CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 2

Akçan, Batuhan
e2580181@ceng.metu.edu.tr

Sönmezer, Mert
e2516920@ceng.metu.edu.tr

April 2, 2024

---

1. By looking at the graphs provided in the question, the equation of the functions are as follows:

$$x(t) = u(t+3) - u(t-7)$$
$$h(t) = u(t-1) - u(t-15)$$

Then, $y(n)$ becomes

$$y(n) = x(n) * h(n)$$
$$y(n) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)$$

$x(\tau)$ and $h(t-\tau)$ will

- partially overlap for $-2 \leq t \leq 8$
- fully overlap for $8 < t \leq 12$
- partially overlap for $12 < t \leq 22$

We need to evaluate the convolution sum for these three regions separately.

$$y(t) = \begin{cases} t+2 & -2 \leq t \leq 8 \\ 10 & 8 < t \leq 12 \\ 22-t & 12 < t \leq 22 \\ 0 & otherwise \end{cases}$$

2. (a) Notice that the overlapped part of the convolution process starts at the point $n = -4$ and ends at the point $n = 4$, so for $n \leq -5$ and $n \geq 5$, $y_1[n] = 0$. Now, let's calculate the convolution sum for every n value in the interval.

$$n = -4 \rightarrow x[-4]h[-4] + x[0]h[0] = 0 + 2 = 2$$
$$n = -3 \rightarrow x[-3]h[-3] + x[1]h[1] = 0 + 0 = 0$$
$$n = -2 \rightarrow x[-2]h[-2] + x[2]h[2] = 0 + 4$$
$$n = -1 \rightarrow x[-1]h[-1] + x[3]h[3] = 0 + 0 = 0$$
$$n = 0 \rightarrow x[0]h[0] + x[4]h[4] = 1 - 6 = -5$$
$$n = 1 \rightarrow x[1]h[1] + x[5]h[5] = 0 + 0 = 0$$
$$n = 2 \rightarrow x[2]h[2] + x[6]h[6] = 2 + 0 = 2$$
$$n = 3 \rightarrow x[3]h[3] + x[7]h[7] = 0 + 0 = 0$$
$$n = 4 \rightarrow x[4]h[4] + x[8]h[8] = 0 - 3 = -3$$

Then, the graph of $y_1[n]$ will be like in Figure 1

(b) We know that the system is LTI. That's why sliding in the input should exactly be reflected to the output.

$$x[n] \xrightarrow{LTI} y_1[n]$$
$$x[n+2] \xrightarrow{LTI} y_1[n+2] = y_2[n]$$

Therefore, the output $y_2[n]$ will be as in Figure 2:

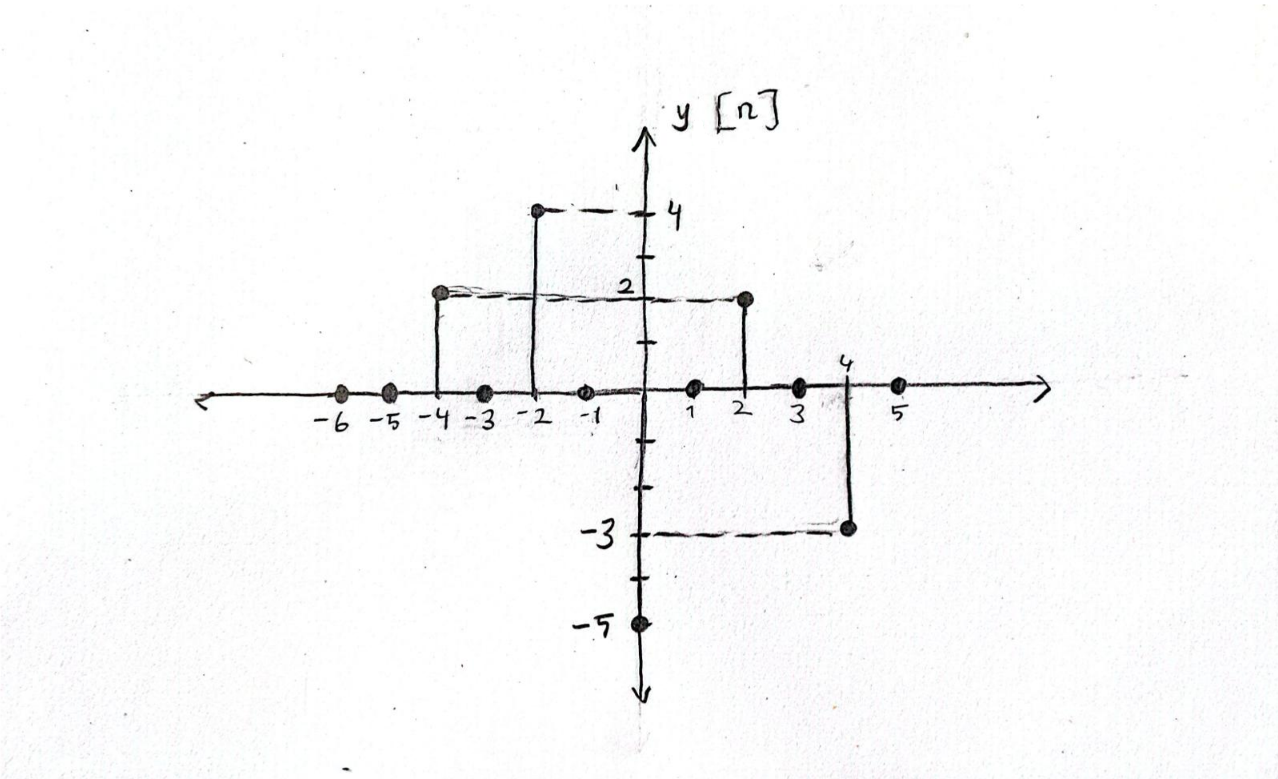

Figure 1: The graph of $y_1[n]$.

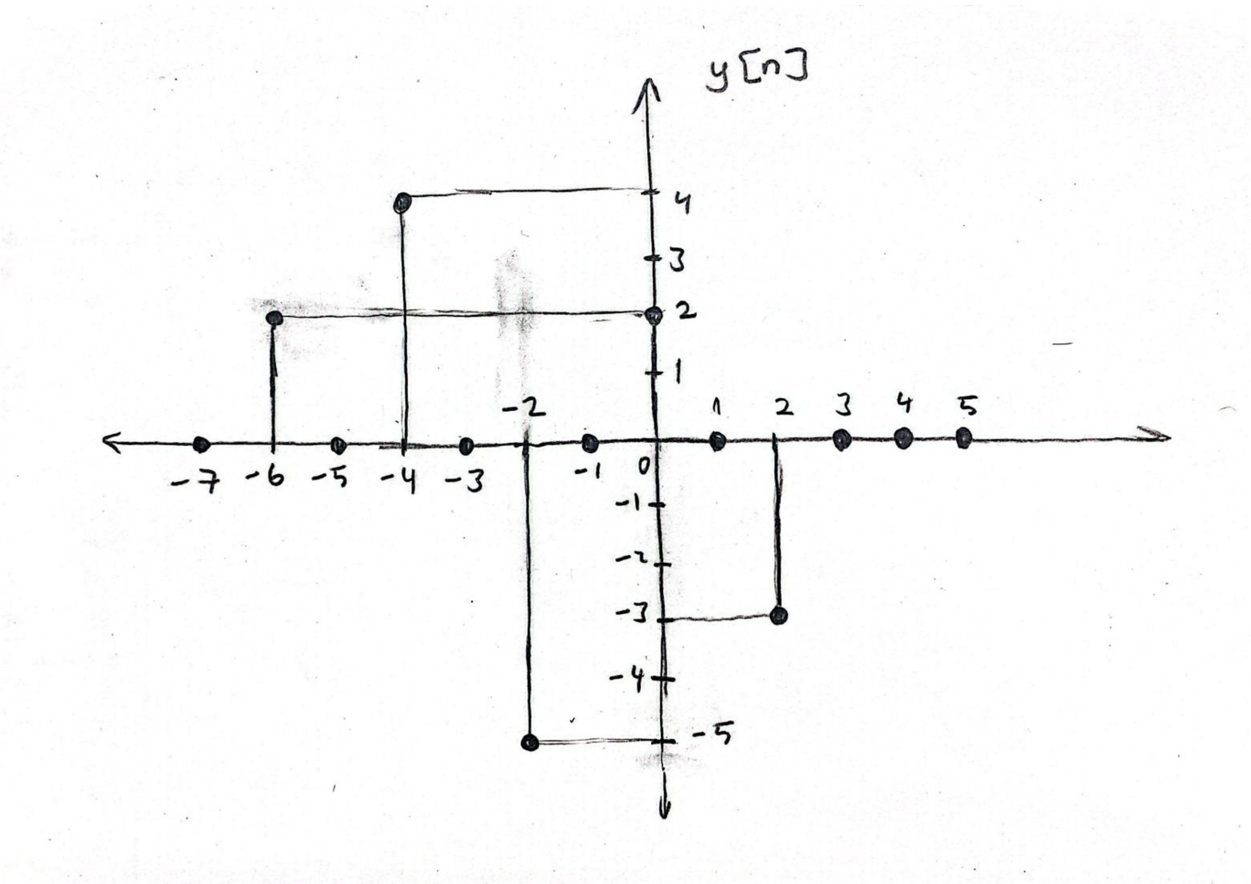

Figure 2: The graph of $y_2[n]$.

(c) To compute $y_3[n]$, we first need to find $x[n+2] * h[n-2]$.

$$y_3[n] = x[n+2] * h[n-2] = \sum_{k=-\infty}^{\infty} x[k+2]h[n-k-2]$$

In the equation above, we can use the change of variable method. Let's define $m = k - 2$. Then, the equation above will be as follows:

$$y_3[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Notice that by looking at this equation, $y_3[n]$ is exactly same as $y_1[n]$. Therefore, the graph of $y_3[n]$ will be the same as Figure 1

3. (a) We need to replace $x[n]$ with $\delta[n]$. Then, $y[n]$ becomes $h[n]$, which is the impulse response of the system.

$$h[n] = \frac{1}{5}\delta[n-1] + \delta[n]$$

(b) To find the output $y[n]$ defined in the question, we need to replace $x[n]$ with $\delta[n-2]$.

$$y[n] = \frac{1}{5}\delta[n-3] + \delta[n-2]$$

(c) This system is BIBO stable because for any bounded input $x[n] < B$, we will get a bounded output. From another perspective, its impulse response is summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |\frac{1}{5}\delta[k-1] + \delta[k]| = \frac{1}{5} + 1 = \frac{6}{5} < \infty$$

(d) This system has memory since $h[n] = K\delta[n]$. In other words, the first term, which is $\frac{1}{5}\delta[n-1]$, violates the memoryless property.

(e) To determine whether the system is invertable, we first need to fined the inverse of the impulse response.

4. (a) We know that

$H(\lambda) = \frac{2\lambda}{\lambda^2-2\lambda+1} = \frac{\sum_{k=0}^{M} b_k \lambda^k}{\sum_{k=0}^{N} a_k \lambda^k}$.

From the equation above, we get

$b_1 = 2$.
$a_2 = 1$, $a_1 = -2$, $a_0 = 1$.

We also know the equation

$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$.

Putting the constants into the equation, we get

$\frac{d^2 y(t)}{dt^2} - 2\frac{dy(t)}{dt} + y(t) = 2\frac{dx(t)}{dt}$, or simply

$y''(t) - 2y'(t) + y(t) = 2x'(t)$.

(b) When $x(t) = 0$, there is no particular solution. We will only find the homogeneous solution, which will be equal to $y(t)$. Also, since $x(t) = 0$, $x'(t) = 0$.

Therefore,

$y''(t) - 2y'(t) + y(t) = 0$.

Assume $y(t) = Ce^{st}$.

Then $y'(t) = Cse^{st}$, $y''(t) = Cs^2e^{st}$.

So, $Cs^2e^{st} - 2Cse^{st} + Ce^{st} = 0$.

$\rightarrow Ce^{st}(s^2 - 2s + 1) = 0$.

$\rightarrow s_1 = s_2 = 1$.

$\rightarrow y(t) = C_1 e^t + C_2 t e^t.$

Since the system is initially at rest, we know that

$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0.$

Thus, $y(0) = C_1 = 0.$

Since $y'(t) = C_1 e^t + C_2(e^t + te^t),\ y'(0) = C_1 + C_2(1+0) = C_2 = 0.$

Hence, we conclude that $y(t) = 0.$

(c) We know that $x'(t) = 2u(t).$

So, $y''(t) - 2y'(t) + y(t) = 4u(t).$

Assume $y_h(t) = Ce^{st}$. We know from part b that $y_h(t) = 0$. So, $y(t) = y_p(t).$

Assume $y_p(t) = Kx(t) = K(2t+1)u(t).$

Then $y_p'(t) = 2Ku(t),\ y_p''(t) = 0.$

Putting into the differential equation, we get

$0 - 4Ku(t) + K(2t+1)u(t) = 4u(t).$

$\rightarrow K(2t-3) = 4 \rightarrow K = \frac{4}{2t-3}.$

So, $y_p(t) = \frac{4}{2t-3}(2t+1)u(t).$

Hence, we conclude that

$y(t) = y_h(t) + y_p(t) = \frac{4}{2t-3}(2t+1)u(t).$

5. (a) Since we are asked to find impulse response, the input is unit impulse input, i.e, $x[n] = \delta[n]$. We will recursively try some values to see a pattern.

$n = 0 \rightarrow y[0] = 0$ (because of the initially at rest condition).

$n = 1 \rightarrow y[1] = \frac{1}{5}y[0] + 2x[-1] = 0$ (since x[n]=1 only if n=0).

$n = 2 \rightarrow y[2] = \frac{1}{5}y[1] + 2x[0] = 2x[0] = 2.$

$n = 3 \rightarrow y[3] = \frac{2}{5}x[0] + 2x[1] = \frac{2}{5}.$

$n = 4 \rightarrow y[4] = \frac{2}{5^2}x[0] + \frac{2}{5}x[1] + 2x[2] = \frac{2}{5^2}.$

As we can see, $y[n] = \frac{2}{5^{n-2}}u[n-2]$ (we multiplied with $u[n-2]$ because when $n = 1$, $y[n]$ does not comply with the pattern).

(b) Arranging the terms in the equation, we get

$y[n] - \frac{1}{5}y[n-1] = 2x[n-2].$

We know the equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$

Therefore, $a_0 = 1,\ a_1 = \frac{-1}{5},\ b_2 = 2.$

We also know the equation $H(\lambda) = \frac{\sum_{k=0}^{M} b_k \lambda^k}{\sum_{k=0}^{N} a_k \lambda^k}.$

Putting constants into the equation, we get

$H(\lambda) = \frac{2\lambda^2}{\frac{-1}{5}\lambda + 1} = \frac{-10\lambda^2}{\lambda - 5}.$

(c) The solution is uploaded as an image in Figure 3, at page 5.

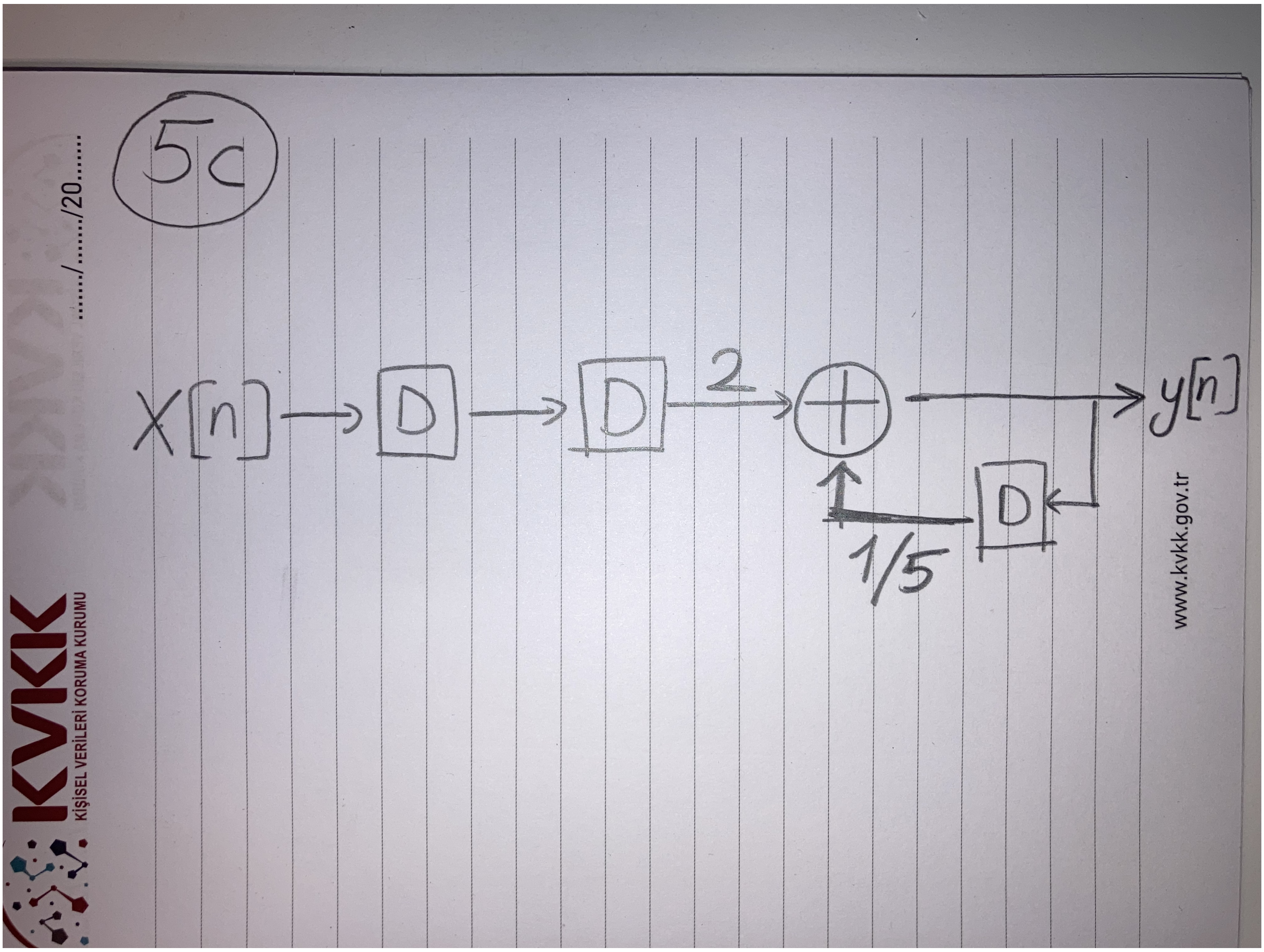

Figure 3: Solution of Question 5 part c.

6. (a) The solution is uploaded as an image in Figure 4, at page 6.

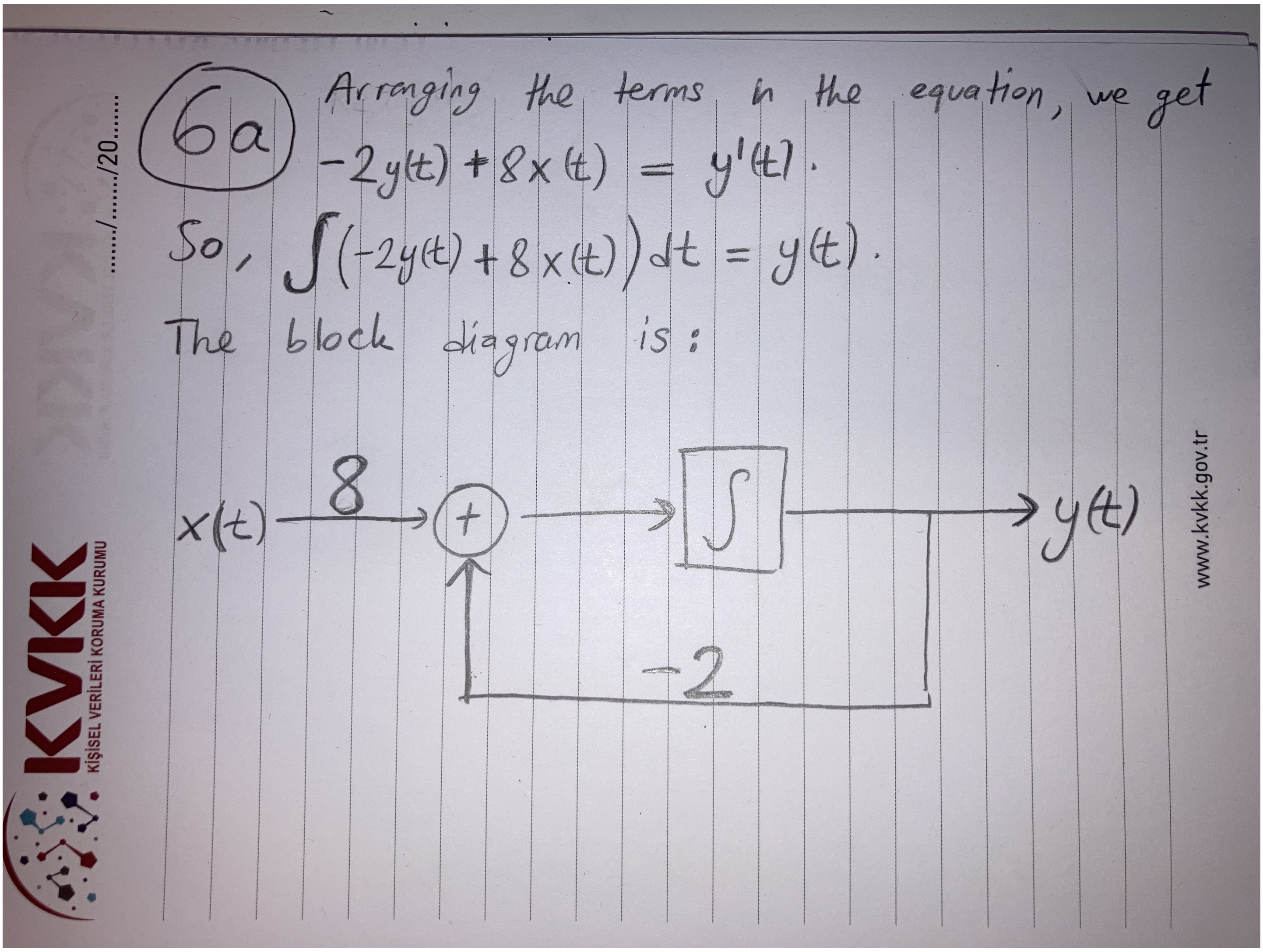

Figure 4: Solution of Question 6 part a.

(b) The solution is uploaded as an image in Figure 5, at page 7.

7. Here is the code solution of the introduced task in the question:

```
import matplotlib.pyplot as plt

def x(n):
    return 1 if n == 1 else 0

def y(n):
    if n < 0:
        return 0
    return 0.25 * y(n-1) + x(n)

n_samples = 5
ys = [y(i) for i in range(n_samples)]

plt.scatter(range(n_samples), ys)
plt.xlabel('n')
plt.ylabel('y[n]')
plt.title('Output signal y[n]')
plt.grid(True)
plt.show()
```

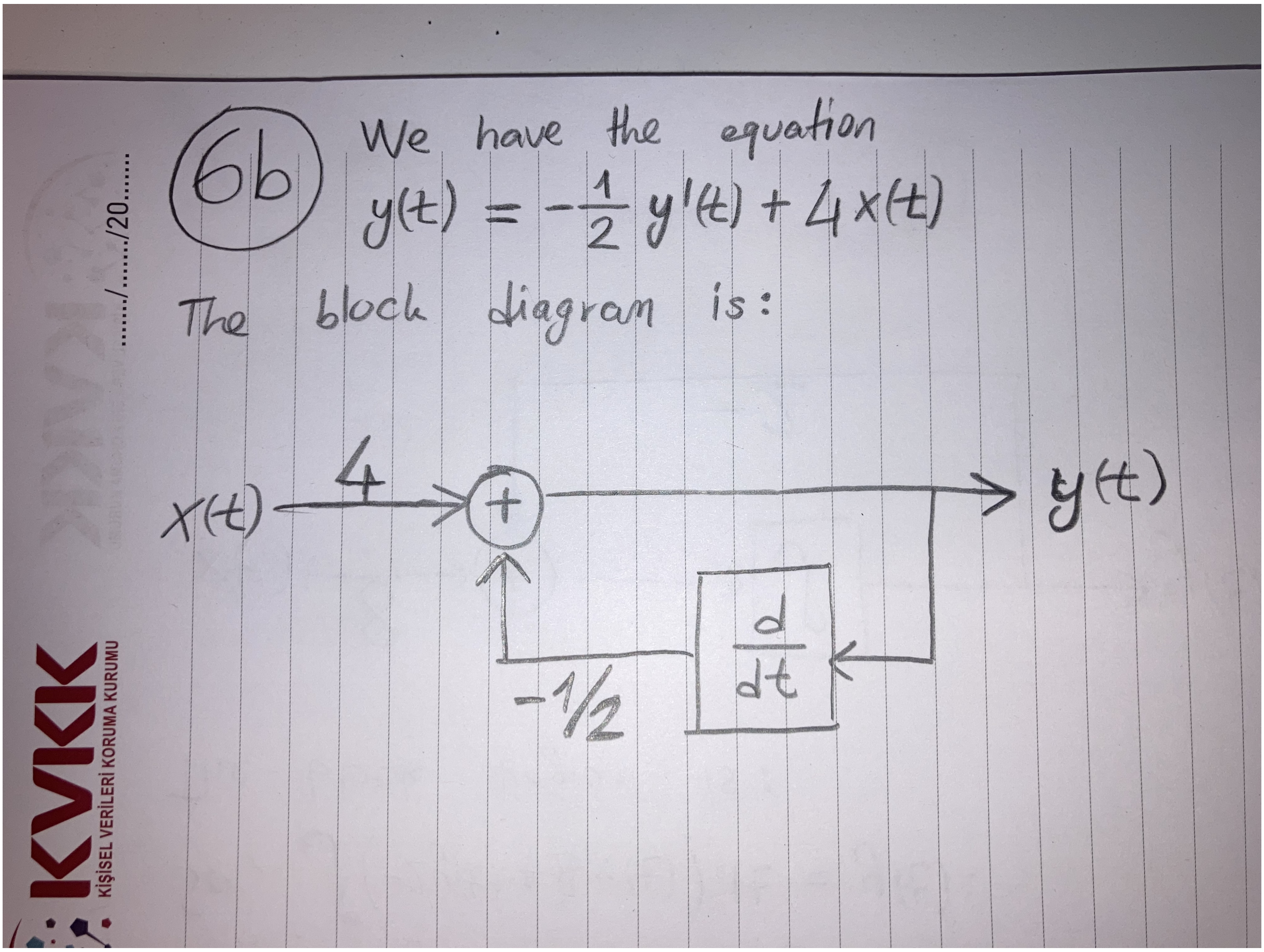

Figure 5: Solution of Question 6 part b.