Student Information

Full Name : Batuhan Akçan
Id Number : 2580181

Answer 1

a) Pick $-2, 2 \in \mathbb{R}$. $f(-2) = f(2) = 4 \in \mathbb{R}$. So, f is not injective.

Pick $-2 \in \mathbb{R}$. $f(x) \neq -2 \quad \forall x \in \mathbb{R}$. So, f is not surjective.

b) $(f(x) = f(y)) \to (x = y) \quad \forall x, y \in \mathbb{R}^+$ So, f is injective.

Pick $-2 \in \mathbb{R}$. $f(x) \neq -2 \quad \forall x \in \mathbb{R}^+$. So, f is not surjective.

c) Pick $-2, 2 \in \mathbb{R}$. $f(-2) = f(2) = 4 \in \mathbb{R}^+$. So, f is not injective.

$\forall y \in \mathbb{R}^+ \quad \exists x$ such that $f(x) = y$. So, f is surjective.

d) $(f(x) = f(y)) \to (x = y) \quad \forall x, y \in \mathbb{R}^+$ So, f is injective.

$\forall y \in \mathbb{R}^+ \quad \exists x$ such that $f(x) = y$. So, f is surjective.

Answer 2

a) $\forall \varepsilon \in \mathbb{R} \quad \exists \delta \in \mathbb{Z} \quad \forall x \in A \quad (||x - x_0|| < \delta \to ||f(x) - f(x_0)|| < \varepsilon)$. Therefore, f is continuous.

b) Assume f is not a constant function. Then
$\exists \varepsilon \in \mathbb{Z} \quad \neg\exists \delta \in \mathbb{R} \quad \forall x \in A \quad (||x - x_0|| < \delta \to ||f(x) - f(x_0)|| < \varepsilon)$
So f must be a constant function in order to be continuous.

Answer 3

a) BASIS: n=2. $X_2 = A_1 \times A_2$ is countable since A_1 and A_2 are countable.
IND. STEP: Assume that X_k is countable. Then $X_{k+1} = A_1 \times A_2 \times ... \times A_k \times A_{k+1} = X_k \times A_{k+1}$ is countable since X_k and A_{k+1} are countable. Hence, X_{k+1} is countable.

b) Let $S = X \times X \times \ldots$. Suppose S is countable. Let $E_n : n \in \mathbb{N}$ be an enumeration of S . For each n , pick two points $a_n, b_n \in E_n$. Then define a function $F \in S$ as:

$$F(n) = \left\{ \begin{array}{ll} b_n, & \text{if } E_n(n) = a_n \\ a_n, & \text{otherwise} \end{array} \right\} \tag{1}$$

So, $E_n \neq F$ and $F \in S$ which is a contradiction. Hence, S is uncountable.

Answer 4

$(n!)^2, \quad 5^n, \quad 2^n, \quad n^{51} + n^{49}, \quad n^{50}, \quad \sqrt{n}\log n, \quad (\log n)^2$

a) Let $\sum a_n = \frac{5^n}{(n!)^2}$. Apply ratio test:
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{5^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{5^n}| = \lim_{n\to\infty} |\frac{5}{(n+1)^2}| = 0 < 1$. So the series $\sum a_n$ is absolutely convergent thus convergent. Therefore, $5^n = O((n!)^2)$.

b) $\lim_{x\to\infty} \frac{2^x}{5^x} = \lim_{x\to\infty} (\frac{2}{5})^x = 0 \to 2^n = O(5^n)$.

c) $\lim_{x\to\infty} \frac{x^{51}+x^{49}}{2^x} = \ldots = \lim_{x\to\infty} \frac{(51!)x}{2^x(\ln 2)^{50}} = \lim_{x\to\infty} \frac{51!}{2^x(\ln 2)^{51}} = 0 \to n^{51} + n^{49} = O(2^n)$.

d) $\lim_{x\to\infty} \frac{x^{50}}{x^{51}+x^{49}} = \lim_{x\to\infty} \frac{1}{x} = 0 \to n^{50} = O(n^{51} + n^{49})$.

e) $\lim_{x\to\infty} \frac{\sqrt{x}\log x}{x^{50}} = \lim_{x\to\infty} \frac{\log x}{x^{99/2}} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 10}}{\frac{99}{2}x^{97/2}} = \lim_{x\to\infty} \frac{99x^{95/2}}{2\ln 10} = 0 \to \sqrt{n}\log n = O(n^{50})$.

f) $\lim_{x\to\infty} \frac{(\log x)^2}{\sqrt{x}\log x} = \lim_{x\to\infty} \frac{\log x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x\ln 10}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln 10} = 0 \to (\log n)^2 = O(\sqrt{n}\log n)$.

Answer 5

a) $\gcd(94, 134) = \gcd(134, 94) = \gcd(94, 40) = \gcd(40, 14) = \gcd(14, 12) = \gcd(12, 2) = \gcd(2, 0) = 2$

b) Let P: Every even integer greater than 2 is the sum of two primes.
Let Q: Every integer greater than 5 is the sum of three primes.
We must prove that both $P \to Q$ and $Q \to P$ holds.
1) $P \to Q$
Assume P. Let n be an even integer and $n > 2$. Then, $n = x + y$ where x,y are prime. Add 3 to the equation: $n + 3 = x + y + 3$ where $n + 3 > 5$. So, every integer $a = n + 3 > 5$ except 6 is the sum of three primes $x, y, 3$. Also, $6 = 2 + 2 + 2$. Therefore, $P \to Q$.
2) $Q \to P$
Assume Q. Let $n > 5$ be an integer. Then, $n = x + y + z$ where x,y,z are prime. Then,

$b = n - z = x + y$ where $b > n - z$. Pick $z = 3$. Then the statement $b = x + y$ is equivalent to P for every n except 6. Proof for 6: $6 = 3 + 3$. Therefore, $Q \to P$.
Hence, $P \equiv Q$.