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The Burkholder Functional in Nonlinear Elasticity Theory

Semester Paper

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Abstract

This work aims to contextualise and analyse the paper [5] "Burkholder integrals, Morrey's problem and quasiconformal mappings" by Kari Astala, Tadeusz Iwaniec, István Prause and Eero Saksman.

With the goal of shedding light on the long-standing Morrey's Conjecture, the authors tackle the celebrated Burkholder functional in two dimensions. Despite its relevance in the field, it is still unknown whether such functional is quasiconvex. Either the positive or negative answer would provide significant information. Nonetheless, the authors manage to prove quasiconvexity of the Burkholder functional in an important particular case: at the identity and for nonnegative integrands. Because of this last assumption, quasiconformal mappings naturally come into play. By relying on the complex variable function theory, the authors come up with a new interpolation lemma, together with a sharp bound that unleashes plenty of sharp estimates for Burkholder functionals and quasiconformal mappings.

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Chapter 1

Introduction

We present a brief physical motivation and reasons for the mathematical interest on the topic, mainly inspired by [17].

1.1 The two-gradient problem

When modelling and studying mathematically elastic materials, a usual variational formulation is the following. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, representing the physical space occupied by the material. Let $W \in C(GL_+(n, \mathbb{R}), \mathbb{R})$ be the stored energy density of the homogeneous elastic material. This is, given a function $u\colon \Omega \to \mathbb{R}^n$ representing a deformation of the elastic material, W takes its differential matrix at every point and associates a potential energy density due to the deformation. Then, the functional

$$I(u) = \int_\Omega W(Du(x))\, dx \tag{1.1}$$

is interpreted as the total stored energy in an elastic material (the only dependency on Du and neither u nor x is due to the assumption of elasticity), therefore one would wish to minimise (1.1) in a suitable space of functions (deformations) to find stable configurations of the material. From the empirical observation that not only do many elastic materials minimise (1.1) as a functional but also they minimise its lagrangian density pointwise, a related problem arises:

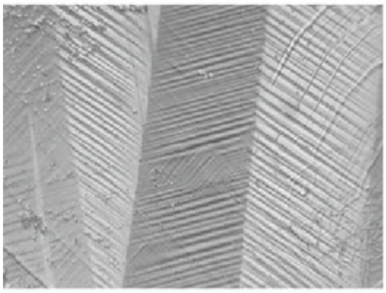
One assumes that the stored energy density is a nonnegative quantity $W \geq 0$. Provided W attains its infimum, it is customary to normalise it so that $\min W(F) = 0$ for $F \in GL_+(n, \mathbb{R})$. Let $K := W^{-1}(0)$ be the set of minimisers. This yields the problem:

Find $u \in W^{1,\infty}(\Omega)$ such that

$$Du(x) \in K \qquad \text{a.e. } x \in \Omega, \tag{1.2}$$

usually subject to some boundary conditions.

For simplicity, we particularise to the case when K is a set of two matrix elements, $K = \{A, B\}$. The physical meaning of this condition is explained in Fig. 1.1.



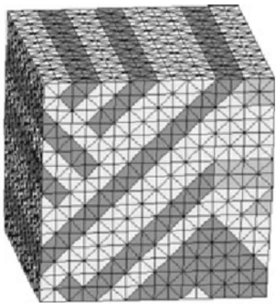


Figure 1.1: On the left, a closeup picture at microscopic scale of a microstructure. Realise that the material has straps, within which an alternating pattern of two gradients in the material arises. This corresponds to the two-gradient problem and the condition $Du(x) \in K = \{A, B\}$. On the right, a discretised cubic representation of the two-gradient microstructure. Picture by Marcel Arndt (Rheinische Friedrichs-Wilhelms Univ., Bonn).

For u in a dense class, assume that we have a flat interface between two regions where Du takes the value A in one and B in the other. Given that Du is a gradient, we may impose that it has no curl also at the interface. One easily checks using Stokes's Theorem that

$$(B - A)\,\tau = 0,$$

for any tangent vector τ to the $(n-1)$-dimensional interface, meaning that the matrices A and B are rank-one connected ($A - B$ is a rank-one matrix). With this, we want to illustrate the importance of rank-one directions when studying functionals modelling elastic materials.

1.2 Notions of convexity

When aiming to minimise functionals, a highly appreciated feature is convexity. However, this notion is rather strict. It is possible to relax such assumption though. In what concerns us, we work with functionals over matrices.

First, set the following notation: For $F \in \mathbb{R}^{m\times n}$, let $M(F)$ be the vector of all the minors of F.

Definition 1.1 *A function $f : \mathbb{R}^{m\times n} \to (-\infty, +\infty]$ is*

(a) convex if, for $\lambda \in [0, 1]$ and $A, B \in \mathbb{R}^{m\times n}$

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B).$$

(b) *polyconvex if one can write* $f(F) = g(M(F))$ *for some convex function* $g : \mathbb{R}^{\mathrm{length}(M(F))} \to (-\infty, +\infty]$.

(c) *quasiconvex if for every open and bounded set* U *with* $|\partial U| = 0$ *it holds that*

$$\int_U f(F + Du) \geq f(F)|U|, \tag{1.3}$$

for all $F \in \mathbb{R}^{m\times n}$ *and* $u \in W_0^{1,\infty}(U, \mathbb{R}^m)$.

(d) *rank-one convex if* f *is convex along rank-one lines. This is, for* $\lambda \in [0, 1]$ *and* $A, B \in \mathbb{R}^{m\times n}$ *with* $\mathrm{rk}(A - B) = 1$

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B).$$

Naturally, one has analogous definitions for the concave counterparts by swapping the direction of the inequalities in the definitions above.

The two-gradient problem is a good justification for defining rank-one convexity (d), which is clearly a weaker property that the usual convexity (a). Of upmost importance for us is the notion of quasiconvexity (or quasiconcavity). It was first introduced by Morrey in 1952 [16], and turned out to be a suitable property for the study of vectorial variational problems. Since then, the topic has been increasingly luring attention, to the extent that a myriad of interesting connections gave rise to a whole field.

Our point is that all these different generalisations of the concept of convexity come in handy when dealing with functionals over matrices, particularly because they are all closely related in the following way. Following up Definition 1.1:

Lemma 1.2 *Let* $n \geq 2$ *and* $m \geq 2$*. Then,*

$$f \text{ convex} \Rightarrow f \text{ polyconvex} \Rightarrow f \text{ quasiconvex} \overset{f<\infty}{\Rightarrow} f \text{ rank-one convex.}$$

Conversely,

$$f \text{ rank-one convex} \overset{m\geq 3}{\not\Rightarrow} f \text{ quasiconvex} \not\Rightarrow f \text{ polyconvex} \not\Rightarrow f\text{convex.}$$

Besides, if f *takes finite values in* $\mathbb{R}$*, and* $n = 1$ *or* $m = 1$*, then the four notions are equivalent.*

We would like to draw our attention on the case $m = 2$ and whether rank-one convexity implies quasiconvexity. Perhaps surprisingly, this remains as an important open problem:

Conjecture 1.3 (Morrey's conjecture) *Let* $n \geq 2$*. Rank-one convex functions* $f\colon \mathbb{R}^{2\times n} \to \mathbb{R}$ *are quasiconvex.*

Sverák gave in 1992 [19] a counterexample that rank-one convex functionals are not quasiconvex for $m \geq 3$. The counterexample is based on the superposition of three plane waves. The counterexample cannot, nonetheless, be extended to two dimensions. On top of that, a number of partial results are driving mathematicians in the field to thinking that Morrey's conjecture may be true.

1.3 The Burkholder functional

Among the mystery that Morrey's conjecture conveys, here is a functional on which many eyes are on.

Definition 1.4 (Burkholder functional) *Define the Burkholder functional* $\mathbf{B}_p : \mathbb{R}^{2\times 2} \to \mathbb{R}$ *by*

$$\mathbf{B}_p(A) := \left(\frac{p}{2}\det A + \left(1 - \frac{p}{2}\right)|A|^2\right)|A|^{p-2}, \quad p \geq 2. \tag{1.4}$$

The notation $|\cdot|$ *is reserved for the operator norm of a matrix.*

The Burkholder functional can be normalised in different ways. Here, $\mathbf{B}_p(\mathrm{Id}) = 1$. This functional finds its origin in the theory for the martingale transform [8], although it gained an enormous importance in nonlinear elasticity theory and related fields. It is the most famous two-dimensional rank-one concave functional [3]. However, it is still open to check whether it is quasiconcave. This last unknown is what the paper of our concern [5] tries to shed light on.

Besides this connection, the interest in the Burkholder functional goes beyond. Certainly, if one proved the Burkholder functional is not quasiconcave, then Morrey's conjecture would be proven false. On the contrary, if it indeed was quasiconvex, even though Morrey's conjecture would still remain unsolved, Iwaniec's conjecture [11] would be shown true:

Definition 1.5 *The following Calderón-Zygmund type singular integral*

$$\mathbf{S}f := -\frac{1}{\pi}\mathrm{p.v.}\int_{\mathbb{C}} \frac{f(\xi)d\xi}{(z-\xi)^2} \tag{1.5}$$

is referred to as the Beurling-Ahlfors transform.

More details on the Beurling-Ahlfors transform can be found in [3]. Perhaps, one of the most remarkable properties of this operator is that it interchanges the complex partial derivatives of Sobolev functions $f \in W^{1,2}(\mathbb{C})$.

$$\mathbf{S}f_{\bar{z}} = f_z - 1 \tag{1.6}$$

Equally remarkable is the fact that $\mathbf{S}$ defines an isometry from $L^2(\mathbb{C})$ to $L^2(\mathbb{C})$.

Conjecture 1.6 (Iwaniec's conjecture) *The explicit sharp L^p operator norm of* $\mathbf{S}$ *is*

$$\|\mathbf{S}\|_{L^p(\mathbb{C})\to L^p(\mathbb{C})} = \begin{cases} p-1, & \text{if } 2 \leq p < \infty \\ \frac{1}{p-1}, & \text{if } 1 < p \leq 2. \end{cases} \tag{1.7}$$

Indeed, the following useful bound found by Burkholder [8], establishes a connection between the full quasiconcavity of the Burkholder functional and Iwaniec's conjecture.

Set $C_p = p\left(1-\frac{1}{p}\right)^{1-p}$ and let $p \geq 2$. Then,

$$C_p \cdot (|f_z|^p - (p-1)^p |f_{\bar{z}}|^p) \leq (|f_z| - (p-1)|f_{\bar{z}}|) \cdot (|f_z| + |f_{\bar{z}}|)^{p-1} = \mathbf{B}_p(Df). \tag{1.8}$$

Chapter 2

Preliminaries

The paper we are reviewing takes advantage of the fact that, when studying the Burkholder functional acting on 2×2-sized differentials of functions f, one can view those functions as maps $f : \mathbb{C} \to \mathbb{C}$ of the complex plane, and thus exploit the complex variable function theory. In this direction, let us introduce some notation and well-known facts. Further details on what is presented can be found in [3].

Remark 2.1 *In terms of complex partial derivatives, the following identities hold:*

$$|Df(z)| = |f_z| + |f_{\bar{z}}| \qquad \det Df(z) = J(z, f) = |f_z|^2 - |f_{\bar{z}}|^2.$$

In view of this, the Burkholder functional is rewritten as reads:

$$\mathbf{B}_p(Df) = (|f_z| - (p-1)|f_{\bar{z}}|)(|f_z| + |f_{\bar{z}}|)^{p-1}. \tag{2.1}$$

2.1 Quasiconformal maps and the Beltrami equation

We move on to another essential concept.

Definition 2.2 *A map $f \in W^{1,2}(\Omega, \Omega')$ is said to be quasiconformal if f defines a homeomorphism between the domains $\Omega \subset \mathbb{C}$ and $\Omega' \subset \mathbb{C}$ and*

$$|Df(z)|^2 \leq K(z)J(z, f) \tag{2.2}$$

for some bounded function $K : \mathbb{C} \to \mathbb{R}$.

The smallest function $K(z) \geq 1$ for the quasiconformal map f is called the distortion function of f. When the hypothesis of f being a homeomorphism is dropped from the definition, we rather use the term "quasiregular map". However, this property is usually kept because when modelling elastic materials, a homeomorphism respects the principle of non-interpenetration of matter, besides the mathematical interest of the subsequent theory.

Let f be quasiconformal. Define, for those $z \in \Omega$ such that $f_z \neq 0$

$$\mu(z) := \frac{f_{\bar{z}}}{f_z}$$

and fix, for instance, $\mu(z) = 0$ otherwise. With this, the quasiconformal map solves the following complex variable PDE

$$f_{\bar{z}} = \mu(z) f_z. \tag{2.3}$$

This is the so-called Beltrami equation. Combining Remark 2.1 and equations (2.2) and (2.3), one reaches the condition

$$(1+\mu)^2 \leq K(z)(1-\mu^2) \implies (1+\mu) \leq K(z)(1-\mu) \implies \mu \leq \frac{K(z)-1}{K(z)+1} < 1.$$

In an abuse of notation, set $K := \operatorname{ess\,sup} K(z)$ and define $k := \|\mu\|_\infty = \frac{K-1}{K+1} < 1$. The function μ is referred to as the dilation function, and so we denote by k its essential supremum.

However, the interesting part comes when one tries to solve a Beltrami equation. Astonishingly, under the mild assumption that μ is a measurable bounded function, the associated Beltrami equation yields a unique solution under appropriate normalisation.

Theorem 2.3 (Wellposedness of the Beltrami equation, see [3]) *Suppose that $0 \leq k < 1$ and that $|\mu| \leq k\chi_{\mathbb{D}}(z)$, $z \in \mathbb{C}$. Then, there is a unique $f \in W^{1,2}_{loc}(\mathbb{C})$ such that*

$$\begin{aligned} &f_{\bar{z}} = \mu(z) f_z \quad \textit{and} \\ &f(z) = z + O\left(\frac{1}{z}\right) \quad \textit{as } z \to \infty. \end{aligned}$$

Such unique solutions are called principal solutions to the Beltrami equation. If we further demand some regularity on the dilation function, the principal solution rewards us back with similar regularity, as these two theorems present.

Theorem 2.4 ($C^{1,\alpha}$ regularity, see [3]) *The principal solution of the Beltrami equation (2.3) in which $\mu \in C^\alpha(\mathbb{C})$, $0 < \alpha < 1$ is a $C^{1,\alpha}(\mathbb{C})$-diffeomorphism. In particular, $|f_z|^2 \geq J(z,f) > 0$, everywhere.*

Theorem 2.5 (Smooth approximation, see [3]) *Suppose the Beltrami coefficients $\mu_l \in C_0^\infty(\Omega)$ satisfy $|\mu_l(z)| \leq k < 1$, for all $l = 1, 2 \ldots$, and converge almost everywhere to μ. Then, the associated principal solutions $f^l : \mathbb{C} \to \mathbb{C}$ are C^∞-smooth diffeomorphisms converging in $W^{1,2}_{loc}(\mathbb{C})$ to the principal solution of the limit equation $f_{\bar{z}} = \mu(z) f_z$.*

As a last preliminary, a lemma concerning principal solutions on which the proof of the main theorem heavily relies on.

Lemma 2.6 (Area inequality, see [3]) *The area of the image of the unit disk under a principal solution in $\mathbb{D}$ does not exceed π. It equals π if and only if the solution is the identity map outside the disk.*

After this presentation of preliminaries, we are ready to tackle the main theorem of the paper of interest.

Chapter 3

Main Theorem and consequences

3.1 Sharp L^p inequality

The central theorem of the paper of focus [5] is the following:

Theorem 3.1 (Sharp L^p-inequality) *Let $f : \mathbb{C} \to \mathbb{C}$ be the principal solution of a Beltrami equation*

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \qquad |\mu(z)| \leq k\, \chi_{\mathbb{D}}(z), \qquad 0 \leq k < 1, \tag{3.1}$$

in particular, conformal outside the unit disk $\mathbb{D}$. Then, for all exponents $2 \leq p \leq 1 + \frac{1}{k}$, we have

$$\int_{\mathbb{D}} \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|} \right) (|f_z(z)| + |f_{\bar{z}}(z)|)^p \, dz \leq \pi. \tag{3.2}$$

Equality occurs for some fairly general piecewise radial mappings discussed later on, see Chapter 6.

At first sight, one might think that this theorem only provides a cumbersome integral bound. However, it turns out that this result is highly meaningful in a number of different directions. Let us analyse them.

First remark, if one does the algebra for (3.2) to get rid of the denominator, and taking into account the Beltrami equation (3.1), one discovers the Burkholder functional (2.1). Equivalently,

$$\int_{\mathbb{D}} \mathbf{B}_p(Df) \leq \pi = \int_{\mathbb{D}} \mathbf{B}_p(\mathrm{Id}).$$

What this says, roughly, is that the Burkholder functional is quasiconcave at the identity, in the direction of differentials of principal solutions of a Beltrami equation, and for a certain range of exponents p.

Second remark, (3.2) is written in a weighted integral form: The weight

$$1 - \frac{p|\mu(z)|}{1+|\mu(z)|} \geq 0$$

is nonnegative thanks to the condition $p \leq 1 + \frac{1}{k}$. Moreover, the only instance when the weight degenerates to zero is at the endpoint $p = 1 + \frac{1}{k}$ and at those points where $|\mu(z)| = k$. Observe, at the same time, that $|Df| = |f_z(z)|+|f_{\bar{z}}(z)|$ (see Remark 2.1). These facts give (3.2) the nature of a weighted L^p bound. For $p < 1 + \frac{1}{k}$, the weight is bounded below by a positive constant, yielding the L^p integrability of Df (this was, however, already established in [2]). On the other hand, at the endpoint, we cannot do better than the weighted L^p integrability. In addition, in this direction, the authors get a sharp bound for such L^p bounds, see Corollary 3.6.

Third remark, one can also discover interesting sharp inequalities when taking the limits of p towards the endpoints. Besides, it is possible to extend some of the results to a larger range of values for p (like for some negative values) relying on a duality principle for functionals acting on matrices.

Fourth remark, the result is sharp in the sense that there exists a class of functions for which equality in the bound (3.2) is attained. Because of the sharpness, one can conclude that such class of functions constitute local maxima for the Burkholder functional. We discuss this fact deeper in Chapter 6.

3.2 Consequences

Let us make precise all these remarks in the shape of theorems.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and denote by $\mathrm{Id} : \Omega \to \mathbb{R}^2$ the identity map. Assume that $f \in \mathrm{Id} + C_0^\infty(\Omega)$ satisfies $\mathbf{B}_p(Df(x)) \geq 0$ for $x \in \Omega$. Then,*

$$\int_\Omega \mathbf{B}_p(Df)\,dx \leq \int_\Omega \mathbf{B}_p(\mathrm{Id})\,dx = |\Omega|, \qquad p \geq 2, \tag{3.3}$$

or, written explicitly,

$$\int_\Omega \left(\frac{p}{2}J(z,f) + \left(1 - \frac{p}{2}\right)|Df|^2\right)|Df|^{p-2} \leq |\Omega|. \tag{3.4}$$

This is, the Burkholder functional is quasiconvex at the identity in those directions that keep the integrand nonnegative. Unpacking this last inequality, $\mathbf{B}_p(Df(x)) \geq 0$, the equivalent condition $|Df(z)|^2 \leq \frac{p}{p-2}J(z,f)$ pops up. This asks nothing else but that f be $\frac{p}{p-2}$-quasiconformal (see (2.2)). Therefore, one can rewrite Theorem 3.2 in the language of quasiconformal maps, instead of demanding for nonnegative integrands:

Theorem 3.3 *Let $f : \Omega \to \Omega$ be a K-quasiconformal map of a bounded open set $\Omega \subset \mathbb{C}$ onto itself, extending continuously up to the boundary, where it coincides with the identity map* $\mathrm{Id}(z) = z$*. Then,*

$$\int_\Omega \mathbf{B}_p(Df)\,dz \le \int_\Omega \mathbf{B}_p(\mathrm{Id})\,dz = |\Omega|, \qquad \text{for all } 2 \le p \le \frac{2K}{K-1}. \tag{3.5}$$

Further, the equality occurs for a class of (expanding) piecewise radial mappings discussed in Chapter 6.

When taking the limit $p \to 2$ in the bound of Theorem 3.2, we reach the following sharp $L \log L(\Omega)$ bound. Such local integrability was already known, see [9] and Theorem 8.6.1 in [14]. The novelty is the sharpness.

Corollary 3.4 *Given a bounded domain $\Omega \subset \mathbb{C}$ and a homeomorphism $f : \Omega \to \Omega$ such that*

$$f(z) - z \in W_0^{1,2}(\Omega),$$

we then have

$$\int_\Omega \left(1 + \log|Df(z)|^2\right) J(z,f)\,dz \le \int_\Omega |Df(z)|^2\,dz. \tag{3.6}$$

Equality occurs for the identity map, as well as for a number of piecewise radial mappings discussed in Chapter 6.

On the other hand, the limit $p \to \infty$ computed on Theorem 3.2 leads to this other corollary.

Corollary 3.5 *Denote by $\mathbf{S}$ the Beurling-Ahlfors operator (defined in* (1.5)*) and assume that μ is a measurable function with $|\mu(z)| \le \mathbb{1}_{\mathbb{D}}(z)$ for every $z \in \mathbb{C}$. Then,*

$$\int_{\mathbb{D}} (1 - |\mu(z)|) e^{|\mu(z)|} |\exp(\mathbf{S}\mu(z)|\,dz \le \pi. \tag{3.7}$$

Equality occurs for an extenive class of piecewise radial mappings discussed in Chaper 6.

A partial result for the integrability of $\exp(\mathrm{Re}\mathbf{S}\mu)$ had already been known from [2].

With regards to L^p integrability, the following gives a sharp $W^{1,p}$ bound for quasiconformal maps on bounded domains.

Corollary 3.6 *Suppose $\Omega \subset \mathbb{C}$ is any bounded domain and $f : \Omega \to \Omega$ is a K-quasiconformal mapping, continuous up to $\partial\Omega$, with $f(z) = z$ for $z \in \partial\Omega$. Then,*

$$\frac{1}{|\Omega|} \int_\Omega |Df(z)|^p\,dz \le \frac{2K}{2K - p(K-1)}, \qquad \text{for } 2 \le p < \frac{2K}{K-1}. \tag{3.8}$$

The estimate holds as an equality for $f(z) = z|z|^{1/K-1}$, $z \in \mathbb{D}$, as well as for a family of more complicated maps described in Chapter 6.

Corollary 3.6 is a direct consequence of Theorem 3.3: for any $p < \frac{2K}{K-1}$, we can bound the weight (in this case, not degenerate) pointwise, using $k = \frac{K-1}{K+1}$ as

$$\left(1 - \frac{p|\mu(z)|}{1+|\mu(z)|}\right) \geq \left(1 - \frac{pk}{1+k}\right) = 1 - \frac{K-1}{2K}p = \frac{2K - p(K-1)}{2K}.$$

Rearranging terms gives the desired inequality.

If the goal is to assume minimal regularity on the mappings f, a satisfactory result is the following, concerning maps of integrable distortion.

Corollary 3.7 *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and suppose $h \in W^{1,2}(\Omega)$ is a homeomorphism $h : \overline{\Omega} \to \overline{\Omega}$ such that $h(z) = z$ for $z \in \partial\Omega$. Assume h satisfies the distortion inequality*

$$|Dh(z)|^2 \leq K(z)J(z,h), \qquad \textit{a.e. in } \Omega,$$

where $1 \leq K(z) < \infty$ almost everywhere in Ω. The smallest such function, denoted by $K(z,h)$, is assumed to be integrable. Then

$$2\int_\Omega [\log|Dh| - \log J(z,h)]\,dz \leq \int_\Omega [K(z,h) - J(z,h)]\,dz. \tag{3.9}$$

In particular, $\log J(z,h)$ *is integrable. Again there is a wealth of functions, to be described in Chapter 6, satisfying* (3.9) *as an identity.*

Of course, the smallest K function for each h in this corollary is

$$K(z,h) = \begin{cases} \frac{|Dh(z)|^2}{J(z,h)}, & \text{if } J(z,h) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

In order to make the most of these estimates, there is a way of extending the definition of the Burkholder functional and its integrability properties to a larger range of exponents p. As usual, the way to achieve this is by some sort of duality argument.

In Definition 1.1, the different types of convexity properties are stated for functionals $f : \mathbb{R}^{m\times n} \to (-\infty, +\infty]$. One can of course reduce the domain of these functionals in order to study such convexity properties in smaller regions of the domain. Particularising to our two-dimensional setting, let $\mathcal{O} \subset \mathbb{R}^{2\times 2}$ be an open subset of the space of 2×2 matrices. A functional f is said to be rank-one convex on $\mathcal{O}$ if, for any $A \in \mathcal{O}$ and any rank-one matrix $X \in \mathbb{R}^{2\times 2}$, $f(A+tX)$ is convex as a function of $t \in \mathbb{R}$, for t close to 0. In the same fashion, f is quasiconvex in $\mathcal{O}$ if it is quasiconvex for maps $f = A + C_0^\infty(\Omega)$ such that $Df(z) \in \mathcal{O}$ for all $z \in \Omega$.

Given a functional on matrices $E : \mathbb{R}^{2\times 2} \to \mathbb{R}$, define its inverse functional as

$$\hat{E}(A) := E(A^{-1})\det(A), \qquad \text{for } A \in \mathcal{O} = \mathbb{R}^{2\times 2}_+. \tag{3.10}$$

Notice that the domain of the inverse functional is the space of 2×2 matrices with positive determinant, so that the definition makes sense. Compute

$$\hat{\hat{E}}(A) = \hat{E}(A^{-1}) \det(A) = E((A^{-1})^{-1}) \det(A^{-1}) \det(A) = E(A),$$

thus in this sense, $\hat{E}$ is the inverse of E. As a side note, the map $E \to \hat{E}$ is sometimes referred to as the Shield transform. Realise that, given two domains Ω, Ω' and two diffeomorphism between them, $d_1 : \Omega \to \Omega'$ and $d_2 : \Omega' \to \Omega$, then it holds

$$\int_\Omega E(Df)\, dx = \int_{\Omega'} E((Dg)^{-1}) \det(Dg)\, dx' = \int_{\Omega'} \hat{E}(Dg)\, dx'$$

by the change of variables formula. This constitutes another manifestation of the duality nature of $\hat{E}$.

Recall the definition of the Burkholder functional (1.4):

$$\mathbf{B}_p(A) := \left(\frac{p}{2} \det A + \left(1 - \frac{p}{2}\right) |A|^2\right) |A|^{p-2}, \qquad p \geq 1,$$

this time written for all $p \geq 1$, not only $p \geq 2$ (which still makes sense although it is not compatible with the results presented above). Let us compute its inverse transform, for $A \in \mathbb{R}^{2\times 2}_+$.

$$\begin{aligned}\hat{\mathbf{B}}_p(A) &:= \left(\frac{p}{2} \det A^{-1} + \left(1 - \frac{p}{2}\right) |A^{-1}|^2\right) |A^{-1}|^{p-2} \det A \\ &= \left(\frac{p}{2}(\det A)^{-1} + \left(1 - \frac{p}{2}\right) (|A|(\det A)^{-1})^2\right) (|A|(\det A)^{-1})^{p-2} \det A \\ &= \left(\frac{p}{2} \det A + \left(1 - \frac{p}{2}\right) |A|^2\right) |A|^{p-2} (\det A)^{1-p}.\end{aligned}$$

We used that, for $A \in \mathbb{R}^{2\times 2}_+$, $|A|^{-1} = |A|(\det A)^{-1}$, which is easy to prove by means of the singular value decomposition. Let now $q := p - 2$ and rewrite

$$\hat{\mathbf{B}}_{p(q)}(A) = \left(\left(1 - \frac{q}{2}\right) \det A + \frac{q}{2}|A|^2\right) |A|^{-q} (\det A)^{q-1}. \tag{3.11}$$

Therefore, we extend the definition of the Burkholder functional to all of the real values of p, by convenience, by

$$\mathbf{B}_p(A) = \left(\frac{p}{2}|A|^2 + \left(1 - \frac{p}{2}\right) \det A\right) |A|^{-p} (\det A)^{p-1}, \qquad \text{for } p \leq 1,\ \det A > 0. \tag{3.12}$$

This functional and the Burkholder functional (for $p \geq 1$) are alike, because they are p-homogeneous and continuous on p. Moreover, accounting for the previous computation and the involutivity of $\hat{E}$, we have that $\hat{\mathbf{B}}_p = \mathbf{B}_q$ if $p + q = 2$. This fact opens a door towards the extension of the quasiconcavity of the Burkholder functional to negative exponents. But first, let us remark the following results.

Firstly, in [7] it is shown that the inverse functional preserves rank-one convexity, quasiconvexity and polyconvexity. Secondly, because of this, the following proposition on the rank-one convexity of the Burkholder functional holds.

Proposition 3.8 *In the space of matrices* $A \in \mathbb{R}^{2\times 2}_{+}$*,*

$$A \mapsto \mathbf{B}_p(A) \ \textit{is} \ \begin{cases} \textit{rank-one convex} & \textit{if } 0 \le p \le 2, \\ \textit{null-Lagrangian} & \textit{if } p = 0 \ \textit{or } p = 2, \\ \textit{rank-one concave} & \textit{if } p \le 0 \ \textit{or } p \ge 2. \end{cases} \tag{3.13}$$

The proof for the case $p \ge 1$ is due to Burkholder [8]. If $p = 2$, $\mathbf{B}_p(A) = \det A$, which is well-known to be null-Lagrangian[1] [17]. For $p = 0$, $\mathbf{B}_p(A) = 1$. The rest of the cases follow from the inverse transformation relation.

With regards to the paper we are studying, by an application of the inverse transform acting on the Burkholder functional as in Theorem 3.3, and keeping track of the transformed exponents $p + q = 2$, one directly gets the following theorem.

Theorem 3.9 *Let* $f : \Omega \to \Omega$ *be a* K*-quasiconformal map of a bounded open set* $\Omega \subset \mathbb{C}$ *onto itself, extending continuously up to the boundary, where it coincides with the identity map* $\mathrm{Id}(z) = z$*. Then,*

$$\int_\Omega \mathbf{B}_p(Df)\, dz \le \int_\Omega \mathbf{B}_p(\mathrm{Id})\, dz, \qquad \textit{for all } -\frac{2}{K-1} \le p \le 0. \tag{3.14}$$

Further, the equality occurs for a class of (compressing) piecewise radial mappings discussed in Chapter 6.

Of course, this theorem may look weird since L^p spaces are not Banach spaces for $p < 1$. Still, bounds of the kind in Theorem 3.9 give us information on the rate at which the integrands approach 0.

[1] We say that a functional is null-Lagrangian if it is both quasiconcave and quasiconvex. Namely, inequality (1.3) holds with equality for all smooth compactly supported functions, which in particular means that the integral $\int_U f(F + Du)$ depends only on the boundary values, i.e., on F. The name "null-Lagrangian" comes from the fact that the Euler-Lagrange equations stemming from such functionals are satisfied for all test functions.

Chapter 4

Proof of the Main Theorem

4.1 Interpolation lemma

Notice that Theorem 3.1 claims a certain integral estimate for a range of exponents p. Therefore, it is a suitable strategy to use an interpolation theorem. Instead of invoking one of such well-established theorems, the authors come up with a new result which resembles the Riesz-Thorin interpolation theorem for analytic families of operators. However, in comparison, the paper's interpolation result allows for a very weak hypothesis for one of the endpoint exponents, with the trade-off of requiring that the analytic family of functions be nonvanishing in the following sense.

Consider two domains, $\Omega, U \subset \mathbb{C}$ and a family of functions $f_\lambda \colon \Omega \times U \to \mathbb{C}$. This is, for every given $\lambda \in U$, $f_\lambda(z)$ is a complex-variable function of $z \in \Omega$. At the same time, for fixed $z \in \Omega$, we demand that $f_\lambda(z)$ is an analytic function of $\lambda \in U$ (this is what we call an analytic family of functions hereinafter).

Definition 4.1 *Let f_λ be an analytic family on some pair of domains $\Omega \times U$. We say that the family f_λ is nonvanishing if*

$$f_\lambda(z) \neq 0, \qquad a.e.\ z \in \Omega,\ \forall\, \lambda \in U.$$

Since there will be measure spaces other than the complex plane equipped with the Lebesgue measure, let $\mathcal{M}(\Omega, \sigma)$ denote the class of complex-valued measurable functions on the domain $\Omega \subset \mathbb{C}$ with a real measure σ on the Borelians of $\mathbb{C}$.

Having agreed on these, here is the novel interpolation result.

Lemma 4.2 (Interpolation lemma) *Let $\mathbb{H}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ be the right hand side half complex plane. Let $0 < p_0, p_1 \leq \infty$ and consider $\{\Phi_\lambda : \lambda \in \mathbb{H}_+\} \subset \mathcal{M}(\Omega, \sigma)$ to be an analytic and nonvanishing family with* *complex parameter* $\lambda \in \mathbb{H}_+$. *Assume further that for some* $a \geq 0$,

$$M_1 := \|\Phi_1\|_{p_1} < \infty \quad \textit{and} \quad M_0 := \sup_{\lambda \in \mathbb{H}_+} e^{-a\mathrm{Re}\lambda} \|\Phi_\lambda\|_{p_0} < \infty. \tag{4.1}$$

Then, letting $M_\theta := \|\Phi_\theta\|_{p_\theta}$ *with* $\dfrac{1}{p_\theta} = (1-\theta) \cdot \dfrac{1}{p_0} + \theta \cdot \dfrac{1}{p_1}$, *we have for every* $0 < \theta < 1$,

$$M_\theta \leq M_0^{1-\theta} M_1^\theta < \infty. \tag{4.2}$$

Remark 4.3 *The nonvanishing condition cannot be removed. This can be seen by taking* $a = 0$, $p_0 = 1$, $p_1 = \infty$ *and the family* $\Phi_\lambda(z) = \left(\frac{1-\lambda}{1+\lambda}\right) g(z)$, *with* $\lambda \in \mathbb{H}_+$ *and* $z \in \Omega$. *We demand* $g \in L^1(\Omega, \sigma) \setminus \left(\bigcup_{p>1} L^p(\Omega, \sigma)\right)$. *Under these choices,* Φ_λ *is an analytic family which vanishes at least at* $\lambda = 1$. *We also have* $M_0 = \sup_{\lambda \in \mathbb{H}_+} \|g\|_1 < \infty$, $M_1 = \|0 \cdot g\|_\infty = 0$. *However, since* $p_\theta = \frac{1}{1-\theta} > 1$, $M_\theta = \frac{1-\theta}{1+\theta} \|g\|_{p_\theta} = +\infty$. *This violates inequality* (4.2).

Although the format of Lemma 4.2 makes it easy to compare it with the most standard version of analytic Riesz-Thorin [18], it will be more handy for us to translate the given lemma from $\mathbb{H}_+$ to the disk $\mathbb{D}$.

Lemma 4.4 (Interpolation lemma for the disk) *Let* $0 < p_0, p_1 \leq \infty$ *and* $\{\Phi_\lambda : |\lambda| < 1\} \subset \mathcal{M}(\Omega, \sigma)$ *be an analytic and nonvanishing family with complex parameter in the unit disk* $\lambda \in \mathbb{D}$. *Suppose*

$$M_0 := \|\Phi_0\|_{p_0} < \infty, \quad M_1 := \sup_{|\lambda|<1} \|\Phi_\lambda\|_{p_1} < \infty \quad \textit{and} \quad M_r := \sup_{|\lambda|=r} \|\Phi_\lambda\|_{p_r}, \tag{4.3}$$

where $\dfrac{1}{p_r} = \dfrac{1-r}{1+r} \cdot \dfrac{1}{p_0} + \dfrac{2r}{1+r} \cdot \dfrac{1}{p_1}$. *Then, for every* $0 \leq r < 1$, *we have*

$$M_r \leq M_0^{\frac{1-r}{1+r}} M_1^{\frac{2r}{1+r}} < \infty. \tag{4.4}$$

It is in fact easy to switch between Lemma 4.2 and Lemma 4.4: just apply the following Möbius transformation $\varphi : \mathbb{H}_+ \to \mathbb{D}$ to the analytic parameter domain:

$$\varphi(\lambda) = \frac{1-\lambda}{1+\lambda}. \tag{4.5}$$

One quickly checks that its inverse $\varphi^{-1} : \mathbb{D} \to \mathbb{H}_+$ takes the same shape,

$$\varphi^{-1}(w) = \frac{1-w}{1+w}. \tag{4.6}$$

The following elementary computations show that the domains are well defined. For $\lambda \in \mathbb{H}_+$ and $w \in \mathbb{D}$,

$$|\varphi(\lambda)|^2 = \frac{1 + |\lambda|^2 - 2\mathrm{Re}\,\lambda}{1 + |\lambda|^2 + 2\mathrm{Re}\,\lambda} < 1,$$

$$\mathrm{Re}\,\{\varphi^{-1}(w)\} = \mathrm{Re}\,\frac{1-w}{1+w} \cdot \frac{1+\bar{w}}{1+\bar{w}} = \frac{1-|w|^2}{|1+w|^2} > 0.$$

The point of using a Möbius transformation of a domain in the complex plane is that these kind of transformations are conformal and preserve analyticity. It is straightforward to check that, up to renaming of the exponents, the transformation (4.5) applied to λ in Lemma 4.2 yields Lemma 4.4. Notice, in particular, that 0 lands on 1 and vice versa, by φ.

Let us move on to the proof of Lemma 4.2.

Proof (of Lemma 4.2) By normalising the family f_λ to $e^{-a\lambda} f_\lambda / M_0$, we may assume $M_0 = 1$ and $a = 0$. We may also restrict ourselves to the case $\sigma(\Omega) < \infty$[1].

The general idea of the proof is, first, to get the inequality from, first, a convex function-linear function inequality from the exponents to the norms of the analytic family, and second, to embed the linear function into a family of analytic functions that allows the usage of Harnack's inequality.

For the sake of cleanliness, we first assume $0 < p_0, p_1 < \infty$ and that there exists a constant $A > 1$ such that

$$\frac{1}{A} \leq |\Phi_\lambda(x)| \leq A, \qquad \text{a.e. } z \in \Omega, \forall\, \lambda \in \mathbb{H}_+, \tag{4.7}$$

and then get rid of these hypothesis at the end of the proof.

Consider $\theta \in (0,1)$ and the function $\frac{1}{p} \to \log\|\Phi_\theta\|_p$, which is seen to be convex if one takes logarithm to the standard Riesz-Thorin interpolation inequality. We wish to find the support line of this convex function at $\frac{1}{p_\theta}$. This is, find a slope I and an offset $u_\infty(\theta)$ (both independent of p) such that

$$u_p(\theta) := \frac{1}{p} I + u_\infty(\theta) \leq \log\|\Phi_\theta\|_p \quad \text{ and } \quad u_{p_\theta}(\theta) = \log\|\Phi_\theta\|_{p_\theta}. \tag{4.8}$$

See Fig. 4.1. Now, consider any new probability space $(\Omega, \mathcal{B}, \mathcal{P})$ such that $\mathcal{P}$ has a density with respect to σ, which we call $\mathcal{P}(z)$ in an abuse of notation. Jensen's inequality applied to the concave function log yields

$$\frac{1}{p} \int_\Omega \mathcal{P}(z) \log\left(\frac{|\Phi_\theta(z)|^p}{\mathcal{P}(z)}\right) d\sigma(z) \leq \log\|\Phi_\theta\|_p\,, \tag{4.9}$$

for any $0 < p < \infty$. Define the following probability density:

$$\mathcal{P}_t(z) := \frac{|\Phi_\theta(z)|^{p_\theta}}{\int_\Omega |\Phi_\theta(y)|^{p_\theta} d\sigma(y)}. \tag{4.10}$$

Note that this density is well defined thanks to our assumption on the analytic family being uniformly bounded (4.7) and the space of finite measure. Moreover,

[1] Here, the authors make no further comment. One can make such an assumption if the measure space is σ-finite and a density argument is available. Anyhow, for the application of the lemma to the main theorem, not only is this the case but also $\sigma(\Omega) < \infty$.

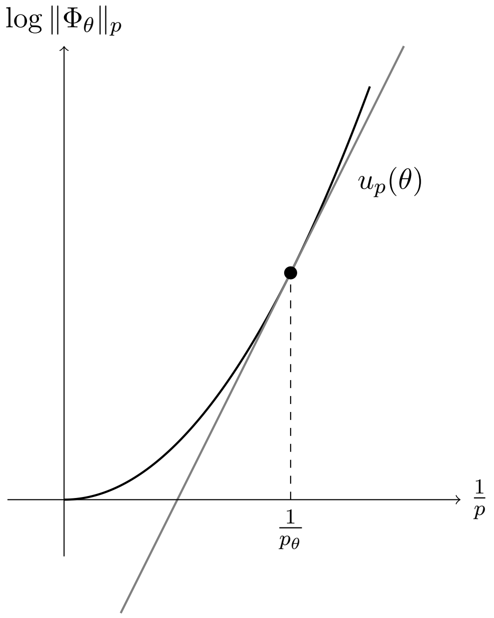


Figure 4.1: Plot depicting the properties in (4.8). Given $\theta \in (0,1)$, the black function represents $\log\|\Phi_\theta\|$ as a function of $1/p$, while the grey line is the support line (or tangent) of the previous function at $1/p_\theta$. This grey linear function corresponds to $u_p(\theta)$.

realise that the probability density $\mathcal{P}_t(z)$ that we defined (4.10) attains equality in (4.9) at the point $p = p_\theta$. Therefore, unpacking (4.9), we see that the right choices are

$$I := \int_\Omega \mathcal{P}_t(z) \log\left(\frac{1}{\mathcal{P}_t(z)}\right) d\sigma(z) \quad \text{and} \quad u_\infty(\theta) := \int_\Omega \mathcal{P}_t(z) \log|\Phi_\theta(z)| d\sigma(z). \tag{4.11}$$

Now that we have determined the support line (4.8), we upgrade $\theta \in (0,1)$ to a complex-valued $\lambda \in \mathbb{H}_+$. This does not break the inequality we obtained,

$$u_p(\lambda) := \frac{1}{p} I + u_\infty(\lambda) \leq \log\|\Phi_\lambda\|_p \qquad \text{and} \qquad u_{p_\theta}(\theta) = \log\|\Phi_\theta\|_{p_\theta}. \tag{4.12}$$

Note the following smart trick: the dependency on λ of $u_p(\lambda)$ relies only on the offset $u_\infty(\lambda)$, which in turn entails the dependency as the logarithm of the modulus of the analytic function. Importantly, thanks to the nonvanishing condition for the family Φ_λ (which is analytic in λ), $\log|\Phi_\lambda|$ is a harmonic function (because the real logarithm is), and so the entire family $u_p(\lambda)$ is harmonic in λ. Accounting for our assumption that $M_0 = 1$, for $p = p_0$ we have $u_{p_0}(\lambda) \leq \log\|\Phi_\lambda\|_{p_0} \leq \log M_0 = 0$, $\forall\, \lambda \in \mathbb{H}_+$. This enables the application of Harnack's inequality for nonpositive harmonic functions in $\mathbb{H}_+$, which for $\theta \in (0,1)$ takes the form[2]

$$u_{p_0}(\theta) \leq \theta u_{p_0}(1) \quad \text{for } \theta \in (0,1). \tag{4.13}$$

[2] A reference for Harnack's inequality on the disk is [15]. For a nonpositive harmonic

The inequalities we elaborated lead to

$$\log\|\Phi_\theta\|_{p_\theta} \overset{(1)}{=} u_{p_\theta} \overset{(2)}{=} u_{p_0}(\theta) + \left(\frac{1}{p_\theta} - \frac{1}{p_0}\right) I$$
$$\overset{(3)}{\leq} \theta u_{p_0}(1) + \theta \left(\frac{1}{p_1} - \frac{1}{p_0}\right) I \overset{(4)}{=} \theta u_{p_1}(1) \overset{(5)}{\leq} \theta \log M_1.$$

Equality (1) holds by the contact point of the support line, equality (2) by linearity of the latter, step (3) by Harnack's inequality and the definition of $\frac{1}{p_\theta}$ in the statement of the lemma. Step (4) is again linearity of the support line, and finally the bound (5) is by the convexity inequality (4.8). After taking exponentials and recalling our assumption $M_0 = 1$, proves the reduced case.

We can quickly deal with the case $p_0 = \infty$ or $p_1 = \infty$ by normalising the measure space $\sigma(\Omega) = 1$. In this case, for $f \in L^\infty(\Omega)$, the function $p \to \|f\|_p$ from $\mathbb{R} \to \mathbb{R}$ is increasing. Since it is also bounded by $\|f\|_\infty$, it has a limit, which can be no other that $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$. Hence, inequality (4.2) for $p = \infty$ follows by the finite exponent case after taking the limit $p \to \infty$ in the mentioned inequality.

Lastly, we get rid of the uniform boundedness assumption (4.7). For every $k \in \mathbb{N}$, let

$$\Omega_k := \{z \in \Omega : |\Phi_\lambda(z)| \in [1/k, k],\ \forall\, \lambda \in \mathbb{H}_+\}. \tag{4.14}$$

These sets are measurable[3] and fill the full domain $\Omega = \bigcup_k \Omega_k$. Since for every $k \in \mathbb{N}$,

$$\Phi_\lambda(z), \qquad z \in \Omega_k,\ \lambda \in \mathbb{H}_+$$

satisfies the uniform bound (4.7) almost everywhere in Ω_k and everywhere in $\mathbb{H}_+$, we get by the reduced case that

$$\|\Phi_\lambda(z)\mathbb{1}_{\Omega_k}\|_{p_\theta} \leq M_1^\theta,$$

where we may take the limit $k \to \infty$ and use Fatou's lemma

$$\|\Phi_\lambda(z)\|_{p_\theta} \leq \liminf_{n\to\infty}\|\Phi_\lambda(z)\mathbb{1}_{\Omega_k}\|_{p_\theta} \leq M_1^\theta,$$

which concludes the full proof. □

function u on the disc, $w \in \mathbb{D}$, we have $u(w) \leq \frac{1-|w|}{1+|w|}u(0)$. Consider the Möbius transformation $\varphi : \mathbb{H}_+ \to \mathbb{D}$ from (4.5) and write, for $\varphi(\lambda) = w$, $u(\varphi(\lambda)) \leq \varphi(|w|)u(\varphi(\lambda))$. Considering the new nonpositive harmonic function $v = u \circ \varphi$ gives $v(\lambda) \leq \varphi(|w|)v(\lambda)$. Taking $\lambda = \theta \in (0, 1)$ forces $w \in \mathbb{R}$ and so by the previous study on φ, $\varphi(|w|) = \theta$ yieds.

[3] For given $\lambda \in \mathbb{H}_+$, $\{z \in \Omega : |\Phi_\lambda(z)| \in [1/k, k]\}$ is measurable because it is the preimage of a closed set by a continuous function. Since Φ_λ is continuous in λ, one can intersect the previous sets over $\lambda \in \mathbb{H}_+ \cap (\mathbb{Q} \times \mathbb{Q})$ to recover Ω_k (taking $(1/k, k)$ open instead of closed, if necessary).

4.2 Proof of the Sharp L^p inequality

Once proven this lemma, we drive towards the proof of the main theorem, Theorem 3.1. One may appreciate that the proof of the lemma was elegant and neat, and so is the proof of the main theorem. However, unlike the argument for the proof of the lemma, which was also clear and explainable in words, the proof of the theorem is rather mysterious. Certainly, the logic works. Nonetheless, one usually struggles to filter the main ideas and to extract the narrative.

Proof (of Theorem 3.1) As we were saying in advance, we are shooting for applying the interpolation lemma to a suitable analytic family of functions. To grant the bound for the "hard" endpoint, we rely on the area inequality. The most tricky part of the proof is the choice of the analytic family, which surely required tons of inspiration from the authors.

To begin with, we can assume $\mu \in C^\infty(\mathbb{D})$ by smooth approximation, Theorem 2.5, and a direct bound involving Fatou's lemma on the final integral bound (3.2). Fix $2 \leq p \leq 1 + 1/k$, k being the essential supremum of μ. We consider a first analytic family of dilation functions, and their corresponding principal solutions, assuming the following shape:

$$F^\lambda_{\bar{z}} = \mu_\lambda(z) F^\lambda_z, \qquad \mu_\lambda = \tau_\lambda(z) \cdot \frac{\mu(z)}{|\mu(z)|}. \tag{4.15}$$

We will unmask τ_λ later on, which is of course going to be analytic in λ and bounded by 1, $|\tau_\lambda(z)| < 1$ so that the Beltrami equation admits a unique principal solution F^λ. Besides, we also have to choose appropriate exponents to which apply the interpolation lemma on the disk. Taking into account that the targeted range is $2 \leq p \leq 1 + 1/k$, and that the relation between exponents in Lemma 4.4 is

$$\frac{1}{p_r} = \frac{1-r}{1+r} \cdot \frac{1}{p_0} + \frac{2r}{1+r} \cdot \frac{1}{p_1},$$

it looks reasonable to choose

$$p_0 = \infty, \qquad p_1 = 2,$$

so that

$$p_r = 1 + \frac{1}{r}.$$

This way, since we want to match

$$M_r = \sup_{|\lambda|=r} \|\Phi_\lambda\|_{p_r}$$

with the integral to be bounded (3.2), we should strive for enabling $p_r = p$ (possible since $0 \leq r < 1$ gives $2 < p < \infty$) and $r = \lambda_0$ such that $|\Phi_{\lambda_0}| = |Df|$.

Having chosen the values for the exponents, it would also be helpful to give a privilege to the values of the analytic parameter $\lambda \in \mathbb{D}$ that appear in the hypotheses of Lemma 4.4. First, for $\lambda = 0$, we choose $\tau_0(z) \equiv 0$, yielding $\mu_\lambda = 0$ and thus $F^0_{\bar{z}} = 0$. By uniqueness of principal solutions (Theorem 2.3), $F^0 = z$. For the special point λ_0 pointed out above, we need to recover $f = F^{\lambda_0}$, meaning that we have to take $\tau_{\lambda_0}(z) = |\mu(z)|$ so that at λ_0 we recover $\mu_{\lambda_0} = \mu$ as well as the principal solution in consideration.

Define $\varphi : \mathbb{D} \to \{\mathrm{Re}(z) < \frac{1}{2}\}$ to be the following Möbius transformation:

$$\varphi(z) = \frac{z}{1+z},$$

which is present in the weight of the target inequality. Indeed, φ is well defined: For $z \in \mathbb{D}$,

$$\begin{aligned}\mathrm{Re}(\varphi(z)) = \mathrm{Re}\left(\frac{z}{1+z}\cdot\frac{1+\bar{z}}{1+\bar{z}}\right) &= \frac{|z|^2+\mathrm{Re}(z)}{1+|z|^2+2\mathrm{Re}(z)} \\ &= \frac{|z|^2+\mathrm{Re}(z)}{(1-|z|^2)+2|z|^2+2\mathrm{Re}(z)} < \frac{|z|^2+\mathrm{Re}(z)}{2|z|^2+2\mathrm{Re}(z)} = \frac{1}{2}.\end{aligned}$$

Conversely, the inverse of the Möbius transformation is $\varphi^{-1}(\omega) = \frac{\omega}{1-\omega}$. For $\mathrm{Re}(\omega) < \frac{1}{2}$,

$$|\varphi^{-1}(\omega)|^2 = \frac{|\omega|^2}{|1-\omega|^2} = \frac{|\omega|^2}{1+|\omega|^2-2\mathrm{Re}(\omega)} < \frac{|\omega|^2}{|\omega|^2} = 1.$$

We have proved that φ is invertible form $\mathbb{D}$ to $\{\mathrm{Re}(z) < \frac{1}{2}\}$. For the analytic family of dilation functions, we choose

$$\mu_\lambda(z) = \tau_\lambda(z)\cdot\frac{\mu(z)}{|\mu(z)|}, \quad \text{with } \varphi(\tau_\lambda(z)) = p\cdot\varphi(|\mu(z)|)\cdot\varphi(\lambda). \tag{4.16}$$

Compute

$$\begin{aligned}\mathrm{Re}\left(p\cdot\varphi(|\mu(z)|)\cdot\varphi(\lambda)\right) &= p\cdot\varphi(|\mu(z)|)\cdot\mathrm{Re}(\varphi(\lambda)) \\ &< \left(\frac{1+k}{k}\right)\cdot\left(\frac{|\mu(z)|}{1+|\mu(z)|}\right)\cdot\frac{1}{2} \leq \frac{1}{2}.\end{aligned}$$

We used the fact that $|\mu(z)| \leq k$ and that φ is an increasing function, as a function of one real variable. We have seen that (4.16) uniquely determines $\tau_\lambda(z)$, for $\lambda \in \mathbb{D}$, $z \in \mathbb{D}$ and moreover, $|\tau_\lambda(z)| < \mathbb{1}_{\mathbb{D}}(z)$ as required. Furthermore, by the early smooth approximation reduction, $\mu_\lambda \in C^\infty(\mathbb{C}) \subset C^\alpha(\mathbb{C})$ is Hölder continuous for all $0 < \alpha \leq 1$. By Theorem 2.4, each Beltrami equation with dilation functions as in (4.16) admits a unique principal solution F^λ, which is a $C^{1,\alpha}$-diffeomorphism. Let us represent F^λ as a power series in λ in order to check that it defines an analytic function of λ. Recall that in (1.6) we mentioned that for the Beurling-Ahlfors transform,

$$\mathbf{S}F^\lambda_{\bar{z}} = F^\lambda_z - 1,$$

which, combined with the Beltrami equation, gives

$$(\mathrm{Id} - \mu_\lambda \mathbf{S}) F^\lambda_{\bar{z}} = \mu_\lambda.$$

Since $|\mu| \le k < 1$ and the fact that $\mathbf{S}$ gives an isometry from L^2 on itself, the operator $(\mathrm{Id} - \mu\mathbf{S})$ is invertible and can be expanded in a Neumann series that converges in the $\|\cdot\|_{L^2 \to L^2}$ norm. Namely,

$$F^\lambda_{\bar{z}} = (\mathrm{Id} - \mu_\lambda \mathbf{S})^{-1} \mu_\lambda = \left(\sum_{k \ge 0} (\mu_\lambda \mathbf{S})^k \right) \mu_\lambda. \tag{4.17}$$

The dependency of the last expression on λ relies on μ_λ. From (4.16) we realise that μ_λ (as well as τ_λ) is analytic in λ because it is a double Möbius transformation of such parameter. Therefore, input the Neumann series, we conclude the analyticity of $F^\lambda_{\bar{z}}$, and thus the one of F^λ and F^λ_z. Importantly, we have

$$|F^\lambda_z|^2 \ge |F^\lambda_z|^2 - |F^\lambda_{\bar{z}}|^2 = J(z, F^\lambda) > 0$$

in the entire domain, since F^λ is a diffeomorphism and so its jacobian does not vanish. All in all, F^λ_z defines an analytic nonvanishing family. Not only this, also $\tau_0(z) \equiv 0$ hence $F^0(z) = z$; and $\tau_{\lambda_0} = |\mu(z)|$ (since we chose $p = p_r = 1 + \frac{1}{\lambda}$), implying $\mu_{\lambda_0} = \mu$ and $F^{\lambda_0} = f$.

F^λ_z is not yet be our definite choice for analytic nonvanishing family, but instead

$$\Phi_\lambda(z) := F^\lambda_z (1 + \tau_\lambda(z)), \tag{4.18}$$

which also possesses such properties since $\tau_\lambda(z) < 1$ and since τ_λ is analytic in λ, as justified above. This choice gives us the following pleasant equality for $\lambda = \lambda_0$.

$$|\Phi_{\lambda_0}(z)| = |f_z|(1 + |\mu(z)|) = |f_z| + |f_{\bar{z}}| = |Df| \tag{4.19}$$

by Remark 2.1. Once the differential of the principal solution f has been recovered, let us choose an appropriate measure space so that we account for the weight in (3.2). Consider the space of measurable functions on the disk $\mathcal{M}(\mathbb{D}, \sigma)$ endowed with the measure

$$d\sigma(z) := \frac{1}{\pi} \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|} \right) dz,$$

which is well defined (this is, nonnegative) thanks to the bound $|\mu| \le k$ a.e. and the range $2 \le p \le 1 + 1/k$.

Let us first check the endpoint of $\lambda = 0$ and M_0. In such case, recall that $\tau_0 \equiv 0$ and $F^0(z) = z$, so $F^0_z = 1$. This gives $\Phi_\lambda \equiv 1$, reaching $M_0 = \|\Phi_0\|_{p_0} = \|\Phi_0\|_\infty = 1 < \infty$.

Let us elaborate more in order to bound M_1. Compute

$$\begin{aligned} J(z,F^\lambda) &\overset{(1)}{=} |F^\lambda_z|^2 - |F^\lambda_{\bar z}|^2 = |F^\lambda_z|^2(1-|\mu_\lambda(z)|^2) \\ &\overset{(2)}{=} |\Phi_\lambda(z)|^2 \cdot \frac{1-|\tau_\lambda|^2}{|1+\tau_\lambda|^2} \overset{(3)}{=} |\Phi_\lambda(z)|^2 \left(1 - 2\mathrm{Re}\frac{\tau_\lambda(z)}{1+\tau_\lambda(z)}\right) \\ &= |\Phi_\lambda(z)|^2 \left(1 - p\frac{|\mu(z)|}{1+|\mu(z)|}\mathrm{Re}\frac{2\lambda}{1+\lambda}\right) \overset{(4)}{\geq} |\Phi_\lambda(z)|^2 \left(1 - p\frac{|\mu(z)|}{1+|\mu(z)|}\right). \end{aligned}$$

(1) is due to Remark 2.1, (2) is by definition of Φ_λ, (3) because of the following algebraic identity

$$\frac{1-|z|^2}{|1+z|^2} = 1 - 2\mathrm{Re}\frac{z}{1+z},$$

and (4) thanks to $\lambda \in \mathbb{D}$ and the previous study of the Möbius transformation φ. Altogether,

$$|\Phi_\lambda(z)|^2 d\sigma(z) \leq \frac{1}{\pi} J(z,F^\lambda)dz, \qquad \forall\, \lambda \in \mathbb{D}.$$

By a direct application of the area inequality, Lemma 2.6, we deduce

$$M_1 = \sup_{|\lambda|<1} \|\Phi_\lambda\|_2 \leq \sup_{|\lambda|<1} \left(\int_{\mathbb{D}} \frac{1}{\pi} J(z,F^\lambda)dz\right)^{\frac{1}{2}} = \sup_{|\lambda|<1} \left(\int_{F^\lambda(\mathbb{D})} \frac{1}{\pi} dz\right)^{\frac{1}{2}} \leq 1 < \infty.$$

The final step is recalling the choice $r = \frac{1}{p-1} = \lambda_0$ and substituting in the interpolation inequality (4.2):

$$\|\Phi_{\lambda_0}\|_{(p_r = \frac{1+r}{r})} \leq \sup_{|\lambda|=r} \|\Phi_\lambda\|_{\frac{1+r}{r}} = M_r \leq M_0^{\frac{1-r}{1+r}} M_1^{\frac{2r}{1+r}} \leq 1,$$

or, explicitly,

$$\int_{\mathbb{D}} \left(1 - \frac{p|\mu(z)|}{1+|\mu(z)|}\right) |Df|^p \, dz = \pi \int_{\mathbb{D}} |\Phi_{\lambda_0}(z)|^{\frac{1+r}{r}} d\sigma(z) \leq \pi.$$

This concludes the proof. □

One important detail to note is that, while the consequences of the main theorem assume linear boundary conditions (Theorem 3.2 and 3.3 assume identity boundary conditions), the main theorem, Theorem 3.1 does *not*. It only assumes the mappings to be principal solutions (therefore it assumes asymptotic normalisation) and conformal outside the disk. Indeed, it is crucial in the proof of the main theorem to use nonlinear boundary conditions, as manifest in the family F^λ. In our favour, the area inequality still works in the setting of principal solutions without linear boundary conditions. Realise also, for such an inequality, the importance of working on the disk $\mathbb{D}$. All in all, to get the quasiconformality results, we increased the difficulty by working with nonlinear boundary conditions.

Chapter 5

Proofs of the consequences

Having proven the main theorem, Theorem 3.1, the consequent results discussed in Chapter 3 boil down to corollaries. It is now essentially a matter of matching hypotheses.

Proof (of Theorem 3.3) We have a K-quasiconformal map $f : \Omega \to \Omega$ that coincides with the identity on the boundary of Ω. We extend f by the identity to all of $\mathbb{C}$. Since f is quasiconformal in Ω, $f \in W^{1,2}(\Omega)$, and clearly the extension of the function satisfies $f \in W^{1,2}_{\text{loc}}(\Omega)$. Thus, if one defines a dilation function by the quotient of weak derivatives $\mu(z) := \frac{f_{\bar{z}}}{f_z}$ whenever the denominator is nonzero (and set $\mu(z) = 0$ elsewhere), by the fact that f is K-quasiconformal, one gets that $\|\mu\|_\infty := k \leq \frac{K-1}{K+1} < 1$.

Consider now the centered disk $\mathbb{D}_R$ of radius R, large enough so that $\Omega \subset \mathbb{D}_R$ (recall Ω is a bounded domain). Accounting for the extension by the identity outside Ω, we have that $\|\mu\|_\infty \leq k < 1$, that $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$, that f solves the Beltrami equation associated to μ and that f has the appropriate normalization at infinity. By Theorem 2.3, f is the unique principal solution to such equation. With this, Theorem 3.1 applies (with the appropriate scaling in order to move from $\mathbb{D}$ to $\mathbb{D}_R$). Hence,

$$\int_{\mathbb{D}_R} \mathbf{B}_p(Df)\, dz \leq \int_{\mathbb{D}_R} \mathbf{B}_p(\mathrm{Id})\, dz = |\mathbb{D}_R|. \tag{5.1}$$

However, since f is the identity in $\mathbb{D}_R \setminus \Omega$,

$$\int_{\mathbb{D}_R \setminus \Omega} \mathbf{B}_p(Df)\, dz = \int_{\mathbb{D}_R \setminus \Omega} \mathbf{B}_p(\mathrm{Id})\, dz, \tag{5.2}$$

and so subtracting (5.2) from (5.1), one reaches

$$\int_{\Omega} \mathbf{B}_p(Df)\, dz \leq \int_{\Omega} \mathbf{B}_p(\mathrm{Id})\, dz = |\Omega|, \tag{5.3}$$

which was our goal estimate. Lastly, let us check about which range of p we are speaking. From Theorem 3.1, $2 \leq p \leq 1 + \frac{1}{k}$. Substituting the relation between the norm of the dilation function and the quasiconformal parameter $k = \frac{K-1}{K+1}$, we get $2 \leq p \leq \frac{2K}{K-1}$. $\square$

Theorem 3.3 unlocks Theorem 3.2, as follows.

Proof (of Theorem 3.2) The hypotheses $\Omega \subset \mathbb{R}^2$ bounded, $f \in \mathrm{Id} + C_0^\infty(\Omega)$ and $\mathbf{B}_p(Df(x)) \geq 0$ from Theorem 3.2 imply those of Theorem 3.3: $f : \Omega \to \Omega$ K-quasiconformal with identity boundary values[1]. Let $p \geq 2$. Since the quaisconformal constant for f is $K = \frac{p}{p-2}$ (coming from $\mathbf{B}_p(Df(x)) \geq 0$) if we apply the latter theorem at the borderline $p' = \frac{2K}{K-1} = p$, we get the bound

$$\int_\Omega \mathbf{B}_p(Df)\, dz \leq \int_\Omega \mathbf{B}_p(\mathrm{Id})\, dz,$$

for all $p \geq 2$. $\square$

For the proof of Corollary 3.4, we need an approximation lemma. We use the notation for homeomorphisms $W_{id}^{1,2}(\mathbb{C})$, for those which lie in the local Sobolev space $W_{\mathrm{loc}}^{1,2}(\mathbb{C})$ and coincide with the identity mapping outside a compact set.

Lemma 5.1 (Approximation lemma, see Theorem 1 in [13]) *Given any homeomorphism $f \in W_{id}^{1,2}(\mathbb{C})$, one can find C^∞-smooth diffeomorphism $f^l \in W_{id}^{1,2}(\mathbb{C})$, $l \geq 1$, such that*

$$\|f^l - f\|_\infty + \|D(f^l - f)\|_{L^2(\mathbb{C})} \to 0, \qquad \text{as } l \to \infty.$$

Passing to a subsequence if necessary, we may ensure that $Df^l \to Df$ almost everywhere.

Proof (of Corollary 3.4) We are given a bounded domain $\Omega \subset \mathbb{C}$ and a homeomorphism f on it such that $f(z) - z \in W_0^{1,2}(\Omega)$. Extend f by the identity outside Ω to all of $\mathbb{C}$. Borrow the approximating (sub)sequence from Lemma 5.1, $f^l \to f$, and build as usual the dilation functions μ^l that give rise to each Beltrami equation, the principal solution of which are the f^l. Namely,

$$f_{\bar{z}}^l = \mu_l(z) f_z^l, \quad |\mu_l(z)| \leq k_l < 1, \;\; \mu_l(z) = 0 \text{ for } |z| \geq R, \tag{5.4}$$

for a fixed R such that the domain lies in the centered disk of radius R, $\Omega \subset \mathbb{D}_R$. Now, each of the f^l satisfy the hypotheses of Theorem 3.3, which gives us

$$\int_{\mathbb{D}_R} \mathbf{B}_p(Df^l)\, dz \leq |\mathbb{D}_R| = \int_{\mathbb{D}_R} \mathbf{B}_2(Df^l)\, dz,$$

[1]The only implication that is not clear at first is the fact that f is a homeomorphism, especially that f is injective. This fact is a result of degree theory for mappings of finite distortion, and it is well-known in the field. Theorem 3.27 in [10].

for $2 \le p \le 1 + \frac{1}{k_l}$, $\|\mu_l\|_\infty = k_l$, where we used that $\mathbf{B}_2(Df^l)(z) = J(z, f^l)$ based on (1.4), and the change of variables theorem. We rewrite the above inequality and divide by $p-2$ to prepare for taking the limit $p \to 2$.

$$\lim_{p \searrow 2} \frac{1}{p-2} \left[\int_{\mathbb{D}_R} \mathbf{B}_p(Df^l)\, dz - \int_{\mathbb{D}_R} \mathbf{B}_2(Df^l)\, dz \right] \le 0. \tag{5.5}$$

Since the integrand is continuous in both p and z, and continuously differentiable with respect to p (as the calculation will show), we may differentiate under integral sign. By elementary derivation,

$$\begin{aligned}
&\lim_{p \searrow 2} \frac{\left(\frac{p}{2} J_{f^l} + \left(1 - \frac{p}{2}\right) |Df^l|^2\right) |Df^l|^{p-2} - J_{f^l}}{p-2} \\
&\quad = \lim_{p \searrow 2} \frac{\frac{p}{2}|Df^l|^{p-2} - 1}{p-2} J_{f^l} + \lim_{p \searrow 2} \frac{2-p}{2} \frac{|Df^l|^p}{p-2} \\
&\quad = \lim_{p \searrow 0} \frac{\frac{p+2}{2}|Df^l|^{p} - 1}{p} J_{f^l} + \lim_{p \searrow 2} \frac{2-p}{2} \frac{|Df^l|^p}{p-2} \\
&\quad = \lim_{p \searrow 0} \frac{p}{2} \frac{|Df^l|^p}{p} J_{f^l} + \lim_{p \searrow 0} \frac{|Df^l|^p - 1}{p} J_{f^l} + \lim_{p \searrow 2} \frac{2-p}{2} \frac{|Df^l|^p}{p-2} \\
&\quad = \frac{J_{f^l}}{2} + \log |Df^l| J_{f^l} - \frac{|Df^l|^2}{2} = \frac{1}{2} \left((1 + \log |Df^l|^2) J_{f^l} - |Df^l|^2 \right),
\end{aligned}$$

where for the middle term we recognised the derivative of the exponential function a^p, for $a \in \mathbb{R}_+$ at $p = 0$. Introducing this calculation in (5.5), we reach

$$\int_{\mathbb{D}_R} (1 + \log |Df^l|^2) J_{f^l}\, dz \le \int_{\mathbb{D}_R} |Df^l|^2\, dz. \tag{5.6}$$

What is left is just a limiting argument to get back to the original principal solution f.

$$\begin{aligned}
&\int_{\mathbb{D}_R} (1 + \log |Df|^2) J_f\, dz \\
&\qquad = \int_{\mathbb{D}_R} (1 + \log(1 + |Df|^2)) J_f\, dz - \int_{\mathbb{D}_R} (\log(1 + |Df|^{-2})) J_f\, dz \\
&\le \liminf_{l \to \infty} \int_{\mathbb{D}_R} (1 + \log(1 + |Df^l|^2)) J_{f^l}\, dz - \lim_{l \to \infty} \int_{\mathbb{D}_R} (\log(1 + |Df^l|^{-2})) J_{f^l}\, dz,
\end{aligned}$$

where Fatou's lemma applied to the first term, and the dominated convergence theorem to the second, since the integrand is almost everywhere convergent ($|Df|$ is, by the approximation lemma) and uniformly bounded, $\log(1 + |Df^l|^{-2}) J_{f^l} \le |Df^l|^{-2} J_{f^l} \le 1$ because of Hadamard's inequality ($|\det A| \le |A|^2$ for $A \in \mathbb{R}^{2\times 2}$). Notice that Fatou's lemma for a direct bound isn't allowed since the logarithm may take negative values. On the other hand, Fatou's lemma on the second term wouldn't suffice due to the minus sign. Bringing the terms together again,

$$\begin{aligned}\int_{\mathbb{D}_R}(1+\log|Df|^2)J_f\,dz &= \liminf_{l\to\infty}\int_{\mathbb{D}_R}(1+\log|Df^l|^2)J_{f^l}\,dz\\ &\le \liminf_{l\to\infty}\int_{\mathbb{D}_R}|Df^l|^2\,dz = \int_{\mathbb{D}_R}|Df|^2\,dz, \quad (5.7)\end{aligned}$$

this last equality by the approximation lemma.

Realise that, in $\mathbb{D}_R \setminus \Omega$, the function f equals the identity map, since we extended f as such. By an analogous argument to that of (5.2), the identity map locally attains equality in (5.7), hence the bound is also true on the initial Ω,

$$\int_\Omega (1+\log|Df|^2)J_f\,dz \le \int_\Omega |Df|^2\,dz,$$

as desired. □

Corollary 3.7 follows from Corollary 3.4.

Proof (of Corollary 3.7) In [4], Theorem 9.1, they show that if $h:\Omega\to\Omega$ is an onto homeomorphism in $W^{1,2}_{\text{loc}}(\Omega)$ and h has integrable distortion $K(z,h)\in L^1(\Omega)$ for the minimal function K in the statement of the corollary, then the inverse of h, $h^{-1}:\Omega\to\Omega$ belongs to the better space $W^{1,2}(\Omega)$ and the integral identity

$$\int_\Omega |D(h^{-1})|^2\,dz = \int_\Omega K(z,h)\,dz \quad (5.8)$$

holds.

According to the previous paragraph, h^{-1} is such that $h^{-1}(z)-z\in W^{1,2}_0(\Omega)$ and a homeomorphism. Therefore, the inequality in Corollary 3.4 applies to h^{-1} and so

$$\begin{aligned}\int_\Omega\left(1+2\log\left(\frac{|Dh|}{J(z,h)}\right)\right)dz &= \int_\Omega\left(1+\log\left(|(Dh)^{-1}|^2\right)\right)dz\\ &= \int_\Omega(1+\log|D(h^{-1})|^2)J_h\,dz \le \int_\Omega|D(h^{-1})|^2\,dz = \int_\Omega K(z,h)\,dz.\end{aligned}$$

In the first equality, we used that, for $A\in\mathbb{R}^{2\times2}_+$, $|A|^{-1}=|A|(\det A)^{-1}$, just as we did when we defined the Burkholder functional for $p\le 1$. The second equality relies on a change of variables and the inverse function theorem. Rearranging terms,

$$2\int_\Omega\left(\log\left(|Dh|-\log J(z,h)\right)\right)dz = \int_\Omega(K(z,h)-1)\,dz = \int_\Omega(K(z,h)-J(z,h))\,dz,$$

follows once more by a change of variables on the integral of the function 1. This proves the corollary. Further, if we apply Hadamard's inequality ($|\det A| \leq |A|^2$, for $A \in \mathbb{R}^{2\times 2}$) on $\log |Dh|$, we reach

$$
\begin{aligned}
-2\int_\Omega \log J(z,h)\,dz \leq \int_\Omega (K(z,h)-1)\,dz - \int_\Omega \log |Dh|^2\,dz \\
\leq \int_\Omega (K(z,h)-1)\,dz - \int_\Omega \log J(z,h)\,dz,
\end{aligned}
$$

which leads to

$$
-\int_\Omega \log J(z,h)\,dz \leq \int_\Omega (K(z,h)-1)\,dz.
$$

This amounts to the integrability of $\log J(z,h)$ (since one can also bound it above by $J(z,h)-1$, which is integrable). $\square$

Directly from the main theorem, in the limit of small norm of the dilation function k, i.e. $p \to \infty$, the proof of Corollary 3.5 follows.

Proof (of Corollary 3.5) We have $|\mu(z)| \leq \mathbb{1}_{\mathbb{D}}(z)$ brought from the statement of the corollary. For $0 < \varepsilon < 1$, the Beltrami equations

$$
f_{\bar z} = \varepsilon \mu f_z, \tag{5.9}
$$

as we know, admit a unique principal solution, to which we can apply Theorem 3.1 with $k = \varepsilon$ and $p = 1 + 1/\varepsilon$ (this is, the endpoint). The outcome is

$$
\begin{aligned}
\int_{\mathbb{D}} \left(1 - \frac{1+\varepsilon}{\varepsilon}\frac{\varepsilon|\mu(z)|}{1+\varepsilon|\mu(z)|}\right) |Df(z)|^{1+\frac{1}{\varepsilon}}\,dz \\
= \int_{\mathbb{D}} \left(\frac{1-|\mu(z)|}{1+\varepsilon|\mu(z)|}\right) |Df(z)|^{1+\frac{1}{\varepsilon}}\,dz \leq \pi
\end{aligned}
\tag{5.10}
$$

Besides this latter bound, recall we can expand $f_{\bar z}$ in a Neumann series (4.17) and use the relation (1.6) to reach

$$
f_z = \sum_{k\geq 0} (\mathbf{S}(\varepsilon\mu))^k.
$$

Taking L^2 norms and using the fact that $\mathbf{S}$ defines an isometry on L^2, we get

$$
\|f_z\|_2 \leq \sum_{k\geq 0} \varepsilon^k \|(\mathbf{S}\mu)^k\|_2 \leq \sum_{k\geq 0} (\varepsilon\|\mu\|_\infty)^k,
$$

which is a geometric series. Therefore, the series converges if $\varepsilon\|\mu\|_\infty < 1$. This implies that the radius of convergence of the series in ε is ≥ 1. In particular, for small ε,

$$
f_z = 1 + \varepsilon \mathbf{S}\mu + O(\varepsilon^2), \qquad \text{a.e. } z \in \mathbb{D}.
$$

This enables the following pointwise computation.

$$\begin{aligned}
\left(1+\frac{1}{\varepsilon}\right)\log|Df| &= \left(\frac{1+\varepsilon}{\varepsilon}\right)(\log(|f_z|+|f_{\bar z}|)) \\
&= \left(\frac{1+\varepsilon}{\varepsilon}\right)(\log(1+\varepsilon|\mu|)+\log|f_z|) \\
&= \left(\frac{1+\varepsilon}{\varepsilon}\right)(\varepsilon|\mu|+\mathrm{Re}\{\log f_z\}+O(\varepsilon^2)) \\
&= \left(\frac{1+\varepsilon}{\varepsilon}\right)(\varepsilon|\mu|+\mathrm{Re}\{f_z-1\}+O(\varepsilon^2)) \\
&= |\mu|+\mathrm{Re}\{\mathbf{S}\mu\}+O(\varepsilon).
\end{aligned}$$

Thus, after exponentiation,

$$|Df|^{1+\frac{1}{\varepsilon}} = \exp(|\mu|+\mathrm{Re}\{\mathbf{S}\mu\})+O(\varepsilon).$$

Finally, Fatou's lemma on the bound (5.10),

$$\begin{aligned}
\int_{\mathbb{D}}(1-|\mu(z)|)e^{|\mu(z)|}|\exp(\mathbf{S}\mu(z))|\,dz &= \int_{\mathbb{D}}(1-|\mu(z)|)e^{|\mu(z)|+\mathrm{Re}\{\mathbf{S}\mu\}}\,dz \\
&\leq \liminf_{\varepsilon\to 0}\int_{\mathbb{D}}\left(\frac{1-|\mu(z)|}{1+\varepsilon|\mu(z)|}\right)|Df(z)|^{1+\frac{1}{\varepsilon}}\,dz \leq \pi,
\end{aligned}$$

finalises the proof. $\square$

Chapter 6

Optimality of the results

6.1 Piecewise radial maps

In this last chapter, we introduce a class of fairly complicated functions that attain equality in Theorem 3.1, showing its sharpness. It is also possible to adapt this class of functions to show sharpness of the consequent corollaries that stem from it. These maps are not a new contribution of the paper in review, but they can be found in the earlier works [6] and [12]. Let us construct such maps.

First of all, we want to build a radial function $g : \mathbb{D}_R \to \mathbb{D}_R$. Consider

$$g(z) = \rho(|z|)\frac{z}{|z|}. \tag{6.1}$$

It remains to specify the real variable function ρ. Given $0 \leq r \leq R$, we demand

- $\rho : [0, R] \to [0, R]$, with $\rho(0) = 0$ and $\rho(R) = R$,
- ρ is absolutely continuous,
- ρ is strictly increasing,
- ρ is linear on $[0, r]$ and
- $\frac{\rho(t)}{t} \geq \dot{\rho}(t) \geq 0$, for $t \in (r, R)$, which we refer to as the "expanding assumption".

See Fig. 6.1 for a typical plot of a function respecting these properties. Given these hypotheses on ρ we aim to compute the Burkholder functional on g. Firstly, given the function in (6.1), it is easy to compute the complex partial derivatives of g.

$$g_z(z) = \frac{1}{2}\left(\dot{\rho}(|z|) + \frac{\rho(|z|)}{|z|})\right),$$
$$g_{\overline{z}}(z) = \frac{1}{2}\left(\dot{\rho}(|z|) - \frac{\rho(|z|)}{|z|})\right)\frac{z}{\overline{z}}.$$

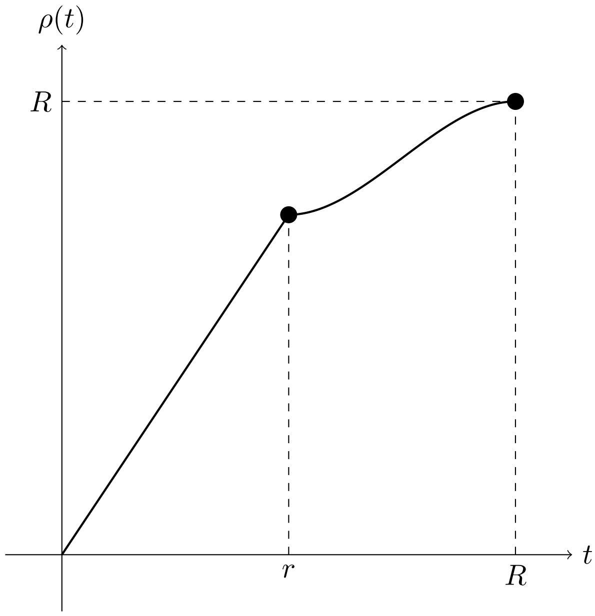


Figure 6.1: An example of how the profile of the function ρ could look like. It is linear in $[0, r]$, starts at $(0, 0)$ and ends at (R, R), it is absolutely continuous, strictly increasing and verifies the expanding assumption.

Now, from the complex partial derivatives representation on the Burkholder functional (2.1), we substitute and change to polar coordinates $|z| \to t$.

$$\begin{aligned}\int_{\mathbb{D}_R} \mathbf{B}_p(Dg)\,dz &= \int_{\mathbb{D}_R} (-p|g_{\bar{z}}| + |g_z| + |g_{\bar{z}}|)(|g_z| + |g_{\bar{z}}|)^{p-1}\,dz \\ &= 2\pi \int_0^R \frac{1}{2}\left(\left[p \cdot \dot{\rho}(t) + \frac{\rho(t)}{t}(p-2)\right] \frac{(\rho(t))^{p-1}}{t^{p-2}}\right) dt \\ &= \int_0^R \left(\frac{(\rho(t))^p}{t^{p-2}}\right)' dt = \pi \frac{(\rho(R))^p}{R^{p-2}} - \lim_{t \to 0^+} \pi \frac{(\rho(t))^p}{t^{p-2}} \\ &= \pi R^2 = |\mathbb{D}_R| = \int_{\mathbb{D}_R} \mathbf{B}_p(\mathrm{Id})\,dz,\end{aligned} \tag{6.2}$$

where we wrote the integrand as the derivative of some function and used the fundamental theorem of calculus. Also, notice that when substituting the derivatives in the Burkholder functional, we took into account the expanding assumption. Further, to compute the limit, we used that the function ρ is linear close to 0^+. Otherwise, if $r = 0$, we would need to further assume that ρ decays fast enough when $t \to 0^+$, namely, $\rho(t) = o(t^{1-\frac{2}{p}})$ as $t \to 0$. We have just built a radial function with identity boundary values on $\mathbb{D}_R$ such that if it perturbs the identity function the Burkholder integral remains the same. Let us reproduce this building block to achieve a much richer class of functions with this property.

Take a complex linear map $f_0(z) = az + b$, $a, b \in \mathbb{C}$ on a bounded domain Ω, and as before consider two radii $0 \le r \le R$ and a centre $z_0 \in \Omega$ such that the larger ball fits inside the domain, $\mathbb{D}_R(z_0) \subset \Omega$. Now, in that ball, we modify the function f_0 using the radial mapping (6.1). Define

$$f_1(z) = \begin{cases} f_0(z) & \text{if } z \notin \mathbb{D}_R(z_0), \\ ag(z - z_0) + (az_0 + b) & \text{if } z \in \mathbb{D}_R(z_0). \end{cases} \tag{6.3}$$

Observe that f_1 is continuous, including at the boundary of $\mathbb{D}_R(z_0)$ thanks to $\rho(R) = R$. In the selected disk, it holds that

$$\int_{\mathbb{D}_R(z_0)} \mathbf{B}_p(Df_1)\, dz = \int_{\mathbb{D}_R(z_0)} \mathbf{B}_p(Df_0)\, dz,$$

just because of a scaled version of (6.2). Of course, outside such disk $f_0 = f_1$, therefore

$$\int_\Omega \mathbf{B}_p(Df_1)\, dz = \int_\Omega \mathbf{B}_p(Df_0)\, dz,$$

and the Burkholder integral is not altered by this surgery. Hence, we can argue inductively to keep on modifying disks of an initial complex linear function while preserving the Burkholder integral. We have to be careful with the overlapping of the modified disks. However, in $\mathbb{D}_r \subset \mathbb{D}_R$ where the function is “radially linear”, we are allowed to insert balls that fit inside $\mathbb{D}_r$, during our inductive process. This is, we are rather nonlinearising annuli and radially linearising the inner disks. We can for sure choose different values for $0 \le r < R$ at each iteration step. Fig. 6.2 depicts how such domains look like.

Consider that we built a sequence of maps $(f_n)_{n\in\mathbb{N}}$ on top of an initial complex linear mapping $f_0(z) = az + b$, inductively with the previous procedure. By induction,

$$\int_\Omega \mathbf{B}_p(Df_n)\, dz = a^p|\Omega|, \qquad \forall\, n \in \mathbb{N}. \tag{6.4}$$

We may also consider not only the elements in the sequence f_n but also its limit, understood in some suitable space of maps.

Definition 6.1 *Let $p \ge 1$ and let $\Omega \subset \mathbb{C}$ be any nonempty bounded domain. The class $\mathcal{A}^p(\Omega)$ consists of piecewise radial mappings $f : \Omega \to \Omega$ obtained from the previous construction, starting with $f_0(z) = z$, and, in the case of being a limit map of a sequence, the convergence $f = \lim_{n\to\infty} f_n$ takes place in $W^{1,p}(\Omega)$ and the expanding assumption $\rho(t)/t \ge \dot\rho(t) \ge 0$, for $t \in (r, R)$ is in force.*

Proposition 6.2 *For any $p \ge 1$ and $f \in \mathcal{A}^p(\Omega)$ we have*

$$\int_\Omega \mathbf{B}_p(Df)\, dz = |\Omega|. \tag{6.5}$$

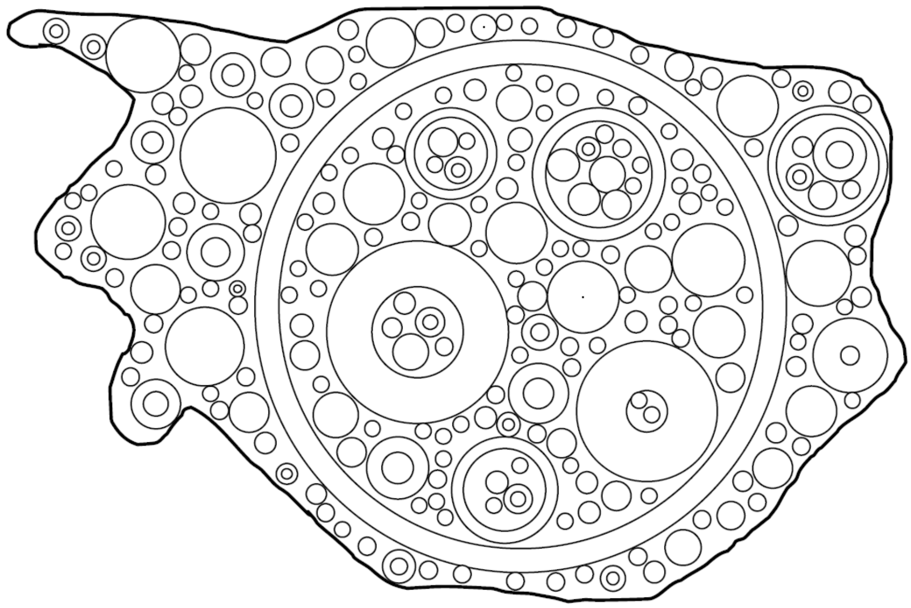


Figure 6.2: Annular packing for a domain of an initial linear function. Inside each annulus, the function is nonlinear and radial, according to ρ. Elsewhere, the function is linear or radially linear (although it may not be linear according to the same function). This construction preserves the Burkholder integrals. Picture from the article in consideration [5].

The veracity of this proposition is clear from (6.5) and the convergence in $W^{1,p}(\Omega)$. Hence, we discovered that the Burkholder functional is null-Lagrangian in the directions of functions in $\mathcal{A}^p(\Omega)$.

6.2 Optimal classes of maps

From Proposition 6.2 we already see why this class proves sharpness for most of the discussed results. However, we may need to restrict such class in order to adapt to the hypotheses of each result. This is not a major task, so let us go through some of the points to take into account. For example, in the setting of $\mathbf{B}_p(Df) \geq 0$, we need to assume an extra condition on the class $\mathcal{A}^p(\Omega)$ in order for such maps to respect the positivity of the Burkholder integrand. Having a glance at the computation in (6.2), we find the new condition, which altogether with the expanding assumption reads as

$$\frac{\rho(t)}{t} \geq \dot{\rho}(t) \geq \left(1 - \frac{2}{p}\right)\frac{\rho(t)}{t}. \tag{6.6}$$

On the other hand, paying attention to Theorem 3.2, since the mapping f is required to be smooth, the class of functions that attain equality is restricted to the subclass satisfying (6.6) and such that ρ is a smooth function, with smooth extension by the identity.

When looking at the case $p < 1$, in order to make the computation in (6.2) work out, one needs to replace the expanding assumption by the"compressing assumption", swapping the inequality:

$$\frac{\rho(t)}{t} \leq \dot{\rho}(t), \qquad \text{for } t \in (r, R),$$

as well as the condition $\lim_{t\to 0^+} \rho(t)/t^{1-2/p} = \infty$ in case we chose $r = 0$ and $p < 0$, to make the limit in the computation be 0. Besides this, if we let the iterative construction go to infinity and consider its limit, we need to make sure the limit makes sense, since it is not clear anymore which Banach space to pick when $p < 1$.

From what we constructed, we get the following corollary regarding maximiser functions of the Burkholder functional.

Corollary 6.3 *The Burkholder functional acting on $C^1_{id}(\Omega)$ functions ($C^1(\Omega)$ functions with identity boundary values outside a compact) attains its local maximum at every C^1-smooth piecewise radial map in $\mathcal{A}^p(\Omega)$ for which condition* (6.6) *is reinforced to:*

$$\frac{\rho(t)}{t} \geq \dot{\rho}(t) \geq \frac{1}{K}\frac{\rho(t)}{t}, \qquad K < \frac{p}{p-2}. \tag{6.7}$$

This corollary follows from checking that functions under such conditions are K-quasiconformal (reuse the first lines in computation (6.2) to check), then apply Theorem 3.3 for the upper bound and Proposition 6.2 to show that this class of functions indeed maximises.

As one can see, by correctly adapting the class $\mathcal{A}^p(\Omega)$ to the statements, optimality for such subclasses follows. Let us emphasise on Corollary 3.5, since in the statement there appears no map of the quasiconformal kind, but rather just a dilation function μ. The following lemma characterises an optimal class of functions μ for such corollary.

Lemma 6.4 *Let $\alpha : (0, 1) \to [0, 1]$ be measurable and with the property*

$$\int_0^1 \frac{1-\alpha(t)}{t}\, dt = \infty. \tag{6.8}$$

Set

$$\mu(z) = -\frac{z}{\bar{z}}\alpha(|z|) \quad \textit{for } |z| < 1, \qquad \mu(z) = 0 \quad \textit{for } |z| \geq 1. \tag{6.9}$$

Then there is equality in Corollary 3.5, i.e.

$$\int_{\mathbb{D}} (1 - |\mu(z)|)e^{|\mu(z)|}|\exp(\mathbf{S}\mu(z)|\, dz = \pi. \tag{6.10}$$

Proof Let $\varphi(z) = 2z\int_{|z|}^{1} \frac{\alpha(t)}{t}\,dt$ for $|z| < 1$ and set $\varphi(z) = 0$ elsewhere. By computing the complex derivatives of φ, we check that it is the "primitive" of two quantities of interest in $\mathbb{D}$:

$$\varphi_z(z) = -\frac{z}{\overline{z}}\alpha(|z|) = \mu(z),$$
$$\varphi_{\overline{z}}(z) = 2\int_{|z|}^{1} \frac{\alpha(t)}{t}\,dt - \alpha(|z|) = \mathbf{S}\mu(z).$$

Hence, we can substitute these expressions in the integral (6.10), and compute it in polar coordinates.

$$\int_{\mathbb{D}} (1 - |\mu(z)|)e^{|\mu(z)|}|\exp(\mathbf{S}\mu(z)|\,dz$$
$$= \int_0^1 2\pi t(1-\alpha(t))\exp\left(\alpha(t) + 2\int_s^1 \frac{\alpha(s)}{s}\,ds - \alpha(t)\right)dt = \pi,$$

where we identified the derivative

$$\frac{d}{dt}\left(t^2 \exp\left[2\int_t^1 \frac{\alpha(s)}{s}\,ds\right]\right) = 2t(1-\alpha(t))\exp\left(2\int_t^1 \frac{\alpha(s)}{s}\,ds\right).$$

When evaluating at the integral limits, at $t = 1$ we get 1 for the derived function on the left hand side, whereas for the limit at $t \to 0^+$, we can rewrite the function making t^2 hop to an exponent inside a logarithm, write it in terms of an integral, and then the null limit follows from assumption (6.8). $\square$

Lastly, let us revise the proof of Corollary 3.6 to find optimisers. According to the lower bound for the weight, equality is attained if and only if $|\mu(z)| = k$ almost everywhere; this is, $K(z,f) = K$ almost everywhere in Ω. For functions of the shape (6.1), such condition is fulfilled, for example, with the choice of radial functions

$$\rho_K(t) = R^{1-\frac{1}{K}}t^{\frac{1}{K}}, \quad \text{for } r < t < R,$$

and extending it linearly up to the origin on $[0, r]$. With these radial functions, we can give birth to a class of maps $\mathcal{A}_K^p(\Omega)$, for $2 \le p < \frac{2K}{K-1}$, by annuli packing of the domain. It is important though, to require that there is no radially linear subdisk in the domain otherwise in that subdisk $K(z,f) = K$ a.e. is violated. Just as before, equality in Theorem 3.1 is attained for the class $\mathcal{A}_K^p(\Omega)$, and thus, as a consequence of the proof of the corollary and Proposition 6.2, the class $\mathcal{A}_K^p(\Omega)$ attains equality in (3.8).

Chapter 7

Conclusions

The paper we reviewed is both innovative and satisfactory. It introduces new mathematical tools as well as showing that results are sharp under the assumptions. Who knows if the tools they developed may come in handy in the quest of showing the full quasiconvexity of the Burkholder functional. However, this famous functional and Morrey's conjecture leave room for other mathematical challenges.

Although all the study was set in two dimensions, one can of course wonder about the higher dimensional case. The techniques used to prove the presented results do not apply in such expanded scenario, since they are heavily relying on the complex plane structure. We need new machinery to tackle objects and problems like the following.

Definition 7.1 *The Burkholder functional acting on $n \times n$ matrices may be defined as follows.* $\mathbf{B}_p^n : \mathbb{R}^{n\times n} \to \mathbb{R}$,

$$\mathbf{B}_p^n = \left(\frac{p}{n}\det A + \left(1 - \frac{p}{n}\right)|A|^n\right)|A|^{p-n}, \qquad p \geq n. \tag{7.1}$$

These higher dimensional functionals are already known to be rank-one convex [12], but it is not known whether they are quasiconvex. Getting inspiration from the exposed results in two dimensions, the authors of the paper conjecture the following sharp bounds in higher dimensions.

Conjecture 7.2 *Suppose $f : \mathbb{B} \to \mathbb{B}$ is a K-quasiconformal mapping of the unit n-dimensional ball onto itself that is equal to the identity on $\partial\mathbb{B}$. Then,*

$$⨍_{\mathbb{B}} \left(n - p + \frac{p}{K(x)}\right)|Df(x)|^p\,dx \leq n, \qquad \textit{for } n \leq p \leq \frac{nK}{K-1}. \tag{7.2}$$

Therefore,

$$⨍_{\mathbb{B}} |Df(x)|^p\,dx \leq \frac{nK}{nK - p(K-1)}. \tag{7.3}$$

There are also sharp estimates with the critical upper exponent $p = \frac{nK}{K-1}$,

$$\fint_{\mathbb{B}} \left(\frac{1}{K(x)} - \frac{1}{K} \right) |Df(x)|^{\frac{nK}{K-1}} \, dx \leq 1 - \frac{1}{K}. \tag{7.4}$$

Hence

$$\int_{\mathbb{E}} |Df(x)|^{\frac{nK}{K-1}} \, dx \leq |\mathbb{B}|, \qquad \textit{where } \mathbb{E} = \{x \in \mathbb{B} : K(x) = 1\}. \tag{7.5}$$

All the above equalities are sharp; n-dimensional radial and power mappings produce examples for the equalities.

Up to the date of this report, Conjecture 7.2 remains open.

We remark that the authors of the paper already realised and noted about the potential of the new techniques they created, as they comment a bit before they introduce their new interpolation lemma. Indeed, the paper cleared the path for further research. For instance, we refer to the very recent paper [1] on the quasiconvexity of the local Burkholder functional.

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