should be on the real axis. Thus,

$$\boxed{0 \leq K \leq 0.36}$$

(At $K = 0.36$, poles $s=-4$ and $s=-2$ become a double pole, which is on the real axis. So, equality is included.)

Q2e We need to put $s = -4 \mp j\omega$ into the equation $\bar{D}(s) + K\bar{N}(s) = 0$.

$$\bar{D}(s) + K\bar{N}(s) = s^4 + 7s^3 + 14s^2 + 8s + K(s^2-2s+2) = 0.$$

Put s into the equation:

$$s^4 + 7s^3 + 14s^2 + 8s + K(s^2-2s+2)\Big|_{s=-4-j\omega} = 0.$$

$$\Rightarrow K(-\omega^2 + 10j\omega + 26) + \omega^4 - 9j\omega^3 - 26\omega^2 + 24j\omega = 0.$$

$$\Rightarrow \omega^4 - (26+K)\omega^2 + 26K - j\left(9\omega^3 - (24+10K)\omega\right) = 0.$$

Let $a = \omega^2$. Then

$$a^2 - (26+K)a + 26K = 0.$$

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Q2e $\Delta = K^2+52K+676-104K = K^2-52K+676$

$$a = \frac{26+K \mp \sqrt{K^2-52K+676}}{2}$$

$$\Rightarrow w = \sqrt{\frac{26+K - \sqrt{K^2-52K+676}}{2}}$$

From the other equation, $9w^3 = (24+10K)w$. Ignore $w=0$.

$\Rightarrow 9w^2 = 24+10K$. Put the above w in this equation.

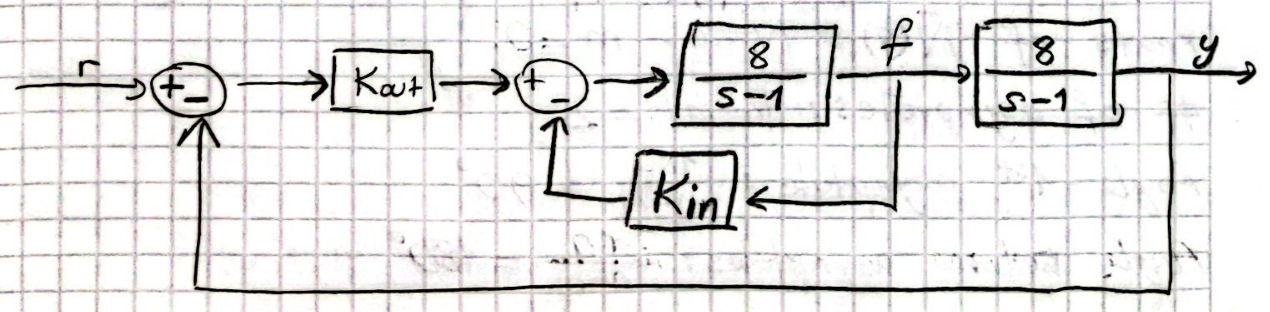
$\Rightarrow \boxed{K=21}$, $\boxed{w = 5.1 \text{ rad/sec}}$

Q2f We should put $s = \gamma \mp j3$ into the equation $\bar{D}(s) + K\bar{N}(s) = 0$, i.e., we should calculate

$$s^4+7s^3+14s^2+8s+K(s^2-2s+2)\Big|_{s=\gamma\mp j3} = 0.$$

From this equation, we will have a real part and an imaginary part. We will equate both of them to zero. Thus, with 2 equations, we can find the 2 variables which are γ and K. γ is the real part of the corresponding poles.

Q3a $c_1 = 8$, $c_2 = 1$, $c_3 = 8$, $c_4 = 1$.



$$(K_{out} - fK_{in})\frac{8}{s-1} = f \Rightarrow \frac{8K_{out}}{s-1} = f\left(\frac{8K_{in}+s-1}{s-1}\right)$$

$$\Rightarrow f = \frac{8K_{out}}{8K_{in}+s-1}$$

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Q3a Thus, $G(s) = f \cdot \frac{8}{s-1} = \frac{8K_{out}}{8K_{in}+s-1} \cdot \frac{8}{s-1} = \frac{64K_{out}}{(8K_{in}+s-1)(s-}$

Hence, $G_{CL}(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{64K_{out}}{(8K_{in}+s-1)(s-1)+64K_{out}}$

$D(s) = s^2+(8K_{in}-2)s + 64K_{out}-8K_{in}+1$

Form the Routh array:

s^2	$1 \quad 64K_{out}-8K_{in}+1$
s	$8K_{in}-2$
1	$64K_{out}-8K_{in}+1$

For stability, we must have

$8K_{in}-2>0 \Rightarrow \boxed{K_{in}>0.25}$

$64K_{out}-8K_{in}+1>0 \Rightarrow \boxed{K_{out}>\frac{8K_{in}-1}{64}}$

Q3b Pick $K_{in}=0.5$. Then $K_{out}>3/64$. $D(s)$ becomes

$D(s) = s^2+2s+64K_{out}-3 = \underbrace{s^2+2s-3}_{\bar{D}(s)} + K_{out}\cdot\underbrace{64}_{\bar{N}(s)} = 0.$

roots of $\bar{D}(s)$: $s=-3$, $s=1$ } $n=2$

roots of $\bar{N}(s)$: None. $m=0$.

# of asymptotes: $|n-m|=2$

Angle 1st asymptote: $\frac{180}{n-m} = 90°$

Angle between asymptotes: $\frac{360}{n-m} = 180°$

$\sigma_0 = \frac{\sum p_i - \sum z_i}{n-m} = \frac{-2}{2} = -1.$

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Q3b) Break away/in points: $\bar{N}'\bar{D} - \bar{N}\bar{D}' = 0.$

$\bar{N}' = 0. \quad \bar{D}' = 2s+2$

$\Rightarrow -64(2s+2) = 0 \Rightarrow s = -1$ is the break-away pt.

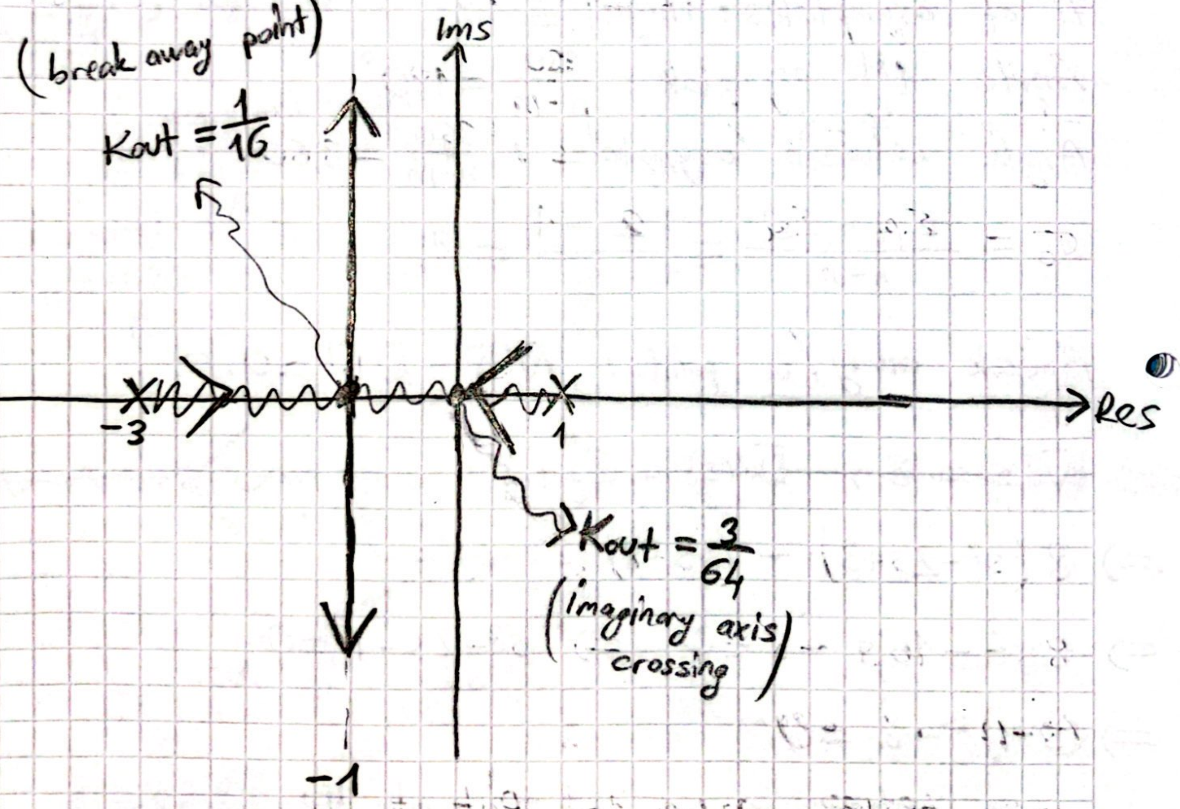
Imaginary axis crossings: $\bar{D}(s) + K\bar{N}(s)\Big|_{s=j\omega} = 0.$

$\Rightarrow s^2+2s-3+64K_{out}\Big|_{s=j\omega} = 0.$

$\Rightarrow -\omega^2+2j\omega-3+64K_{out} = 0.$

$\Rightarrow -\omega^2+64K_{out}-3+j(2\omega) = 0.$

$\Rightarrow \omega = 0, \; K_{out} = \frac{3}{64}.$



Put $s=-1$ in $\bar{D}(s) + K\bar{N}(s) = 0. \Rightarrow s^2+2s-3+64K_{out}\Big|_{s=-1} = 0.$

$\Rightarrow K_{out} = \frac{1}{16}$ is the gain at the break-away point.

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Q3c For any $K_{out} > \frac{1}{64}$, K_{in} is greater than 0.25 and thus the system is stable. So,

Pick $K_{out} = 1/32$.

D(s) becomes

$$D(s) = s^2 + (8K_{in} - 2)s - 8K_{in} + 3$$

$$= \underbrace{s^2 - 2s + 3}_{\bar{D}(s)} + K_{in}\underbrace{(8s - 8)}_{\bar{N}(s)} = 0$$

roots of $\bar{D}(s)$: $\bar{D}(s) = (s-1)^2 + 2$

$\Rightarrow s_1 = 1 + j\sqrt{2}$, $s_2 = 1 - j\sqrt{2}$ $\}$ $n=2$

roots of $\bar{N}(s)$: $s = 1$ $\}$ $m = 1$.

# of asymptotes: $|n-m| = 1$.

Angle 1^{st} asymptote: $\frac{180}{n-m} = 180^\circ$

Angle between asymptotes: $\frac{360}{n-m} = 360^\circ$

$$\sigma_0 = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{2-1}{1} = 1.$$

Break away/in points: $\bar{N}'\bar{D} - \bar{N}\bar{D}' = 0$.

$\bar{N}'(s) = 8$, $\bar{D}'(s) = 2s - 2$.

$\Rightarrow 8(s^2 - 2s + 3) - (8s-8)(2s-2) = 0$

$\Rightarrow 8s^2 - 16s - 8 = 0 \Rightarrow s^2 - 2s - 1 = 0$.

$\Rightarrow (s-1)^2 - 2 = 0$.

$\Rightarrow s_1 = 1 + \sqrt{2} = 2.41$
$\quad s_2 = 1 - \sqrt{2} = -0.41$ $\}$ Put into the equation $\bar{D}(s) + K\bar{N}(s) = 0$.

$\Rightarrow s^2 - 2s + 3 + K_{in}(8s-8)\Big|_{s=2.41} = 0 \Rightarrow K_{in_1} = -0.35$.

Similarly, $K_{in_2} = 0.35$. We pick only s_2 because $K_{in} > 0$.

Q3c

Imaginary axis crossings: $\bar{D}(s) + K\bar{N}(s) \Big|_{s=j\omega} = 0.$

$\Rightarrow s^2 - 2s + 3 + K_{in}(8s - 8) \Big|_{s=j\omega} = 0.$

$\Rightarrow -\omega^2 - 2j\omega + 3 + 8K_{in}j\omega - 8K_{in} = 0.$

$\Rightarrow -\omega^2 - 8K_{in} + 3 + j(8K_{in}\omega - 2\omega) = 0$

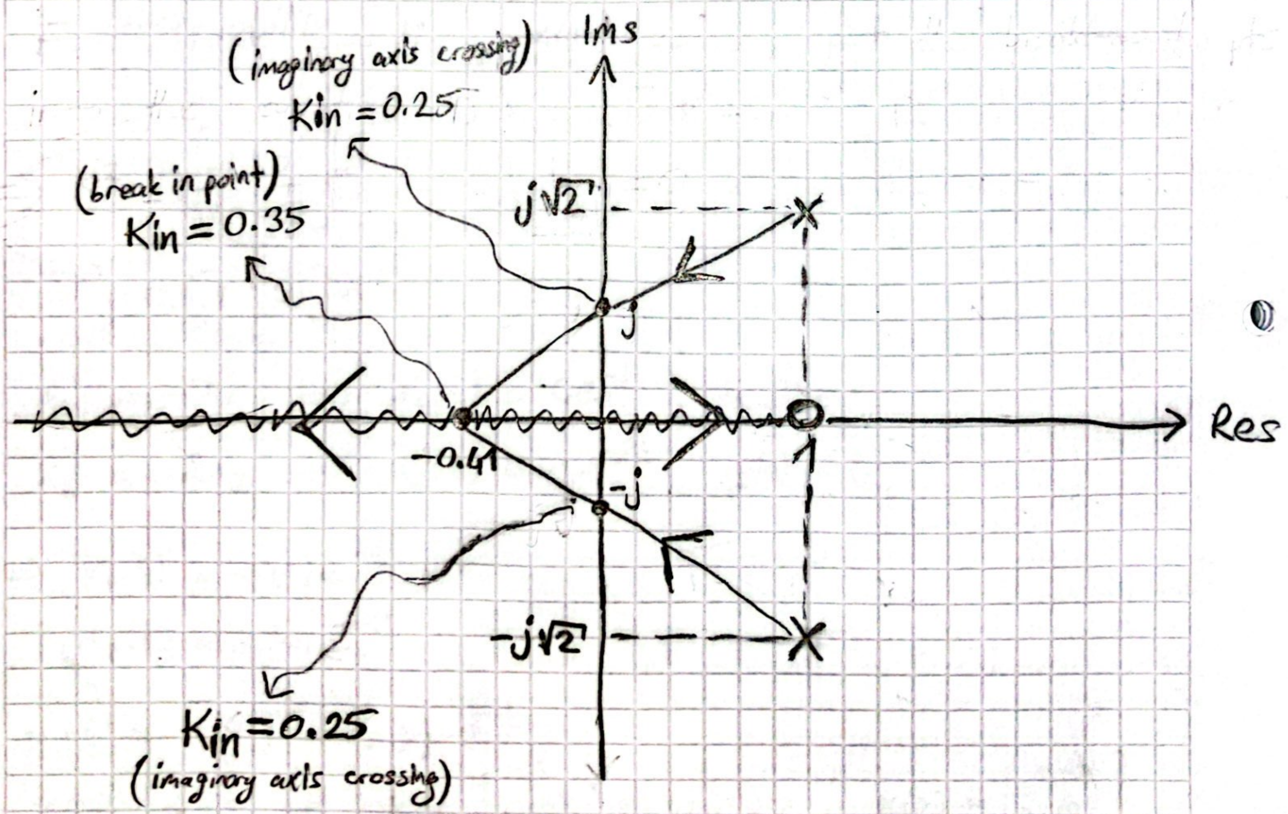
$\Rightarrow \omega^2 = -8K_{in} + 3 \quad , \quad (8K_{in} - 2)\omega = 0.$

$\Rightarrow \omega = \mp\sqrt{3 - 8K_{in}}$. Put in the other equation.

$(8K_{in} - 2)\sqrt{3 - 8K_{in}} = 0.$

$K_{in_1} = \frac{3}{8}, \quad \omega_1 = 0$

$K_{in_2} = \frac{1}{4}, \quad \omega_2 = \mp\sqrt{1} = \mp 1$ rad/sec



The trajectories of poles can be vice versa. Both of the solutions are correct.