HW1 SOLUTIONS

Q1) a) $$\frac{s^2-6s-19}{(s+3)^2(s^2+2s+5)} = \frac{As+B}{(s+3)^2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{(As+B)(s^2+2s+5)+(Cs+D)(s^2+6s+9)}{(s+3)^2(s^2+2s+5)}$$

$$= \frac{As^3+2As^2+5As+Bs^2+2Bs+5B+Cs^3+6Cs^2+9Cs+Ds^2+6Ds+9D}{(s+3)^2(s^2+2s+5)}$$

$$= \frac{(A+C)s^3+(2A+B+6C+D)s^2+(5A+2B+9C+6D)s+5B+9D}{(s+3)^2(s^2+2s+5)}$$

$\Rightarrow$ $A+C=0 \Rightarrow A=-C$

$2A+B+6C+D=1 \Rightarrow B+4C+D=1$

$5A+2B+9C+6D=-6 \Rightarrow 2B+4C+6D=-6$ $\Big\} \Rightarrow B+5D=-7$

$5B+9D=-19$ $\qquad 5B+9D=-19 \qquad 5B+9D=-19$

$B=-7-5D \Rightarrow -35-25D+9D=-19$

$-16D=16 \Rightarrow D=-1$

$\Rightarrow B=-7+5=-2$

$4C=4 \rightarrow C=1 \qquad A=-1$

$$\Rightarrow \frac{s^2-6s-19}{(s+3)^2(s^2+2s+5)} = -\frac{s+2}{(s+3)^2} + \frac{s-1}{s^2+2s+5}$$

$$\frac{s+2}{(s+3)^2} = \frac{s+3-1}{(s+3)^2} = \frac{1}{s+3} - \frac{1}{(s+3)^2} \Rightarrow e^{-3t} - te^{-3t}, \; t \geq 0.$$

$$\frac{s-1}{s^2+2s+5} = \frac{s-1}{(s+1)^2+4} = \frac{s+1-2}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$\Rightarrow e^{-t}\cos 2t - e^{-t}\sin 2t \,,\quad t \geqslant 0$$

$$f(t) = -\left(e^{-3t} - te^{-3t}\right) + e^{-t}\left(\cos 2t - \sin 2t\right) \,,\ t \geqslant 0$$

b-) $$\frac{s^2}{(s+1)^3} = \frac{s^2+2s+1-2s-1}{(s+1)^3} = \frac{(s+1)^2-2(s+1-1)-1}{(s+1)^3}$$

$$= \frac{(s+1)^2-2(s+1)+1}{(s+1)^3} = \frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}$$

$$f(t) = \left(1 - 2t + \frac{t^2}{2}\right)e^{-t} \,,\ t \geqslant 0$$

c-) $$\frac{s^3}{(s^2+1)^2} = \frac{s(s^2+1-1)}{(s^2+1)^2} = \frac{s}{(s^2+1)} - \frac{s}{(s^2+1)^2}$$

$$f(t) = \cos t - \frac{1}{2} t \sin t \,,\ t \geqslant 0$$

Q2-) We can write the DE representation as

$$M_1\ddot{x}_1 = -K_2 x_1 - B_2\dot{x}_1 - B_1(\dot{x}_1 - \dot{x}_3) - K_3(x_1 - x_2)$$

$$M_2\ddot{x}_2 = -K_3(x_2 - x_1) - B_3(\dot{x}_2 - \dot{x}_3) + f$$

$$M_3\ddot{x}_3 = -K_1 x_3 - B_1(\dot{x}_3 - \dot{x}_1) - B_3(\dot{x}_3 - \dot{x}_2)$$

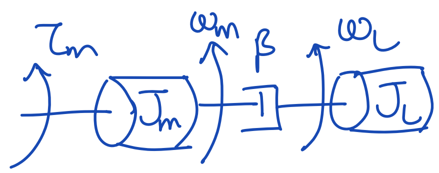
Define the state

$$x = \begin{bmatrix} x_1 & \dot{x}_1 & x_2 & \dot{x}_2 & x_3 & \dot{x}_3 \end{bmatrix}^T \quad u = f \quad y = x_3$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_2+K_3}{M_1} & -\frac{B_1+B_2}{M_1} & \frac{K_3}{M_1} & 0 & 0 & \frac{B_1}{M_1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_3}{M_2} & 0 & -\frac{K_3}{M_2} & -\frac{B_3}{M_2} & 0 & \frac{B_3}{M_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{B_1}{M_3} & 0 & \frac{B_3}{M_3} & -\frac{K_1}{M_3} & -\frac{B_1+B_3}{M_3} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x$$

Q3-) a-)



input: τ_m

output: ω_L

$$J_m \dot{\omega}_m = \tau_m - \beta(\omega_m - \omega_L)$$
$$J_L \dot{\omega}_L = -\beta(\omega_L - \omega_m)$$

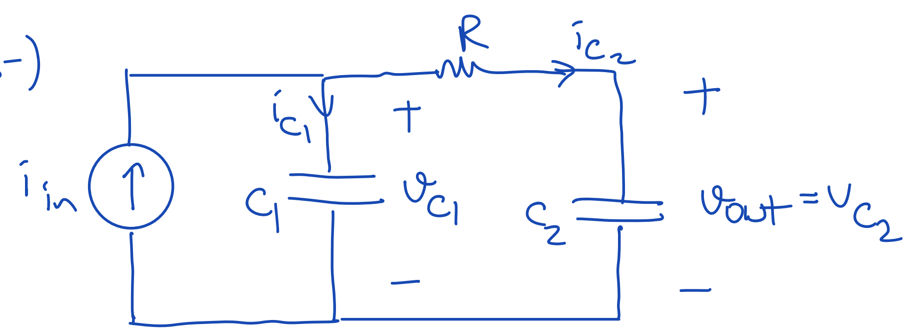
$\Rightarrow$ Pass the eqn's into Laplace domain

$$(J_m s + \beta)\Omega_m = T_m + \beta\Omega_L$$
$$(J_L s + \beta)\Omega_L = +\beta\Omega_m$$

$$(J_m s + \beta)(J_L s + \beta)\Omega_L = \beta(J_m s + \beta)\Omega_m$$
$$(J_m s + \beta)(J_L s + \beta)\Omega_L = \beta(T_m + \beta\Omega_L)$$

$$\Rightarrow G(s) \triangleq \frac{\Omega_L}{T_m} = \frac{\beta}{(J_m s + \beta)(J_L s + \beta) - \beta^2}$$

b-)



input: i_{in}

output: V_{c_2}

$$C_1 \dot{V}_{c_1} = i_{c_1} = i_{in} - i_{c_2} = i_{in} - \frac{V_{c_1} - V_{c_2}}{R}$$
$$C_2 \dot{V}_{c_2} = i_{c_2} = \frac{V_{c_1} - V_{c_2}}{R}$$

Pass the eqn's into Laplace domain,

$$\left(C_1 s + \frac{1}{R}\right) V_{C_1} = I_{in} + \frac{V_{C_2}}{R}$$

$$\left(C_2 s + \frac{1}{R}\right) V_{C_2} = \frac{V_{C_1}}{R}$$

$$\left(C_1 s + \frac{1}{R}\right)\left(C_2 s + \frac{1}{R}\right) V_{C_2} = \frac{1}{R}\left(C_1 s + \frac{1}{R}\right) V_{C_1}$$

$$\left(C_1 s + \frac{1}{R}\right)\left(C_2 s + \frac{1}{R}\right) V_{C_2} = \frac{1}{R}\left(I_{in} + \frac{V_{C_2}}{R}\right)$$

$$G(s) \triangleq \frac{V_{C_2}}{I_{in}} = \frac{1/R}{(C_1 s + 1/R)(C_2 s + 1/R) - 1/R^2}$$

c-) System have the same transfer functions when we let

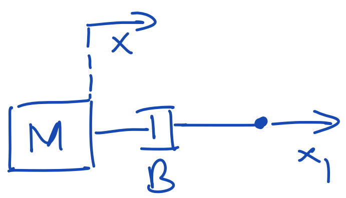
$$C_1 \leftarrow J_m \quad C_2 \leftarrow J_e \quad \beta \leftarrow 1/R$$

In some sense, inertias are equivalent to capacitors, which are both storing energy while damper is equivalent to resistor, which both dissipate/spend energy.

Q4-) Note that when we apply a torque/force directly to a damper or spring, the torque/force is immediately transferred to the other side.

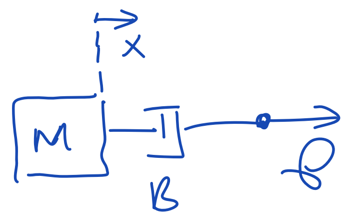
i.e., the following systems are different.

System 1



output: x
input: x_1

System 2



output: x
input: f

The DE representation for system-1 can be written as

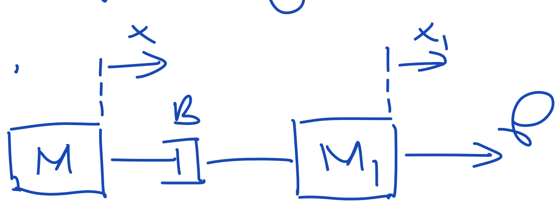
$$M\ddot{x} = -B(\dot{x}-\dot{x}_1)$$

The DE representation for system-2 can be written as

$$M\ddot{x} = f$$

i.e., System-2 DE is independent of B.

This surprising fact can be explained as follows. Suppose that we define a hypothetical mass $M_1=0$ for System-2 as follows.



Since M_1 is zero, the new system is equivalent to system-2.
Let us write the DE representation for the new system.

$$M\ddot{x} = -B(\dot{x} - \dot{x}_1)$$
$$M_1\ddot{x}_1 = -B(\dot{x}_1 - \dot{x}) + f$$

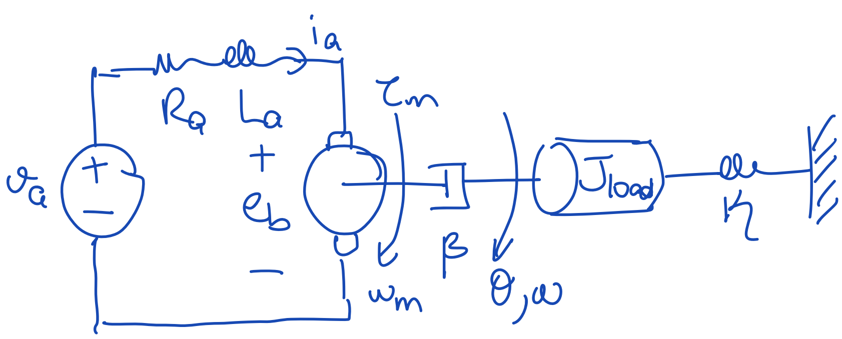
Now substitute $M_1 = 0$ in the second eqn, to get

$$0 = -B(\dot{x}_1 - \dot{x}) + f \implies f = B(\dot{x}_1 - \dot{x}) = -B(\dot{x} - \dot{x}_1)$$

Now substitute this in the first eqn. to get

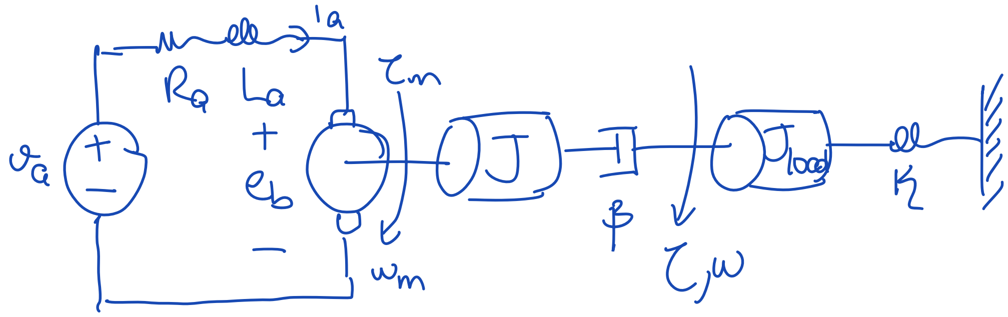
$$M\ddot{x} = f$$

In Q4 the torque τ_m is applied directly to a damper. Hence it is transferred to the inertia J_{load} without any change.



$$L_a\dot{i}_a + R_a i_a = v_a - e_b$$
$$e_b = K\omega_m \qquad \tau_m = K i_a$$
$$J_{load}\dot{\omega} = \tau_m - K\theta$$
$$\dot{\theta} = \omega$$
$$\tau_m = \beta(\omega_m - \omega)$$

In order to obtain these eqn's, you may again use a hypothetical inertia $J=0$ if you like as follows.



$$L_a \dot{i}_a + R_a i_a = v_a - e_b$$

$$e_b = K\omega_m \qquad \tau_m = K i_a$$

$$J\dot{\omega}_m = \tau_m - \beta(\omega_m - \omega)$$

$$J_{load}\dot{\omega} = -\beta(\omega - \omega_m) - k\theta$$

$$\dot{\theta} = \omega$$

Set $J=0$ to get $\Rightarrow$

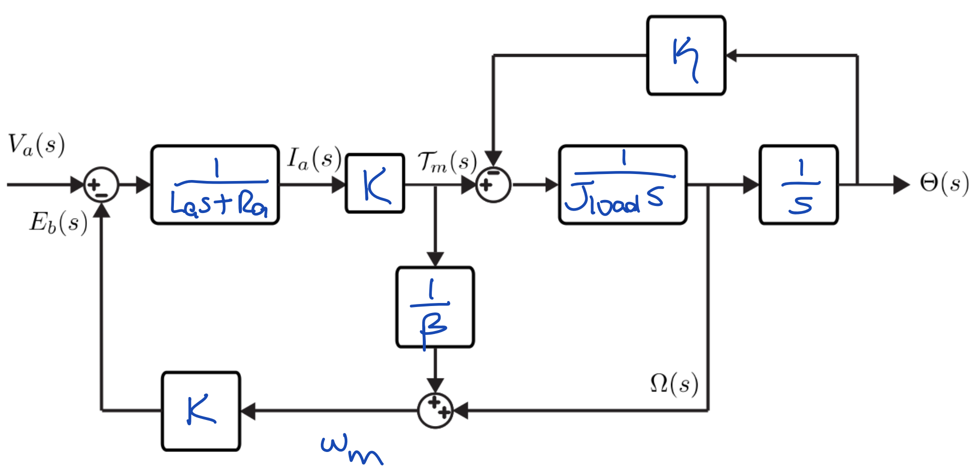
$$L_a \dot{i}_a + R_a i_a = v_a - e_b$$

$$e_b = K\omega_m \qquad \tau_m = K i_a$$

$$J_{load}\dot{\omega} = \tau_m - k\theta$$

$$\dot{\theta} = \omega$$

$$\tau_m = \beta(\omega_m - \omega)$$



b-) Note that we have

$$\tau_m = \beta(\omega_m - \omega) \Rightarrow \frac{1}{\beta}\tau_m = \omega_m - \omega$$

$$\Rightarrow \quad \omega_m = \omega + \frac{1}{\beta}\tau_m = \omega + \frac{K}{\beta} i_a$$

Let us write th state eqn's. with $u = v_a$, $y = \theta$

$$\dot{\theta} = \omega$$

$$J_{load}\dot{\omega} = \tau_m - k\theta = K i_a - k\theta$$

$$L_a \dot{i}_a = v_a - e_b - R_a i_a$$

$$= v_a - K\omega_m - R_a i_a$$

$$= v_a - K\left(\omega + \frac{K}{\beta} i_a\right) - R_a i_a$$

$$L_a \dot{i}_a = v_a - K\omega - \left(\frac{K^2}{\beta} + R_a\right) i_a$$

$$\Rightarrow$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{J_{load}} & 0 & \frac{K}{J_{load}} \\ 0 & -\frac{K}{L_a} & -\frac{\frac{K^2}{\beta} + R_a}{L_a} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$