CENG 384 - Signals and Systems for Computer Engineers
Spring 2024
Homework 4

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May 19, 2024

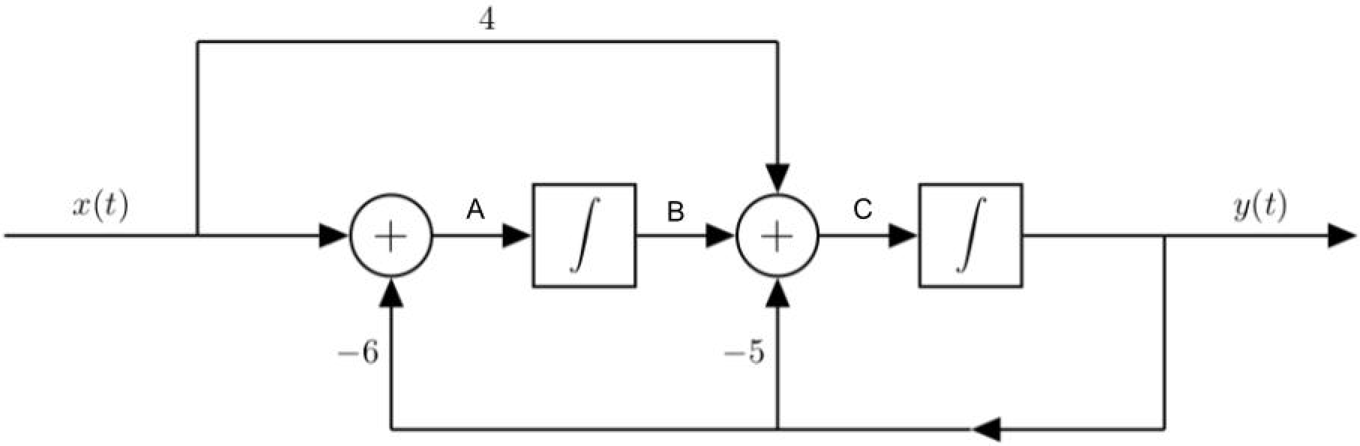

Figure 1: Question 1 part a

1. (a) To find the differential equation of the system represented in Figure 1, we need to find the expressions corresponding to each letter.

$$A = x(t) - 6y(t)$$
$$B = \int A dt = \int (x(t) - 6y(t))dt$$
$$C = 4x(t) - 5y(t) + B = 4x(t) - 5y(t) + \int (x(t) - 6y(t))dt$$
$$y(t) = \int C dt = \int \Big(4x(t) - 5y(t) + \int (x(t) - 6y(t))dt\Big)dt$$

As you see, we obtained $y(t)$, but it is still not in the desired form. Now, let's convert it into the constant coefficient differential equation form.

$$y'(t) = 4x(t) - 5y((t) + \int (x(t) - 6y(t))dt$$
$$y'(t) + 5y(t) - 4x(t) = \int (x(t) - 6y(t))dt$$
$$y''(t) + 5y'(t) - 4x'(t) = x(t) - 6y(t)$$

After some basic algebraic manipulations applied to the equation above, it can be obtain the following equation:

$$y''(t) + 5y'(t) + 6y(t) = x(t) + 4x'(t)$$

(b) Frequency response of the system can be found by leveraging the linearity property of the Fourier Transform. Firstly, starting with taking the Fourier Transform of both sides will be a reasonable choice.

$$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = X(j\omega) + 4(j\omega)X(j\omega)$$

Then, let us replace the input by an impulse function and its Fourier transform, which is,

$$x(t) = \delta(t) \to X(j\omega) = 1$$

Utilizing the equation above, let's try to hit the frequency response of the system defined in the question.

$$\begin{aligned}((j\omega)^2 + 5(j\omega) + 6)Y(j\omega) &= 4(j\omega) + 1 \\ H(j\omega) &= \frac{4(j\omega)+1}{(j\omega)^2 + 5(j\omega) + 6}\end{aligned}$$

(c) Finding the inverse Fourier Transform of the frequency response gives us the impulse response of the system.

$$H(j\omega) = \frac{4(j\omega)+1}{(j\omega)^2 + 5(j\omega) + 6} \to h(t)$$

To find the inverse Fourier Transform, we can utilize the partial fraction method.

$$\begin{aligned}\frac{4(j\omega)+1}{(j\omega)^2 + 5(j\omega) + 6} &= \frac{A}{2+j\omega} + \frac{B}{3+j\omega} \\ Aj\omega + 3A + Bj\omega + 2B &= 4j\omega + 1\end{aligned}$$

This gives us the following equations:

$$\begin{aligned}A + B &= 4 \\ 3A + 2B &= 1 \\ A = -7 \text{ and } B &= 11\end{aligned}$$

Namely,

$$H(j\omega) = \frac{-7}{2+j\omega} + \frac{11}{3+j\omega}$$

Now, let's calculate the inverse Fourier Transform.

$$h(t) = (-7e^{-2t} + 11e^{-3t})u(t)$$

(d) We can use the following formula to get the output of the system:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

We first need to find the inverse Fourier Transform of the input $x(t)$.

$$x(t) = \frac{1}{4}e^{\frac{-t}{4}}u(t) \to X(j\omega) = \frac{1}{1+4j\omega}$$

When multiplying the $H(j\omega)$ and $X(j\omega)$,

$$H(j\omega)X(j\omega) = \frac{1+4j\omega}{((j\omega)^2 + 5j\omega + 6)}\frac{1}{(1+4j\omega)} = \frac{1}{(j\omega)^2 + 5j\omega + 6} = \frac{1}{2+j\omega} + \frac{1}{3+j\omega} = Y(j\omega)$$

Then, finding the inverse Fourier Transform of the expression above hits $y(t)$.

$$Y(j\omega) = \frac{1}{2+j\omega} + \frac{1}{3+j\omega} \to y(t) = (e^{-2t} + e^{-3t})u(t)$$

2. (a) We have

$H(jw) = \frac{jw+4}{(jw)^2+5jw+6} = \frac{Y(jw)}{X(jw)}$

Hence

$jwX(jw) + 4X(jw) = (jw)^2Y(jw) + 5jwY(jw) + 6Y(jw)$

According to tables 4.1 and 4.2, taking the inverse Fourier Transform, we get

$\frac{d}{dt}x(t) + 4x(t) = \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t)$.

(b) We have

$H(jw) = \frac{jw+4}{(jw+3)(jw+2)} = \frac{A}{jw+3} + \frac{B}{jw+2} = \frac{(A+B)jw+2A+3B}{(jw+3)(jw+2)}$

We have the equations

$A + B = 1$ and $2A + 3B = 4$

Therefore,

$A = -1,\ B = 2.$

Hence,

$H(jw) = \frac{-1}{jw+3} + \frac{2}{jw+2}.$

According to tables 4.1 and 4.2, taking the inverse Fourier Transform, we get

$h(t) = (-e^{-3t} + 2e^{-2t})u(t).$

(c) According to tables 4.1 and 4.2, Fourier Transform of the given $x(t)$ is

$X(jw) = \frac{1}{4+jw} - \frac{1}{(4+jw)^2} = \frac{3+jw}{(4+jw)^2}.$

We know that

$Y(jw) = X(jw)H(jw) = \frac{jw+4}{(jw+3)(jw+2)} \cdot \frac{3+jw}{(4+jw)^2} = \frac{1}{(jw+2)(jw+4)}.$

(d) We have

$Y(jw) = \frac{1}{(jw+2)(jw+4)} = \frac{A}{jw+2} + \frac{B}{jw+4} = \frac{(A+B)jw+4A+2B}{(jw+2)(jw+4)}.$

We have the equations

$A + B = 0$ and $4A + 2B = 1.$

Hence, we have

$A = \frac{1}{2},\ B = \frac{-1}{2}$

Thus, we can write

$Y(jw) = \frac{1}{2}\frac{1}{jw+2} - \frac{1}{2}\frac{1}{jw+4}.$

According to tables 4.1 and 4.2, taking the inverse Fourier Transform, we get

$y(t) = (\frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t})u(t).$

3. (a) Recall the following formula:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

The Fourier Transform of the input-output pair should be found separately.

$$x[n] = \left(\frac{2}{3}\right)^n u[n] \longleftrightarrow X(e^{j\omega}) = \frac{1}{1 - \frac{2}{3}e^{-j\omega}}$$

The expression above represents the Fourier Transform of the input $x[n]$. Now, let's turn to $y[n]$

$$y[n] = n\left(\frac{2}{3}\right)^{n+1} u[n] = \frac{2}{3}n\left(\frac{2}{3}\right)^n u[n] = \frac{2}{3}(n+1-1)\left(\frac{2}{3}\right)^n u[n] = \frac{2}{3}\left((n+1)\left(\frac{2}{3}\right)^n u[n] - \left(\frac{2}{3}\right)^n u[n]\right)$$

Now, we can utilize the linearity property of Fourier Transform.

$$y[n] = n\left(\frac{2}{3}\right)^{n+1} u[n] \longleftrightarrow Y(e^{j\omega}) = \frac{2}{3}\left(\frac{1}{(1 - \frac{2}{3}e^{-j\omega})^2} - \frac{1}{1 - \frac{2}{3}e^{-j\omega}}\right)$$

Now, let's find the frequency response,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{2}{3}\left(\frac{1}{(1 - \frac{2}{3}e^{-j\omega})^2} - \frac{1}{1 - \frac{2}{3}e^{-j\omega}}\right)\left(1 - \frac{2}{3}e^{-j\omega}\right)$$

After equating the denominators and some algebraic manipulations, we obtain $H(e^{j\omega})$ as follows:

$$H(e^{j\omega}) = \frac{\frac{4}{9}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega}}$$

(b) To find the impulse response of the system, we need to find the inverse Fourier Transform of the frequency response. Firstly, let's apply the partial fraction method.

$$\begin{aligned}
&H(e^{j\omega}) = \frac{\frac{4}{9}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega}} = \frac{4e^{-j\omega}}{9-6e^{-j\omega}} \\
&\frac{4e^{-j\omega}}{9-6e^{-j\omega}} = \frac{A}{3} + \frac{B}{3-2e^{-j\omega}} \\
&3A - 2Ae^{-j\omega} + 3B = 4e^{-j\omega} \\
&A = -2 \text{ and } B = 2
\end{aligned}$$

Then, we get the following equation:

$$H(e^{j\omega}) = \frac{-2}{3} + \left(\frac{2}{3}\right)\frac{1}{1-\frac{2}{3}e^{-j\omega}}$$

Now, we can find the inverse Fourier Transform of each term and sum them up.

$$H(e^{j\omega}) \longleftrightarrow h[n] = \frac{-2}{3}\delta[n] + \left(\frac{2}{3}\right)^{n+1}u[n]$$

(c) We know that

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\frac{4}{9}e^{-j\omega}}{1-\frac{2}{3}e^{-j\omega}}$$

Hence, we get the following equation:

$$\frac{4}{9}e^{-j\omega}X(e^{j\omega}) = Y(e^{j\omega}) - \frac{2}{3}e^{-j\omega}Y(e^{j\omega})$$

When applying the inverse Fourier Transform to the equation above, the difference equation is obtained as follows:

$$y[n] = \frac{2}{3}y[n-1] + \frac{4}{9}x[n-1]$$

(d) Figure 2 illustrates the block diagram of the system.

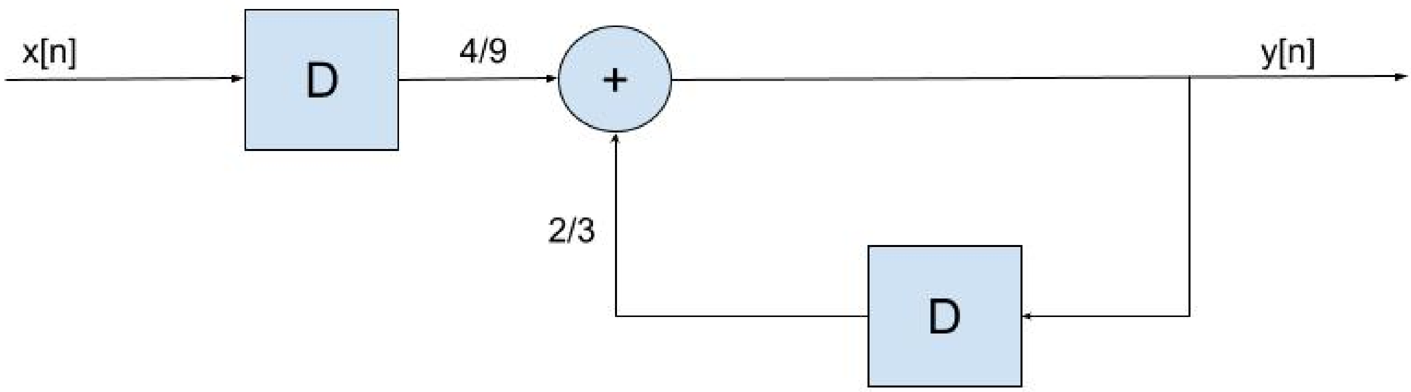

Figure 2: Question 3 part d

4. (a) We have

$2x[n] - \frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] = y[n]$

Rearranging the terms, we get

$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$

(b) Taking the Fourier Transform of the difference equation, we get

$Y(e^{jw}) - \frac{3}{4}e^{-jw}Y(e^{jw}) + \frac{1}{8}e^{-2jw}Y(e^{jw}) = 2X(e^{jw})$

$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{2}{1-\frac{3}{4}e^{-jw}+\frac{1}{8}e^{-2jw}}.$

(c) We have

$H(e^{jw}) = \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})} = \frac{A}{1-\frac{1}{2}e^{-jw}} + \frac{B}{1-\frac{1}{4}e^{-jw}} = \frac{(\frac{-A}{4}+\frac{-B}{2})e^{-jw}+A+B}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})}$

We get the equations

$\frac{-A}{4} + \frac{-B}{2} = 0$ and $A + B = 2$

Hence, we have

$A = 4,\ B = -2.$

So, we get

$H(e^{jw}) = \frac{4}{1-\frac{1}{2}e^{-jw}} + \frac{-2}{1-\frac{1}{4}e^{-jw}}.$

According to Table 5.2, taking the inverse Fourier Transform, we get

$h[n] = (4 \cdot (\frac{1}{2})^n - 2 \cdot (\frac{1}{4})^n) \cdot u[n]$

(d) According to Table 5.2, Fourier Transform of the input is

$X(e^{jw}) = \frac{1}{1-\frac{1}{4}e^{-jw}}$

We know that

$Y(e^{jw}) = X(e^{jw})H(e^{jw}) = \frac{2}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})^2} = \frac{A}{1-\frac{1}{2}e^{-jw}} + \frac{Be^{-jw}+C}{(1-\frac{1}{4}e^{-jw})^2} = \frac{(\frac{A}{16}+\frac{-B}{2})e^{-2jw}+(\frac{-A}{2}+B+\frac{-C}{2})e^{-jw}+A+C}{(1-\frac{1}{2}e^{-jw})(1-\frac{1}{4}e^{-jw})^2}$

We have the equations

$\frac{A}{16} + \frac{-B}{2} = 0,\ \frac{-A}{2} + B + \frac{-C}{2} = 0,\ A + C = 2$

Solving the equations, we get

$A = 8,\ B = 1,\ C = -6$

Hence, we have

$Y(e^{jw}) = \frac{8}{1-\frac{1}{2}e^{-jw}} + \frac{e^{-jw}-6}{(1-\frac{1}{4}e^{-jw})^2} = \frac{8}{1-\frac{1}{2}e^{-jw}} + \frac{e^{-jw}-4}{(1-\frac{1}{4}e^{-jw})^2} + \frac{-2}{(1-\frac{1}{4}e^{-jw})^2} = \frac{8}{1-\frac{1}{2}e^{-jw}} - \frac{4}{1-\frac{1}{4}e^{-jw}} + \frac{-2}{(1-\frac{1}{4}e^{-jw})^2}$

Taking the Inverse Fourier Transform, according to the tables 5.1 and 5.2, we get

$y[n] = (-4 \cdot (\frac{1}{4})^n - 2 \cdot (n+1) \cdot (\frac{1}{4})^n + 8 \cdot (\frac{1}{2})^n) \cdot u[n]$

5. As we can see from the block diagram,

$y[n] = x[n] * h_1[n] + x[n] * h_2[n]$

Using Linearity Property, converting to the frequency domain, we get

$Y(e^{jw}) = X(e^{jw})H_1(e^{jw}) + X(e^{jw})H_2(e^{jw}) = X(e^{jw})H(e^{jw})$

So, we have

$H(e^{jw}) = H_1(e^{jw}) + H_2(e^{jw})$

According to Table 5.2, taking the Fourier Transform of $h_1[n]$, we get

$H_1(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}} = \frac{3}{3-e^{-jw}}$

We know that

$H_2(e^{jw}) = H(e^{jw}) - H_1(e^{jw}) = \frac{5e^{-jw}-12}{e^{-2jw}-7e^{-jw}+12} - \frac{3}{3-e^{-jw}} = \frac{5e^{-jw}-12}{(e^{-jw}-4)(e^{-jw}-3)} + \frac{3}{e^{-jw}-3}$

$= \frac{8e^{-jw}-24}{(e^{-jw}-4)(e^{-jw}-3)} = \frac{8}{e^{-jw}-4} = \frac{-2}{1-\frac{1}{4}e^{-jw}}$

Taking the Inverse Fourier Transform, by the Table 5.2, we get

$h_2[n] = -2(\frac{1}{4})^n u[n]$.

6.

```python
import numpy as np
import matplotlib.pyplot as plt

# Parameters
n_min = -50
n_max = 50
n = np.arange(n_min, n_max + 1)
x_n = (1/2)**np.abs(n)

omega = np.linspace(-3*np.pi, 3*np.pi, 400)
X_omega = np.zeros(len(omega), dtype=complex)

for i, w in enumerate(omega):
    X_omega[i] = np.sum(x_n * np.exp(-1j * w * n))

plt.plot(omega, np.abs(X_omega))
plt.grid(True)
plt.show()
```

Listing 1: DTFT