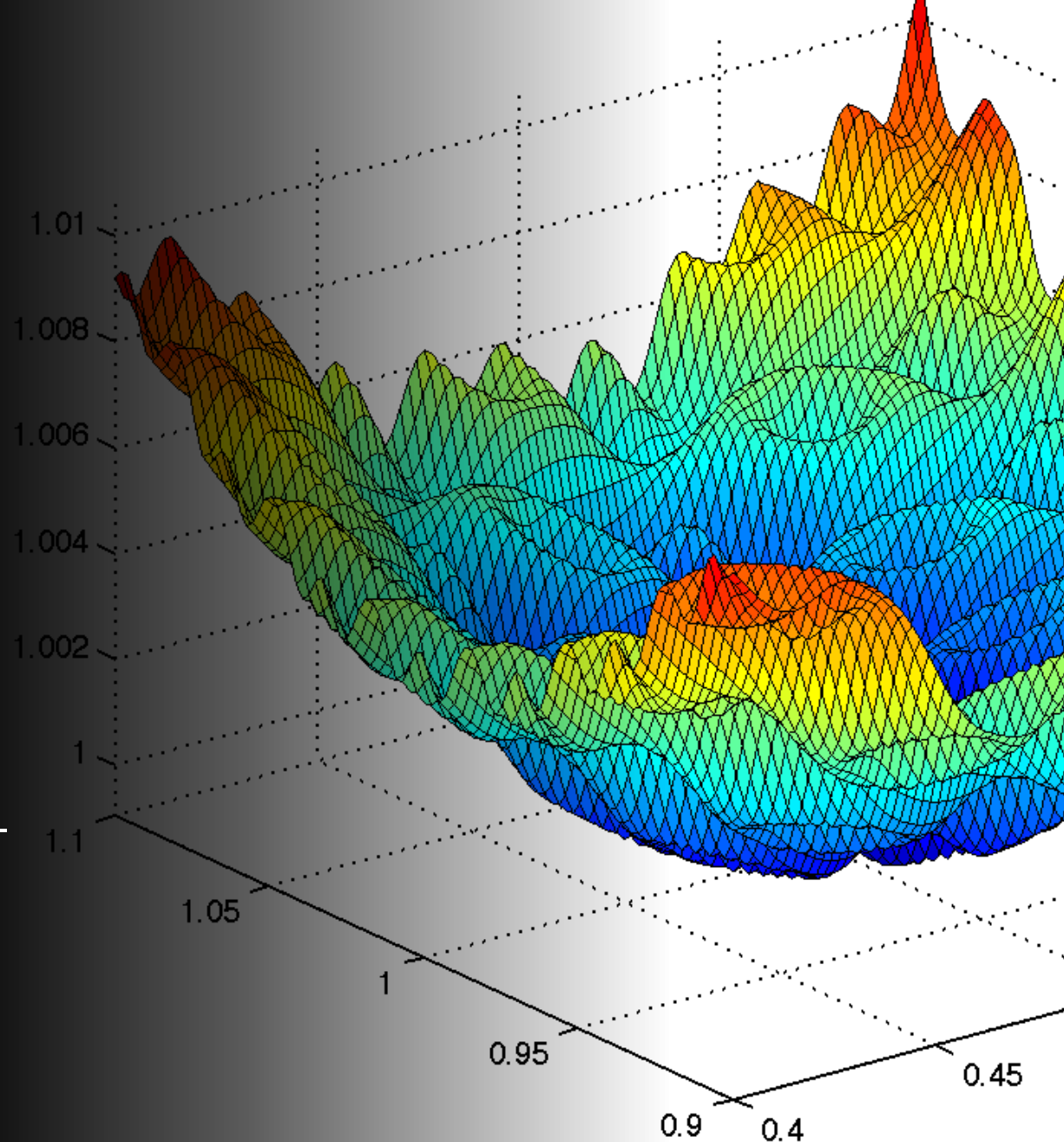# CS 5/7320
Artificial Intelligence

Local Search
AIMA Chapters 4.1 & 4.2

Slides by Michael Hahsler
based on slides by Svetlana Lazepnik
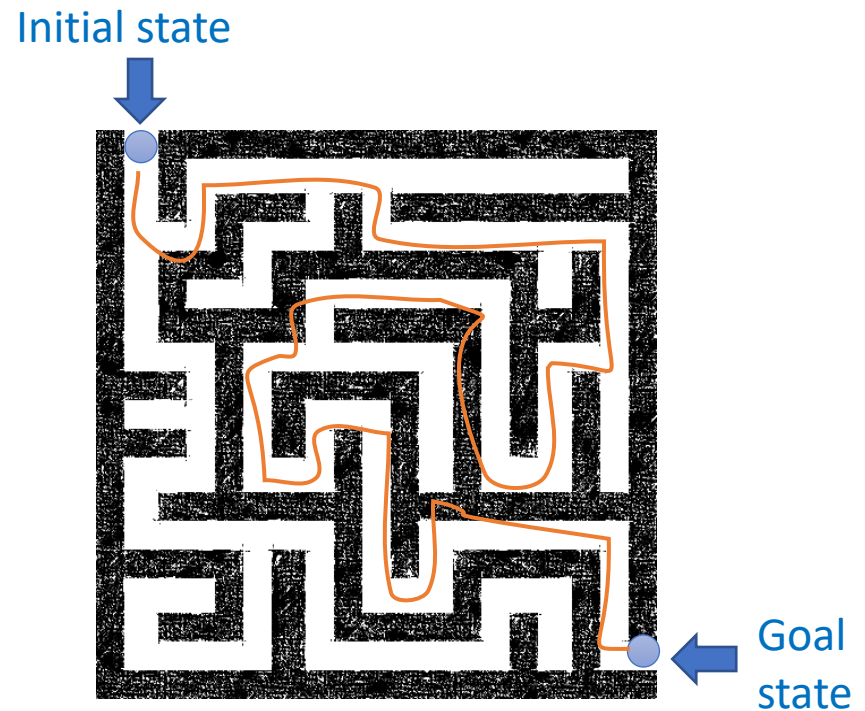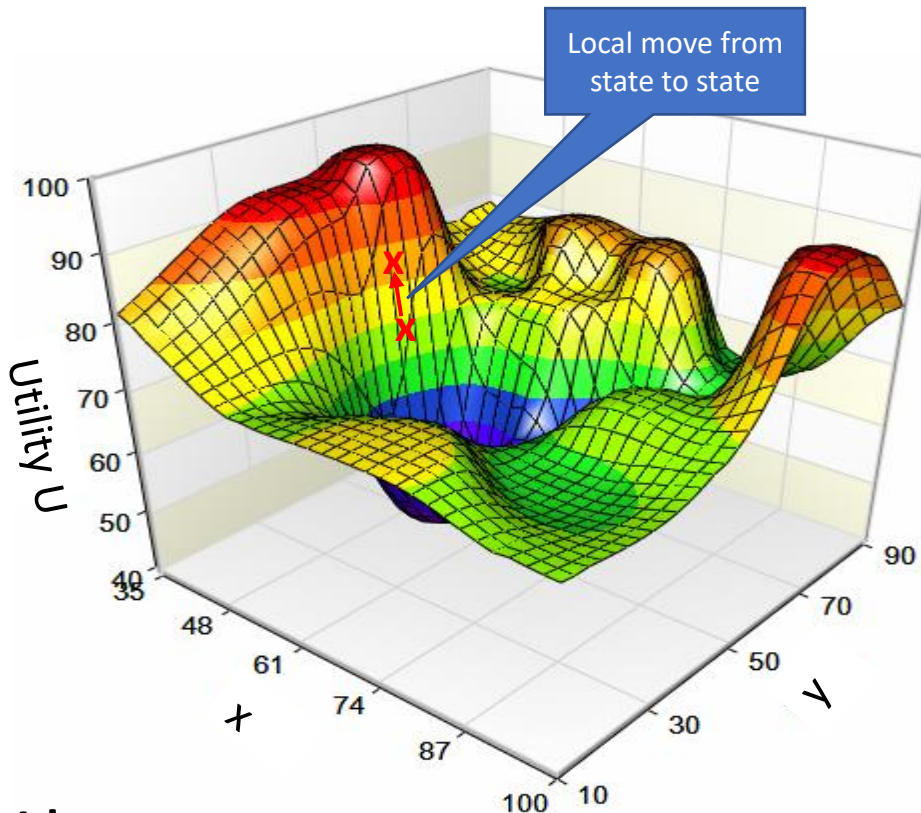with figures from the AIMA textbook.

Recap: Uninformed and Informed Search

Tries to find the
best path
from a
given initial state
to a
given goal state.

- Typically searches a large portion of the search space (needs time and memory).
- Often comes with optimality guarantees.

Initial state

Goal state

Local Search Algorithms

Local move from state to state

- What if we do not know the goal state, but the utility of different states is given by a utility function
$$U = u(s)?$$

- We use a factored state description. Here $s = (x, y)$

- We could try to identify the best or a at least a "good" state?

- This is the **optimization problem**:
$$s^* = \operatorname*{argmax}_{s \in S} u(s)$$

- We need a fast and memory-efficient way to find the best/a good state.

Idea:

- Start with a current solution (a state) and improve the solution by moving from the current state to a "neighboring" better state (a.k.a. performing a series of local moves).

Local Search Algorithms

Difference to search from the previous chapter:

a) **Goal state is unknown,** but we know or can calculate the utility for each state. We want to identify the state with the highest utility.

b) Often no explicit initial state + **path to goal and path cost are not important.**

c) **No search tree**. Just stores the current state and move to a "better" state if possible.

Use in AI

- **Goal-based agent**: Identify a good goal state with a good utility before planning the path to that state.

- **Utility-based agent**: Always move to neighboring higher utility states. A simple greedy method used for complicated/large state spaces or online search.

- **General optimization**: $u(s)$ can be replaced by a general objective function. Local search is an effective heuristic to find good solutions in large or continuous search spaces. E.g., gradient descend to train neural networks.
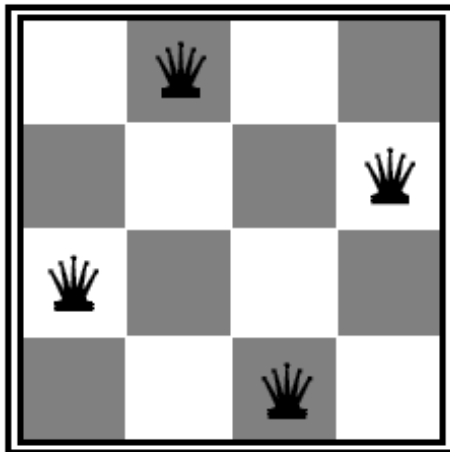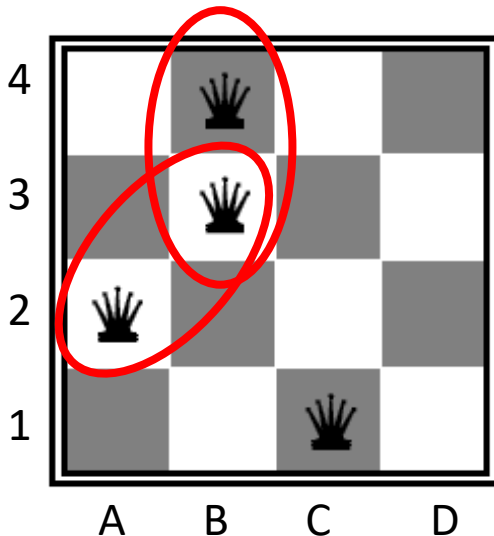
states
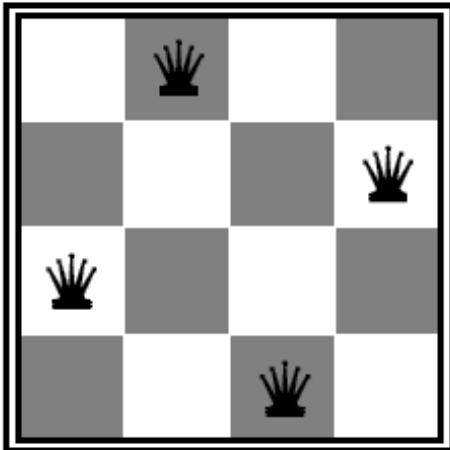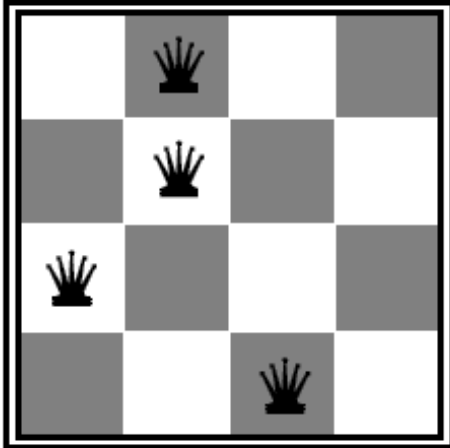
Example: n-Queens Problem

Goal: Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal.

Defining the search problem:

- **State space:** All possible n-queen configurations. How many are there?

- **State representation:** How do we define a structured representation?

- **Objective function**: What is a possible utility function given the state representation?

- **Local neighborhood:** What states are close to each other?

Example: n-Queens Problem

2 conflicts = utility of -2

0 conflicts = utility of 0

Defining the search problem:

- **State space:** All possible n-queen configurations. How many are there?
  4-queens problem: $\binom{16}{4} = 1820$

- **State representation:** How do we define a structured representation?
  E.g. $(A2, B3, B4, C1)$

- **Objective function**: What is a possible utility function given the state representation?
  Maximizing utility means minimize the number of pairwise conflicts based on the state representation.

- **Local neighborhood:** What states are close to each other?
  Move a single queen.

Example: Traveling salesman problem

- **Goal**: Find the shortest tour connecting a given set of cities

- **State space:** all possible tours (states are not individual cities!)

- **State representation:** Order of cities in the tour.

- **Objective function:** minimize the length of the tour

- **Local neighborhood:** Change the order of visiting a few cities.

Note: We have solved a different problem with uninformed/informed search! Each city was defined as a state and the path was the solution.

Hill-Climbing Search
aka Greedy Local Search

Idea: keep a single "current" state and try to find better neighboring states.
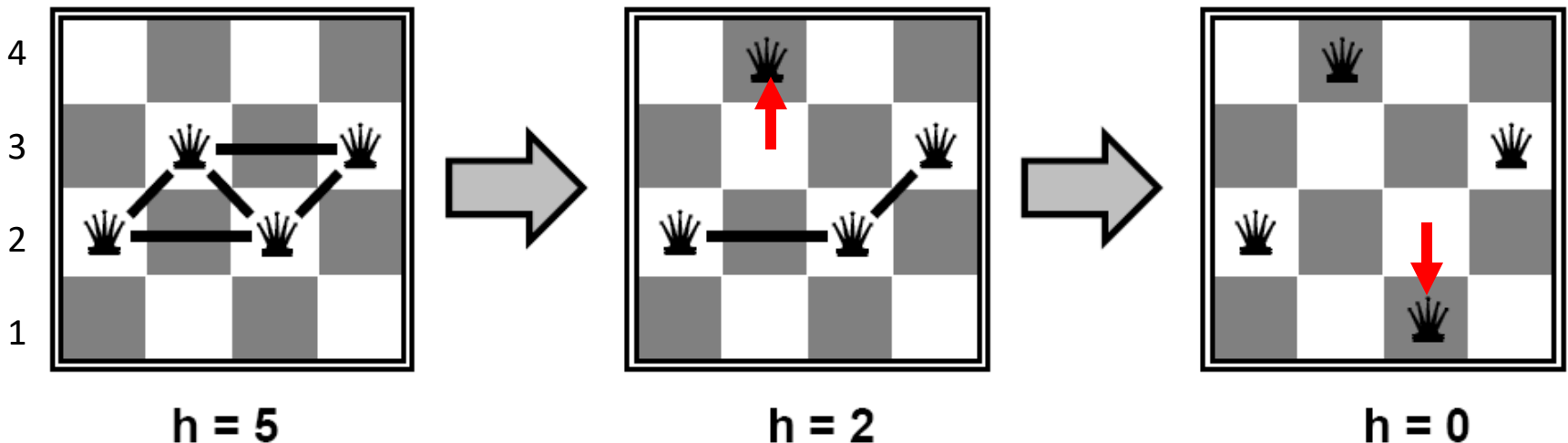
Example: *n*-Queens Problem

- **Goal:** Put *n* queens on an *n* × *n* board with no two queens on the same row, column, or diagonal.

- **State space:** all possible *n*-queen configurations. We can restrict the state space: Only one queen per column.

- **State representation:** row position of each queen in its column (e.g., 2, 3, 2, 3)

- **Objective function:** minimize the number of pairwise conflicts.

- **Local neighborhood:** Move one queen anywhere in its column.

> State space is reduced from 1820 to $4^4 = 256$

Improvement strategy

- Find a local neighboring state (move one queen within its column) to reduce conflicts

h = 5   h = 2   h = 0

Example: n-Queens Problem

To find the best local move, we have to evaluate all local neighbors (moving a queen in its column) and calculate the objective function.

Objective value after moving the queen to this square

18	12	14	13	13	12	14	14
14	16	13	15	12	14	12	16
14	12	18	13	15	12	14	14
15	14	14	♛	13	16	13	16
♛	14	17	15	♛	14	16	16
17	♛	16	18	15	♛	15	♛
18	14	♛	15	15	14	♛	16
14	14	13	17	12	14	12	18

Current objective value: $h = 17$
best local improvement has $h = 12$

Notes:
- There are many options with $h = 12$. We have to choose one!
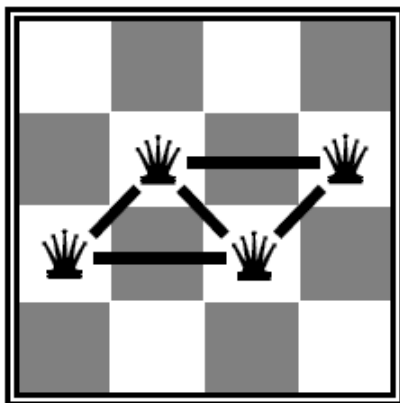- Calculating all the objective values may be expensive!

Example: n-Queens Problem

Optimization problem: find the best state s^* representing an arrangement of queens.
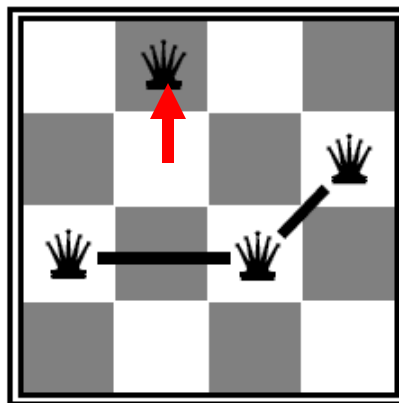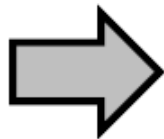
$$s^* = \text{argmin}_{s \in S} \ \text{conflicts}(s)$$

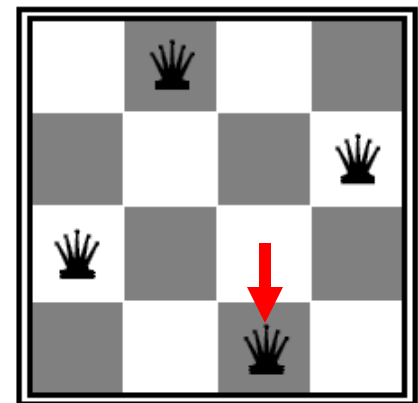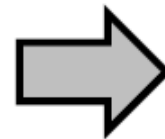subject to: s has one queen per column

Remember: This makes the problem a lot easier.

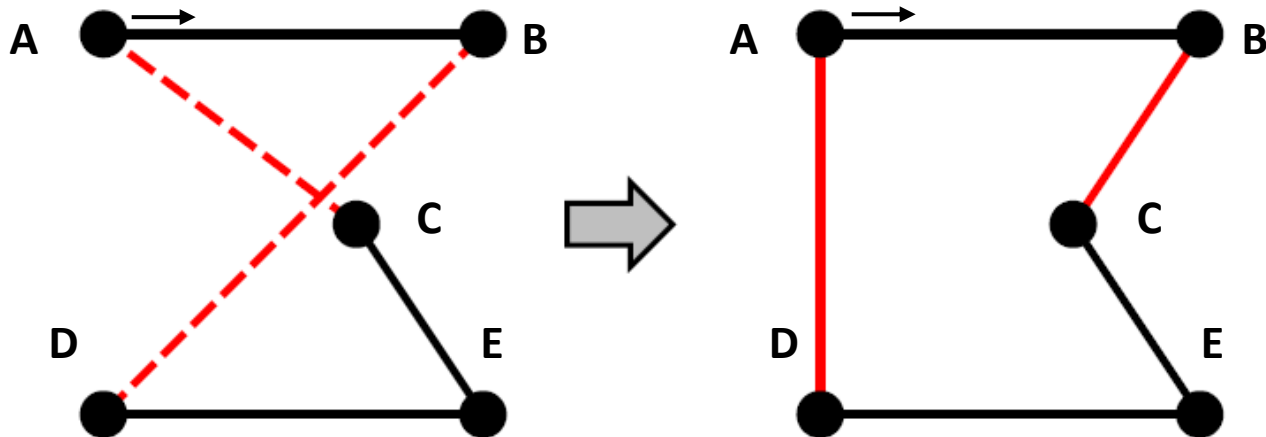

h = 5   h = 2   h = 0

Example: Traveling Salesman Problem

- **Goal:** Find the shortest tour connecting n cities
- **State space:** all possible tours
- **State representation:** tour (order in which to visit the cities) = a permutation
- **Objective function:** length of tour
- **Local neighborhood:** reverse the order of visiting a few cities

Local move to reverse the order of cities C, E and D:

State representation: **AB****DEC**   **AB****CED**
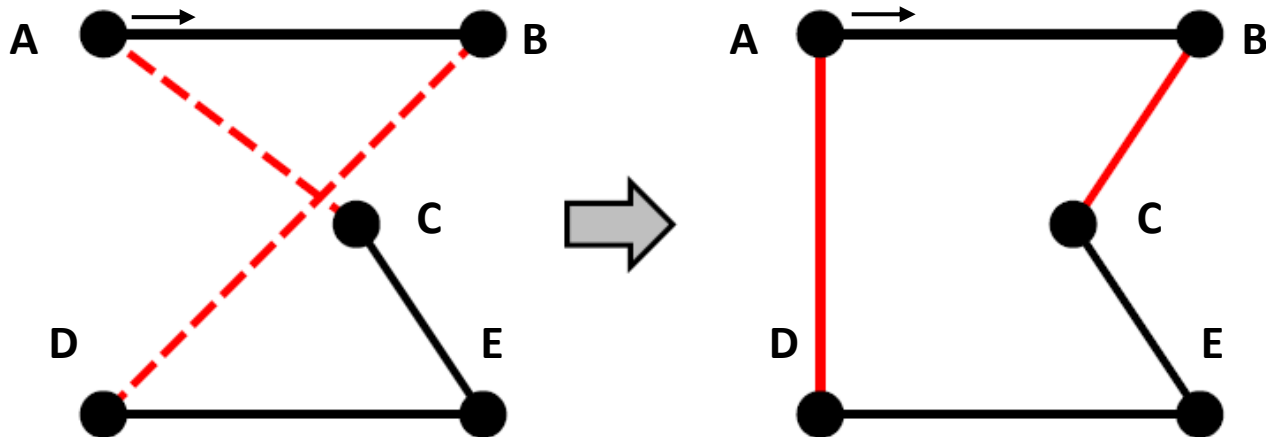
Example: Traveling Salesman Problem

Optimization problem: Find the best tour π

$$\pi^* = \text{argmin}_\pi \; \text{tourLength}(\pi)$$

s.t. π is a valid permutation (i.e., sub-tour elimination)

Local move to reverse the order of cities C, E and D:

State representation: **AB**D**E**C   **AB**C**ED**

Hill-Climbing Search (= Greedy Local Search)

function HILL-CLIMBING(*problem*) **returns** a state that is a local maximum
    current ← *problem*.INITIAL   Typically, we start with a random state
    while *true* **do**
        neighbor ← a highest-valued successor state of *current*
        if VALUE(*neighbor*) ≤ VALUE(*current*) **then return** *current*
        current ← *neighbor*

Variants:

Steepest-ascend hill climbing

- Check all possible successors and choose the highest-valued successors.

Stochastic hill climbing

- choose randomly among all uphill moves, or

- generate randomly one new successor at a time until a better one is found = first-choice hill climbing – **the most popular variant**, this is what people often mean when they say "stochastic hill climbing"

Local Optima

Hill-climbing search is like greedy best-first search with the objective function as a (maybe not admissible) heuristic and no frontier (just stops in a dead end).

Is it complete/optimal?

- No – can get stuck in local optima
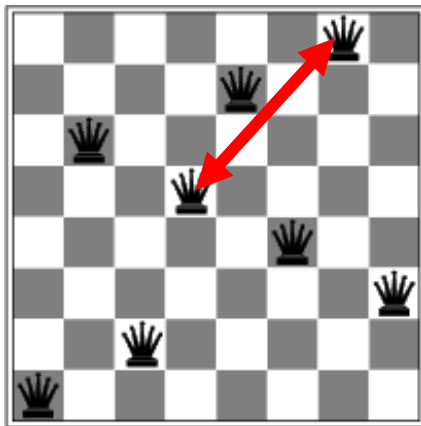
$h = 1$

Example: local optimum for the 8-queens problem. No single queen can be moved within its column to improve the objective function.
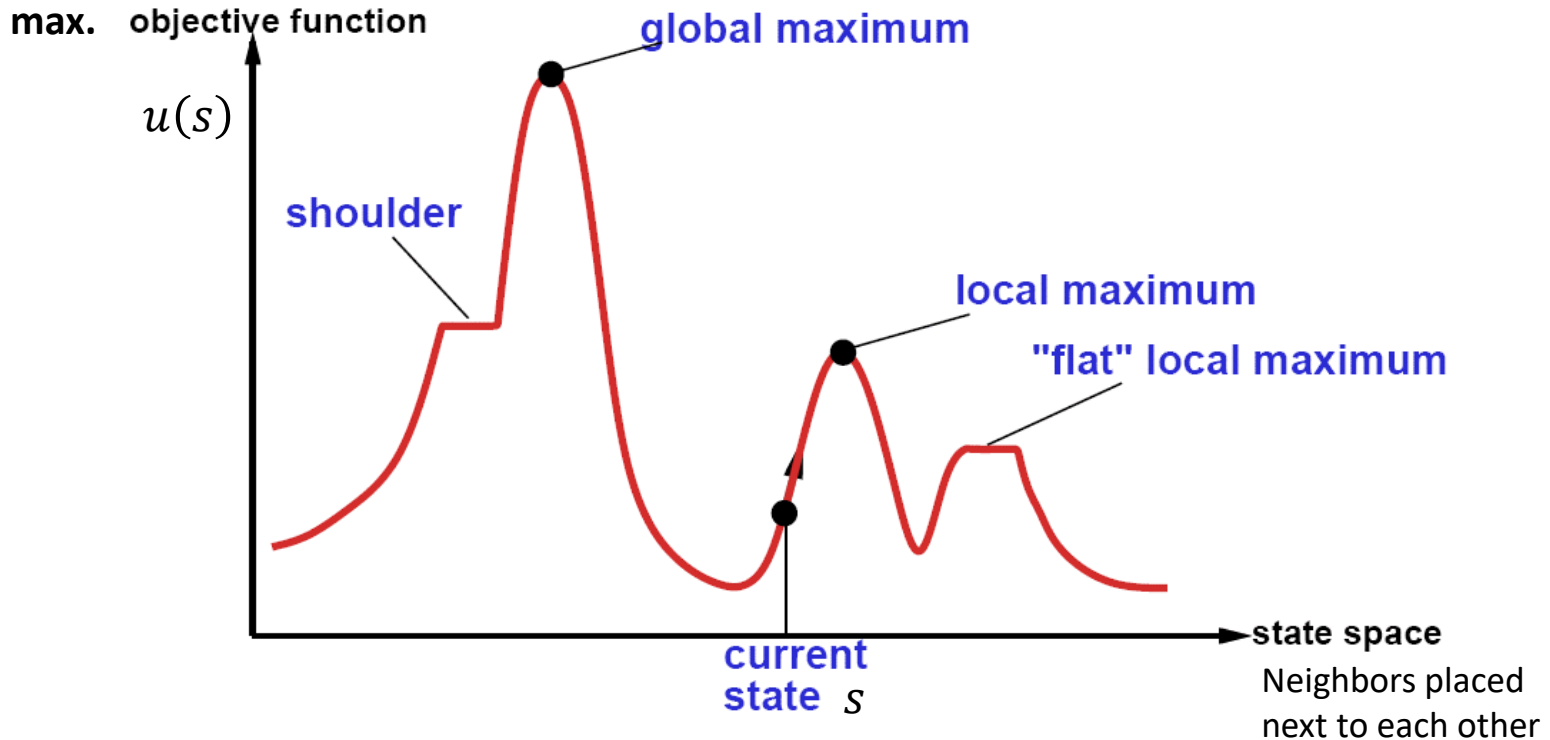
Simple approach that can help with local optima:

Random-restart hill climbing

- Restart hill-climbing many times with random initial states and return the best solution.

The State Space "Landscape"

We can get the utility (objective function value) from the state description using $U = u(s)$.

How to escape local maxima?

→ Random restart hill-climbing can help.

What about "shoulders" (called "ridges" in higher dimensional space)?

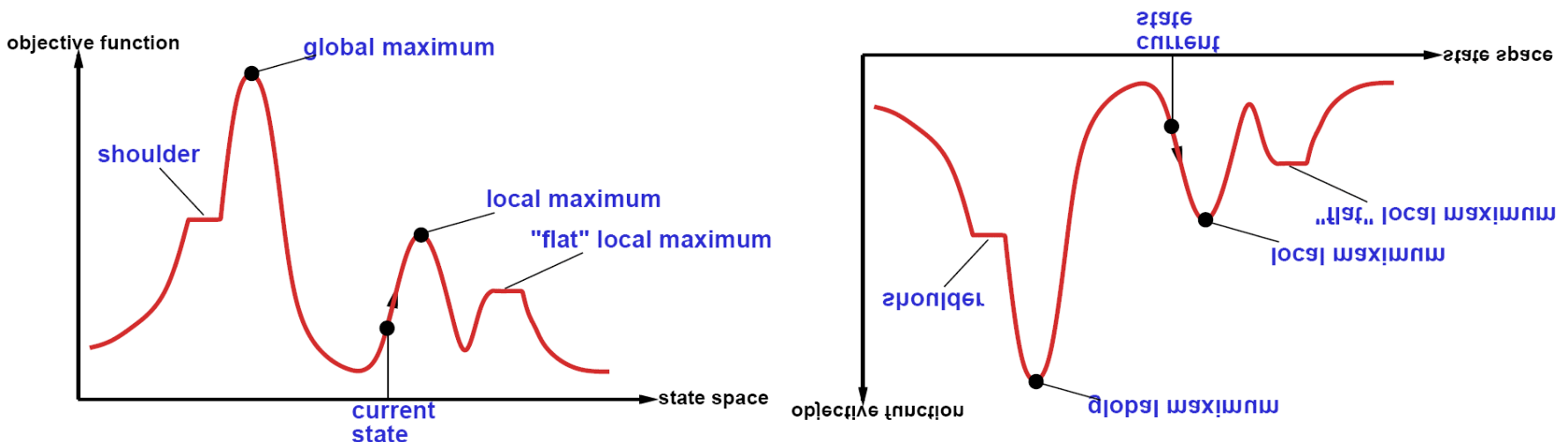→ Hill-climbing that allows sideways moves and uses momentum.

Minimization vs. Maximization

- The name hill climbing implies **maximizing a function**.
- Optimizers like to state problems as **minimization problems** and call hill climbing gradient descent instead.
- Both types of problems are equivalent:

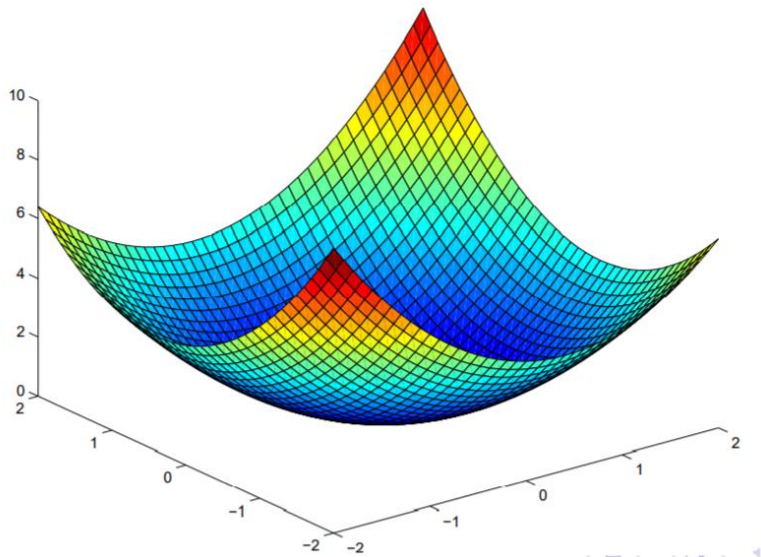$$\max x \big(f(x) \big) \qquad \Longleftrightarrow \qquad \min \big(-f(x) \big)$$

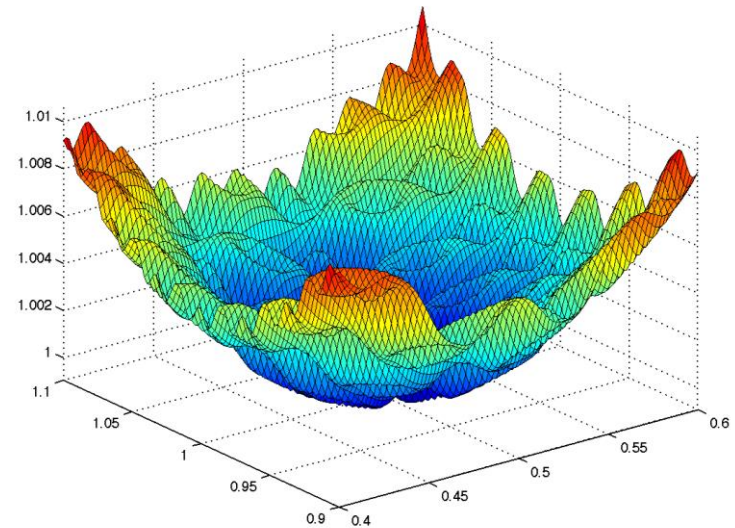# Convex vs. Non-Convex Optimization Problems

Minimization problems

Convex Problem

One global optimum + smooth function → calculus makes it easy

Non-convex Problem

Many local optima → hard

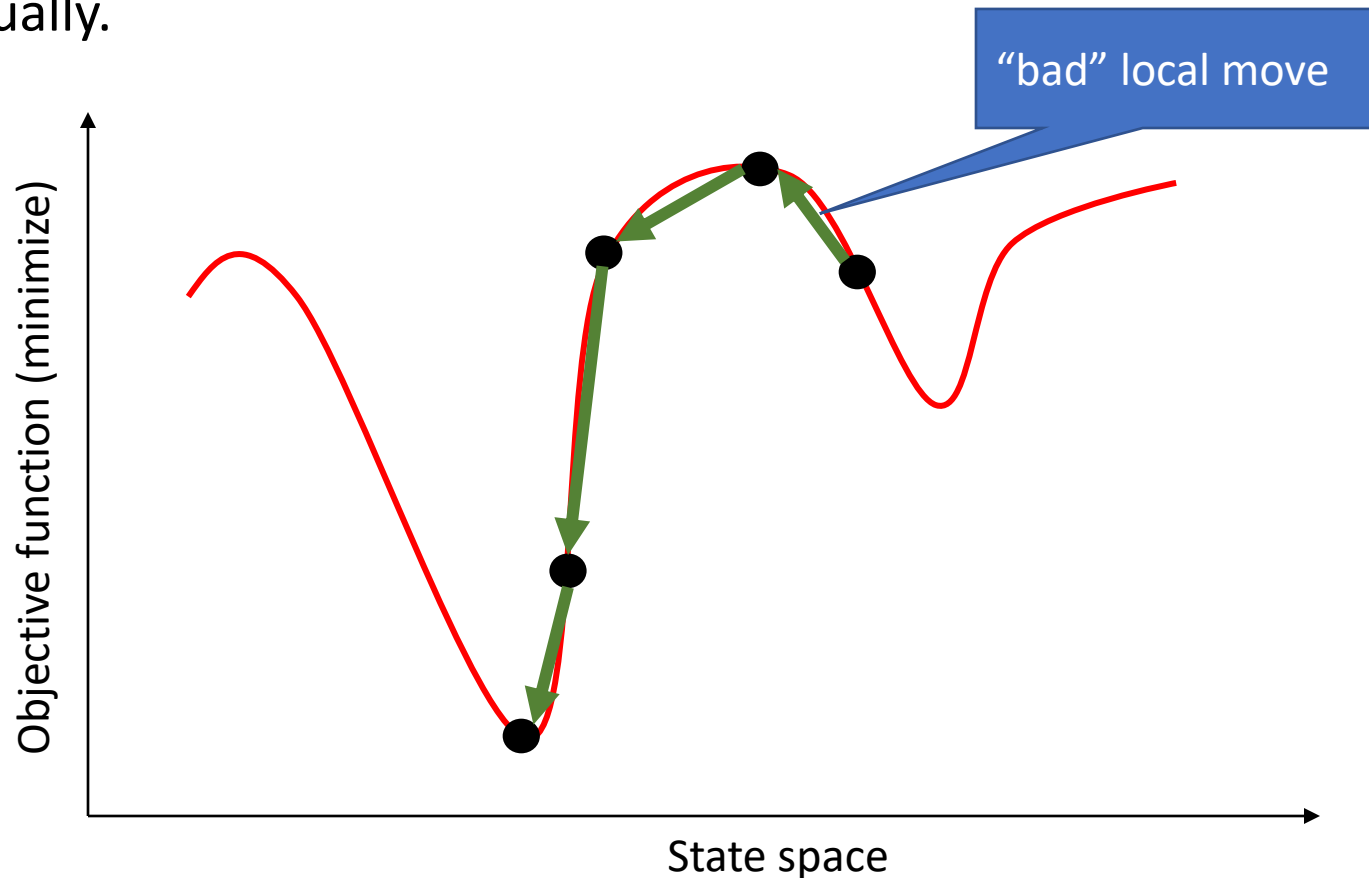Many discrete optimization problems are like this.

Simulated Annealing

Use heat to escape local optima…

Simulated Annealing

- **Idea**: First-choice stochastic hill climbing + escape local minima by **allowing some "bad" moves** but gradually decrease their frequency.

- Inspired by the process of tempering or hardening metals by decreasing the temperature (chance of accepting bad moves) gradually.

"bad" local move

Objective function (minimize)

State space

Simulated Annealing

- **Idea**: First-choice stochastic hill climbing + escape local minima by allowing some "bad" moves but gradually decreasing their frequency as we get closer to the solution.

- The probability of accepting "bad" moves follows **an annealing schedule** that reduces the temperature T over time t.

function SIMULATED-ANNEALING($problem, schedule$) **returns** a solution state
    $current \leftarrow problem$.INITIAL   Typically, we start with a random state
    for $t = 1$ **to** ∞ **do**
        $T \leftarrow schedule(t)$
        if $T = 0$ **then return** $current$
        $next \leftarrow$ a randomly selected successor of $current$
        $\Delta E \leftarrow$ VALUE($next$) – Value($current$)
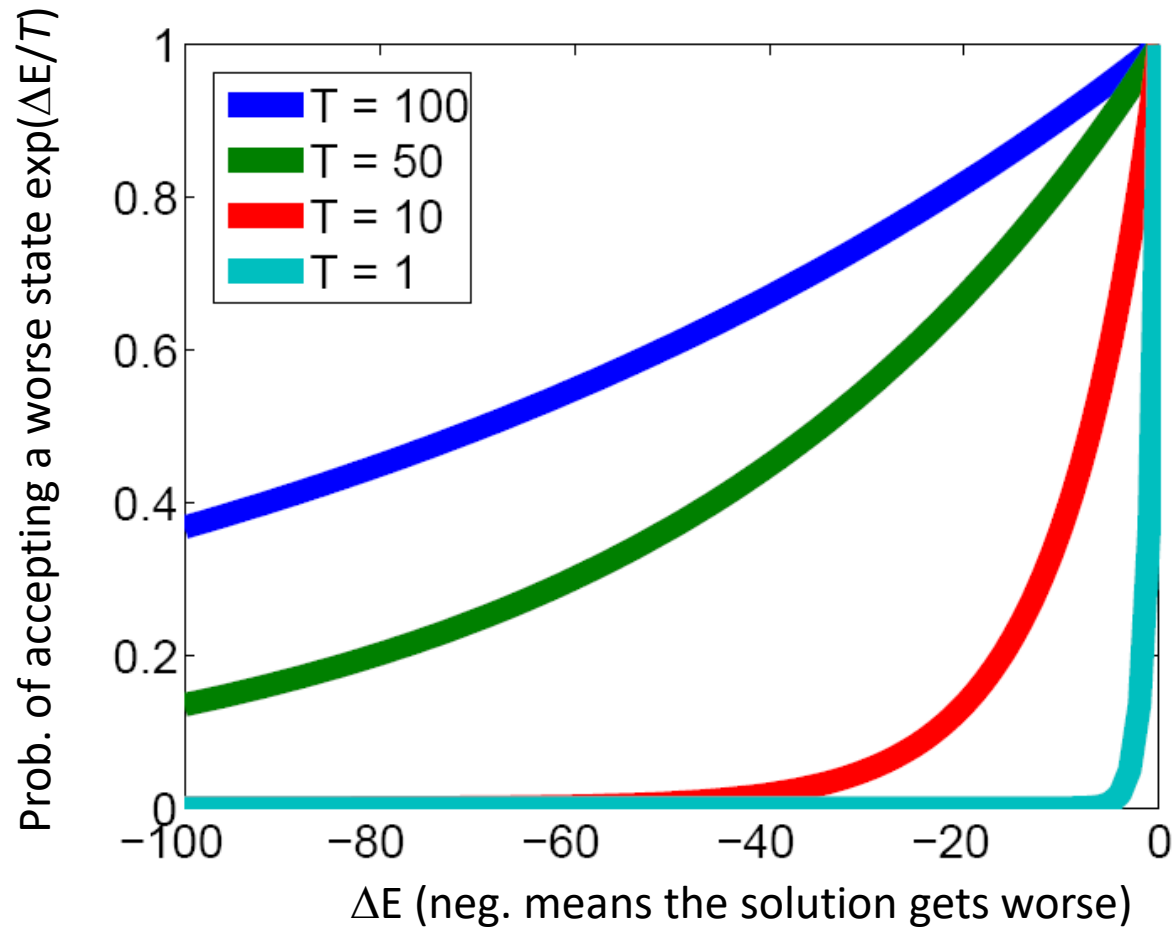   **if** $\Delta E > 0$ **then** $current \leftarrow next$
        else $current \leftarrow next$ only with probability $e^{-\Delta E/T}$

Always do good moves

Uses the Metropolis acceptance criterion to accept "bad" moves

Note: Use VALUE(current) – VALUE(next) for minimization

The Effect of Temperature

The lower the temperature, the less likely the algorithm will accept a worse state.

Cooling Schedule

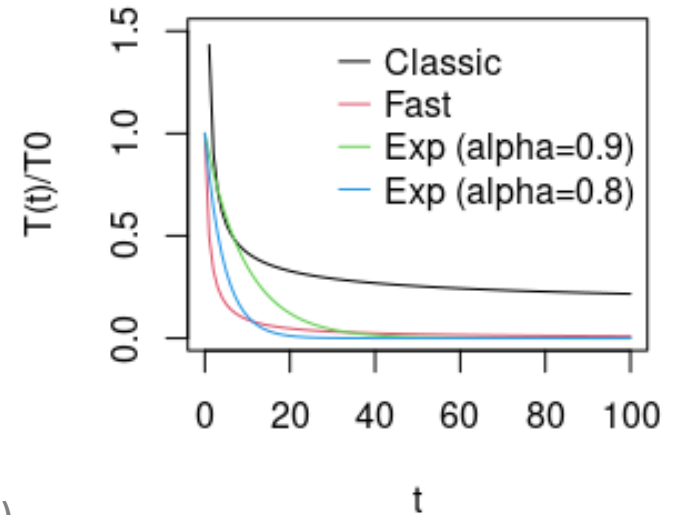
The cooling schedule is very important.
Popular schedules for the temperature at time t:

- **Classic simulated annealing:** $T_t = T_0 \dfrac{1}{\log(1+t)}$

- **Fast simulated annealing** (Szy and Hartley; 1987)

$$T_t = T_0 \frac{1}{1+t}$$

- **Exponential cooling** (Kirkpatrick, Gelatt and Vecchi; 1983)

$$T_t = T_0 \alpha^t \quad \text{for} \quad 0.8 < \alpha < 1$$

Notes:

- The best schedule is typically determined by trial-and-error.

- Choose T_0 to provide a high probability that any move will be accepted at time $t = 0$.

- T_t will not become 0 but very small. Stop when $T < \epsilon$ (ϵ is a very small constant).

Simulated Annealing Search

Guarantee: If temperature decreases **slowly enough**, then simulated annealing search will find a global optimum with probability approaching one.

However:
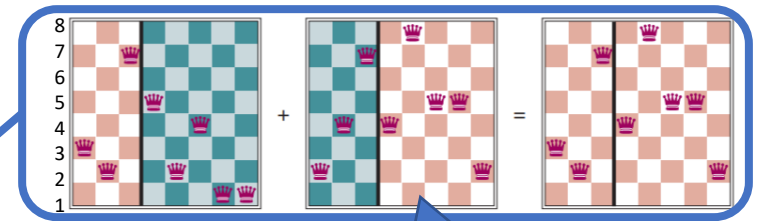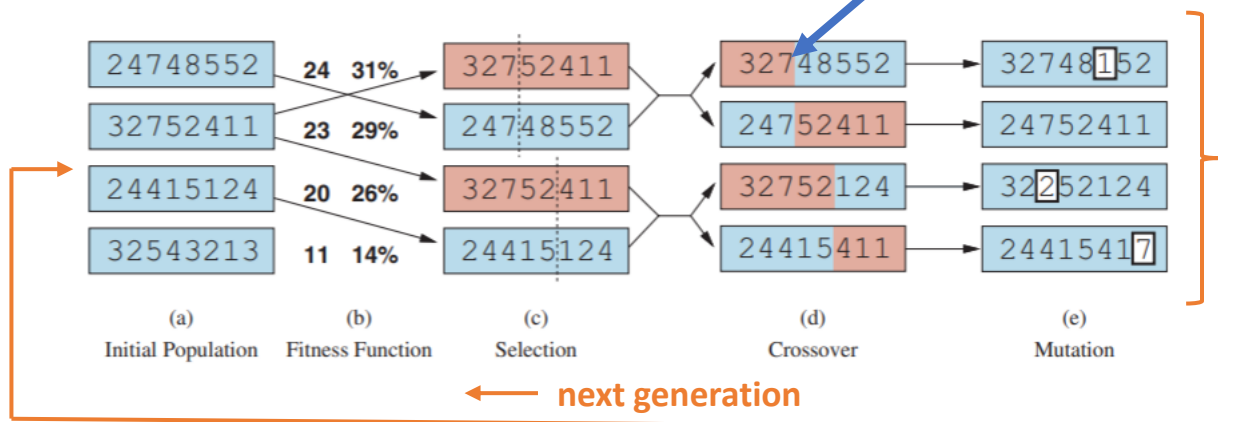- This usually takes impractically long.
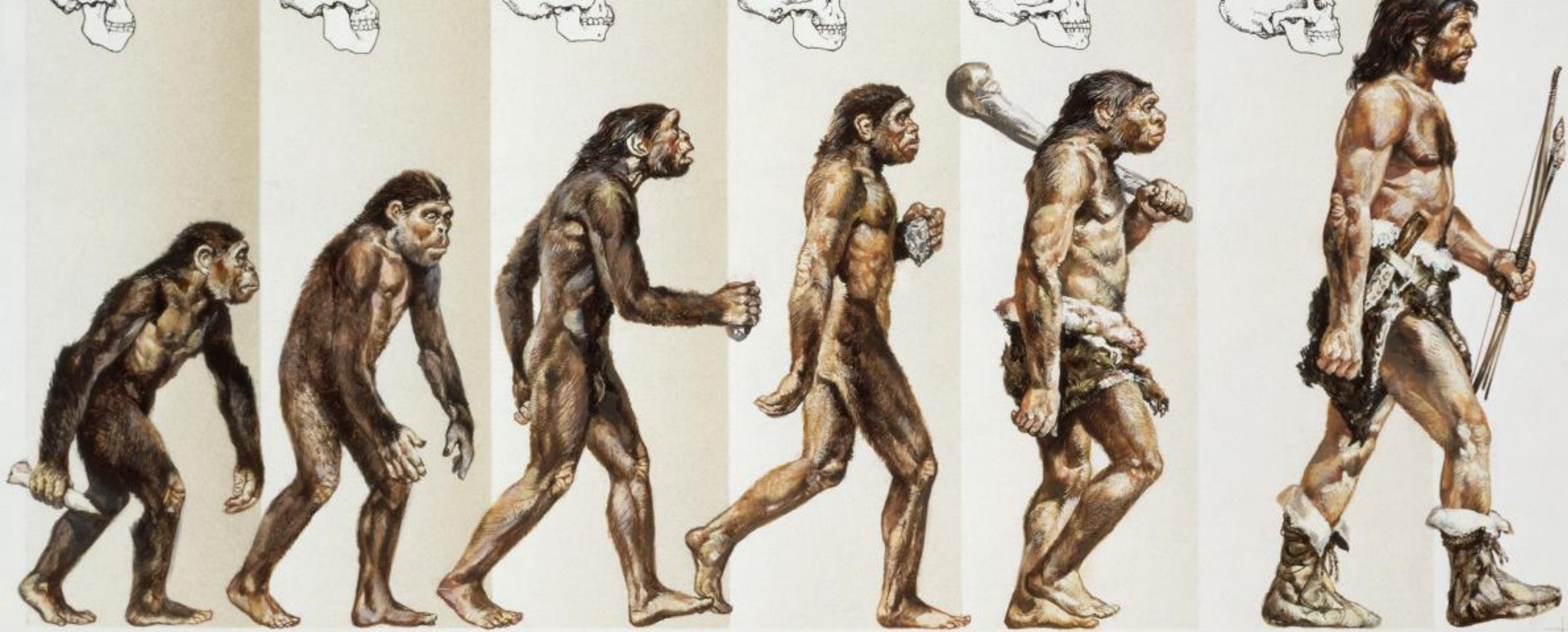
Evolutionary Algorithms

A Population-based Metaheuristics

Evolutionary Algorithms / Genetic Algorithms

- A metaheuristic for **population-**based optimization.

- Uses mechanisms inspired by biological evolution (genetics):
  - Reproduction: Random selection with probability based on a **fitness** function.
  - Random recombination (crossover)
  - Random mutation
 - Repeated for many generations

- Example: 8-queens problem

24748552	24 31%	32752411	32748552	32748**1**52
32752411	23 29%	24748552	24752411	24752411
24415124	20 26%	32752411	32752124	32**2**52124
32543213	11 14%	24415124	24415411	244154**7**
(a) Initial Population	(b) Fitness Function	(c) Selection	(d) Crossover	(e) Mutation

← **next generation**

Individual = state **representation** as a chromosome: row of the queen in each column
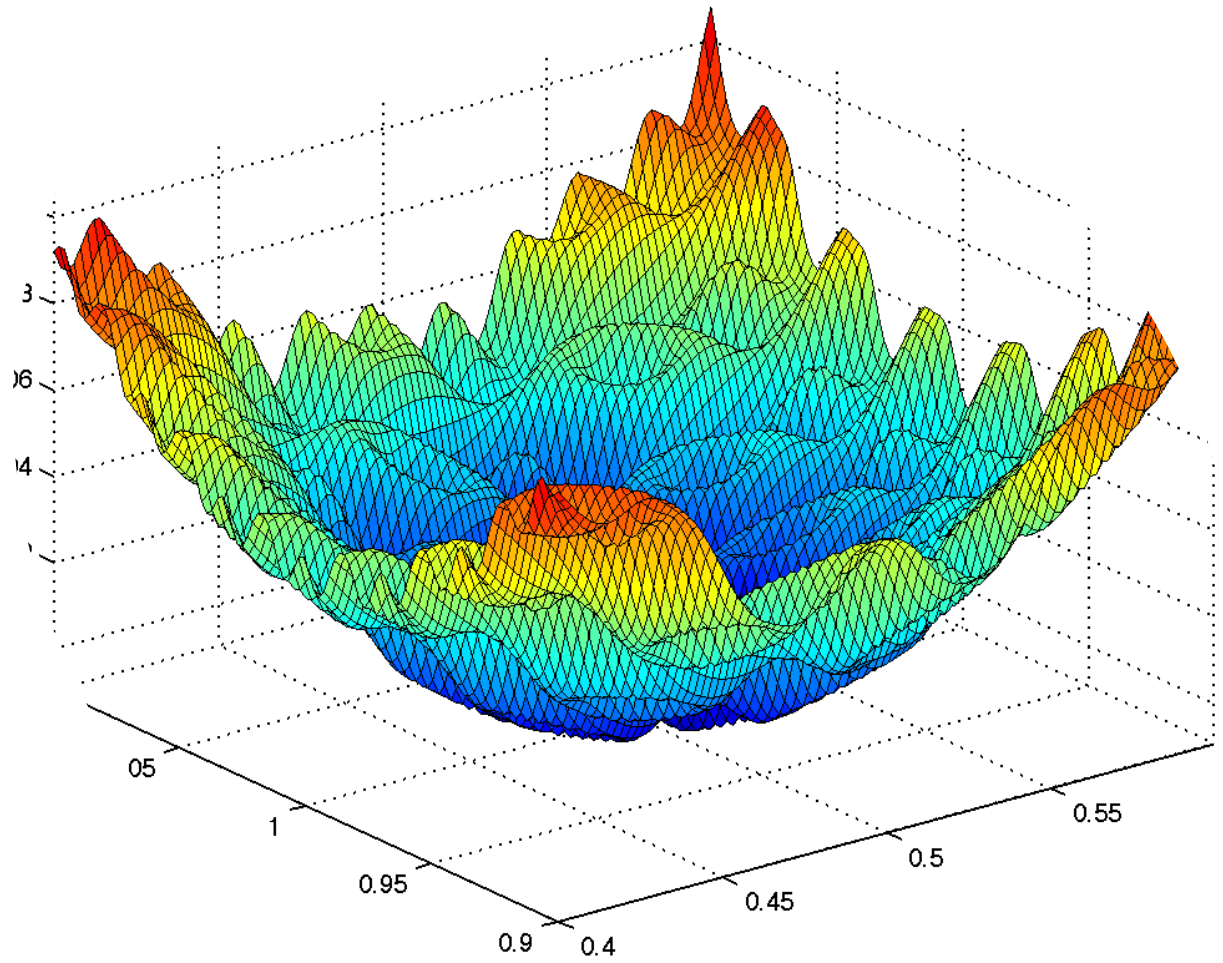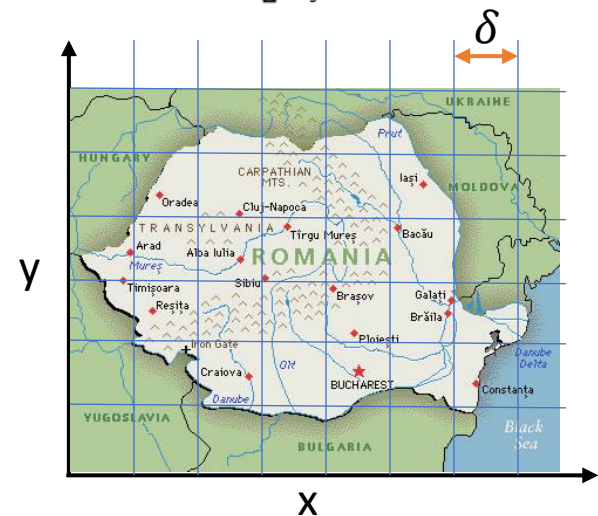
Search in Continuous Spaces

Discretization of Continuous Space

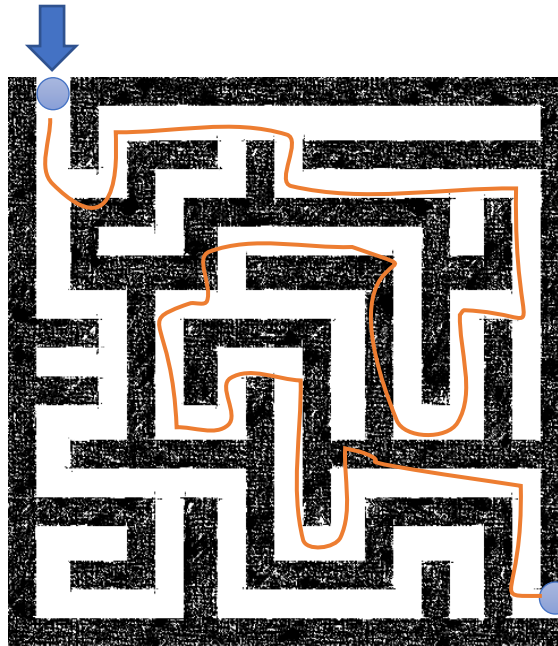- Use atomic states and create a graph as the transition function.
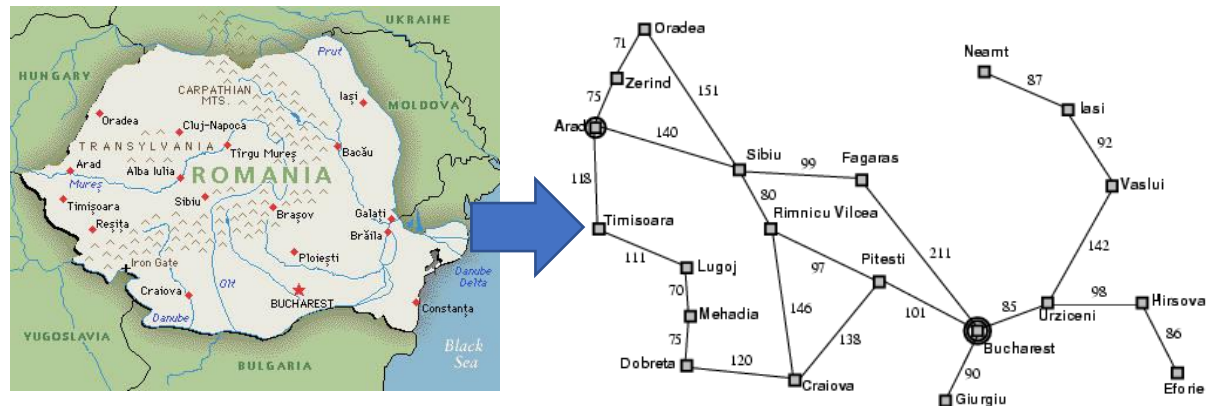
- Use a grid with spacing of size δ

  Note: You probably need a way finer grid!

Discretization of Continuous Space

How did we discretize this space?

Search in Continuous Spaces: Gradient Descent

$f(x)$

State space: infinite

State representation: $x = (x_1, x_2, \ldots, x_k)$

Objective function: min $f(x) = f(x_1, x_2, \ldots, x_k)$

Local neighborhood: small changes in $x_1, x_2, \ldots, x_k$

Gradient at point x: $\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_k} \right)$

   (=evaluation of the Jacobian matrix at x)
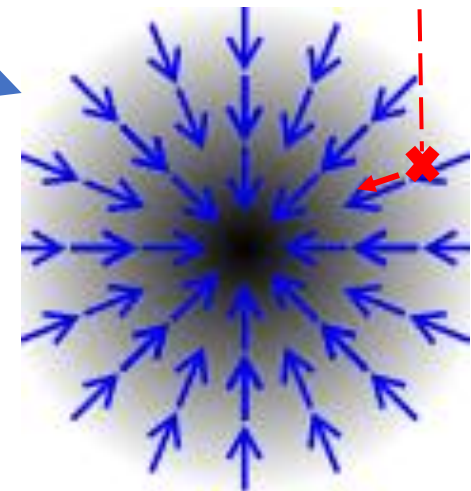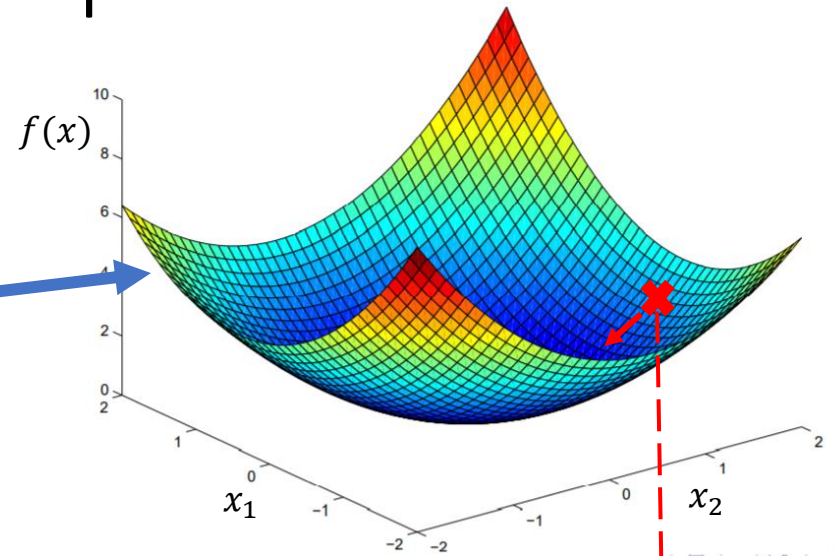
Find optimum by solving: $\nabla f(x) = 0$

- **Gradient descent (= Steepest-ascend hill climbing for minimization)** with step size α

$$\text{Repeat: } x \leftarrow x - \alpha \nabla f(x)$$

- **Newton-Raphson method**
  uses the inverse of the Hessian matrix (second-order partial derivative of $f(x)$)
  $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for the step size α

$$\text{Repeat: } x \leftarrow x - H_f^{-1}(x)\nabla f(x)$$

x_1 x_2

Note: May get stuck in a local optima if the search space is non-convex! Use simulated annealing, momentum or other methods to escape local optima.

Search in Continuous Spaces: Empirical Gradient Methods

- What if the mathematical formulation of the objective function is not known?

- We may have objective values at fixed points, called the **training data**.

- In this case, we can estimate the gradient using the data and use **empirical gradient search.**

→ We will talk more about search in continuous spaces with loss functions using gradient descend when we talk about **parameter learning for machine learning.**